Two-level domain decomposition methods

Victorita Dolean

Contents

1 Coarse Space Corrections

Let us establish first some scalability metrics for our methods.

Strong vs Weak Scalability

- **Strong scalability (Amdahl)**: How solution time varies with processor count for a fixed total problem size.
- **Weak scalability (Gustafson)**: How solution time varies with processor count when problem size per processor is constant.

Scalability Breakdown in One-Level Methods Let us run some experiments

# Subdomains	8	16	32	64
ASM Iterations	18	35	66	128

Figure 1: Iteration growth with subdomain count (1-level ASM)

Iteration count increases with the number of subdomains, indicating lack of scalability.

Numerical Experiment: Toy Problem Let us perform another experiment to test the influence of the overlap for the iterative and Krylov solver.

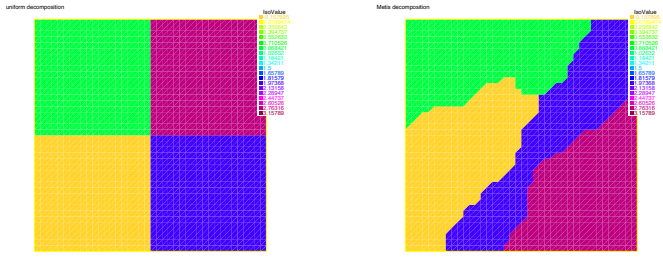


Figure 2: Uniform and METIS-based domain decompositions

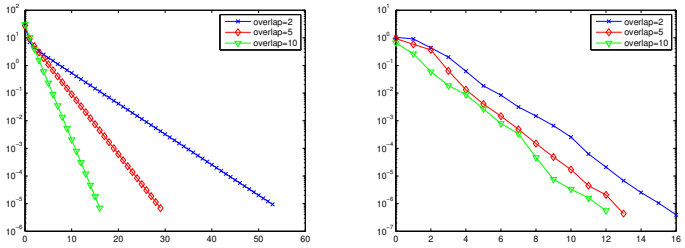


Figure 3: Schwarz convergence as solver (left) and as preconditioner (right)

1.1 Condition Number Bounds and Weak Scalability

How we can quantify all this? First we need a generic condition number bound.

Lemma 1. *If there exist constants $C_1, C_2 > 0$ such that*

$$C_1 \left(M_{AS}\mathbf{x}, \mathbf{x}\right) \leq \left(A\mathbf{x}, \mathbf{x}\right) \leq C_2 \left(M_{AS}\mathbf{x}, \mathbf{x}\right), \quad \forall \mathbf{x} \in \mathbb{R}^n,$$

then the eigenvalues of $M_{AS}^{-1}A$ satisfy

$$\lambda_{\min} \geq C_1, \quad \lambda_{\max} \leq C_2, \quad \Rightarrow \quad \kappa(M_{AS}^{-1}A) \leq \frac{C_2}{C_1}.$$

If $\kappa(M_{\text{AS}}^{-1}A)$ is independent of N, then solution time is independent of the number of subdomains $\Rightarrow$ weak scalability is achieved.

and then a theoretical estimate of eigenvalues. Let us start with $\lambda_{\max}$.

Lemma 2. *Let $col(j) \in \{1, \ldots, \mathcal{N}_c\}$ be a coloring such that*

$$col(k) = col(l) \Rightarrow (AR_k^T\mathbf{x}_k, R_l^T\mathbf{x}_l) = 0.$$

Then:

$$\lambda_{\max}(M_{AS}^{-1}A) \leq \mathcal{N}_c$$

Proof. Using a result from the literature:

$$(M_{\text{AS}}\mathbf{x}, \mathbf{x}) = \min_{\substack{\mathbf{x}_j \in \mathbb{R}^{n_j} \\ \mathbf{x}=\sum_{j=1}^N R_j^T\mathbf{x}_j}} \sum_{j=1}^{N} (A_j\mathbf{x}_j, \mathbf{x}_j), \quad A_j = R_j A R_j^T$$

Choose a minimizing set $\{\mathbf{x}_j\}$:

$$\begin{aligned}(M_{\text{AS}}\mathbf{x}, \mathbf{x}) &= \sum_{j=1}^{N} (AR_j^T\mathbf{x}_j, R_j^T\mathbf{x}_j) \\ &= \sum_{c=1}^{\mathcal{N}_c} \left(A \sum_{\text{col}(i)=c} R_i^T\mathbf{x}_i, \sum_{\text{col}(i)=c} R_i^T\mathbf{x}_i \right) \\ &\geq \frac{1}{\mathcal{N}_c}(A\mathbf{x}, \mathbf{x})\end{aligned}$$

This gives the desired upper bound. □

Numerical Illustration: Poisson Problem For the lower bound we don't have a theoretical estimate for now but the decrease of the lowest eigenvalue can be seen from the following experiments. We can see the lack of scalability on this simple problem (beheviour degrading with the number of subdomains)

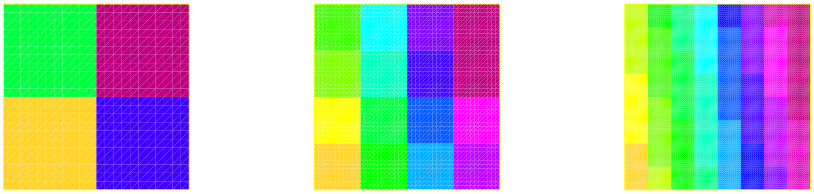


Figure 4: Increasing domain decomposition granularity

# Subdomains	2×2	4×4	8×8
ASM Iterations	20	36	64

Table 1: Poisson problem with 20×20 grid and 2-layer overlap

Root Cause of Stagnation

- A few small eigenvalues dominate the preconditioned spectrum.
- These modes represent global effects not handled by local subdomain solves.

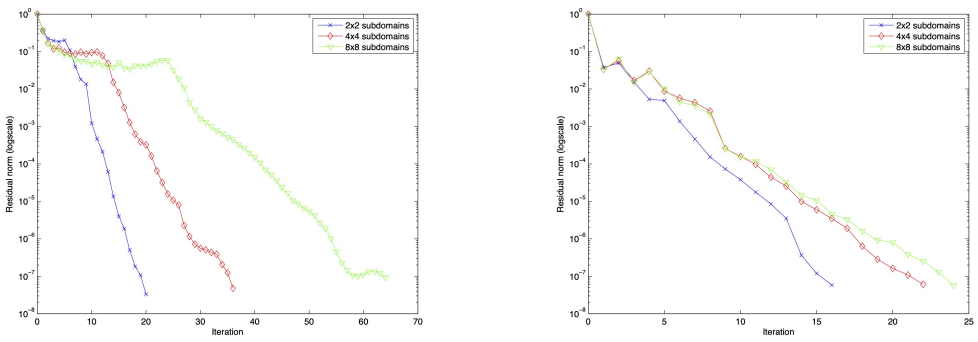


Figure 5: Spectral behavior of one-level ASM

Classical Remedy

Introduce a **coarse space** that aggregates global modes and communicates across all subdomains. This recovers weak scalability.

1.2 Targeting Slow Convergence Modes and definition of the two-level method

We now address the slow convergence in one-level methods by enriching the solver with a coarse space that captures the difficult global modes.

We assume the preconditioned system has the form:

$$M^{-1}A\,\mathbf{x} = M^{-1}\mathbf{b}$$

Examples of Slow Modes

- Constant functions — in the null space of the Laplace operator.
- Rigid body motions — in linear elasticity problems.

Notation

Let Z be a rectangular matrix whose columns span the space of slow modes.

In order to identify a generic form of the coarse correction we seek to correct a current approximation $\mathbf{y}$ by adding a linear combination of the coarse basis while minimising the residual:

$$\min_{\beta} \|A(\mathbf{y} + Z\beta) - \mathbf{b}\|_{A^{-1}}$$

This is equivalent to:

$$\min_{\beta\in\mathbb{R}^{n_c}} \; 2(A\mathbf{y} - \mathbf{b}, Z\beta) + (AZ\beta, Z\beta)$$

The minimizer is:

$$\beta = (Z^TAZ)^{-1}Z^T(\mathbf{b} - A\mathbf{y})$$

Thus, the coarse space correction is:

$$Z\beta = Z(Z^TAZ)^{-1}Z^T(\mathbf{b} - A\mathbf{y}) =: P_0\,\mathbf{r}$$

Galerkin Correction

The operator $P_0 := Z(Z^TAZ)^{-1}Z^T$ defines the **Galerkin projection** onto the coarse space.

Let $R_0 := Z^T$ and $\mathbf{r} := \mathbf{b} - A\mathbf{y}$. Then the coarse correction is:

$$Z\beta = R_0^T(R_0AR_0^T)^{-1}R_0\,\mathbf{r}$$

This yields the two-level preconditioner:

$$M^{-1}_{\mathrm{AS},2} = \underbrace{R_0^T(R_0AR_0^T)^{-1}R_0}_{\text{Coarse solve}} + \underbrace{\sum_{i=1}^{N} R_i^T(R_iAR_i^T)^{-1}R_i}_{\text{One-level ASM}}$$

Remarks

- $M_{\text{AS},2}^{-1}$ mimics the one-level ASM structure.
- The choice of coarse basis R_0 (or Z) is flexible.
- The coarse problem is small — cost is negligible in parallel.

1.3 The Nicolaides Coarse Space

The simplest and most classical choice of a coarse space in two-level additive Schwarz methods is the one proposed by R. Nicolaides in 1987. It is based on the idea that *constant functions* often represent the global error modes that slow down convergence in problems like the Poisson equation.

The Nicolaides coarse space constructs one global basis vector per subdomain. These vectors are designed to be constant on each subdomain, smoothly combined across overlaps via a partition of unity.

Definition of the Basis Vectors

Let R_i be the restriction operator to subdomain i, and D_i the local partition-of-unity weight matrix (diagonal). Then the coarse vector associated with subdomain i is:

$$Z_i := R_i^T D_i R_i \mathbf{1}, \quad 1 \leq i \leq N$$

where $\mathbf{1}$ is the all-ones vector on the degrees of freedom of the full domain. Each Z_i is supported in subdomain i and smoothly tapered across the overlap.

Global Coarse Basis Structure

Stacking all local vectors column-wise gives the global coarse basis matrix Z, which has a block structure:

$$Z = \begin{bmatrix} D_1 R_1 \mathbf{1} & 0 & \cdots & 0 \\ 0 & D_2 R_2 \mathbf{1} & \ddots & \\ \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & D_N R_N \mathbf{1} \end{bmatrix}$$

This matrix defines a space of functions that are locally constant on each subdomain and sum to a globally meaningful approximation due to the partition-of-unity property:

$$\sum_{i=1}^{N} R_i^T D_i R_i = I$$

Convergence Guarantee Although simple, this coarse space is already sufficient to ensure good scalability for problems with smooth coefficients. The following result provides a quantitative convergence bound.

Widlund–Dryja Theorem

Let $M_{\text{AS},2}^{-1}$ be the two-level additive Schwarz preconditioner using the Nicolaides coarse space. Then:

$$\kappa(M_{\text{AS},2}^{-1} A) \leq C \left(1 + \frac{H}{\delta}\right)$$

where H is the diameter of subdomains and δ is the overlap width.

This bound shows that the condition number can be controlled independently of the number of subdomains, provided the overlap is sufficiently large relative to the subdomain size.

Numerical Results and Limitations The table below compares the iteration counts of a one-level and a two-level method using the Nicolaides coarse space. The improvement is substantial and notably independent of the number of subdomains.

# Subdomains	8	16	32	64
ASM (1-level)	18	35	66	128
ASM + Nicolaides	20	27	28	27

Table 2: Iteration counts with and without coarse space correction

Despite this strong performance for homogeneous problems, the Nicolaides coarse space is known to break down in the presence of high heterogeneity or multiscale features.

Limitation of the Nicolaides Coarse Space

The Nicolaides space is not robust in the face of strong heterogeneity. To address this, we must:

- Use more expressive coarse spaces,
- Adapt them to the local coefficients, e.g., through spectral enrichment (as in GenEO).

2 Coarse Spaces for Heterogeneous Problems

Many scientific and engineering problems involve solving large linear systems arising from the discretization of partial differential equations (PDEs). These problems become particularly challenging when the underlying medium exhibits *strong heterogeneity*, as is common in geophysical flows, composite materials, and biological tissues.

A classical example is the **Darcy pressure equation**, discretized using P^1 finite elements:

$$Au = f, \qquad \mathrm{cond}(A) \sim \frac{\alpha_{\max}}{\alpha_{\min}} \cdot h^{-2}$$

Here, the condition number grows not only with mesh refinement ($h \to 0$) but also with the contrast $\alpha_{\max}/\alpha_{\min}$ in the material coefficients. This can lead to severe ill-conditioning.

Such problems typically display:

- **High contrast**: material parameters vary by orders of magnitude.
- **Multiscale features**: localized inclusions or fine-scale layers that interact with coarse-scale structures.

These characteristics call for the design of *robust preconditioners*—methods that retain efficiency as both the mesh is refined and the heterogeneity increases.

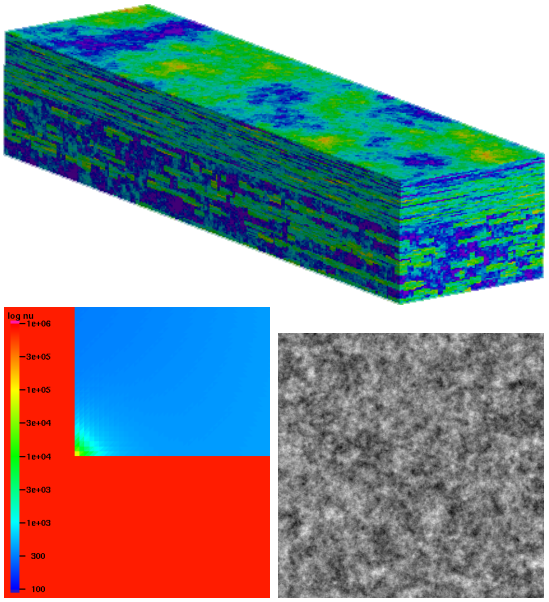
Goal: Robust Solvers

Design iterative solvers that:

- Remain effective as $h \to 0$
- Handle arbitrarily large coefficient jumps

Applications include:

- Flow in layered or stochastic porous media
- Structural mechanics with stiff inclusions
- Electromagnetic scattering in composite materials



2.1 GenEO: Adaptive Spectral Coarse Space

In the presence of strong heterogeneity, simple coarse spaces like the Nicolaides basis may fail to provide satisfactory convergence. This motivates the construction of *adaptive* coarse spaces that reflect the underlying coefficient distribution. A prominent example of such a method is the GenEO (Generalized Eigenproblems in the Overlap) approach.

Main Idea

GenEO builds the coarse space by solving localized generalized eigenvalue problems on each subdomain. The dominant eigenvectors correspond to the modes that are hardest to eliminate by local interactions alone. By assembling these slow-to-converge components into a global basis, the coarse space directly targets the performance bottlenecks of the iterative solver.

Application Areas

GenEO has been successfully applied to:

- **Darcy problems** with highly variable permeability,
- **Linear elasticity**, especially in nearly incompressible or composite materials.

Local Spectral Problem

For each subdomain j, we define a local generalized eigenvalue problem:

$$A_j^{\text{Neu}}\,\phi_{j,k} = \lambda_{j,k} D_j R_j A R_j^T D_j \cdot \phi_{j,k}$$

where:

- R_j is the restriction to subdomain j,
- D_j is the local partition-of-unity weight,
- A_j^{Neu} is the Neumann operator on subdomain j,
- $\phi_{j,k}$ is the k-th eigenvector and $\lambda_{j,k} \geq 0$ the corresponding eigenvalue.

The eigenvectors $\phi_{j,k}$ are defined in the local space and capture oscillatory modes in the overlap region that are weakly controlled by local solves.

Mode Selection and Coarse Space Construction

From each subdomain, we select eigenmodes whose eigenvalues exceed a threshold $\tau > 0$:

$$\lambda_{j,k} \leq \tau \quad \Rightarrow \quad \text{include } R_j^T D_j \phi_{j,k} \text{ in the coarse space}$$

This criterion has the following implications:

- When $\tau = 0$, we recover the Nicolaides space (based on constant functions).
- Smaller values of τ lead to a smaller number of global modes, improving efficiency.
- The threshold τ can be tuned to match the contrast and geometry of the coefficients.

Theoretical Convergence Guarantee

The effectiveness of GenEO is supported by a convergence result from Spillane et al. (2014), which bounds the condition number of the two-level preconditioned system.

Spillane et al. (2014)

Under mild assumptions, the preconditioned system satisfies:

$$\kappa(M_{\text{AS},2}^{-1} A) \leq (1 + k_0)\,[2 + k_0(2k_0 + 1)(1 + 1/\tau)]$$

where k_0 is the maximal number of overlapping subdomains at a point.
A practical choice for the threshold is:

$$\tau := \left(\max_j \frac{H_j}{\delta_j} \right)^{-1}$$

with H_j the subdomain diameter and δ_j the overlap width.

2.2 Numerical Results

To evaluate the effectiveness of GenEO, we consider a challenging Darcy problem with highly heterogeneous permeability:

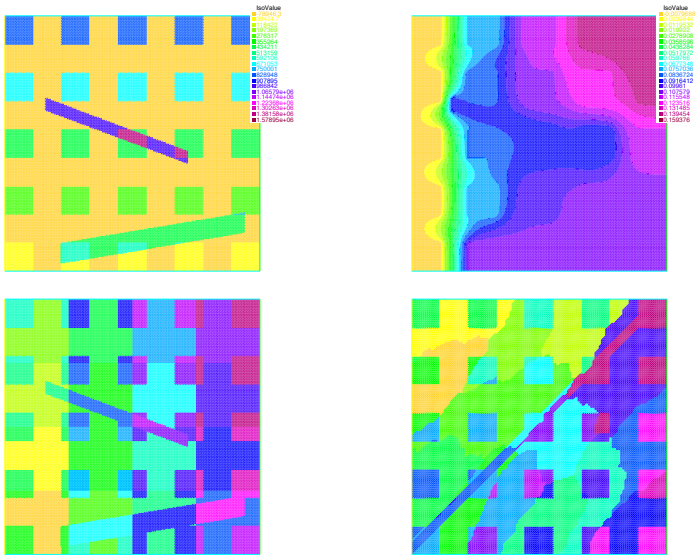
$$1 \leq \alpha(x) \leq 1.5 \times 10^6$$

Such a range of contrast is representative of real-world porous media, such as layered geological formations or fractured materials. The system is discretized using P^1 finite elements, and a two-level additive Schwarz method is applied with GenEO as the coarse space.

Two types of domain decomposition are used:

- **Uniform**: regular block decomposition

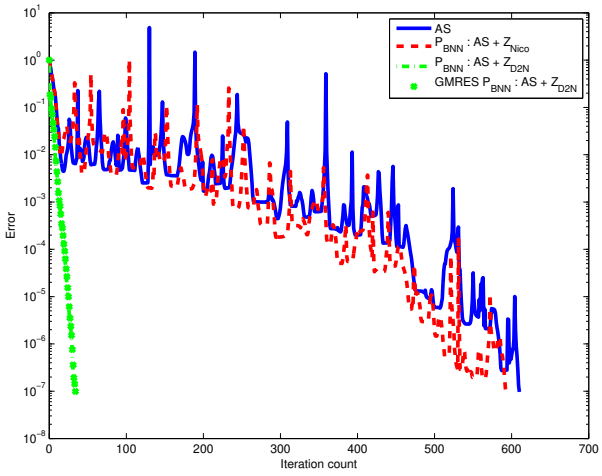
- **METIS**: graph-based partitioning optimized for load balancing and minimal communication



Top row: heterogeneous permeability $\alpha(x)$ and resulting solution $u(x)$. Bottom row: uniform vs METIS decomposition.

Convergence Behavior in Elasticity To further stress-test the GenEO approach, a nearly incompressible elasticity problem is considered. In such problems, standard iterative methods often stagnate due to near-null modes associated with rigid body motion or incompressibility constraints.

The figure below shows the convergence of GMRES applied with different coarse spaces. The GenEO-enhanced preconditioner demonstrates rapid and stable convergence, in contrast to the poor performance of ASM and Nicolaides-based methods.



GenEO leads to robust convergence even in the presence of extreme heterogeneity.

The GenEO space is formed by selecting local eigenvectors above a given spectral threshold. These eigenvectors correspond to the difficult error modes that span across subdomains.

Quantitative Comparison

The following table compares the number of eigenvectors per subdomain selected under three configurations:

- No coarse space (pure ASM)
- Nicolaides (1 vector per subdomain)
- GenEO (adaptive spectral selection)

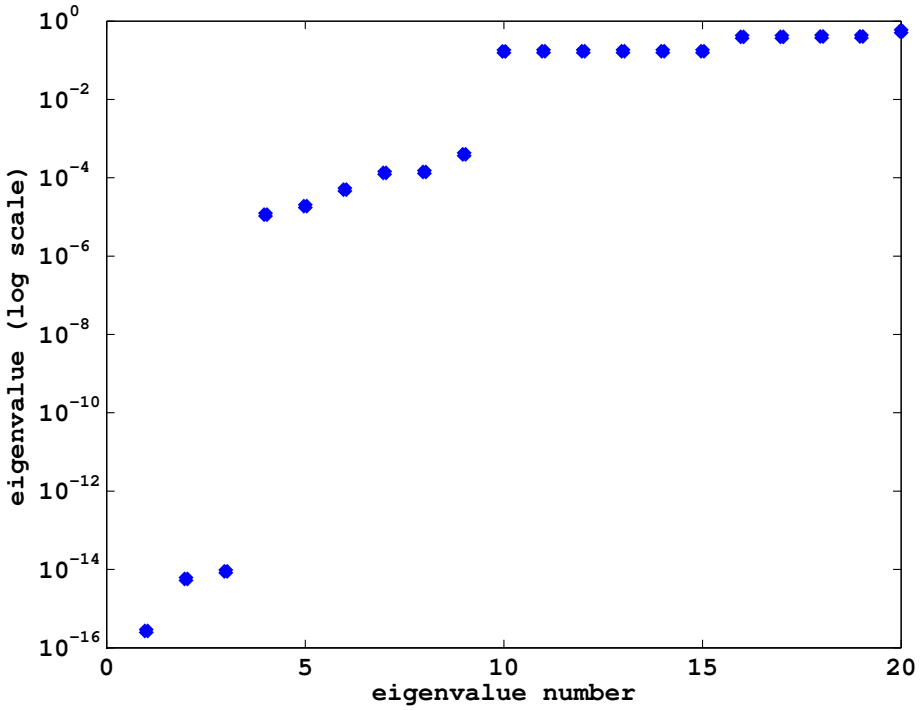
Coarse Space Type	ASM	ASM + Z_{Nico}	ASM + Z_{GenEO}
$\max(m_i - 1, 1)$	–	–	273
m_i (average)	614	543	**36**
$m_i + 1$	–	–	32

Number of local coarse basis functions m_i chosen automatically based on eigenvalue threshold.

This reduction in iteration count highlights GenEO's ability to represent difficult modes efficiently thus leading to a better convergence.

Spectral Distribution

The eigenvalue spectrum confirms that only a few modes dominate. These are precisely the ones retained by GenEO. The rest decay rapidly and do not affect convergence significantly.



Logarithmic scale

3 Theoretical Background

3.1 The Fictitious Space Lemma - an abstract framework

To analyze the behavior of Schwarz-type preconditioners, it is helpful to reformulate them in abstract functional terms.

Let $\mathcal{H}_0 = \mathbb{R}^{\#\mathcal{N}}$ be the global finite-dimensional space where the PDE is discretized, and let A be the global system matrix. We define the global bilinear form:

$$a(\mathbf{U}, \mathbf{V}) := \mathbf{V}^T A \mathbf{U}$$

In contrast, let $\mathcal{H}_P := \prod_{i=1}^{N} \mathbb{R}^{\#\mathcal{N}_i}$ be the product space of local unknowns (one for each subdomain), and define the local bilinear form:

$$b(\mathcal{U}, \mathcal{V}) := \sum_{i=1}^{N} \mathbf{V}_i^T A_i \mathbf{U}_i \qquad \text{where} \quad A_i := R_i A R_i^T$$

To connect these spaces, we introduce the *assembly operator*:

$$\mathcal{R}_{\text{AS}} : \mathcal{H}_P \to \mathcal{H}_0, \quad \mathcal{R}_{\text{AS}}(\mathcal{U}) := \sum_{i=1}^{N} R_i^T \mathbf{U}_i$$

Then, the additive Schwarz preconditioner takes the form:

$$M_{\text{AS}}^{-1} = \mathcal{R}_{\text{AS}}\, B^{-1}\, \mathcal{R}_{\text{AS}}^*$$

where B is the block-diagonal matrix made of local matrices A_i.

3.2 Application to one-level methods

The Fictitious Space Lemma (FSL) provides a powerful framework to analyze preconditioners of the form $M^{-1} = \mathcal{R} B^{-1} \mathcal{R}^*$. It ensures spectral equivalence between the preconditioned and original systems under three conditions:

- **(1) Surjectivity:** The operator $\mathcal{R}$ is surjective:

$$\forall u \in \mathcal{H}_0, \quad \exists u_P \in \mathcal{H}_P \text{ such that } \mathcal{R} u_P = u$$

- **(2) Continuity:** There exists $c_R > 0$ such that:

$$a(\mathcal{R} u_P, \mathcal{R} u_P) \leq c_R \cdot b(u_P, u_P) \quad \forall u_P \in \mathcal{H}_P$$

- **(3) Stable Decomposition:** There exists $c_T > 0$ such that:

$$\forall u \in \mathcal{H}_0, \quad \exists u_P \in \mathcal{H}_P \text{ with } \mathcal{R} u_P = u, \quad \text{and} \quad c_T \cdot b(u_P, u_P) \leq a(u, u)$$

If these conditions are satisfied, the spectrum of the preconditioned system satisfies the bounds:

$$\text{Spec}(M^{-1} A) \subset [c_T, c_R]$$

Estimates for the One-Level methods The FSL framework gives insight into the conditioning of classical one-level methods. Specifically, for additive Schwarz (ASM):

- The continuity constant is bounded by the maximum number of neighbors per subdomain:

$$c_R = \mathcal{N}_c$$

- The stability constant is bounded using a worst-case local Rayleigh quotient:

$$\tau_1 := \min_i \min_{\mathbf{U}_i \neq 0} \frac{\mathbf{U}_i^T A_i^{\text{Neu}} \mathbf{U}_i}{\mathbf{U}_i^T D_i R_i A R_i^T D_i \mathbf{U}_i} \quad \Rightarrow \quad c_T := \frac{\tau_1}{\mathcal{M}_c}$$

  where $\mathcal{M}_c$ is the maximum overlap multiplicity.

This yields the spectral estimate:

$$\frac{\tau_1}{\mathcal{M}_c} \leq \lambda(M_{\text{AS}}^{-1} A) \leq \mathcal{N}_c$$

Caution: In heterogeneous problems, the value τ_1 can become very small, leading to poor performance of one-level methods.

Extension to Other Schwarz Variants The same framework applies to variants such as RAS and SORAS. We summarize the formulations below:

- **RAS:**

$$M_{\text{RAS}}^{-1} = \sum_{i=1}^{N} R_i^T D_i A_i^{-1} R_i$$

- **ASM:**

$$M_{\text{AS}}^{-1} = \sum_{i=1}^{N} R_i^T A_i^{-1} R_i$$

- **Optimized (ORAS):**

$$M_{\text{ORAS}}^{-1} = \sum_{i=1}^{N} R_i^T D_i B_i^{-1} R_i$$

Symmetric variants:

$$M_{\text{OAS}}^{-1} = \sum_{i=1}^{N} R_i^T B_i^{-1} R_i \quad \text{(not recommended)}$$

$$M_{\text{SORAS}}^{-1} = \sum_{i=1}^{N} R_i^T D_i B_i^{-1} D_i R_i \quad \text{(stable)}$$

Here, B_i is the matrix of a Robin-type subdomain problem that improves convergence compared to standard Dirichlet subproblems.

Spectral Bounds for SORAS Let A be the global matrix, and define the bilinear forms:

$$a(\mathbf{U}, \mathbf{V}) := \mathbf{V}^T A \mathbf{U}, \quad b(\mathcal{U}, \mathcal{V}) := \sum_{i=1}^{N} \mathbf{V}_i^T B_i \mathbf{U}_i$$

with operator:

$$\mathcal{R}_{\text{SORAS}}(\mathcal{U}) := \sum_{i=1}^{N} R_i^T D_i \mathbf{U}_i \quad \Rightarrow \quad M_{\text{SORAS}}^{-1} = \mathcal{R}_{\text{SORAS}} B^{-1} \mathcal{R}_{\text{SORAS}}^*$$

The spectral bounds are then given by:

- Upper bound:

$$\gamma_1 := \max_i \max_{\mathbf{U}_i \neq 0} \frac{(R_i^T D_i \mathbf{U}_i)^T A (R_i^T D_i \mathbf{U}_i)}{\mathbf{U}_i^T B_i \mathbf{U}_i} \quad \Rightarrow \quad c_R = \mathcal{N}_c \cdot \gamma_1$$

- Lower bound:

$$\tau_1 := \min_i \min_{\mathbf{U}_i \neq 0} \frac{\mathbf{U}_i^T A_i^{\text{Neu}} \mathbf{U}_i}{\mathbf{U}_i^T B_i \mathbf{U}_i} \quad \Rightarrow \quad c_T = \frac{\tau_1}{\mathcal{M}_c}$$

SORAS Spectral Estimate

$$\frac{\tau_1}{\mathcal{M}_c} \;\leq\; \lambda(M_{\text{SORAS}}^{-1} A) \;\leq\; \mathcal{N}_c \cdot \gamma_1$$

4 Numerical Results and Conclusion

A critical test for the robustness of preconditioners is the simulation of nearly incompressible elastic materials. In such settings, standard iterative methods often face severe difficulties due to the ill-conditioning introduced by the incompressibility constraint.

We consider a two-material model composed of steel and rubber:

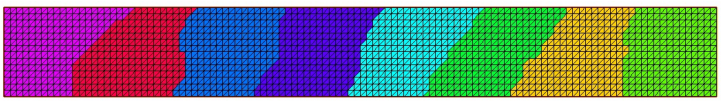
- Steel: $(E_1, \nu_1) = (210 \times 10^9, 0.3)$
- Rubber: $(E_2, \nu_2) = (0.1 \times 10^9, 0.4999)$

The resulting PDE system is discretized using Taylor–Hood mixed finite elements, i.e., $\mathbb{P}_2^d$–$\mathbb{P}_1$ pairs for displacement and pressure. The weak formulation is saddle-point structured, leading to a symmetric but indefinite matrix system:

$$A = \begin{bmatrix} H & B^T \\ B & -C \end{bmatrix}$$

Figure 6: 2D sandwich: steel core and rubber layers.

The domain is partitioned using METIS for optimal subdomain layout:



Comparison of Preconditioners

We test various preconditioners with and without coarse spaces. The results below show the number of GMRES iterations required for convergence, along with the dimension of the coarse space used in each case:

Table 3: GMRES iteration counts for 2D elasticity with increasing subdomains

DOFs	Subdomains	AS	SORAS	AS+ZEM	dim	SORAS+ZEM	dim	AS+GenEO	dim	SORAS+GenEO-2	dim
35841	8	150	184	117	24	79	24	110	184	13	145
70590	16	276	337	170	48	144	48	153	400	17	303
141375	32	497	>1000	261	96	200	96	171	800	22	561
279561	64	>1000	>1000	333	192	335	192	496	1600	24	855
561531	128	>1000	>1000	329	384	400	384	>1000	2304	29	1220
1077141	256	>1000	>1000	369	768	>1000	768	>1000	3840	36	1971

Interpretation

The results clearly demonstrate:

- **Standard ASM and SORAS fail** to converge beyond 64 subdomains.
- Adding a coarse space (ZEM or GenEO) drastically reduces iteration counts.
- GenEO delivers both robustness and scalability even at 256 subdomains.

4.1 Scalability Tests in 2D and 3D

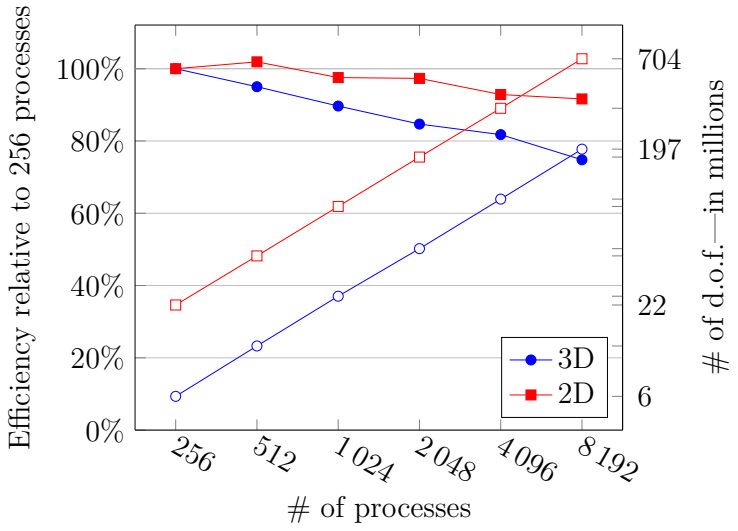
We now test the strong and weak scalability of the proposed methods on large-scale problems.

Strong Scalability: Stokes in 3D We consider a driven cavity problem for the Stokes system with automatic mesh partitioning.

	N	Factorization	Deflation	Solution	# of it.	Total	# of d.o.f.
3D	1 024	79.2 s	229.0 s	76.3 s	45	387.5 s	$50.63 \cdot 10^6$
	2 048	29.5 s	76.5 s	34.8 s	42	143.9 s	
	4 096	11.1 s	45.8 s	19.8 s	42	80.9 s	
	8 192	4.7 s	26.1 s	14.9 s	41	56.8 s	
2D	1 024	5.2 s	37.9 s	51.5 s	51	95.6 s	$100.13 \cdot 10^6$
	2 048	2.4 s	19.3 s	22.1 s	42	44.5 s	
	4 096	1.1 s	10.4 s	10.2 s	35	22.6 s	
	8 192	0.5 s	4.6 s	6.9 s	38	12.7 s	

Peak performance: 50 million degrees of freedom (DoFs) in 3D solved in 57 seconds.
Hardware: IBM/Blue Gene Q with 1.6 GHz Power A2 processors (IDRIS-GENCI allocation).

Weak Scalability: Heterogeneous Elasticity in 3D We also test weak scalability using a steel–rubber sandwich geometry and automatic mesh partitioning.



(a) Timings of various simulations

200 million unknowns solved in 200 seconds in 3D.

Hardware: IBM/Blue Gene Q, demonstrating excellent weak scalability with GenEO-based methods.

4.2 Conclusion

Throughout these notes, we have developed a comprehensive framework for understanding and implementing **two-level domain decomposition methods** based on *spectral coarse spaces*. These methods address the core challenge of scalability in solving large, heterogeneous PDE systems.

What We Achieved

We started by identifying the limitations of classical one-level additive Schwarz methods, particularly in terms of weak scalability and sensitivity to coefficient heterogeneity. To overcome these, we introduced **coarse space corrections**—a global enhancement mechanism designed to efficiently transmit long-range information across the domain.

We examined in detail:

- The **Nicolaides coarse space** — simple, fast, but insufficient for high-contrast problems.
- The **GenEO method** — a robust spectral approach based on local generalized eigenproblems, capturing slow-to-converge modes adaptively.
- The **Fictitious Space Lemma** — a unifying theoretical tool to analyze convergence and design variants such as SORAS and ORAS.

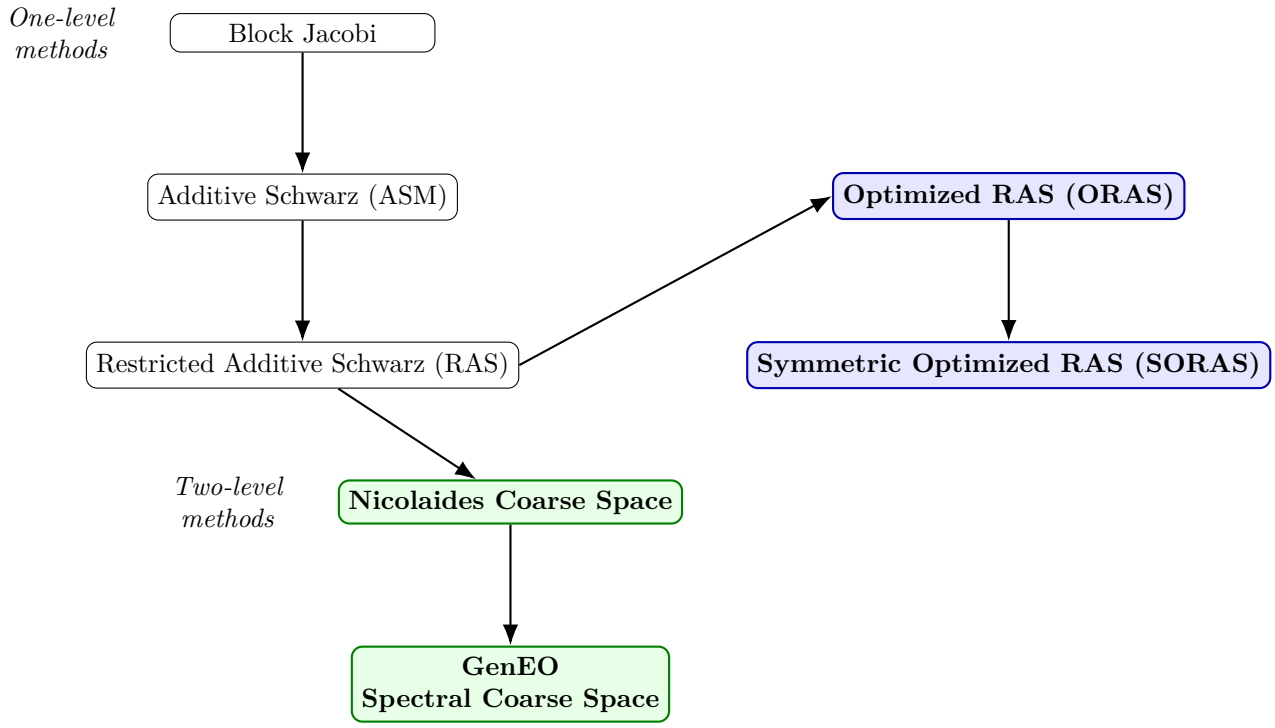


Figure 7: Hierarchy of Schwarz-based methods and their two-level extensions.

Key Takeaways

- **Spectral coarse spaces are essential** for ensuring robustness and scalability in the presence of multiscale and high-contrast features.

- **GenEO adapts automatically** to material heterogeneity, reducing iteration counts and enabling convergence even at massive scales.
- **Convergence guarantees** are rigorously established for a wide class of methods, including additive Schwarz, optimized Schwarz, and Balancing Neumann–Neumann (BNN).

Software Ecosystem

The algorithms discussed are not just theoretical—they are fully implemented and publicly available:

- **HPDDM** (High-Performance Domain Decomposition Methods): a C++/MPI library with scalable solvers and spectral coarse space tools.
- **FreeFem++** plugin `ffddm`: a high-level interface allowing fast prototyping and integration with finite element simulations.

Final Reflection

Domain decomposition is not just a computational trick—it is a mathematical principle of localization and reconstruction. The move from purely local solvers to globally informed methods like GenEO marks a significant step forward in scalable scientific computing.

Scalability is not just about splitting the work—it's about connecting the pieces.

Further Reading

Many of the concepts, algorithms, and theoretical results presented in these notes are covered in greater depth in the book:

> *An Introduction to Domain Decomposition Methods: Algorithms, Theory, and Parallel Implementation*,
> Victorita Dolean, Pierre Jolivet, Frédéric Nataf, SIAM, 2015.

The book provides a comprehensive introduction to both the practical and mathematical foundations of domain decomposition methods, including parallel implementation strategies and convergence analysis. It is freely available at:

`https://www.ljll.fr/nataf/OT144DoleanJolivetNataf_full.pdf`

If you refer to this book in your work, please cite it using the following BibTeX entry:

```
@book{Dolean:2015:DDM,
  title={An Introduction to Domain Decomposition Methods: Algorithms, Theory,
  and Parallel Implementation},
  author={Dolean, Victorita and Jolivet, Pierre and Nataf, Fr{\'e}d{\'e}ric},
  volume={144},
  year={2015},
  publisher={Society for Industrial and Applied Mathematics},
  doi={10.1137/1.9781611974058}
}
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