Computational Linear Algebra: Floating Point Representation and Rounding Error

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- The leading 1 is not stored, it is implied

IEEE Standard Floating Point Representation (cont.)

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Precision	Sign	Exponent	Mantissa	Total
single	1	8	23	32
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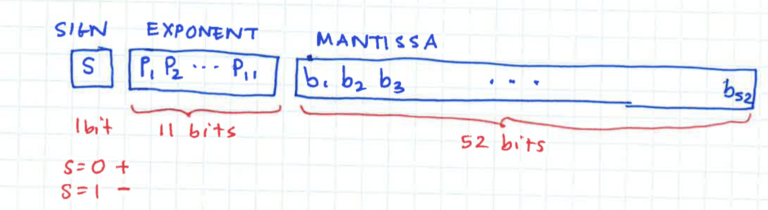
- We will almost always be using the 64-bit "double" precision:



- The sign is 0 for positive numbers and 1 for negative numbers.

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- Back to our example:

$$23.171875 = 10111.001011 = +1.0111001011 * 2^4$$

- The sign is 0
- The mantissa is

0111001011000000000000000000000000000000000000000000

or separated into four bit chunks:

0111 0010 1100 0000 0000 0000 0000 0000 0000 0000 0000 0000 0000

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- We need both positive and negative exponents, and we keep two integers for special cases, so we have 2046 left to cover integers from -1022 to 1023:

binary	00000000000	00000000001	00000000010	$\cdots$	11111111110	11111111111
decimal	0	1	2	$\cdots$	2046	2047
exponent	special	-1022	-1021	$\cdots$	1023	special

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- 1023 is the exponent bias. We add this amount to the actual exponent, and then store that number (note: this is the same as adding 1024 and subtracting 1)
- Subtract 1023 after storing (or subtract 1024 and add 1)
- Thus $+1.0111001011 * 2^4$ is

0	10000000011	$011100101100 \cdots 0$

Special Cases and More Examples

Special cases:

- Exponent of 2047 is equal to ∞ if the mantissa is all zeros, and equal to NaN (Not a Number) otherwise
- Exponent of 0 used to represent subnormal (very small) numbers. See the book as we will not harp on these

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$$-1.0111000101100 * 2^{50} \quad \text{and} \quad +1.0111000101100 * 2^{-33}$$

Rounding

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If we always chop (drop/ignore/truncate) the bits after b_{52} then we will systematically be rounding down. This can accumulate, so we use rounding:

- If what is left over is $< 1/2$:

  $$1.\boxed{b_1 b_2 b_3 \cdots b_{51} b_{52}}\ \underbrace{0\cdots\cdots\cdots\cdots}_{\text{round down by truncating}}$$

  just drop the bits beyond position 52

- If what is left over is $> 1/2$:

  $$1.\boxed{b_1 b_2 b_3 \cdots b_{51} b_{52}}\ \underbrace{1\cdots\cdots\cdots\cdots}_{\text{round up by adding 1 to } b_{52}}$$

  add 1 to bit b_{52} carrying as necessary

- If what is left over is $= 1/2$:

  $$1.\boxed{b_1 b_2 b_3 \cdots b_{51} b_{52}}1\ \underbrace{0\cdots\cdots\cdots\cdots}_{\text{all 0's}}$$

  if $b_{52} = 1$: (round up)
  add 1 to b_{52}
  if $b_{52} = 0$: (round down)
  do nothing

  - in this way we always end up with $b_{52} = 0$
  - these two possibilities are equally likely

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- Find gap between 2^{20} and next smallest representable number

Rounding Error

Example:

- $$(9.4)_{10} = (1001.\overline{0110})_2$$

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- $$\text{fl}(9.4) = +1.\boxed{0010\ 1100\ 1100\ 1100\ 1100\ 1100\ 1100\ 1100\ 1100\ 1100\ 1100\ 1100\ 1101} * 2^3$$
- So we truncated the tail by $.\overline{1100} * 2^{-52} * 2^3 = .\overline{0110} * 2^{-51} * 2^3 = 0.4 * 2^{-48} = 0.8 * 2^{-49}$
- And by rounding up, we added $2^{-52} * 2^3 = 1.0 * 2^{-49}$
- Therefore $\text{fl}(9.4) = 9.4 + 0.2 * 2^{-49}$

Absolute Rounding Error:

- $|x_c - x| = |\text{fl}(9.4) - 9.4| = 0.2 * 2^{-49}$

Relative Rounding Error:

- $\frac{|x_c - x|}{|x|} = \frac{|\text{fl}(9.4) - 9.4|}{9.4} = \frac{0.2 * 2^{-49}}{9.4} = \frac{8}{47} * 2^{-52}$
- The relative rounding error is always less than or equal to $\frac{\epsilon_{mach}}{2}$

Addition and Subtraction of Floating Point Numbers

- There is a register dedicated to doing computations so that the additions can be done with higher precision
- To add/subtract two numbers:
  1. Line up decimals
  2. Do the addition/subtraction (in higher precision, so you don't need to worry about rounding yet)
  3. Once the arithmetic is complete, round to convert back to floating point form