1.

Let $f : A \to X$ and $g : B \to X$ be two objects in the slice category $\mathcal{C}/X$. Their product $f \times g$ is a map $P \to X$ with two projection $\pi_1 : f \times g \to f$, $\pi_2 : f \times g \to g$ are indeed maps $P \to A$ and $P \to B$ with $g \circ \pi_2 = f \circ \pi_1$. By definition of product, for any object Z with $z_1 : Z \to A$, $z_2 : Z \to B$ satisfying $f \circ z_1 = g \circ z_2$, we have a unique map $Z \to P$. However, this property is the definition of pullback, concluding that P satisfies the definition of pullback. Since pullback is unique, they coincide.

2.

(a) ($\Longrightarrow$) Let $f, g : Z \to M$ be arrows satisfying $m \circ f = m \circ g$. Since m is monic, $f = g$ and therefore, the arrow $Z \to M$ is unique, concluding that M is a pullback. ($\Longleftarrow$) Suppose M is a pullback and $m \circ f = m \circ g$. Since M is a pullback, there is a unique $h : Z \to M$ with $1_M \circ h = f$ and $1_M \circ h = g$, concluding that $f = g$. Thus m is monic.

(b) We can make a cube

$$\begin{tikzcd}
& (A \times_X B) \times_A (A \times_X Y) \arrow[rr] \arrow[dl] \arrow[dd] & & A \times_X Y \arrow[dl] \arrow[dd] \\
A \times_X B \arrow[rr] \arrow[dd] & & A \arrow[dd] & \\
& B \times_X Y \arrow[rr] \arrow[dl] & & Y \arrow[dl] \\
B \arrow[rr] & & X &
\end{tikzcd}$$

Each faces are pullback squares, concluding that the back side is also a pullback. Now when considering a pullback functor f^*, we get $(A \times_X B \to X) \mapsto (A \times_X B \times_Y Y \to Y)$ and hence it preserves the pullback.

(c) Suppose m is monic. Then by (a), the square

$$\begin{tikzcd}
M \arrow[r, "1_M"] \arrow[d, "1_M"] & M \arrow[d, "m"] \\
M \arrow[r, "m"] & A
\end{tikzcd}$$

is a pullback square. By (b), we have another pullback square

$$\begin{tikzcd}
M' \arrow[r] \arrow[d] & M' \arrow[d, "m'"] \\
M' \arrow[r, "m'"] & A'
\end{tikzcd} .$$

Again by (a), m' is a mono.

3.

Consider a diagram

$$\begin{tikzcd}
Z \arrow[r, shift left, "h"] \arrow[r, shift right, "k"'] & M' \arrow[r, "f'"] \arrow[d, "m'"] & M \arrow[d, "m"] \\
& A' \arrow[r, "f"] & A
\end{tikzcd} .$$

Suppose $m'h = m'k$. Then $fm'h = fm'k$ and by definition of pullback, $mf'h = mf'k$. Since m is monic, we have $f'h = f'k$. We have $mf'h = fm'h$ and hence again by the definition of pullback, there is a unique $q : Z \to M'$ satisfying $fm'q = mf'q$. But since h, k both satisfy the property, we can conclude that $q = h = k$. Thus m' is monic.

4.

($\Longrightarrow$) If $M \subseteq N$, then there is a mono $s : M \to N$ such that $ns = m$. Suppose $z \in_A M$. Then there is $f : Z \to M$ such that $z = mf = nsf$. Therefore, $fs : Z \to N$ satisfies the property, concluding that $z \in_A N$. ($\Longleftarrow$) Suppose $z \in_A M \Rightarrow z \in_A N$ for all Z. By taking $z = m$, we can see that $m \in_A N$, meaning that there is $s : M \to N$ with $m = ns$. Since m and n are both monic, we conclude that s is monic, i.e., $M \subseteq N$.

5.

By 4, $M \subseteq N$ and $N \subseteq M$. Then there are two monos s, r with $s : M \hookrightarrow N$ and $r : N \hookrightarrow M$. Then $nsr = n \Rightarrow sr = 1_N$, $mrs = m \Rightarrow rs = 1_M$, concluding that $M \cong N$. Thus $M = N$.

6.

Firstly, by the property of pullback, $fe = ge$. Let $z : Z \to A$ be such that $fz = gz$. Take $z = z_1 : Z \to A$ and $fz = z_2 : Z \to B$. Then $\langle f, g\rangle z = \langle fz, fz\rangle$, concluding that $\Delta z_2 = \langle f, g\rangle z_1$. By UMP of a pullback, there is a unique $u : Z \to E$ making a commutative diagram. Thus we conclude that e is an equalizer of f, g.

7.

Proposition 5.25

8.

(identity) For $A \in \mathrm{Ob}(\mathcal{C})$, take $(A, 1_A)$.

(Associativity) Let $U \subseteq A, V \subseteq B, W \subseteq C$. Consider a map

$$\begin{tikzcd}
 & & U \times_B V \arrow[dl] \arrow[dr] & & V \times_C W \arrow[dl] \arrow[dr] & & \\
 & U \arrow[dl] \arrow[dr, "|f|"] & & V \arrow[dl] \arrow[dr, "|g|"] & & W \arrow[dl] \arrow[dr, "|h|"] & \\
A \arrow[rr, "f"] & & B \arrow[rr, "g"] & & C \arrow[rr, "h"] & & D
\end{tikzcd}$$

The map $h \circ (g \circ f)$ is a pair $(|h \circ (g \circ f)|, (|h|^*(W)) = (|h \circ (g \circ f)|, (U \times_B V) \times_C W) = (|(h \circ g) \circ f|, U \times_B (V \times_C W)) = (|(h \circ g) \circ f|, |f|^*(|g|^* W) = (h \circ g) \circ f$. Thus it is associative, concluding that $\mathbf{Par}(\mathcal{C})$ is a category.

9.

Define $\varprojlim_{\mathbf{J}} :$ **Diagrams(J,C)→C** by $\varprojlim_{\mathbf{J}} F = \varprojlim F$ and $\varprojlim_{\mathbf{J}}(\theta : F \to G) = \varprojlim F \to \varprojlim G$ by $\gamma_i \varprojlim_{\mathbf{J}} \theta = \theta_i \beta_i$ where $\beta_i : \varprojlim F \to F_i$ and $\gamma_i : \varprojlim G \to G_i$ are product maps. Then $\varprojlim_{\mathbf{J}} 1_F = 1_{\varprojlim F}$ since for all i, $1_{F_i}\beta_i = \beta_i \varprojlim_{\mathbf{J}} \theta$. Also, from the definition of arrow, we have $\varprojlim_{\mathbf{J}}(\phi \circ \theta) = \varprojlim_{\mathbf{J}} \phi \circ \varprojlim_{\mathbf{J}} \theta$ where $\phi : G \to H$.

Now consider $\mathbf{Sets}^I$ for a set I. It is a set of functions $A : I \to \mathbf{Sets}$. As above, we can define a category of such objects and define a product functor

$$\prod_{i \in I} : \mathbf{Sets}^I \to \mathbf{Sets}$$

defined by $A \to \prod_{i \in I} A_i$.

10.

(a) Pushout: Given any arrows f, g with $\mathrm{dom}(f) = \mathrm{dom}(g) = C$, the pushout of f and g consists of arrows $c_1 : A \to Q$, $c_2 : B \to Q$ such that $c_1 f = c_2 g$ and the universal property as follows: Given any $z_1 : A \to Z, z_2 : B \to Z$ with $z_1 f = z_2 g$, there is a unique $v : Q \to Z$ with $z_1 = v c_1$ and $z_2 = v c_2$.

$$\begin{tikzcd}
Z & & \\
& Q \arrow[ul, "\exists! v"'] & B \arrow[l, "c_2"] \arrow[ull, "z_2"'] \\
& A \arrow[u, "c_1"] \arrow[uul, "z_1"] & C \arrow[l, "f"'] \arrow[u, "g"']
\end{tikzcd}$$

(b) skip

11.

Note that q is a coequalizer of $r_1, r_2 : R \to X$. By using the fact that $\mathcal{P}(X) \cong 2^X$, we define $\mathcal{P}q : 2^Q \to 2^X$ by $\mathcal{P}q([a])(x) = [a]([x])$ and $\mathcal{P}r_i : 2^X \to 2^R$ by $\mathcal{P}r_1(a)(x, y) = a(x)$ and $\mathcal{P}r_2(a)(x, y) = a(y)$ where $a : X \to 2$ and $[a] : Q \to 2$ is induced from a. Then it satisfies the diagram

$$\mathcal{P}Q \xrightarrow{\mathcal{P}q} \mathcal{P}X \xrightarrow{\mathcal{P}r_i} \mathcal{P}R.$$

Suppose $z : Z \to \mathcal{P}X$ satisfies $\mathcal{P}r_1 z = \mathcal{P}r_2 z$. Then for all $(x, y) \in R$, $a(x) = a(y)$ where a is in the image of z. Since R is an equivalence relation, for all $x \in [x]$, $a(x)$ has the same value, meaning that there is a unique map $Z \to \mathcal{P}Q$.

12.

limit: Consider a cone $\zeta_n : Z \to n$. If $n = 0$, then ζ_0 is zero map and for all n, ζ_n is factored through ζ_0, concluding that ζ_n is zero. The limit is a terminal object in the category of cone, which is object 0 with inclusion $0 \hookrightarrow n$.

colimit: Consider a co-cone $\psi_n : n \to Y$. By the property, ψ_n is a restriction of ψ_{n+m} and since Y is also a poset, $\psi_n(m) \leq \psi_n(n)$ for $m < n$, concluding that ψ_n is monotone. Hence we conclude that a map $\theta : \mathbb{N} \to Y$ uniquely defines a cone. Thus ω with inclusion $n \subseteq \omega$ defines the colimit.

13.

(a) Suppose M_k, N_k are abelian groups. Note that $M_0 \to \cdots \to M_k \to \cdots$, $N_0 \leftarrow \cdots \leftarrow N_k \leftarrow \cdots$. Then we can see that $\varprojlim_k M_k = M_0$. $\varinjlim_k M_k$ is indeed the direct limit of M_k, which is $\bigoplus_k M_k / \sim$ where $m_i \sim \mu_{ik}(m_i)$ for $m_i \in M_i, \mu_{ik} : M_i \to M_k$. For N_k, $\varinjlim_k N_k = N_0$ and $\varprojlim N_k$ is the inverse limit of abelian groups: $\mathcal{S} = \{g \in \prod N_k \mid \text{if } i \leq j, \text{ then } \nu_{ji}(g_j) = g_i\}$.

(b) All elements of colimit of M_k and limit of N_k may have an elemnt of infinite order.(right????)