Computational Linear Algebra:

Low Rank Approximations with the Singular Value Decomposition

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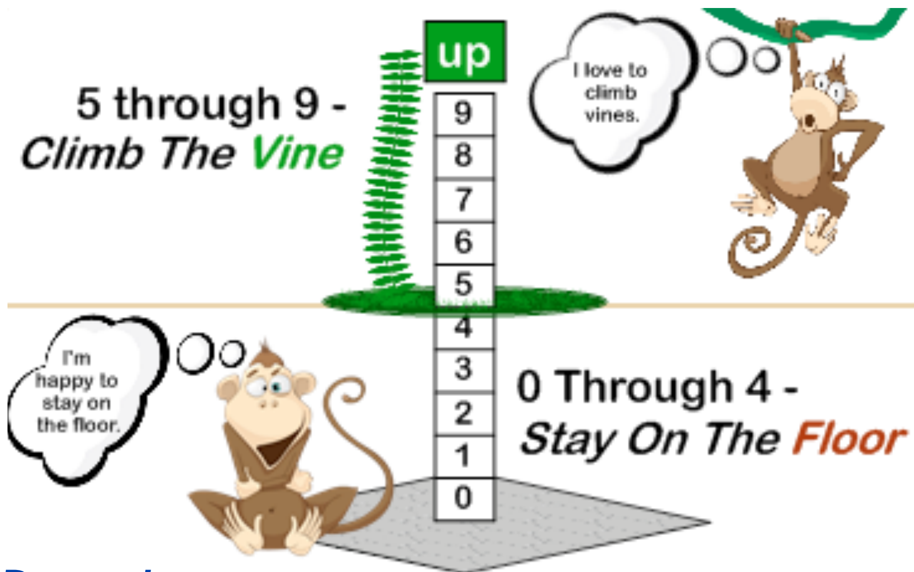
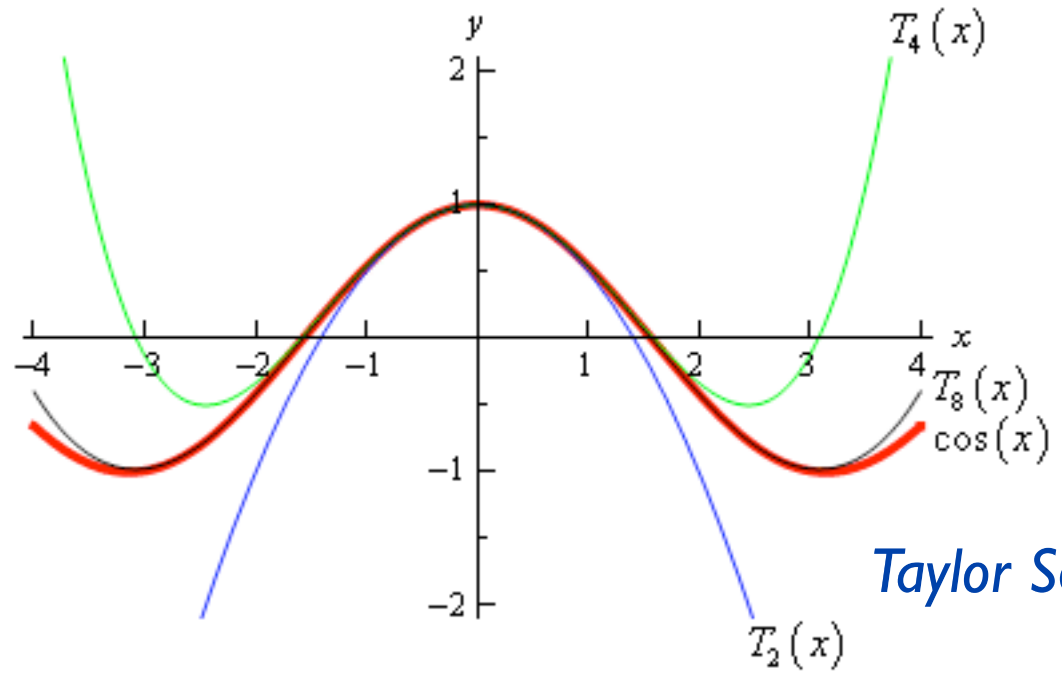
MACALESTER

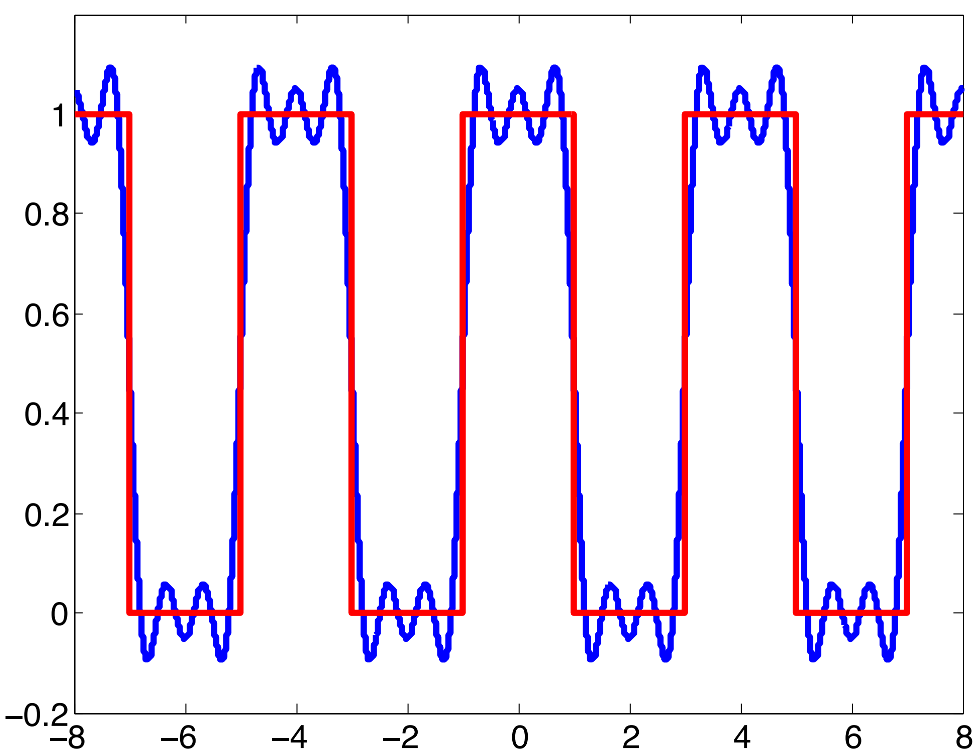
Approximations



Rounding



Taylor Series Approximation



Fourier Analysis

Matrix Approximation

0	1	2	5	0	1	4	0	2	0	1	9	0	15	0	1	0	10	5	36	1	8	0	0	1	0
1	0	0	0	5	0	0	0	1	0	0	1	0	0	1	0	0	2	0	0	2	0	0	0	1	0
12	0	0	0	4	0	0	2	1	0	0	0	0	0	7	0	0	4	0	1	0	0	0	0	0	0
2	0	1	6	14	0	0	3	13	0	0	0	0	0	4	1	0	0	4	4	1	1	4	0	0	0
16	3	8	26	3	5	2	7	6	0	0	4	5	10	5	4	1	22	9	12	2	4	8	0	3	0
3	0	0	1	0	1	0	0	5	0	0	0	0	0	10	0	0	3	0	3	1	0	0	0	0	0
5	1	0	0	5	0	1	4	0	0	0	1	0	0	3	1	0	6	0	0	0	0	1	0	0	0
24	0	0	0	32	2	0	0	7	0	0	1	0	0	8	0	0	0	0	5	1	0	0	0	0	0
0	1	8	1	3	0	2	0	0	0	0	2	0	16	9	0	0	2	9	8	0	7	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0
3	0	0	4	6	0	0	1	6	0	0	8	2	3	3	0	0	1	0	1	0	1	1	0	2	0
2	1	0	0	7	0	0	0	1	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0
10	0	5	9	4	1	9	0	2	0	0	3	2	4	12	0	0	0	4	8	1	1	0	0	2	0
1	3	1	3	0	6	1	1	0	0	0	1	4	20	2	5	0	17	3	13	7	2	3	0	0	0
0	0	0	0	5	0	0	0	0	0	0	4	0	0	4	0	0	2	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
8	0	0	1	26	4	3	0	1	0	1	3	0	1	6	2	0	0	5	12	3	0	3	0	0	0
4	2	2	0	10	2	1	6	1	0	1	0	0	1	4	0	0	1	0	8	1	0	0	0	0	0
4	1	4	1	11	5	1	47	18	0	0	3	0	2	11	1	0	2	0	9	0	0	5	0	1	0
1	0	0	0	0	0	3	0	0	0	0	2	0	3	0	0	0	5	5	2	0	0	0	0	0	0
2	0	0	0	17	0	0	0	3	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0
2	1	0	0	11	0	0	8	1	0	0	0	0	1	2	0	0	0	0	1	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	1	1	0	1	1	0	0	0	0	0	1	0	0	0	1	0	1	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Q1. Rank of an outer product

Let u, v be non-zero vectors in $\mathbb{R}^3$.
What is the rank of the 3×3 matrix $A = uv^\top$?

(a) 0

(b) 1

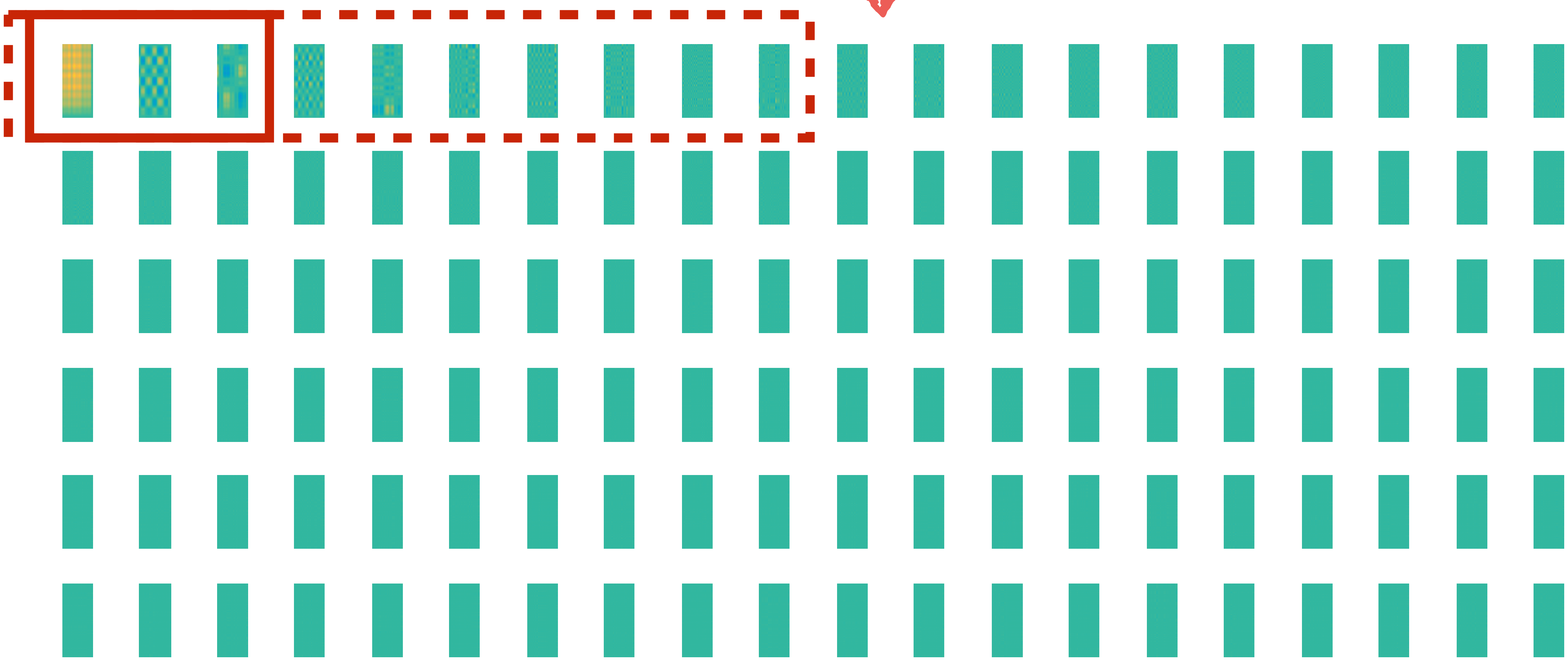
(c) 2

(d) 3

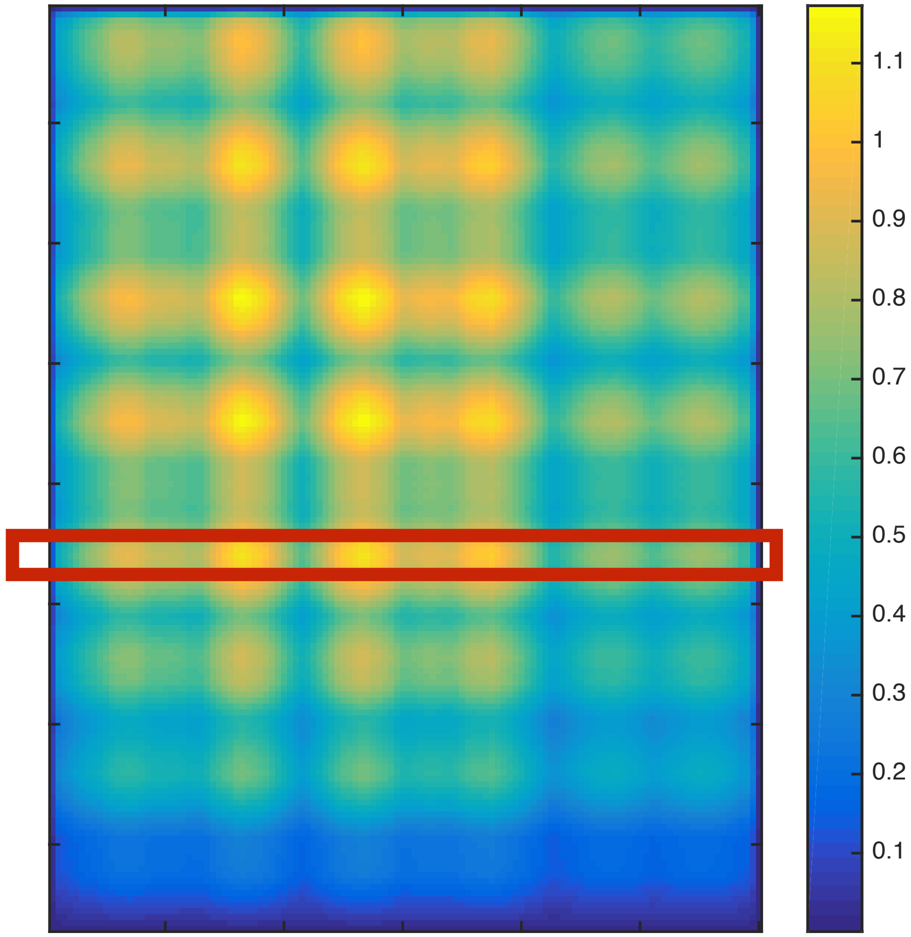
(e) it depends on your choice of u and v

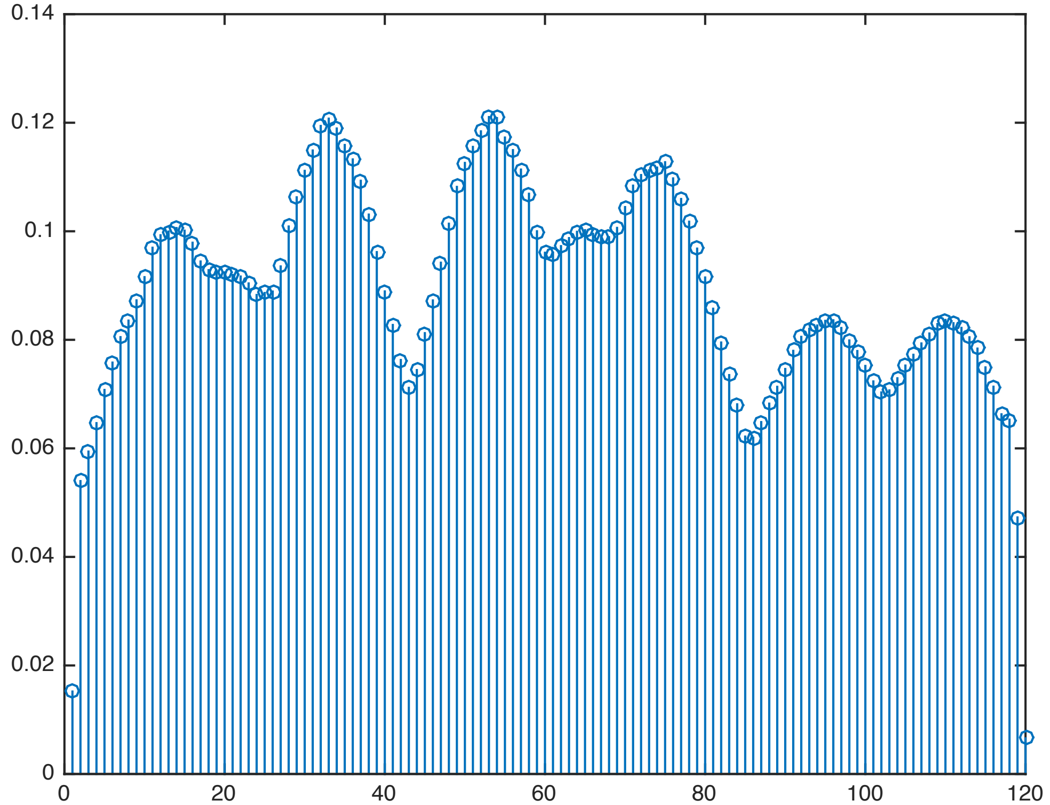
Singular value decomposition

Write as the sum of 120 rank 1 matrices

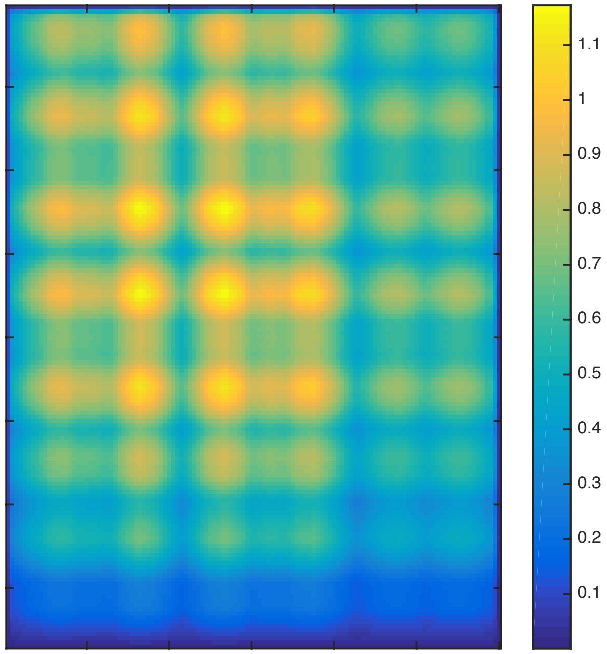


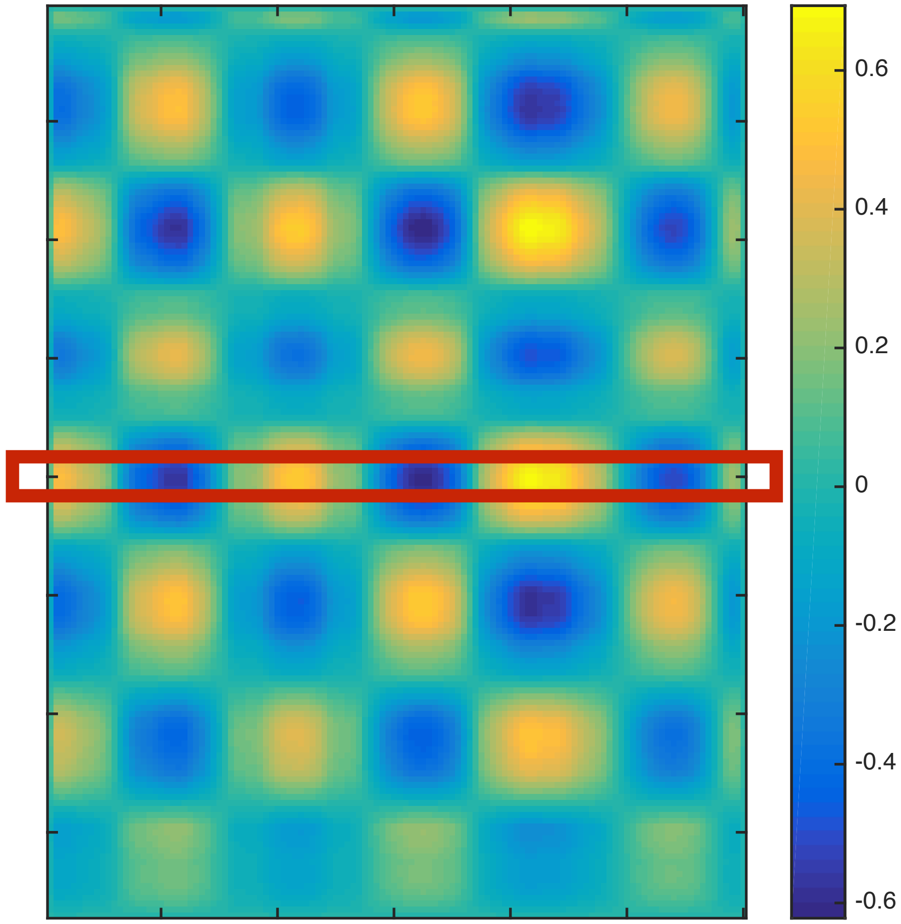
Best rank 1 approximation: $\sigma_1 u_1 v_1^T$

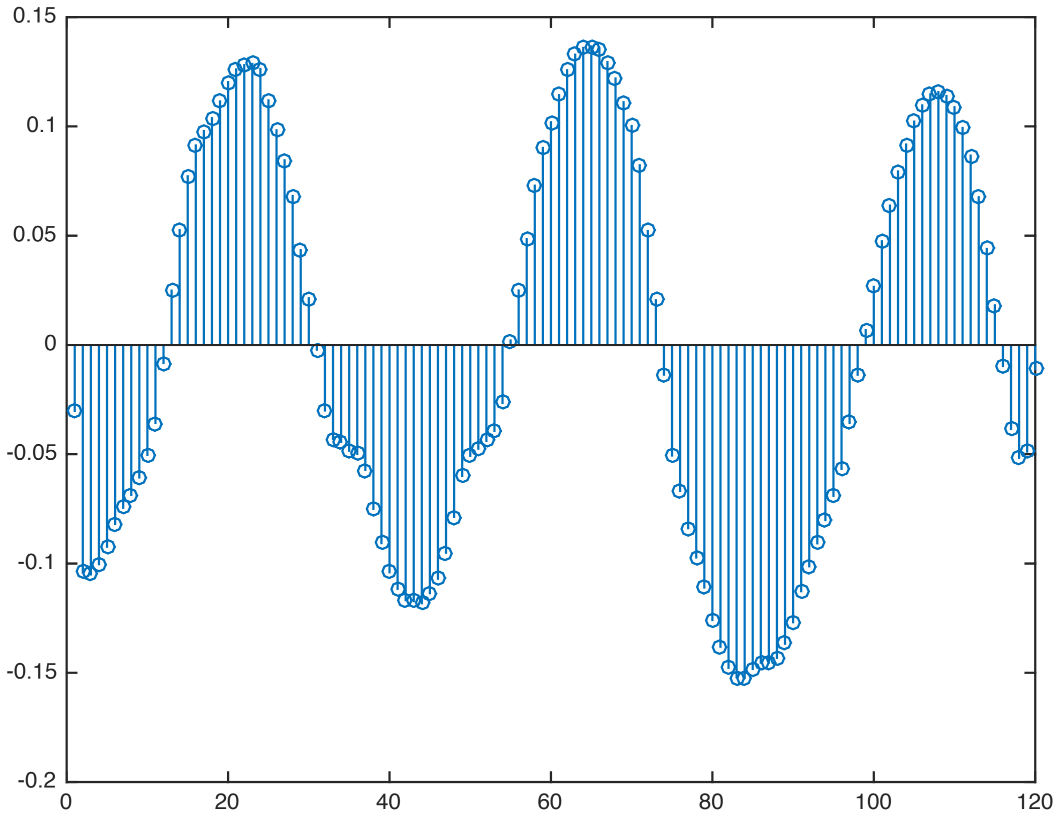




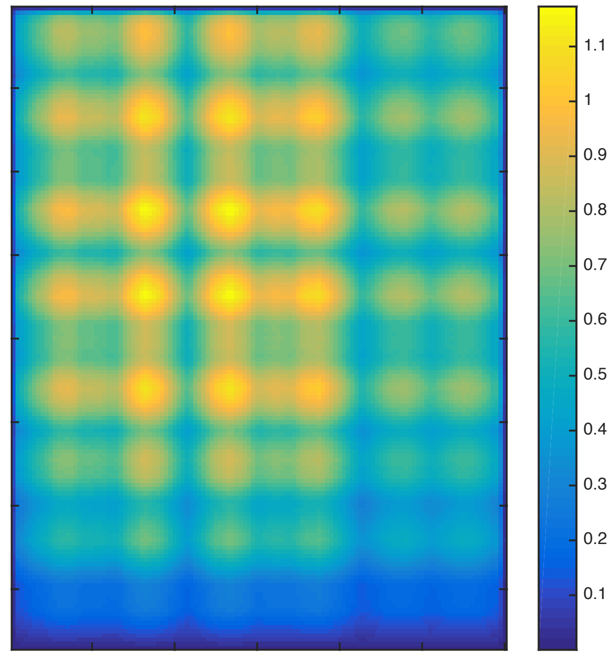
Second rank 1 component: $\sigma_2 u_2 v_2^T$

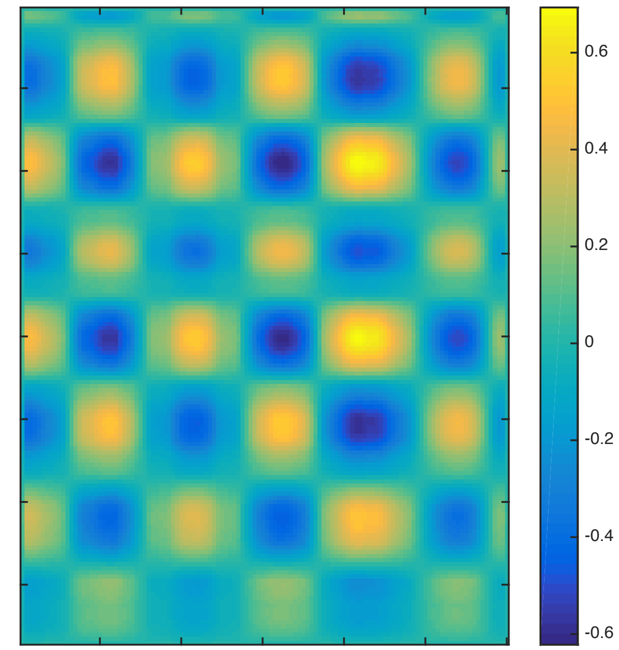


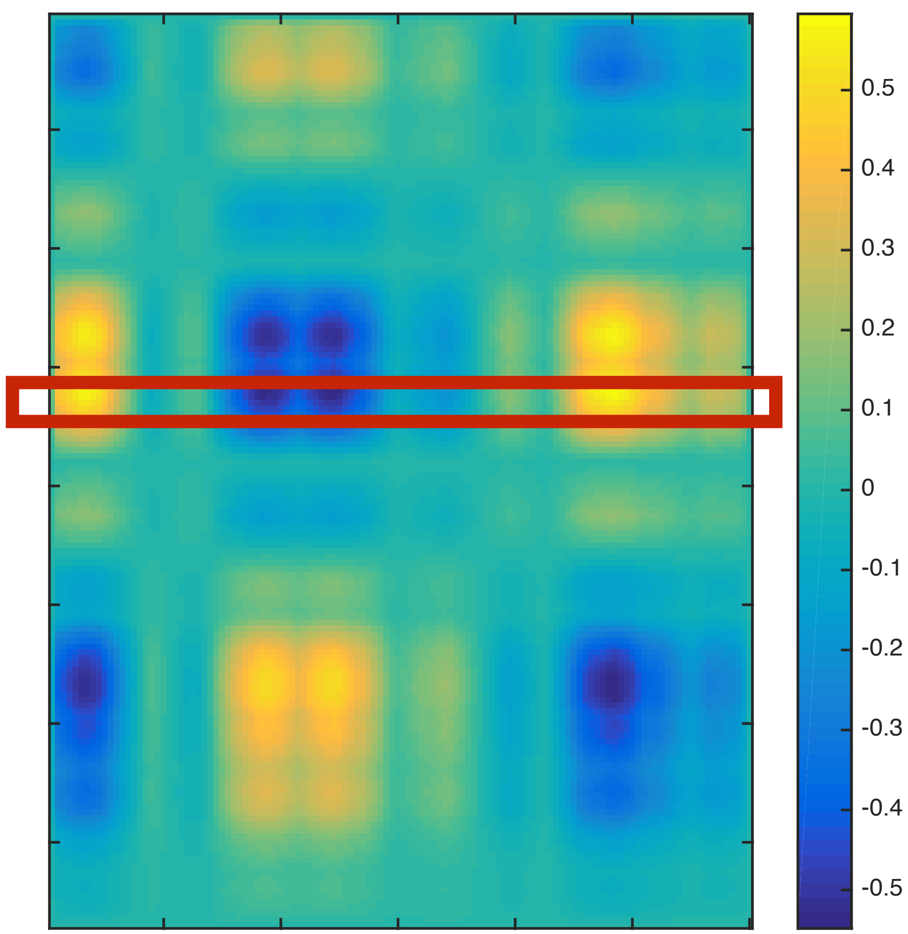


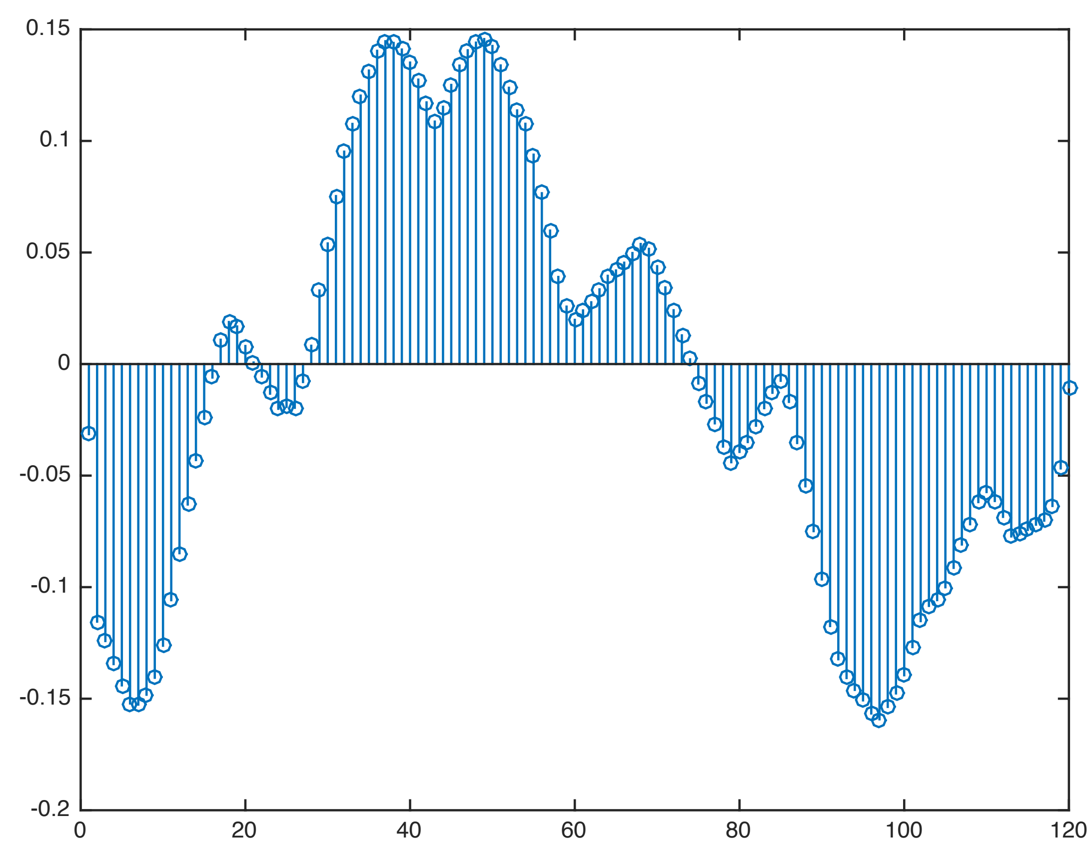


Third rank 1 component: $\sigma_3 u_3 v_3^T$



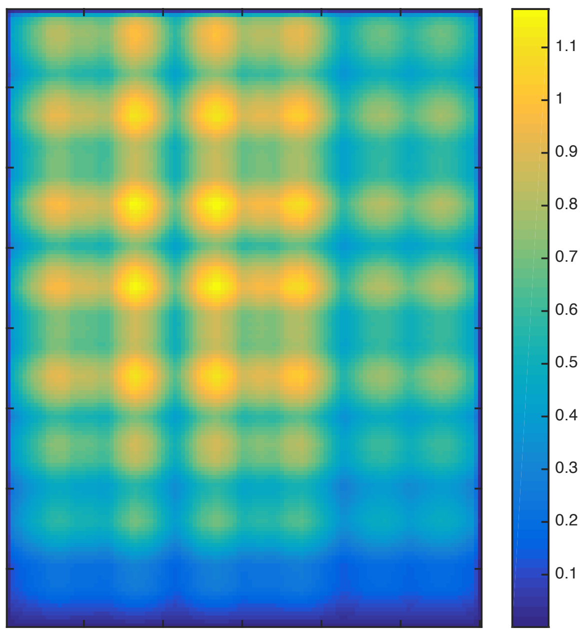






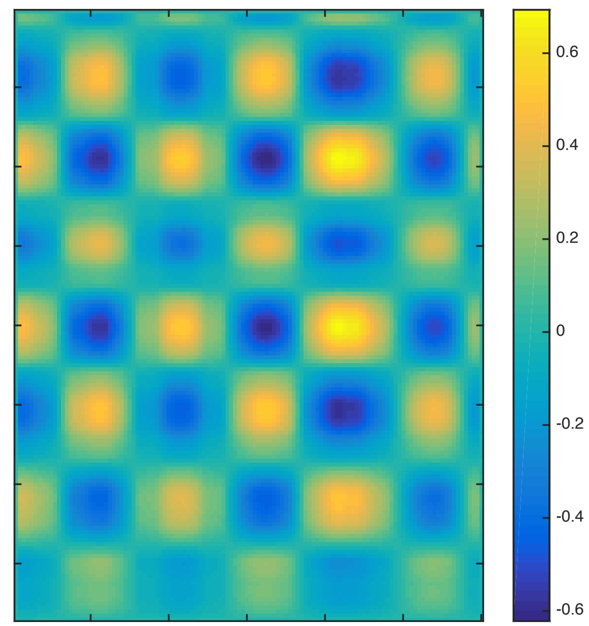
Best rank 3 approximation

$\sigma_1 u_1 v_1^T$



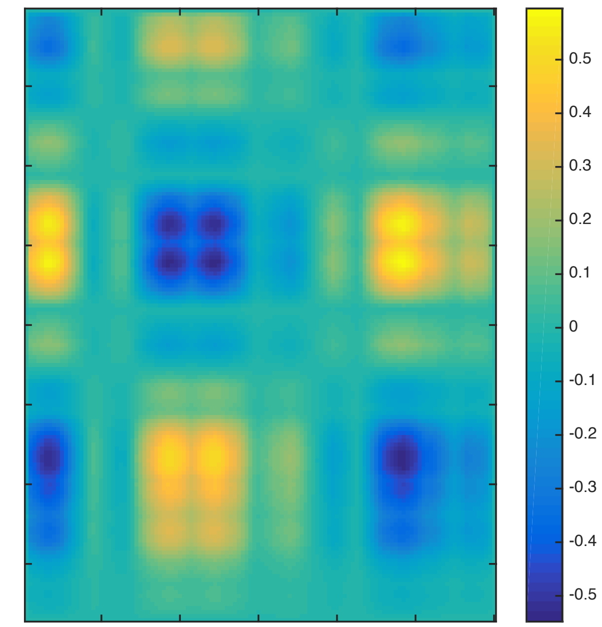
+

$\sigma_2 u_2 v_2^T$



+

$\sigma_3 u_3 v_3^T$



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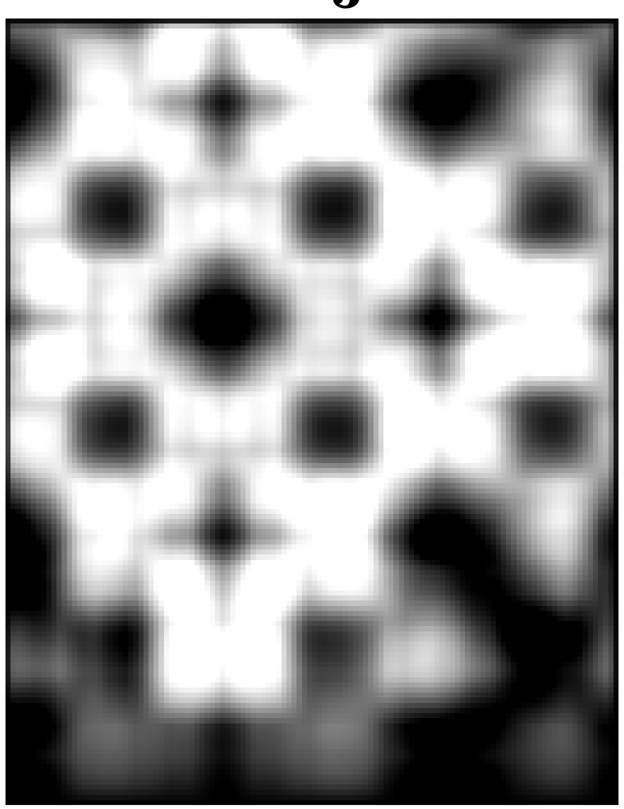
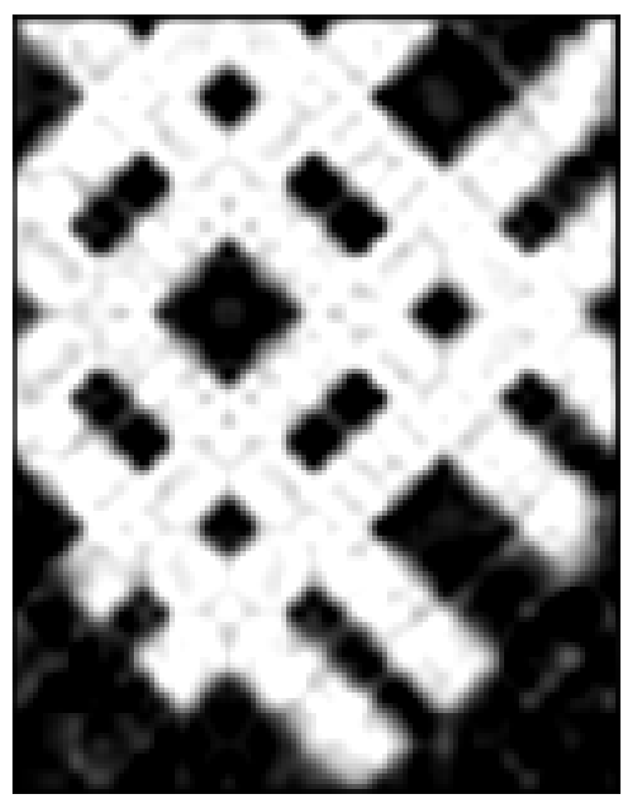
A_3



A_{10}



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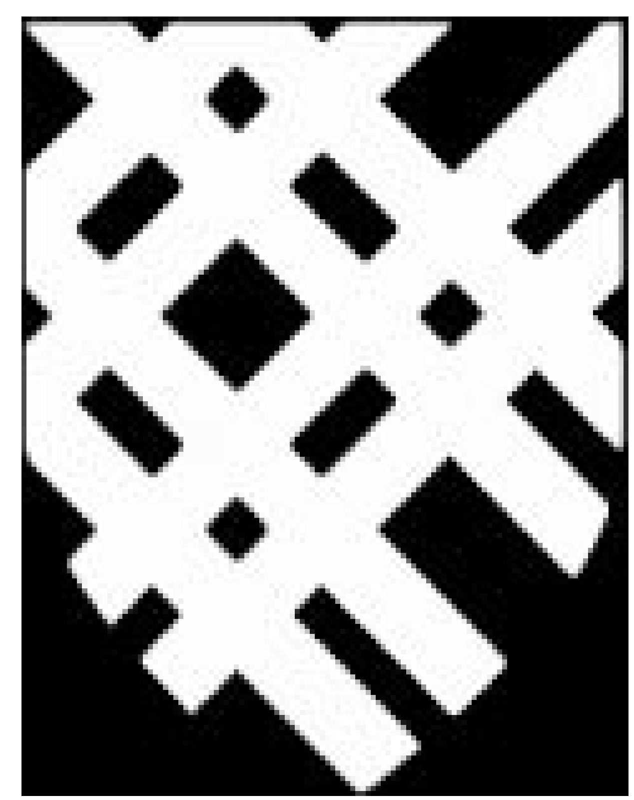

Further reading

- D. Kalaman, “A singularly valuable decomposition: The SVD of a matrix,” *College Math. Journal*, vol. 27, no. 1, Jan. 1996, pp. 2-23.

- G. Strang, “The fundamental theorem of linear algebra,” *Amer. Math. Monthly*, vol. 100, no. 9, Nov. 1993, pp. 848-855.