Morse Theory and linkages

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MATH075

Morse Theory

- H.C. Marston Morse in the 1930's

- Idea: Study the topology of a space by looking at differentiable functions on that space.

- Idea: Interesting things only happen at the critical points.

Higher Dimensions

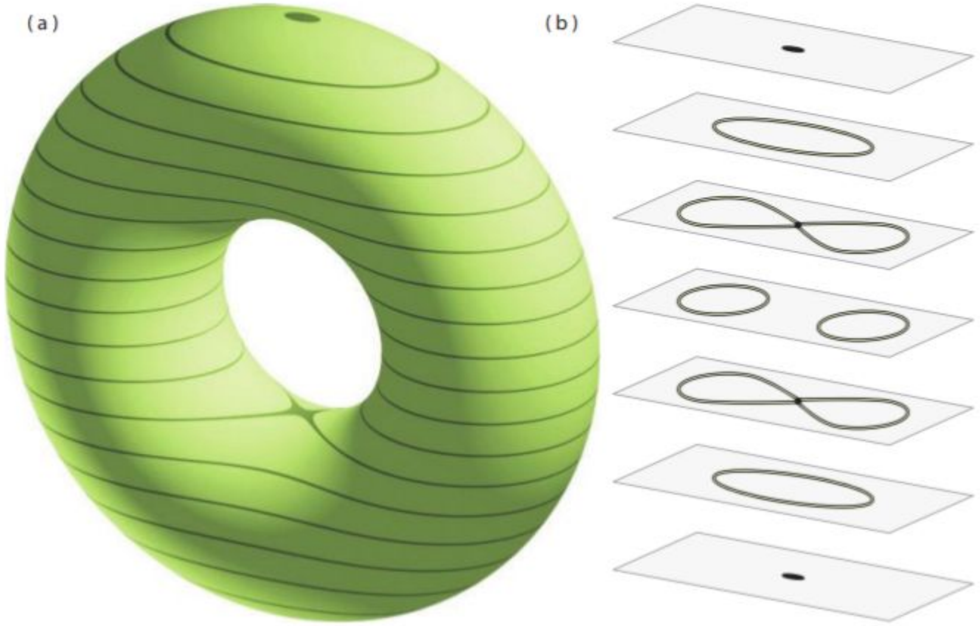

Figure 7.21. (a) Slicing a torus at different heights. (b) Certain slices by the plane Π_z, where every other height depicted here is a critical point.

Devadoss and Rourke, page 234

Different things happen to the topology at each critical point...

The Index: Classifying Critical Points

- Second derivatives?

- What’s the pattern? The amount of “dimensions” that the function is decreasing in.
  - Mins - 0
  - Maxs - 2
  - Saddle points - 1

- Formal: dimension of the negative-definite submatrix of the Hessian.

Okay, so what does each index do?

- Index 0: Removing a circle
- Index 1: Merging or disconnecting
- Index 2: Adding a circle

Higher Dimensions: The index.

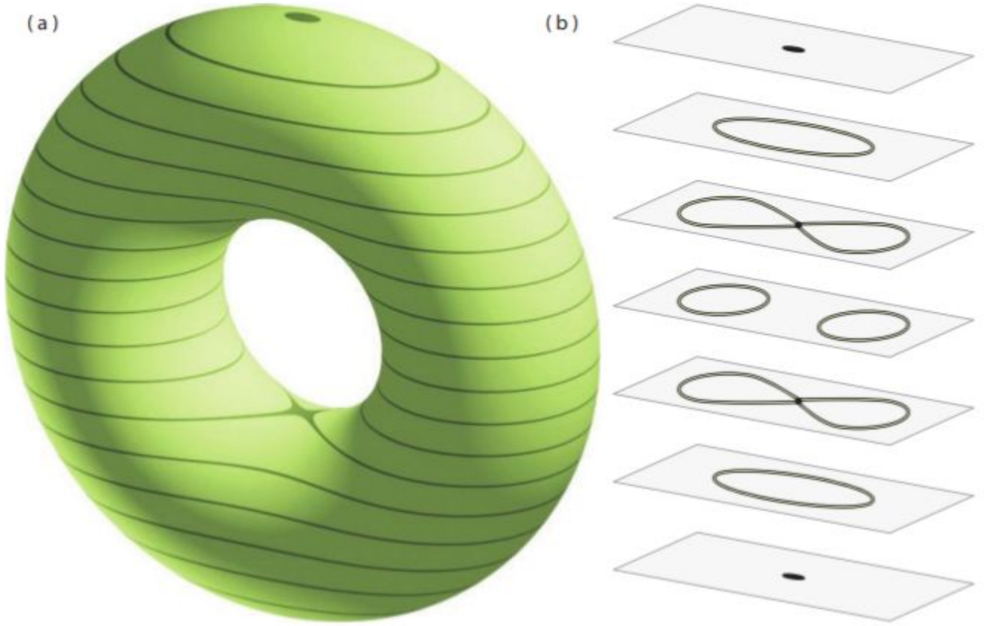

Figure 7.21. (a) Slicing a torus at different heights. (b) Certain slices by the plane Π_z, where every other height depicted here is a critical point.

Index 2

Index 1

Index 1

Index 0

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Recall: Configuration Spaces of linkages

These are manifolds, and we always want to know about the topology of them… Can we use Morse Theory?

- Need a function that has the Configuration Space as a level set.
  - Allow the first length to vary.
  - Level set when the first length is L1

- Theorem: The critical points of this map are the straight-line configurations

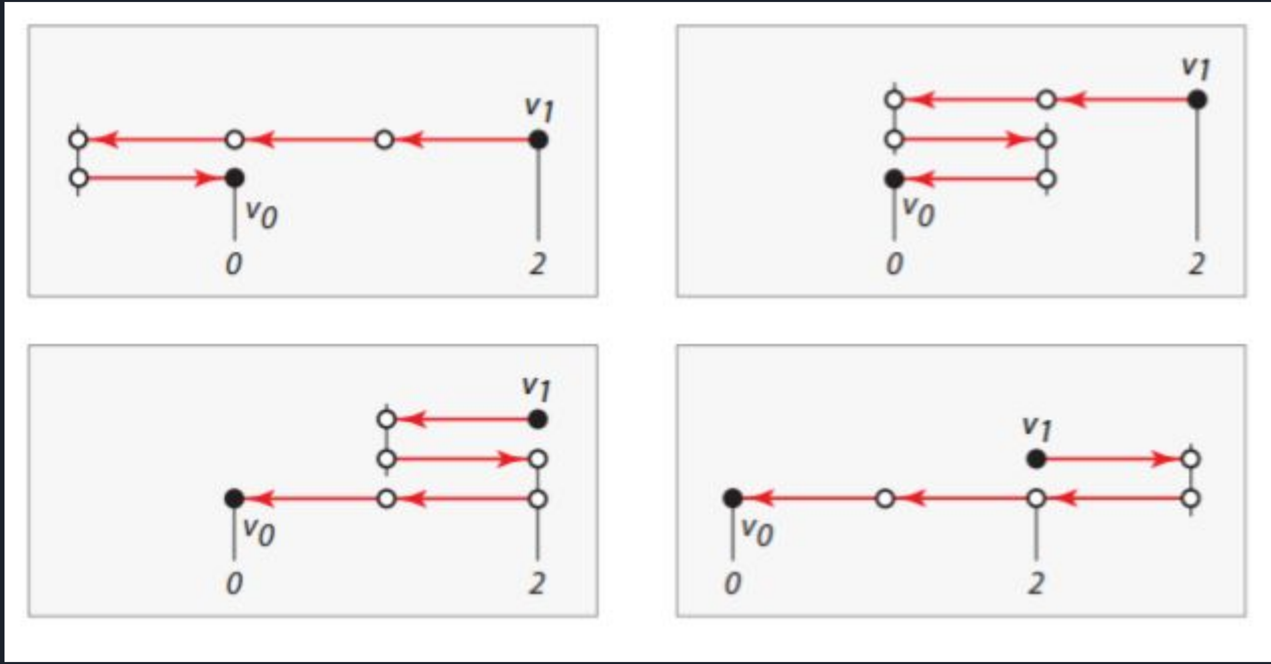
The Equilateral Pentagon

All lengths are the same.

Cinderella

Only one critical point at $z = 4$.

Critical Points at 2



Each of index 2. Devadoss and Rourke page 235

What happens to at the index in higher dimensions?

- Index 3: Sphere appears
- Index 2: Attach a handle
- Index 1: Detach a handle
- Index 0: A sphere disappears

So the topology is...

- Add a sphere (z = 4)
- Add 4 handles (z = 2)
- That’s it! Surface of genus 4.