

1.

Let $f : A \rightarrow X$ and $g : B \rightarrow X$ be two objects in the slice category \mathcal{C}/X . Their product $f \times g$ is a map $P \rightarrow X$ with two projection $\pi_1 : f \times g \rightarrow f$, $\pi_2 : f \times g \rightarrow g$ are indeed maps $P \rightarrow A$ and $P \rightarrow B$ with $g \circ \pi_2 = f \circ \pi_1$. By definition of product, for any object Z with $z_1 : Z \rightarrow A$, $z_2 : Z \rightarrow B$ satisfying $f \circ z_1 = g \circ z_2$, we have a unique map $Z \rightarrow P$. However, this property is the definition of pullback, concluding that P satisfies the definition of pullback. Since pullback is unique, they coincide.

2.

- (a) (\implies) Let $f, g : Z \rightarrow M$ be arrows satisfying $m \circ f = m \circ g$. Since m is monic, $f = g$ and therefore, the arrow $Z \rightarrow M$ is unique, concluding that M is a pullback. (\impliedby) Suppose M is a pullback and $m \circ f = m \circ g$. Since M is a pullback, there is a unique $h : Z \rightarrow M$ with $1_M \circ h = f$ and $1_M \circ h = g$, concluding that $f = g$. Thus m is monic.

- (b) We can make a cube

$$\begin{array}{ccccc}
 & & (A \times_X B) \times_A (A \times_X Y) & \longrightarrow & A \times_X Y \\
 & \swarrow & \downarrow & & \swarrow \downarrow \\
 A \times_X B & \longrightarrow & A & & \\
 \downarrow & & \downarrow & & \downarrow \\
 & & B \times_X Y & \longrightarrow & Y \\
 & \swarrow & \downarrow & & \swarrow \downarrow \\
 B & \longrightarrow & X & &
 \end{array}$$

Each faces are pullback squares, concluding that the back side is also a pullback. Now when considering a pullback functor f^* , we get $(A \times_X B \rightarrow X) \mapsto (A \times_X B \times_Y Y \rightarrow Y)$ and hence it preserves the pullback.

- (c) Suppose m is monic. Then by (a), the square

$$\begin{array}{ccc}
 M & \xrightarrow{1_M} & M \\
 \downarrow 1_M & & \downarrow m \\
 M & \xrightarrow{m} & A
 \end{array}$$

is a pullback square. By (b), we have another pullback square

$$\begin{array}{ccc}
 M' & \longrightarrow & M' \\
 \downarrow & & \downarrow m' \\
 M' & \xrightarrow{m'} & A'
 \end{array}$$

Again by (a), m' is a mono.

3.

Consider a diagram

$$\begin{array}{ccccc}
 Z & \xrightarrow{h} & M' & \xrightarrow{f'} & M \\
 & \searrow_k & \downarrow m' & & \downarrow m \\
 & & A' & \xrightarrow{f} & A
 \end{array}$$

Suppose $m'h = m'k$. Then $fm'h = fm'k$ and by definition of pullback, $mf'h = mf'k$. Since m is monic, we have $f'h = f'k$. We have $mf'h = fm'h$ and hence again by the definition of pullback, there is a unique $q : Z \rightarrow M'$ satisfying $fm'q = mf'q$. But since h, k both satisfy the property, we can conclude that $q = h = k$. Thus m' is monic.

4.

(\implies) If $M \subseteq N$, then there is a mono $s : M \rightarrow N$ such that $ns = m$. Suppose $z \in_A M$. Then there is $f : Z \rightarrow M$ such that $z = mf = nsf$. Therefore, $fs : Z \rightarrow N$ satisfies the property, concluding that $z \in_A N$. (\impliedby) Suppose $z \in_A M \Rightarrow z \in_A N$ for all Z . By taking $z = m$, we can see that $m \in_A N$, meaning that there is $s : M \rightarrow N$ with $m = ns$. Since m and n are both monic, we conclude that s is monic, i.e., $M \subseteq N$.

5.

By 4, $M \subseteq N$ and $N \subseteq M$. Then there are two monos s, r with $s : M \hookrightarrow N$ and $r : N \hookrightarrow M$. Then $nsr = n \Rightarrow sr = 1_N$, $mrs = m \Rightarrow rs = 1_M$, concluding that $M \cong N$. Thus $M = N$.

6.

Firstly, by the property of pullback, $fe = ge$. Let $z : Z \rightarrow A$ be such that $gz = fz$. Take $z = z_1 : Z \rightarrow A$ and $gz = z_2 : Z \rightarrow B$. Then $\langle f, g \rangle z = \langle fz, gz \rangle$, concluding that $\Delta z_2 = \langle f, g \rangle z_1$. By UMP of a pullback, there is a unique $u : Z \rightarrow E$ making a commutative diagram. Thus we conclude that e is an equalizer of f, g .

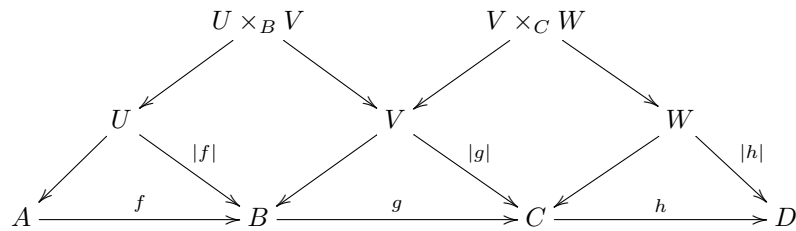
7.

Proposition 5.25

8.

(identity) For $A \in \text{Ob}(\mathcal{C})$, take $(A, 1_A)$.

(Associativity) Let $U \subseteq A, V \subseteq B, W \subseteq C$. Consider a map



The map $h \circ (g \circ f)$ is a pair $(|h \circ (g \circ f)|, (|h|^*(W))) = (|h \circ (g \circ f)|, (U \times_B V) \times_C W) = (|(h \circ g) \circ f|, U \times_B (V \times_C W)) = (|(h \circ g) \circ f|, |f|^*(|g|^*W)) = (h \circ g) \circ f$. Thus it is associative, concluding that $\mathbf{Par}(\mathcal{C})$ is a category.

9.

Define $\lim_{\leftarrow \mathbf{J}} : \mathbf{Diagrams}(\mathbf{J}, \mathbf{C}) \rightarrow \mathbf{C}$ by $\lim_{\leftarrow \mathbf{J}} F = \lim_{\leftarrow \mathbf{J}} F$ and $\lim_{\leftarrow \mathbf{J}} (\theta : F \rightarrow G) = \lim_{\leftarrow \mathbf{J}} F \rightarrow \lim_{\leftarrow \mathbf{J}} G$ by $\gamma_i \lim_{\leftarrow \mathbf{J}} \theta = \theta_i \beta_i$ where $\beta_i : \lim_{\leftarrow \mathbf{J}} F \rightarrow F_i$ and $\gamma_i : \lim_{\leftarrow \mathbf{J}} G \rightarrow G_i$ are product maps. Then $\lim_{\leftarrow \mathbf{J}} 1_F = 1_{\lim_{\leftarrow \mathbf{J}} F}$ since for all i , $1_{F_i} \beta_i = \beta_i \lim_{\leftarrow \mathbf{J}} \theta$. Also, from the definition of arrow, we have $\lim_{\leftarrow \mathbf{J}} (\phi \circ \theta) = \lim_{\leftarrow \mathbf{J}} \phi \circ \lim_{\leftarrow \mathbf{J}} \theta$ where $\phi : G \rightarrow H$.

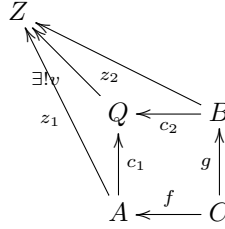
Now consider \mathbf{Sets}^I for a set I . It is a set of functions $A : I \rightarrow \mathbf{Sets}$. As above, we can define a category of such objects and define a product functor

$$\prod_{i \in I} : \mathbf{Sets}^I \rightarrow \mathbf{Sets}$$

defined by $A \rightarrow \prod_{i \in I} A_i$.

10.

- (a) Pushout: Given any arrows f, g with $\text{dom}(f) = \text{dom}(g) = C$, the pushout of f and g consists of arrows $c_1 : A \rightarrow Q$, $c_2 : B \rightarrow Q$ such that $c_1 f = c_2 g$ and the universal property as follows: Given any $z_1 : A \rightarrow Z$, $z_2 : B \rightarrow Z$ with $z_1 f = z_2 g$, there is a unique $v : Q \rightarrow Z$ with $z_1 = v c_1$ and $z_2 = v c_2$.



- (b) skip

11.

Note that q is a coequalizer of $r_1, r_2 : R \rightarrow X$. By using the fact that $\mathcal{P}(X) \cong 2^X$, we define $\mathcal{P}q : 2^Q \rightarrow 2^X$ by $\mathcal{P}q([a])(x) = [a]([x])$ and $\mathcal{P}r_i : 2^X \rightarrow 2^R$ by $\mathcal{P}r_1(a)(x, y) = a(x)$ and $\mathcal{P}r_2(a)(x, y) = a(y)$ where $a : X \rightarrow 2$ and $[a] : Q \rightarrow 2$ is induced from a . Then it satisfies the diagram

$$\mathcal{P}Q \xrightarrow{\mathcal{P}q} \mathcal{P}X \xrightarrow{\mathcal{P}r_i} \mathcal{P}R.$$

Suppose $z : Z \rightarrow \mathcal{P}X$ satisfies $\mathcal{P}r_1 z = \mathcal{P}r_2 z$. Then for all $(x, y) \in R$, $a(x) = a(y)$ where a is in the image of z . Since R is an equivalence relation, for all $x \in [x]$, $a(x)$ has the same value, meaning that there is a unique map $Z \rightarrow \mathcal{P}Q$.

12.

limit: Consider a cone $\zeta_n : Z \rightarrow n$. If $n = 0$, then ζ_0 is zero map and for all n , ζ_n is factored through ζ_0 , concluding that ζ_n is zero. The limit is a terminal object in the category of cone, which is object 0 with inclusion $0 \hookrightarrow n$.

colimit: Consider a co-cone $\psi_n : n \rightarrow Y$. By the property, ψ_n is a restriction of ψ_{n+m} and since Y is also a poset, $\psi_n(m) \leq \psi_n(n)$ for $m < n$, concluding that ψ_n is monotone. Hence we conclude that a map $\theta : \mathbb{N} \rightarrow Y$ uniquely defines a cone. Thus ω with inclusion $n \subseteq \omega$ defines the colimit.

13.

- (a) Suppose M_k, N_k are abelian groups. Note that $M_0 \rightarrow \dots \rightarrow M_k \rightarrow \dots$, $N_0 \leftarrow \dots \leftarrow N_k \leftarrow \dots$. Then we can see that $\varprojlim_k M_k = M_0$. $\varinjlim_k M_k$ is indeed the direct limit of M_k , which is $\bigoplus_k M_k / \sim$ where $m_i \sim \mu_{ik}(m_i)$ for $m_i \in M_i, \mu_{ik} : M_i \rightarrow M_k$. For N_k , $\varinjlim_k N_k = N_0$ and $\varprojlim_k N_k$ is the inverse limit of abelian groups: $\mathcal{S} = \{g \in \prod N_k \mid \text{if } i \leq j, \text{ then } \nu_{ji}(g_j) = g_i\}$.
- (b) All elements of colimit of M_k and limit of N_k may have an elemnt of infinite order.(right????)