On the complexity of the edge guarding problem ☆

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Abstract

We revisit the complexity of the edge guarding problem on polyhedral terrains. We prove that it is NP-hard to decide whether there exists an edge set of size k that covers all of the faces of an n-vertex triangulated terrain. To such end, we introduce the notion of (F, H)-*transversals*. Also, we present a family of maximal planar graphs on n vertices that require at least $(n-2)/3$ edge guards to be covered. This reduces the gap between the previously known lower and upper bounds on the minimum edge guard set for such graphs.

1 Introduction

Since its formulation by Victor Klee in the 1970s, the *art gallery* problem has stimulated an increasing number of researchers, notably from the computational geometry community. The original question asks for the minimum number of guards that can patrol the interior of a gallery. In its standard version, a gallery is represented by a simple polygon and guards are placed at fixed points belonging to this polygon. Chvátal [4] was the first to prove that $\lfloor n/3 \rfloor$ guards are always sufficient and sometimes necessary. By using the fact that the triangulation of a simple polygon is 3-colorable, Fisk [7] designed a simpler and elegant demonstration for the same problem.

Several generalizations have also been studied, such as considering polygons with holes and orthogonal polygons, or allowing guards to patrol along areas of different shapes. We are concerned with the problem of guarding polyhedral terrains. A polyhedral terrain T can be viewed as the graph of a polyhedral function $z = F(x, y)$, defined over the xy-plane [5]. Given two points u and v on T, we say that u is *visible* from v if the line segment $\overline{uv}$ does not intersect any point strictly below T. The *visibility region* of a point u is defined by the set of points on T visible from u. If guards are supposed to have fixed positions, i.e., if they cannot move during the surveillance, they are called *point guards*. If we further restrict their positions to the terrain vertices only, we call them *vertex guards*. In another interesting variety, called *edge guards*, they are allowed to patrol along a straight line, usually the terrain edges.

Early results on the hardness of guarding polyhedral terrains were presented by Cole and Sharir [5], who showed that it is NP-complete to determine the minimum number of vertex guards that collectively see the whole terrain. Based on such proof, Zhu [13] showed that it is also NP-complete to compute the smallest edge guard set. Both works, however, require the construction of elaborate terrains where a reduction from 3-SAT is carried out. Employing recent complexity achievements for the cycle transversal problem on planar graphs, we have succeeded in developing an improved polynomial reduction that constructs a simpler terrain for demonstrating the NP-hardness of the edge guarding problem.

Regarding lower and upper bounds on the number of edge guards, Everett and Rivera-Campo [6] and Bose et al. [2] have simultaneously provided the best known bounds so far. While any terrain can be guarded using at most $\lfloor n/3 \rfloor$ edge guards [2, 3, 6], the corresponding lower bound is only $\lfloor (4n-4)/13 \rfloor$ [2]. More recently, Kaučič et al. [9] have claimed to find an inconsistency in Bose et al. [2]'s lower bound demonstration, which would weaken it to the value of $\lfloor (2n-4)/7 \rfloor$. In response to [9], Bose [1] has presented a more detailed proof ensuring that his previous result [2] was in fact correct. Here, we also consider reducing the gap between the best known lower and upper bounds so far. In this direction, we show a family of maximal planar graphs on n vertices whose minimum edge guard set is of size $(n-2)/3$.

2 Preliminaries

Let $G = (V, E)$ be an arbitrary graph, and let G_1, G_2 be two subgraphs of G. If G_1 and G_2 are disjoint and there is no edge connecting both, then we say that G_1 and G_2 are *independent*. A collection of subgraphs of G is independent if their members are pairwise independent.

Planar 3-SAT$_{2+1}$. Given a 3-SAT formula φ in conjunctive normal form, its *incidence graph* is a bipartite graph $G[\varphi] := (V, E)$, with partitions (V_c, V_v), where the sets V_c and V_v correspond to clauses and

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variables in φ, respectively. The edges in $G[\varphi]$ denote inclusion between clauses and variables. If the graph $G[\varphi]$ is planar, we say φ is a *planar* 3-SAT instance. The problem of deciding whether such a formula is satisfiable is known to be NP-complete [10].

The 3-SAT$_{2+1}$ problem is a 3-SAT variation characterized by clauses with 2 or 3 literals, whose variables occur exactly 3 times, twice positively and once negatively. The standard decision problem defined over these formulas is still NP-complete [11]. Based on arguments from [10] and [11], we can state that Planar 3-SAT$_{2+1}$ remains hard to solve:

Lemma 1 *Planar 3-SAT$_{2+1}$ is NP-complete.*

Triangle transversal. Let G be an arbitrary graph with no loops and multiple edges, and let H be any given family of graphs. An *H-subgraph* of G is an induced subgraph of G isomorphic to an element of H. Let F be another fixed family of graphs. Then, a collection X of F-subgraphs of G is an *(F, H)-transversal* of G if every H-subgraph in G is intersected by members of X. In case F is composed by vertices only, the set X is simply called an *H-transversal*.

Given a graph G and an integer $k > 0$, the problem of deciding whether G has an H-transversal of size at most k was proved to be NP-complete [12]. Dichotomy results about C_k-transversals were presented in [8] for bounded degree graphs. In this paper, we are interested on the case where H consists only of triangles, and F is composed by edges, i.e., we deal with (C_2, C_3)-transversals. For obvious reasons, these are termed *edge-triangle-transversals*.

3 Main results

We use the fact that if the polyhedral surface is convex then the visibility region of any vertex is limited to its incident faces. Thus, the computation of an edge guard set can be reduced to an edge-triangle-transversal query on planar triangulations. First, we show that the decision version of the edge-triangle-transversal problem restricted to planar graphs is as hard to solve as Planar 3-SAT$_{2+1}$, which is guaranteed to be NP-complete by Lemma 1.

Theorem 2 *The edge-triangle-transversal problem for planar graphs is NP-complete.*

Proof. Clearly, it is in NP. To prove its NP-hardness, let F be an instance of planar 3-SAT$_{2+1}$ with variables $x_1, x_2, \ldots, x_n$ and clauses $C_1, C_2, \ldots, C_m$.

Variables. For each variable x_i, we associate a subgraph G_i in G, as illustrated in Fig. 1a. Notice that any transversal of G_i has at least 3 edges because it has a maximum of three independent triangles. Observe that if either b_i or b'_i belongs to a transversal

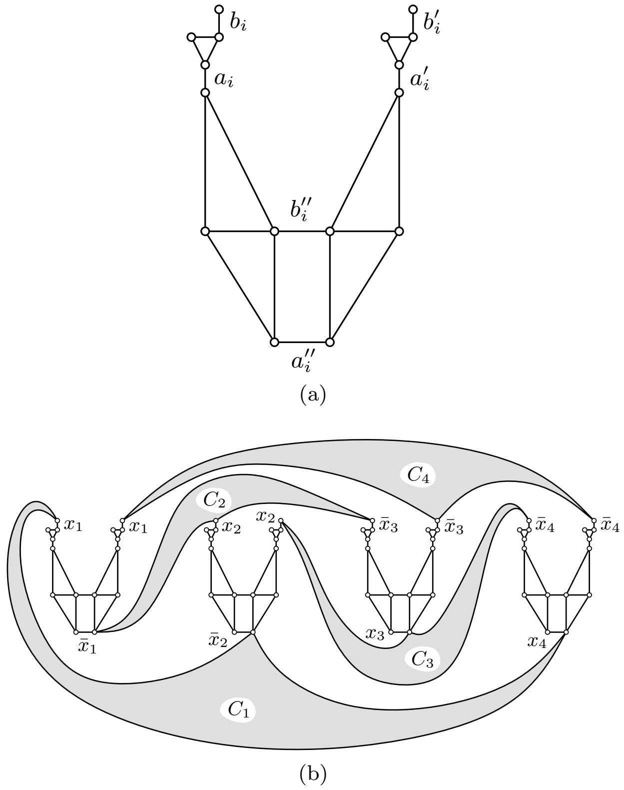


Figure 1: (a) Subgraph G_i associated with variable x_i. (b) A schematic view of the resulting graph associated to the Planar 3-SAT$_{2+1}$ formula $(x_1 \vee \bar{x}_2 \vee x_4) \wedge (x_1 \vee \bar{x}_3 \vee \bar{x}_4) \wedge (x_2 \vee x_3 \vee \bar{x}_4) \wedge (\bar{x}_1 \vee x_2 \vee \bar{x}_3)$.

together with a''_i, then it will be necessary to use an additional edge, e.g., b'_i, so as to cover the remaining untouched triangles. Furthermore, if a_i is in the same transversal as b''_i, then we can replace a_i with b_i without changing the transversal size. The same holds for a'_i and b'_i. Thus it is always possible to select either sets $\{a_i, a'_i, a''_i\}$ or $\{b_i, b'_i, b''_i\}$.

The upper vertices of edges b_i and b'_i, and any vertex of edge a''_i are used as connectors between G_i and the F clauses. Furthermore, if x_i (resp. $\bar{x}_i$) occurs twice in F then edges b_i and b'_i correspond to these occurrences and edge a''_i to the occurrence of its negation $\bar{x}_i$ (resp. x_i).

Clauses. For each clause C_j, $j = 1, 2, \ldots, m$, construct a triangle whose vertices represent its literals. When the clause has only two literals, we simply add an artificial vertex, since it has no influence on any assignment of *true* values. To finish the reduction, let $k = 3n$. Figure 1b illustrates the whole reduction. Next, it is shown that F is satisfiable if and only if G has an edge-triangle-transversal of size exactly $3n$.

First, suppose F is satisfiable. Let X be an initially empty transversal. If the variable x_i occurs twice with value *true* in F, we insert $\{b_i, b'_i, b''_i\}$ into X. Otherwise, we pick $\{a_i, a'_i, a''_i\}$. It is easy to see that every triangle in G is covered by X and that $|X| = 3n$.

Now, suppose X is an edge-triangle-transversal of G of size $3n$. Since each G_i has at most 3 independent triangles and G has n independent copies of G_i, the set X is irreducible. The smallest transversal of G_i can be either $\{a_i, a_i', a_i''\}$ or $\{b_i, b_i', b_i''\}$. If the first subset is selected, we assign *true* to the literals x_i occurring only once. Otherwise, we assign *true* to the literals x_i with double occurrence. □

Actually, Theorem 2 implies a stronger result that says the edge-triangle-transversal problem remains NP-complete even for planar graphs of maximum degree five. It is equally interesting that it remains a hard computational task when restricted to triangulations.

Theorem 3 *The edge-triangle-transversal problem is NP-complete for maximal planar graphs.*

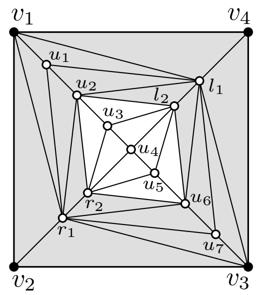
Proof. Let G be an arbitrary planar graph. The proof consists in transforming G into a maximal planar graph G' whose edge-triangle-transversal is trivially determined from any transversal for G.

The triangles in G are kept untouched, while all the other faces are triangulated as follows. Let $f = (v_1, v_2, \ldots, v_s)$ be a face in G of size $s > 3$. We begin by splitting f into two cycles towards a path joining the vertices v_1 and $v_{\lfloor s/2 \rfloor+1}$. This path is composed by 8 edges, namely, (v_1, u_1), (u_1, u_2), $\ldots$, $(u_7, v_{\lfloor s/2 \rfloor+1})$. This gives rise to two cycles C_r and C_l containing $\lfloor n/2 \rfloor + 8$ and $n - \lfloor n/2 \rfloor + 8$ vertices each. In the interior of cycle C_r (resp. C_l), we insert vertices r_1 and r_2 (resp. l_1 and l_2), which are then connected to the vertices $\{v_1, v_2, \ldots, v_{\lfloor s/2 \rfloor+1}\} \cup \{u_1, u_2, u_6, u_7\}$ and $\{v_{\lfloor s/2 \rfloor+1}, v_{\lfloor s/2 \rfloor+2}, \ldots, v_1\} \cup \{u_2, u_3, \ldots, u_6\}$, respectively. Figure 2 illustrates this construction for 4- and 5-faces. All vertices, edges, and faces created during this step are called *false*. Otherwise, we call such entities *true*.

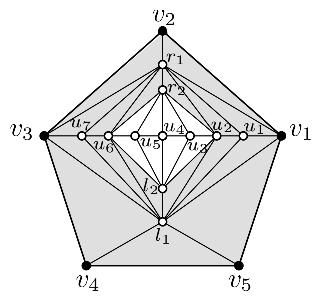
Let ℓ denote the number of non-triangular faces in G, and let G' be a maximal planar graph resulting from the construction process just described. Given any positive integer k, we claim that G has an edge-triangle-transversal of size at most k if and only if G' has an edge-triangle-transversal of size not exceeding $k + 2\ell$.

Suppose X is an edge-triangle-transversal for G of size k. Let $X' = X \cup E^*$, where E^* is the set of all edges of type (r_1, r_2) and (l_1, l_2) (see Fig. 2). We claim that X' is an edge-triangle-transversal of G'. First, observe that every false face has a vertex in E^*, while the true ones are touched by elements of X. Moreover, the following holds: $|X'| \leq |X| + |E^*| \leq k + 2\ell$.

We shall now assume that there exists an edge-triangle-transversal X' for G' of size at most $k + 2\ell$. Clearly, even if all true edges are selected, there will be uncovered faces (see unshaded areas in Fig. 2). The key observation is that, in any circumstance, we can always choose the edges belonging to E^*. Thus, true faces must be covered only by true edges. Hence $X = X' \setminus E^*$ is a transversal of G. Since E^* has exactly 2ℓ edges, the inequality $|X| \leq k$ holds. □



(a) 4-face.



(b) 5-face.

Figure 2: Examples of constructions of triangulations for (a) 4-faces and (b) 5-faces. The shaded regions indicate the sole triangles covered by edges with end points at the outer cycle.

Now, the computational complexity of the edge guarding problem over polyhedral terrains turns out to be easily characterizable:

Theorem 4 *The edge guard problem for triangulated polyhedral terrains is NP-hard.*

Proof. Given a maximal planar graph G, we construct a terrain T as follows. For each face f in G, insert a point p at its centroid and connect it to the boundary vertices of f. Then, slightly translate p by h along the z direction, with $h < 0$, forming a small pit. For convenience, the new edges are labeled *false*. Otherwise they are called *true*.

Let X be an edge-triangle-transversal of G with size $k > 0$. We claim that the collection X is an edge-triangle-transversal of G if and only if it is also an edge guard set for T.

Since every pit in T can be entirely seen from its rim, the edges in X are always sufficient to cover the whole terrain T constructed as above. Suppose now that X' is an arbitrary guard set for T returned by some algorithm, for example, a standard greedy one. Mark the edges in X' as true or false. Note that any true edge in X' covers at least the same number of faces as a false one. Thus, it is likely that an optimal solution for T consists only of true edges. Otherwise, observe that we can always replace a false edge in X' by any true edge incident to it. Hence $X = X'$ is an edge-triangle-transversal of G. □

Lower bound. In Refs. [2] and [3], it was argued whether it would be possible to reduce the gap between sufficiency and necessity for edge guarding triangulated terrains. Alternatively, whether there

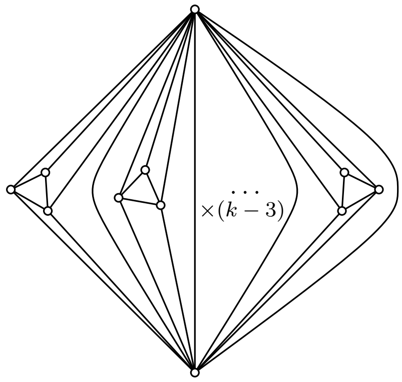


Figure 3: Planar triangulation with $n = 3k+2$ vertices that requires $(n-2)/3$ edge guards, where k is the number of independent triangles.

would be a planar triangulation on 9 vertices requiring exactly 3 edge guards, because the gap between $\lfloor(4n-4)/13\rfloor$ and $\lfloor n/3\rfloor$ is only stressed for graphs with more than 8 vertices. A brute-force solution would be to enumerate all planar triangulations on n vertices, to compute all possible edge guard sets for each one of them, and check if their sizes are all above $\lfloor(4n-4)/13\rfloor$. We have observed, however, that the two-connected planar graph presented in [2] for proving the best known lower bound so far could be triangulated without the addition of new vertices, and thus extending this result to planar triangulations:

Theorem 5 *There exists a maximal planar graph on n vertices, with $n \equiv 2 \mod 3$, that requires $\lfloor n/3\rfloor$ edge guards.*

Proof. We proceed by modifying the graph presented in [2, Fig. 6]. It is composed by k disjoint triangles arranged side-by-side, and two vertices, one above and the other below the base line where these triangles are placed. Additionally, we insert $k-1$ edges linking the upper and the lower vertices, passing through the regions between pairs of consecutive triangles. The resulting maximal planar graph G is composed by $3k+2$ vertices, as shown in Fig. 3. Since G has a maximum of k independent triangles, the size of any edge guard set is at least $(n-2)/3$. □

4 Conclusion

Recent results in graph theory [8] have motivated us to provide a purely combinatorial proof for the edge guarding problem. In fact, our results also extend to the vertex guard version, after contracting some edges in the gadgets we have designed.

In the proof of Theorem 2, we have produced a planar graph whose maximum degree does not exceed 5. An interesting open question is whether there exists an equivalent bound for maximal planar graphs.

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