Vectors in $\boldsymbol{R}^n$, a basic guide

Standard matrix-vector notation

Matrices are capital Roman letters

Identity matrix is I

$$C = \begin{pmatrix} 2 & 5 & 5 & 6 \\ 0 & 2 & -8 & 3 \\ -1 & -1 & 6 & -3 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Double subscript: matrix entry (scalar) $C_{23} = -8$

Single subscript: matrix column (vector) $C_2 = \begin{pmatrix} 5 \\ 2 \\ -1 \end{pmatrix}$

Vectors are lower-case Roman letters

-> note, always column vectors!

$$v = \begin{pmatrix} 2 \\ \pi \\ \sqrt{6} \end{pmatrix}$$

Single subscript: vector entry (scalar) $v_3 = \sqrt{6}$

(but sometimes v_3 is actually the third vector in a list of vectors)

Standard basis for $\mathbb{R}^n$ is $\{e_i\}_{i=1}^n$

$$\text{In } \mathbb{R}^4,\ e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Scalars are lower-case Greek letters $\alpha = 2.4$

Standard matrix-vector notation

Matrix-vector multiplication: very common

-> Try to develop a column attitude

$$\begin{pmatrix} -1 & 2 & 1 \\ 2 & 7 & -3 \\ 1 & -3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} =$$

Standard matrix-vector notation

Matrix-vector multiplication: very common

-> Try to develop a column attitude

$$\begin{pmatrix} -1 & 2 & 1 \\ 2 & 7 & -3 \\ 1 & -3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} = 1 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix} + 5 \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ -27 \\ \end{pmatrix}$$

Standard matrix-vector notation

Matrix-vector multiplication: very common

-> Try to develop a column attitude

$$Av = \sum_j A_j v_j$$

$$(Av)_i = \sum_j A_{ij} v_j$$

$$\begin{pmatrix} -1 & 2 & 1 \\ 2 & 7 & -3 \\ 1 & -3 & 4 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} = 1 \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix} - 2 \begin{pmatrix} 2 \\ 7 \\ -3 \end{pmatrix} + 5 \begin{pmatrix} 1 \\ -3 \\ 4 \end{pmatrix} = \begin{pmatrix} 0 \\ -27 \\ 27 \end{pmatrix}$$

Standard matrix-vector notation

Superscript T means transpose.

$$C = \begin{pmatrix} 2 & 5 & 5 & 6 \\ 0 & 2 & -8 & 3 \\ -1 & -1 & 6 & -3 \end{pmatrix} \qquad C^T = \begin{pmatrix} 2 & 0 & -1 \\ 5 & 2 & -1 \\ 5 & -8 & 6 \\ 6 & 3 & -3 \end{pmatrix}$$

Use T if you need a row vector.

$$v = \begin{pmatrix} 2 \\ \pi \\ \sqrt{6} \end{pmatrix} \qquad v^T = \begin{pmatrix} 2 & \pi & \sqrt{6} \end{pmatrix}$$

$(A+B)^T = A^T + B^T$

Important: $(Av)^T = v^T A^T, \quad (AB)^T = B^T A^T$

``the transpose of a product is the reversed product of the transposes.''

Similar property for matrix inverses: $(AB)^{-1} = B^{-1} A^{-1}$

"The inverse of the transpose is the transpose of the inverse," $(A^T)^{-1} = (A^{-1})^T = A^{-T}$

beware: $(A+B)^{-1} \neq A^{-1} + B^{-1}$

Standard matrix-vector notation

Suppose we have two vectors in $\mathbb{R}^4$:

$$w = \begin{pmatrix} 0 \\ 3 \\ -4 \\ 1 \end{pmatrix} \quad v = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \end{pmatrix}$$

Which of these are scalars and which are matrices?

$$v^T w \qquad w^T v \qquad v w^T \qquad w v^T$$

Standard matrix-vector notation

Suppose we have two vectors in $\mathbb{R}^4$: $w = \begin{pmatrix} 0 \\ 3 \\ -4 \\ 1 \end{pmatrix}$ $v = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \end{pmatrix}$

$$v^T w = 4 = w^T v$$

Which of these are scalars and which are matrices?

$v^T w$	$w^T v$	$v w^T$	$w v^T$
Sc	Sc	Ma	Ma

$$v w^T = \begin{pmatrix} 0 & 6 & -8 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Which expressions are valid?

$$v^T v v w^T v$$

$$v v^T w w^T w$$

$$v v w^T v$$

$$v w^T w v^T$$

Standard matrix-vector notation

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Which expressions are valid?

$$v^T v v w^T v \qquad \to \text{ a vector}$$

$$v v^T w w^T w \qquad \to \text{ a vector}$$

$$v v w^T v \qquad \to \text{ invalid}$$

$$v w^T w v^T \qquad \to \text{ a matrix}$$

Standard matrix-vector notation

$$v = \begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \end{pmatrix} \qquad w = \begin{pmatrix} 0 \\ 3 \\ -4 \\ 1 \end{pmatrix}$$

Inner product (very common, a.k.a ``dot product'')

-> a scalar from two vectors $\qquad v^T w = 4$

-> symmetry (a scalar is its own transpose) $\qquad v^T w = w^T v$

-> defines length of a vector

other defn's of length are possible

this is the ``2-norm'' or standard length

``unit vector'' if length one

$$|v| = \|v\| = \|v\|_2 = \sqrt{v^T v} \qquad \leftrightarrow \qquad |v|^2 = v^T v$$

-> defines angle between two nonzero vectors

right angle ($v \perp w$) if $v^T w = 0$

$$\cos\theta = \frac{v^T w}{|v||w|}$$

Outer product (common)

-> a matrix from two vectors

-> also known as a ``rank one matrix''

-> defined even if vectors have different numbers of entries

$$vw^T = \begin{pmatrix} 0 & 6 & -8 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & -3 & 4 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$wv^T = (vw^T)^T$$

Standard matrix-vector notation

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Is this } e_2 e_3^T \text{ or } e_3 e_2^T?$$

$$\textit{Hint:} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

Standard matrix-vector notation

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Is this $e_2 e_3^T$ ✗ or $e_3 e_2^T$ ✓?

Hint: $$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

$$e_2 e_3^T e_2 = 0$$
$$e_3 e_2^T e_2 = e_3$$

Vector Norms

Vector Norms

Goal: measure the length of a vector.

So far we have used: $\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$ for $x \in \mathbb{R}^n$

This is the most important way to measure length, but there are others.

What do we mean by “length” anyway?

Vector Norms

Let V be a vector space over the real or complex numbers.
A norm is a function $\|\cdot\|$ carrying V to $\mathbb{R}$ such that:

1) Lengths aren't negative $\|x\| \geq 0$
2) Nonzero vectors have nonzero length $\|x\| = 0 \text{ only for } x = 0$
3) "Homogeneity" or reasonable scaling behavior $\|\lambda x\| = |\lambda|\|x\|$
4) Triangle inequality $\|x + y\| \leq \|x\| + \|y\|$

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Examples of norms on our favorite vector space, $\mathbb{R}^n$:

$$\|x\|_2 = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

Euclidean norm (can omit the _2)

$$\|x\|_1 = |x_1| + |x_2| + \cdots + |x_n|$$

One-norm

$$\|x\|_\infty = \max_{1 \leq i \leq n} |x_i|$$

Infinity-norm or max-norm

$$\|x\|_p = \left(|x_1|^p + |x_2|^p + \cdots + |x_n|^p\right)^{1/p}$$

p-norm;
needs $p \in (1, \infty)$;
p=2 is special

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For $x^T = (0, 1, 1, 1, 0, 1, 1, 1, 0, -1, -1)$, find:

$$\|x\|_1 =$$
$$\|x\|_2 =$$
$$\|x\|_3 =$$
$$\|x\|_\infty =$$

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p-norm;
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For $x^T = (0, 1, 1, 1, 0, 1, 1, 1, 0, -1, -1)$, find:

$$\|x\|_1 = 8$$
$$\|x\|_2 = 2\sqrt{2} \approx 2.8284$$
$$\|x\|_3 = 2$$
$$\|x\|_\infty = 1$$

How would you scale $x = (1, 2, 4)$ so that it becomes a unit vector in the 1-norm?

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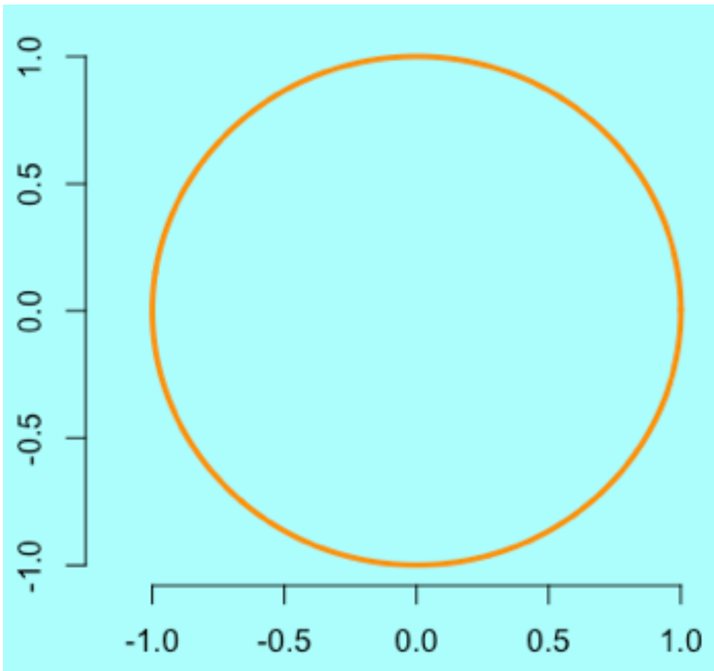
(1/7, 2/7, 4/7)

How would you scale $x = (1, 2, 4)$ so that it becomes a unit vector in the ∞-norm?

(1/4, 2/4, 1)

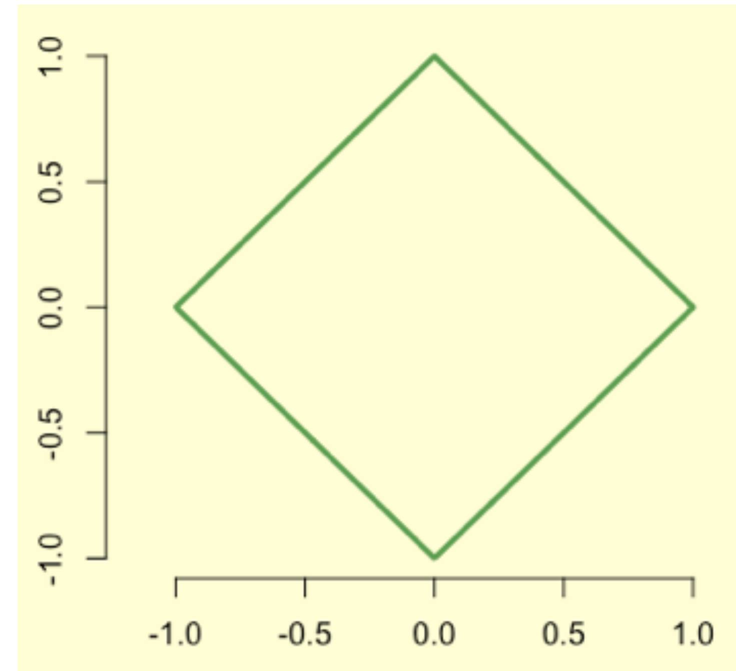
Draw unit “balls” for various norms in two dimensions

Euclidean




```r
t <- 0:629 / 100 # numbers from 0 to 2*pi
x <- cos(t)
y <- sin(t)
par(bg = '#acfffc') # background color
plot(x,y,'l',lwd=3,asp=1,col = "#ff9408",
     xlim=c(-1,1),ylim=c(-1,1),bty='n')
```

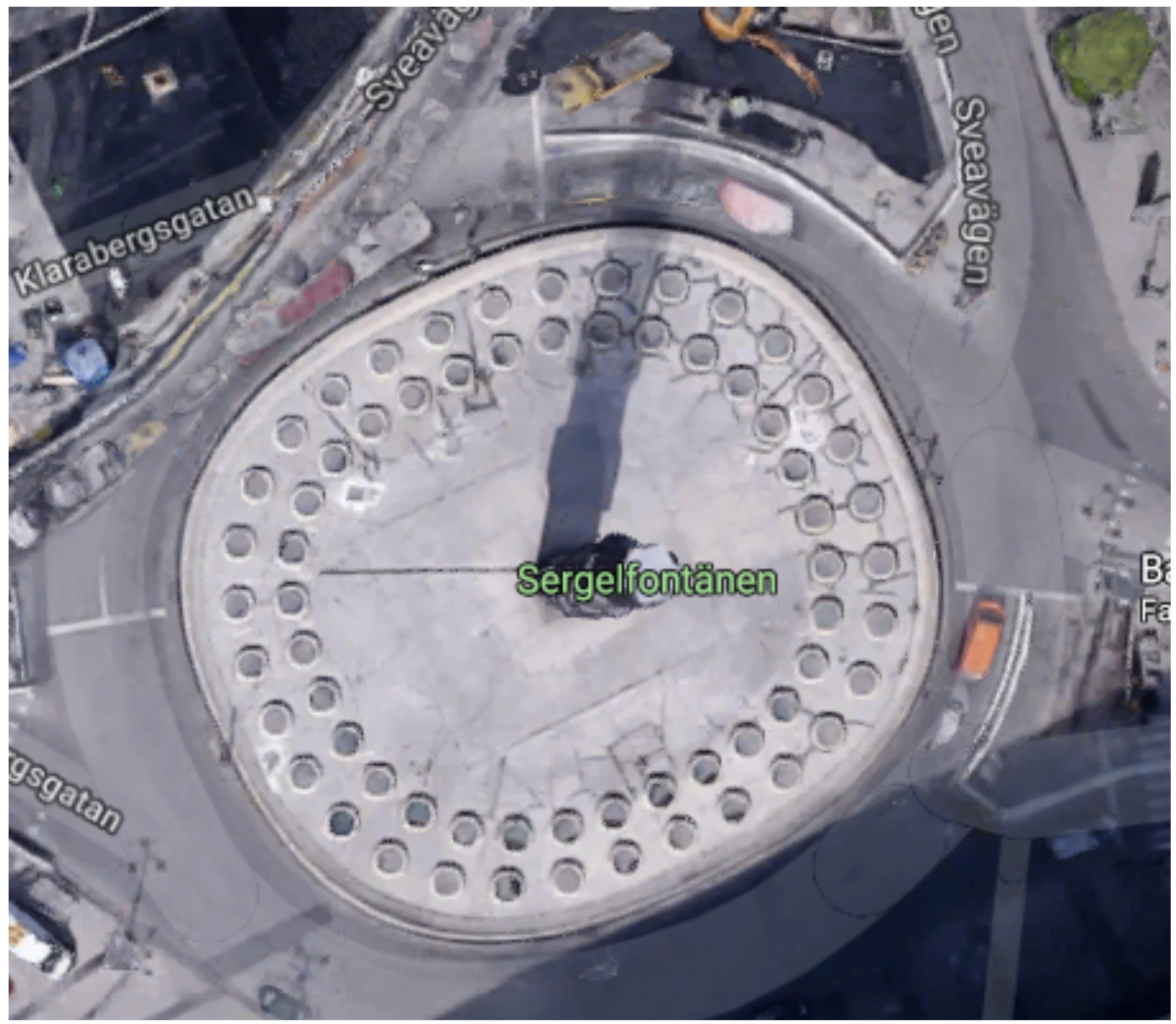
One-norm




```r
t <- 0:629 / 100 # numbers from 0 to 2*pi
x <- cos(t)
y <- sin(t)
n1 <- abs(x) + abs(y) # the one-norm
x <- x / n1
y <- y / n1
par(bg = '#ffffd4') # background color
plot(x,y,'l',lwd=3,asp=1,col = "#5fa052",
     xlim=c(-1,1),ylim=c(-1,1),bty='n')
```

You try the infinity norm and the four-norm

(for nice colors see https://xkcd.com/color/rgb/



Sveavägen
Sveavägen
Klarabergsgatan
Sergelfontänen

Vector Norms

Theorem. On a finite-dimensional vector space, all norms are equivalent.

If $\|\cdot\|_*$ and $\|\cdot\|_\#$ are any two norms on $\mathbb{R}^n$,
then there exist positive constants c and C such that

$$c\|x\|_* \leq \|x\|_\# \leq C\|x\|_*$$

for all $x \in \mathbb{R}^n$.

“Rank” of a matrix

A matrix of all zeros is "rank zero."

If u and v are nonzero, $A = uv^T$ is rank one.

In general, the rank of a matrix A is the smallest k such that we can write:

$$A = \sum_{i=1}^{k} u_i v_i^T$$

where u_i, v_i are any vectors.

Many interesting matrices are “nearly low-rank.”

Low-rank approximation saves a huge amount of storage and can greatly speed up modern computations.

$$\begin{pmatrix} 1 & 1 & 1 & -2 \\ 2 & 2 & 2 & -4 \\ 4 & 4 & 4 & -8 \\ 1 & 1 & 1 & -2 \end{pmatrix} = uv^T$$

What are u, v?

$$\begin{pmatrix} 1 & 3 & 4 & 10 \\ 2 & 6 & 8 & 20 \\ 0 & -3 & -4 & -10 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 3 & 4 & 10 \\ 2 & 6 & 8 & 20 \\ 0 & -3 & -4 & -10 \end{pmatrix} = u_1 v_1^T + u_2 v_2^T$$

What are u_1, u_2, v_1, v_2?

Suppose A is a million-by-million matrix.

-> one trillion entries (!)

If you approximate A by a rank-ten matrix,

$$A \approx \sum_{i=1}^{10} u_i v_i^T$$

Now you only have to store the u_i, v_i vectors.
How much storage is this?