1.

(a) Object of **Rel**: sets, Arrow of **Rel**: $R : A \to B$ means $R \subseteq A \times B$ with composition: if $S : B \to C$, then $S \circ R : A \to C$ is a set $S \circ R = \{(a, c) \in A \times C \mid \exists b \text{ with } (a, b) \in R, (b, c) \in S\}$. We can define $1_A : A \to A$ by $1_A = \{(a, a) \mid a \in A\}$ as an ideneity. It remains to show associativity and unit property. (i) Let $R : A \to B$, $S : B \to C$, $T : C \to D$. Then $T \circ (S \circ R) = \{(a, d) \in A \times D \mid \exists b \in B, c \in C \text{ satisfying } (a, b) \in R, (b, c) \in S, (c, d) \in T\} = (T \circ S) \circ R$. (ii) Let $R : A \to B$. Then $R \circ 1_A = \{(a, b) \mid \exists a \in A \text{ such that } (a, a) \in 1_A, (a, b) \in S\} = R = 1_A \circ R$. Thus **Rel** is a category.

(b) Define $G :$ **Sets**$\to$ **Rel** by $G(C) = C$ for $C \in \mathrm{Ob}(\textbf{Sets})$, $G(f) = \{(a, f(a)) \mid a \in A\}$ for $f : A \to B$. Then $G(f) : G(A) \to G(B)$, $G(1_A) = \{(a, a) \mid a \in A\} = 1_{G(A)}$, $G(g \circ f) = \{(a, g \circ f(a)) \mid a \in A\} = G(g) \circ G(f)$. Hence G is a functor.

(c) Define $C : \textbf{Rel}^{\mathrm{op}} \to \textbf{Rel}$ by $C(A) = A$ and $C(R^c) = R : A \to B$. Then $C(R^c) = C(A) \to C(B)$, $C(1^c_A) = 1_{C(A)} = 1_A$ and $C(S^c \circ R^c) = C(\{(c, a) \mid \exists b \text{ such that } (b, a) \in R^c, (c, b) \in S^c\}) = C(S) \circ C(R)$. Hence C is a functor.

2.

(a) Define $C : \textbf{Rel}^{\mathrm{op}} \to \textbf{Rel}$ as Exercise 1 and $D :$ **Rel**$\to$ $\textbf{Rel}^{\mathrm{op}}$ similar way. Then $C \circ D(A) = A = D \circ C(A)$ and $C \circ D(R : A \to B) = C(R^c) = R$ and $C \circ D(R^c) = R^c$, constructing the isomorphism.

(b) Suppose there is an isomorphism. Let $C : \textbf{Sets} \to \textbf{Sets}^{\mathrm{op}}$,$D : \textbf{Sets}^{\mathrm{op}} \to \textbf{Sets}$ be two isomorphisms. Let A be a set and $B = \{*\}$ is a singleton. In **Sets**, there is only one arrow from A to B. On the other hand, we can find many arrows of $A \to B$ in $\textbf{Sets}^{\mathrm{op}}$ and hecne $C \circ D$ cannot be an identity functor, contradiction. Hence they are not isomorphic

(c) Define a functor $F : P(X) \to P(X)^{\mathrm{op}}$ by $F(A) = A$, $F(f : A \to B) = f^{\mathrm{op}} : B \to A$ if $A \subseteq B$. Then F is an isomorphism.

3.

(a),(b) Obvious

(c) Let $A = \{a, b, c\}$ with $a \leq c, a \leq b$ and $B = \{x, y, z\}$ with $x \leq y \leq z$. Then A is partially ordered whereas B is totally ordered. Define $f : A \to B$ defined by $a \mapsto x, b \mapsto y$, and $c \mapsto z$. Then f is bijective homomorphism since it preserve the ordering, but not an isomorphism since A is not totally ordered.

4.

Obviously $x \leq x$. Suppose $x \leq y$ and $y \leq z$. Let O be an open set containing z. Then from the definition $y \in O$ and in conclusion, $x \in O$. Therefore, specialization is a preorder. To prove it is poset, it is enough to show the antisymmetry. Assume T_0 and suppose $x \leq y$ and $y \leq x$. If $x \neq y$, then there is two open $O_x \ni x$ and $O_y \ni y$ such that $y \notin O_x$, $x \notin O_y$, contradicting the assumption. Hence $x = y$, concluding that $\leq$is a poset. If X is T_1, obviously it is trivial.

5.

$U : \mathcal{C}/C \to \mathcal{C}$ is defined by $U(f) = \mathrm{dom} f$ for $f \in \mathrm{Ob}(\mathcal{C}/C)$, $U(a : f \to f')$ is a map $a : \mathrm{dom} f \to \mathrm{dom} f'$ in $\mathrm{Ar}(\mathcal{C})$ such that $f' = f \circ a$. Now define $F : \mathcal{C}/C \to \mathcal{C}^{\to}$ by $F(f) = f$ for $f \in \mathrm{Ob}(\mathcal{C}/C)$ and $F(a : f \to f') = (f, f')$ for $a \in \mathrm{Ar}(\mathcal{C}/C)$. Then clearly $\textbf{dom} \circ F = U$.

6.

We can consider a coslice category $C/\mathcal{C}$ as $\mathcal{C}^{\mathrm{op}}/C$.

7.

No, it isn't since F is not surjective. Indeed, the preimage of $(f^{-1}(a), f^{-1}(b))$ does not contain (X, Y) where $X \cap Y \neq \varnothing$. But we can make an equivalence of categories. Define a functor $G : \mathbf{Sets}\times\mathbf{Sets} \to \mathbf{Sets}/2$ by $G(A, B) = f : A \coprod B \to 2$ where $A \coprod B = A \times \{1\} \cup B \times \{2\}$ and $f(A) = a$, $f(B) = b$. Then 2 categories are equivalent. If we take $1 = \{a\}$ instead 2, if we take F as the previous case, then the preimage is just (X, X), meaning that F is not surjective. Hence also it is not an isormophism.

8.

Let $P : \mathbf{Cat} \to \mathbf{Pre}$ defined by $P(\mathcal{C}) = \mathcal{C}$. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor. Then by the property of functor, if we have an arrow $A \to B$ in $\mathcal{C}$, then we have an arrow $F(A) \to F(B)$ in $\mathcal{D}$, meaning that P is a functor by definition of P.

9,10. See the solution

11.

(a) M maps a set X to a set of words of X X^* with concatenation and a function $f : X \to Y$ to $M(X) \to M(Y)$ where $M(f)(a_1 \cdots a_k) = f(a_1) \cdots f(a_k)$. Obviously it is a functor.

(b) Let $f : X \to Y$. We can induce a function $f' : X \to |M(Y)|$ from f. Then by UMP, there is a unique monoid homomorphism $\bar{f}' : M(X) \to M(Y)$ such that $|\bar{f}'| \circ i = f'$. We can see that it defines a map of morphisms in M, concluding that M is a functor.

12.

Let $\bar{h} : \mathcal{C}(G) \to \mathcal{D}$ defined by $U(\bar{h}) \circ i = h$. Then obviously $\bar{h}$ is completely determined by h, and is unique. Also, it is a functor since it preserves composition and $\bar{h}(\epsilon_A) = 1_A$ where ϵ_A is an empty path on A.

13.

A category is small if the collection of objects and arrows are all sets. By Theorem 1.6, it is isomorphic to one in which the objects are sets and the arrows are function, a concrete category.