1.

Let $n = |N|, m = |M|$. Note that $N^M = \{f : M \to N\}$. The result is trivial.

2.

$(A \times B)^C \cong A^C \times B^C$: Note that $1_{(A\times B)^C} : (A \times B)^C \to (A \times B)^C$. Then $\bar{1}_{(A\times B)^C} = \epsilon \circ (1 \times C)$: $(A \times B)^C \times C \to A \times B$. Thus $\widetilde{\pi_1 \circ \bar{1}} : (A \times B)^C \to A^C$ and $\widetilde{\pi_2 \circ \bar{1}} : (A \times B)^C \to B^C$ and hence we set the function

$$f = (\widetilde{\pi_1 \circ \bar{1}}, \widetilde{\pi_2 \circ \bar{1}}) : (A \times B)^C \to A^C \times B^C.$$

Now Define $g = \widetilde{(\bar{\pi}_1, \bar{\pi}_2)} : A^C \times B^C \to (A \times B)^C$. Then we can see that $g = f^{-1}$.

$(A^B)^C \cong A^{B\times C}$: Define $\alpha_Z : Z \times (B \times C) \to (Z \times C) \times B$ be a canonical isomorphism. In this case, note that the evaluation function is of the form $\epsilon : (A^B)^C \times (C \times B) \to A$ defined by $\epsilon(f, (c, b)) = f(c)(b)$. Then $\epsilon \circ \alpha_{(A^B)^C} : (A^B)^C \times (C \times B) \to A$. Thus we can define $f = \epsilon \circ \widetilde{\alpha_{(A^B)^C}}$: $(A^B)^C \to A^{(B\times C)}$. On the other hand, from $A^{B\times C}$, we can see $\epsilon : A^{B\times C} \times (B \times C) \to A$. Then we can set $g = \epsilon \circ \widetilde{\widetilde{\alpha^{-1}_{A^{(B\times C)}}}}$.

3.

For $\epsilon : B^A \times A \to B$, $\tilde{\epsilon} : B^A \to B^A$. Then the evaluation of ev: $\tilde{\epsilon} : \epsilon \circ (\tilde{\epsilon} \times 1_A)$ defined by $\mathrm{ev}(\tilde{\epsilon}(g), a) = \tilde{\epsilon}(g)(a)$. Since the transpose is unique and 1_{B^A} can play the role of $\tilde{\epsilon}$, we conclude that $\tilde{\epsilon} = 1_{B^A}$. For $1 : A \times B \to A \times B$, $\tilde{1} : A \to (A \times B)^B$ is defined by $(\tilde{1}(a))(b) = (a, b)$. Finally, $\widetilde{\epsilon \circ \tau} : A \to B^{B^A}$ maps $a \in A$ to $\alpha : B^A \to B$ defined by $\alpha(g) = g(a)$ for $g \in B^A$.

4.

Suppose **Mon** is cartesian closed. Then $X \times 0 \in$ **Mon** and we can see $\mathrm{Hom}(X \times 0, Y) \cong \mathrm{Hom}(X, Y^0) = \{*\}$. Thus we conclude that $X \times 0$ is an initial object. From the fact that initial and termianl object is identical in **Mon**, we can see that $\mathrm{Hom}(X, Y) \cong \mathrm{Hom}(1 \times X, Y) \cong \mathrm{Hom}(1, Y^X) = \{*\}$ and therefore, all hom-set is trivial, which is false in **Mon**. As a result, we conclude that **Mon** is not cartesian closed.

5.

Obviously the category **Graph** is CCC. We will determine $\mathbf{2}^G$ for an arbitrary graph G. As in the book, the vertex in $\mathbf{2}^G$ consists of $\phi : G_v \to \{v_0, v_1\}$ and the edge $\theta : G_e \to (v_0 \to v_1)$ can be defined by the commutative diagram

$$\begin{tikzcd}
G_v \arrow[d, "\phi"'] & G_e \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "\theta"'] & G_v \arrow[d, "\psi"'] \\
\{v_0, v_1\} & (v_0 \to v_1) \arrow[l, "s"'] \arrow[r, "t"] & \{v_0, v_1\}
\end{tikzcd}.$$

Thus we conclude that in $\mathbf{2}^G$, two vertices ϕ and ψ adjacent with an edge θ is ϕ maps all sources in G_e to v_0 and ψ maps all targets in G_e to v_1. Now suppose G is a directed graph of the form

$$\begin{tikzcd}
a \arrow[r] & b & c \arrow[l] \\
& d \arrow[u] &
\end{tikzcd}.$$

Then the exponential $\mathbf{2}^G$ consists of 16 vertices which are arbitrary maps $G_v \to \{v_0, v_1\}$ and there is an edge between 2 maps φ and ψ where φ maps $\{a, c, d\}$ to v_0 and ψ maps b to v_1.

6.

It's easy to see that (A,R) can be identified as an undirected graph G where $A = G_v, R = G_e$. An undirected graph is a special case of graph and indeed it is a subcategory of the category of graph. Thus by previous exercise, we conclude that such category is CCC.

7.

Consider $f : (A,P) \to (B,Q)$. Then we can see the function as a composition of two functions: $f_1 : A/P \to B/Q$ and $f_2 : P \to Q$. Now consider a slice category $\mathbf{Sets}/2$. We can see the function f as an arrow in $\mathbf{Sets}/2$: For two objects $a : A \to 2$ and $b : B \to 2$, define $P = a^{-1}(1)$ and $Q = b^{-1}(1)$. Then f is just an arrow $a \to b$. Now it is obvious that $\mathbf{Sets}/2$ is identical to $\mathbf{Sets} \times \mathbf{Sets}$: $a \mapsto (a^{-1}(0), a^{-1}(1))$ and $(A,B) \mapsto f$ where $f(A) = 0, f(B) = 1$. Since $\mathbf{Sets}$ is a CCC, so is $\mathbf{Sets} \times \mathbf{Sets}$ and thus the category $\mathbf{Sub}$ is CCC.

8.

Initial and terminal points of the category of pointed sets are identical: (a,a). But then as in exercise 4, this category cannot have an exponential object, concluding that this category is not CCC.

9.

Note that $1 \times A \cong A$. From this, we can get $\mathrm{Hom}(A,B) \cong \mathrm{Hom}(1 \times A, B) \cong \mathrm{Hom}(1, B^A)$ where the map is usual transpose map in CCC. Thus they are bijective.

10.

ωCPO : Let A, B be ωCPO's. Define the product $A \times B = \{(a,b) \mid (a,b) \leq (c,d) \text{ iff } a \leq c, b \leq d\}$. Obviously it is ωCPO. For exponential, define B^A be a set of monotone continuous functions from A to B with ordering $f \leq g$ iff $f(a) \leq g(a)$ for all $a \in A$. It is still ωCPO and hence ωCPO is CCC.

strictωCPO: An ωCPO category is called a strict ωCPO if the arrow preserves the least element, $\bot$. Consider an object $\{\bot\}$. Then it is ωCPO and it is both initial and terminal. Therefore, category of strict ωCPO is not a CCC.

11.

Note that $a \Rightarrow b$ is equivalent to $\neg a \vee b$.

(a) By definition of $\Rightarrow$, the formula is always true, i.e., the value is $\top$:

$$\begin{aligned}((p \vee q) \Rightarrow r) \Rightarrow ((p \Rightarrow r) \wedge (q \Rightarrow r)) &\Leftrightarrow \neg((p \vee q) \Rightarrow r) \vee ((p \Rightarrow r) \wedge (q \Rightarrow r)) \\ &\Leftrightarrow \neg(\neg(p \vee q) \vee r) \vee ((\neg p \vee r) \wedge (\neg q \vee r)) \\ &\Leftrightarrow ((p \vee q) \wedge \neg r) \vee ((\neg p \wedge \neg q) \vee r) \\ &\Leftrightarrow ((p \vee q) \wedge \neg r) \vee \neg((p \vee q) \wedge \neg r).\end{aligned}$$

(b) Join an meet are coproduct and product in poset category, resp. Hence we can generalize it as

$$C^{A+B} \to C^A \times C^B.$$

Indeed, by the definition of coproduct, there are two arrrows $A \to A+B$ and $B \to A+B$. Since $C^{(-)}$ is a contravariant functor, we have two arrows $C^{A+B} \to C^A$ and $C^{A+B} \to C^B$. As a result, by definition of product, there is a unique arrow $C^{A+B} \to C^A \times C^B$.

12.

Let $\alpha : A \to B$ be an arrow in a CCC and C be an object in the same category. Then we have a map $f : C^B \times A \to C$ by $f(g, a) = \epsilon \circ (g, \alpha(a))$. Thus it induces $\tilde{f} : C^B \to C^A$. Now define F such that $F(A) = C^A$ and $F(\beta) = \tilde{f}$ where $\beta : A \to B$ and $\tilde{f} : C^B \to C^A$ induced from β. It is obvious that F is a contravariant functor.

13.

First, note the canonical isomorphisms:

$$\begin{aligned} \mathrm{Hom}(A \times C + B \times C, X) &\cong \mathrm{Hom}(A \times C, X) \times \mathrm{Hom}(B \times C, X) \\ &\cong \mathrm{Hom}(A, X^C) \times \mathrm{Hom}(B, X^C) \\ &\cong \mathrm{Hom}(A + B, X^C) \\ &\cong \mathrm{Hom}((A + B) \times C, X). \end{aligned}$$

Set $X = (A+B) \times C$ and $A \times C + B \times C$, resp. Then an identity map corresponds to an isomorphism, concluding $(A + B) \times C \cong A \times C + B \times C$.

14.

We show that the only possible D is 1. If $D = \emptyset$, the initial object, then $D^D \cong 1$ and there is no arrow $s : D^D \to D$. If $D \cong 1$, then $D^D \cong 1$ and there is unique $s : D^D \to D$ since D is terminal. Finally, if $|D| \geq 2$, then $|D^D| \geq |2^D| = |\mathcal{P}(D)|$. Therefore, $s : D^D \to D$ cannot be monic. But since rs is an identity map, s must be a mono, contradiction. Hence D is a terminal object.

15.

(a) 2^I is a Heyting Algebra
Define $f \leq g$ iff $f(i) \leq g(i)$ for all $i \in I$. Then we can define meet and join pointwise: $f \wedge g : I \to 2$ such that $(f \wedge g)(i) \leq f(i)$ and $g(i)$ for all i, $f \vee g$ similar to the meet. Also, define 0,1 as constant maps:$I \to \bot$, $I \to \top$. For exponential, define $p \Rightarrow q$ by $(p \Rightarrow q)(i) = \top$ iff $p(j) \leq q(j)$ for all $j \geq i$. Then $a \leq b \Rightarrow c$ iff $\forall i$, $a(i) \leq (b \Rightarrow c)(i)$ iff $\forall i$, $a(i) = \bot$ or $(b \Rightarrow c)(i) = \top$ iff $\forall i \forall j$, $a(i) = \bot$ or $j \geq i \Longrightarrow b(j) \leq c(j)$ iff $\forall i, a \wedge b(i) \leq c(i)$ iff $a \wedge b \leq c$. Thus it is a Heyting Algebra.

(b) By definition, a map $y : \boldsymbol{A} \to 2^{\boldsymbol{A}^{\mathrm{op}}}$ is defined by $y(a) : \boldsymbol{A}^{\mathrm{op}} \to 2$ where $y(a)(x) = \top$ iff $a \leq x$ and $y(a)(x) = \bot$ iff $a > x$. For monotonicity, suppose $a \leq b$. If $x \in \boldsymbol{A}$ satisfies $y(a)(x) = \top$, then $x \leq a$ and hence $x \leq b$ by transitivity, meaning that $y(b)(x) = \top$. Thus we conclude that $y(a) \leq y(b)$, proving monotonicity. For injectivity, consider two elements $a \neq b$. If they are comparable, then $a < b$ or $b < a$. If $a < b$, then $y(a)(b) = \bot$ whereas $y(b)(b) = \top$, concluding that $y(a) \neq y(b)$. If they are not comparable, then we can conclude that $y(a)(b) = \bot$ $y(b)(a) = \bot$. Thus again, $y(a) \neq y(b)$ and hence y is injective.
Don't know how to prove that y preserves CCC structure.

16.

We define $A \times B$ pointwise, i.e., for each $i \in I$, $(A \times B)_i = A_i \times B_i$. Then obviously $A \times B$ satisfies the condition of product.