Take Home Final Exam

Computational Linear Algebra
Tuesday, May 3, 2022 — Monday, May 9, 2022 (Noon sharp)

Please upload your answers to Moodle

- The exam is open notes and open book. You may use the internet and library to look things up, but of course you should cite any reference you use that is not the textbook.
- **It is not okay to talk to each other or anyone else about the exam! This of course includes discussing how you solve problems, but it even includes things like asking, "how it is going?" or "which problem you are working on?" or making comments like "problem 5 is easy."**
- Please show and explain all of your work so that I can give you as much partial credit as possible.
- You are welcome to use any of the `R` functions in `365Functions.r`
- The most convenient method to turn in your answers would be to do them in a single R Markdown file and submit the html file containing your code and your solutions. I have provided a starter file `2022.365.E3.Rmd` in case you would like to do that. **Make sure to insert your name and type out the academic honesty statement at the top of your Markdown file.**
- Hints: I may give a small hint or a little help with R for free. If you are stuck, I may sell a hint for points so that you can move forward. I will warn you of the point value before I sell you anything.

Problem	Point Value	Your Score
1	10	
2	16	
3	6	
4	13	
Total	45	
EC	2	

Please sign the following (if you are able) and be sure to turn in this cover sheet with your exam: *I pledge my honor that I have not participated in any dishonest work on this exam, nor do I know of dishonest work done by other students on this exam.*

____
(signature)

As an alternative to turning this in, you may also type out this pledge as part of your exam.

1. (10 points)

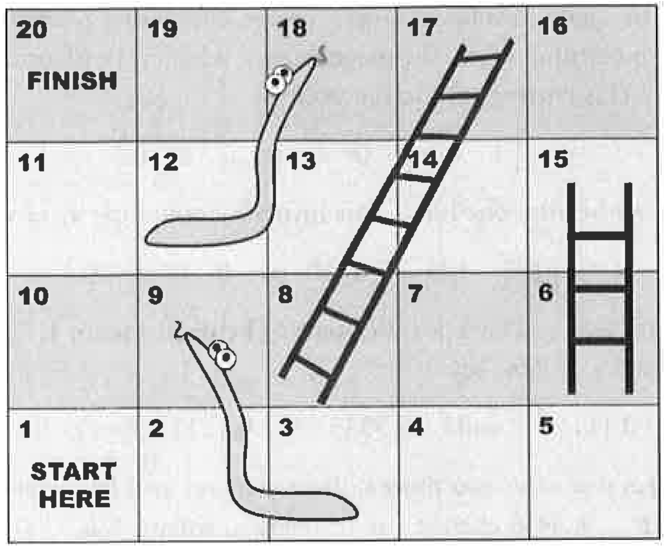

Figure 1: A Chutes and Ladders game board variant.

In this problem, we are going to model a variant of the Chutes and Ladders game as a Markov chain. We'll analyze some of the game dynamics **assuming there is just one player**. Here are some more details of the movement rules on the game board shown above in Figure 1:

- In this variant of the game, the single player roles a **three-sided** die each turn, which yields the possible values of 0, 1, or 2, each with probability equal to one third. The player starts at square 1, and moves the number of spaces rolled on the die. The direction of movement is according to the order indicated in the top left corner of each square. For example, if the player starts on square 5 and rolls a 2, she ends up on square 7. If she rolls a 0, she stays on square 5.
- If the player's roll leads to a space that is the bottom of the ladder, she is **instantaneously** transferred to the top of the ladder. For example, if she starts on square 5 and rolls a 1, she ends the turn on square 15.
- On the other hand, if the player's roll leads to a space that is the top of a chute, she instantaneously falls to the bottom of the slide. For example, if she starts on square 7 and rolls a 2, she ends the turn on square 3.
- Once she reaches square 20, she stays there and the game is over. If the player rolls 2 while starting on square 19, she moves to square 20 and the game is over.

(a) (2 points) Let the state of the Markov chain be the square on which the player starts her turn. First draw a state diagram and label the edges with the transition probabilities. Then write down the transition matrix P.

(b) (3 points) Compute the probability that after 20 turns the player has reached the end (square 20).

(c) (3 points) Now let's change the board slightly and add a chute from square 20 back to square 1, so the game never ends![1] You play the game for a very long time (e.g., while you are contemplating the beauty of linear algebra), and then I walk into the room. What is the probability that I find you on square 19 at the beginning of your next turn? Howabout on square 10? On which square(s) am I most likely to find you?

(d) (2 points) Briefly discuss why the method you used to compute your answer to part (c) lead to an answer for the question that was asked.

[1] If you roll a 2 on 19, you also go back to square 1.

2. (16 points)

For all parts of this problem, let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 5 & -4 & 2 \\ -2 & 2 & -1 \\ 3 & 2 & 3 \end{bmatrix}.$$

(a) (3 points) Find two non-zero, orthogonal vectors $f, g \in \mathbb{R}^3$ such that the vector Af is orthogonal to the vector Ag in $\mathbb{R}^4$. Demonstrate in your `R` code that Af and Ag are orthogonal by computing their dot product, and that f and g are orthogonal by computing their dot product.

(b) (3 points) Use the SVD of A to find a vector $h \in \mathbb{R}^4$ of length 2 (in the 2-norm) such that $A^\top h = 0$.

(c) (4 points) Is the matrix $AA^\top$ diagonalizable? Is it full rank? *Briefly* justify both of your answers.

(d) (3 points) Continuing with the same A from above, we execute the following code in `R`:

```
> out <- svd(A)
> U <- out$u
> S <- U%*%t(U)
```

TRUE or FALSE (circle one). There exists a non-zero vector $y \in \mathbb{R}^4$ such that $Sy = y$. *Briefly* justify your answer.

(e) (3 points) Let $P = 2S - I$, where S is the matrix computed above and I is a 4×4 identity matrix. Explain in words what multiplying a vector $y \in \mathbb{R}^4$ by the matrix P does to the vector y. It might help to show a picture as well.

3. (6 points) Throughout this problem, for any two real-valued continuous functions, f and g, defined on the interval $[-1, 1]$, use the following inner product:

$$\langle f, g \rangle := \int_{-1}^{1} f(x)g(x)\ dx,$$

and the norm $||f|| = \sqrt{\langle f, f \rangle}$. Let $\mathcal{P}_2$ be the space of all degree 2 polynomials (i.e., quadratic functions of the form $ax^2 + bx + c$).

Note: For this problem, if you don't like computing integrals by hand, you may (i) take 5 seconds to bask in the shame of not liking integration, and compute all integrals with Wolfram Alpha or R, or (ii) come up with a clever way of doing the problem without any integration at all.

(a) (3 points) Let $h(x) = 2x^3 + 3x^2 - 2x + 3$. Find the degree 2 polynomial $\hat{h}(x)$ that satisfies

$$\hat{h} = \underset{f \in \mathcal{P}_2}{\operatorname{argmin}} \Big\{ ||h - f|| \Big\}.$$

(b) (3 points) Write down a formula for all degree 3 polynomials g such that

$$\underset{f \in \mathcal{P}_2}{\operatorname{argmin}} \Big\{ ||g - f|| \Big\} = 3x^2 + 6x - 5.$$

Hint: No integration is necessary for this part of the problem.

4. (13 points) Cryptograms

In a substitution cipher, each letter is replaced by a different one according to a random permutation. For example, if the cipher is given by:

$$\begin{gathered} a \leftarrow r \\ b \leftarrow z \\ c \leftarrow p \\ d \leftarrow b \\ e \leftarrow n \\ \vdots \end{gathered}$$

then the word "bee" is encoded as "znn". A cryptogram is a short piece of encrypted text, and in this problem, we examine cryptograms generated from substitution ciphers.

(a) (0 points) **Fun but totally optional warmup:** If you'd like to get familiar with cryptograms generated from substitution ciphers, you can try to solve the following cryptogram. Think about what approaches you are using, how you are taking advantage of structure in the English language, and how you might automate what you are doing.

sou suy-fdppmw ndjyfkyi nmsouw lksodjs m nmsouw
ids m pds nmwsouw er ldwvkyi m pds omwfuw,
er eukyi m pds qzmwsuw, er eukyi m qupn-qsmwsuw.
er ndjwsuuy, sour tpmcuf okz ky comwiu dn m swmfkyi comwsuw.

sou qokt kq ky sou omwedw ydl,
quu kn rdj cmy qtds okz.
mydsouw kzzkiwmys, cdzky' jt nwdz sou edssdz.
okq uyuzkuq fuqswdruf okq wut, mzuwkcm ndwids okz.

lu ndjios lkso okz.
zu? k fkuf ndw okz.
zu? k swjqsuf okz.
zu? k pdbuf okz.
myf zu? k'z sou fmzy nddp soms qods okz.

souwu'q m zkppkdy sokyiq k ombuy's fdyu, ejs hjqs rdj lmks!
loms'q rdjw ymzu, zmy?
mpugmyfuw omzkpsdy!

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R	S	T	U	V	W	X	Y	Z
A	0	1	2	5	0	1	4	0	2	0	1	9	0	15	0	1	0	10	5	36	1	8	0	0	1	0
B	1	0	0	0	5	0	0	0	1	0	0	1	0	0	1	0	0	2	0	0	2	0	0	0	1	0
C	12	0	0	0	4	0	0	2	1	0	0	0	0	0	7	0	0	4	0	1	0	0	0	0	0	0
D	2	0	1	6	14	0	0	3	13	0	0	0	0	0	4	1	0	0	4	4	1	1	4	0	0	0
E	16	3	8	26	3	5	2	7	6	0	0	4	5	10	5	4	1	22	9	12	2	4	8	0	3	0
F	3	0	0	1	0	1	0	0	5	0	0	0	0	0	10	0	0	3	0	3	1	0	0	0	0	0
G	5	1	0	0	5	0	1	4	0	0	0	1	0	0	3	1	0	6	0	0	0	0	1	0	0	0
H	24	0	0	0	32	2	0	0	7	0	0	1	0	0	8	0	0	0	0	5	1	0	0	0	0	0
I	0	1	8	1	3	0	2	0	0	0	0	2	0	16	9	0	0	2	9	8	0	7	0	0	0	0
J	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
K	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	1	0	0	0
L	3	0	0	4	6	0	0	1	6	0	0	8	2	3	3	0	0	1	0	1	0	1	1	0	2	0
M	2	1	0	0	7	0	0	0	1	0	0	0	0	0	0	0	0	0	0	2	0	0	0	0	0	0
N	10	0	5	9	4	1	9	0	2	0	0	3	2	4	12	0	0	0	4	8	1	1	0	0	2	0
O	1	3	1	3	0	6	1	1	0	0	0	1	4	20	2	5	0	17	3	13	7	2	3	0	0	0
P	0	0	0	0	5	0	0	0	0	0	0	4	0	0	4	0	0	2	0	0	0	0	0	0	0	0
Q	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
R	8	0	0	1	26	4	3	0	1	0	1	3	0	1	6	2	0	0	5	12	3	0	3	0	0	0
S	4	2	2	0	10	2	1	6	1	0	1	0	0	1	4	0	0	1	0	8	1	0	0	0	0	0
T	4	1	4	1	11	5	1	47	18	0	0	3	0	2	11	1	0	2	0	9	0	0	5	0	1	0
U	1	0	0	0	0	0	3	0	0	0	0	2	0	3	0	0	0	5	5	2	0	0	0	0	0	0
V	2	0	0	0	17	0	0	0	3	0	0	0	0	0	2	0	0	0	0	0	0	0	0	0	0	0
W	2	1	0	0	11	0	0	8	1	0	0	0	0	1	2	0	0	0	0	1	0	0	1	0	0	0
X	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Y	2	0	0	1	1	0	1	1	0	0	0	0	0	1	0	0	0	1	0	1	0	0	1	0	0	0
Z	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Figure 2: The digram frequency matrix for Lincoln's Gettysburg address.

In `2022.365.E3.Rmd`, you will find a 26×26 matrix G that represents the *digram frequency matrix* for Lincoln's Gettysburg address.[2] You can see this matrix in Figure 2 above. The ij entry of this matrix is g_{ij} = the number of times that the ith letter is followed immediately by the jth letter. Blank spaces and punctuation, if present, are ignored, and the first letter of the text is assumed to follow the last letter. For example, the pair "th" occurs 47 times in this text.

(b) (2 points) In `2022.365.E3.Rmd`, I have computed the vector $f = G1$, where 1 is the column vector of all ones. Describe what this vector tells us about the text.

(c) (3 points) Compute the SVD of the matrix G. Use it to give the numerical rank of G. Explain how the numerical rank "makes sense" from the given data.

(d) (4 points)

(i) Compute the best rank one approximation of G; i.e., find G_1 such that

$$||G - G_1||_2 = \inf_{B \in \mathbb{R}^{26\times 26} \text{ s.t. } \operatorname{rank}(B) \leq 1} \{||G - B||_2\}.$$

Note: roughly speaking, the ij entry of G_1 is proportional to the product of the frequency of letter i with the frequency of letter j. It does not really account for the combinations/orders in which letters tend to appear most frequently.

(ii) Also compute the best rank two approximation of G; i.e., find G_2 such that

$$||G - G_2||_2 = \inf_{B \in \mathbb{R}^{26\times 26} \text{ s.t. } \operatorname{rank}(B) \leq 2} \{||G - B||_2\}.$$

(e) (2 points) Computational linguists have noted that rank two approximations of digram frequency matrices help distinguish consonants from vowels.[3] Let $x = u_2$ and $y = v_2$ (where these are the u's and the v's from the SVD), and make a graph that plots the ith letter at the position (x_i, y_i). Describe how the vowels are grouped in this graph.[4]

(f) (2 points) This technique can be used in cryptography. Here is a simple example. In `2022.365.E3.Rmd`, I also give you the digram frequency matrix S from another text, but in this text the letters have been permuted with a substitution cipher so as to secretly encrypt the message. Use the SVD to see if you can figure out which letters in this scrambled text are the vowels.

Hint: remember that if $U\Sigma V^T$ is an SVD of a matrix, then $(-U)\Sigma(-V^T)$ is also an SVD of the same matrix.

Extra credit (2 points). Unscramble the following excerpt from the encrypted text and explain how you did it:

d lbyz b crzbt ulbu ti njvr xduuxz sldxcrzw hdxx jwz cbi xdyz dw b wbudjw hlzrz ulzi hdxx wju kz mvcozc ki ulz sjxjr jn ulzdr qadw kvu ki ulz sjwuzwu jn ulzdr slbrbsuzr

[2] Four score and seven years ago our fathers brought forth on this continent a new nation, conceived in liberty, and dedicated to the proposition that all men are created equal....

[3] If you are interested in learning more about this, let me know after the exam and I will give you some further reading. One important point underlying the analysis is that vowels tend to follow consonants more often than other vowels.

[4] Note: the vowel "U" is a little harder to classify as it often follows other vowels, such as in the "OU" combination.