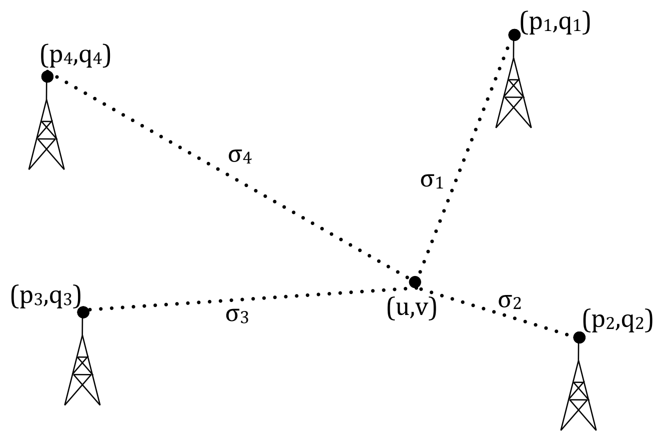


1. While walking in the woods, I got lost! Luckily I have a device that lets me compute the distances (as the bird flies) between my location and four different communication towers. Let's say that the distance between my new unknown location (u, v) and the tower located at position (p_i, q_i) is σ_i, where $i = 1, 2, 3, 4$. This gives me a system of four **nonlinear** equations in two variables (u and v):

$$\sqrt{(u - p_i)^2 + (v - q_i)^2} = \sigma_i, \;\; i = 1, 2, 3, 4 \tag{1}$$

Nonlinear equations are yucky though, so the first thing I'm going to do is linearize these equations around my starting coordinates $(0, 0)$. If I assume that I started at coordinates $(0, 0)$ and did not walk too far relative to the distances to the towers, then I can approximate the left-hand side of (1) as follows:

$$\sqrt{(u - p_i)^2 + (v - q_i)^2} \approx \sqrt{p_i^2 + q_i^2} - \frac{p_i u + q_i v}{\sqrt{p_i^2 + q_i^2}}.$$

This gives me the following four **linear** equations in the two variables u and v:

$$\boxed{\frac{-p_i}{\sqrt{p_i^2+q_i^2}} u + \frac{-q_i}{\sqrt{p_i^2+q_i^2}} v = \sigma_i - \sqrt{p_i^2+q_i^2}}$$

$$\sqrt{p_i^2 + q_i^2} - \frac{p_i u + q_i v}{\sqrt{p_i^2 + q_i^2}} = \sigma_i, \;\; i = 1, 2, 3, 4 \tag{2}$$

Due to the linearization and measurement errors, I cannot solve these four equations exactly. I would like you to setup and solve a least squares problem based on (2) in order to find my location, using the following data:

$(p_1, q_1) = (12, 0)$ $\qquad$ $\sigma_1 = 9.62$
$(p_2, q_2) = (-10, 2)$ $\qquad$ $\sigma_2 = 12.50$
$(p_3, q_3) = (3, 8)$ $\qquad$ $\sigma_3 = 6.52$
$(p_4, q_4) = (-2, -8)$ $\qquad$ $\sigma_4 = 10.50$

Note: The picture above is conceptual only and does not represent the actual distances measured or locations of the towers.

You can use the next page for any work, but please write down your A matrix, b vector, and answer for the location here:

$$A = \begin{bmatrix} \frac{-12}{\sqrt{12^2+0^2}} & 0 \\ \frac{10}{\sqrt{10^2+2^2}} & \frac{-2}{\sqrt{10^2+2^2}} \\ \frac{-3}{\sqrt{3^2+8^2}} & \frac{-8}{\sqrt{3^2+8^2}} \\ \frac{2}{\sqrt{2^2+8^2}} & \frac{8}{\sqrt{2^2+8^2}} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \qquad b = \begin{bmatrix} 9.62 - \sqrt{12^2+0^2} \\ 12.50 - \sqrt{(-10)^2+2^2} \\ 6.52 - \sqrt{3^2+8^2} \\ 10.50 - \sqrt{(-2)^2+(-8)^2} \end{bmatrix}$$

$\sigma_i - \sqrt{p_i^2+q_i^2}$

My location: $\begin{bmatrix} 2.497 \\ 1.455 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$

2. Let $\mathbf{u}$ be a vector in $\mathbb{R}^3$ with $||\mathbf{u}||_2 = 1$. Define the matrix $H = I - 2\mathbf{u}\mathbf{u}^\top$, where I is the 3×3 identity matrix.

   (a) Show that $H^2 = I$.

   (b) What does H do when it is applied to the vector $\mathbf{u}$? That is, what is $H\mathbf{u}$?

   (c) If $\mathbf{w}$ is orthogonal to $\mathbf{u}$, then what does H do when it is applied to the vector $\mathbf{w}$? That is, what is $H\mathbf{w}$?

   (d) Geometrically, what does H do to vectors in $\mathbb{R}^3$? Draw a picture.

   (e) Prove that $||H\mathbf{x}||_2 = ||\mathbf{x}||_2$ for all $\mathbf{x} \in \mathbb{R}^3$. Also confirm that this is true on your picture from part (d). Hint: you can use your answer from part (a) in the proof.

3. Interpolation. Consider the function $f(x) = \cos(4x)$ on the interval $[0, 1]$. I would like you to make a good polynomial approximation to this function in three ways:

   (a) Sample points from the function on $[0, 1]$ and use them to construct the 6th degree interpolating polynomial through them.

   (b) Sample 50 points from the function on $[0, 1]$ and use them to construct a least squares 5th degree polynomial that fits the data.

   (c) Make a spline of the function that uses 6 cubic splines.

   In each case, give a plot of the function and the approximation. Compare the three approximations and discuss the tradeoffs.

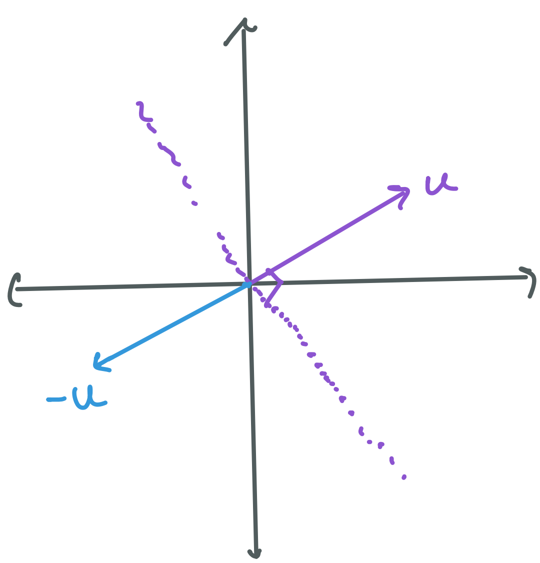
2

a) $$H^2 = (I - 2uu^T)(I - 2uu^T) = I^2 - 4uu^T + 4u\underbrace{\boxed{u^Tu}}_{1\ = u\cdot u = ||u||_2^2 = 1}u^T$$

$$= I - 4uu^T + 4uu^T = \boxed{I}$$

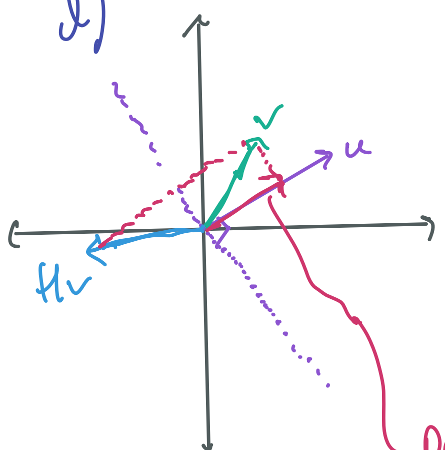
b) $$Hu = I\cdot u - 2uu^Tu = u - 2\underbrace{\langle u,u\rangle u}_{\text{projection of } u \text{ onto } u \text{ is just } u}$$

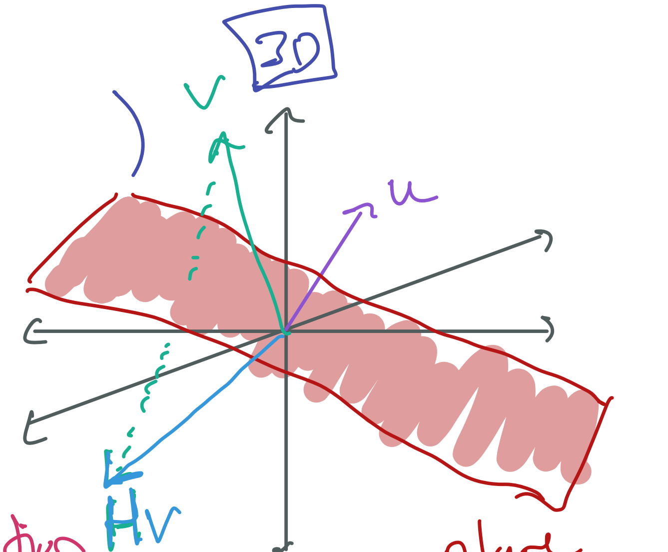
$$= u - 2u = -u$$



c) $Hw = w - 2uu^Tw = w - 2\underbrace{\langle u,w\rangle}_{0 \text{ since } u\perp w} u = w$

d)





H reflects v across the orthogonal complement of u. In 2D, this is the line perpendicular to u. In 3D, it is the plane for which u is the normal vector.

e) $H^T = (I - 2uu^T)^T = I^T - 2(uu^T)^T$

$= I - 2u^{T^T}u^T$

$= I - 2uu^T = H$

$\|Hx\|_2^2 = x^T H^T H x = x^T H^2 x = x^T x = \|x\|_2^2$

(a)

$\Rightarrow \|Hx\|_2 = \|x\|_2$