Definitions

- Pivot position.
- Elementary row operations.
- The definition of $A\mathbf{x}$ in both words and symbols.
- Linear combination.
- Span$(\mathbf{v}_1, ..., \mathbf{v}_k)$. In particular, you should be able to visualize Span$(\mathbf{v})$, Span$(\mathbf{u}, \mathbf{v})$ and give geometric interpretations of these sets in $\mathbb{R}^2$ or $\mathbb{R}^3$.
- Linearly independent and linearly dependent.
- Linear transformation (and the superposition principle).
- Domain, codomain, image, range, onto, and one-to-one.
- The transpose of a matrix, the inverse of a matrix, invertible matrix.
- Vector space and subspace.
- Null space, column space, and row space of a matrix.
- Kernel and range of a linear transformation.
- Basis and dimension.
- Rank.
- Eigenvalue, eigenvector, eigenspace.
- Characteristic polynomial and characteristic equation.
- Diagonalizable matrix.
- Similar matrices.
- Dot product, length of a vector, angle between vectors.
- Orthogonal vectors, orthogonal spaces.
- Orthogonal complement of a subspace.
- Orthogonal basis, orthonormal basis, orthogonal matrix.
- Orthogonal projection, least squares solutions.

Theorems

- Theorem 1.2 (Existence and uniqueness theorem).
- Theorem 1.3 (Matrix equation, vector equation, system of linear equations).
- Theorem 1.4 (When do the columns of A span $\mathbb{R}^m$?).
- Theorem 1.5 (Properties of the matrix-vector product $A\mathbf{x}$).
- Theorem 1.6 (Relation between solutions to $Ax = b$ and $Ax = 0$).

- Theorems 1.7, 1.8, 1.9 (Properties of linearly dependent sets).
- Theorem 1.10 (Finding the standard matrix of a linear transformation).
- Theorem 1.11 (Characterization of one-to-one linear transformation).
- Theorem 1.12 (Characterization of onto linear transformation).
- Theorems 2.8, 3.4, and extension on p. 235 in Section 4.6 (Invertible Matrix Theorems).
- Theorem 4.1 (On generating subspaces).
- Theorem 4.5 (Spanning Set Theorem).
- Theorem 4.7 (Unique Representation Theorem).
- Theorem 4.10 (Number of vectors in a basis).
- Theorem 4.12 (The Basis Theorem).
- Theorem 4.8 (Any n-dimensional vector space looks like $\mathbb{R}^n$).
- Theorem 4.14 (The Rank Theorem).
- Theorem 5.2 (On distinct eigenvalues and linearly independent eigenvectors).
- Theorem 5.5 (Diagonalization).
- Theorem 6.3 (Orthogonal pairs of fundamental subspaces of matrices).
- Theorem 6.5 (Finding the representation of a vector with respect to an orthogonal basis).
- Theorem 6.7 (Matrices with orthonormal columns preserve geometry).
- Theorem 6.8 (Orthogonal Decomposition Theorem).
- Theorem 6.9 (Best Approximation Theorem).
- Theorems 6.13 and 6.14 (Normal equations and finding a least squares solution for an invertible A)

Important Skills (partial list)

- Form an augmented matrix and reduce a matrix or augmented matrix into row echelon or reduced row echelon form. Determine whether a given matrix is in either of those forms. Determine whether a particular form of a matrix is a possible row echelon or reduced echelon form.
- Determine whether a system is consistent and if it has a unique solution. Write the general solution in parametric vector form. Describe the set of solutions geometrically.
- Determine values of parameters that make a system consistent, or make the solution unique. Describe existence or uniqueness of solutions in terms of pivot positions. Determine when a homogeneous system has a nontrivial solution. Construct examples of matrices or equations of given dimensions with solution sets with desired properties (e.g., unique, many solutions, forming a plane).
- Interpret a system of equations in terms of both (i) the constraints imposed by each equation ("the row picture") and (ii) identifying one vector (b) as a linear combination of other vectors (the columns of A).
- Determine when a vector is in a subset spanned by specified vectors. Exhibit a vector as a linear combination of specified vectors. Determine whether a specified vector is in the range of a linear transformation.

- Determine whether the columns of an $m \times n$ matrix span $\mathbb{R}^m$. Determine whether the columns are linearly independent.
- Write the equation of a line or a plane in parametric vector form.
- Compute a matrix-vector product, and interpret it as a linear combination of the columns of A. Use linearity of matrix multiplication to compute $A(\mathbf{u}+\mathbf{v})$ or $A(c\mathbf{u})$.
- Find the standard matrix of a linear transformation.
- Determine whether a transformation is linear. Determine whether a linear transformation $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one or onto, using the properties of the matrix A.
- Use the Invertible Matrix Theorem to prove results about square matrices (see, e.g., activity A13 and p. 112)
- Perform matrix operations such as scalar multiplication, matrix multiplication, matrix addition, matrix transpose, and matrix inverse, for example to solve matrix equations.
- Use matrices and matrix operations to model applied problems (e.g., case study 2)
- Compute the determinant of a small (2x2 or 3x3) matrix.
- Determine whether a subset of vectors is a subspace, and either prove it is a subspace or identify which property is not satisfied with a counterexample.
- Determine whether a set of vectors is linearly independent and whether it is a basis for some subspace or vector space. Find a basis for a vector space, or for the null space or column space of a matrix, or for the kernel or range of a linear transformation. Find the dimension of a vector space or subspace. Find the dimension of the null space (number of free variables) and column space (number of pivot columns) of a matrix.
- Manipulate and interpret abstract vector spaces such as infinite dimensional sequences, polynomials, and 2×2 matrices.
- Use coordinate vectors to determine whether vectors in abstract vector spaces are linearly independent and to find bases for subspaces of abstract vector spaces.
- Find and interpret the rank of a matrix.
- Calculate the characteristic equation, eigenvalues, and eigenvectors of a square matrix. Find eigenvectors for a specific eigenvalue. Check if a vector is an eigenvector of a given matrix.
- Determine whether a square matrix is diagonalizable. Factor a diagonalizable matrix into $A = PDP^{-1}$, where D is a diagonal matrix.
- Use eigenvalues and eigenvectors to analyze the asymptotic behavior of discrete dynamical systems.
- Find the length of a vector, the distance between two vectors, or the angle between two vectors.
- Determine whether a set of vectors are orthogonal. Determine whether a vector is orthogonal to a subspace or whether two subspaces are orthogonal, by checking whether their basis vectors are orthogonal.
- Find the orthogonal projection of (i) one vector onto another vector, or (ii) one vector onto a subspace (using an orthogonal basis for the subspace). Find the distance between a vector and a space (by computing the residual).
- Given a set of vectors, use the Gram-Schmidt process to find an orthogonal (or orthonormal) set of vectors with the same span.

- Interpret orthogonal projections geometrically. Draw abstract geometric pictures representing projections (even if the vector spaces can't be drawn like $\mathbb{R}^{23}$ or more abstract spaces like the space of polynomials of a given degree).
- Set up the matrix equation to find the "best-fitting" function to a set of data using least squares. Interpret the normal equations $A^\top Ax = A^\top b$. Find the least squares approximation by solving the normal equations $\hat{x} = (A^\top A)^{-1}A^\top b$.

Most Important Theorems For Understanding Linear Algebra Beyond This Course

- Theorem 2.8, 3.4: The Invertible Matrix Theorem, on pages 112, 235, 275
- Theorem 4.8: Any n-dimensional vector space looks like $\mathbb{R}^n$
- Theorem 4.14: $\dim(\mathrm{Row}(A)) + \dim(\mathrm{Nul}(A)) = n$
- Theorem 5.5 Diagonalization Theorem
- Theorem 6.3: $(\mathrm{Row}(A))^\perp = \mathrm{Nul}(A)$ and $(\mathrm{Col}(A))^\perp = \mathrm{Nul}(A^\top)$
- Theorem 6.9: Best Approximation Theorem
- Theorem 7.3 Spectral Theorem for Symmetric Matrices

Most Important Topic We Did Not Cover

- Singular value decomposition (Section 7.4)

If you take MATH/COMP 365, we'll cover this in depth. If not, I encourage you to have a quick read of Section 7.4 in the book.