**1.**

$$
\begin{aligned}
(\mathcal{F}(h) \circ \phi_A)(a) &= \mathcal{F}(h)(\phi_A(a)) \\
&= \mathcal{F}(h)(\{U \in \mathrm{Ult}(A) \mid a \in U\}) \\
&= \mathcal{P}(\mathrm{Ult}(h))(\{U \in \mathrm{Ult}(A) \mid a \in U\}) \\
&= (\mathrm{Ult}(h))^{-1}(\{U \in \mathrm{Ult}(A) \mid a \in U\}) \\
&= \{V \in \mathrm{Ult}(B) \mid h(a) \in V\} \\
&= (\phi_B \circ h)(a).
\end{aligned}
$$

Therefore, we have a commutative diagram

$$
\begin{tikzcd}
A \arrow[r, "\phi_A"] \arrow[d, "h"'] & \mathcal{F}(A) \arrow[d, "\mathcal{F}(h)"] \\
B \arrow[r, "\phi_B"'] & \mathcal{F}(B)
\end{tikzcd}
.
$$

**2.**

**(i)** $\phi_B$ is a homomorphism: $\phi_B(0) = \{V \in \mathrm{Ult}(B) \mid 0 \in V\} = \varnothing$. Also, $\phi_B(a \wedge b) = \{V \in \mathrm{Ult}(B) \mid a \wedge b \in V\} = \{V \in \mathrm{Ult}(B) \mid a \in V \text{ and } b \in V\} = \phi_B(a) \cap \phi_B(b)$ and $\phi_B(a \vee b) = \{V \in \mathrm{Ult}(B) \mid a \vee b \in V\} = \{V \in \mathrm{Ult}(B) \mid a \in V \text{ or } b \in V\} = \phi_B(a) \cup \phi_B(b)$.

**(ii)** $\phi_B$ is injective: Suppose $a \nleq b$. Then $a \wedge (\neg b) \neq 0$ and hence $\uparrow (a \wedge \neg b)$ is a nontrivial filter. Therefore, $\phi_B(a \wedge (\neg b)) = \phi_B(a) \cap \phi_B(\neg b)$ is not empty by Boolean prime ideal theorem. Let $U$ be one of the member. Then $a \in U$ and $\neg b \in U$, hence $b \notin U$ since $U$ is an ultrafilter, resulting that $\phi_B(a) \neq \phi_B(b)$. The other case is analogous.

**3.**

**(i)**$A(B) \cong \mathbf{Ult}(B)$ : Let $a \in A(B)$ be an atom. Then obviously $\uparrow (a) \in \mathrm{Ult}(B)$. Conversely, let $U \in \mathrm{Ult}(B)$. We will show that $\wedge_{x \in U} x \in A(B)$. Suppose $b \leq \wedge x$. Since $\wedge x \in U$ is minimal, $b \notin U$ and hence $\neg b \in U$. Then $(\wedge x) \vee \neg b = \neg b$ but $b \vee \neg b = 1 \leq \neg b$, meaning that $b = 0$.

**(ii)Lemma7.33:** (a) $b = \bigvee\{a \in A(B) \mid a \leq b\} = y$ : Obviously $b \geq y$. Now suppose $b \nleq y$. Then $b \wedge \neg y > 0$. Since $B$ is finite, there is an atom $x \leq b \wedge \neg y$. Since $x \leq b$, it is a member of the set and hence $x \leq y$. Also $x \leq \neg y$ and hence $x \leq y \wedge \neg y = 0$, contradiction. Thus $b = y$.
(b) $a \in A(B), a \leq b \vee b' \to a \leq b$ or $a \leq b'$ : Suppose $b, b' > 0$ and $a \nleq b, a \nleq b'$. Since $a$ is an atom, $a \wedge b = a \wedge b' = 0$ but then $a \wedge (b \vee b') = (a \wedge b) \vee (a \wedge b') = 0$, contradiction.

**(iii)**$\beta : B \cong \mathcal{P}(A(B))$ : Note that $\beta$ is defined by $\beta(b) = \{a \in A(B) \mid a \leq b\}$. Then the inverse is defined by $\beta^{-1}(X) = \vee_{a \in X} a$ by (ii). Now suppose $S \subseteq A(B)$. Obviously $S \subseteq \beta(\vee_{x \in S} x)$. For $y \in \beta(\vee_{x \in S} x)$, $y \in A(B)$ and $y \leq \vee x$. Then by (ii)(b), there is $x_i \in S$ such that $y \leq x_i$, meaning that $y = x_i$. Therefore, $\beta(\vee x) \subseteq S$ and hence $\beta$ is onto.

**4.**

$$\mathbf{Groups} \xrightarrow{U} \mathbf{Mon} \xrightarrow{V} \mathbf{Sets}$$

$U$ : Faithful: obvious

Full: If $A, B$ are groups and $h : A \to B$ is a monoid homomorphism, then $h(ab) = h(a)h(b)$, $h(e) = h(aa^{-1}) = h(a)h(a)^{-1} = e$, meaning that $h(a^{-1}) = h(a)^{-1}$ and hence a group homomorphism.

Injective on arrows, objects: Obvious
Not surjective on arrows, objects: $\mathbb{N}$ is an example

$V$ : Faithful: obvious
Not full, surjective on arrows: There is a function which is not a monoid homomorphism.
Surjective on objects: By assigning multiplication, we can make any set $X$ a monoid.
Not injective on arrows, objects: Obvious.

**5.**

Define a functor

$$A : \mathbf{Pos} \to \mathbf{Top}$$

by $(X, \leq) \mapsto (X, \mathcal{T})$ where $U \in \mathcal{T}$ iff for all $x \in U$, $x \leq y \Rightarrow y \in U$ for objects and $f \mapsto f$ for arrows.

**(i)** Obviously $\mathcal{T}$ is a topology. Let $f : X \to Y$ and $V \subseteq Y$ be an open set. Let $a \in f^{-1}(V)$. If $a \leq b$ in $X$, then $f(a) \leq f(b)$ since $f$ is an arrow in **Pos**. By the definition of topology, we can get $f(b) \in V$, and hence $b \in f^{-1}(V)$ consequently, meaning that $f^{-1}(V)$ is open in $X$. Therefore, $A$ is a well-defined functor.

**(ii)** $A$ is trivially faithful. Now let $f : X \to Y$ be continuous. Define $U = \{y \mid f(x) \leq y\} \subseteq Y$. By definition, $f^{-1}(U)$ is also closed upward. Thus for $x \in f^{-1}(U)$, if $x \leq z$, then $z \in f^{-1}(U)$ and therefore, $f(x) \leq f(z)$. Thus $f$ is monotone and hence $A$ is full.

**6.**

$$\begin{tikzcd} \mathbf{C} \arrow[rr, "F"] \arrow[dr, "E"] & & \mathbf{D} \\ & \mathbf{E} \arrow[ur, "D"] & \end{tikzcd}$$

**(a)** Define $\mathbf{E}$ as follows: For object, define Ob($\mathbf{E}$)=Ob($\mathbf{C}$). Now define a relation on arrows of $\mathbf{C}$ by $f \sim g$ iff $F(f) = F(g)$. Let $[f]$ be the arrow of $\mathbf{E}$. Then obviously $E$ is bijective on objects and full. By defining $D$ canonically, we can construct such factorization.

**(b)** Object of $\mathbf{E}$: Image of $F$, arrows: corresponding arrows in $\mathbf{D}$.

**7.**

Suppose $\alpha : F \to G$ be a natural isomorphism. Then there is $\alpha^{-1} : G \to F$ the inverse of $\alpha$, satisfying $\alpha \circ \alpha^{-1} = 1_G$ and $\alpha^{-1} \circ \alpha = 1_F$. Then for any object $C$, $\alpha_C \circ (\alpha_C)^{-1} = 1_{FC}$ and hence $\alpha_C^{-1} = (\alpha^{-1})_C$, meaning that all components of $\alpha$ is an isormorphism. Conversely, suppose all components of $\alpha$ are isormophisms. Choose $\alpha^{-1}$ such that all components are inverse of the components of $\alpha$. We will show that $\alpha^{-1}$ is a natural transformation. Let $f : C \to D$. We know that by definition, $Gf \circ \alpha_C = \alpha_D \circ Ff$. By composoing $\alpha_D^{-1}$ on the left side, $\alpha_D^{-1} \circ Gf \circ \alpha_C = Ff$. Similarly, we get $\alpha_D^{-1} \circ Gf = Ff \circ \alpha_C^{-1}$, showing that $\alpha^{-1}$ is a natural transformation.

**8.**

Let $F, G \in \mathbf{D}^{\mathbf{C}}$ be two functors. Define $F \times G$ by $(F \times G)(C) = FC \times GC$ and $(F \times G)f = Ff \times Gf$. Then we have canonical natural transformations $\pi_1, \pi_2$. Now take a functor $H$ and natural transformations $\alpha : H \to F$, $\beta : H \to G$. By UMF, for any object $C$, there is a unique arrow $h_C : HC \to FC \times GC$

such that $\pi_{1C} \circ h_C = \alpha_C$ and $\pi_{2C} \circ h_C = \beta_C$. Now it remains to prove the arrow case. Consider a diagram

$$\begin{tikzcd} HC \arrow[r, "h_C"] \arrow[d, "Hf"] & FC \times GC \arrow[r, "\pi_{1C}"] \arrow[d, "Ff \times Gf"] & FC \arrow[d, "Ff"] \\ HD \arrow[r, "h_D"] & FD \times GD \arrow[r, "\pi_{1D}"] & FD \end{tikzcd} .$$

Since $\pi_1$ is natural, we get

$$\begin{aligned} \pi_{1D} \circ (Ff \times Gf) \circ h_C &= Ff \circ \pi_{1C} \circ h_C \\ &= Ff \circ \alpha_C \\ &= \alpha_D \circ Hf \\ &= \pi_{1D} \circ h_D \circ Hf. \end{aligned}$$

We can show the similar case for $\pi_2$ and those results show that by UMP, the diagram commutes, proving that $h$ is natural.

**9.**

For $f : A \to B$, define $\mathcal{PP}f : \mathcal{PP}A \to \mathcal{PP}B$ by $\{U \subseteq A \mid a \in U\} \mapsto \{V \subseteq B \mid f(a) \in V\}$. Then the map $\eta_A : A \to \mathcal{PP}A$ is a natural transformation.

**10.**

It is enough to show that the map Hom satisfies the condition of bifunctor lemma. So we only need to show that the diagram

$$\begin{tikzcd} \mathrm{Hom}(C,D) \arrow[r, "\mathrm{Hom}(f,D)"] \arrow[d, "\mathrm{Hom}(C,g)"'] & \mathrm{Hom}(C',D) \arrow[d, "\mathrm{Hom}(C',g)"] \\ \mathrm{Hom}(C,D') \arrow[r, "\mathrm{Hom}(f,D')"] & \mathrm{Hom}(C',D') \end{tikzcd}$$

commutes where $f : C \to C'$ in $\mathbf{C^{op}}$ and $g : D \to D'$ in $\mathbf{D}$. But for $h \in \mathrm{Hom}(C,D)$, either paths maps it to $g \circ h \circ f : C' \to D'$, concluding that Hom is a functor.

**11.**

Obviously $\mathbf{1}$ is the identity groupoid. For two groupoids $A, B$, we can define $A \times B$ canonically. For exponential, define $B^A$ be a functor category. For $F, G : A \to B$, Take $C, D \in A$ and consider a component of natrural transformation $\alpha$, $\alpha_C : FC \to GC$. Since $\alpha_C$ is an arrow in $B$, meaning that $\alpha_C^{-1}$ exists. Thus by exercise 7, $\alpha$ is a natural isomorphism, concluding that $B^A$ is a groupoid.

**12.**

Suppose $\boldsymbol{C} \cong \boldsymbol{D}$. Then there are two functors $F : \boldsymbol{C} \to \boldsymbol{D}$ and $G : \boldsymbol{D} \to \boldsymbol{C}$ such that $\alpha : 1_{\boldsymbol{D}} \to FG$ and $\beta : GF \to 1_{\boldsymbol{C}}$ are natural isomorphisms. Suppose $\boldsymbol{C}$ has binary products and let $D, D' \in \boldsymbol{D_0}$. Then we have $GD \times GD'$ in $\boldsymbol{C}$. Let $Z \in \boldsymbol{D}$ and $a : Z \to D$, $a' : Z \to D'$. Then there is a unique arrow $h : GZ \to GD \times GD'$ such that $\pi_1 \circ h = Ga$ and $\pi_2 \circ h = Ga'$. Now consider $F(GD \times GD')$ and the map

$$Z \overset{\alpha_Z}{\to} FGZ \overset{Fh}{\to} F(GD \times GD') \overset{F\pi_1}{\to} FGD \overset{\alpha_D^{-1}}{\to} D.$$

Then we can see

$$\begin{aligned}\alpha_D^{-1} \circ F\pi_1 \circ Fh \circ \alpha_Z &= \alpha_D^{-1} \circ F(\pi_1 \circ h) \circ \alpha_Z \\ &= \alpha_D^{-1} \circ FGa \circ \alpha_Z \\ &= \alpha_D^{-1} \circ \alpha_D \circ a \\ &= a\end{aligned}$$

The map $Fh \circ \alpha_Z : Z \to F(GD \times GD')$ is unique since $h$ is unique and therefore we can regard $\alpha_D^{-1} \circ F\pi_1$ as a projection $F(GD \times GD') \to D$, meaning that $F(GD \times GD')$ is the product of $D$ and $D'$.

**13.**

Don't understand the meaning of the problem.

**14.**

pass

**15.**

Two categories $\boldsymbol{C}, \boldsymbol{D}$ are equivalent if there are two functors $F, G$ such that $FG \cong 1_{\boldsymbol{D}}$ and $GF \cong 1_{\boldsymbol{C}}$. Obviously the equivalence is reflexive and symmetry. For trainsivity, let $\boldsymbol{E}$ be another category and suppose $\boldsymbol{D} \cong \boldsymbol{E}$. with $H : \boldsymbol{D} \to \boldsymbol{E}$ and $J : \boldsymbol{E} \to \boldsymbol{D}$. Then the map $GJHF \cong G1_{\boldsymbol{D}}F = GF \cong 1_{\boldsymbol{C}}$ and $HFGJ \cong H1_{\boldsymbol{D}}J = HJ \cong 1_{\boldsymbol{E}}$ and hence $\boldsymbol{C} \cong \boldsymbol{E}$, proving that it is an equivalence relation.

**16.**

Let $\boldsymbol{C}$ be a category and define a relation $\sim$ where $A \sim B$ if they are isomorphic. By choose one object from each isomorphism class, we get a subcategory $\boldsymbol{D}$. Define an inclution functor $I : \boldsymbol{D} \to \boldsymbol{C}$ and define $F : \boldsymbol{C} \to \boldsymbol{D}$ by $F(A) = B$ where $B \in [A]$ a class and $F(f : C \to D) = i_D \circ f \circ i_C^{-1}$ where $i_C : C \to E \in [C]$ an isomorphism. Then $F$ is a functor and $IF \cong 1_{\boldsymbol{C}}, FI \cong 1_{\boldsymbol{D}}$, concluding the assertion.

**17.**

Skip

**18.**

Define $A \otimes B = \pi_1(A \times B) = A$ and $I$ as a terminal object. Then we can see it is a monoidal category. Obviously, by converting it to the opposite category, we can see the same ressult for coproducts.