

# Computational Linear Algebra

## Orthogonal Polynomials

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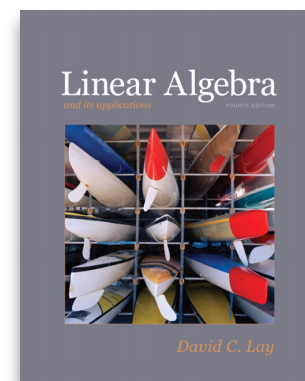
# Vector Space

## DEFINITION

A **vector space** is a nonempty set  $V$  of objects, called *vectors*, on which are defined two operations, called *addition* and *multiplication by scalars* (real numbers), subject to the ten axioms (or rules) listed below.<sup>1</sup> The axioms must hold for all vectors  $\mathbf{u}$ ,  $\mathbf{v}$ , and  $\mathbf{w}$  in  $V$  and for all scalars  $c$  and  $d$ .

1. The sum of  $\mathbf{u}$  and  $\mathbf{v}$ , denoted by  $\mathbf{u} + \mathbf{v}$ , is in  $V$ .
2.  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ .
3.  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ .
4. There is a **zero** vector  $\mathbf{0}$  in  $V$  such that  $\mathbf{u} + \mathbf{0} = \mathbf{u}$ .
5. For each  $\mathbf{u}$  in  $V$ , there is a vector  $-\mathbf{u}$  in  $V$  such that  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ .
6. The scalar multiple of  $\mathbf{u}$  by  $c$ , denoted by  $c\mathbf{u}$ , is in  $V$ .
7.  $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ .
8.  $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ .
9.  $c(d\mathbf{u}) = (cd)\mathbf{u}$ .
10.  $1\mathbf{u} = \mathbf{u}$ .

<sup>1</sup>Technically,  $V$  is a *real vector space*. All of the theory in this chapter also holds for a *complex vector space* in which the scalars are complex numbers. We will look at this briefly in Chapter 5. Until then, all scalars are assumed to be real.



# Inner Product Space

## DEFINITION

An **inner product** on a vector space  $V$  is a function that, to each pair of vectors  $\mathbf{u}$  and  $\mathbf{v}$  in  $V$ , associates a real number  $\langle \mathbf{u}, \mathbf{v} \rangle$  and satisfies the following axioms, for all  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  in  $V$  and all scalars  $c$ :

1.  $\langle \mathbf{u}, \mathbf{v} \rangle = \langle \mathbf{v}, \mathbf{u} \rangle$
2.  $\langle \mathbf{u} + \mathbf{v}, \mathbf{w} \rangle = \langle \mathbf{u}, \mathbf{w} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle$
3.  $\langle c\mathbf{u}, \mathbf{v} \rangle = c \langle \mathbf{u}, \mathbf{v} \rangle$
4.  $\langle \mathbf{u}, \mathbf{u} \rangle \geq 0$  and  $\langle \mathbf{u}, \mathbf{u} \rangle = 0$  if and only if  $\mathbf{u} = \mathbf{0}$

A vector space with an inner product is called an **inner product space**.

