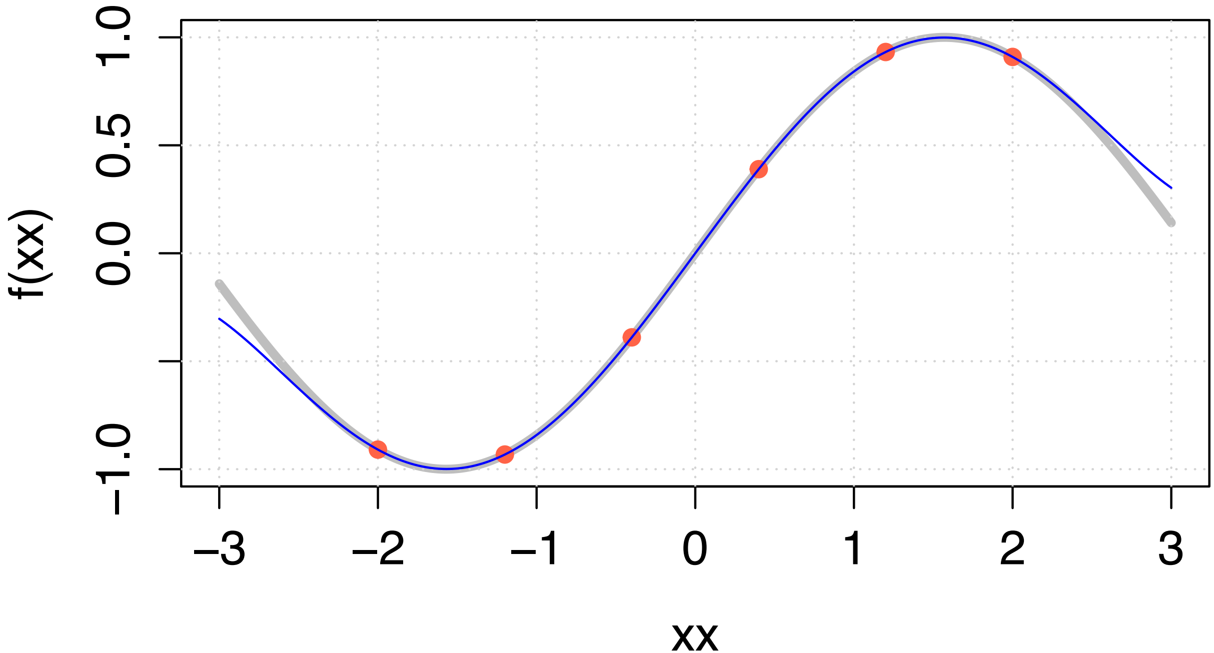
Computational Linear Algebra: Best Approximation and Chebyshev Interpolation

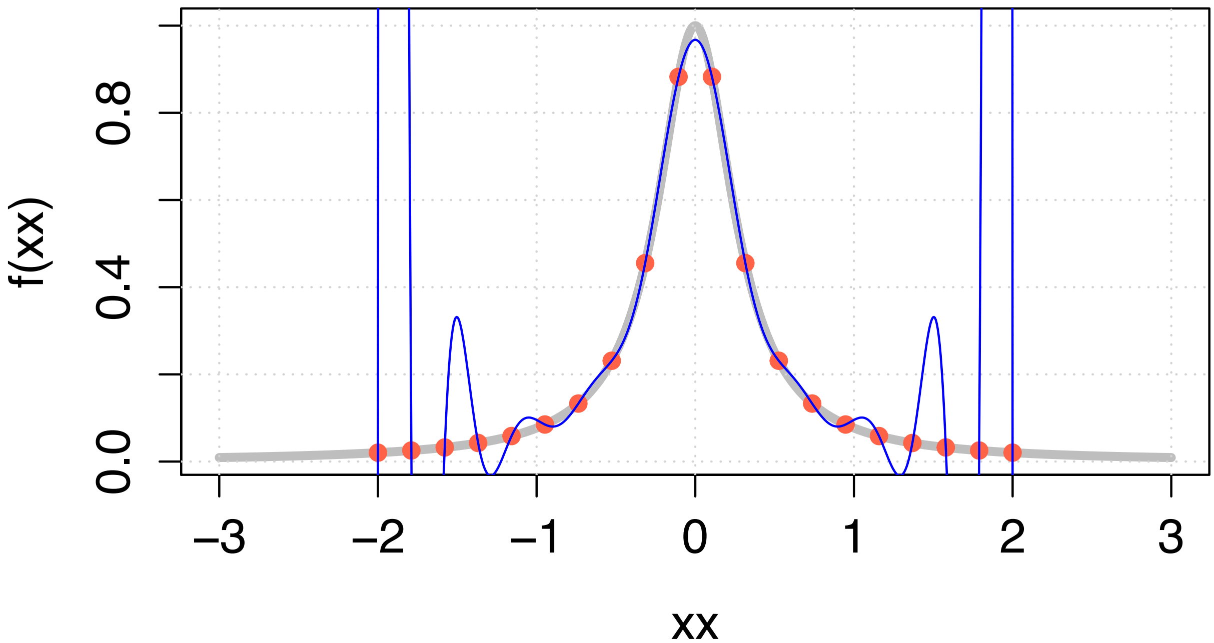
Function Approximation by Sampling and then Performing Polynomial Interpolation

$f(x) = \sin(x)$: few interpolating points yields a good approximation



Function Approximation by Sampling and then Performing Polynomial Interpolation

$f(x) = \frac{1}{1+12x^2}$: a less good approximation



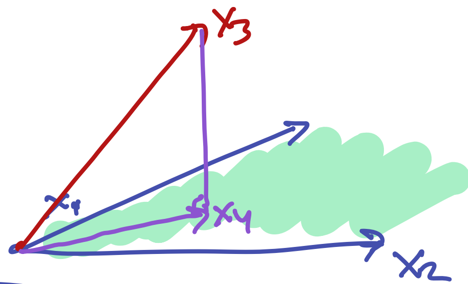
Approximation

- Two main components of approximation problems:
  - Approximation space from which to select the approximation
  - Distance notion

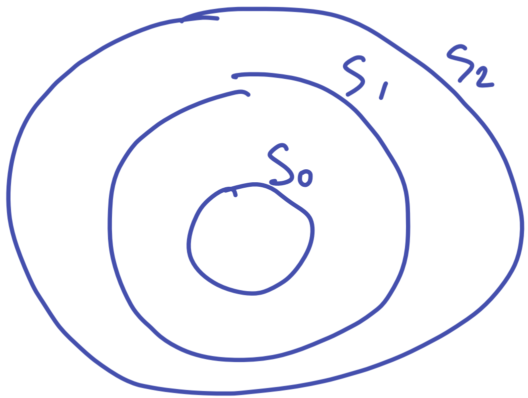
$\boxed{\ldots, -1, 0, 1, 2, 3, \ldots}$

4.32

4



$A = \text{span}\{x_1, x_2\}$



$S_0 \subset S_1 \subset S_2$

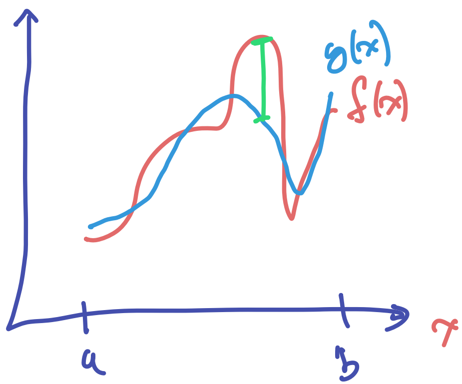
Distances for Function Approximation

- How do we measure the distance between two functions?
  - The norm of the difference between the two functions
- Most commonly used norms for function spaces are the L^p norms:

$$||f||_p = \left(\int_S |f(x)|^p dx \right)^{\frac{1}{p}}$$

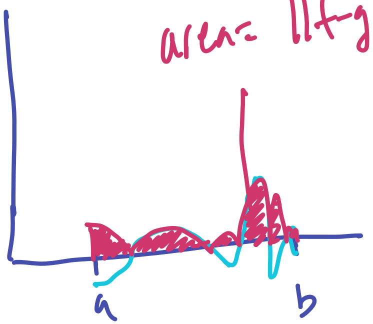
- Examples of distances between functions f and g (over an interval $[a, b]$):
  - $\boxed{\|f - g\|_1 = \int_a^b |f(x) - g(x)| dx}$
  - $\|f - g\|_2 = \left(\int_a^b |f(x) - g(x)|^2 dx \right)^{\frac{1}{2}} = \sqrt{\langle f - g, f - g \rangle}$
  - $\|f - g\|_\infty := \sup_{x \in [a,b]} |f(x) - g(x)|$

Distances for Function Approximation: Examples

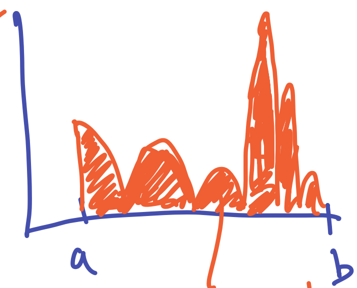


$|f(x)-g(x)| = |f-g|$

area $= \|f-g\|_1$



$[f(x)-g(x)]^2$



$\|f-g\|_\infty =$

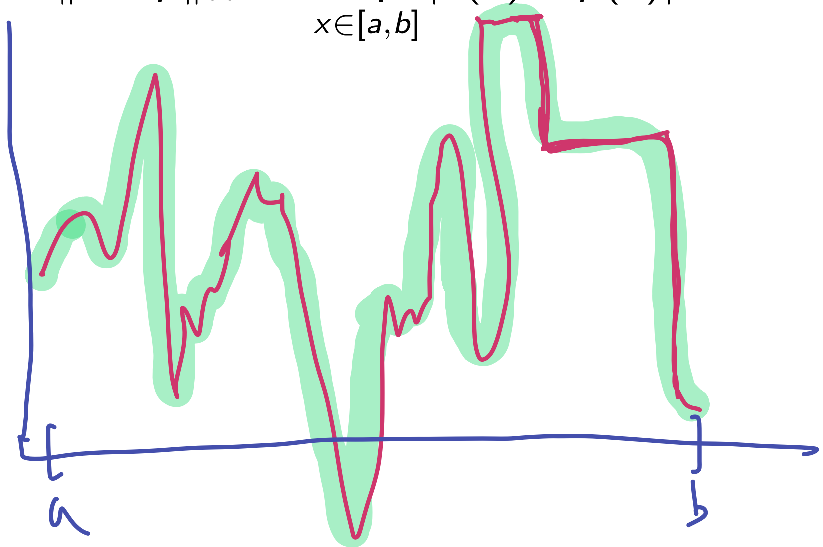
area $= \int_a^b [f(x)-g(x)]^2 dx = \|f-g\|_2$

Best Minimax Approximation

Weierstrass Approximation Theorem

For any continuous function f on $[a, b]$ and any $\epsilon > 0$, there exists a polynomial p such that the approximation error

$$\|f - p\|_\infty := \sup_{x \in [a,b]} |f(x) - p(x)| < \epsilon.$$



$\epsilon = .01$

Best Minimax Approximation

Weierstrass Approximation Theorem

For any continuous function f on $[a, b]$ and any $\epsilon > 0$, there exists a polynomial p such that the approximation error

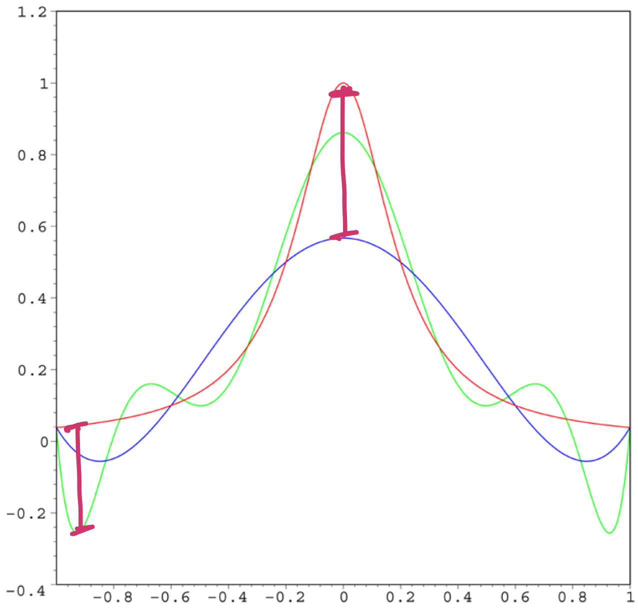
$$\|f - p\|_\infty := \sup_{x \in [a,b]} |f(x) - p(x)| < \epsilon.$$

- Catch: The degree of the approximating polynomial may be large
- What is the best we can do when the degree of the approximating polynomial is bounded?

1885: Über die analytische Darstellbarkeit sogenannter willkürlicher Functionen einer reellen Veränderlichen

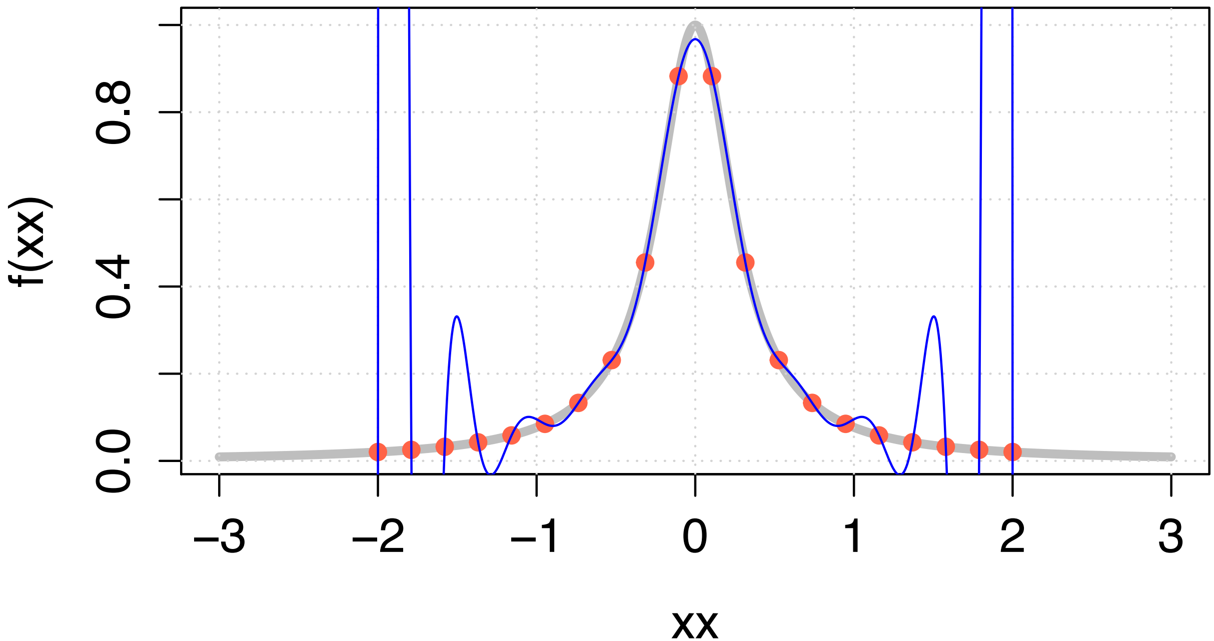
The Runge Phenomenon

- Fix $n+1$ points in $[-1,1]$
- Unique polynomial of degree n passing through those points
- If you pick $n+1$ points uniformly, max error may increase with n (despite Weierstrass theorem)



Red is function to be approximated, blue is fifth order approx., green is ninth order approx. Source: Wikipedia.

The Runge Phenomenon



- The Runge phenomenon is a problem of oscillation at the edges of an interval that occurs when using polynomial interpolation with polynomials of high degree
- This shows that going to higher degrees does not always improve accuracy

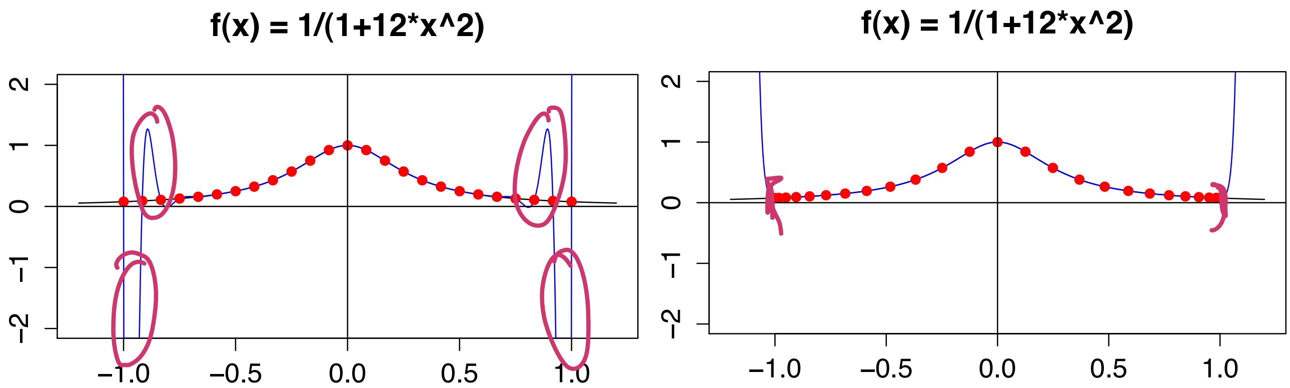
Interpolation Error

Question: Can we choose better sampling points: $x_1, x_2, \ldots, x_n$ to minimize the error?

Interpolation Error

Question: Can we choose better sampling points: $x_1, x_2, \ldots, x_n$ to minimize the error?

Check this out:



How were those points chosen?

Chebyshev polynomials

Pafnuty Lvovich Chebyshev (Russian: Пафну́тий Льво́вич Чебышёв, IPA: [pɐfˈnutʲɪj ˈlʲvovʲɪtɕ tɕɪbɨˈʂof]) (16 May [O.S. 4 May] 1821 – 8 December [O.S. 26 November] 1894)[1] was a Russian mathematician. His name can be alternatively transliterated as *Chebysheff*, Chebychov, *Chebyshov*; or *Tchebychev*, *Tchebycheff* (French transcriptions); or *Tschebyschev*, *Tschebyschef*, *Tschebyscheff* (German transcriptions). *Chebychev*, mixture between English and French transliterations, is sometimes erroneously used.

Contents [hide]

Pafnuty Chebyshev

Pafnuty Lvovich Chebyshev

Chebyshev polynomials

Trig definition

$$T_n(x) = \cos(n \arccos(x))$$

wait, this is a polynomial???

Recursion definition

$$T_0(x) = 1$$
$$T_1(x) = x$$
$$T_2(x) = 2x^2 - 1$$
$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

Every polynomial of degree *n* has a unique Chebyshev expansion:

$$p(x) = \sum_{j=0}^{n} c_j T_j(x)$$

example: if $p(x) = 2 - 5x + 4x^2$, then $p(x) = 4T_0(x) - 5T_1(x) + 2T_2(x)$

the constants c_n are called the "Chebyshev coefficients."

Chebyshev polynomials

Trig definition

$$T_n(x) = \cos(n \arccos(x))$$

wait, this is a polynomial???

Recursion definition

$$T_0(x) = 1$$

$$T_1(x) = x$$

$$T_2(x) = 2x^2 - 1$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

Exercises:

(1) Use the trig definition to illustrate the first eight Chebyshev polynomials in R. Put them all on the same plot in different colors. Plot them on the interval [-1,1].

(2) Use the recursion definition to find expressions for the third and fourth Chebyshev polynomials (in the usual monomial basis).

(3) Express the polynomial $p(x) = 40x^4 - 5$ as a sum of Chebyshev polynomials.

Computational Benefits of Chebyshev Polynomials

Recurrence Relations

- $T_0(x) = 1$
  $T_1(x) = x$
  $T_k(x) = 2xT_{k-1}(x) - T_{k-2}(x) \quad \text{for } k \geq 2$

- $T_k(x)T_{k'}(x) = \frac{1}{2}\left[T_{k+k'}(x) + T_{|k-k'|}(x)\right]$

A=1
B=5

Shifted Chebyshev Polynomials

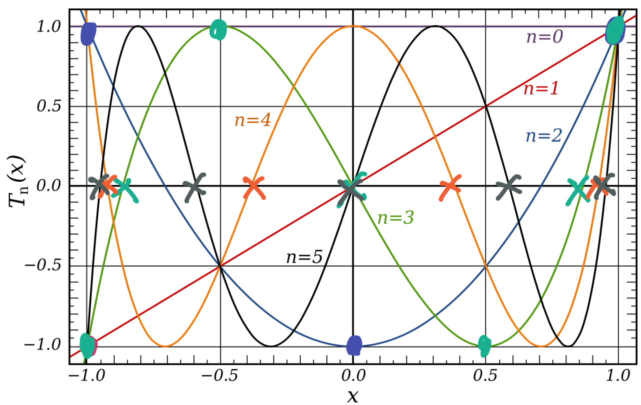
- To shift the domain from [-1,1] to [A,B], define

$$\overline{T}_k(x) := T_k\left(\underbrace{\boxed{\frac{2}{B-A}}}_{\frac{2}{4}}\left(x - \underbrace{\boxed{\frac{(A+B)}{2}}}_{3}\right)\right);$$

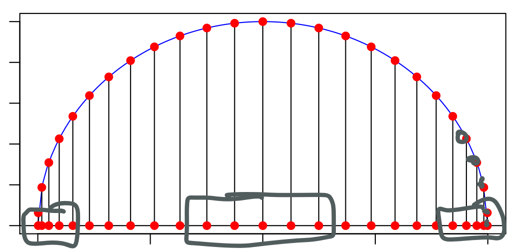
i.e, stretch and shift

Chebyshev Polynomials

- $T_n(x) := \cos\big(n \arccos(x)\big),\ \ x \in [-1, 1],\ \ n = 0, 1, 2, \ldots$
- $T_n(x)$ has $n + 1$ extrema at $\cos\left(\frac{k\pi}{n}\right)$, $k = 0, 1, \ldots, n$
- Maximum magnitude alternates between 1 and -1 at these $n + 1$ points
- Chebyshev nodes: $T_n(x) = 0$ at $x_i = \cos\left(\frac{2i-1}{2n}\pi\right)$, $i = 1, 2, \ldots, n$
  - These are the x-coordinates of evenly spaced points around the circle (right picture below)



Source: Wikipedia



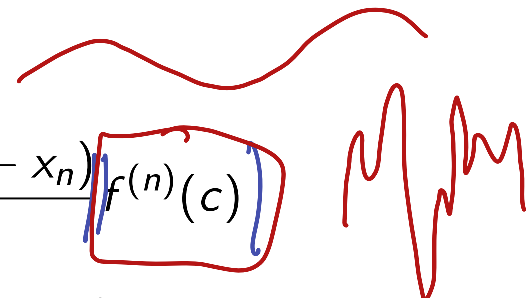
Upper Bound on the Interpolation Error

Theorem (3.4)

Assume f is an n-times continuously differentiable function. If n points $x_1, \ldots, x_n$ are sampled from a function

$$\to y_1 = f(x_1), y_2 = f(x_2), \ldots, y_n = f(x_n),$$

then the interpolation error is given by



$$\left|f(x) - p(x)\right| = \left|\frac{(x - x_1)(x - x_2)\cdots(x - x_n)}{n!}\right| \boxed{\left|f^{(n)}(c)\right|}$$

where c lies between the smallest and the largest of the numbers $x, x_1, \ldots, x_n$

- Thus the error at x is governed by
  - the distance from x to the points
  - the size of the nth derivative $f^{(n)}(c)$
- And we can use this to bound the error

Chebyshev Nodes Minimize the Upper Bound

Theorem

The choice of nodes $-1 \le x_1, \ldots, x_n \le 1$ *that minimizes*

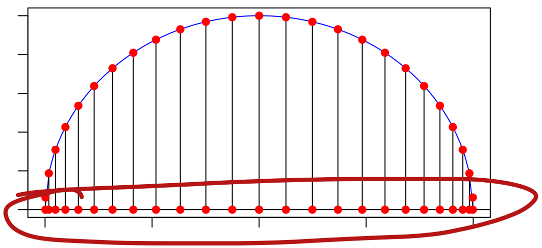
$$\max_{-1\le x\le 1} |(x-x_1)(x-x_2)\cdots(x-x_n)|$$

are

$$\boxed{x_i = \cos\left(\frac{(2i-1)\pi}{2n}\right),} \qquad 1 \le i \le n.$$

and the maximum value of this error is $\frac{1}{2^{n-1}}$.

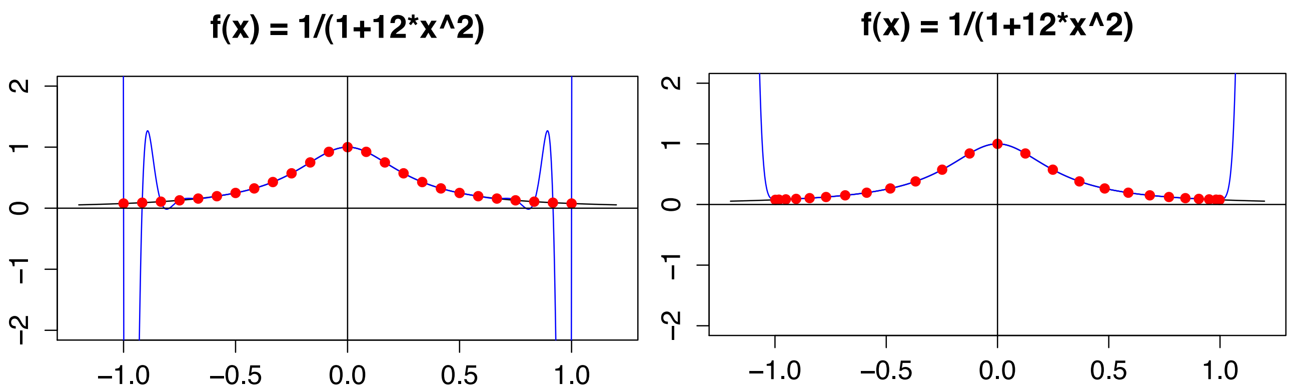
- Exactly the roots of $T_n(x)$
- The Chebyshev nodes are the x-coordinates of evenly spaced points around the circle



Sampling at the Chebyshev Nodes Leads to Better Approximation Error than Sampling at Evenly Spaced Nodes

Question: Can we choose better sampling points: $x_1, x_2, \ldots, x_n$ to minimize the error?

Check this out:

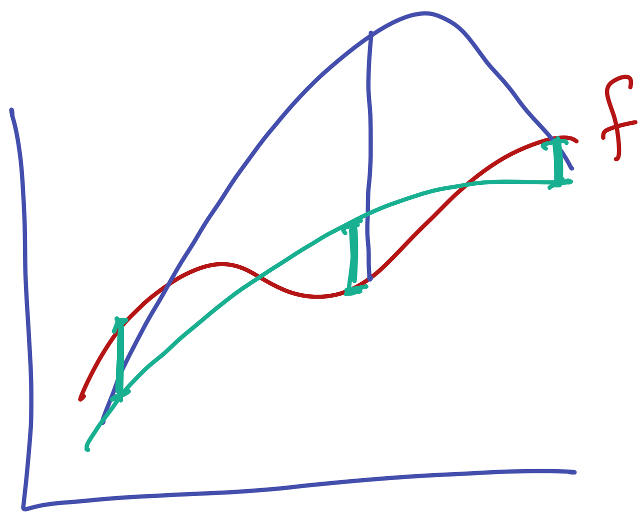


How were those points chosen?

The Minimax Property of Chebyshev Polynomials

Necessary and sufficient conditions for $\|f - p_n^*\|_\infty = \inf_{p_n \in \mathcal{P}_n} \|f - p_n\|_\infty$?

- There exist $n+2$ distinct points $x_1 < x_2 < \ldots < x_{n+2}$ such that:
  - $|f(x_i) - p_n^*(x_i)| = \|f - p_n^*\|_\infty$, $i = 1, 2, \ldots, n+2$
  - Residuals at these points alternate signs



The Minimax Property of Chebyshev Polynomials

Necessary and sufficient conditions for $\|f - p_n^*\|_\infty = \inf_{p_n \in \mathcal{P}_n} \|f - p_n\|_\infty$?

- There exist $n+2$ distinct points $x_1 < x_2 < \ldots < x_{n+2}$ such that:
  - $|f(x_i) - p_n^*(x_i)| = \|f - p_n^*\|_\infty$, $i = 1, 2, \ldots, n+2$
  - Residuals at these points alternate signs
- Application: $\arg\min_{p_{n-1} \in \mathcal{P}_{n-1}} \|\boxed{x^n} - p_{n-1}\|_\infty = x^n - \boxed{\frac{1}{2^{n-1}} T_n(x)}$

The Minimax Property of Chebyshev Polynomials

Necessary and sufficient conditions for $\|f - p_n^*\|_\infty = \inf_{p_n \in \mathcal{P}_n} \|f - p_n\|_\infty$?

- There exist $n+2$ distinct points $x_1 < x_2 < \ldots < x_{n+2}$ such that:
  - $|f(x_i) - p_n^*(x_i)| = \|f - p_n^*\|_\infty$, $i = 1, 2, \ldots, n+2$
  - Residuals at these points alternate signs
- Application: $\arg\min_{p_{n-1} \in \mathcal{P}_{n-1}} \|x^n - p_{n-1}\|_\infty = x^n - \frac{1}{2^{n-1}} T_n(x)$

How do we (approximately) compute p_n^*?

- Polynomial interpolation with the $n+1$ points chosen to be the Chebyshev nodes (zeros) of $T_{n+1}(x)$
- Puts more of the interpolation points towards the ends than uniform choice
- Near-optimal and the error decreases as you consider higher degree polynomials
- Can iterate by setting new interpolation points to be those with the largest magnitude of error in previous round (c.f., Remez algorithm)