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$\boxed{1}$ Show $\underbrace{\max_{x\neq 0}\left\{\frac{\|Ax\|_p}{\|x\|_p}\right\}}_{\text{LHS}} = \underbrace{\max_{\|x\|_p=1}\left\{\|Ax\|_p\right\}}_{\text{RHS}}$

<u>Step 1:</u> RHS $\leq$ LHS

$$\max_{\|x\|_p=1}\left\{\|Ax\|_p\right\} = \max_{\|x\|_p=1}\left\{\frac{\|Ax\|_p}{\|x\|_p}\right\} \leq \max_{x\neq 0}\left\{\frac{\|Ax\|_p}{\|x\|_p}\right\}$$

because $\|x\|_p=1$ for all x in the set

"such that"

Let $S_1 := \{x\in\mathbb{R}^n : \|x\|_p=1\}$

$S_2 := \{x\in\mathbb{R}^n : x\neq 0\}$

"defined"

Logic:

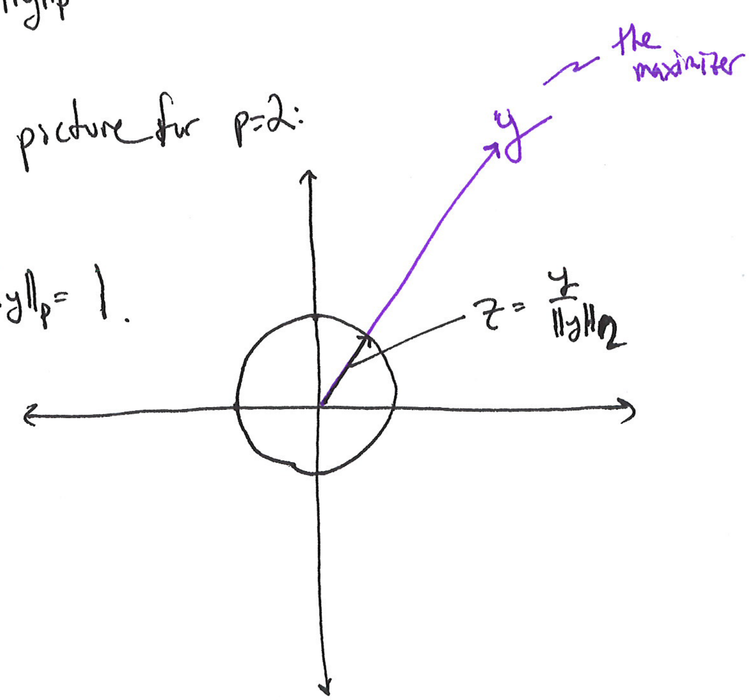
$$S_1 \subset S_2 \Rightarrow \max_{x\in S_1}\{f(x)\} \leq \max_{x\in S_2}\{f(x)\}$$

<u>Step 2:</u> LHS $\leq$ RHS

Let $y = \underset{x\in\mathbb{R}^n : x\neq 0}{\arg\max}\left\{\frac{\|Ax\|_p}{\|x\|_p}\right\}$.

This is equivalent to saying $\frac{\|Ay\|_p}{\|y\|_p} = \max_{x\in\mathbb{R}^n : x\neq 0}\left\{\frac{\|Ax\|_p}{\|x\|_p}\right\}$.

Now let $z = \frac{y}{\|y\|_p}$. Here's a picture for $p=2$:



Note that $\|z\|_p = \left\|\frac{y}{\|y\|_p}\right\|_p = \frac{1}{\|y\|_p}\cdot\|y\|_p = 1$.

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$\boxed{1 \text{ (cont.)}}$

We have

$$\text{LHS} = \max_{x \neq 0} \left\{ \frac{\|Ax\|_p}{\|x\|_p} \right\} = \frac{\|Ay\|_p}{\|y\|_p} = \frac{\|A(\|y\|_p \cdot z)\|_p}{\|(\|y\|_p \cdot z)\|_p} = \frac{\|y\|_p}{\|y\|_p} \cdot \frac{\|Az\|_p}{\|z\|_p} \cdots$$

$y = \|y\|_p \cdot z$

$$\cdots = \frac{\|Az\|_p}{\|z\|_p} \leq \underbrace{\max_{x \in \mathbb{R}^n : \|x\|_p = 1}}_{S_2} \left\{ \frac{\|Ax\|_p}{\|x\|_p} \right\} = \max_{x \in \mathbb{R}^n : \|x\|_p = 1} \left\{ \|Ax\|_p \right\}$$

b/c denominator $= 1 \ \forall x$ in S_2.

$z \in S_2$, so by same argument as above

$$f(z) \leq \max_{x \in S_2} \{ f(x) \}$$

$\boxed{2}$

a) $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \ldots x_n^2}$

Let $k = \underset{i \in \{1,2,\ldots n\}}{\operatorname{argmax}} |x_i|$ so that $\|x\|_\infty = |x_k|$.

Then $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \ldots x_n^2} \leq \sqrt{x_k^2 + x_k^2 + \ldots x_k^2} = \sqrt{n(x_k)^2} = \sqrt{n} \cdot |x_k|$

$= \sqrt{n} \cdot \|x\|_\infty$ (1)

Also, $\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \ldots x_n^2} \geq \sqrt{x_k^2} = |x_k| = \|x\|_\infty$ (2)

Put (1) and (2) together to get

$$\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n} \|x\|_\infty$$

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$\boxed{2\ b}$

$$\|A\|_\infty = \max_{x\neq 0}\left\{\frac{\|Ax\|_\infty}{\|x\|_\infty}\right\} \le \max_{x\neq 0}\left\{\frac{\|Ax\|_2}{\|x\|_\infty}\right\} \le \max_{x\neq 0}\left\{\frac{\|Ax\|_2}{\frac{\|x\|_2}{\sqrt{n}}}\right\} \cdots$$

From (a), $\|Ax\|_\infty \le \|Ax\|_2$

From (a), $\|x\|_\infty \ge \frac{\|x\|_2}{\sqrt{n}}$

Note: making the denominator smaller makes the overall quantity larger.

$$\cdots = \max_{x\neq 0}\left\{\frac{\sqrt{n}\,\|Ax\|_2}{\|x\|_2}\right\} = \sqrt{n}\max_{x\neq 0}\left\{\frac{\|Ax\|_2}{\|x\|_2}\right\}$$

$$= \sqrt{n}\,\|A\|_2,$$

so $\|A\|_\infty \le \sqrt{n}\,\|A\|_2$.

For the first inequality,

$\|Ax\|_2 \le \sqrt{n}\|Ax\|_\infty$ by part (a)

$$\frac{\|A\|_2}{\sqrt{n}} = \frac{1}{\sqrt{n}}\max_{x\neq 0}\left\{\frac{\|Ax\|_2}{\|x\|_2}\right\} = \max_{x\neq 0}\left\{\frac{\|Ax\|_2}{\sqrt{n}\,\|x\|_2}\right\} \le \max_{x\neq 0}\left\{\frac{\sqrt{n}\,\|Ax\|_\infty}{\sqrt{n}\,\|x\|_2}\right\} \cdots$$

$$\le \max_{x\neq 0}\left\{\frac{\|Ax\|_\infty}{\|x\|_\infty}\right\} = \|A\|_\infty,$$

so $\frac{\|A\|_2}{\sqrt{n}} \le \|A\|_\infty$.

$\|x\|_\infty \le \|x\|_2$ by (a)