Easily inverted matrices

Easily inverted matrices

The main math problem in our course: Ax=b

Given: $A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^n$

Find: $x \in \mathbb{R}^n$ such that $Ax = b$

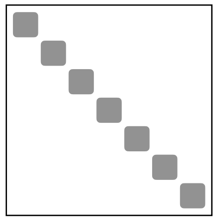
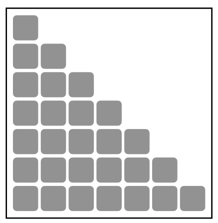
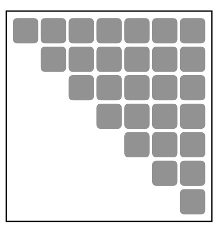
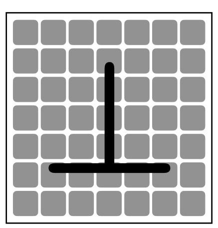
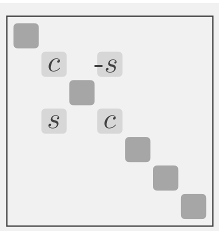
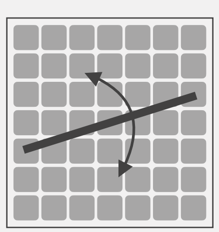
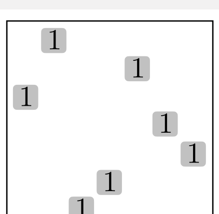
Math 236 idea: Find A^{-1} and then multiply to get $x = A^{-1}b$
This is nice in principle… but a horrible idea in practice.
Finding an explicit matrix inverse is not just time-consuming, it can lead to an inaccurate answer!
More on this later when we talk about machine arithmetic.

Instead, find x directly without first getting A^{-1}

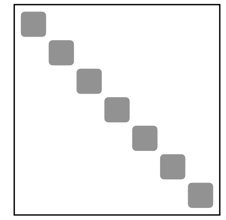
The best way to do this depends on the structure of the matrix

Today: some especially easy ones

Easily inverted matrices

Type		Solution of $Ax = b$	Flops
	diagonal	division, $x_i = b_i / A_{ii}$	n
	lower triangular	forward substitution	n^2
	upper triangular	backward substitution	n^2
	orthogonal	multiplication by transpose	$2n^2$
	Givens rotation	$x = G^T b$	6
	Householder reflection $H = I - 2vv^T$	$x = Hb = b - 2(v^T b)v$	$4n$
	permutation	reordering, $x = P^T b$	0

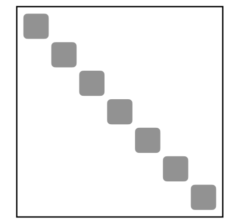
Easily inverted matrices



diagonal

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0.5 \end{pmatrix} \begin{pmatrix} \quad \\ \\ \\ \\ \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 2 \\ 6 \end{pmatrix}$$

Easily inverted matrices

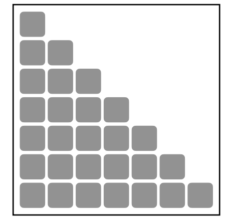


diagonal

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0.5 \end{pmatrix} \begin{pmatrix} 3.5 \\ 1 \\ -2 \\ 12 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 2 \\ 6 \end{pmatrix}$$

$Ax = b$ with A diagonal:
Just take $x_i = b_i / A_{ii}$.
Fails if any $A_{ii} = 0$ (but in that case A is not invertible anyway)
Total work: n divisions

Easily inverted matrices

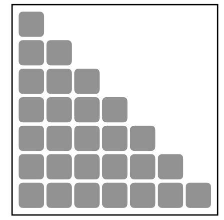


lower triangular

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0.25 & 2 & -1 & 0 \\ 3 & -1 & 1 & 0.5 \end{pmatrix} \begin{pmatrix} \quad \\ \quad \\ \quad \\ \quad \end{pmatrix} = \begin{pmatrix} 8 \\ -8 \\ 2 \\ 0 \end{pmatrix}$$

Idea: get entries of the solution one at a time, starting at the top.
"Forward Substitution"

Easily inverted matrices



lower triangular

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0.25 & 2 & -1 & 0 \\ 3 & -1 & 1 & 0.5 \end{pmatrix} \begin{pmatrix} 4 \\ \\ \\ \\ \end{pmatrix} = \begin{pmatrix} 8 \\ -8 \\ 2 \\ 0 \end{pmatrix}$$

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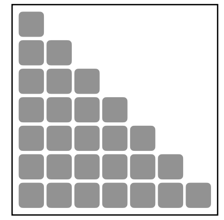
Easily inverted matrices



lower triangular

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0.25 & 2 & -1 & 0 \\ 3 & -1 & 1 & 0.5 \end{pmatrix} \begin{pmatrix} 4 \\ -3 \\ \\ \end{pmatrix} = \begin{pmatrix} 8 \\ -8 \\ 2 \\ 0 \end{pmatrix}$$

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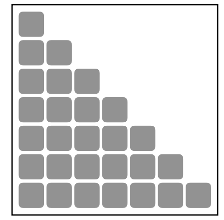
Easily inverted matrices

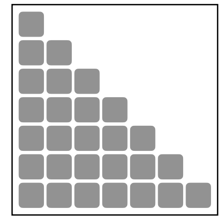


lower triangular

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0.25 & 2 & -1 & 0 \\ 3 & -1 & 1 & 0.5 \end{pmatrix} \begin{pmatrix} 4 \\ -3 \\ -7 \\ \end{pmatrix} = \begin{pmatrix} 8 \\ -8 \\ 2 \\ 0 \end{pmatrix}$$

Idea: get entries of the solution one at a time, starting at the top.
"Forward Substitution"

Easily inverted matrices

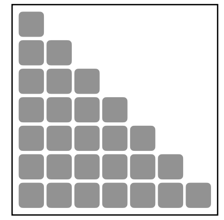


lower triangular

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0.25 & 2 & -1 & 0 \\ 3 & -1 & 1 & 0.5 \end{pmatrix} \begin{pmatrix} 4 \\ -3 \\ -7 \\ -16 \end{pmatrix} = \begin{pmatrix} 8 \\ -8 \\ 2 \\ 0 \end{pmatrix}$$

Idea: get entries of the solution one at a time, starting at the top.
"Forward Substitution"

Easily inverted matrices



lower triangular

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0.25 & 2 & -1 & 0 \\ 3 & -1 & 1 & 0.5 \end{pmatrix} \begin{pmatrix} 4 \\ -3 \\ -7 \\ -16 \end{pmatrix} = \begin{pmatrix} 8 \\ -8 \\ 2 \\ 0 \end{pmatrix}$$

Arithmetic count

-> to get b_i once you have b_{i-1},

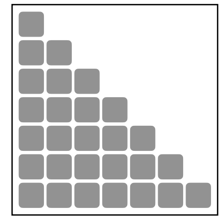
there are (i-1) multiplications,

there are (i-1) additions,

and there is one division

-> total is 2i - 1 *flops* to get b_i

Easily inverted matrices



lower triangular

$$\begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0.25 & 2 & -1 & 0 \\ 3 & -1 & 1 & 0.5 \end{pmatrix} \begin{pmatrix} 4 \\ -3 \\ -7 \\ -16 \end{pmatrix} = \begin{pmatrix} 8 \\ -8 \\ 2 \\ 0 \end{pmatrix}$$

Arithmetic count

-> to get b_i once you have b_{i-1},

there are (i-1) multiplications,

there are (i-1) additions,

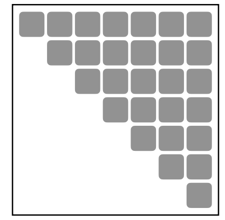
and there is one division

-> total is 2i - 1 *flops* to get b_i

`$=\mathcal{O}(n^2)$`

$$\begin{aligned} S &= \sum_{i=1}^{n} \Big((i-1) + (i-1) + 1 \Big) \\ &= \sum_{i=1}^{n} \Big(2i - 1 \Big) \\ &= \left(2 \sum_{i=1}^{n} i \right) - n \\ &= 2\frac{n(n-1)}{2} - n \\ &= n^2 - 2n \\ &= \mathcal{O}(n^2) \end{aligned}$$

Easily inverted matrices



upper triangular

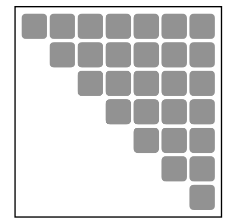
$$\begin{pmatrix} -2 & 1 & 1 & 3 \\ 0 & 4 & 1 & -3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} \\ \\ \\ \end{pmatrix} = \begin{pmatrix} -11 \\ -15 \\ 2 \\ 1 \end{pmatrix}$$

Idea:

Slogan:

Arithmetic count:

Easily inverted matrices



upper triangular

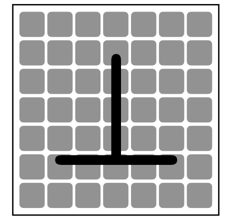
$$\begin{pmatrix} -2 & 1 & 1 & 3 \\ 0 & 4 & 1 & -3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 4 \\ -3 \\ -1.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} -11 \\ -15 \\ 2 \\ 1 \end{pmatrix}$$

Idea: start with the last entry of x and work up

Slogan: “backward substitution”

Arithmetic count: $\mathcal{O}(n^2)$

Easily inverted matrices



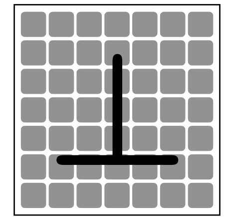
orthogonal

$$Q = \begin{pmatrix} \frac{-3}{5} & \frac{16}{25} & \frac{12}{25} \\ \frac{4}{5} & \frac{12}{25} & \frac{9}{25} \\ 0 & \frac{3}{5} & \frac{-4}{5} \end{pmatrix}$$

The columns are all ______ __________.

Each column is ______________ to the others.

Easily inverted matrices



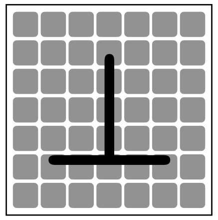
orthogonal

$$Q = \begin{pmatrix} \frac{-3}{5} & \frac{16}{25} & \frac{12}{25} \\ \frac{4}{5} & \frac{12}{25} & \frac{9}{25} \\ 0 & \frac{3}{5} & \frac{-4}{5} \end{pmatrix}$$

The columns are all unit vectors.

Each column is perpendicular to the others.

Easily inverted matrices



orthogonal

orthogonal matrix: a square matrix whose columns are unit vectors, perpendicular
(with complex-valued matrices, the name is unitary matrix)

Why is this nice?

Recall: $(AB)_{ij} = \sum_k A_{ik} B_{kj}$ "$(AB)_{ij}$ is the i-th row of A dotted with the j-th column of B"

Now modify this: $(Q^T Q)_{ij}$ is the i-th column of Q dotted with the j-th column of Q.

$(Q^T Q)_{ij} = Q_i^T Q_j$

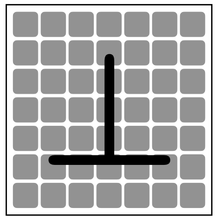
If Q is an orthogonal matrix, then $Q^T Q = I$ (the transpose is a left inverse)

Fact: If Q is square and $Q^T Q = I$, then also $QQ^T = I$ (rows of A are also unit vectors, perpendicular)
The transpose of a square orthogonal matrix is its inverse (!!!)

$$\boxed{Qx = b \quad \longleftrightarrow \quad x = Q^T b}$$

-> this is the one place where we use an explicit matrix inverse
('computing' the inverse is just transposition, very easy)

Easily inverted matrices



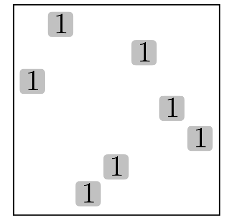
orthogonal

What to remember about orthogonal matrices:

1) Definition: square matrix whose columns are unit vectors, perpendicular
2) Equivalent: any square matrix Q with $Q^TQ = I$
3) Inverse is transpose: $Q^{-1} = Q^T$
4) Solving $Qx = b$ is easy: just take $x = Q^Tb$

Orthogonal matrices are wonderful.

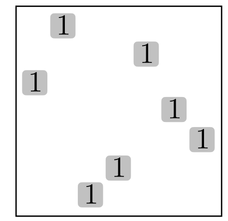
Easily inverted matrices



permutation

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \\ \\ \\ \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 2 \\ 6 \end{pmatrix}$$

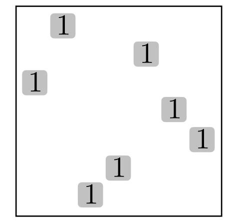
Easily inverted matrices



permutation

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \\ 7 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 2 \\ 6 \end{pmatrix}$$

Easily inverted matrices



permutation

Permutation matrices

Definition: a square matrix with a single 1 in each row and column, 0 otherwise

A permutation matrix is a special kind of _________ matrix.

Pv : reorders entries of v

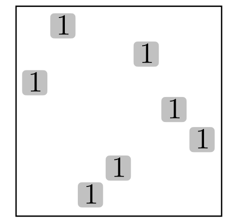
PA: left multiplication reorders __________ of A

AP: right multiplication reorders ___________ of A

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \\ 7 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 2 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 4 \\ 2 & 4 & 2 & 8 \\ 0 & 0 & 0 & 0 \\ 3 & 6 & 3 & 12 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 4 & 2 & 8 \\ 3 & 6 & 3 & 12 \\ 1 & 2 & 1 & 4 \end{pmatrix}$$

Easily inverted matrices



permutation

Permutation matrices

Definition: a square matrix with a single 1 in each row and column, 0 otherwise

A permutation matrix is a special kind of orthogonal matrix.

Pv : reorders entries of v

PA: left multiplication reorders rows of A

AP: right multiplication reorders columns of A

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \\ 7 \\ 2 \end{pmatrix} = \begin{pmatrix} 7 \\ 4 \\ 2 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 2 & 1 & 4 \\ 2 & 4 & 2 & 8 \\ 0 & 0 & 0 & 0 \\ 3 & 6 & 3 & 12 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 4 & 2 & 8 \\ 3 & 6 & 3 & 12 \\ 1 & 2 & 1 & 4 \end{pmatrix}$$