Risk Parity Model

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大綱

- 前言
- 模型推導
- 實作
- 結論

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前言

Risk Parity Model

- 著重在風險的配置（ allocation of risk ），而非資本配置

	風險配置	資產配置
目標	使投資組合內個別資產風險貢獻達到某一目標水準	在某一預期報酬率，極小化投資組合波動度

Markowitz model : Global Minimum Variance Portfolio

- **Markowitz model**

$$\min_{w} \quad \boldsymbol{w}^{\boldsymbol{T}}\Sigma\boldsymbol{w}$$

$$\text{subject to} \quad \boldsymbol{1}^{\boldsymbol{T}}\boldsymbol{w} = 1$$
$$\boldsymbol{w}^{\boldsymbol{T}}E(r) = \mu$$

- **Risk Parity**

$$\boldsymbol{w}_{\boldsymbol{i}}(\Sigma\boldsymbol{W})_{\boldsymbol{i}} = \frac{\boldsymbol{1}}{\boldsymbol{N}} * \boldsymbol{W}'\Sigma\boldsymbol{W}$$

缺點

- 只考慮極小化整個portfolio的風險
- portfolio風險來源可能只來自少數資產

Risk Parity:
“Leverage Aversion and Risk Parity”, 2012

股債比 60/40 基金

風險分散？

88%的風險都來自於股票

市值權重投資組合

1926年-2010年
股、債之平均權重分別為
68%、32%

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模型推導

Homogeneous function

<u>Def</u> If $f(x,y)$ is a homogeneous function of n, then we have:

$$f(tx,ty) = t^n * f(x,y)$$

- Now we define the portfolio variance as $Var(w_1,\dots,w_N) = W'\Sigma W$
  - Since $Var(tw_1,\dots,tw_N) = t^2 * Var(w_1,\dots,w_N)$
  - $Var(w_1,\dots,w_N)$ is a homogeneous function of n = 2

- 為了方便計算，我們設計了 $\sigma(w_1,\dots,w_N) = \sqrt{\mathrm{W}'\Sigma\mathrm{W}}$
  - Since $\sigma(tw_1,\dots,tw_N) = t * \sigma(w_1,\dots,w_N)$
  - $\sigma(w_1,\dots,w_N)$ is a homogeneous function of n = 1

Homogeneous function

$f(tx,ty) = t^n * f(x,y)$ ⋯ 將兩邊對t取偏微分

$$n * t^{n-1} * f(x,y) = \frac{\partial f}{\partial tx} * \frac{\partial tx}{\partial t} + \frac{\partial f}{\partial ty} * \frac{\partial ty}{\partial t} = x * \frac{\partial f}{\partial tx} + y * \frac{\partial f}{\partial ty}$$

- 則當$t = 1$ & $n = 1$時，$f(x,y) = x * \frac{\partial f}{\partial x} + y * \frac{\partial f}{\partial y}$
- 因此我們可以將$\sigma(w_1, \ldots, w_N) = \sqrt{\mathrm{W}'\Sigma\mathrm{W}}$ 改寫為:

$$\sigma(\mathrm{W}) = \sum_{i=1}^{N} w_i * \frac{\partial \sigma}{\partial w_i}$$

Risk Parity Model

- $\sigma(\mathrm{W}) = \sqrt{\mathrm{W}'\Sigma\mathrm{W}} = \sum_{i=1}^{N} w_i * \frac{\partial\sigma}{\partial w_i}$ …因為Σ是symmetric
  - $\frac{\partial\sigma}{\partial w_i} = \frac{1}{2} * (W'\Sigma W)^{-0.5} * 2(\Sigma W)_i$
- 因此$\sigma(\mathrm{W})$可以改寫為:

$$\sigma(\mathrm{W}) = \sum_{i=1}^{N} \frac{w_i(\Sigma W)_i}{\sqrt{W'\Sigma W}}$$

- 可以發現投資組合的波動度可以被拆分為個別資產的風險貢獻之和

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$$\sigma(\mathrm{W}) = \sum_{i=1}^{N} \frac{w_i(\Sigma W)_i}{\sqrt{W'\Sigma W}}$$

Def Risk Contribution（風險貢獻）

$$\mathrm{RC_i} = \frac{w_i(\Sigma W)_i}{\sqrt{W'\Sigma W}}$$

$$\sum_{i=1}^{N} \mathrm{RC_i} = \sum_{i=1}^{N} \frac{w_i(\Sigma W)_i}{\sqrt{W'\Sigma W}} = \sigma(\mathrm{W})$$

Risk Parity Model

Def Relative Risk Contribution

$$b_i = \frac{w_i(\Sigma W)_i}{W'\Sigma W}$$

$$\sum_{i=1}^{N} b_i = \sum_{i=1}^{N} w_i(\Sigma W)_i * \frac{1}{W'\Sigma W} = 1$$

註: $\sum_{i=1}^{N} \mathrm{RC_i} = \sum_{i=1}^{N} \frac{w_i(\Sigma W)_i}{\sqrt{W'\Sigma W}} = \sigma(\mathrm{W}) = \frac{W'\Sigma W}{\sqrt{W'\Sigma W}} = \sqrt{W'\Sigma W}$

Risk Parity Model

$$b_i = \frac{w_i(\Sigma W)_i}{W'\Sigma W}$$

- Risk Parity的核心: **使個別資產的風險貢獻相同**

$$b_i = b_j = \frac{1}{N} \quad for\ i \neq j$$

- 我們也可以指定要分派給各個資產的風險貢獻:
  - Example: 有3個資產，let $b_1 = 0.3, b_2 = 0.5, b_3 = 0.2$
  - 註: $\sum_{i=1}^{N} \mathrm{b_i} = 1$

Risk Parity Model

$$b_i = \frac{w_i(\Sigma W)_i}{W'\Sigma W}\ , b_i * W'\Sigma W = w_i(\Sigma W)_i$$

- In general, our goal is to let $\boldsymbol{w_i(\Sigma W)_i} = b_i * \boldsymbol{W'\Sigma W}\quad for\ i = 1, \ldots, N$
  - 註1: b_i是研究者自行給定的
  - 註2:$b_i = \frac{1}{N}$時，代表我們希望每個資產的風險貢獻相同

Risk Parity Model

Goal: $w_i(\Sigma \boldsymbol{w})_i = b_i * \boldsymbol{w}'\Sigma \boldsymbol{w} \quad for\ i = 1, \ldots, N$

$s.t. \quad \boldsymbol{w} \geq \boldsymbol{0}, \quad \boldsymbol{1}'\boldsymbol{w} = 1$

➢ 變數變換，令$\boldsymbol{x} = \frac{\boldsymbol{w}}{\sqrt{\boldsymbol{w}'\Sigma \boldsymbol{w}}}, \quad i.e., \quad x_i = \frac{w_i}{\sqrt{\boldsymbol{w}'\Sigma \boldsymbol{w}}}$

1) $\because \boldsymbol{x} = \frac{\boldsymbol{w}}{\sqrt{\boldsymbol{w}'\Sigma \boldsymbol{w}}} \quad \Rightarrow \quad \Sigma \boldsymbol{x} = \frac{\Sigma \boldsymbol{w}}{\sqrt{\boldsymbol{w}'\Sigma \boldsymbol{w}}}, \quad \therefore (\Sigma \boldsymbol{x})_i = \frac{(\Sigma \boldsymbol{w})_i}{\sqrt{\boldsymbol{w}'\Sigma \boldsymbol{w}}}$

Risk Parity Model

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2) $w_i(\Sigma\boldsymbol{w})_i = b_i * \boldsymbol{w}'\Sigma\boldsymbol{w} \quad \Rightarrow \quad \frac{w_i(\Sigma\boldsymbol{w})_i}{\boldsymbol{w}'\Sigma\boldsymbol{w}} = b_i \quad \Rightarrow \quad \frac{w_i}{\sqrt{\boldsymbol{w}'\Sigma\boldsymbol{w}}}\frac{(\Sigma\boldsymbol{w})_i}{\sqrt{\boldsymbol{w}'\Sigma\boldsymbol{w}}} = b_i$

Risk Parity Model

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2) $w_i(\Sigma \boldsymbol{w})_i = b_i * \boldsymbol{w}'\Sigma \boldsymbol{w} \Rightarrow \frac{w_i(\Sigma \boldsymbol{w})_i}{\boldsymbol{w}'\Sigma \boldsymbol{w}} = b_i \Rightarrow \frac{w_i}{\sqrt{\boldsymbol{w}'\Sigma \boldsymbol{w}}} \frac{(\Sigma \boldsymbol{w})_i}{\sqrt{\boldsymbol{w}'\Sigma \boldsymbol{w}}} = b_i$

$\Rightarrow x_i(\Sigma \boldsymbol{x})_i = b_i \Rightarrow (\Sigma \boldsymbol{x})_i = \frac{b_i}{x_i}, \quad or\ in\ vector\ form: \Sigma \boldsymbol{x} = \frac{\boldsymbol{b}}{\boldsymbol{x}}$

Risk Parity Model

- 限制式 $\boldsymbol{w} \geq \mathbf{0}, \quad \mathbf{1}'\boldsymbol{w} = 1$ 亦需轉換

Risk Parity Model

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1) $\because x_i = \dfrac{w_i}{\sqrt{\boldsymbol{w}'\Sigma\boldsymbol{w}}}$, $\therefore \boldsymbol{1}'\boldsymbol{x} = \sum_{i=1}^{n} x_i = \dfrac{\sum_{i=1}^{n} w_i}{\sqrt{\boldsymbol{w}'\Sigma\boldsymbol{w}}} = \dfrac{1}{\sqrt{\boldsymbol{w}'\Sigma\boldsymbol{w}}}$

$\Rightarrow x_i = w_i * \boldsymbol{1}'\boldsymbol{x} \Rightarrow \boldsymbol{w} = \dfrac{\boldsymbol{x}}{\boldsymbol{1}'\boldsymbol{x}}$

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2) $\boldsymbol{w} = \frac{\boldsymbol{x}}{\mathbf{1}'\boldsymbol{x}} \geq \mathbf{0} \quad \Rightarrow \quad \boldsymbol{x} \geq \mathbf{0}$

Note: $\mathbf{1}'\boldsymbol{w} = \frac{\mathbf{1}'\boldsymbol{x}}{\mathbf{1}'\boldsymbol{x}} = 1$

Risk Parity Model

- 限制式 $\boldsymbol{w} \geq \mathbf{0}$, $\mathbf{1}'\boldsymbol{w} = 1$ 亦需轉換

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     $\Rightarrow x_i = w_i * \mathbf{1}'\boldsymbol{x} \Rightarrow \boldsymbol{w} = \dfrac{\boldsymbol{x}}{\mathbf{1}'\boldsymbol{x}}$

  2) $\boldsymbol{w} = \dfrac{\boldsymbol{x}}{\mathbf{1}'\boldsymbol{x}} \geq \mathbf{0} \Rightarrow \boldsymbol{x} \geq \mathbf{0}$

     Note: $\mathbf{1}'\boldsymbol{w} = \dfrac{\mathbf{1}'\boldsymbol{x}}{\mathbf{1}'\boldsymbol{x}} = 1$

- 經過變數變換後，新目標式： $\sum \boldsymbol{x} = \dfrac{\boldsymbol{b}}{\boldsymbol{x}}$, $s.t.$ $\boldsymbol{x} \geq \mathbf{0}$

Risk Parity Model

New Goal: $\sum \boldsymbol{x} = \frac{\boldsymbol{b}}{\boldsymbol{x}}, \quad s.t. \quad \boldsymbol{x} \geq \boldsymbol{0}$

- 上述目標式仍無從解起，因此設計出下列 convex optimization problem:

$$\min_{\boldsymbol{x} \geq \boldsymbol{0}} \frac{1}{2} \boldsymbol{x}' \sum \boldsymbol{x} - \sum_{i=1}^{n} b_i \log x_i$$

Risk Parity Model

New Goal: $\sum \boldsymbol{x} = \frac{\boldsymbol{b}}{\boldsymbol{x}}, \quad s.t. \quad \boldsymbol{x} \geq \boldsymbol{0}$

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$$\min_{\boldsymbol{x} \geq \boldsymbol{0}} \frac{1}{2} \boldsymbol{x}' \sum \boldsymbol{x} - \sum_{i=1}^{n} b_i \log x_i$$

- **原因：**

$$\text{Let} \quad f(x) = \frac{1}{2} \boldsymbol{x}' \sum \boldsymbol{x} - \boldsymbol{b}' \log \boldsymbol{x}$$

$$\Rightarrow \quad f'(x) = \sum \boldsymbol{x} - \frac{\boldsymbol{b}}{\boldsymbol{x}} = \boldsymbol{0} \quad \Rightarrow \quad \sum \boldsymbol{x} = \frac{\boldsymbol{b}}{\boldsymbol{x}}, \quad \textit{equal to our new goal!}$$

Risk Parity Model

New Goal: $\sum \boldsymbol{x} = \frac{\boldsymbol{b}}{\boldsymbol{x}}, \quad s.t. \quad \boldsymbol{x} \geq \boldsymbol{0}$

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- 若 $f(x)$ 存在 global minimum，則最小化 $f(x)$ 的解，同等於目標式的解
- 接著需證明 $f(x)$ 確實存在 global minimum

Risk Parity Model

證明 $f(x)$ 存在 global minimum.

- Step 1: We prove that $f(x)$ $is\ strictly\ convex.\quad If\ it's\ done,$ $then\ by\ definition,\ f(x)\ has\ at\ most\ one\ critical\ point.$

- Step 2: We prove that $f(x)$ $has\ at\ least\ one\ critical\ point.$

Risk Parity Model

證明 $f(x)$ 存在 global minimum.

- Step 1: We prove that $f(x)$ *is strictly convex*. *If it's done, then by definition,* $f(x)$ *has at most one critical point.*

  1) $f(x) = \frac{1}{2}\boldsymbol{x}'\sum\boldsymbol{x} - \boldsymbol{b}'\log\boldsymbol{x}$

  $\Rightarrow \quad f'(x) = \sum\boldsymbol{x} - \frac{\boldsymbol{b}}{\boldsymbol{x}}$

  $\Rightarrow \quad f''(x) = \sum + diag(\frac{\boldsymbol{b}}{\boldsymbol{x}'\boldsymbol{x}})$

Risk Parity Model

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1) $f(x) = \frac{1}{2}\boldsymbol{x}'\sum\boldsymbol{x} - \boldsymbol{b}'\log\boldsymbol{x}$

$\Rightarrow \quad f'(x) = \sum\boldsymbol{x} - \frac{\boldsymbol{b}}{\boldsymbol{x}}$

$\Rightarrow \quad f''(x) = \sum + diag(\frac{\boldsymbol{b}}{\boldsymbol{x}'\boldsymbol{x}})$

2) $\sum = E[\,(\boldsymbol{z} - E(\boldsymbol{z}))(\boldsymbol{z} - E(\boldsymbol{z}))'\,]$

$Let \quad \boldsymbol{a} = \boldsymbol{z} - E(\boldsymbol{z}), \quad \because \boldsymbol{aa}' \text{ } is \text{ } P.S.D, \quad \therefore E(\boldsymbol{aa}') = \sum \text{ } is \text{ } also \text{ } P.S.D$

Risk Parity Model

證明 $f(x)$ 存在 global minimum.

- Step 1: We prove that $f(x)$ *is strictly convex*. *If it's done, then by definition,* $f(x)$ *has at most one critical point.*

  3) $\because \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}, \quad \therefore diag\left(\frac{\boldsymbol{b}}{\boldsymbol{x}'\boldsymbol{x}}\right)$ *is* $P.D$

Risk Parity Model

證明 $f(x)$ 存在 global minimum.

- Step 1: We prove that $f(x)$ *is strictly convex*. *If it's done, then by definition,* $f(x)$ *has at most one critical point.*

3) $\because \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}, \quad \therefore diag\left(\frac{\boldsymbol{b}}{\boldsymbol{x}'\boldsymbol{x}}\right)$ *is P.D*

4) Note that $f''(x) = \sum + diag(\frac{\boldsymbol{b}}{\boldsymbol{x}'\boldsymbol{x}})$

$\therefore P.S.D\ matrix + P.D\ matrix = P.D\ matrix,$

$\therefore f(x)\ is\ strictly\ convex \quad \Rightarrow \quad f(x)\ has\ at\ most\ one\ critical\ point.$

Risk Parity Model

證明 $f(x)$ 存在 global minimum.

- Step 2: We prove that $f(x)$ *has at least one critical point.*

1) $f(x) = \frac{1}{2}\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x} - \sum_{i=1}^{n} b_i \log(x_i)$

2) *Assumption:* $\boldsymbol{\Sigma}$ *is full column rank,* *then* $\boldsymbol{\Sigma}$ *is* $P.D$ *matrix.*

Risk Parity Model

證明 $f(x)$ 存在 global minimum.

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1) $f(x) = \frac{1}{2}\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x} - \sum_{i=1}^{n} b_i \log(x_i)$

2) *Assumption:* $\boldsymbol{\Sigma}$ *is full column rank, then* $\boldsymbol{\Sigma}$ *is P.D matrix.*
Let λ_1 *be the lowest eigenvalue of* $\boldsymbol{\Sigma}$.
By assumption, $\lambda_1 > 0$, *we have* $\boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x} \geq \lambda_1 \sum_{i=1}^{n} x_i^2$

Lemma 1.8

Definition: $\boldsymbol{\Sigma}\boldsymbol{x} = \lambda\boldsymbol{x}$
$\Rightarrow \boldsymbol{x}'\boldsymbol{\Sigma}\boldsymbol{x} = \lambda\boldsymbol{x}'\boldsymbol{x}$

Risk Parity Model

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By assumption, $\lambda_1 > 0$, *we have* $\boldsymbol{x}'\boldsymbol{\Sigma x} \geq \lambda_1 \sum_{i=1}^{n} x_i^2$

Lemma 1.8

$\Rightarrow \frac{1}{2}\boldsymbol{x}'\boldsymbol{\Sigma x} \geq \frac{1}{2}\lambda_1 \sum_{i=1}^{n} x_i^2 \Rightarrow f(x) \geq \frac{1}{2}\lambda_1 \sum_{i=1}^{n} x_i^2 - \sum_{i=1}^{n} b_i \log(x_i)$

$\Rightarrow f(x) \geq \sum_{i=1}^{n}\left(\frac{1}{2}\lambda_1 x_i^2 - b_i \log(x_i)\right) = \sum_{i=1}^{n} g_i(x_i)$

Definition: $\boldsymbol{\Sigma x} = \lambda \boldsymbol{x}$

$\Rightarrow \boldsymbol{x}'\boldsymbol{\Sigma x} = \lambda \boldsymbol{x}'\boldsymbol{x}$

Risk Parity Model

證明 $f(x)$ 存在 global minimum.

➢ Step 2: We prove that $f(x)$ *has at least one critical point.*

3) $g_i(t) \coloneqq \frac{1}{2}\lambda_1 t^2 - b_i \log(t)\,, \quad t > 0, \quad and\ it's\ convex$

$\Rightarrow \quad It\ has\ a\ global\ minimum\ at\ t_i^* = \sqrt{\frac{b_i}{\lambda_1}}$

- $g''_i(t) = \lambda_1 + \frac{b_i}{t^2}$
- $g'_i(t) = \lambda_1 t - \frac{b_i}{t} = 0$

$\Rightarrow \quad t_i^* = \sqrt{\frac{b_i}{\lambda_1}}$

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$$\lim_{t\to+\infty} g_i(t) = \lim_{t\to 0^+} g_i(t) = +\infty$$

- $g''_i(t) = \lambda_1 + \frac{b_i}{t^2}$
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$$\Rightarrow\ t_i^* = \sqrt{\frac{b_i}{\lambda_1}}$$

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$\Rightarrow\ t_i^* = \sqrt{\frac{b_i}{\lambda_1}}$

$\because g(\cdot)$ *has a global minimum at* t_k^*

4) $\because f(x) \geq \sum_{i=1}^{n} g_i(x_i),\ \therefore f(x) \geq g_i(x_i) + \sum_{k\neq i} g_k(x_k) \geq g_i(x_i) + \sum_{k\neq i} g_k(t_k^*)$

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- $g''_i(t) = \lambda_1 + \frac{b_i}{t^2}$
- $g'_i(t) = \lambda_1 t - \frac{b_i}{t} = 0$

$\Rightarrow\ t_i^* = \sqrt{\frac{b_i}{\lambda_1}}$

$\because g(\cdot)$ *has a global minimum at* t_k^*

4) $\because f(x) \geq \sum_{i=1}^{n} g_i(x_i),\ \therefore f(x) \geq g_i(x_i) + \sum_{k\neq i} g_k(x_k) \geq g_i(x_i) + \sum_{k\neq i} g_k(t_k^*)$

$\Rightarrow\quad f(x) \geq g_i(x_i) + \mu,\quad where\ \mu = \min_i \sum_{k\neq i} g_k(t_k^*)$

選一個 i 使得 $\sum_{k\neq i} g_k(t_k^*)$ 是最小的

Risk Parity Model

證明 $f(x)$ 存在 global minimum.

- Step 2: We prove that $f(x)$ *has at least one critical point.*

5) $Let \quad B > 0 \quad an\ arbitrary\ threshold.$

$Whenever\ t \notin [\alpha, \beta], \quad \exists\, t \quad s.t. \quad g_i(t) = \infty, \quad then \quad g_i(t) > B - \mu$

Risk Parity Model

證明 $f(x)$ 存在 global minimum.

- Step 2: We prove that $f(x)$ *has at least one critical point.*

5) $Let \quad B > 0 \quad an\ arbitrary\ threshold.$

$Whenever\ t \notin [\alpha, \beta], \quad \exists\ t \quad s.t. \quad g_i(t) = \infty, \quad then \quad g_i(t) > B - \mu$

6) $\forall\ i, \quad x_i \notin [\alpha, \beta], \quad then \quad f(x) \geq g_i(x_i) + \mu > B - \mu + \mu = B,$

$i.e., \quad f(x) > B, \quad f(x)\ has\ at\ least\ one\ critical\ point.$

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- Step 2: We prove that $f(x)$ *has* *at least one critical point.*

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   $Whenever\ t \notin [\alpha, \beta], \quad \exists\, t \quad s.t. \quad g_i(t) = \infty, \quad then \quad g_i(t) > B - \mu$

6) $\forall\, i, \quad x_i \notin [\alpha, \beta], \quad then \quad f(x) \geq g_i(x_i) + \mu > B - \mu + \mu = B,$

   $i.e., \quad f(x) > B, \quad f(x)\ has\ at\ least\ one\ critical\ point.$

- $f(x)\ has\ at\ most\ one\ critical\ point\ \&\ at\ least\ one\ critical\ point.$

  $\Rightarrow \quad f(x)\ has\ global\ minimum.$

Risk Parity Model

透過牛頓法求極值，不斷迭代找到一階微分為零的點

- 一維問題的解法：

  1) $f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2$

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2) $Let\ \Delta x = x - x_0 \quad \Rightarrow \quad \Delta x = -\frac{f'(x_0)}{f''(x_0)}$

$x_1 = x_0 + \Delta x = x_0 - \frac{f'(x_0)}{f''(x_0)}$

Risk Parity Model

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$x_1 = x_0 + \Delta x = x_0 - \frac{f'(x_0)}{f''(x_0)} \quad \Rightarrow \quad x_{n+1} = x_n + \Delta x = x_n - \frac{f'(x_n)}{f''(x_n)}$

Risk Parity Model

透過牛頓法求極值，不斷迭代找到一階微分為零的點

- 一維問題的解法：

1) $f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2$

$\Rightarrow \quad f'(x) = f'(x_0) + f''(x_0)(x - x_0) = 0$

2) $Let\ \Delta x = x - x_0 \quad \Rightarrow \quad \Delta x = -\frac{f'(x_0)}{f''(x_0)}$

$x_1 = x_0 + \Delta x = x_0 - \frac{f'(x_0)}{f''(x_0)} \quad \Rightarrow \quad x_{n+1} = x_n + \Delta x = x_n - \frac{f'(x_n)}{f''(x_n)}$

迭代直到$\Delta x \longrightarrow 0, \quad i.e., \quad f'(x_n) \longrightarrow 0$

Risk Parity Model

- 多維問題的解法：

1) $$f(\boldsymbol{x}) = f(\boldsymbol{x_0}) + \nabla f(\boldsymbol{x_0})' (\boldsymbol{x} - \boldsymbol{x_0}) + \frac{1}{2} (\boldsymbol{x} - \boldsymbol{x_0})' \nabla^2 f(\boldsymbol{x_0}) (\boldsymbol{x} - \boldsymbol{x_0})$$

$$\Rightarrow \quad \nabla f(\boldsymbol{x}) = \nabla f(\boldsymbol{x_0}) + \nabla^2 f(\boldsymbol{x_0}) (\boldsymbol{x} - \boldsymbol{x_0})$$

2) $$Let\ \Delta\boldsymbol{x} = \boldsymbol{x} - \boldsymbol{x_0} \quad \Rightarrow \quad \Delta\boldsymbol{x} = -\left(\nabla^2 f(\boldsymbol{x_0})\right)^{-1} \cdot \nabla f(\boldsymbol{x_0})$$

$$\Rightarrow \quad \boldsymbol{x_1} = \boldsymbol{x_0} + \Delta\boldsymbol{x} = \boldsymbol{x_0} - \left(\nabla^2 f(\boldsymbol{x_0})\right)^{-1} \cdot \nabla f(\boldsymbol{x_0})$$

$$\Rightarrow \quad \boldsymbol{x_{n+1}} = \boldsymbol{x_n} + \Delta\boldsymbol{x} = \boldsymbol{x_n} - \underbrace{\left(\nabla^2 f(\boldsymbol{x_n})\right)^{-1}}_{\text{Hessian matrix}} \cdot \underbrace{\nabla f(\boldsymbol{x_n})}_{\text{gradient}} \cdot \underbrace{r}_{\text{step size of each iteration}}, \quad r \text{ 近於 } 0^+$$

迭代直到$\| \nabla f(\boldsymbol{x_n}) \| < \boldsymbol{\varepsilon}, \quad where\ \boldsymbol{\varepsilon}\ is\ arbitrary\ small$

3

實作

標的:股+債
回測環境

- **標的**

公債 : 3到7年公債ETF(IEI)、7到10年公債ETF(IEF)

股票 : APPLE(AAPL)、GOOGLE(GOOG)

- **日期**

回測時間:2014/1/1~2020/12/13

移動窗格:240個交易日

更新時間:40個交易日

- **比較對象**

1. Risk parity portfolio vs. Markowitz portfolio
2. Risk parity portfolio vs. S&P500
3. ~~Risk parity portfolio vs. 加條件的Risk parity portfolio~~

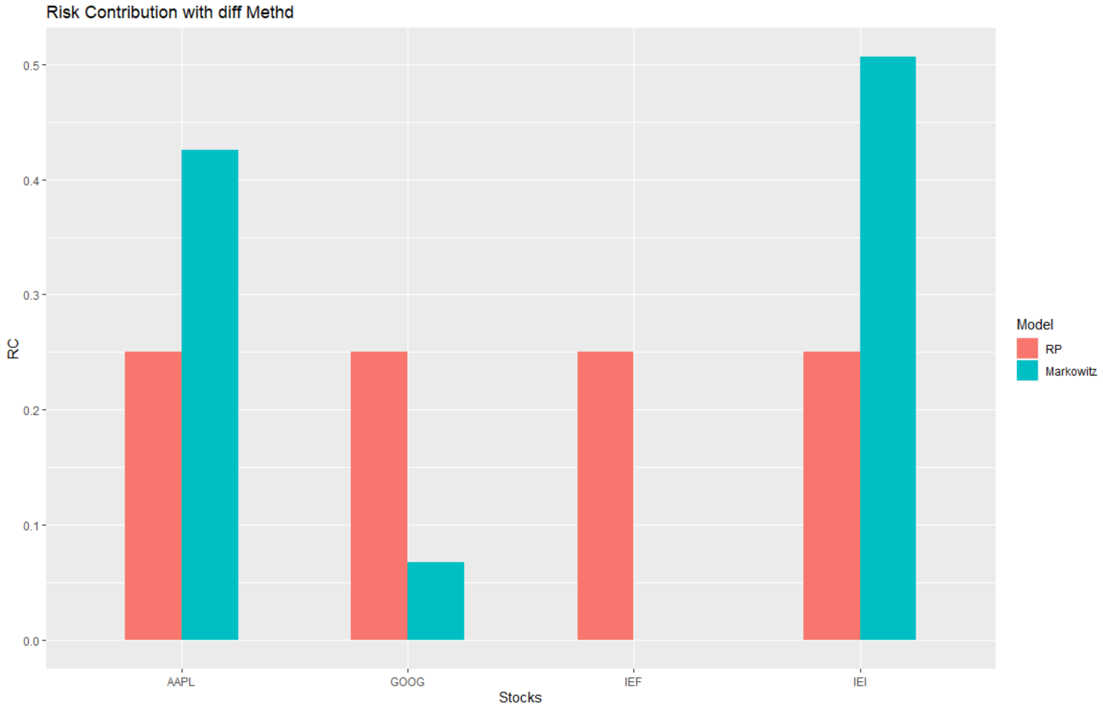
- **R library**

riskParityPortfolio

標的:股+債
風險貢獻度比例

Markowitz:
RC集中於”IEI”
及”AAPL”
分布較極端

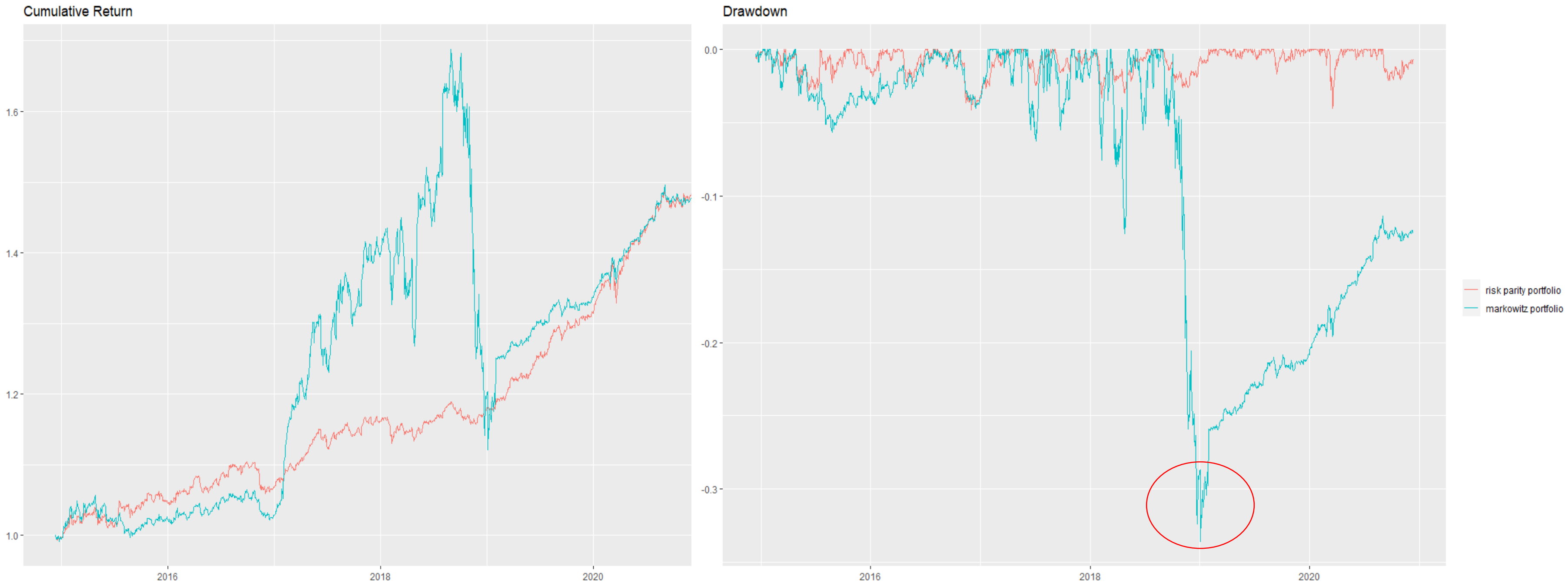
Risk parity(4種條件下):
平均貢獻給兩檔債券和兩
檔股票



標的:股+債
報酬累積及回落圖

Risk parity: 報酬較低，但穩定不易有急遽回落

Markowitz: 累積報酬與Risk parity相仿，但易受系統風險影響，吃到較大回落



標的:股+債
績效檢視比較

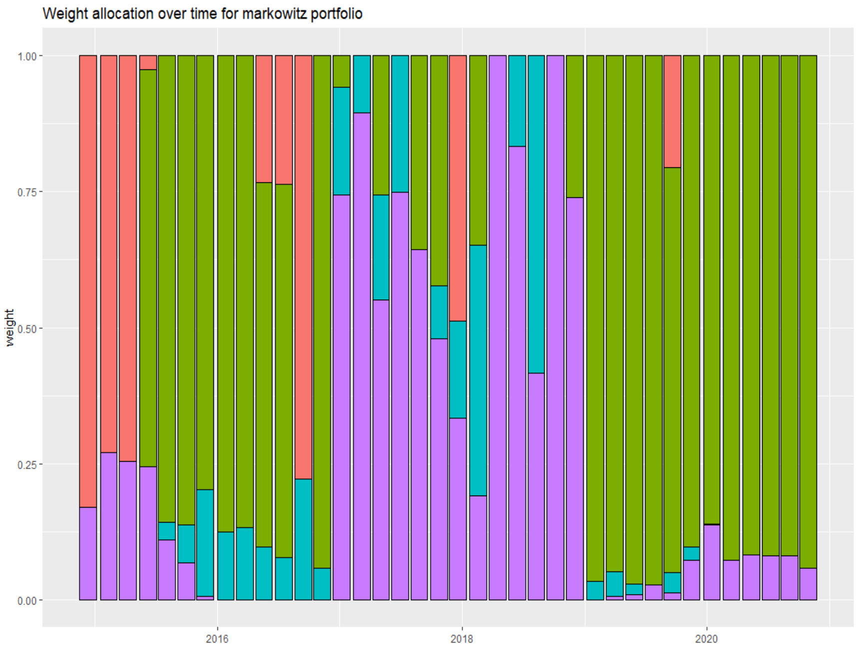
risk parity投組與markowitz投組資產配置後的績效比較

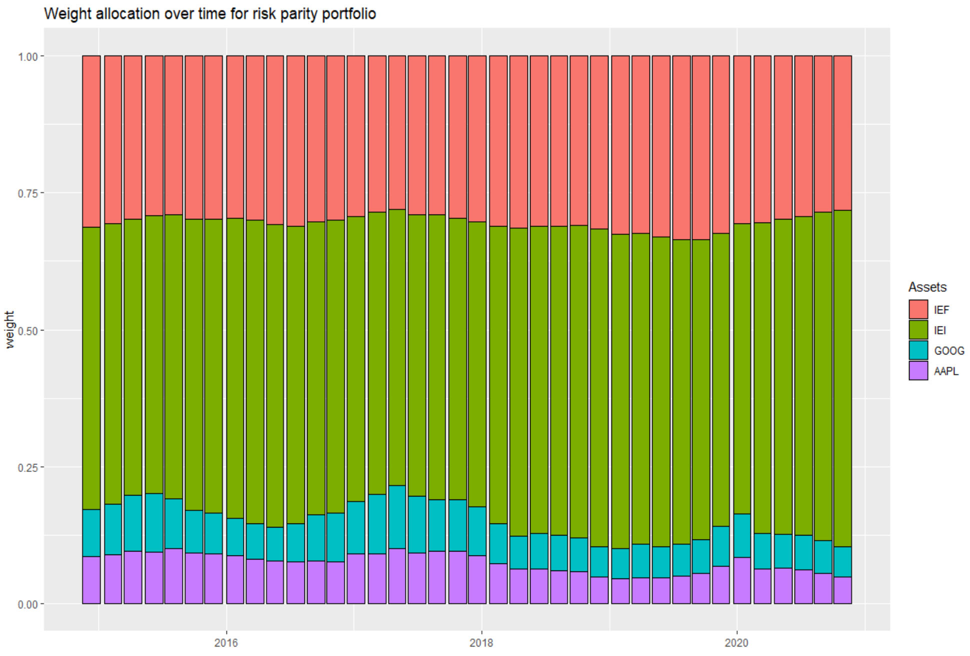
```
> backtestSummary(BT)$performance
                    risk parity portfolio markowitz portfolio
Sharpe ratio                 1.646209e+00        5.430581e-01
max drawdown                 4.142583e-02        3.358196e-01
annual return                6.815089e-02        6.767854e-02
annual volatility            4.139868e-02        1.246248e-01
Sterling ratio               1.645130e+00        2.015324e-01
Omega ratio                  1.315226e+00        1.146704e+00
ROT (bps)                    3.751849e+03        2.176314e+02
VaR (0.95)                   3.997888e-03        9.231557e-03
CVaR (0.95)                  5.719619e-03        2.029539e-02
```

標的:股+債
資產配置差異

Risk parity:投組建構比例穩定，債券成分較重

Markowitz:隨著市場波動而定時調整投組構造，
較Risk parity多出許多交易成本





標的:股+債
績效檢視比較

risk parity投組與S&P500的績效比較

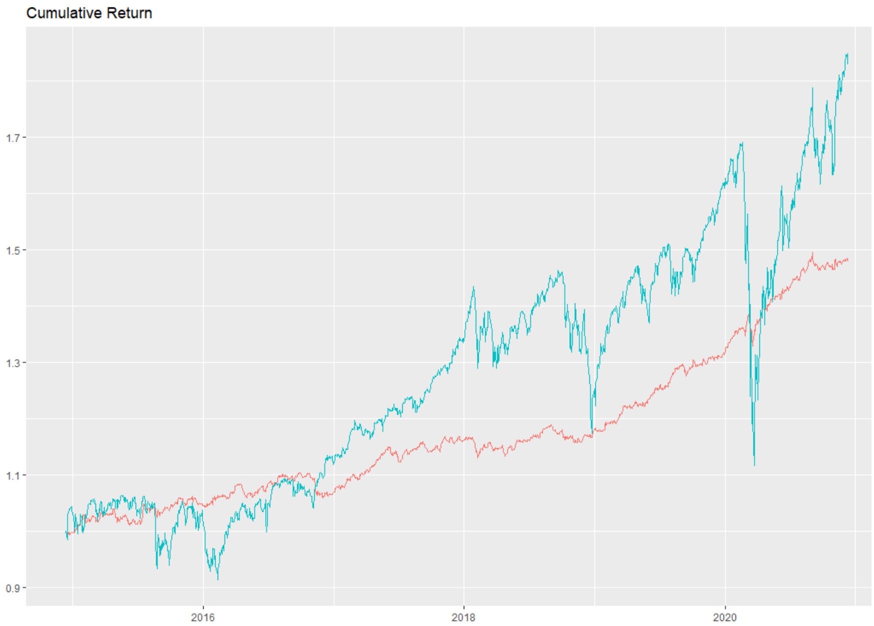
- 以2014年到今年的數據顯示，Risk parity夏普比率較低，但在年化波動度上遠低於S&P500

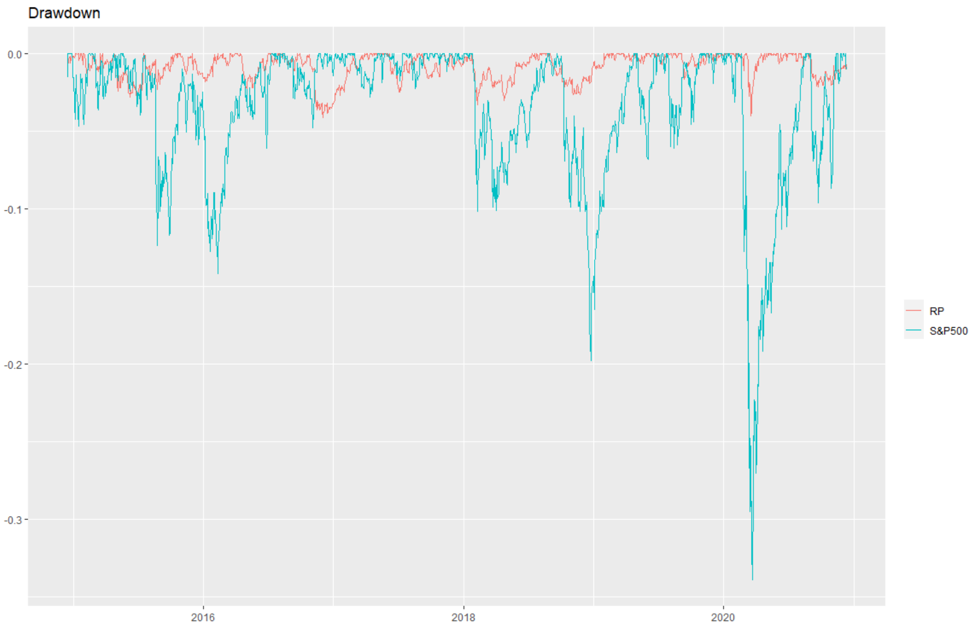
```
> backtestSummary(BT2)$performance
                              RP      S&P500
Sharpe ratio        1.646209e+00 0.56710397
max drawdown        4.142583e-02 0.33924960
annual return       6.815089e-02 0.10607323
annual volatility   4.139868e-02 0.18704371
Sterling ratio      1.645130e+00 0.31267018
Omega ratio         1.315226e+00 1.14110802
ROT (bps)           3.751849e+03        Inf
VaR (0.95)          3.997888e-03 0.01701222
CVaR (0.95)         5.719619e-03 0.02946877
```

標的:股+債
報酬累積及回落圖

Risk parity: 報酬較低，但穩定不易有急遽回落

S&P500: 有較高報酬，但易受系統風險影響，吃到較大回落





標的:美股十大成分股
回測環境

- **標的**

2014年的美股十大成分股

'AAPL','MSFT','AMZN','FB','GOOGL','GOOG','BRK-B','JNJ','JPM','V'

- **日期**

回測時間:2014/1/1~2020/12/13

移動窗格:240個交易日

更新時間:40個交易日

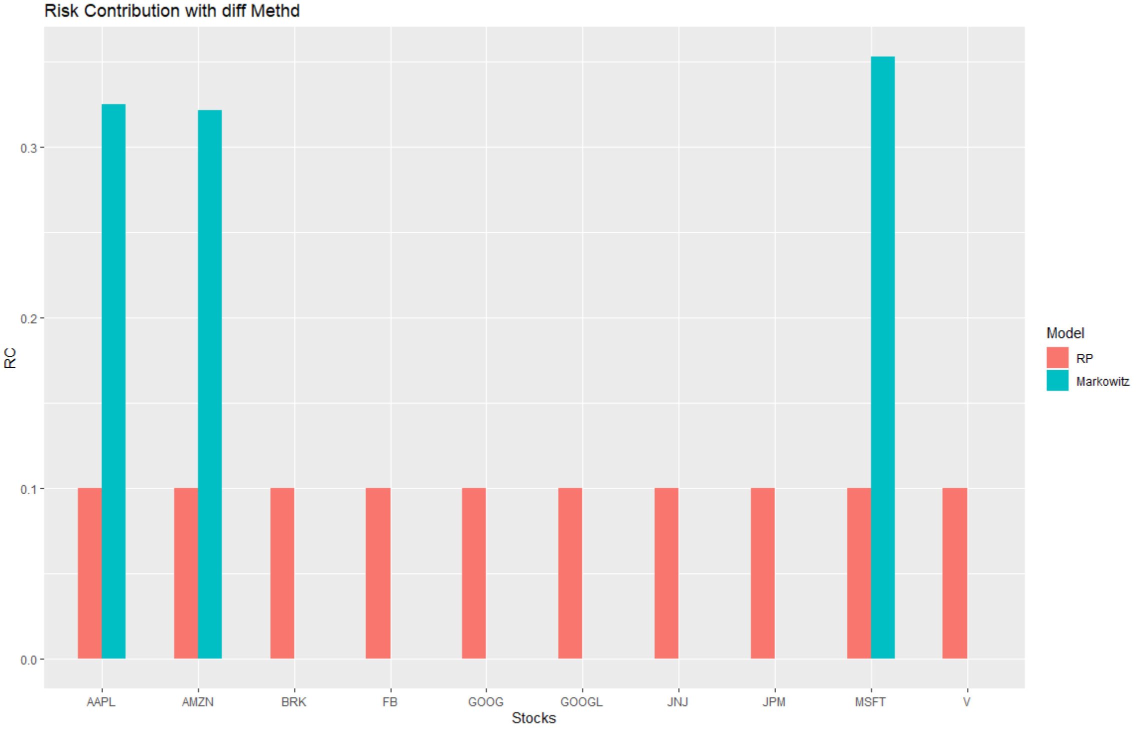
- **比較對象**

1. Risk parity portfolio vs. Markowitz portfolio
2. Risk parity portfolio vs. S&P500

標的:美股十大成分股風險貢獻度比例

Markowitz:
RC集中於占比較高之三支個股上

Risk parity(4種條件下):
平均貢獻給十檔股票



標的:美股十大成分股
績效檢視比較

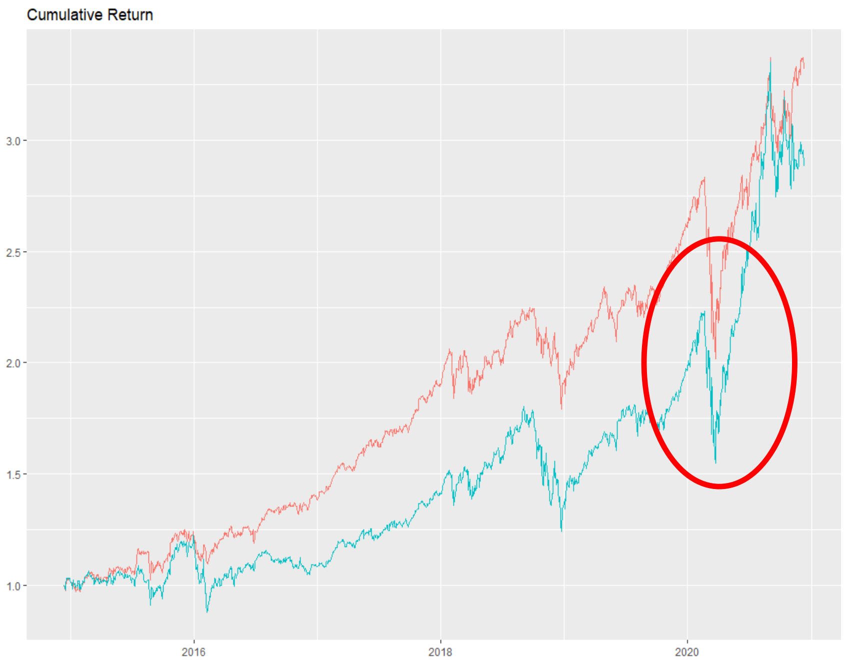
risk parity投組與markowitz資產配置後的績效比較

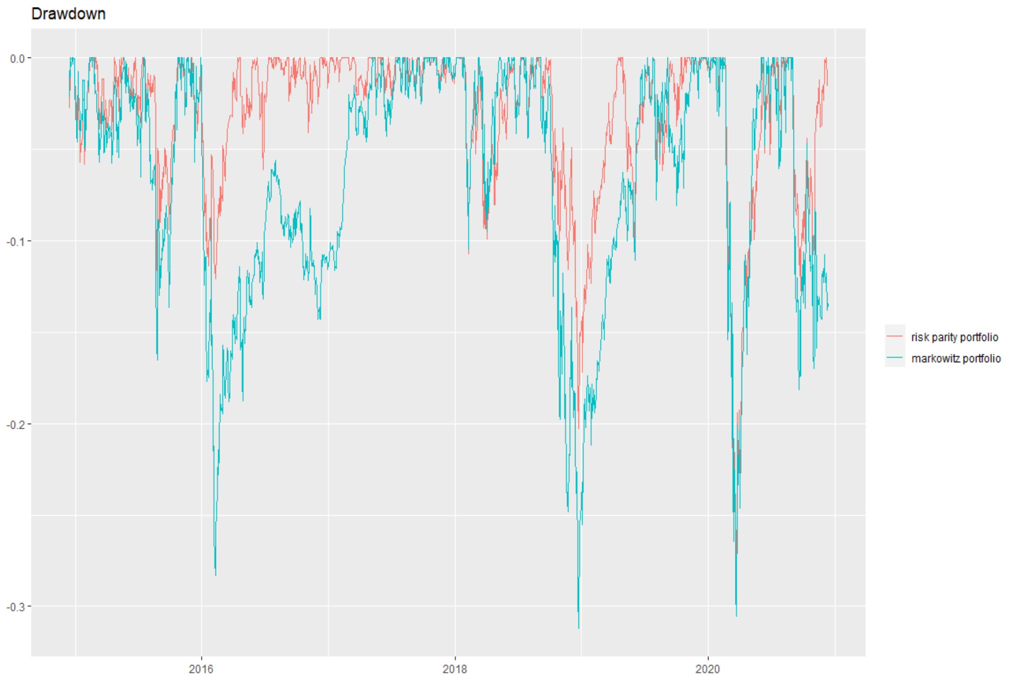
➢ 以2014年到今年的數據顯示，Risk parity投組表現略優一些。

```
> backtestSummary(bt)$performance
                  risk parity portfolio markowitz portfolio
Sharpe ratio               1.098776e+00          0.72856560
max drawdown               2.877405e-01          0.31197900
annual return              2.217428e-01          0.19450962
annual volatility          2.018089e-01          0.26697612
Sterling ratio             7.706346e-01          0.62347023
Omega ratio                1.237786e+00          1.17108101
ROT (bps)                  5.438277e+03        495.77535255
VaR (0.95)                 1.897366e-02          0.02496065
CVaR (0.95)                3.144827e-02          0.04179917
>
```

標的:美股十大成分股 報酬累積及回落圖

➢ Risk parity 投組不論是在報酬上，亦或是風險回落上，都較優於 Markowitz 投組的結果，而因沒加入債券分散風險，會吃到2020疫情影響的虧損。



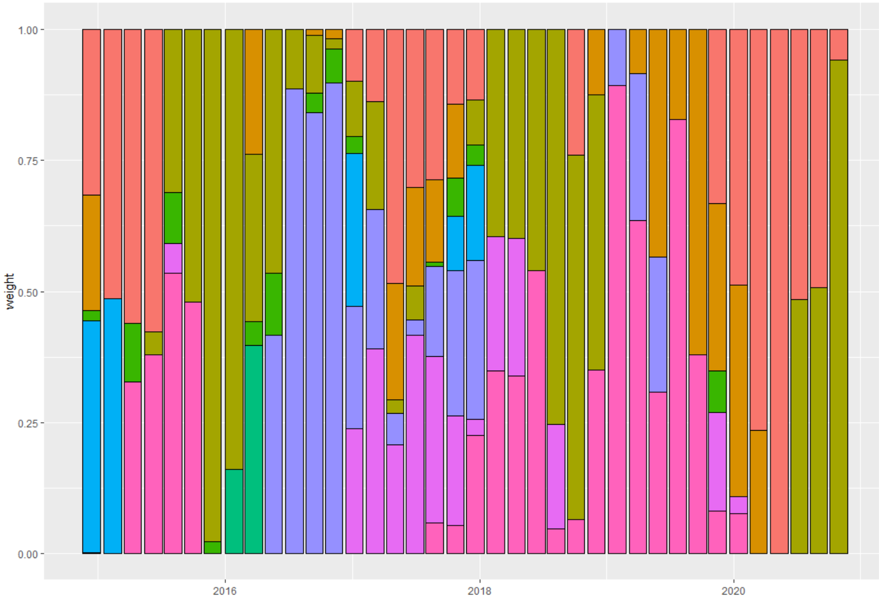


標的:美股十大成分股
資產配置差異

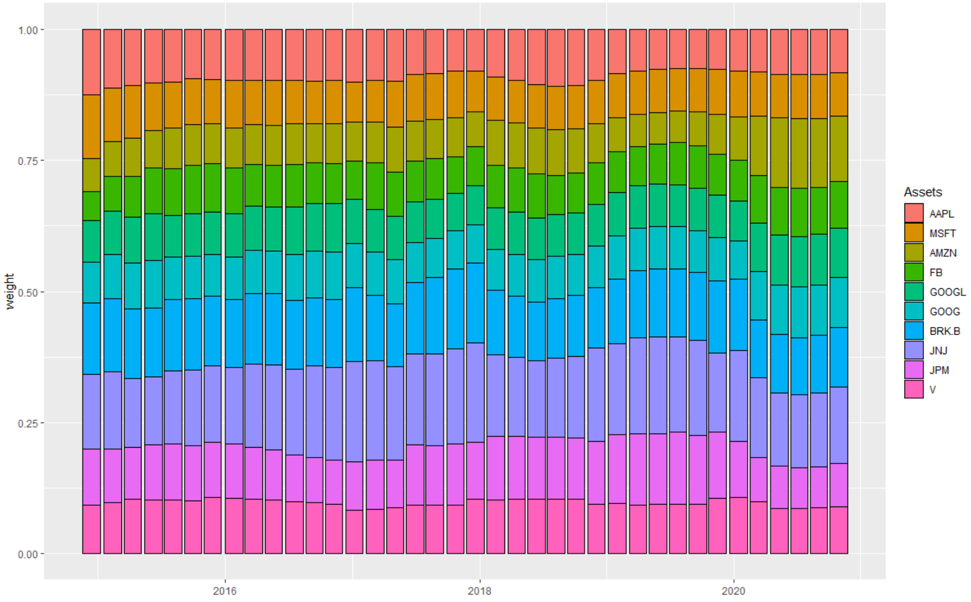
Risk parity: 投組建構比例穩定，架構平均

Markowitz: 隨著市場波動而定時調整投組構造，有時甚至投組內只有一股。投組權重變動劇烈，較Risk parity多出許多交易成本。

Weight allocation over time for markowitz portfolio



Weight allocation over time for risk parity portfolio



標的:美股十大成分股績效檢視比較

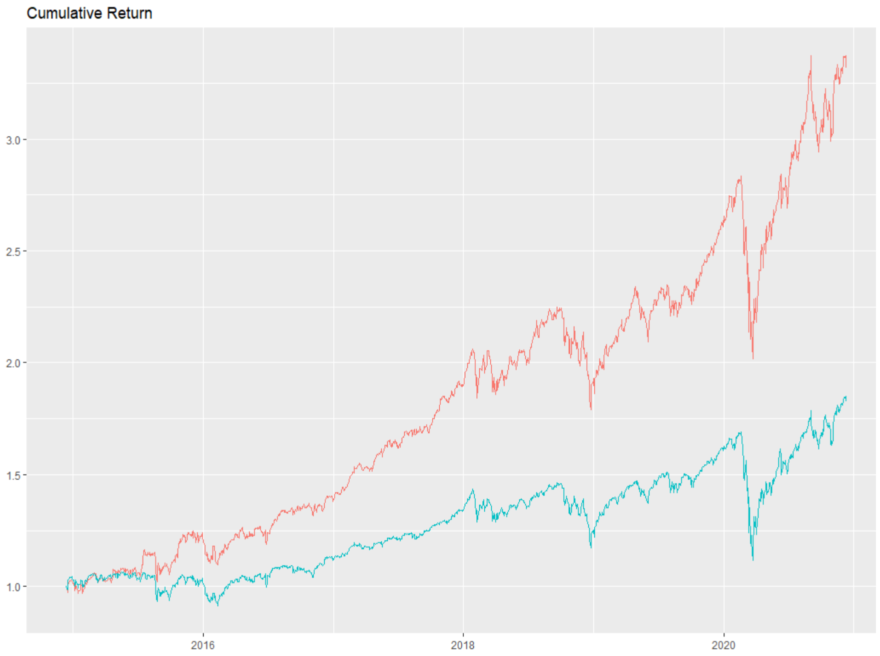
risk parity投組與S&P500的績效比較

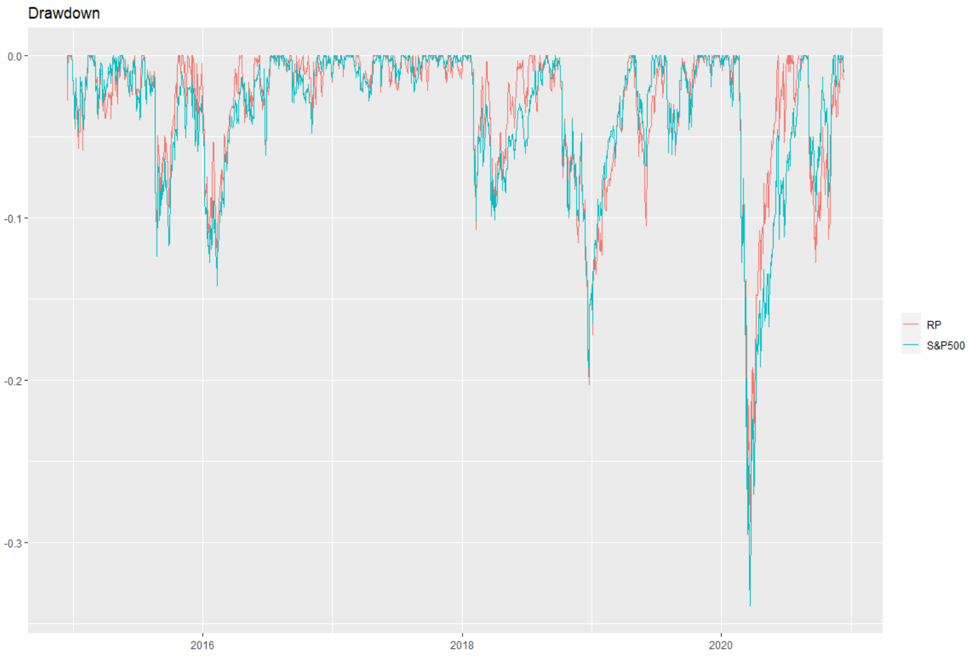
- 以2014年到今年的數據顯示，Risk parity投組表現優於S&P500，但可能是因為沒加入債券避險的原因，年化波動度略高一籌。

```
> backtestSummary(bt2)$performance
                              RP      S&P500
Sharpe ratio        1.098776e+00 0.56710397
max drawdown        2.877405e-01 0.33924960
annual return       2.217428e-01 0.10607323
annual volatility   2.018089e-01 0.18704371
Sterling ratio      7.706346e-01 0.31267018
Omega ratio         1.237786e+00 1.14110802
ROT (bps)           5.438277e+03        Inf
VaR (0.95)          1.897366e-02 0.01701222
CVaR (0.95)         3.144827e-02 0.02946877
```

標的:美股十大成分股
報酬累積及回落圖

- Risk parity投組報酬上遠勝於大盤，但在Drawdown的部分與S&P500相近。





4

結論

Risk Parity

優勢

a） 策略構造穩定
b） 金融海嘯時，維持穩定績效
c） 不會過度集中在特定資產上

劣勢

a） 資產相關性變化
b） 資產波動度變化

未來方向

未來我們可能要再思考如何用適合的商品去套用於Risk parity中，以應變不斷改變的市場。

Reference

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- V. Chopra and W. Ziemba, “The effect of errors in means, variances and covariances on optimal portfolio choice”, Journal of Portfolio Management, 1993

- FLORIN SPINU, “AN ALGORITHM FOR COMPUTING RISK PARITY WEIGHTS”, 2013

- Clifford S. Asness, Andrea Frazzini, and Lasse H. Pedersen, “Leverage Aversion and Risk Parity”, 2012