Computational Linear Algebra: Eigenvalues and Eigenvectors: Applications

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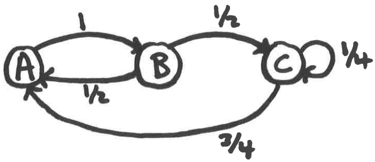
Markov Chains

Mathematical model for stochastic (random) processes

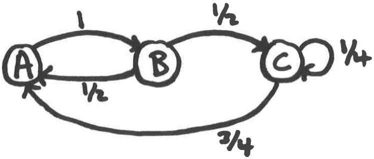


- We'll assume that our process is discrete in time and has a finite number of states, but both of those can be relaxed
- Key property: probability distribution of next state only depends on the current state, not the full history of the process
- Transition matrix P captures these probabilities:

$$P_{ij} = \Pr(X_{t+1} = j \mid X_t = i)$$

  - Rows sum to 1
- Applications throughout natural sciences, computer/information sciences, statistics, social sciences, music composition, and **many** other disciplines

Markov Chain Properties



- A Markov chain is irreducible if you can get from any state to any other state (connected)
- A state is periodic if the greatest common divisor of the number of steps to return to the state is greater than 1
- A Markov chain is aperiodic if no state is periodic
  - If the chain is irreducible, then one state being aperiodic implies all states are aperiodic

The Perron-Frobenius Theorem

Definition

A matrix is nonnegative if all of its entries are greater than or equal to 0.

Definition

A nonnegative $n \times n$ matrix M is primitive if there is a positive integer k so that M^k has all *positive* entries

$\Leftrightarrow$ aperiodic and irreducible communication graph

Definition

A nonnegative $n \times n$ matrix M is stochastic if its columns each sum to 1.

Theorem

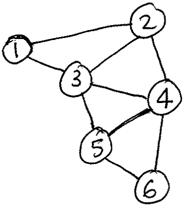
(Perron-Frobenius) *If M is a nonnegative $n \times n$ primitive matrix, then M has a dominant eigenvalue λ_1, and there is a dominant eigenvector v_1 with all positive entries.*

If, in addition, M is stochastic, then $\lambda_1 = 1$, so the dominant eigenvector is a fixed point or stationary vector satisfying $M\mathrm{v}_1 = \mathrm{v}_1$.

Eigenvector Application Example: Graphs and Networks

A transportation network is an (undirected) graph of geographically important entities (like cities) connected by transportation routes. For example, here is a simple transportation network for Minnesota.

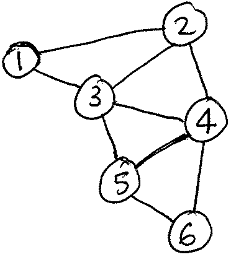
1. *Moorhead*
2. *Duluth*
3. *St. Cloud*
4. *Mpls/St. Paul*
5. *Mankato*
6. *Rochester*



The adjacency matrix of this network is the square matrix A whose i, j-entry is given by

$$a_{ij} = \begin{cases} 1, & \text{if vertices } i \text{ and } j \text{ are connected by an edge} \\ 0, & \text{if vertices } i \text{ and } j \text{ are not connected by an edge} \end{cases}$$

Adjacency Matrix



The augmented adjacency matrix is the matrix $B = A + I$. In the example above,

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

The augmented adjacency matrix represents the ability to travel from one node to another as well as the ability to stay at a node