1.

Let ~ be a categorical equvalent relation. Define $N_\sim = \{g \mid g \sim e\}$ where e an identity arrow. Then it forms a subgroup of G : e is the identity in $N_\sim$, $gh \in N_\sim$ for $g, h \in N_\sim$, and $e \sim gg^{-1} \sim eg^{-1} \sim g^{-1} \to g^{-1} \in N_\sim$. Also, we can see that $xN_\sim = \{xg \mid g \sim e\} = \{x(x^{-1}h) \mid x^{-1}h = g \sim e\} = \{h \mid h \sim x\}$ and $N_\sim x = \{gx \mid g \sim e\} = \{(hx^{-1})x \mid hx^{-1} = g \sim e\} = \{h \mid h \sim x\}$, concluding that $N_\sim$ forms a normal subgroup in G, i.e., a equivalence relation can be expressed as a normal subgroup. Conversely, suppose N is a normal subgroup of G. Define $\sim_N$ by $f \sim g$ iff and $fg^{-1} \in N$. Then since N is a subgroup, $\sim_N$ is an equivalence relation. In addition to that, since N is a normal subgroup, $hfg^{-1}h^{-1} = hgkk^{-1}g^{-1}h^{-1} \in N$ and hence $hfk \sim_N hgk$ for all $h, k \in G$. Therefore, $\sim_N$ is a congruence in the category.

If $g, h \in G$ are in the same coset of N then $fg^{-1} \in G$, concluding that $f \sim_N g$. Therefore, the quotient $G/\sim$ coincide when N corresponds to $\sim$.

2.

Recall slice category: **Sets**$/I$ consists of objects and arrows where objects are functions $f : X \to I$ for some $X \in$ **Sets** and arrows are fucntions of the form $a : X \to Y$ satisfying $g = a \circ f$ where $f : X \to I$ and $g : Y \to I$. Let $f : G \to I$ be a group in this category. Then it has a binary operation $m : f \times f \to f$, identity arrow $i : 1 \to f$, and inverse $i : f \to f$. The identity arrow u is indeed a map $u : 1 \to X$ such that $fu = e_i : 1 \to I$ where $e_i(*) = i$. We can regard $f \times f : X \times X \to I$ as a pullback of f, f and $m : X \times X \to X$ a map from the pullback. Inverse is a map $i : X \to X$ defined by $fi = f$. As a result, we can make the following commutative diagrams:

$$\begin{tikzcd}
 & I & \\
X \times X \arrow[ur, "f \times f"] \arrow[r, "m"] & X \arrow[u, "f"'] & X \arrow[l, "i"'] \arrow[ul, "f"'] \\
 & 1 \arrow[u, "u"'] \arrow[uu, bend left=40, "e_i"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
(X \times X) \times X \arrow[rr, "\cong"] \arrow[d, "m \times 1"] & & X \times (X \times X) \arrow[d, "m \times 1"] \\
X \times X \arrow[dr, "m"] \arrow[ddr, bend right, "f \times f"'] & & X \times X \arrow[dl, "m"'] \arrow[ddl, bend left, "f \times f"] \\
 & X \arrow[d, "f"] & \\
 & I &
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "\langle u, 1_X \rangle"] \arrow[d, "\langle 1_X, u \rangle"] \arrow[dr, "1_X"] & X \times X \arrow[d, "m"] & \\
X \times X \arrow[r, "m"] & X \arrow[r, "f"] & I
\end{tikzcd}
\qquad
\begin{tikzcd}
X \times X \arrow[d, "1_X \times i"] & X \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "u"] & X \times X \arrow[d, "i \times 1_X"] \\
X \times X \arrow[r, "m"] & X \arrow[d, "f"] & X \times X \arrow[l, "m"'] \\
 & I &
\end{tikzcd}$$

Fix $i \in I$ and set f_i, X such that $f_i(X) = i$. When replacing f by f_i, the function becomes trivial, and hence the form of X must by the ordinary group in **Sets**. Therefore, we can conclude that the group f consists of ordinary groups $f^{-1}(i)$ where $i \in I$.

3.

Let G be an abelian group. We will show that it is the object of groups of category of groups. It is enough to show to determine the group homomorphism $m : G \times G \to G$ satisfying the definition of groups in a category. Define $m(a, b) = a + b$, $u : 1 \to G$ where $u(*) = 0_G$, and $i : G \to G$ by $i(a) = -a$. Then all arrows are homomorphism in **Groups** and hence G is a group in a category of groups.

4.

Don't understand the meaning!!!!!!!!

5.

(a) Since cokernel is the coequalizer of f and $u : A \to B$ where $u(a) = 0$ for all $a \in A$, the property $c \circ f = 0$ is obvious. For the other property, define $u : C \to G$ by $u(\bar{b}) = g(b)$. Then it is well-defined: if $\bar{a} = \bar{b}$, then $a - b \in \mathrm{Im} f$. Therefore, $g(a - b) = g(a) - g(b) = 0 \Rightarrow g(a) = g(b)$ and hence $u(\bar{a}) = u(\bar{b})$. Therefore, $g = u \circ c$. If $v : C \to G$ do the same role as u, they coincide for all $\bar{x} \in C$ since they are determined completely by g. Hence $u = v$.

(b) skip

(c) Let $c : B \to C$ be the cokernel of f and $i : K \to B$ the kernel of c. By the definition of coequalizer, $c \circ f = 0$ and by the definition of kernel, $c \circ i = 0$. Then by the definition of equalizer, there is a unique $v : A \to K$ such that $f = i \circ v$. Therefore, f factors through K. Furthermore, also by the definition of coequalizer and quotient, $\mathrm{Im} f \to B \to C$ is zero map and hence again there is a unique map $u : \mathrm{Im} f \to K$ such that $j : \mathrm{Im} f \to B = i \circ u$. Note that v is monic, hence injective, proving that $\mathrm{Im} f \subseteq K$. On the other hand, we can see that $K \subseteq \mathrm{Im} f$, concluding that $\mathrm{Im} f = K$.

6.

skip

7.

By definition, $f \sim f', g \sim g'$ implies that $gf \sim gf', gf' \sim g'f'$. By transitivity, $gf \sim g'f'$.

8.

1. If $f \sim g$, then $HF = HG \to H(f) = H(g)$. Also, if $f \sim g$, then $HF = HG \to H(hfk) = H(h)H(f)H(k) = H(h)H(g)H(k) \to hfk \sim hgk$.

2. Obviously $\sim$ is an equivalence relation.

Therefore, $\sim$ is a congruence. By definition, a category $\mathbf{D}/\sim$ consists of objects: objects in $\mathbf{D}$, arrows: $[f]$. We can assign a functor $\mathbf{D} \to \mathbf{D}/\sim$ as a quotient functor. Suppose $H : \mathbf{D} \to \mathbf{E}$ coequalizes F, G. Then by definition, we can define a functor $K : \mathbf{D}/\sim \to \mathbf{E}$ by $K([f]) = H(f)$. It is well-defined and such functor is unique. Therefore this category is a coequalizer.

9.

Obvious.