1.

Suppose $C \cong C'$. Then there is an isomorphism $f : C \to C'$ with inverse f^{-1}. Thus $Ff : FC \to FC'$ is an isomorphism with inverse Ff^{-1}. Conversely, suppose $FC \cong FC'$. Then there is an isomorphism $h : FC \to FC'$. Since F is full, there are $f : C \to C', g : C' \to C$ such that $Ff = h, Fg = h^{-1}$. Then $1_{FC} = h \circ h^{-1} = Ff \circ Fg = F(f \circ g)$. Since F is faithful, $f \circ g = 1_C$ and thus $g = f^{-1}$, meaning $C \cong C'$.

2.

Let $P, Q \in \mathbf{Sets}^{\mathbf{C}^{\mathrm{op}}}$ and $\varphi, \psi : P \to Q$ such that $\varphi \circ \vartheta = \psi \circ \vartheta$ for $\vartheta : yC \to P$ for each $C \in \mathbf{C}$. We can define a map

$$\varphi_{C*} = \psi_{C*} : \mathrm{Hom}(yC.P) \to \mathrm{Hom}(yC, Q)$$

by $\varphi_{C*}(\vartheta) = \psi_{C*}(\vartheta) = \varphi \circ \vartheta = \psi \circ \vartheta$. By Yoneda lemma, we can deduces $\mathrm{Hom}(yC, P) \cong PC$ and $\mathrm{Hom}(yC, Q) \cong QC$, making a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(yC, P) \arrow[r, "\cong"] \arrow[d, "\psi_{C*}"', "\varphi_{C*}"] & PC \arrow[d, "\psi_C"', "\varphi_C"] \\
\mathrm{Hom}(yC, Q) \arrow[r, "\cong"] & QC
\end{tikzcd}.$$

But since $\varphi_{C*} = \psi_{C*}$, we conclude that $\varphi_C = \psi_C$ for all C, meaning $\varphi = \psi$.

3.

$$\begin{aligned}
\mathrm{Hom}(X, A^B \times A^C) &\cong \mathrm{Hom}(X, A^B) \times \mathrm{Hom}(X, A^C) \\
&\cong \mathrm{Hom}(X \times B, A) \times \mathrm{Hom}(X \times C, A) \\
&\cong \mathrm{Hom}(X \times B + X \times C, A) \\
&\cong \mathrm{Hom}(X \times (B + C), A) \\
&\cong \mathrm{Hom}(X, A^{B+C}).
\end{aligned}$$

By Yoneda lemma, $A^B \times A^C \cong A^{B+C}$. Similarly, $(A \times B)^C \cong A^C \times B^C$.

4.

Let $f : P \to Q$. Then $S(f) = \mathrm{Hom}(-, f) : \mathrm{Hom}(-, P) \to \mathrm{Hom}(-, Q)$ and hence S is a functor. This functor is faithful. From text, simplicial nerve is defined by for any n and a poset P,

$$S(P)_n = \{(p_1, \cdots, p_n) \in P^n \mid p_1 \leq \cdots \leq p_n\},$$

which is just a set of monotone map

$$S(P)(n) = \mathrm{Hom}([n], P).$$

5.

Define $S : \mathbf{Cat} \to \mathbf{Cat}^{\Delta^{\mathrm{op}}}$ by $S(\boldsymbol{C}) = \mathrm{Fun}(-, \boldsymbol{C})$. Then it generalizes the concept of simplicial nerve.

6.

Consider a diagram $\boldsymbol{J} \to \boldsymbol{D}^{\boldsymbol{C}}$. Then for some $C \in \boldsymbol{C_0}$, we have an induced diagram $\boldsymbol{J} \to \boldsymbol{D}$ defined by $i \mapsto F_i(C)$ and $(i \to j) \mapsto (F_i(C) \to F_j(C))$. Since $\boldsymbol{D}$ is complete, we have a limit $\underleftarrow{\lim}_{\mathbf{J}} F_i(C)$. Thus from the diagram, we have the limit $\underleftarrow{\lim}_{\mathbf{J}} F_i$ defined by $(\underleftarrow{\lim}_{\mathbf{J}} F_i)(C) = \underleftarrow{\lim}_{\mathbf{J}}(F_i(C))$. Colimit of $\boldsymbol{D}^{\boldsymbol{C}}$ is the limit of $(\boldsymbol{D}^{\mathrm{op}})^{\boldsymbol{C}^{\mathrm{op}}} = (\boldsymbol{D}^{\boldsymbol{C}})^{\mathrm{op}}$.

7.

For product,

$$\begin{aligned} y(A \times B)(C) &\cong \mathrm{Hom}(C, A \times B) \\ &\cong \mathrm{Hom}(C, A) \times \mathrm{Hom}(C, B) \\ &\cong yA(C) \times yB(C) \\ &\cong (yA \times yB)(C). \end{aligned}$$

Thus, $yA \times yB \cong y(A \times B)$. For exponential,

$$\begin{aligned} (yB)^{(yA)}(C) &\cong \mathrm{Hom}(yC, (yB)^{(yA)}) \\ &\cong \mathrm{Hom}(yC \times yA, yB) \\ &\cong \mathrm{Hom}(y(C \times A), yB) \\ &\cong \mathrm{Hom}(C \times A, B) \\ &\cong \mathrm{Hom}(C, B^A) \\ &\cong y(B^A)(C). \end{aligned}$$

Thus $(yB)^{(yA)} \cong y(B^A)$.

8.

Let $C, C' \in P$ and $AC = x, AC' = x'$. In $\int_P A$, define $(x, C) \leq (x', C')$ if $C \leq C'$ and there is a map $f : x \mapsto x'$. Then it is a poset and the projection π is monotone. Finally, consider a natural transformation $\alpha : A \to B$. Then we can define $\int_P \alpha : \int_P A \to \int_P B$ by $(AC, C) \mapsto (BC, C)$ for all $C \in P$. It clearly defines the functor

$$\int_P : \mathbf{Sets}^{P^{\mathrm{op}}} \to \mathbf{Pos}/P.$$

9,10.

skip

11.

Product of $a : A \to X$ and $b : B \to X$ is a pullback of them. Since X is a set, $\mathbf{Sets}^X$ is equivalent to $\mathbf{Sets}/X$. Therefore, it is enough to find the exponential of $\mathbf{Sets}^X$ to show that $\mathbf{Sets}/X$ is cartesian closed. As a category, X is a discrete category. Yoneda embedding $y : X \to \mathbf{Sets}^X$ is defined by $y(x) = \mathrm{Hom}_X(-, x) : X \to \mathbf{Sets}$ for $x \in X$. Since X is discrete, $y(x)(z) = \varnothing$ for all $z \neq x$ and $y(x)(x) = \{1_x\}$. Now for $P, Q \in \mathbf{Sets}^X$, exponential Q^P can be defined by $Q^P(x) = \mathrm{Hom}(y(x) \times P, Q)$, which exists. For a category $\mathbf{C}$, the subobject classifier is the pair $(\Omega, t : 1 \to \Omega)$ where $\Omega(C)$ is a set of sieves on C and t_C is a total sieve. Since X is discrete, $\Omega(x) = \{1_x\}$ and $t_x = \{1_x\}$ for all $x \in X$.

12.

(a) We know that $\Omega(C) = \{S \mid S \text{ sieve on } C\}$. For $p \in P^{\mathrm{op}}$ where P is a poset, then S is the set of arrows $\{p \leq x\}$, which is identical to $\{x \mid p \leq x\}$. Therefore, for $\boldsymbol{P} = \mathbf{2} = \{0 \leq 1\}$, $\Omega : \mathbf{2}^{\mathrm{op}} \to \mathbf{Sets}$ where $\Omega(0) = \{\{0,1\},\{1\}\}, \Omega(1) = \{\{1\}\}$ and $t : 1 \to \Omega$ where $t_0(*) = \{0,1\}, t_1(*) = \{1\}$. For $\boldsymbol{P} = \boldsymbol{\omega}$, the subobject classifier is defined by

$$\begin{aligned}
\Omega(0) &= \{\{0,1,2,\dots\},\{1,2,3,\dots\},\{2,3,4,\dots\},\dots\} \\
\Omega(1) &= \{\{1,2,3,\dots\},\{2,3,4\dots\},\{3,4,5\dots\},\dots\} \\
&\vdots \\
\Omega(n) &= \{\{n,n+1,\dots\},\{n+1,n+2,\dots\},\dots\}
\end{aligned}$$

and $t_n = \{n, n+1, \dots\}$.

(b) Since all topos operations(pullbacks,exponentials,subobject classifiers) are finite, it is also a topos.

13.

Recall that the category of graphs is equivalent to the functor category $\mathbf{Sets}^{\Gamma^{\mathrm{op}}}$ where Γ is a category

$$C \rightrightarrows D.$$

Thus the category of graphs is a topos. Then the subobject classifier Ω is defined by $\Omega(C) = \{1_C\}$ and $\Omega(D) = \{1_D, s : C \to D, t : C \to D\}$. point $1 \to \Omega$ is the same.