MATH/COMP 365: Ax=b (Part I: Gaussian Elimination and Complexity)

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Linear Algebra

- Linear algebra takes off from the question: Solve the following system of linear equations (given here in 3 dimensions but generally in n dimensions):

$$\begin{array}{rcrcrcr} 2x_1 & + & 4x_2 & - & 2x_3 & = & 8 \\ x_1 & + & 4x_2 & - & 3x_3 & = & 8 \\ -2x_1 & - & 6x_2 & + & 7x_3 & = & -3 \end{array}$$

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- 3 ways to think about this:
  - Vector equation: $x_1 \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 4 \\ -6 \end{pmatrix} + x_3 \begin{pmatrix} -2 \\ -3 \\ 7 \end{pmatrix} = \begin{pmatrix} 8 \\ 8 \\ -3 \end{pmatrix}$

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    Can we get to the RHS as a linear combination of the column vectors on the LHS?
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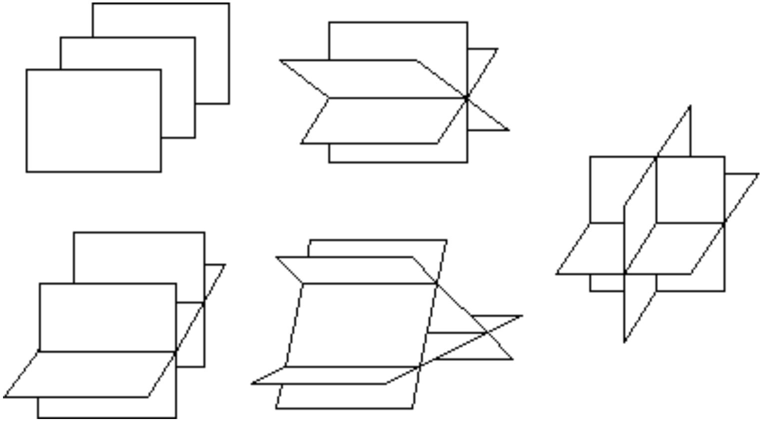
  - Hyperplanes intersecting in $\mathbb{R}^n$
    3 possibilities: 0, 1, ∞

Three Hyperplanes Intersecting



Invertibility

- (identity) The matrix I_n is the $n \times n$ matrix with 1's on the diagonal

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

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- To solve $Ax = b$:

$$\begin{aligned} Ax &= b \\ A^{-1}Ax &= A^{-1}b \\ x &= A^{-1}b \end{aligned}$$

- This is computationally impractical, however

Operation Counts for Naive Gaussian Elimination

Problem: $Ax = b$, where A is an $N \times N$ matrix with no particular structure, and b is an $N \times 1$ vector

- Step I: Elimination: Convert $Ax = b$ to $Ux = c$, where U is upper triangular
  - Number of multiplications/divisions: $\frac{N^3}{3} + \frac{N^2}{2} - \frac{5N}{6}$
  - Number of additions/subtractions: $\frac{N^3}{3} - \frac{N}{3}$
- Step II: Back substitution: Solve $Ux = c$
  - Number of multiplications/divisions: $\frac{N^2}{2} + \frac{N}{2}$
  - Number of additions/subtractions: $\frac{N^2}{2} - \frac{N}{2}$

Uses of Complexity Computations

- Predict computation times
- Compare algorithms
- Identify bottlenecks

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Main considerations:

- Computational complexity (speed)
- Memory
- Convergence
- Conditioning (susceptibility to error magnification) and stability

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- Tolerance: how precisely do you need to solve for x?

Big Linear Algebra Theorem

Theorem. If A is an $n \times n$ matrix, then *the following statements are equivalent*:

- A is nonsingular (we are defining nonsingular here)
- A is invertible, i.e., A^{-1} exists
- $\det(A) \neq 0$
- $Ax = b$ has a unique solution for all $b \in \mathbb{R}^n$
- $Az \neq 0$ for all $z \in \mathbb{R}^n$ except $z = 0$ (and of course $A0 = 0$)
- the columns of A are linearly independent
- the columns of A span $\mathbb{R}^n$ (i.e., $\text{rank}(A) = n$)
- 0 is not an eigenvalue of A
- $\text{rref}(A) = I_n$
- $\text{NullSpace}(A) = \{0\}$
- $\text{Image}(A) = \mathbb{R}^n$