AMath 584 / Math 584
Homework #1
Due 11:00pm PDT
Tuesday, October 11, 2016

Name: Your Name Here

Netid: Your NetID Here

Problem 1. Exercise 1.1 in Trefethen and Bau. Note for example that part (a) is accomplished by multiplying B on the right by the matrix

$$C_1 = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

You can write out analogous matrices for each step and then write the required product of these matrices. Make sure to write them *in the correct order.*

Problem 2. Exercise 2.1 in Trefethen and Bau.

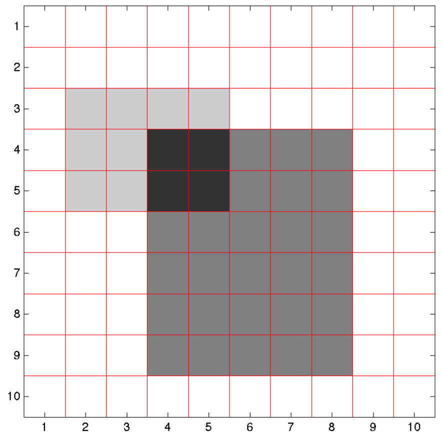
Problem 3. Exercise 2.6 in Trefethen and Bau. In addition, please note the value that α takes in the special case when u and v are orthogonal.

Problem 4. Exercises 3.1, 3.3, and 3.4 in Trefethen and Bau.

Problem 5.

Linear algebra is widely used in image processing. An image that appears on a computer screen, for example, is typically represented as an array of pixels that could be viewed as forming a matrix. If it's a color image then there are typically three arrays (e.g. for the red, green, and blue channels (`[R,G,B]` to describe an arbitrary color) but for simplicity we will consinder only a grayscale image, and assume that a single value ranges from 0 for white to 1 for black. The array below might then represent a 10×10 image with 100 pixels as shown in the following figure.

```
0         0         0         0         0         0         0         0         0         0
0         0         0         0         0         0         0         0         0         0
0    0.3000    0.3000    0.3000    0.3000         0         0         0         0         0
0    0.3000    0.3000    0.9000    0.9000    0.6000    0.6000    0.6000         0         0
0    0.3000    0.3000    0.9000    0.9000    0.6000    0.6000    0.6000         0         0
0         0         0    0.6000    0.6000    0.6000    0.6000    0.6000         0         0
0         0         0    0.6000    0.6000    0.6000    0.6000    0.6000         0         0
0         0         0    0.6000    0.6000    0.6000    0.6000    0.6000         0         0
0         0         0    0.6000    0.6000    0.6000    0.6000    0.6000         0         0
0         0         0         0         0         0         0         0         0         0
```



In Matlab, if C is defined to be the 10×10 matrix above, then the commands below will produce this plot (see the end of the assignment for a Python version):

```
cmap = linspace(1,0,11)' * [1 1 1];    % define grayscale colormap
figure
colormap(cmap)     % use gray scale colormap
image(11*C)        % plot image (indexing into colormap of length 11)

hold on    % now add grid lines for clarity...
x = linspace(0.5,10.5,11);
y = x;
[X,Y] = meshgrid(x,y)
plot(X',Y','r')    % plot horizontal grid lines
plot(X,Y,'r')      % plot vertical grid lines
axis square        % scale x and y axis equally
hold off
```

Note that we scale C by 11 to index into the colormap which is defined here as an 11×3 matrix with `[R,G,B]` values corresponding to 11 shades of gray. (Note that this matrix has been specified as an outer product of a column vector with 11 components and a row vector with 3 components. Print these things out if you are not sure what this matrix looks like.)

The matrix C given above can also be specified fairly easily as a combination of two outer products, $C = 0.3a_1b_1^T + 0.6a_2b_2^T$ where for example $a_1 = [0,0,1,1,1,0,0,0,0,0]^T$ and $b_1 = [0,1,1,1,1,0,0,0,0,0]^T$.

Problem 5a. Compute the matrix $A_1 = a_1b_1^T$ and produce a plot of the image that this describes.

Problem 5b. Determine the vectors a_2, b_2 and the matrix $A_2 = a_2b_2^T$ so that the matrix above can be written as $A = 0.3A_1 + 0.6A_2$.

Note that it takes much less storage to store the image in terms of the vectors forming these outer products than it would to store every element of the matrix. This would be more dramatic if the image were say 1000×1000 (1 megapixel, with 1,000,000 matrix elements) rather than only 10×10. This is a form of image compression.

This works so well because the matrix for this image is a rank 2 matrix and so only 2 outer products are needed. If it were a full rank matrix then this compression might not work so well. But images generally aren't random pixels. Instead they have a lot of structure and so it is sometimes possible to approximate an image well with a low rank approximation.

The singular value decomposition gives a way of writing any matrix as a linear combination of rank one matrices (outer products) and if the singular values decay rapidly (which they often do for real images) then we may be able to truncate the SVD to relatively few terms.

Problem 5c. Use Matlab or Python to determine the SVD of the matrix A above and confirm that only two singular values are nonzero. With this decomposition we see that we can express C as $C = \sigma_1 u_1 v_1^* + \sigma_2 u_2 v_2^*$. Confirm that this is true and print out the singular values and vectors used in this sum, and also the two rank-1 matrices that appear in this sum separately.

Problem 5d. Plot the image given by the matrix $\sigma_1 u_1 v_1^*$. This is the best approximation to the matrix C by a rank-1 matrix.

Problem 5e. Confirm that u_1 and u_2 are orthonormal vectors (whereas a_1 and a_2 were not). In fact u_1 and u_2 form an orthonormal basis for the 2-dimensional space span(a_1, a_2). To check this: In Matlab (or Python), form the matrix with columns $[u_1|u_2|a_1|a_2]$ and confirm that it has rank 2. Explain why this shows that the vectors u_1 and u_2 span the same subspace of $\mathbb{C}^2$ as that spanned by a_1 and a_2.

Note: The SVD is not the best approach for image compression, and the point of this example is primarily to get familiar with the SVD and the idea of low-rank representations of a matrix.

See `https://www.mathworks.com/company/newsletters/articles/professor-svd.html` for an introductory article by Cleve Moler (inventor of Matlab) on the SVD that also shows an example of using it to compress an image of Gene Golub, who did much to popularize the use of the SVD for a wide range of applications.

Plotting an image in Python:

This produces a similar plot to the one shown above, but flipped vertically and also recall that Python indexing starts at 0.

```
imshow(C, cmap='Greys', interpolation='none')
x = linspace(-0.5,9.5,11)
y = x
[X,Y] = meshgrid(x,y)
plot(X.T,Y.T,'r');
plot(X,Y,'r');
axis('scaled')
axis([-0.5, 9.5, -0.5, 9.5])
```