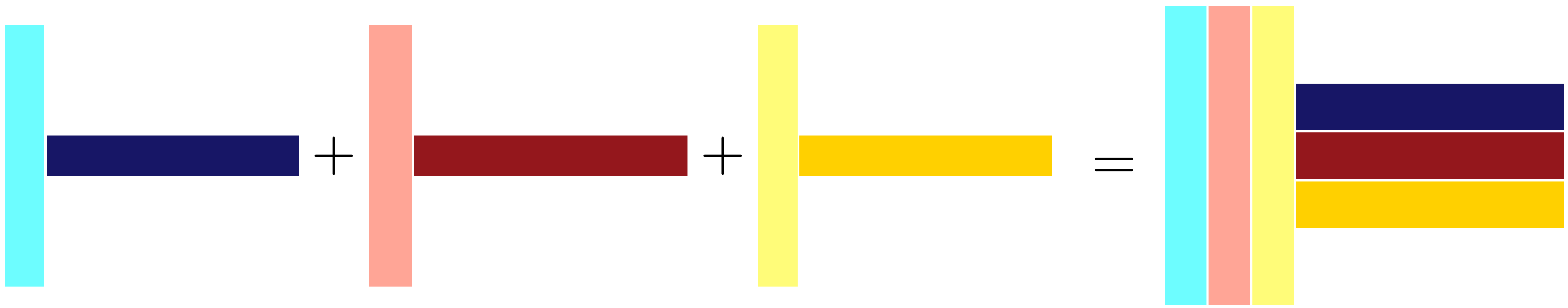
Matrix Factorization: $A=LU$

Matrix Factorization: $A=U^TU$

Wonderful fact: a sum of rank-one matrices has an easy factorization



$$P_1 Q_1^T \quad + \quad P_2 Q_2^T \quad + \quad P_3 Q_3^T \quad = \quad P Q^T$$

```
> p1 <- runif(8)
> p2 <- runif(8)
> p3 <- runif(8)
> q1 <- runif(8)
> q2 <- runif(8)
> q3 <- runif(8)
> P <- cbind(p1,p2,p3)
> Q <- cbind(q1,q2,q3)
> M1 <- P %*% t(Q)
> M2 <- p1 %*% t(q1) + p2 %*% t(q2) + p3 %*% t(q3)
> norm(M1-M2)
[1] 0
```

First idea: **greedy low-rank approximation**

$$A = \begin{pmatrix} 10 & 60 & -40 & 60 \\ 5 & 10 & -10 & 100 \\ 2 & 22 & 12 & -38 \\ -2 & -14 & 19 & 22 \end{pmatrix}$$

1) Find largest entry (in absolute value). This is the **pivot.**
2) Call the pivot row v_1^T and call the pivot column w_1.
3) Divide v_1^T by the pivot: $v_1^T \leftarrow v_1^T/100$

$$A \approx w_1 v_1^T = \begin{pmatrix} 60 \\ 100 \\ -38 \\ 22 \end{pmatrix} (.05 \quad 0.1 \quad -0.1 \quad 1) = \begin{pmatrix} 3 & 6 & -6 & 60 \\ 5 & 10 & -10 & 100 \\ -1.9 & -3.8 & 3.8 & -38 \\ 1.1 & 2.2 & -2.2 & 22 \end{pmatrix}$$

$$\begin{pmatrix} 10 & 60 & -40 & 60 \\ 5 & 10 & -10 & 100 \\ 2 & 22 & 12 & -38 \\ -2 & -14 & 19 & 22 \end{pmatrix} = \begin{pmatrix} 3 & 6 & -6 & 60 \\ 5 & 10 & -10 & 100 \\ -1.9 & -3.8 & 3.8 & -38 \\ 1.1 & 2.2 & -2.2 & 22 \end{pmatrix} + \begin{pmatrix} 7 & 54 & -36 & 0 \\ 0 & 0 & 0 & 0 \\ 3.9 & 25.8 & 8.2 & 0 \\ -3.1 & -16.2 & 21.2 & 0 \end{pmatrix}$$

$$A \quad = \quad \text{rank-one approximation} \quad + \quad \text{remainder}$$

First idea: **greedy low-rank approximation**

$$\begin{pmatrix} 10 & 60 & -40 & 60 \\ 5 & 10 & -10 & 100 \\ 2 & 22 & 12 & -38 \\ -2 & -14 & 19 & 22 \end{pmatrix} = \begin{pmatrix} 3 & 6 & -6 & 60 \\ 5 & 10 & -10 & 100 \\ -1.9 & -3.8 & 3.8 & -38 \\ 1.1 & 2.2 & -2.2 & 22 \end{pmatrix} + \begin{pmatrix} 7 & 54 & -36 & 0 \\ 0 & 0 & 0 & 0 \\ 3.9 & 25.8 & 8.2 & 0 \\ -3.1 & -16.2 & 21.2 & 0 \end{pmatrix}$$

4) Within remainder matrix, find largest entry (in absolute value).
5) Call the pivot row v_2 and call the pivot column w_2 .
6) Divide v_2 by the pivot: $v_2 \leftarrow v_2/54$

$$A \approx \begin{pmatrix} 60 & 54 \\ 100 & 0 \\ -38 & 25.8 \\ 22 & -16.2 \end{pmatrix} \begin{pmatrix} .05 & 0.1 & -0.1 & 1 \\ \frac{7}{54} & 1 & \frac{-2}{3} & 0 \end{pmatrix} = \begin{pmatrix} 10 & 60 & -42 & 60 \\ 5 & 10 & -10 & 100 \\ \frac{13}{9} & 22 & -13.4 & -38 \\ -1 & -14 & 8.6 & 22 \end{pmatrix}$$

$$A \approx w_1 v_1^T + w_2 v_2^T$$

$$\begin{pmatrix} 10 & 60 & -40 & 60 \\ 5 & 10 & -10 & 100 \\ 2 & 22 & 12 & -38 \\ -2 & -14 & 19 & 22 \end{pmatrix} = \begin{pmatrix} 10 & 60 & -42 & 60 \\ 5 & 10 & -10 & 100 \\ \frac{13}{9} & 22 & -13.4 & -38 \\ -1 & -14 & 8.6 & 22 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{5}{9} & 0 & 25.4 & 0 \\ -1 & 0 & 10.4 & 0 \end{pmatrix}$$

$$A \quad = \quad \text{rank-two approximation} \quad + \quad \text{remainder}$$

First idea: **greedy low-rank approximation**

$$\begin{pmatrix} 10 & 60 & -40 & 60 \\ 5 & 10 & -10 & 100 \\ 2 & 22 & 12 & -38 \\ -2 & -14 & 19 & 22 \end{pmatrix} = \begin{pmatrix} 60 & 54 \\ 100 & 0 \\ -38 & 25.8 \\ 22 & -16.2 \end{pmatrix} \begin{pmatrix} .05 & 0.1 & -0.1 & 1 \\ \frac{7}{54} & 1 & \frac{-2}{3} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{5}{9} & 0 & 25.4 & 0 \\ -1 & 0 & 10.4 & 0 \end{pmatrix}$$

$$A \quad = \quad \text{rank-two approximation} \quad + \quad \text{remainder}$$

We could continue to get a rank-three approximation.
Let's quit here though.
The main problem with the greedy idea:
-> finding the max takes a lot of searching (*e.g.* if A has 10000 cols)

Application: image compression

jennajereb99 • Follow
Macalester College

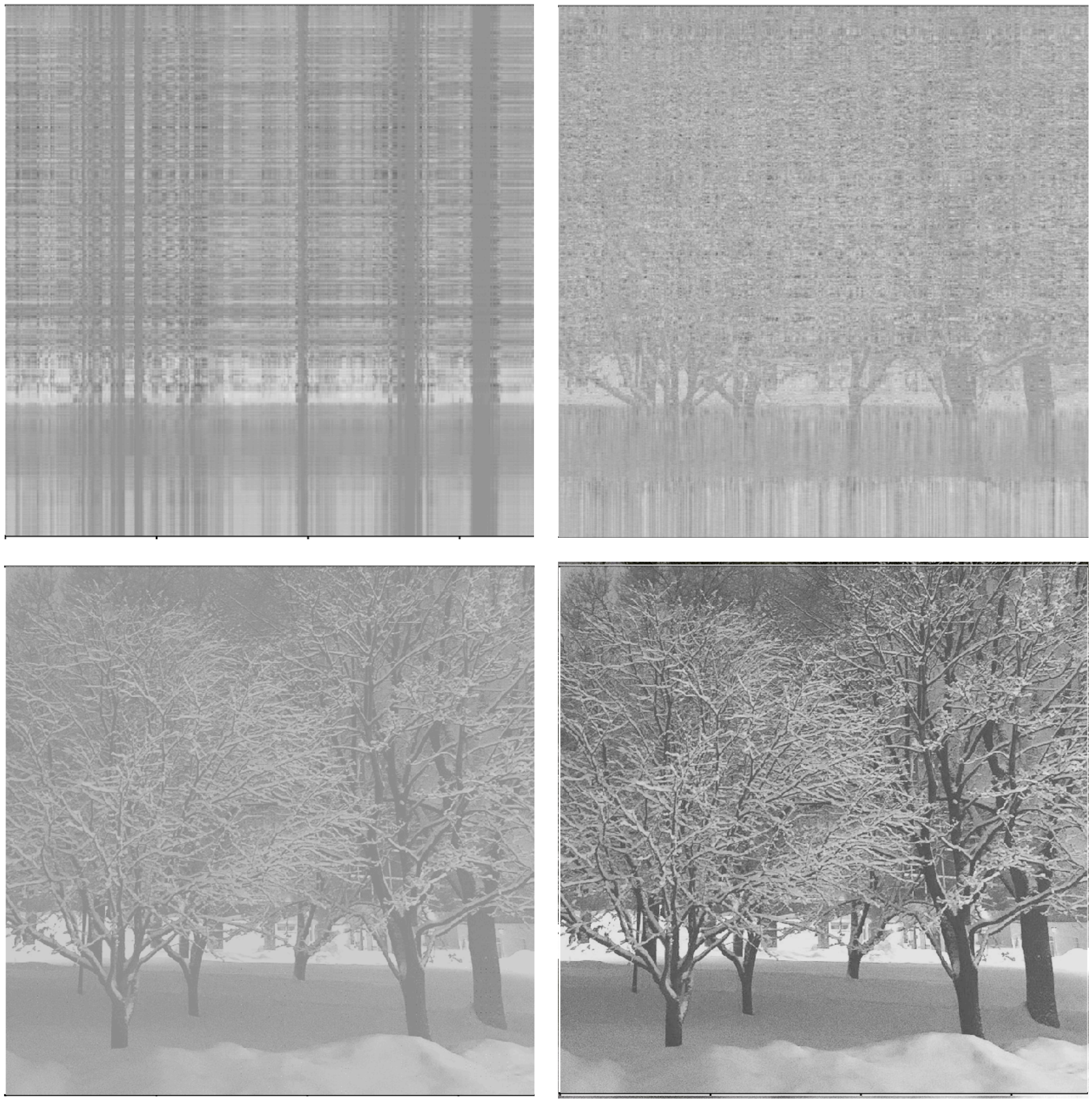
jennajereb99 I've always loved snow on trees just after a beautiful snow; it never gets old. No, not even in March.
.
#heymac

the image is a 700-by-700 matrix (490000 numbers total)
entries are in [0,1]
idea: factor A=PQ, where P and Q are small
e.g. if P is 700-by-10 and Q is 10-by-700, then we can store 1400 numbers instead

Greedy low-rank idea on images

rank 6

rank 66



rank 666

rank 700

$$\begin{pmatrix} 10 & 60 & -40 & 60 \\ 5 & 10 & -10 & 100 \\ 2 & 22 & 12 & -33 \\ -2 & -14 & 19 & 21 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.5 & 1 & 0 & 0 \\ 0.2 & -0.5 & 1 & 0 \\ -0.2 & 0.1 & 0.4 & 1 \end{pmatrix} \begin{pmatrix} 10 & 60 & -40 & 60 \\ 0 & -20 & 10 & 70 \\ 0 & 0 & 25 & -10 \\ 0 & 0 & 0 & 30 \end{pmatrix}$$

$$A \quad = \quad L \quad U$$

Warnings about A=LU

- fails if we encounter a zero pivot
- can give an inaccurate result if we encounter a near-zero pivot

Solution: ``partial pivoting''
-> pivot on largest entry in first nonzero column
-> this still avoids searching the entire matrix for the maximum entry
-> we will discuss this next time

Third idea: Cholesky

$$\begin{pmatrix} 4 & 2 & 0 & -6 \\ 2 & 10 & -6 & 9 \\ 0 & -6 & 29 & -13 \\ -6 & 9 & -13 & 35 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & -2 & 5 & 0 \\ -3 & 4 & -1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & -3 \\ 0 & 3 & -2 & 4 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

How is this different from the A=LU that we just saw?

1) A is symmetric
2) L and U are transposes
3) L does not have ones on diagonal

Cholesky idea: if you start with a symmetric A, try to modify the A=LU idea so that the symmetry is preserved all the way through.

$$A = LU$$

Gauss elimination without pivoting

$$A = LL^T$$

$$A = U^T U$$

Cholesky

Third idea: Cholesky

$$\begin{pmatrix} \boxed{4} & 2 & 0 & -6 \\ 2 & 10 & -6 & 9 \\ 0 & -6 & 29 & -13 \\ -6 & 9 & -13 & 35 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & -2 & 5 & 0 \\ -3 & 4 & -1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & -3 \\ 0 & 3 & -2 & 4 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

1) First pivot: upper left entry

2) Call the pivot row v_1^T and call the pivot column w_1 .

✘ ~~3) Divide v_1^T by the pivot: $v_1^T \leftarrow v_1^T/4$~~ (need symmetry)

✔ 3)

Third idea: Cholesky

$$\begin{pmatrix} \boxed{4} & 2 & 0 & -6 \\ 2 & 10 & -6 & 9 \\ 0 & -6 & 29 & -13 \\ -6 & 9 & -13 & 35 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & -2 & 5 & 0 \\ -3 & 4 & -1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & -3 \\ 0 & 3 & -2 & 4 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

1) First pivot: upper left entry
2) Call the pivot row v_1^T and call the pivot column w_1 .
3) Divide v_1^T by the pivot: $v_1^T \leftarrow v_1^T/4$ (need symmetry)
3) Divide both v_1^T and w_1 by the square root of the pivot

Step 1

$$\begin{pmatrix} 4 & 2 & 0 & -6 \\ 2 & 10 & -6 & 9 \\ 0 & -6 & 29 & -13 \\ -6 & 9 & -13 & 35 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 0 \\ -3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & -3 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 9 & -6 & 12 \\ 0 & -6 & 29 & -13 \\ 0 & 12 & -13 & 26 \end{pmatrix}$$

(remaining steps omitted)

$$\begin{pmatrix} 4 & 2 & 0 & -6 \\ 2 & 10 & -6 & 9 \\ 0 & -6 & 29 & -13 \\ -6 & 9 & -13 & 35 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 \\ 0 & -2 & 5 & 0 \\ -3 & 4 & -1 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 & -3 \\ 0 & 3 & -2 & 4 \\ 0 & 0 & 5 & -1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

Warnings about Cholesky

- fails if we encounter a zero or negative pivot

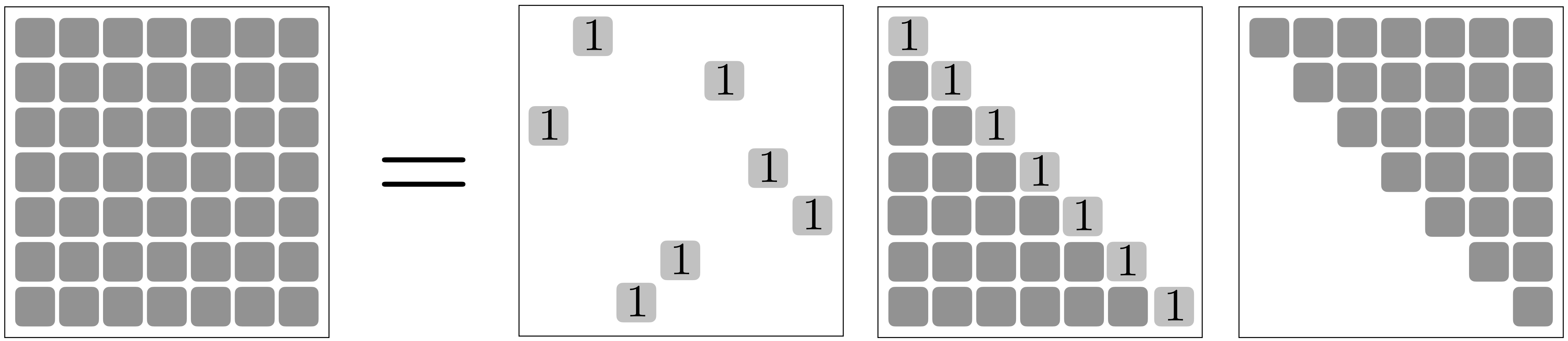
Definition. A square matrix A is *positive-definite* if $x^T Ax > 0$ for all x.
Theorem: A is positive-definite if and only if A has all eigenvalues greater than zero.
Theorem: A is positive-definite if and only if it has a Cholesky factorization.

Plan:
If A is symmetric, give Cholesky a try.
If you meet a negative pivot, abandon Cholesky and try PA=LU or A=QR instead.

If Cholesky fails (nonsymmetric A or negative eigenvalue)…

one popular idea is PA = LU

aka “Gauss elimination with partial pivoting”



$$A = P^T L U$$

Warmup: review of permutation matrices

1) If you multiply A on the left by P, does it reorder the rows or the columns of A?

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -15 & -16 & -14 & -12 \\ 25 & 10 & 40 & 20 \\ 15 & 2 & 46 & 10 \\ 20 & 28 & 72 & -24 \end{pmatrix} =$$

Warmup: review of permutation matrices

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-> reorders the rows.

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-> reorders the rows.

2) Write down a single 4x4 P such that PA is obtained from A by doing this three-part process:
   (a) swap rows 1 and 2
   (b) swap rows 2 and 4
   (c) swap rows 3 and 4

Warmup: review of permutation matrices

1) If you multiply A on the left by P, does it reorder the rows or the columns of A?

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -15 & -16 & -14 & -12 \\ 25 & 10 & 40 & 20 \\ 15 & 2 & 46 & 10 \\ 20 & 28 & 72 & -24 \end{pmatrix} = \begin{pmatrix} 25 & 10 & 40 & 20 \\ 20 & 28 & 72 & -24 \\ 15 & 2 & 46 & 10 \\ -15 & -16 & -14 & -12 \end{pmatrix}$$

-> reorders the rows.

2) Write down a single 4x4 P such that PA is obtained from A by doing this three-part process:
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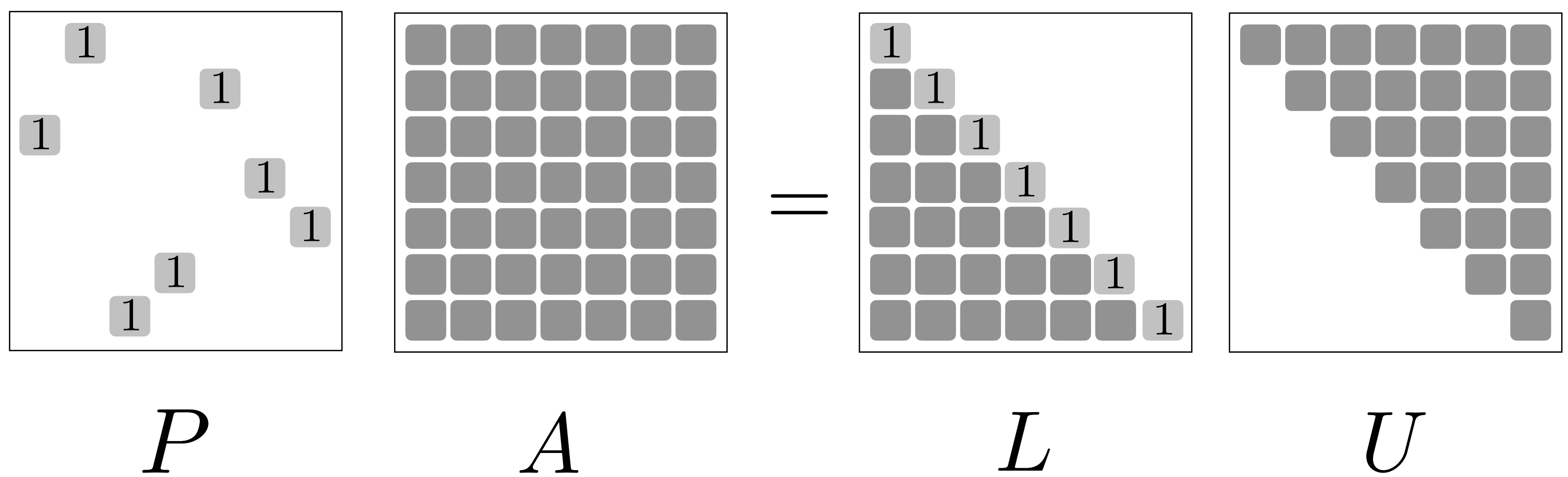
$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

-> product of three transpositions.

$$A = \begin{pmatrix} -15 & -16 & -14 & -12 \\ 25 & 10 & 40 & 20 \\ 15 & 2 & 46 & 10 \\ 20 & 28 & 72 & -24 \end{pmatrix}$$



The *j*-th pivot is the largest (in abs value) entry in column *j*.

$$A = \begin{pmatrix} -15 & -16 & -14 & -12 \\ \boxed{25} & 10 & 40 & 20 \\ 15 & 2 & 46 & 10 \\ 20 & 28 & 72 & -24 \end{pmatrix}$$

The *j*-th pivot is the largest (in abs value) entry in column *j*.
-> first pivot is 25

$$\begin{pmatrix} -15 & -16 & -14 & -12 \\ 25 & 10 & 40 & 20 \\ 15 & 2 & 46 & 10 \\ 20 & 28 & 72 & -24 \end{pmatrix} = \begin{pmatrix} -3/5 \\ 1 \\ 3/5 \\ 4/5 \end{pmatrix} \begin{pmatrix} 25 & 10 & 40 & 20 \end{pmatrix} + \begin{pmatrix} 0 & -10 & 10 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & 22 & -2 \\ 0 & 20 & 40 & -40 \end{pmatrix}$$

$$A = \begin{pmatrix} -15 & -16 & -14 & -12 \\ \boxed{25} & 10 & 40 & 20 \\ 15 & 2 & 46 & 10 \\ 20 & 28 & 72 & -24 \end{pmatrix}$$

The *j*-th pivot is the largest (in abs value) entry in column *j*.
-> first pivot is 25

$$\begin{pmatrix} -15 & -16 & -14 & -12 \\ 25 & 10 & 40 & 20 \\ 15 & 2 & 46 & 10 \\ 20 & 28 & 72 & -24 \end{pmatrix} = \begin{pmatrix} -3/5 \\ 1 \\ 3/5 \\ 4/5 \end{pmatrix} (25 \quad 10 \quad 40 \quad 20) + \begin{pmatrix} 0 & -10 & 10 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & 22 & -2 \\ 0 & 20 & 40 & -40 \end{pmatrix}$$

This column has four nonzero entries, so it must be the first col of L.
However, the 1 should be the first entry.
We will swap rows 1 and 2 by premultiplying both sides by a P.

$$A = \begin{pmatrix} -15 & -16 & -14 & -12 \\ \boxed{25} & 10 & 40 & 20 \\ 15 & 2 & 46 & 10 \\ 20 & 28 & 72 & -24 \end{pmatrix}$$

The *j*-th pivot is the largest (in abs value) entry in column *j*.
-> first pivot is 25

$$\begin{pmatrix} -15 & -16 & -14 & -12 \\ 25 & 10 & 40 & 20 \\ 15 & 2 & 46 & 10 \\ 20 & 28 & 72 & -24 \end{pmatrix} = \begin{pmatrix} -3/5 \\ 1 \\ 3/5 \\ 4/5 \end{pmatrix} (25 \quad 10 \quad 40 \quad 20) + \begin{pmatrix} 0 & -10 & 10 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -4 & 22 & -2 \\ 0 & 20 & 40 & -40 \end{pmatrix}$$

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$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -15 & -16 & -14 & -12 \\ 25 & 10 & 40 & 20 \\ 15 & 2 & 46 & 10 \\ 20 & 28 & 72 & -24 \end{pmatrix} = \begin{pmatrix} 1 \\ -3/5 \\ 3/5 \\ 4/5 \end{pmatrix} (25 \quad 10 \quad 40 \quad 20) + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -10 & 10 & 0 \\ 0 & -4 & 22 & -2 \\ 0 & 20 & 40 & -40 \end{pmatrix}$$

$$\begin{pmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}\begin{pmatrix}-15&-16&-14&-12\\25&10&40&20\\15&2&46&10\\20&28&72&-24\end{pmatrix}=\begin{pmatrix}1\\-3/5\\3/5\\4/5\end{pmatrix}\begin{pmatrix}25&10&40&20\end{pmatrix}+\begin{pmatrix}0&0&0&0\\0&-10&10&0\\0&-4&22&-2\\0&20&40&-40\end{pmatrix}$$

The j-th pivot is the largest (in abs value) entry in column j.
-> second pivot is 20

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -15 & -16 & -14 & -12 \\ 25 & 10 & 40 & 20 \\ 15 & 2 & 46 & 10 \\ 20 & 28 & 72 & -24 \end{pmatrix} = \begin{pmatrix} 1 \\ -3/5 \\ 3/5 \\ 4/5 \end{pmatrix} (25 \quad 10 \quad 40 \quad 20) + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -10 & 10 & 0 \\ 0 & -4 & 22 & -2 \\ 0 & 20 & 40 & -40 \end{pmatrix}$$

The *j*-th pivot is the largest (in abs value) entry in column *j*.
-> second pivot is 20

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -15 & -16 & -14 & -12 \\ 25 & 10 & 40 & 20 \\ 15 & 2 & 46 & 10 \\ 20 & 28 & 72 & -24 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -3/5 & -1/2 \\ 3/5 & -1/5 \\ 4/5 & 1 \end{pmatrix} \begin{pmatrix} 25 & 10 & 40 & 20 \\ 0 & 20 & 40 & -40 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 30 & -20 \\ 0 & 0 & 30 & -10 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -15 & -16 & -14 & -12 \\ 25 & 10 & 40 & 20 \\ 15 & 2 & 46 & 10 \\ 20 & 28 & 72 & -24 \end{pmatrix} = \begin{pmatrix} 1 \\ -3/5 \\ 3/5 \\ 4/5 \end{pmatrix} (25 \quad 10 \quad 40 \quad 20) + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -10 & 10 & 0 \\ 0 & -4 & 22 & -2 \\ 0 & 20 & 40 & -40 \end{pmatrix}$$

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$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -15 & -16 & -14 & -12 \\ 25 & 10 & 40 & 20 \\ 15 & 2 & 46 & 10 \\ 20 & 28 & 72 & -24 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -3/5 & -1/2 \\ 3/5 & -1/5 \\ 4/5 & 1 \end{pmatrix} \begin{pmatrix} 25 & 10 & 40 & 20 \\ 0 & 20 & 40 & -40 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 30 & -20 \\ 0 & 0 & 30 & -10 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

now interchange rows:

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -15 & -16 & -14 & -12 \\ 25 & 10 & 40 & 20 \\ 15 & 2 & 46 & 10 \\ 20 & 28 & 72 & -24 \end{pmatrix} = \begin{pmatrix} 1 \\ -3/5 \\ 3/5 \\ 4/5 \end{pmatrix} (25 \quad 10 \quad 40 \quad 20) + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -10 & 10 & 0 \\ 0 & -4 & 22 & -2 \\ 0 & 20 & 40 & -40 \end{pmatrix}$$

The *j*-th pivot is the largest (in abs value) entry in column *j*.
-> second pivot is 20

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -15 & -16 & -14 & -12 \\ 25 & 10 & 40 & 20 \\ 15 & 2 & 46 & 10 \\ 20 & 28 & 72 & -24 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -3/5 & -1/2 \\ 3/5 & -1/5 \\ 4/5 & 1 \end{pmatrix} \begin{pmatrix} 25 & 10 & 40 & 20 \\ 0 & 20 & 40 & -40 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 30 & -20 \\ 0 & 0 & 30 & -10 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

now interchange rows: 2 and 4

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -15 & -16 & -14 & -12 \\ 25 & 10 & 40 & 20 \\ 15 & 2 & 46 & 10 \\ 20 & 28 & 72 & -24 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 4/5 & 1 \\ 3/5 & -1/5 \\ -3/5 & -1/2 \end{pmatrix} \begin{pmatrix} 25 & 10 & 40 & 20 \\ 0 & 20 & 40 & -40 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 30 & -10 \\ 0 & 0 & 30 & -20 \end{pmatrix}$$

$$\begin{pmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}-15&-16&-14&-12\\25&10&40&20\\15&2&46&10\\20&28&72&-24\end{pmatrix}=\begin{pmatrix}1&0\\4/5&1\\3/5&-1/5\\-3/5&-1/2\end{pmatrix}\begin{pmatrix}25&10&40&20\\0&20&40&-40\end{pmatrix}+\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&30&-10\\0&0&30&-20\end{pmatrix}$$

The *j*-th pivot is the largest (in abs value) entry in column *j*.
-> third pivot is 30 (a tie, so pick the first 30 in third column)

$$\begin{pmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}-15&-16&-14&-12\\25&10&40&20\\15&2&46&10\\20&28&72&-24\end{pmatrix}=\begin{pmatrix}1&0&0\\4/5&1&0\\3/5&-1/5&1\\-3/5&-1/2&1\end{pmatrix}\begin{pmatrix}25&10&40&20\\0&20&40&-40\\0&0&30&-10\end{pmatrix}+\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&-10\end{pmatrix}$$

no row swaps needed.
Last pivot is the lower entry.

$$\begin{pmatrix}0&1&0&0\\0&0&0&1\\0&0&1&0\\1&0&0&0\end{pmatrix}\begin{pmatrix}-15&-16&-14&-12\\25&10&40&20\\15&2&46&10\\20&28&72&-24\end{pmatrix}=\begin{pmatrix}1&0&0&0\\4/5&1&0&0\\3/5&-1/5&1&0\\-3/5&-1/2&1&1\end{pmatrix}\begin{pmatrix}25&10&40&20\\0&20&40&-40\\0&0&30&-10\\0&0&0&-10\end{pmatrix}$$

$$P \qquad A \qquad = \qquad L \qquad U$$