1.

Let $\Phi : \mathrm{Hom}(C, Z) \to \mathrm{Hom}(A, Z) \times \mathrm{Hom}(B, Z)$ be an isomorphism. Let $a \in \mathrm{Hom}(A, Z)$, $b \in \mathrm{Hom}(B, Z)$. Since Φ is an isomorphism, there is $f \in \mathrm{Hom}(C, Z)$ such that $(a, b) = (f \circ c_1, f \circ c_2)$. Also, f is unique since Φ is an isormorphism. Thus C is a coproduct of A and B. By applying it to $\mathcal{C}^{\mathrm{op}}$, we have the dual statement.

2.

Let $i_A : A \to A + B$, $i_B : B \to A + B$, $i_{MA} : MA \to MA + MB$, $i_{MB} : MB \to MA + MB$, and $\eta_A : A \to MA$, $\eta_B : B \to MB$. Take a forgetful functor $U : \mathbf{Mon} \to \mathbf{Sets}$ and let $e : A+B \to U(MA+MB)$ defined by $e = [U(i_{MA}) \circ \eta_A, U(i_{MB}) \circ \eta_B]$. Take any object $Z \in \mathbf{Mon}$ and $f : A + B \to UZ$. Define $h : MA + MB \to Z$ by $Uh \circ e = f$. By UMP, $Uh \circ e = [Uh \circ Ui_{MA} \circ \eta_A, Uh \circ Ui_{MB} \circ \eta_B] = [U(h \circ i_{MA}) \circ \eta_A, U(h \circ i_{MB}) \circ \eta_B] = [f \circ i_A, f \circ i_B]$. By UMP of monoid, there is a unique map $\overline{f \circ i_A} : MA \to Z$ satisfying $U(\overline{f \circ i_A}) \circ \eta_A = f \circ i_A$. Thus we conclude that $U(h \circ i_{MA}) = U(\overline{f \circ i_A})$, i.e., $h \circ i_{MA} = \overline{f \circ i_A}$. Similarly, $h \circ i_{MB} = \overline{f \circ i_B}$. Therefore, h is uniquely determined by the condition $Uh \circ e = f$ and hence $h = [\overline{f \circ i_A}, \overline{f \circ i_B}]$. Therefore, when we consider the UMP of $A + B$, we can conclude that $MA + MB \cong M(A + B)$.

3.

We only need to show that the map $[f, g]$ is unique. By UMP of free monoid, $[f, g]'$ in p.59 is unique. Since $[f, g]$ is induced from $[f, g]'$, it must be unique.

4.

Note that $A + B = A \coprod B$ in **Sets**. Define $p_1 : \mathcal{P}(A + B) \to \mathcal{P}(A)$, $p_2 : \mathcal{P}(A + B) \to \mathcal{P}(B)$ by $p_1(X) = X \cap A$, $p_2(X) = X \cap B$. Consider a boolean algebra Z and $f : Z \to \mathcal{P}(A)$ and $g : Z \to \mathcal{P}(B)$. Then we can find a map $h : Z \to \mathcal{P}(A + B)$ by $h(z) = f(z) \coprod g(z)$. For uniqueness, suppose $h' : Z \to \mathcal{P}(A + B)$ be another map such that $p_1 h' = f, p_2 h' = g$. By definition, $h(z) \cap A = h'(z) \cap A$ and $h(z) \cap B = h'(z) \cap B$. Since $h(z) = (h(z) \cap A) \coprod (h(z) \cap B) = h'(z)$, we conclude $h = h'$.

5.

skip

6.

Obvious

7.

Let P, Q be projective, $e : E \twoheadrightarrow X$ be an epi, and consider $f : P \to X$, $g : Q \to X$. Then there are $\bar{f}, \bar{g}$ such that $f = e \circ \bar{f}, g = e \circ \bar{g}$. By definition of coproduct, there is a unique $[\bar{f}, \bar{g}] : P + Q \to E$ such that $[\bar{f}, \bar{g}] \circ i_P = \bar{f}, [\bar{f}, \bar{g}] \circ i_Q = \bar{g}$. By definition, there is a unique map $[f, g] : P + Q \to X$ defiend by $e \circ [\bar{f}, \bar{g}] = [f, g]$. Thus $P + Q$ is projective.

8.

An object I is injective if for any mono $m : M \hookrightarrow X$ and a map $f : M \to I$, there is $\bar{f} : X \to I$ such that $f = \bar{f} \circ m$. It is obvious that a map of posets is monic iff it is injective on elements. Let I be a poset with a single element. Then trivially I is injective. Now suppose $I = \{1, 2\}$ is a distinct poset, $M = \{a \le b, c\}$, $X = \{a \le b \le c\}$, and $m : M \hookrightarrow X$ trivial. Define $f : M \to I$ by $a, b \mapsto 1$ and $c \mapsto 2$. Since X is well-ordered, the map from X to I must be trivial. Thus I is not injective.

9.

$\bar{h} = h \circ i : M \to N$ where $h : TM \to N$ a homomorphism. $\bar{h}(xy) = h \circ i(xy) = h(xy) = h(x)h(y) = \bar{h}(x)\bar{h}(y)$ since i is injection.

10.

Obvious

11.

Define $R = \{(f(x), g(x)) \mid x \in A\} \cup \{(g(x) \cup f(x)) \mid x \in A\} \cup \{(b, b) \mid b \in B\}$. Then R is reflexive since $(y, y) \in R$, symmetric since if $(a, b) \in R$, then $(b, a) \in R$, and finally transitive, concluding that $R \subseteq B \times B$ is an equivalence relation. Note that this is the smallest equivalence relation. Define $Q := B/R$ be a set of equivalence classes and $q : B \to Q$ a canonical map. Then clearly $qf = qg$. Let Z be a set and $z : B \to Z$ be a map satisfying $zf = zg$. Define $u : Q \to Z$ by $u([b]) = z(b)$. Clearly it is well-defined and satisfies$uq = z$. Suppose v is another map such that $vq = z$. Then $uq = vq$ and since q is epi, $u = v$.

12.

Note that $P + Q$ is a disjoint union of P, Q with the same order. Consider $P + Q/\sim$ where $0_P \sim 0_Q$ and the canonical map $q : P + Q \to P + Q/\sim$. Then we can consider $i_{0P} : P \to P + Q/\sim$ and $i_{0Q} : Q \to P + Q/\sim$. Let Z be a rooted poset and $z_P : P \to Z$, $z_Q : Q \to Z$. Then there is a unique $u : P + Q \to Z$ as a morphism of poset such that $z_P = ui_P$ and $z_Q = ui_Q$. Since P, Q, Z are rooted posets and the morphism between them are arrows in rooted poset, we have $z_P(0_P) = z_Q(0_Q) = 0_Z$. Therefore, $u(0_P) = u(0_Q) = 0_Z$ and it induces unique arrow $\bar{u} : P+Q/\sim \to Z$ defined by $\bar{u}([x]) = z_P(x)$ if $x \in P$ and $z_Q(x)$ if $x \in Q$. Thus we conclude that $P + Q/\sim = P +_0 Q$.

13.

(1) Define the smallest equiavalence relation ~ defined as in (a). If $hf = hg$, then $hfm = hgm$ and hence $fm \sim gm$. Suppose $n \sim n'$ and $m \sim m'$. If $hf = hg$, then $h(nm) = h(n)h(m) = h(n')h(m') = h(n'm')$, concluding $nm \sim n'm'$.

(2) $N/\sim$ is obviously a monoid. Let $h : N \to X$ be a morphism such that $hf = hg$. Then we can induce a unique $\bar{h} : N/\sim \to X$ by $\bar{h} = hq$ where q is a canonical morphism.

14.

(a) Consider $\ker f = \{(a, a') \mid f(a) = f(a')\}$. Then obviously it is an equivalence relation. Consider $h : X \to A \times A$ such that $fp_1h = fp_2h$. Obviously $h(x) \in \ker f$ for all $x \in X$ and hence there is a map $X \to \ker f$, which is uniquely determined by h. Thus $\ker f$ is the equalizer of fp_1 and fp_2.

(b) Let R be an equivalence relation on A. The map $q : A \to A/R$ is defined by $a \mapsto [a]$. Obviously $\ker q = \{(a, a') \mid q(a) = q(a')\} = \{(a, a') \mid a \sim_R a'\} = R$.

(c) Take $f : A \to B$ with $f(a) = f(a')$ if $(a, a') \in R$. By (a), $\ker f$ is an equivalence relation and hence $\langle R \rangle \subseteq \ker f$. By definition, $fp_1 = fp_2$ and there is a unique map $A/\langle R \rangle \to B$.

(d) The coequalizer of the projections of R to A is $A/\langle R \rangle$ with the projection $q : A \to A/\langle R \rangle$, where $\langle R \rangle$ is the kernel.

15.

Let $|q| : |Y| \to |Y/\langle f = g\rangle| = |Q|$ be a coequalizer of $|f|, |g|$. Then given topological space Z and $h : Y \to Z$ with $hf = hg$, there is a unique $|u| : |Q| \to |Z|$ with $|h| = |u||q| = |uq|$. Let $W \subseteq Z$ be an open set. Then $h^{-1}(W) = q^{-1}u^{-1}(W)$ is open in Y. Thus $u^{-1}(W)$ is open in Q, concluding that u is continuous. Thus Q is a coequalizer of f and g.