Computational Linear Algebra Class 3: Iterative Methods for Root-Finding (Part 2)

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MACALESTER

Reminder: Rate of convergence

- Sequence of approximations converges with rate q if

$$\lim_{i\to\infty} \frac{||e_{i+1}||}{||e_i||^q} = C \quad (\text{i.e., } ||e_{i+1}|| \approx C||e_i||^q)$$

q	Convergence rate
q=1	linear
q>1	superlinear
q=2	quadratic

Failure of Newton's Method

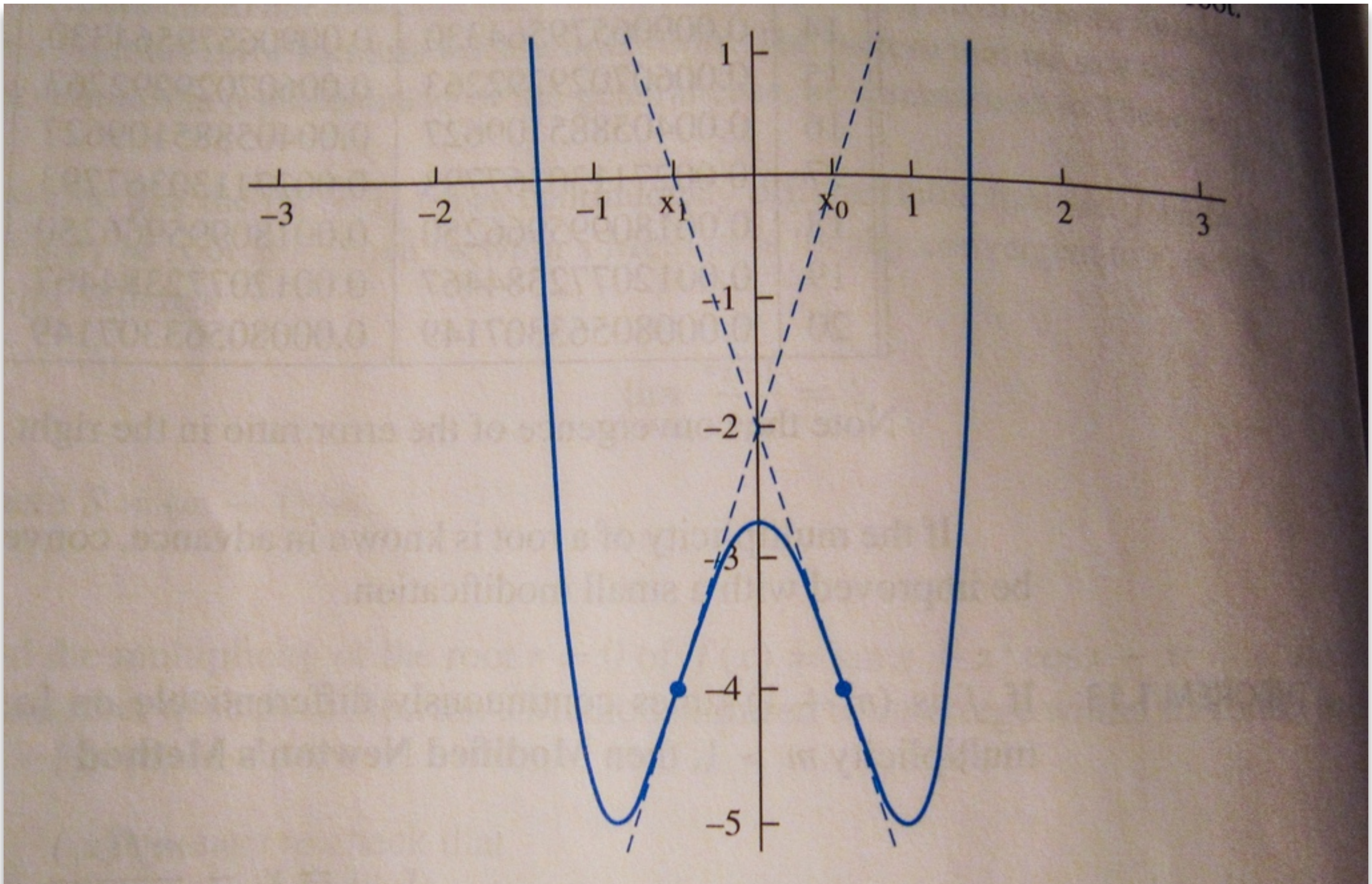


Figure 1.10 Failure of Newton's Method in Example 1.15. The iteration alternates between 1/2 and −1/2, and does not converge to a root.

Rubric for evaluating solution method

- Does sequence of approximations converge to a root?
- How fast does it converge?
- How stable is the solution process to tiny numerical errors?

The ideal method may change from instance to instance:

- There may be a tradeoff between speed of convergence and robustness
- Different problem variants and information available about the problem
  - Reliable bracket *[a,b]* available?
  - *A priori* knowledge about properties of the function? (e.g., continuously differentiable, polynomial, a single zero)
  - Interested in a single root or multiple roots?

Bisection Method: Summary

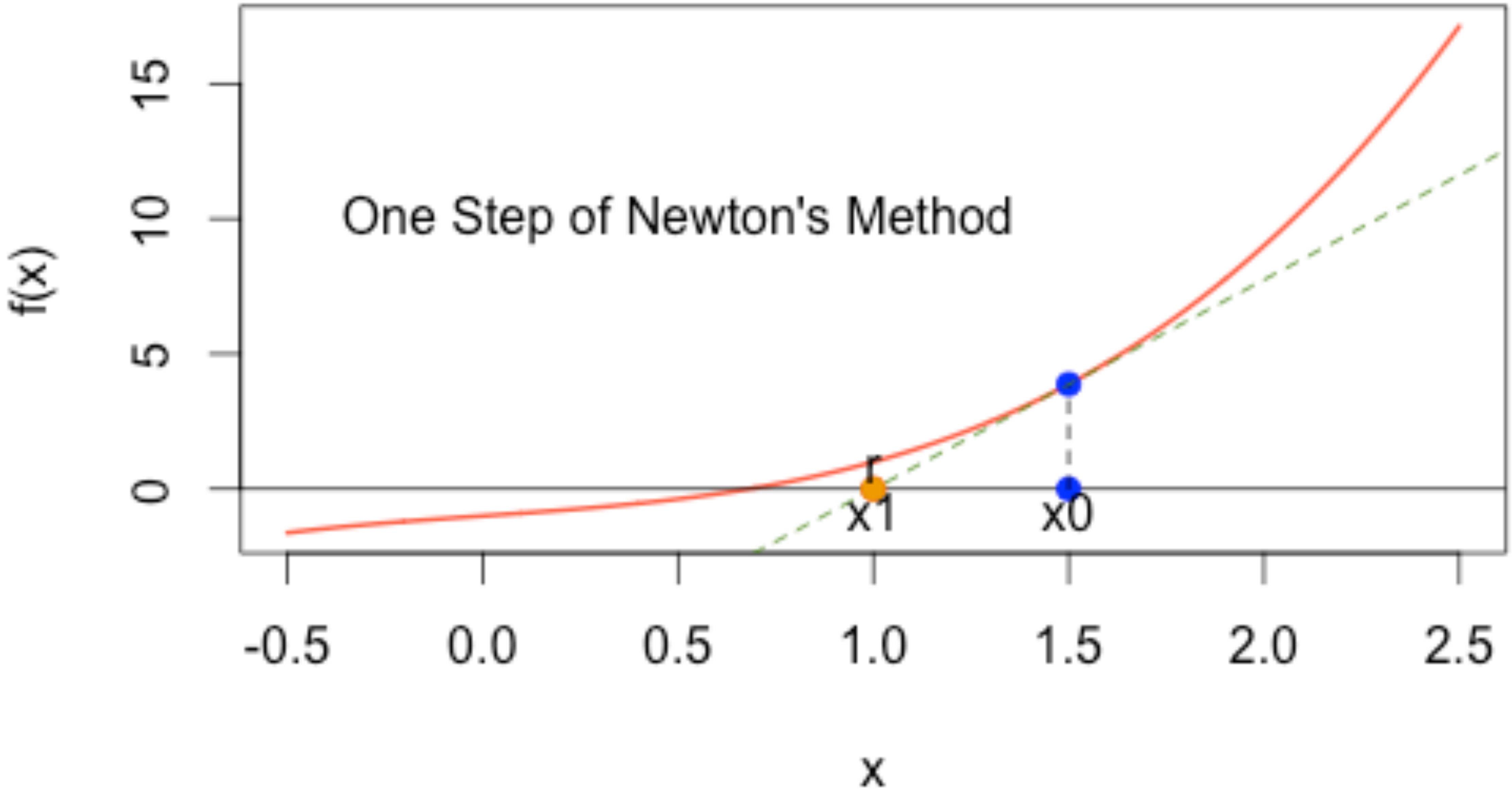
- Pros:
  - Always converges if there is a root in the given interval
  - Error approximately cuts in half each step (one additional bit of accuracy)
- Cons:
  - Only uses signs of the function values, not magnitude of function values or other properties of the function
  - Finds only one root in the interval, not multiple
  - Slow convergence

Newton's Method: Summary

- Pros:
  - Possibility for quadratic convergence
  - Leverages more information about the function
- Cons:
  - Requires the derivative (more function calls to compute leads to longer iterations, issues if function is not continuously differentiable)
  - Can fail (loop or divergence) if initial guess is poor or if

$$f'(x_i) = 0$$

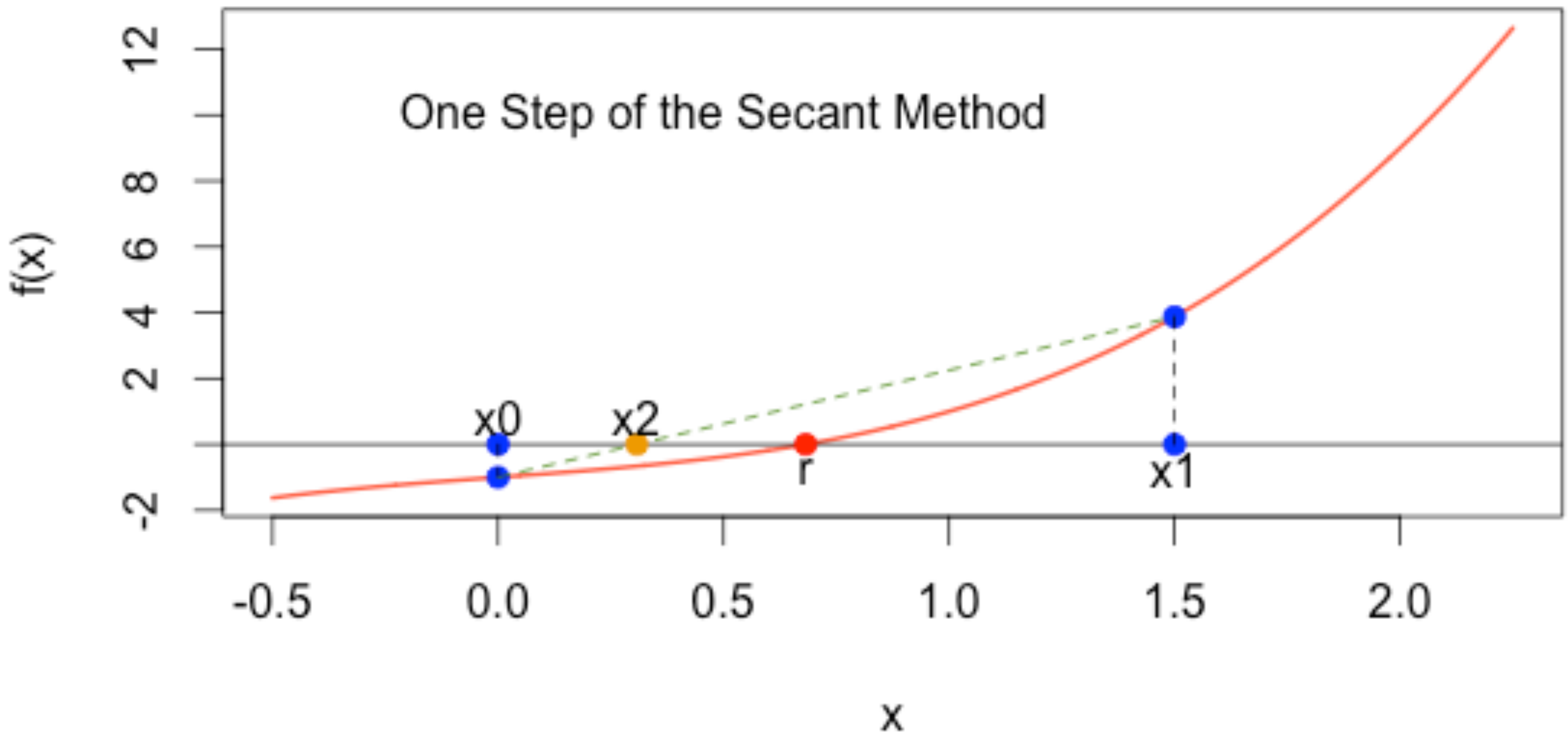
Newton's vs. Secant vs. IQI



- What if we don't have a derivative or don't want to compute it?

Newton's vs. Secant vs. IQI

- Idea: Approximate the tangent line with a secant line

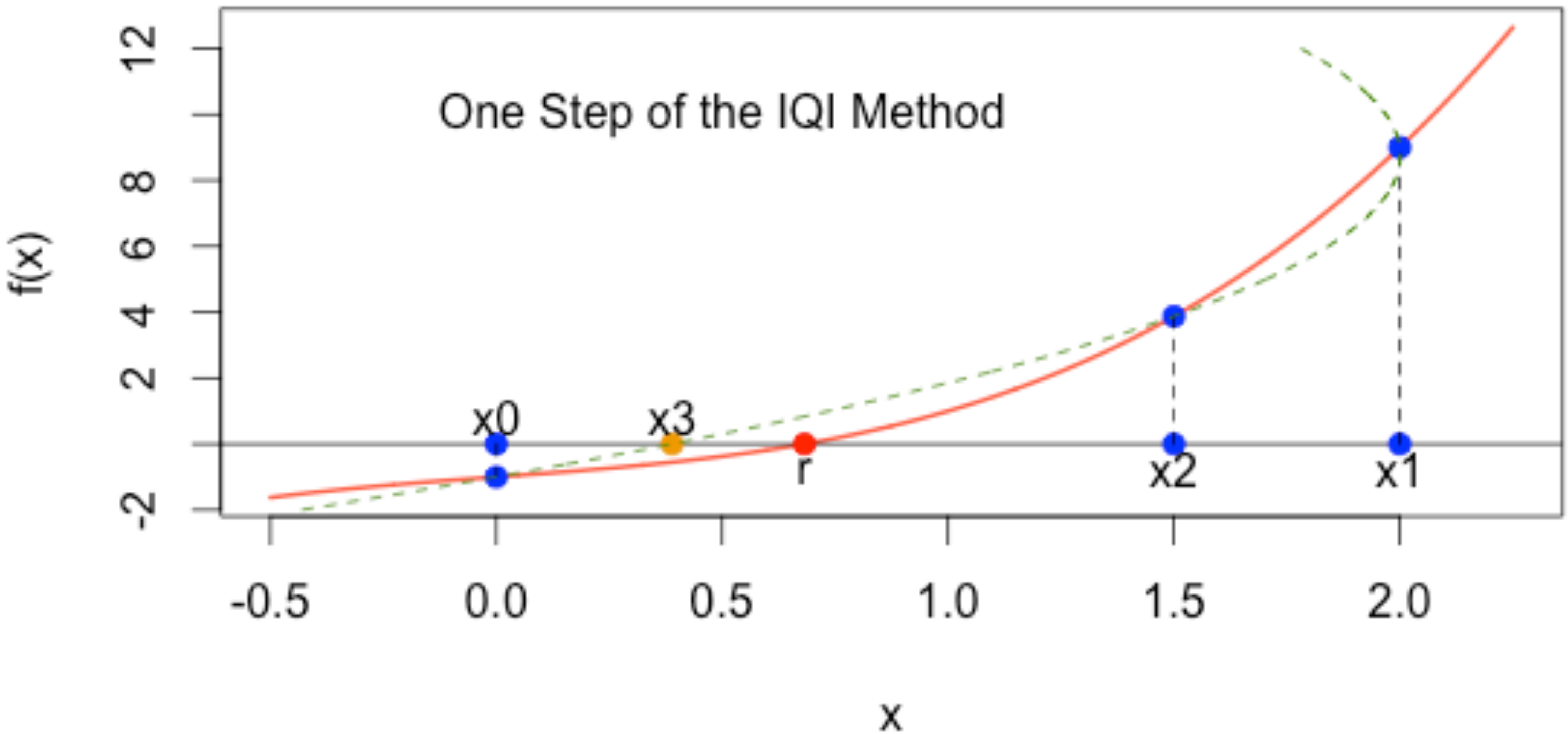


$\boxed{\text{Superlinear convergence: } q \approx 1.62}$

*When it converges and does not have a derivative of 0 at the root

Newton's vs. Secant vs. IQI

- Inverse quadratic interpolation (IQI): Horizontal parabola, goes through three initial points



$\boxed{\text{Superlinear convergence: } q \approx 1.84}$

*When it converges and does not have a derivative of 0 at the root

Brent's Method

- Hybrid method that also starts with a bracketing interval
- Roughly: 1) IQI if bracketing interval is cut at least in half and $/f(x_i)/$ decreases, 2) else try Secant Method with same goal, 3) else use Bisection Method
- See uniroot in R (based on zeroin.c) or fzero in MATLAB