Computational Linear Algebra Class 2: Iterative Methods for Root-Finding (part 1)

David Shuman

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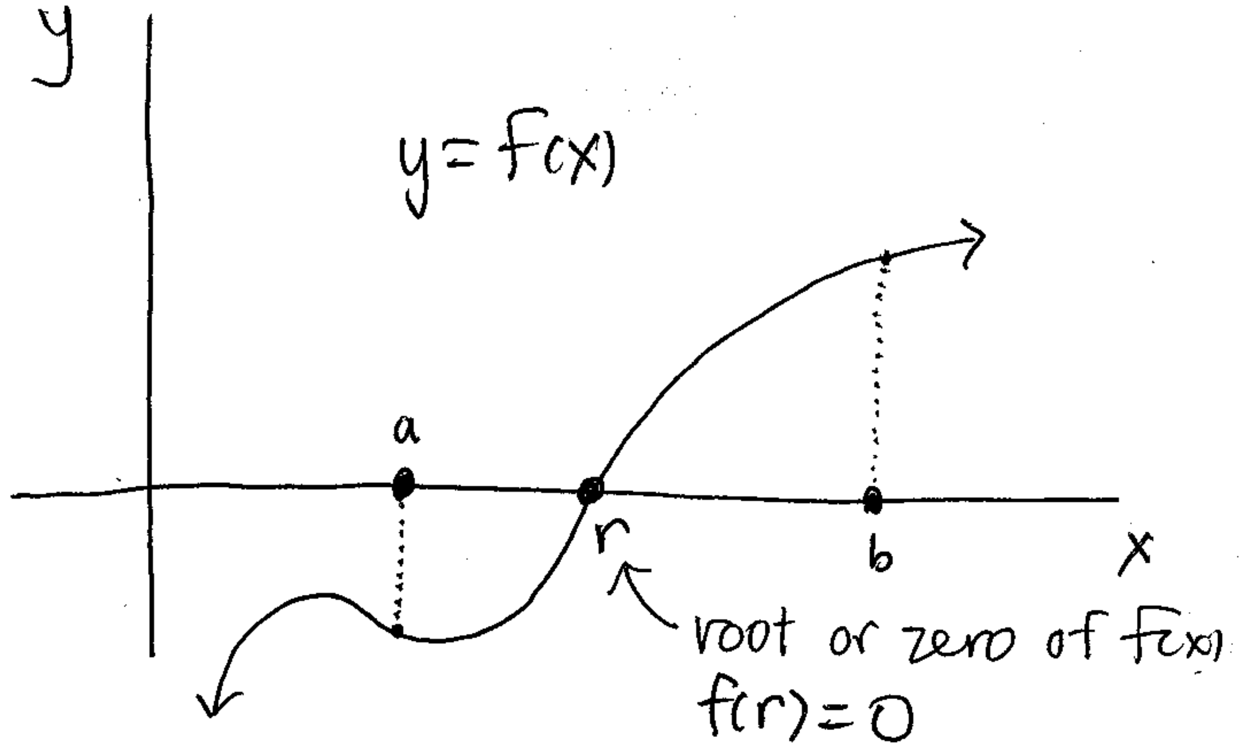
MACALESTER

The one-dimensional root-finding problem

- Find x s.t. $f(x)=0$

- Intermediate Value Theorem (IVT): If $f(x)$ is continuous on an interval $[a,b]$ and $f(a)f(b)<0$, then there exists $c \in [a,b]$ s.t. $f(c)=0$



- Use *iterative methods*: generate a sequence $x^{(1)}$, $x^{(2)}$, $x^{(3)}$,... of approximations that hopefully converges to r

Importance

- Before presenting half a dozen methods to solve this problem, why is it important?
- Some applications
  - In optimization, we can find local minima and maxima by finding the points where the derivative is equal to 0
  - Used all the time in signals and systems to understand and design the behavior of systems
    - e.g., you can filter an audio signal in order to remove certain undesirable frequencies from an audio clip. To understand which frequencies are eliminated, you need to find roots
    - similar idea when designing automatic control policies to stabilize systems (thermostats, flying, etc.)
  - Cryptography and coding theory
  - Fixed points ($f(x)=x$) in economics (e.g., Nash equilibria in game theory)

Rubric for evaluating solution method

- Does sequence of approximations converge to a root?
- How fast does it converge?
- How stable is the solution process to tiny numerical errors?

The ideal method may change from instance to instance:

- There may be a tradeoff between speed of convergence and robustness
- Different problem variants and information available about the problem
  - Reliable bracket *[a,b]* available?
  - *A priori* knowledge about properties of the function? (e.g., continuously differentiable, polynomial, a single zero)
  - Interested in a single root or multiple roots?

Bisection Method: Pseudocode

Bisection Method

Given initial interval $[a, b]$ such that $f(a)f(b) < 0$

while $(b - a)/2 > \text{TOL}$

    $c = (a + b)/2$

    if $f(c) = 0$, **stop**, **end**

    if $f(a)f(c) < 0$

        $b = c$

    else

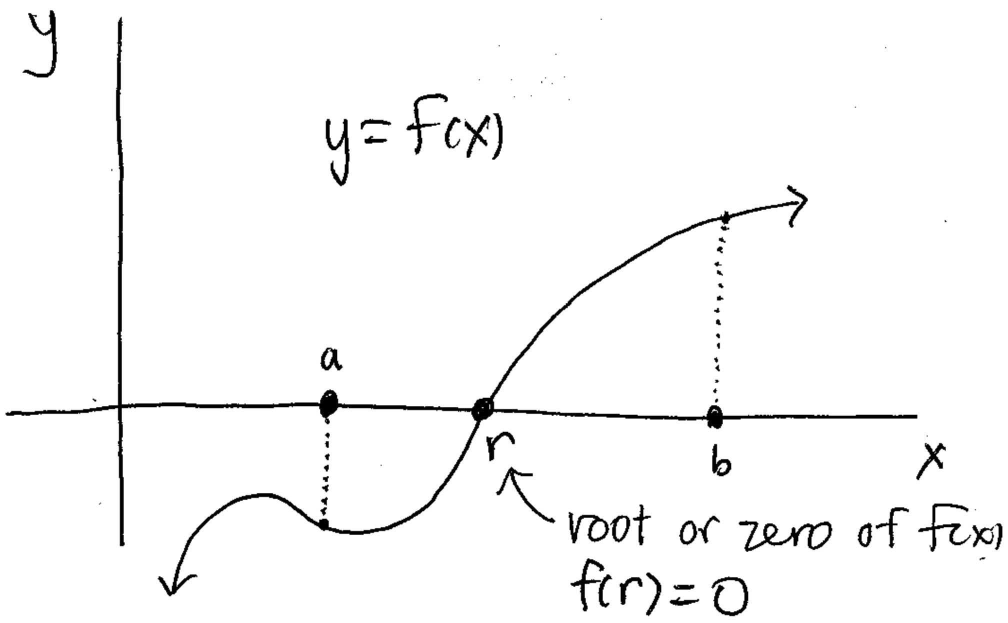
        $a = c$

    end

end

The final interval $[a, b]$ contains a root.

The approximate root is $(a + b)/2$.



Bisection Method: Rate of convergence

- After i steps of the algorithm:
  - $i+2$ function evaluations
  - solution error $= e_i = |x^{(i)}\text{-}r| <= (b\text{-}a)/2^{(i+1)}$
- Sequence of approximations converges with rate q if

$$\lim_{i\to\infty} \frac{||e_{i+1}||}{||e_i||^q} = C \quad \text{(i.e., } ||e_{i+1}|| \approx C||e_i||^q\text{)}$$

q	Convergence rate
q=1	linear
q>1	superlinear
q=2	quadratic

Bisection Method: Summary

- Pros:
  - Always converges if there is a root in the given interval
  - Error approximately cuts in half each step (one additional bit of accuracy)
- Cons:
  - Only uses signs of the function values, not magnitude of function values or other properties of the function
  - Finds only one root in the interval, not multiple
  - Slow convergence