1) a) $f(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \mathcal{F}(k)\, e^{2\pi i \frac{nk}{N}}$

$$f(n) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \left(\frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m)\, e^{-2\pi i \frac{mk}{N}} \right) e^{2\pi i \frac{nk}{N}} = \frac{1}{N} \sum_{m=0}^{N-1} f(m) \sum_{k=0}^{N-1} e^{2\pi i \frac{(n-m)k}{N}}$$

$$f(n) = \sum_{m=0}^{N-1} f(m) \cdot \mathbb{I}\{n=m\} = f(n)$$

$\therefore\ \mathcal{F}^{-1}(\mathcal{F}(k)) = f(n)$

b) $\mathcal{F}[f*g] = \mathcal{F}[f] \cdot \mathcal{F}[g]$

Sé que $f*g := \sum_{m=0}^{N-1} f(m)\, g(n-m)$

$$\mathcal{F}[f*g](k) = \frac{1}{\sqrt{N}} \cdot \sum_{n=0}^{N-1} (f*g)(n)\, e^{-2\pi i \frac{nk}{N}}$$

$$= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} f(m)\, g(n-m)\, e^{-2\pi i \frac{nk}{N}}$$

Sea $n' = n-m$

$$= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m) \sum_{n'=0}^{N-1} g(n')\, e^{-2\pi i \frac{(n'+m)k}{N}}$$

$$= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} f(m)\, e^{-2\pi i \frac{mk}{N}} \cdot \sum_{n'=0}^{N-1} g(n')\, e^{-2\pi i \frac{n'k}{N}}$$

$\therefore\ \mathcal{F}[f*g](k) = \mathcal{F}[f](k) \cdot \mathcal{F}[g](k)$

c) $\mathcal{F}(k) = \mathcal{F}(k+N)$

$$\mathcal{F}(k+N) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n)\, e^{-2\pi i n \frac{(k+N)}{N}}$$

Sé que $e^{-2\pi i n \frac{N}{N}} = 1$

$$= \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n)\, e^{-2\pi i \frac{nk}{N}} = \mathcal{F}(k)$$

d) $\mathcal{F}(-k) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n)\, e^{2\pi i \frac{nk}{N}} = \mathcal{F}^*(k)$

e) Si $f(n)$ es real vale por d, pero como es periódica

$|\mathcal{F}(k)| = |\mathcal{F}(k+N)| = |\mathcal{F}(-k)|$

f) $\mathcal{F}^*(N-k) = \mathcal{F}(k)$

Como $\mathcal{F}(N-k) = \mathcal{F}^*(-k)$, vale $\mathcal{F}^*(N-k) = \mathcal{F}(k)$

g) $\mathcal{F}\left(\frac{N}{2}-k\right) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n)\, e^{-2\pi i n \frac{\left(\frac{N}{2}-k\right)}{N}} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f(n)\, e^{-2\pi i \left(\frac{n}{2} - \frac{nk}{N}\right)}$

$\mathcal{F}^*\left(\frac{N}{2}+k\right) = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} f^*(n)\, e^{2\pi i \left(\frac{n}{2} + \frac{nk}{N}\right)}$ Si $f(n)$ es real $f^*(n) = f(n)$

c) a) i) $a f(m,n) \xrightarrow{DFT} a F(k,l)$

$$F(k,l) = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} f(m,n)\, e^{-2\pi i \frac{(mk+nl)}{N}}$$

$$F'(k,l) = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{M-1} a f(m,n)\, e^{-2\pi i \frac{(mk+nl)}{N}}$$

$$F'(k,l) = \frac{1}{N}\, a \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} f(m,n)\, e^{-2\pi i \frac{(mk+nl)}{N}}$$

$F'(k,l) = a F(k,l)$ $\quad\therefore\ a f(m,n) \xrightarrow{DFT} a F(k,l)$

ii)

$$f(an, bm) \xrightarrow{DFT} \frac{1}{|ab|} F\left(\frac{k}{|a|}, \frac{l}{|b|}\right)$$

$$F'(k,l) = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} f(an, bm)\, e^{-2\pi i \frac{(mk+nl)}{N}}$$

Sea $m' = am$, $n' = bn$

$$F'(k,l) = \frac{1}{|ab| N} \sum_{m'=0}^{N-1} \sum_{n'=0}^{N-1} f(m', n')\, e^{-2\pi i \left(\frac{m'k}{|a|} + \frac{n'l}{|b|}\right) \cdot \frac{1}{N}}$$

$$F'(k,l) = \frac{1}{|ab|} F\left(\frac{k}{|a|}, \frac{l}{|b|}\right)$$

iii) $f(r, \varphi + \varphi_0) \xrightarrow{DFT} F(r, \theta + \varphi_0)$ $\quad x = r\cos\varphi$, $y = r\sin\varphi$ $\quad u = r\cos\theta$, $v = r\sin\theta$

$$f(x', y') = f(r\cos(\varphi + \varphi_0), r\sin(\varphi + \varphi_0))$$

$$x' = r\cos(\varphi + \varphi_0) = r(\cos\varphi\cos\varphi_0 - \sin\varphi\sin\varphi_0) = x\cos(\varphi_0) - y\sin(\varphi_0)$$

$$y' = r\sin(\varphi + \varphi_0) = r(\sin\varphi\cos(\varphi_0) + \cos\varphi\sin(\varphi_0)) = x\sin(\varphi_0) + y\cos\varphi_0$$

$$F(u,v) = \sum_x \sum_y f(x,y)\, e^{-2\pi i (ux + vy)}$$

$$F'(u,v) = \sum_x \sum_y f(x', y')\, e^{-2\pi i (ux + vy)}$$ para $ux + vy = r\cos(\theta)x + r\sin(\theta)y$

b) Para $f(m,n)$ su DFT es $F(k,l) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n)\, e^{-2\pi i \left(\frac{km}{M} + \frac{ln}{N}\right)}$

$$DFT(f(m-m_0, n-n_0)) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m-m_0, n-n_0)\, e^{-2\pi i \left(\frac{km}{M} + \frac{ln}{N}\right)}$$

$m' = m - m_0$, $m = m' + m_0$ y $n' = n - n_0$, $n = n' + n_0$

$$F'(k,l) = \sum_{m'=0}^{M-1} \sum_{n'=0}^{N-1} f(m', n')\, e^{-2\pi i \left(\frac{k(m'+m_0)}{M} + \frac{l(n'+n_0)}{N}\right)} = e^{-2\pi i \left(\frac{km_0}{M} + \frac{ln_0}{N}\right)} \cdot \sum_{m'=0}^{M-1} \sum_{n'=0}^{N-1} f(m', n')\, e^{-2\pi i \left(\frac{km'}{M} + \frac{ln'}{N}\right)}$$

$$F'(k,l) = F(k,l) \cdot e^{-2\pi i \left(\frac{km_0}{M} + \frac{ln_0}{N}\right)}$$

c) La IDFT es $f(m,n) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)\, e^{2\pi i\left(\frac{um}{M}+\frac{vn}{N}\right)}$

$$f'(m,n) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u-u_0, v-v_0)\, e^{2\pi i\left(\frac{um}{M}+\frac{vn}{N}\right)} \qquad \begin{array}{ll} u' = u-u_0 & u = u'+u_0 \\ v' = v-v_0 & v = v'+v_0 \end{array}$$

$$f'(m,n) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u',v')\, e^{2\pi i\left(\frac{u'm}{M}+\frac{v'n}{N}\right)} \cdot e^{2\pi i\left(\frac{u_0 m}{M}+\frac{v_0 n}{N}\right)}$$

$$f'(m,n) = e^{2\pi i\left(\frac{u_0 m}{M}+\frac{v_0 n}{N}\right)} \cdot \frac{1}{MN}\sum_{u'=0}^{M-1}\sum_{v'=0}^{N-1} F(u',v')\, e^{2\pi i\left(\frac{u'm}{M}+\frac{v'n}{N}\right)}$$

$$f'(m,n) = f(m,n)\, e^{2\pi i\left(\frac{u_0 m}{M}+\frac{v_0 n}{N}\right)}$$