Navier Stokes Equations

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Abstract

This project discusses how to solve Navier-Stokes equations by mixed element methods. It begins with the Stokes equations solver, and also its AMG block precondition skills; then a Newton iteration is designed to deal the nonlinear discretized system of Navier-Stokes equations. For large viscosity parameter (≥ 0.02, or RE≤ 50), the solutions of steady Navier-Stokes equations can be solved directly by using an ILU preconditioner, while for the large viscosity (≥ 0.0005, or RE≤ 2000) cases, a time dependent Navier-Stokes solver is implemented to approximate the steady solution by making the computing time sufficient large. A series bentchmark simulations are fulfiled to check the correctness of the algorithms and codes.

1 Stokes Equations

1.1 Weak Form

The Stokes equations we discussed take the form as

$$\begin{array}{rcl} -\nabla^2\vec{u}+\nabla p &=& \vec{0} \\ \nabla\cdot\vec{u} &=& \vec{0} \end{array} \tag{1}$$

The equations (1) are the fundamental model of viscous flow with boundary values considered that

$$\vec{u}=\vec{w}\ on\ \partial\Omega_D, \quad \frac{\partial\vec{u}}{\partial n}-\vec{n}p=\vec{s}\ on\ \partial\Omega_N. \tag{2}$$

Here the variable p is the pressure.

We use the P2-P1 mixed elements, or famous as Taylor-Hood elements [Lee05] to discrete the equations. And the weak form of the equations (1) reads

$$\begin{array}{rcl} -\int_\Omega \nabla^2\vec{u}\cdot\vec{v}+\nabla p &=& 0 \\ \int_\Omega q\nabla\cdot\vec{u} &=& 0 \end{array} \tag{3}$$

It can be reformed by applying Green Theorem as

$$\begin{array}{rcl} \int_\Omega \nabla\vec{u}:\nabla\vec{v}-\int_\Omega p\nabla\cdot\vec{v} &=& \int_{\partial\Omega_N}\vec{s}\cdot\vec{v} \quad for\ all\ \vec{v}\in H^1_{E_0} \\ \int_\Omega q\nabla\cdot\vec{u} &=& 0 \quad for\ all\ q\in L_2(\Omega) \end{array} \tag{4}$$

Assume that the basis functions of the velocity space are $\vec{\phi}_j$ such that

$$\vec{u}_h=\sum_{j=1}^{n_u}u_j\vec{\phi}_j+\sum_{j=n_u+1}^{n_u+n_\partial}u_j\vec{\phi}_j, \tag{5}$$

while the basis functions of the pressure space are ψ_k such that

$$p_h=\sum_{k=1}^{n_p}p_k\psi_k, \tag{6}$$

then the weak form can be expressed by the following linear system:

$$\begin{bmatrix} A & B^T \\ B & 0 \end{bmatrix}\begin{bmatrix} u \\ p \end{bmatrix}=\begin{bmatrix} f \\ g \end{bmatrix} \tag{7}$$

Here the matrix A and B are given by

$$\begin{array}{rcll} A &=& [a_{ij}], & a_{ij} = \int_\Omega \nabla\vec{\phi}_i : \nabla\vec{\phi}_j, \\ B &=& [b_{kj}], & b_{kj} = -\int_\Omega \psi_k \nabla\cdot\vec{\phi}_j, \end{array} \tag{8}$$

respectively.

1.2 Matrix Form

Specially, we set the $\vec{\phi}_i$ a standard basis and $n_u = 2n$, then

$$\{\vec{\phi}_1, \vec{\phi}_2, \cdots, \vec{\phi}_{2n}\} := \{(\phi_1, 0)^T, \cdots, (\phi_n, 0)^T, (0, \phi_1)^T, \cdots, (0, \phi_n)^T\} \tag{9}$$

With $u := ([u_x]_1, \cdots, [u_x]_n, [u_y]_1, \cdots, [u_y]_n)$, system (7) can be rewritten as

$$\begin{bmatrix} A & 0 & B_x^T \\ 0 & A & B_y^T \\ B_x & B_y & 0 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \\ p \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \\ g \end{bmatrix} \tag{10}$$

where the $n \times n$ matrix A is the scalar Laplacian matrix, and the $n_p \times n$ matrices B_x and B_y represent weak derivatives in the x and y direction.

$$\begin{array}{rcll} A &=& [a_{ij}], & a_{ij} = \int_\Omega \nabla\phi_i : \nabla\phi_j \\ B_x &=& [b_{x,ki}], & b_{x,ki} = -\int_\Omega \psi_k \frac{\partial\phi_i}{\partial x} \\ B_y &=& [b_{y,kj}], & b_{y,kj} = -\int_\Omega \psi_k \frac{\partial\phi_j}{\partial y} \end{array} \tag{11}$$

While the basis functions given by mixed finite element methods are piecewise linear, the A is symmetric positive definite and highly sparse, resulting in the linear system symmetric positive defined and highly sparse. So we have lots of methods to solve the linear system (7) there.

1.3 Precondition

Now that we have a symmetric positive definite matrix, some proper precondition is needed to improve computing efficiency. There is a block AMG preconditioner for Stokes matrixes.

$$M = \begin{bmatrix} A & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & Q \end{bmatrix} \tag{12}$$

where Q is a $n_p \times n_p$ pressure quality matrix.

$$Q = [q_{ij}], \quad q_{ij} = \int_\Omega \psi_i \psi_j \tag{13}$$

Specially, as we just need a preconditioner and Q is a diagonally dominant matrix, we can set Q a diagonal matrix. It makes the matrix easier but has a similar effect.

1.4 Result

We consider the question given that: $\Omega = [0, 1] \times [0, 1]$, the boundary value

$$\left\{ \begin{array}{rcll} u_x &=& 1, & on\ \{(x, y) : x \in [0, 1]\ and\ y = 1\} \\ u &=& 0, & others \end{array} \right. \tag{14}$$

The computing environment is personal computer. The mesh relies on easymesh and the finite element methods rely on AFEPack and dealII 8.1.0. We set $h = 0.05$, get the result that

From the Figure 1, we can see that the Stokes system is symmetric and stable. There are two counter-rotating recirculations(called Moffatt eddies). The figure gives us a brief understanding of fluid system.

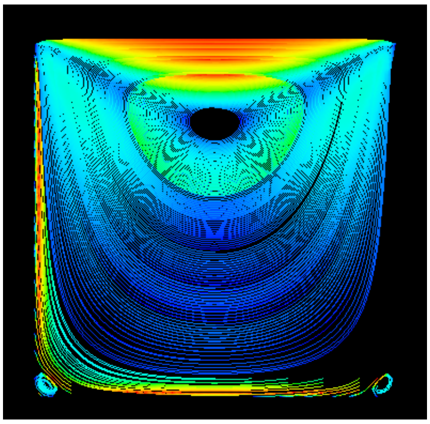

Figure 1: Stokes Solution

2 Navier Stokes

2.1 Weak Form

After solved the Stokes equations, let's consider the Navier-Stokes equations:

$$\begin{array}{rcl} -\nu\nabla^2\vec{u} + \vec{u}\cdot\nabla\vec{u} + \nabla p & = & \vec{F} \\ \nabla\cdot\vec{u} & = & \vec{0} \end{array} \tag{15}$$

As we did before, we also have a standard weak formulation that

$$\begin{array}{rcl} -\nu\int_\Omega \nabla^2\vec{u}\cdot\vec{v} + \int_\Omega \vec{u}\cdot\nabla\vec{u}\cdot\vec{v} + \nabla p & = & \int_\Omega \vec{F}\cdot\vec{v}, \quad for\ all\ \vec{v}\in H^1_{E_0} \\ \int_\Omega q\nabla\cdot\vec{u} & = & 0, \quad for\ all\ q\in L_2(\Omega) \end{array} \tag{16}$$

Now we can't solve the problem directly because of the nonlinear item $\vec{u}\cdot\nabla\vec{u}$. So newton iteration is used:

When we calculate $A(x) = b$, it means we calculate $A(x) - b = 0$. The newton iteration is that $x_{n+1} = x_n - (A(x_n) - b)\cdot J(A(x_n))$. $J(A)$ means A's Jacobi matrix. So we have

$$J(A(x_n))\Delta x = b - A(x_n)$$

Having same basis functions as (9), we get a similar matrix system that

$$\begin{bmatrix} A+N+W_{xx} & W_{xy} & B_x^T \\ W_{yx} & A+N+Wyy & B_y^T \\ B_x & B_y & 0 \end{bmatrix} \begin{bmatrix} \Delta u_x \\ \Delta u_y \\ \Delta p \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \\ g \end{bmatrix} \tag{17}$$

Where A and B are same and the matrix N is the $n\times n$ scalar convection matrix

$$N = [n_{ij}], \quad n_{ij} = \int_\Omega (\vec{u}_h\cdot\nabla\phi_j)\phi_i \tag{18}$$

The $n\times n$ matrices $W_{xx}, W_{xy}, W_{yx}, W_{yy}$ represent weak derivatives of the current velocity in the x and y directions

$$W_{xy} = [w_{xy,ij}], \quad w_{xy,ij} = \int_\Omega \frac{\partial u_x}{\partial y}\phi_u\phi_j \tag{19}$$

In the iteration, the right item is the residual. So we get the f and g value that

$$f = [f_i], \quad f_i = \int_\Omega (\vec{F}\cdot\vec{\phi}_i - \vec{u}_h\cdot\nabla\vec{u}_h\cdot\vec{\phi}_i - \nu\nabla\vec{u}_h : \nabla\vec{\phi}_i + p_h(\nabla\cdot\vec{\phi}_i)) \tag{20}$$

$$g = [g_k], \quad g_k = \int_\Omega \psi_k(\nabla\cdot\vec{u}_h) \tag{21}$$

When we calculate f in x and y directions, it changes that

$$\begin{array}{rcl} f_x & = & [f_{xi}], \quad f_{xi} = \int_\Omega (F_x\cdot\phi_i - \vec{u}_h\cdot\nabla u_x\cdot\phi_i - \nu\nabla u_x : \nabla\phi_i + p_h(\nabla\cdot\phi_i)) \\ f_y & = & [f_{yi}], \quad f_{yi} = \int_\Omega (F_y\cdot\phi_i - \vec{u}_h\cdot\nabla u_y\cdot\phi_i - \nu\nabla u_y : \nabla\phi_i + p_h(\nabla\cdot\phi_i)) \end{array} \tag{22}$$

With a boundary value we can calculate specific problems now.

2.2 Result

First we consider one problem which has an accurate solution that

$$u_x = 1 - y^2; \quad u_y = 0; \quad p = -2\nu x; \tag{23}$$

This is called Poiseuille channel flow. The square domain $\Omega = (-1,1)^2$. We set mesh $h = 0.2,\ 0.1,\ 0.05,\ 0.02$ and $\nu = 1$, then plot the image about numerical error $||u_h - u_0||_2$ and h.

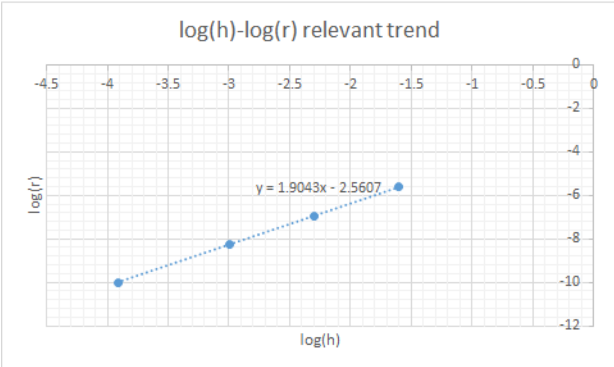

Figure 2: the relevance between log(h) and log(r)

The slope of the figure (2) is 1.9043. We can see our method converges with a about $O(h^2)$ order.

Then under same boundary value as (14), let's see what difference will appear between Navier-Stokes fluid and Stokes fluid when $\nu = 1,\ 0.1,\ 0.01,\ h = 0.05$. (Without precondition we can hardly do matrix calculation, and matrix system is not symmetric again. So we use ilu precondition here provided by dealII.)

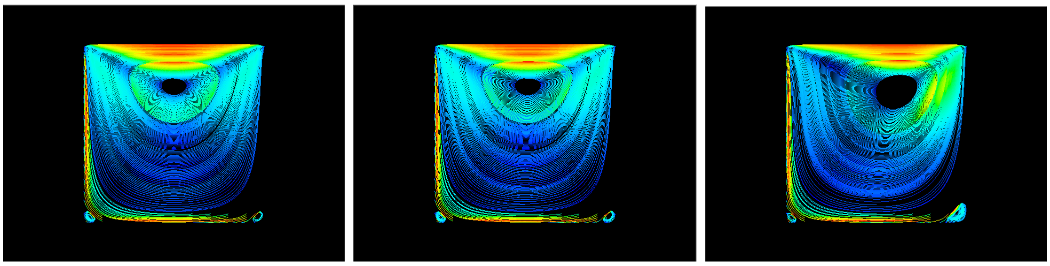

Figure 3: the Navier-Stokes Solution(ν=1, 0.1, 0.01)

We can see that the fluid is not symmetric again. With the viscosity growing, the Moffatt eddies become more unstable. The right side corner eddy catches more energy from the prime eddy. We continue calculating until the viscosity comes to 0.0001. If the viscosity becomes smaller, the process even don't converge. New methods is needed now.

2.3 Time Depending

We consider the equations that

$$\begin{aligned} \frac{\partial \vec{u}}{\partial t} - \nu\nabla^2\vec{u} + \vec{u}\cdot\nabla\vec{u} + \nabla p &= \vec{F} \\ \nabla\cdot\vec{u} &= \vec{0} \end{aligned} \tag{24}$$

The difference is that we add a time-control item to Navier-Stokes equation. When this equations come to converge, it has a same convergence as Navier-Stokes. These two equations with same boundary values will converge to a same steady state.

We use implicit format that $\frac{\partial u}{\partial t} = \frac{u^{n+1}-u^n}{\Delta t}$ in x and y directions. So the iteration becomes that

$$\begin{bmatrix} A+N+W_{xx}+T_x & W_{xy} & B_x^T \\ W_{yx} & A+N+Wyy+T_y & B_y^T \\ B_x & B_y & 0 \end{bmatrix} \begin{bmatrix} \Delta u_x \\ \Delta u_y \\ \Delta p \end{bmatrix} = \begin{bmatrix} f_x + t_x \\ f_y + t_y \\ g \end{bmatrix} \tag{25}$$

The T is the $n \times n$ time developing matrix

$$T = [t_{ij}], \quad t_{ij} = \frac{1}{\Delta t} \int_\Omega \vec{\phi}_i \vec{\phi}_j \tag{26}$$

and the vector t is the former velocity value

$$t = [t_k], \quad t_k = \frac{1}{\Delta t} \int_\Omega \vec{u} \vec{\phi}_i \tag{27}$$

If the time is small enough, the solution must converge to the steady solution. Now let's try to solve the Navier-Stokes equations while the $\nu = 0.00005 (Re = 2000)$. The solution is that from

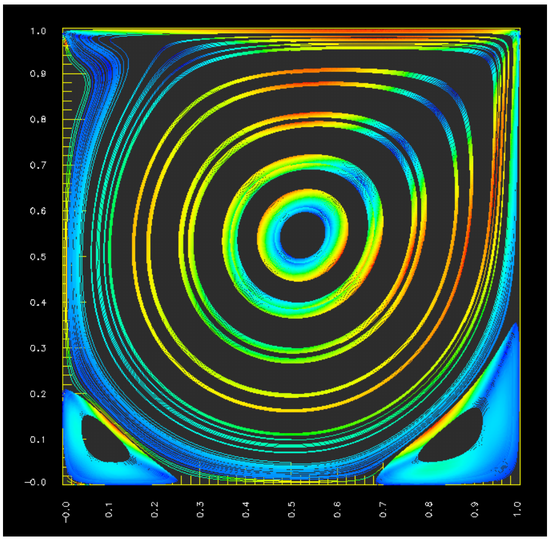

Figure 4: the Navier-Stokes fluid while $\nu = 0.00005$

the figure (4), we can find that new eddy comes to appear and the fluid becomes unstable. When the Re($\frac{1}{\nu}$) grows again, the model can't explain the phenomenon any more. It needs to introduce more theories.

3 Future Work

3.1 3D

Actually we just calculate the problem in the plane. For practical calculating, we need to extend it to 3D space. It doesn't exist essential difficulty. We just change the matrix (17) that

$$\begin{bmatrix} A+N+W_{xx} & W_{xy} & W_{xz} & B_x^T \\ W_{yx} & A+N+Wyy & W_{yz} & B_y^T \\ W_{zx} & W_{zy} & A+N+W_{zz} & B_z^T \\ B_x & B_y & B_z & 0 \end{bmatrix} \begin{bmatrix} \Delta u_x \\ \Delta u_y \\ \Delta u_z \\ \Delta p \end{bmatrix} = \begin{bmatrix} f_x \\ f_y \\ f_z \\ g \end{bmatrix} \tag{28}$$

The difficulty is that we use easymesh to build the 2D mesh. Now we need to find new mesh-building tools to get the mesh.

3.2 Precondition

When the matrix size grows, better precondition is needed. Considering the problem (24), we use semi-implicit scheme that

$$
\begin{array}{rcl}
\frac{\vec{u^{n+1}}-\vec{u^n}}{\Delta t} - \nu\nabla^2\vec{u^{n+1}} + \vec{u^n}\cdot\nabla\vec{u^n} + \nabla p^{n+1} & = & \vec{F} \\
\nabla\cdot\vec{u^{n+1}} & = & \vec{0}
\end{array}
\tag{29}
$$

Thus we change the equations to a time-developinig Stokes equations. Now we can use proper precondition like block AMG precondition given before to make the calculating easier.

References

[Lee05] Barry Lee. *Finite elements and fast iterative solvers : with applications in incompressible fluid dynamics*. 2005.