Computational Linear Algebra
Singular Value Decomposition

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MACALESTER

SVD of a Wide Matrix

$$A = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix} = \underbrace{\begin{bmatrix} -0.95 & -0.32 \\ -0.32 & 0.95 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 18.97 & 0 & 0 \\ 0 & 9.49 & 0 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} -1/3 & -2/3 & -2/3 \\ 2/3 & 1/3 & -2/3 \end{bmatrix}}_{V^\top}$$

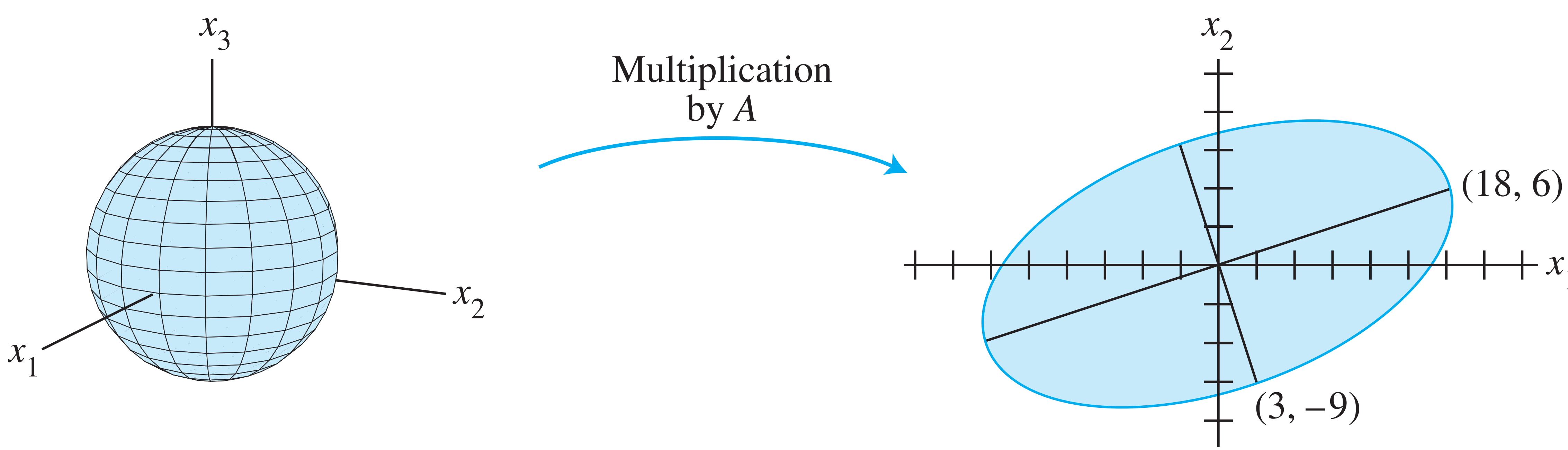
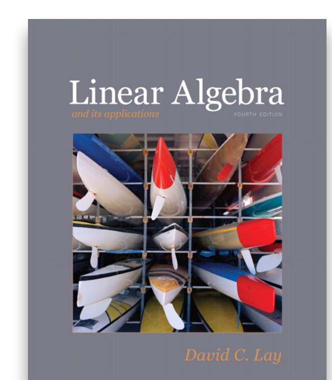
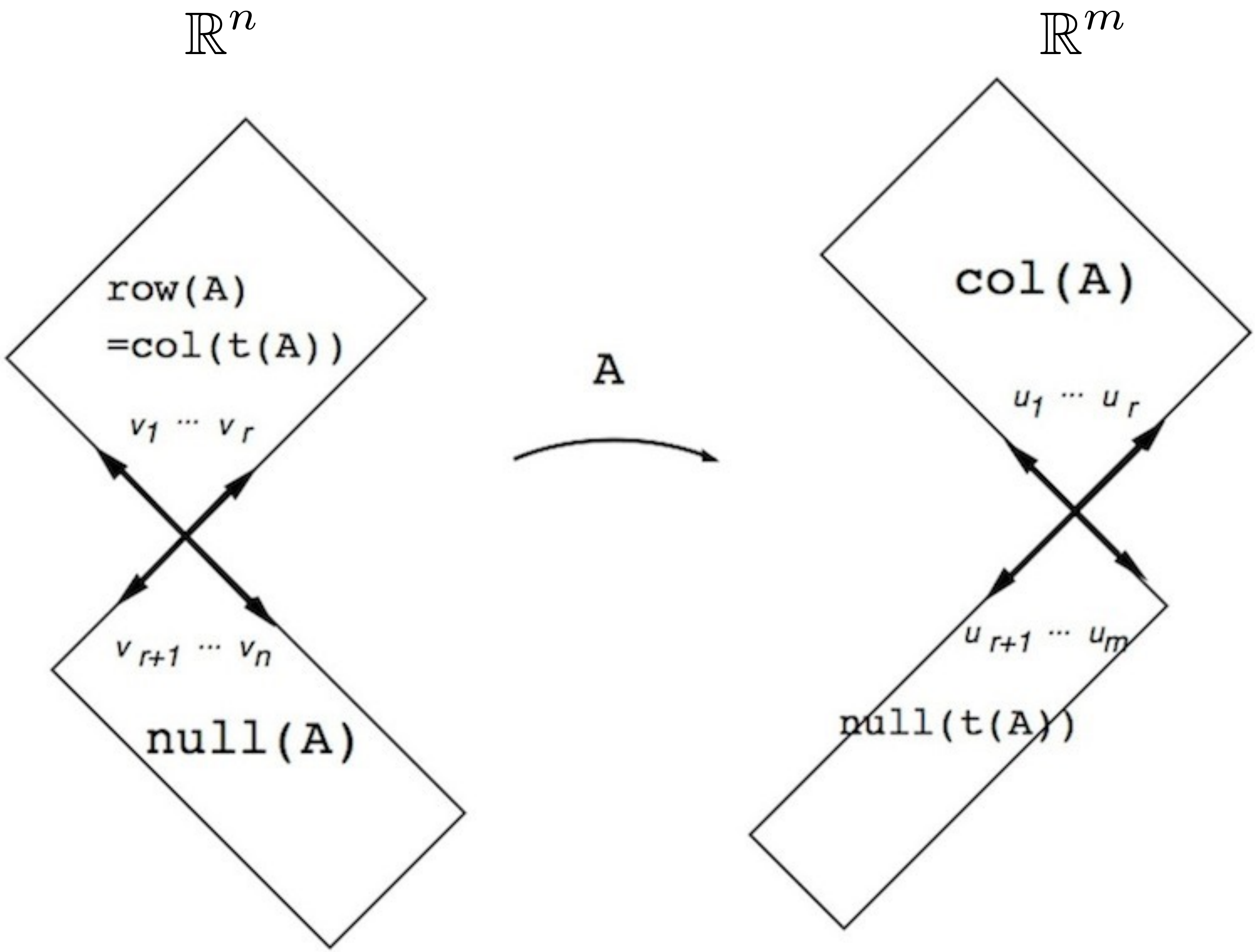


FIGURE 1 A transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$.



Section 7.4

Four Fundamental Subspaces



Source: "Fundamental Theorem of Linear Algebra," by Gilbert Strang (see Moodle)

SVD of a Tall Matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} -2\sqrt{6}/6 & 0 \\ -\sqrt{6}/6 & -\sqrt{2}/2 \\ -\sqrt{6}/6 & \sqrt{2}/2 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \end{bmatrix}}_{\Sigma} \underbrace{\begin{bmatrix} -\sqrt{2}/2 & -\sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix}}_{V^\top}$$

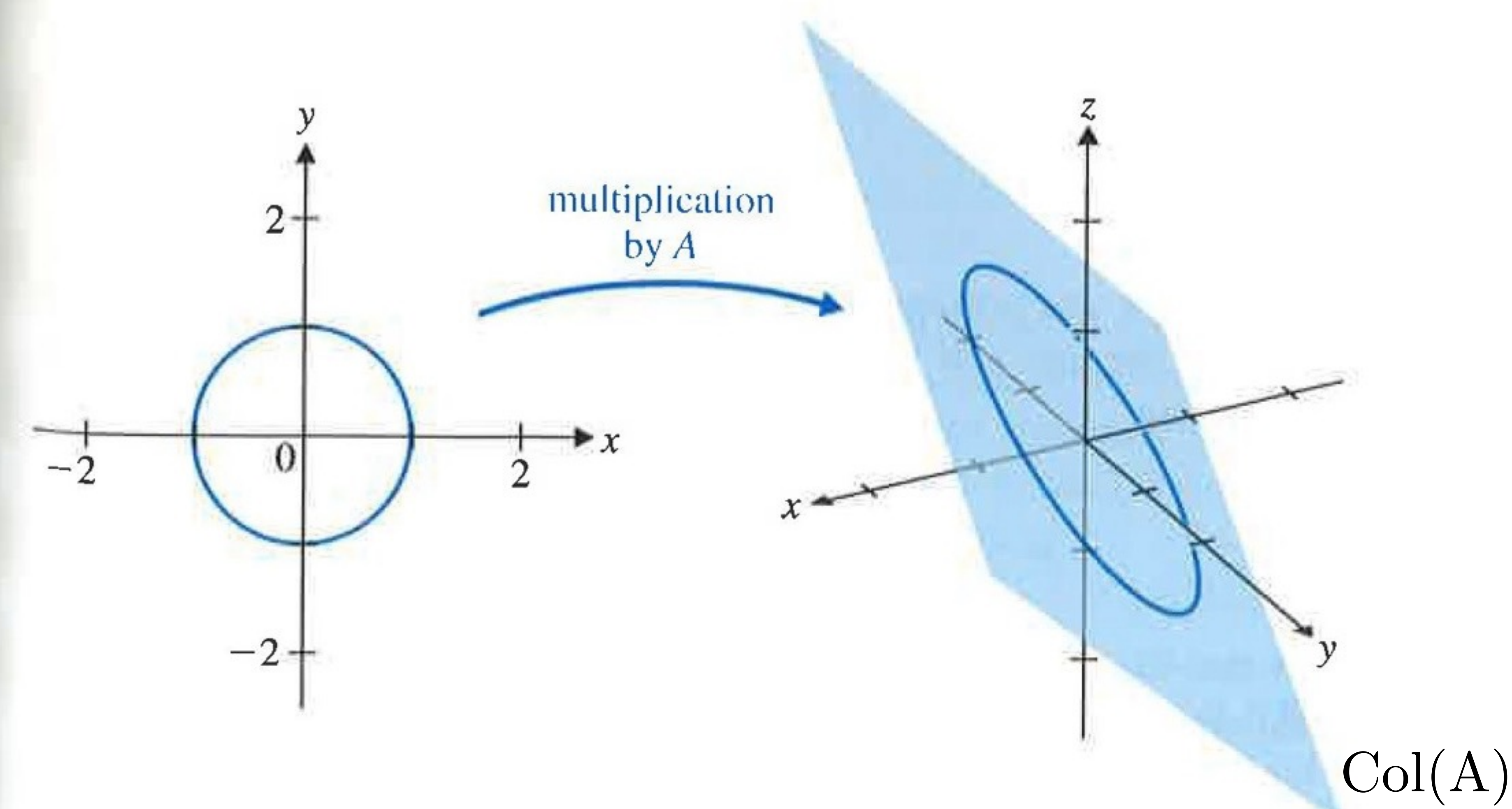


Figure 7.18

The matrix A transforms the unit circle in $\mathbb{R}^2$ into an ellipse in $\mathbb{R}^3$

Source: Linear Algebra, D. Poole, Section 7.4

EVD vs. SVD

- Eigenvalue decomposition
  - Change of basis with a single basis for the domain and range
  - Does not exist for all A (or even all square A's)
  - Basis is orthonormal if and only if A is symmetric
  - Especially useful theoretically and computationally for problems involving iterated applications of A
    - e.g., A^k or e^{tA}
- Singular value decomposition
  - Change of basis with two different orthonormal bases for the domain and range
  - Always exists for any A and bases always orthonormal
  - Especially useful with A or A^{-1} or A^T