1.

($\Longrightarrow$) Suppose $f : A \to B$ is not surjective. Then there is $b \in B$ such that $b \notin f(A)$. Let $i, j : B \to \{0, 1\}$ such that $i(b) = 0$, $j(b) = 1$ and other elements of B go to the same element. Then $if = jf$ but $i \neq j$, concluding that f is not an epi. ($\Longleftarrow$) Suppose f is surjective and $i, j : B \to C$ are different. Then there is an element $b \in B$ such that $i(b) \neq j(b)$, concluding that $if(a) \neq jf(a)$ where $f(a) = b$.

2.

Let $f : p \leq q$ and $i : e_1 \leq p$, $j : e_2 \leq p$ or $i : q \leq e_1, j : q \leq e_2$. Since i, j have the same domain or codomain, mono and epi follow directly.

3.

$g = 1_A \circ g = g \circ f \circ g = g' \circ f \circ g = g' \circ 1_B = g'$.

4.

(a) Firstly, $h^{-1} = g^{-1} \circ f^{-1}$. Secondly, let $hi = hj$. Then $gfi = gfj$ and by the assumption, $i = j$. Thirdly, let $ih = jh$. Then $igf = jgf$. As previous, $i = j$.

(b) Suppose $fi = fj$. Then $gfi = gfj$ for any g. Since $h = gf$ and h is monic, $i = j$.

(c) Suppose $ig = jg$. Then $igf = ih = jh = jgf$, concluding $i = j$.

(d) Let $A = C = \{0\}$, $B = \{0, 1\}$. Then h is monic since all maps to A is constant. But g need not be.

5.

$(a \Rightarrow b, c, d)$: Suppose $f : A \to B$ is an isomorphism. If $fi = fj$, then $f^{-1}fi = i = j = f^{-1}j$ and hence f is a split mono. Similarly, f is a split epi.

$(b \Rightarrow a)$: Since f is split epi. there is g such that $fg = 1_B$. Since f is mono, $(fg)f = f = f(1_A)$, implying $gf = 1_A$. Hence $g = f^{-1}$.

$(c \Rightarrow a)$: Similar to above case.

$(d \Rightarrow b, c)$: It enough to show that split mono(epi) implies mono(epi). Suppose $gf = 1_A$. If $fi = fj$, then $gfi = i = j = gfj$, concluding f is mono. Similar argument holds for epi.

6.

Suppose h is not injective on vertices. Then there is two distinct vertices v, w such that $h(v) = h(w)$. Consider $\mathbf{1}$, a graph of one vertex and $f, g : \mathbf{1} \to G$ two homomorphisms whose images are v, w, respectively. Then $hf = hg$, concluding that h is not mono. Use the same argument to the edge case.

7.

Recall that an object P is projective if for any epi $e : E \to X$ and $f : P \to X$, there is an arrow $f' : P \to E$ such that $ef' = f$. Let A be a retract of P and $g : A \to X$, $a : A \to P$ with right inverse $s : P \to A$. Then $g = (ga)s$ and by the definition of projective, there is $\hat{ga} : P \to E$ such that $e(\hat{ga}) = ga$. Then $e(\hat{ga})s = gas = g$ and hence we can find a map $(\hat{ga})s : A \to E$ satisfying the property, conclding that A is projective.

8.

Let $f : P \to X$ be a function and $e : E \twoheadrightarrow X$ be a surjection. Since e is surjective, for all $x \in X$, we have $E = \bigcup_x e^{-1}(x)$, which is a disjoint union. Define $\hat{f} : P \to E$ such that for each $p \in P$, $\hat{f}(p)$ is an element in $e^{-1}(f(p))$, which is possible by Choice Axiom. Then obviously $e\hat{f}(p) = e(k) = f(p)$ where $e^{-1}(k) \in e^{-1}(f(p))$. Hence P is projective. Since P is arbitrary, all sets are projective.

9.

($\Longleftarrow$) Suppose $f : A \to B$ is surjective and $g, h : B \to C$ with $gf = hf$. Since f is surjective, $f(A) = B$ and hence $g(x) = h(x)$ for all $x \in B$, concluding f is epi. ($\Longrightarrow$) Suppose f is not epi. Then there are $g, h : B \to C$ such that $gf = hf$ but $g \neq h$. Then there is $x \in B$ such that $g(x) \neq h(x)$. If x is in the image of f, then $g(x) = h(x)$ by assumption, contradiction. Hence f is not surjective. Consider $\mathbf{1}$ in **Pos**. Let $f : \mathbf{1} \to X$ for some object X and $e : E \twoheadrightarrow X$ be an epi. Then e is surjective and hence we can find $\hat{f} : \mathbf{1} \to E$ such that $f = e \circ \hat{f}$. Thus $\mathbf{1}$ is projective.

10.

(i) Let A be a set. It can be regarded as a discrete poset in **Pos**. Let $f : A \to X$ be a map and $e : X \twoheadrightarrow Y$ be an epi in **Pos**. For each $y \in Y$, define $x_y \in X$ by an element in X such that $e(x_y) = y$. Define $\hat{f} : A \to X$ by $\hat{f}(a) = x_{f(a)}$. Then A is projective. **(ii)** Consider $P = \{0 \leq 1\}$, $X = \{a, b\}$ a discrete poset, and an epi $e : X \twoheadrightarrow P$ with $e(a) = 0, e(b) = 1$. Consider $\mathbf{1}_P : P \to P$. Since X is discrete, it lose all information of order, meaning that for all $g : P \to X$, we cannot get $e \circ g \neq \mathbf{1}_P$. Thus P is not projective. **(iii)** Let P be projective and $|P|$ be a discretization of P, i.e., a set. Let $|1_P| : |P| \to P$ be an identity map. Sincc P is projective, there is a map $f : P \to |P|$ with $|1_P| \circ f = 1_P$. Then we can see that the only arrow is identity, meaning that P is a set. Hence it is obvious that **Sets** is a subccategory of **Pos**.

11.

Let $f : A \to U(B)$, underlying set of a monoid B. By UMP of free monoids, there is a unique $\bar{f} : M(A) \to B$ such that $f = U(\bar{f}) \circ i$ where $i : A \to U(M(A))$. Let $\eta : A \to U(X)$ be an initial object. Then for any $f : A \to U(B)$, there is a unique morphism $\hat{f}$ such that $f = \hat{f} \circ \eta$. But this is exactly the same as the definition of free monoid $M(A)$.

12.

Let $p : B \to \mathbf{2}$ be a homomorphism of boolean algebras. We will show that $F = p^{-1}(1)$ is an ultrafilter. Firstly suppose $a \in F$ and $a \leq b$. Then $p(a) \leq p(b)$ and hence $b \in F$. Besides, if $a, b \in F$, then $p(a \wedge b) = p(a) \wedge p(b) = 1$, concluding $a \wedge b \in F$. Hence F is a filter. Now let $x \in B$. If $x \in F$, then $p(x \wedge \neg x) = p(0) = 0$, concluding that $\neg x \notin F$, and vice versa. Since p is surjective, either $x \in F$ or $\neg x \in F$. Thus if there is U strictly containing F, then there is $y \in U$ such that $y = \neg a$ for some $a \in F$, concluding that $U = B$. Therefore, $U = B$ and hence F is an ultrafilter.

13.

Let $P = (A \times B) \times C, Q = A \times (B \times C)$, $p_1 : P \to A \times B \to A$, $p_2 : P \to A \times B \to B$, $p_3 : P \to C$ and $q_1 : Q \to A, q_2 : Q \to B \times C \to B, q_3 : Q \to B \times C \to C$. By UMP, there is a unique map $f_1 = p_2 \times p_3 : P \to B \times C$. Therefore, there is $f = p_1 \times (p_2 \times p_3) : P \to Q$ with $q_i = p_i f$. Similarly, there is $g : Q \to P$ such that gf and fg are identities. Therefore, $P \cong Q$.

14.

(a) Let $P = \prod_{i \in I} X_i$, $p_i : P \to X_i$. P is a product if for any object Y with $y_i : Y \to X_i$, there is a unique map $f : Y \to P$ with $y_i = p_i f$.

(b) Define $x_i : X^I \to X_i$ by $x_i(f) = f(i), i \in I$. Then by UMP, there is a map $x : X^I \to \prod_{i \in I} X$ such that $p_i x = x_i$. On the other hand, define $y : \prod_{i \in I} X \to X^I$ by $y(a) = f : I \to X$ defined by $y(a)(i) = p_i(a)$. Then obviously we can see xy, yx are identities.

15.

Obvious by the definition of product.

16.

???

17.

Note that $\pi_1 \circ \Gamma(f) = 1_A$, meaning $\Gamma(f)$ is split mono, implying mono. Define $G : \mathbf{Sets} \to \mathbf{Rel}$ by $G(A) = A$ for an object A and $G(f) = \Gamma(f)$ where $f : A \to B$.

18.

Let $M = 1$. Then $U(M) = \mathrm{Hom}(1, M)$ and $U(f) = \mathrm{Hom}(1, f) : \mathrm{Hom}(1, A) \to \mathrm{Hom}(1, B)$. Hence U is representable. It follows from the corollory that U preserves the product.