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Assignment - 3

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(3) $E(W_s W_t) = \min(s,t)$ for $s,t \geq 0$

assume $0 \leq s \leq t$

we can write
$$W_t = W_s + (W_t - W_s)$$

$$E[W_s W_t] = E[W_s(W_s + (W_t - W_s))]$$
$$= E[W_s^2] + E[W_s(W_t - W_s)]$$

Now

- $E[W_s^2] = Var(W_s) = s$

- Since W_s and $W_t - W_s$ are independent and $E[W_s] = 0$

$$E[W_s(W_t - W_s)] = \underbrace{E[W_s] \times E[W_t - W_s]}_{=0}$$

$$E[W_s W_t] = s = \min(s,t)$$

Now Conclusion $\boxed{E[W_s W_t] = \min(s,t)}$

(4) Let $0 \leq s < t$

$$X = W_t - W_s$$

mean

$$E[W_t - W_s] = E[W_t] - E[W_s] = 0 - 0 = 0$$

Variance

$$var(W_t - W_s) = var(W_t) + var(W_s) - 2Cov(W_t, W_s)$$

But from earlier result $E[W_t W_s] = \min(s,t) = s$

$$\text{So } var(W_t - W_s) = t + s - 2s = t - s$$

$$W_t - W_s \sim N(0, t-s)$$

$\longrightarrow$ Let us take two disjoint intervals

$[t_0, t_1]$ $[t_2, t_3]$

such that $t_0 < t_1 < t_2 < t_3$

we consider $X = W_{t_1} - W_{t_0}$
$Y = W_{t_3} - W_{t_2}$

Brownian motion property $(W_{t_1} - W_{t_0}) \perp\!\!\!\perp (W_{t_3} - W_{t_2})$

Now

$$(W_t - W_s) \sim N(0, t-s) \quad \text{for } 0 < s < t$$

increments over disjoint intervals are independent

5. From Brownian motion Property

$$w_t = w_s + (w_t - w_s)$$

$$E[w_t | F_s] = E[w_s + (w_t - w_s) | F_s]$$

$$= E[w_s | F_s] + E[w_t - w_s | F_s]$$

* $E[w_s | F_s] = w_s$ because w_s is F_s- measurable

* $w_t - w_s$ is independent of F_s, since Brownian motion has independent increments

$$E[w_t - w_s | F_s] = E[w_t - w_s] = 0$$

Now $E[w_t | F_s] = w_s + 0 = w_s$

Here - Brownian motion is a Martingale

* X_t is F_t - adapted
* $E[|X_t|] < \infty$ and
* $E[X_t | F_s] = X_s$ for all $0 \leq s < t$

$\{W_t\}_{t \geq 0}$ is a Martingale with respect to its natural filtration F_t