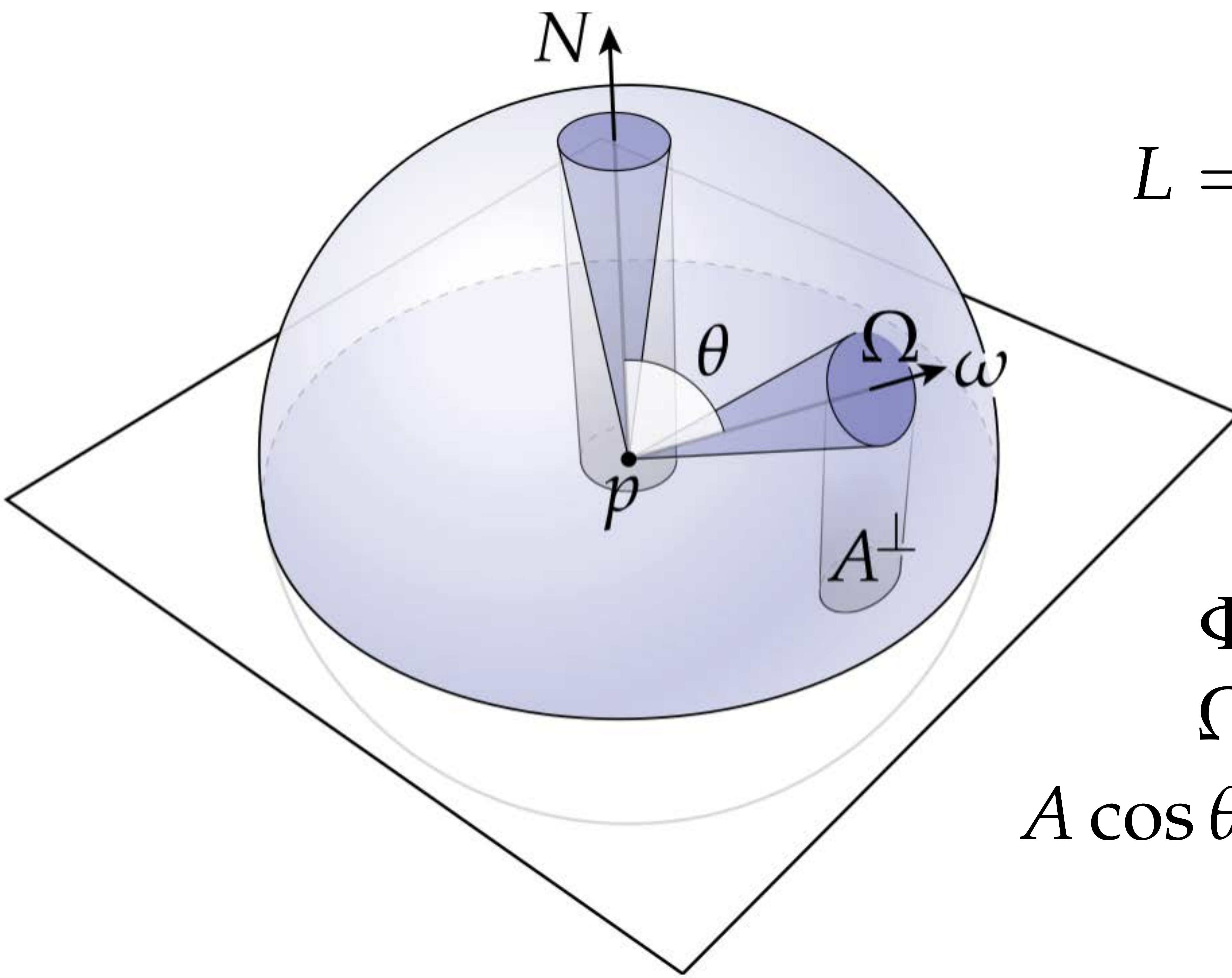
The Rendering Equation

Computer Graphics
CMU 15-462/15-662, Fall 2016

Review: What is *radiance?*

- **Radiance at point p in direction N is radiant energy ("#hits") per unit time, per solid angle, per unit area perpendicular to N.**



$$L = \frac{\partial^2 \Phi}{\partial \Omega \, \partial A \cos\theta}$$

Φ — **radiant energy**

Ω — **solid angle**

$A\cos\theta$ — **projected area**

Intuition: Radiance and Irradiance

irradiance

radiance in direction ω

$$\underbrace{E}_{\text{irradiance}} = \int_{H^2} \underbrace{L(\omega)}_{\text{radiance in direction } \omega} \cos \underbrace{\theta}_{\text{angle between } \omega \text{ and normal}} \, d\omega$$

angle between ω and normal

Incident vs. Exitant Radiance

INCIDENT

EXITANT

In both cases: intensity of illumination is highly dependent on *direction* (not just location in space or moment in time).

The Rendering Equation

- **Core functionality of photorealistic renderer is to estimate radiance at a given point p, in a given direction ω_o**
- **Summed up by the *rendering equation* (Kajiya):**

outgoing/observed radiance

emitted radiance (e.g., light source)

angle between incoming direction and normal

incoming radiance

$$L_o(\mathbf{p}, \omega_o) = L_e(\mathbf{p}, \omega_o) + \int_{\mathcal{H}^2} f_r(\mathbf{p}, \omega_i \rightarrow \omega_o) L_i(\mathbf{p}, \omega_i) \cos\theta \, d\omega_i$$

point of interest

direction of interest

scattering function

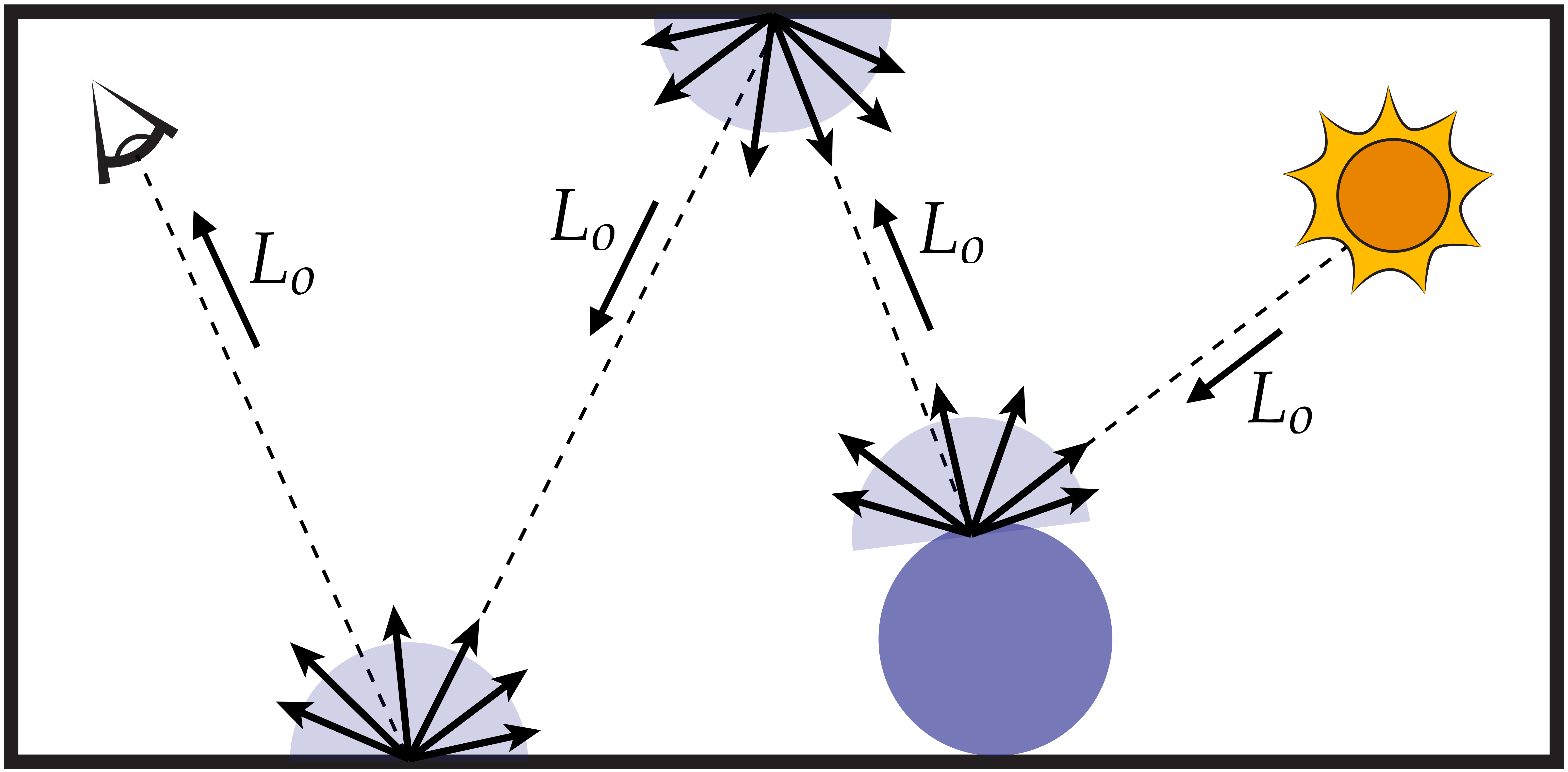
incoming direction

all directions in hemisphere

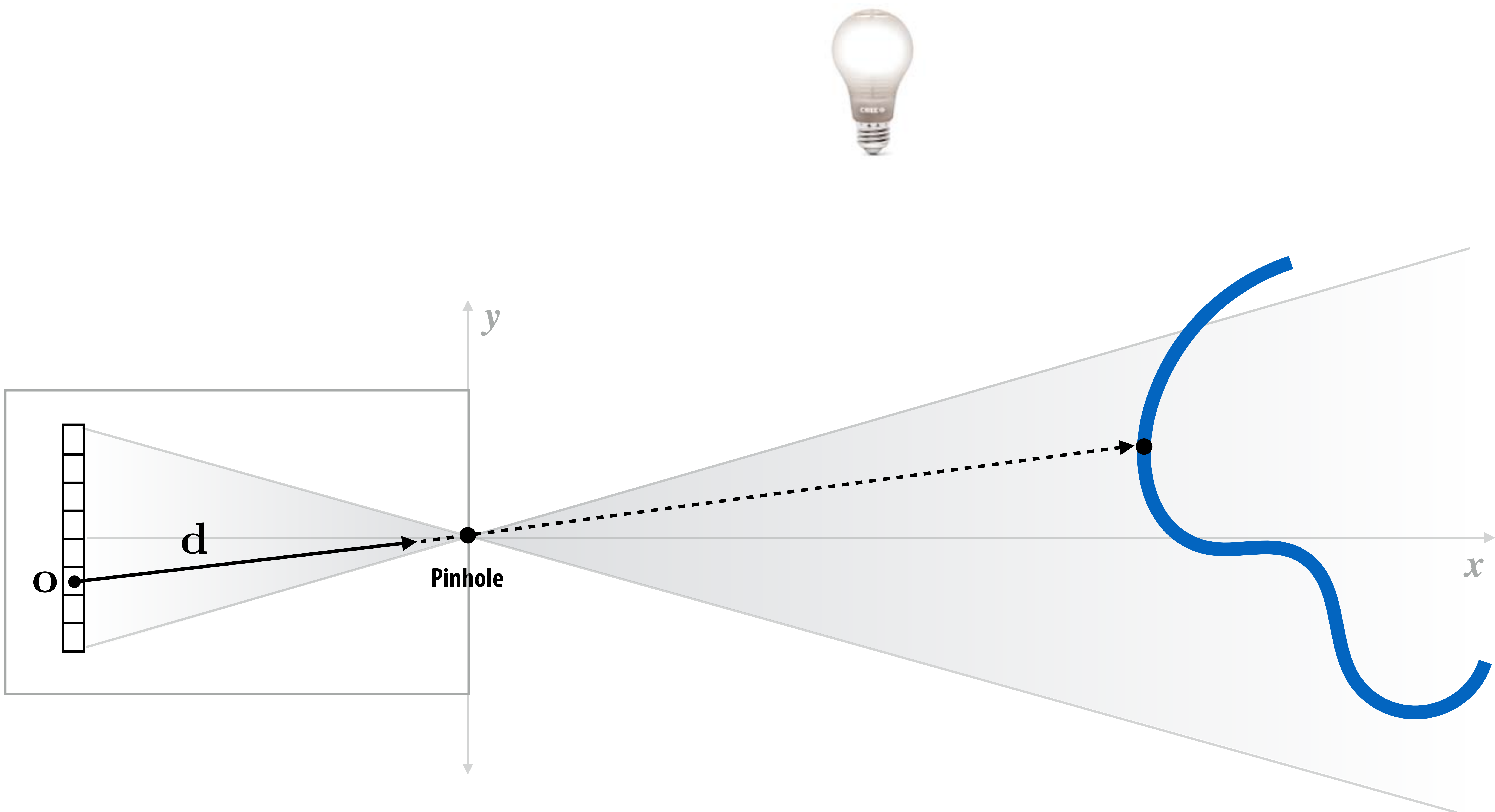
Key challenge: to evaluate incoming radiance, we have to compute yet another integral. I.e., rendering equation is *recursive*.

Recursive Raytracing

- **Basic strategy: recursively evaluate rendering equation!**

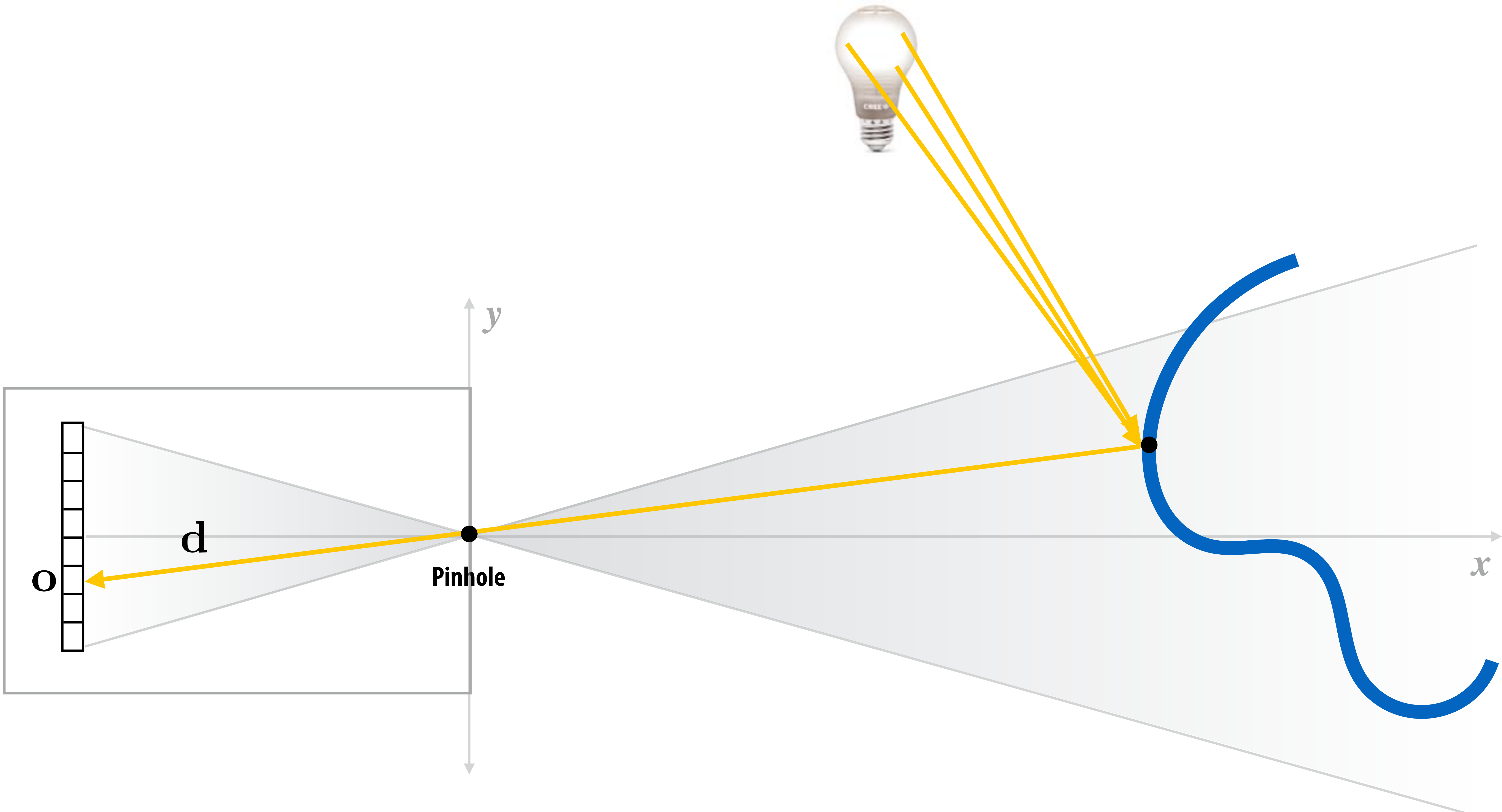


Renderer measures radiance along a ray

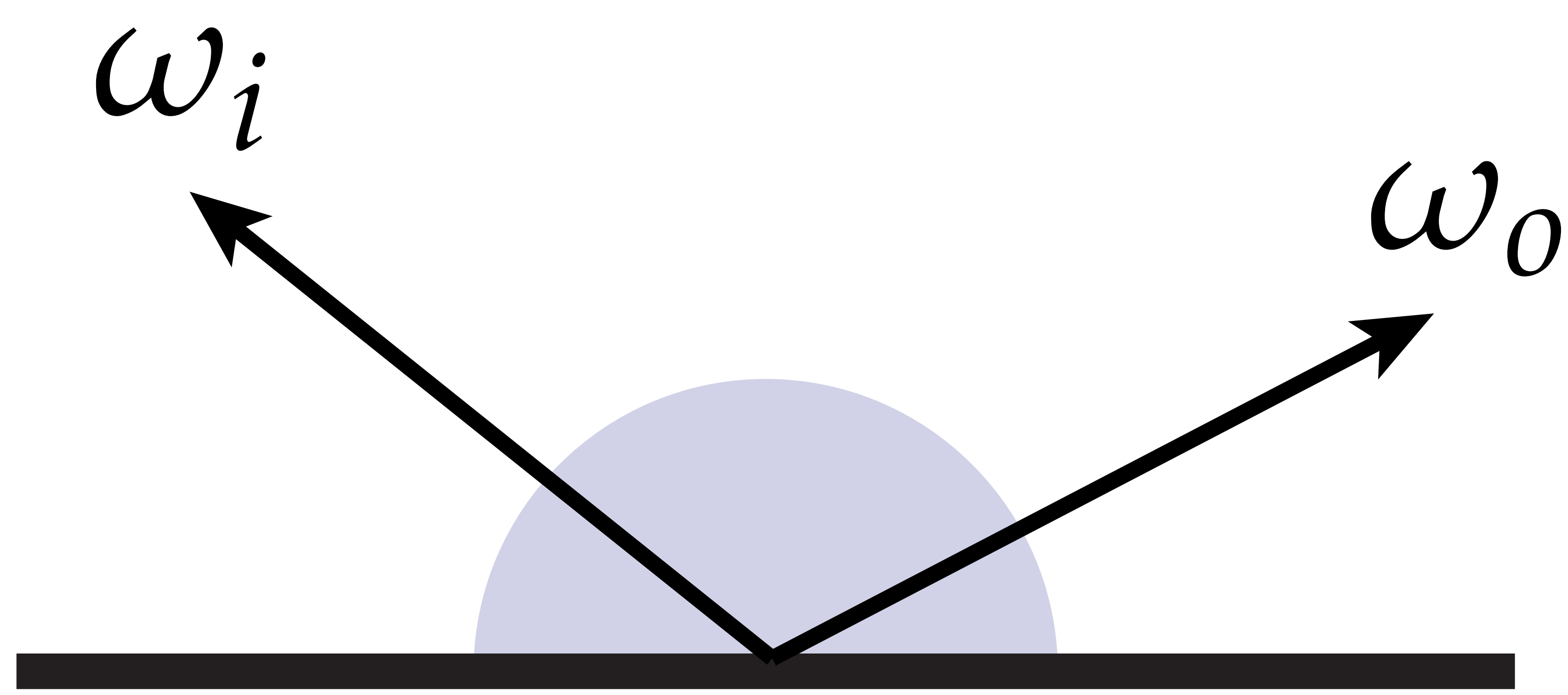


At each "bounce," want to measure radiance traveling in the direction opposite the ray direction.

Renderer measures radiance along a ray



Radiance entering camera in direction *d* = light from scene light sources that is reflected off surface in direction *d*.



How does *reflection* of light affect the outgoing radiance?

$$L_o(\mathbf{p},\omega_o) = \int_{\mathcal{H}^2} f_r(\mathbf{p},\omega_i,\omega_o)\, L_i(\mathbf{p},\omega_i)\cos\theta\,\mathrm{d}\,\omega_i$$

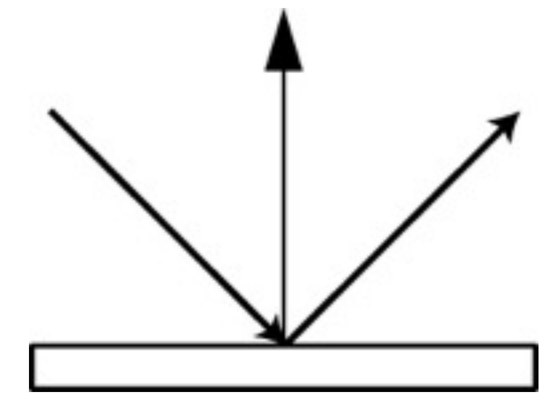
Reflection models

- *Reflection* is the process by which light incident on a surface interacts with the surface such that it leaves on the incident (same) side without change in frequency
- Choice of reflection function determines surface appearance

Categories of reflection functions

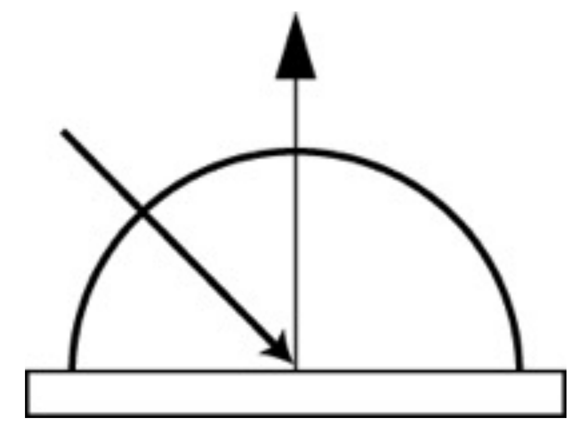
- **Ideal specular**

  Perfect mirror



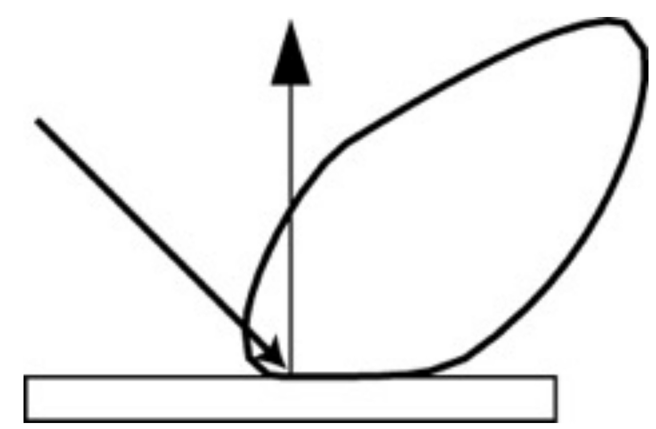
- **Ideal diffuse**

  Uniform reflection in all directions



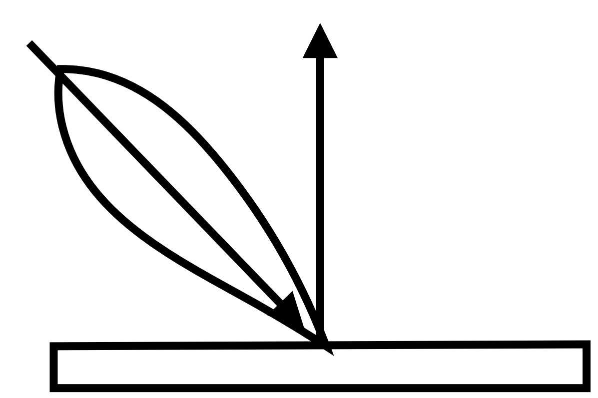

- **Glossy specular**

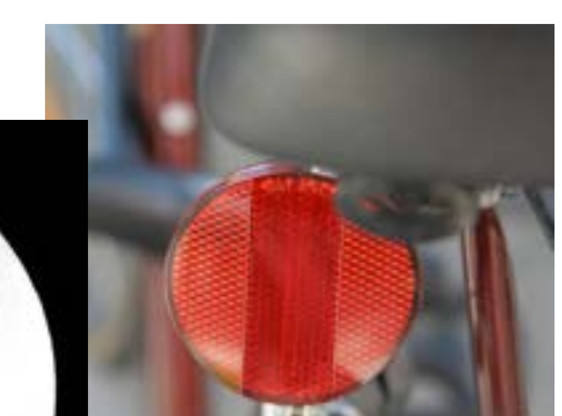
  Majority of light distributed in reflection direction



- **Retro-reflective**

  Reflects light back toward source






Diagrams illustrate how incoming light energy from given direction is reflected in various directions.

Materials: diffuse

Materials: plastic

Materials: red semi-gloss paint

Materials: Ford mystic lacquer paint

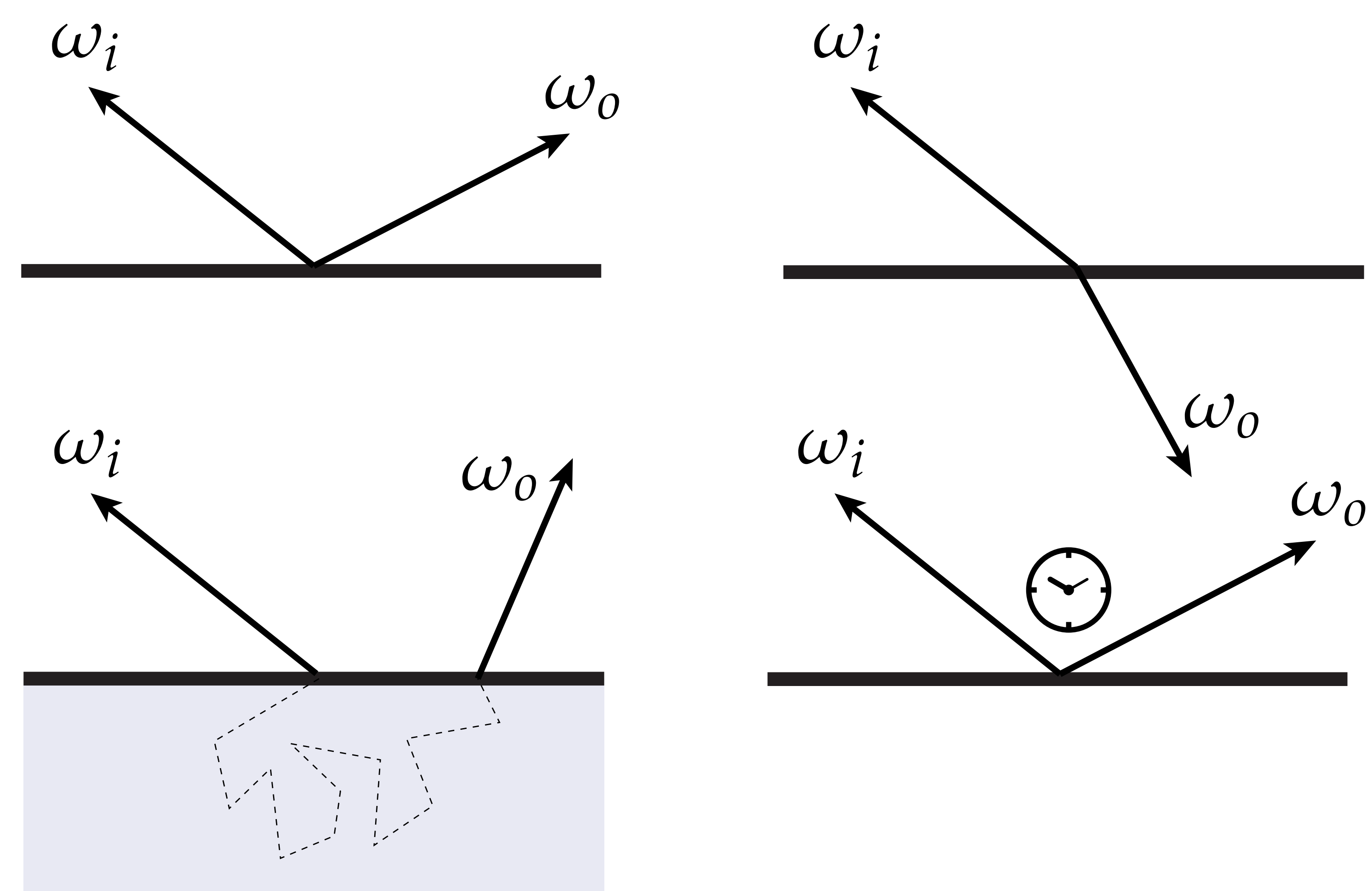
Materials: mirror

Materials: gold

Materials

Models of Scattering

- How can we model "scattering" of light?
- Many different things that could happen to a photon:
  - bounces off surface
  - transmitted through surface
  - bounces around inside surface
  - absorbed & re-emitted
  - ...

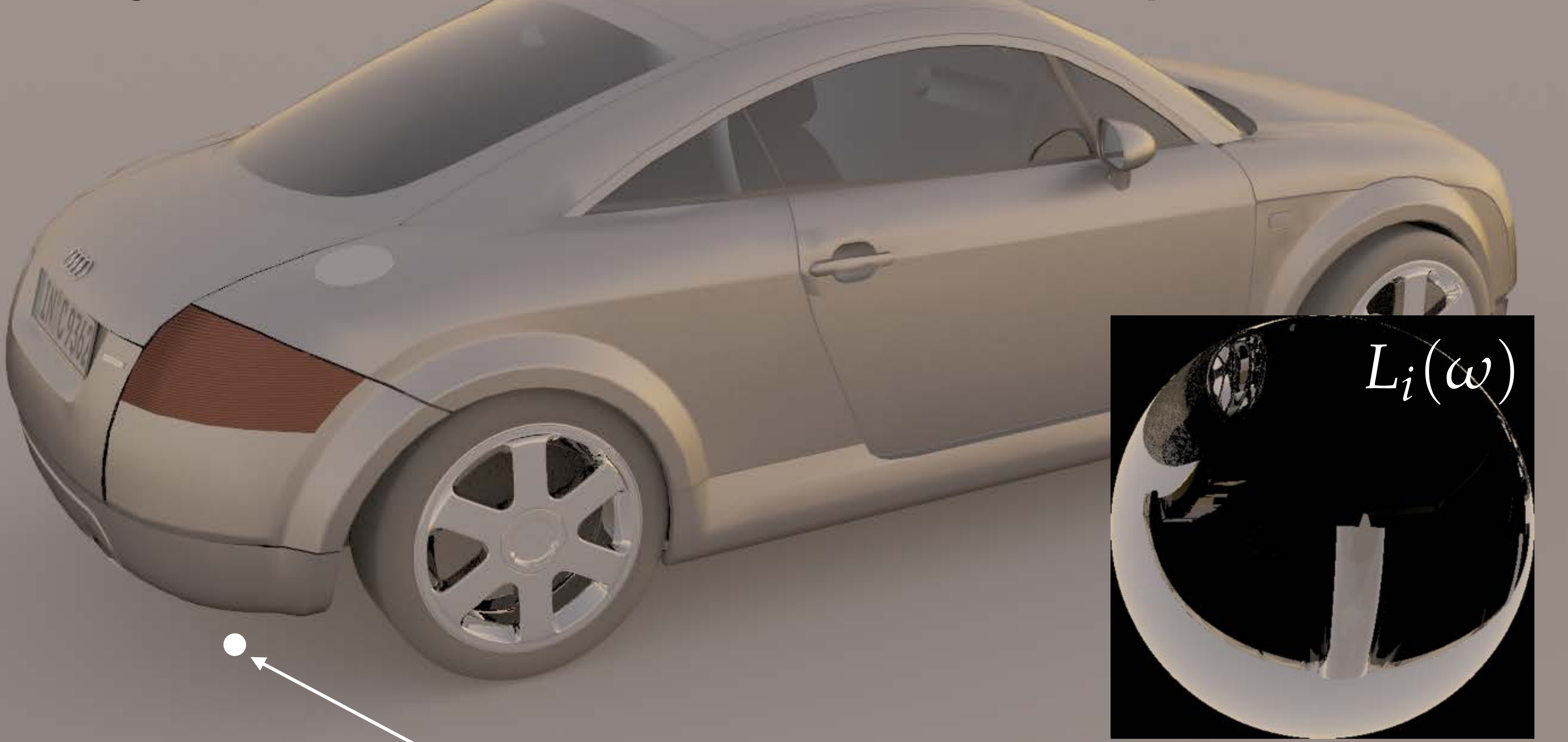


- What goes in must come out! (Total energy must be conserved)
- In general, can talk about *"probability*"* a particle arriving from a given direction is scattered in another direction

*Somewhat more complicated than this, because some light is absorbed!

Hemispherical incident radiance

At any point on any surface in the scene, there's an incident radiance field that gives the directional distribution of illumination at the point

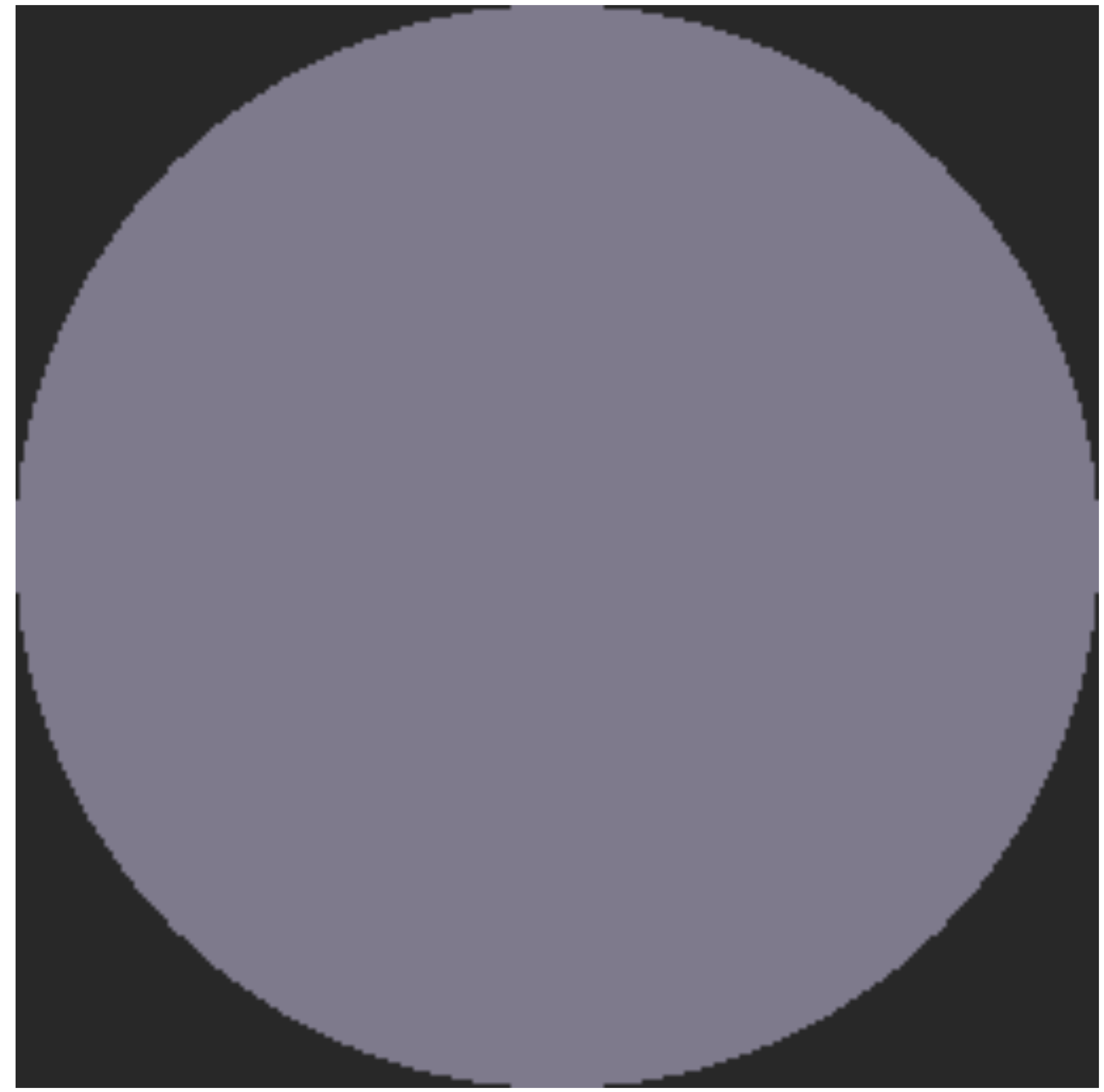


Consider view of hemisphere from this point

Diffuse reflection

Exitant radiance is the same in all directions

Incident radiance

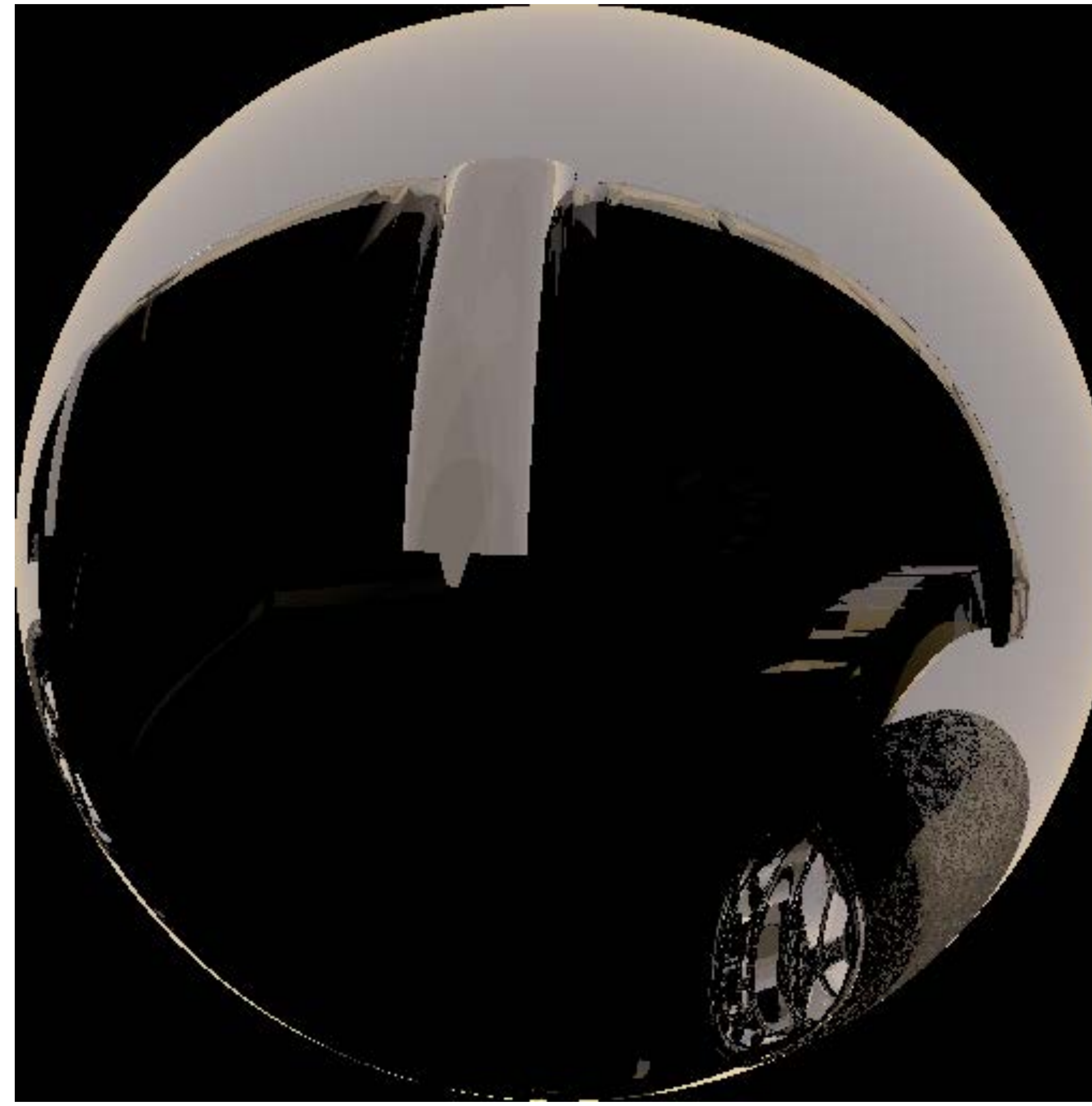


Exitant radiance

Ideal specular reflection

Incident radiance is “flipped around normal” to get exitant radiance

Incident radiance



Exitant radiance

Plastic

Incident radiance gets “flipped and blurred”

Incident radiance

Exitant radiance

Copper

More blurring, plus coloration (nonuniform absorption across frequencies)

Incident radiance

Exitant radiance

Scattering off a surface: the BRDF

- "Bidirectional reflectance distribution function"
- Encodes behavior of light that "bounces off" surface
- Given incoming direction ω_i, how much light gets scattered in any given outgoing direction ω_o?
- Describe as distribution $f_r(\omega_i \rightarrow \omega_o)$

$$f_r(\omega_i \to \omega_o) \geq 0$$

why *less* than or equal?

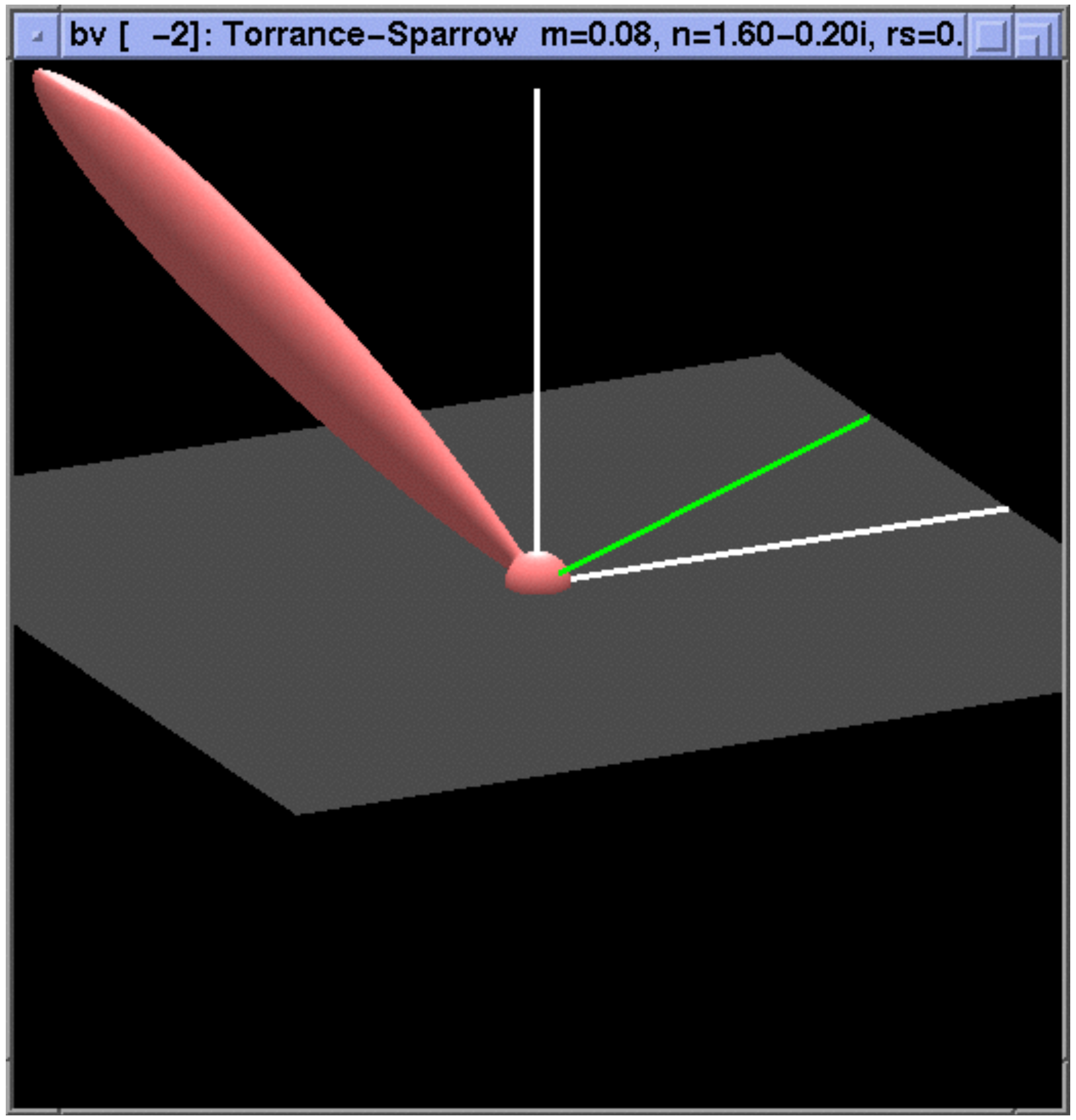
$$\int_{\mathcal{H}^2} f_r(\omega_i \to \omega_o) \ \cos\theta \ d\omega_i \leq 1$$

where did the rest of the energy go?!

$$f_r(\omega_i \to \omega_o) = f_r(\omega_o \to \omega_i)$$

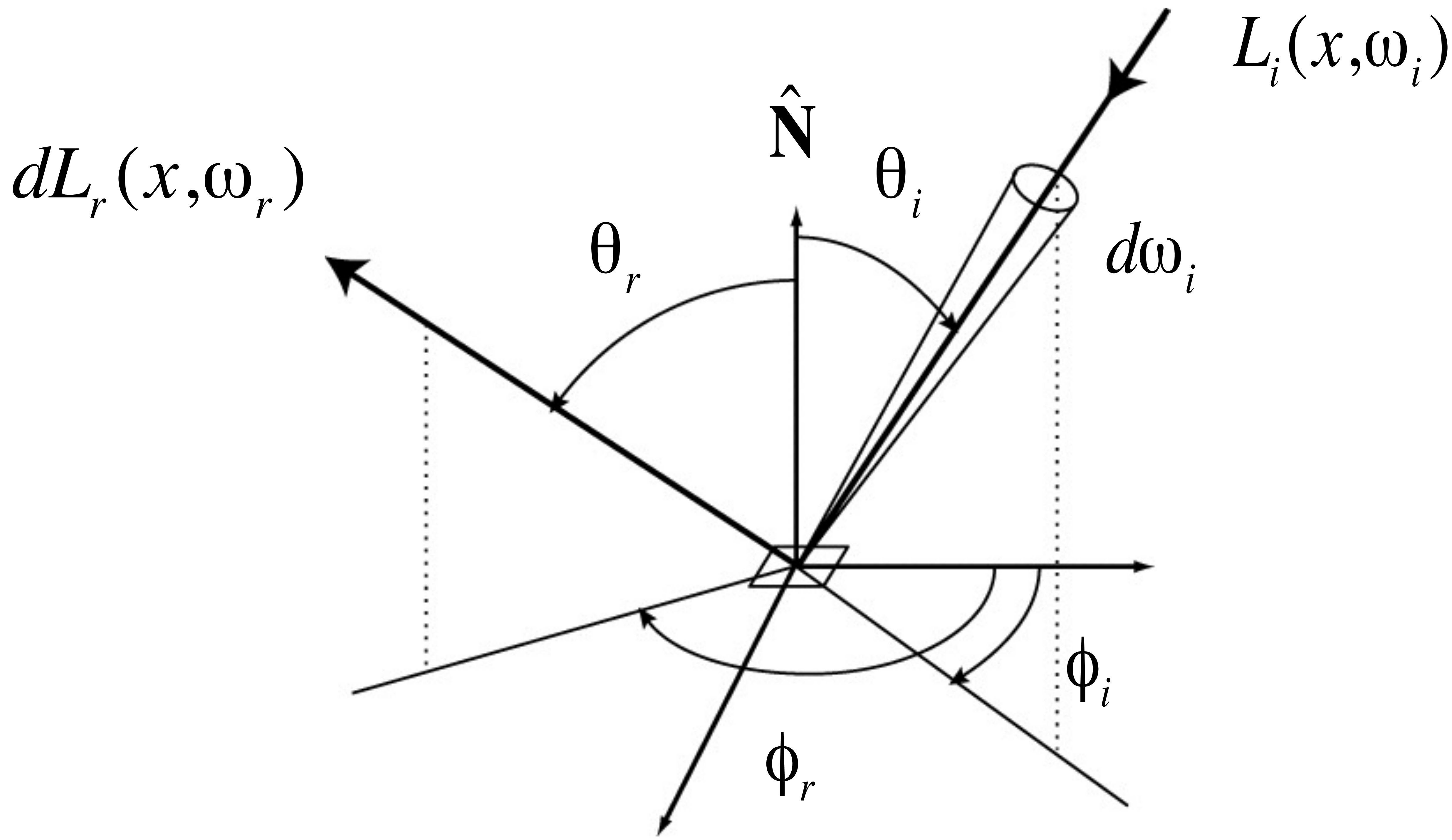
"Helmholtz reciprocity"

Q: Why should Helmholtz reciprocity hold? Think about little mirrors…



bv (Szymon Rusinkiewicz)

Radiometric description of BRDF

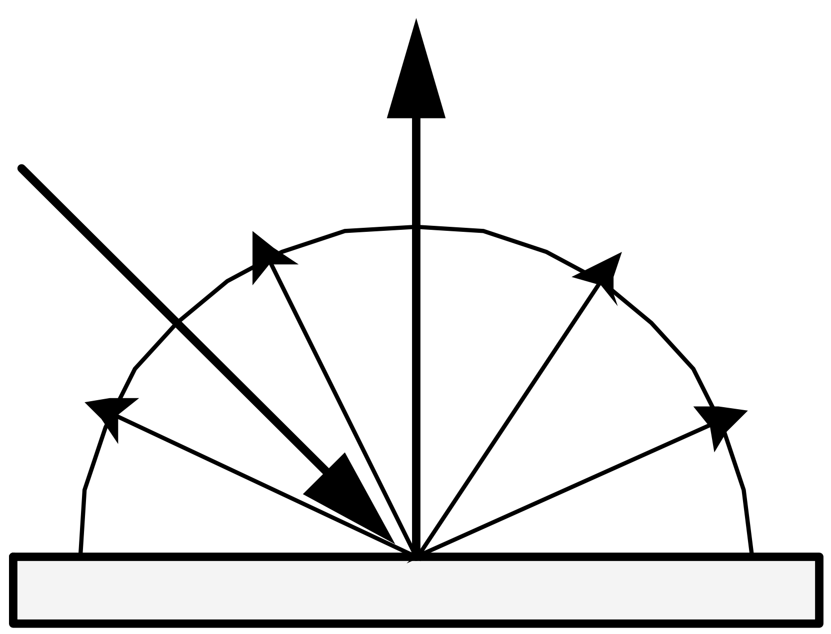


$$f_r(\omega_i \to \omega_o) = \frac{dL_o(\omega_o)}{dE_i(\omega_i)} = \frac{dL_o(\omega_o)}{dL_i(\omega_i)\cos\theta_i} \left[\frac{1}{sr}\right]$$

“For a given change in the incident irradiance, how much does the exitant radiance change?”

Example: Lambertian reflection

Assume light is equally likely to be reflected in each output direction

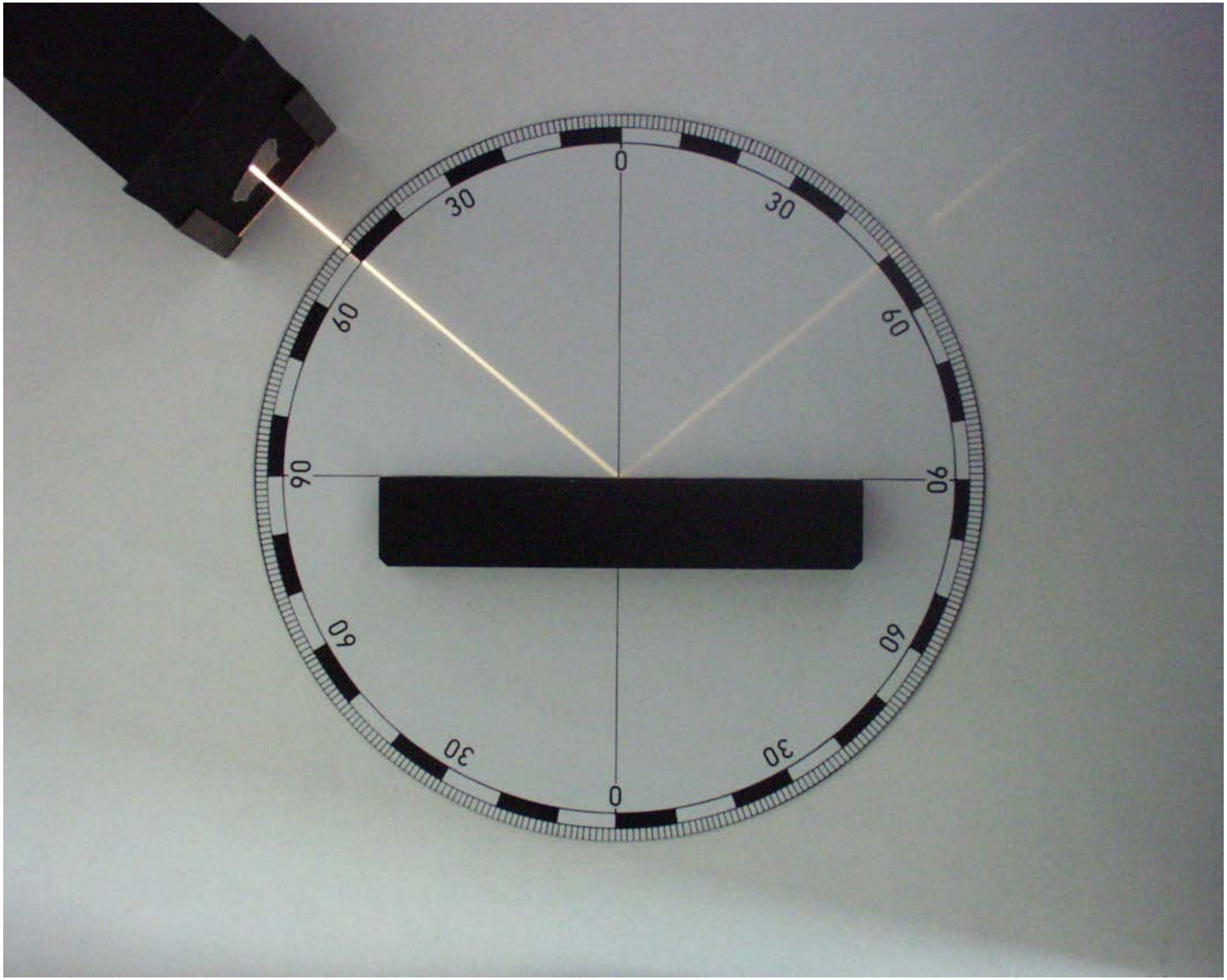


$f_r = c$

$$L_o(\omega_o) = \int_{H^2} f_r \, L_i(\omega_i) \, \cos\theta_i \, \mathrm{d}\omega_i$$
$$= f_r \int_{H^2} L_i(\omega_i) \, \cos\theta_i \, \mathrm{d}\omega_i$$
$$= f_r E$$

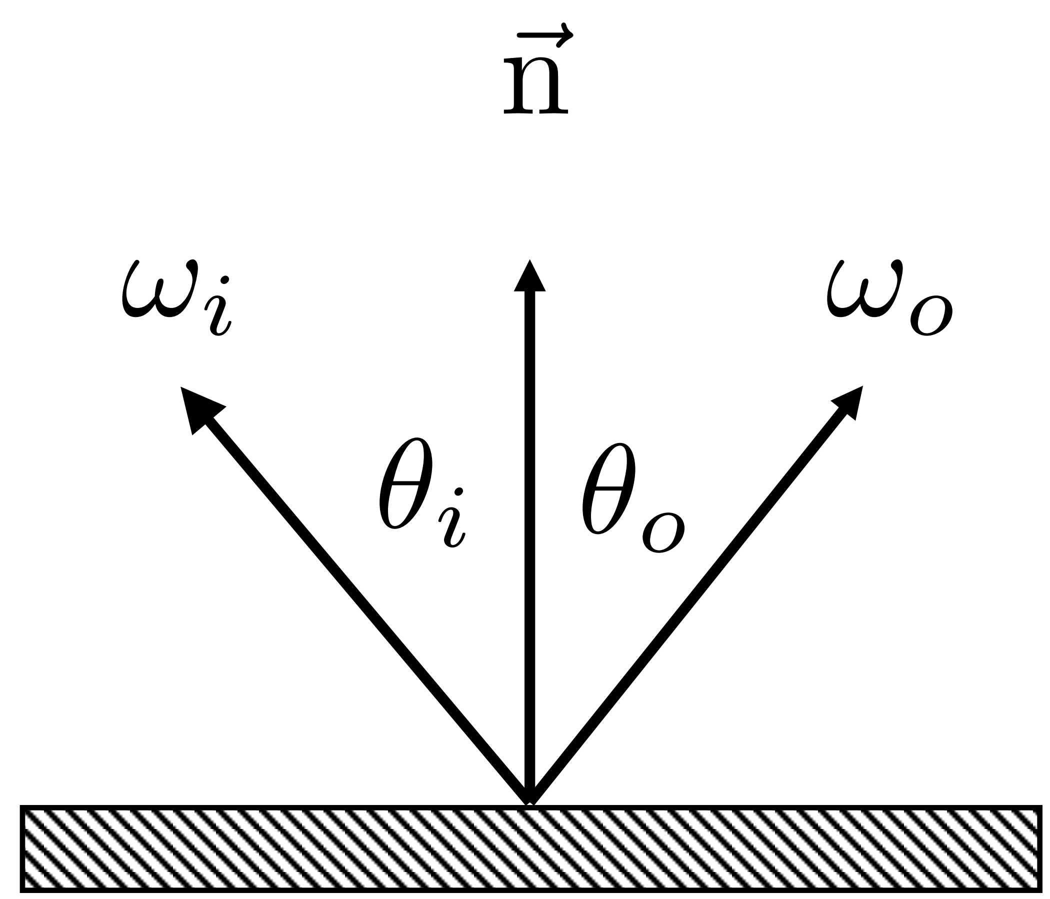
$$f_r = \frac{\rho}{\pi}$$

Example: perfect specular reflection



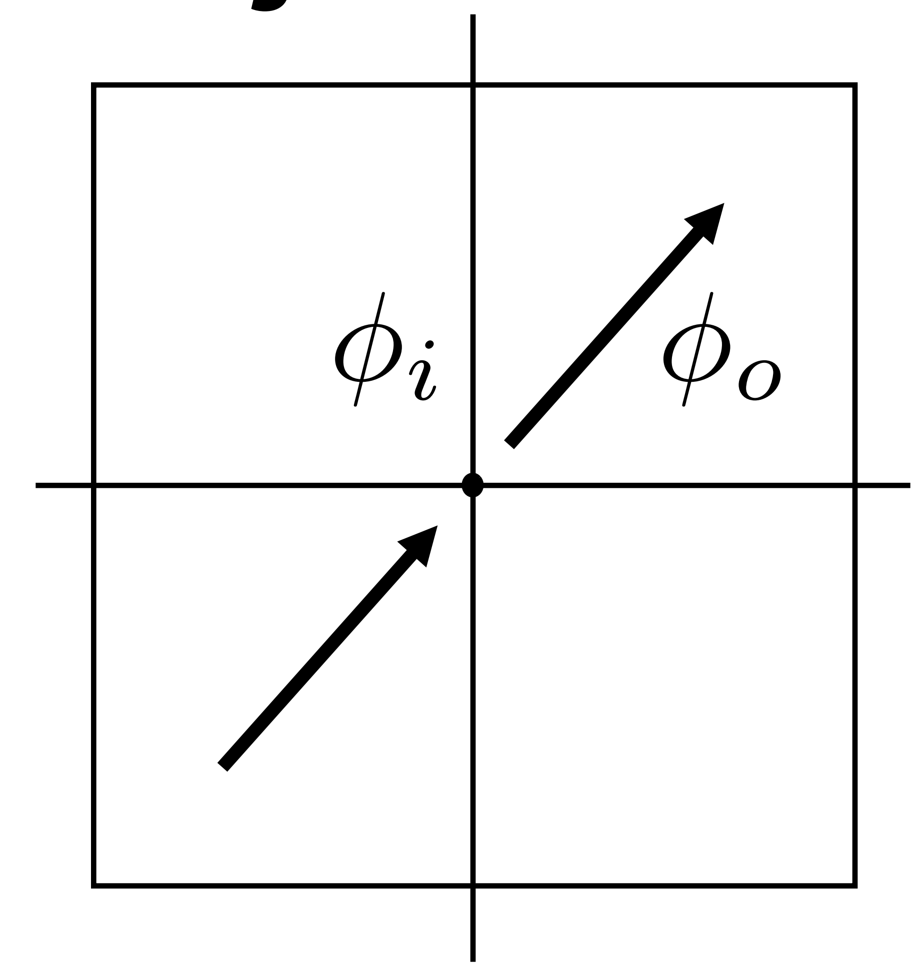
[Zátonyi Sándor]

Geometry of specular reflection



$$\theta = \theta_o = \theta_i$$

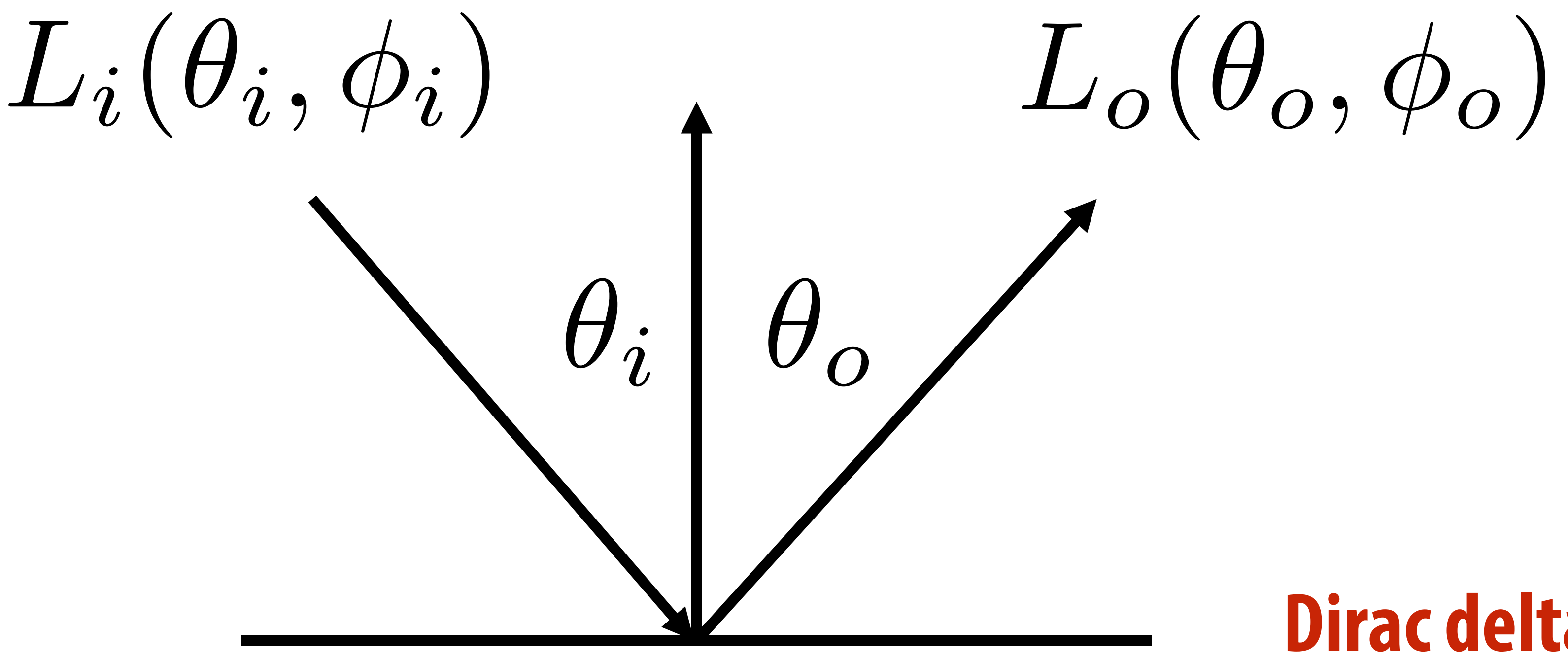
Top-down view
(looking down on surface)



$$\omega_o + \omega_i = 2\cos\theta\,\vec{n} = 2(\omega_i \cdot \vec{n})\vec{n}$$

$$\omega_o = -\omega_i + 2(\omega_i \cdot \vec{n})\vec{n}$$

Specular reflection BRDF



$$L_o(\theta_o, \phi_o) = L_i(\theta_o, \phi_o \pm \pi)$$

Dirac delta

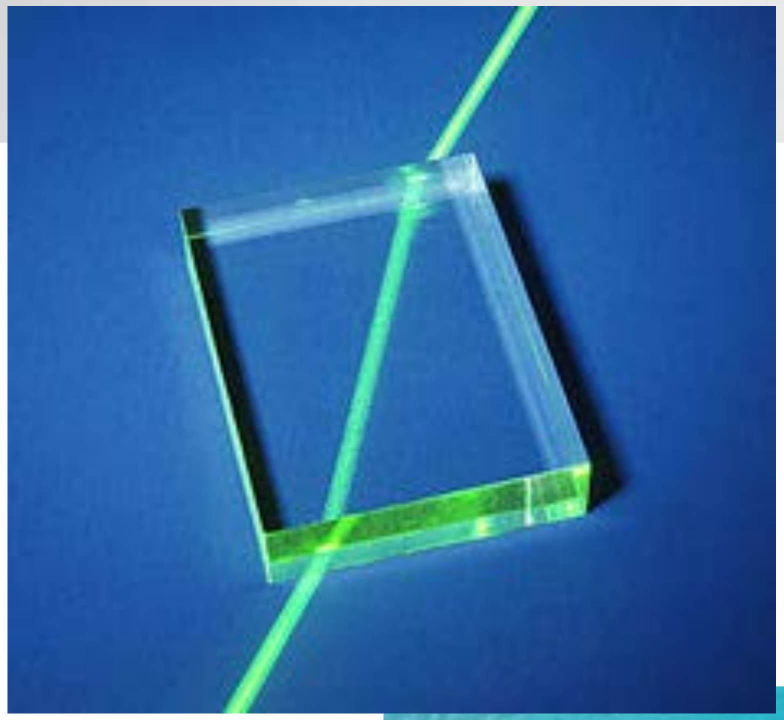
$$f_r(\theta_i, \phi_i; \theta_o, \phi_o) = \frac{\delta(\cos\theta_i - \cos\theta_o)}{\cos\theta_i}\delta(\phi_i - \phi_o \pm \pi)$$

- **Strictly speaking, f_r is a *distribution*, not a function**
- **In practice, no hope of finding reflected direction via random sampling; simply pick the reflected direction!**

Transmission

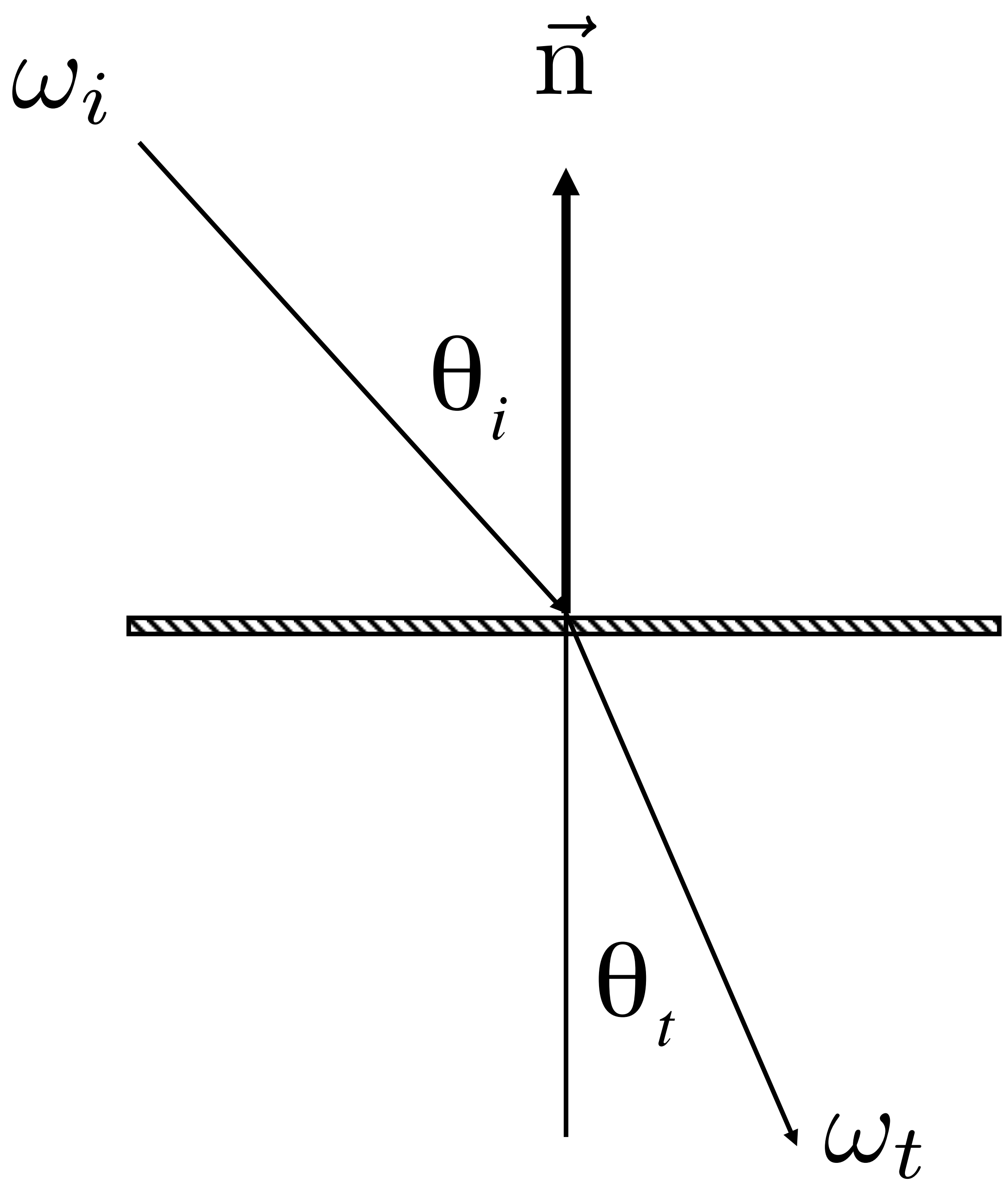
In addition to reflecting off surface, light may be transmitted through surface.

Light refracts when it enters a new medium.

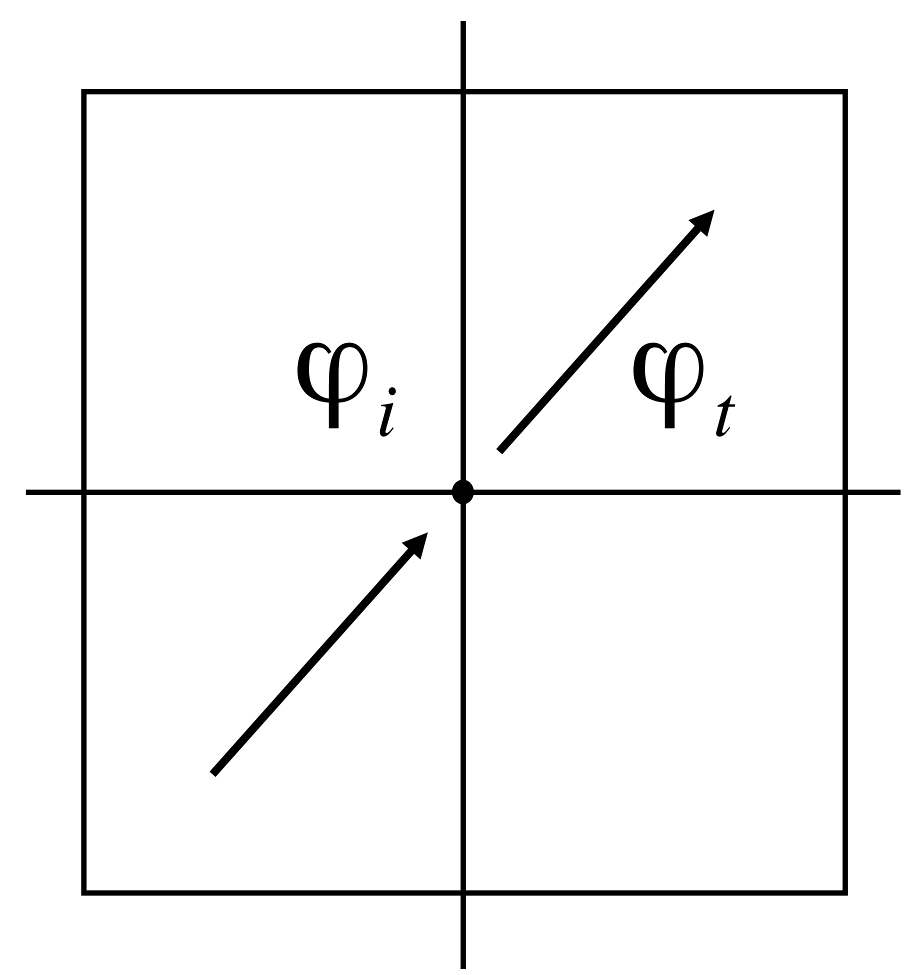


Snell's Law

Transmitted angle depends on index of refraction of medium incident ray is in and index of refraction of medium light is entering.



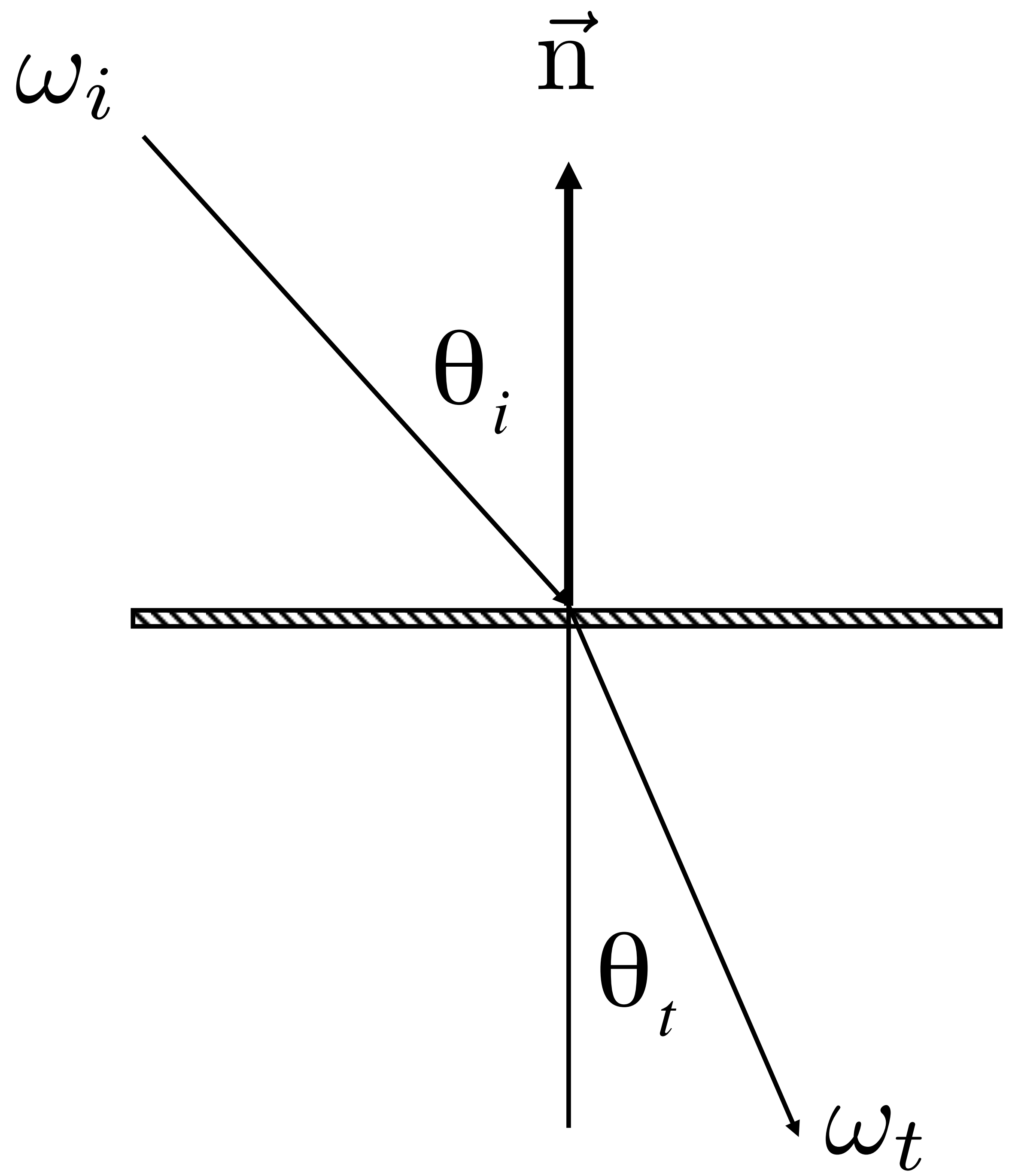
$$\eta_i \sin \theta_i = \eta_t \sin \theta_t$$



Medium	η*
Vacuum	1.0
Air (sea level)	1.00029
Water (20°C)	1.333
Glass	1.5-1.6
Diamond	2.42

*** index of refraction is wavelength dependent (these are averages)**

Law of refraction



$$\eta_i \sin\theta_i = \eta_t \sin\theta_t$$

$$\begin{aligned}
\cos\theta_t &= \sqrt{1 - \sin^2\theta_t} \\
&= \sqrt{1 - \left(\frac{\eta_i}{\eta_t}\right)^2 \sin^2\theta_i} \\
&= \sqrt{1 - \left(\frac{\eta_i}{\eta_t}\right)^2 (1 - \cos^2\ \theta_i)}
\end{aligned}$$

Total internal reflection:

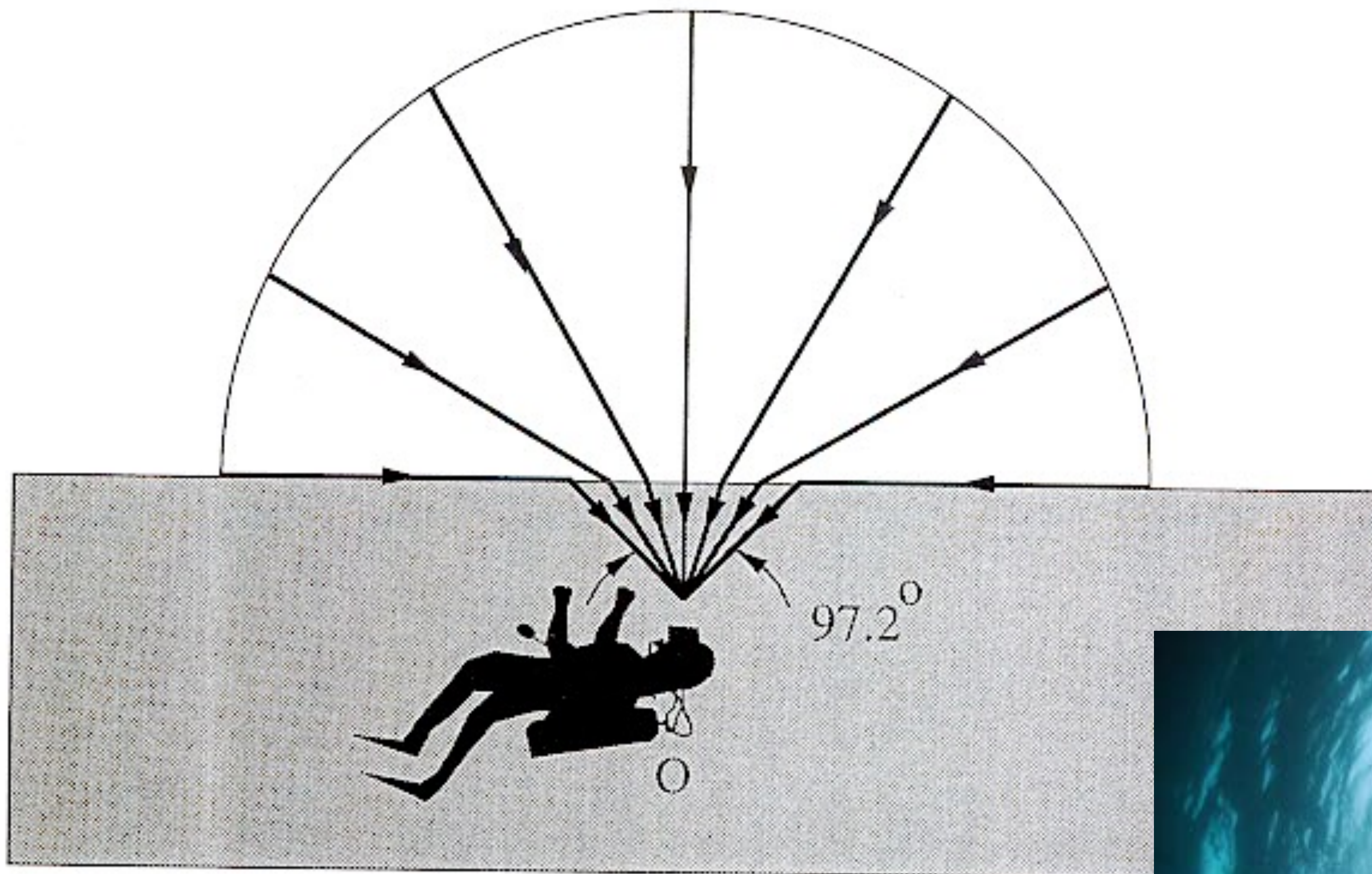
When light is moving from a more optically dense medium to a less optically dense medium: $\frac{\eta_i}{\eta_t} > 1$

$$1 - \left(\frac{\eta_i}{\eta_t}\right)^2 (1 - \cos^2\theta_i) < 0$$

Light incident on boundary from large enough angle will not exit medium.

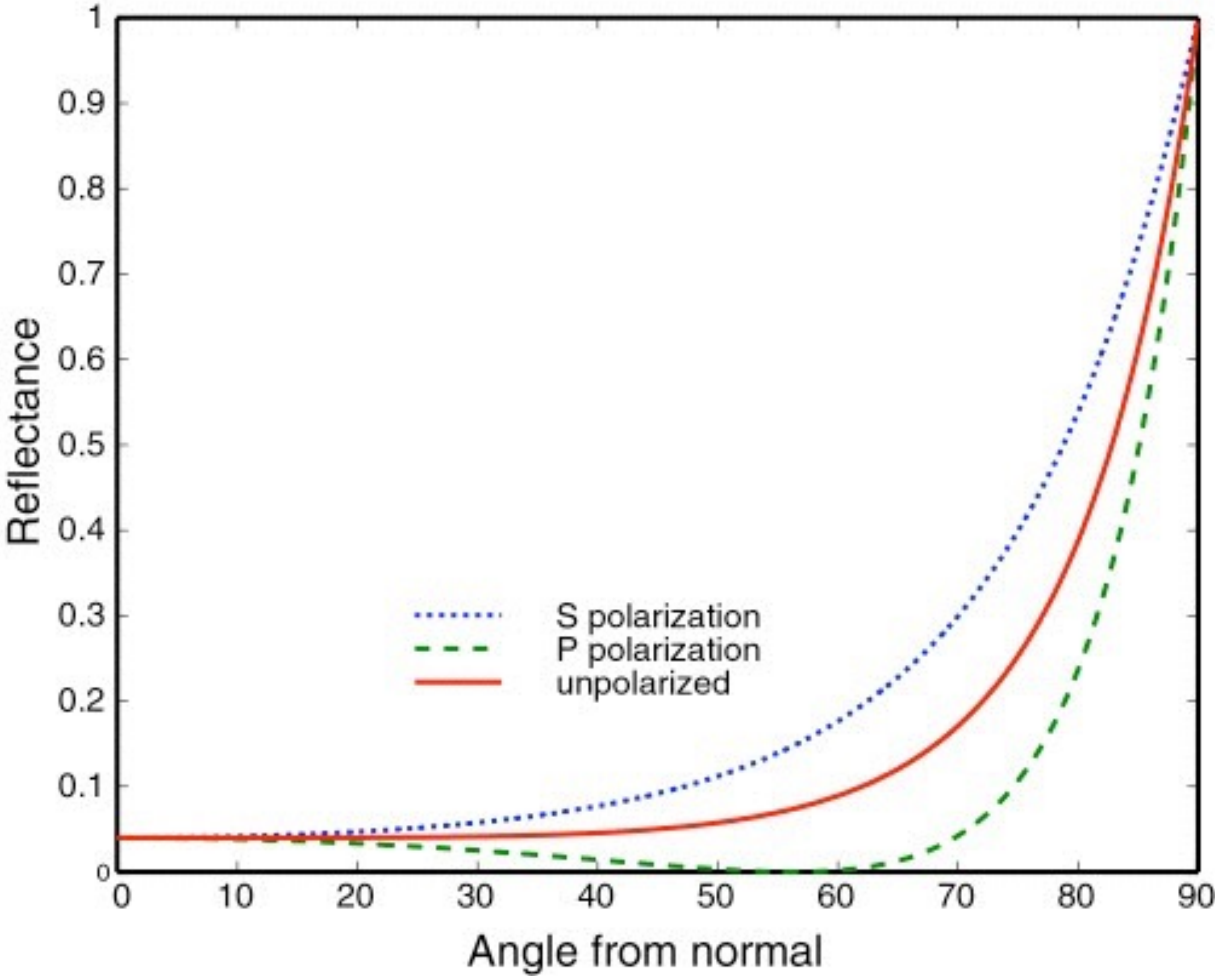
Optical manhole

Only small "cone" visible, due to total internal reflection (TIR)



Fresnel reflection

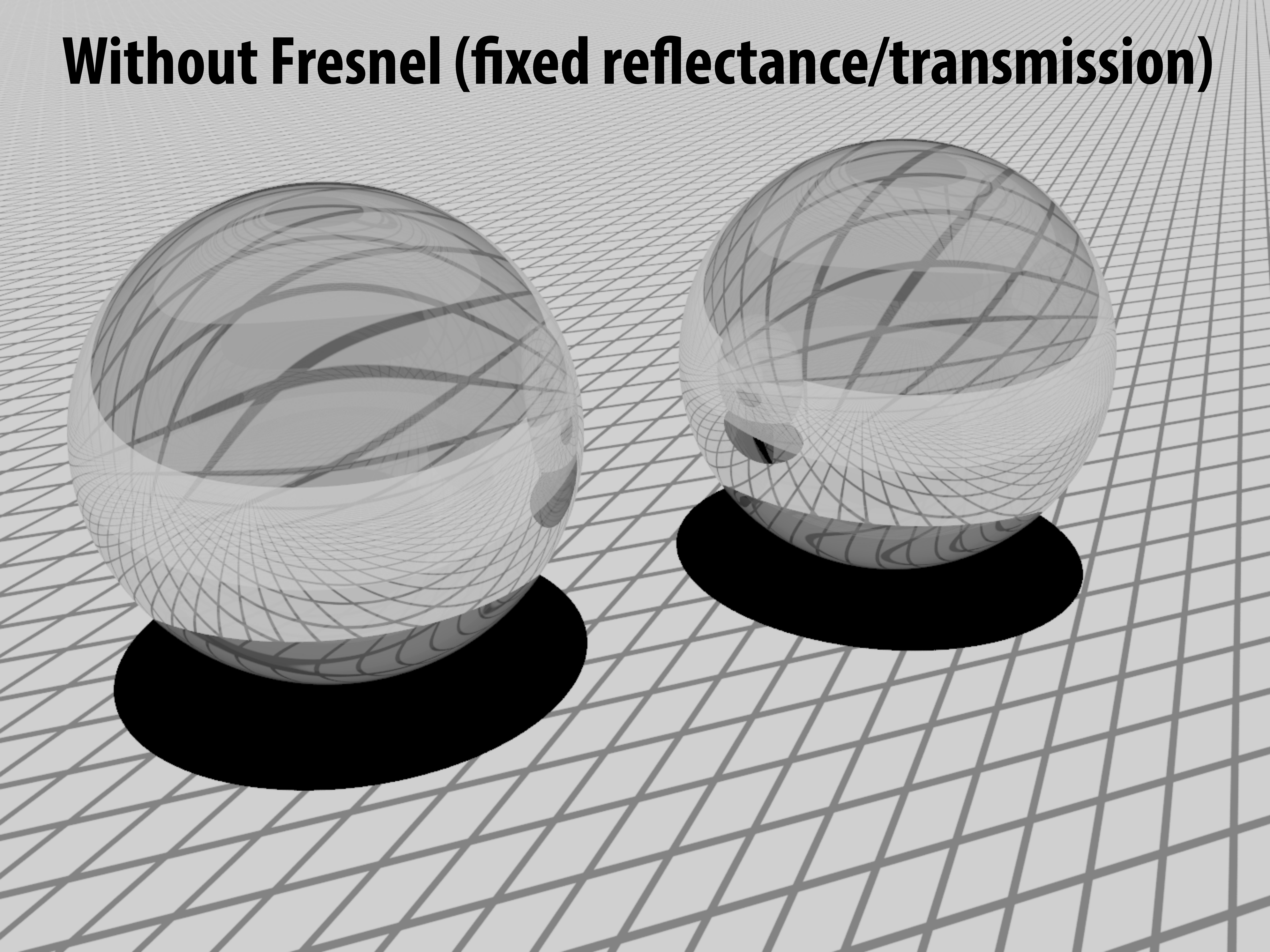
Many real materials: reflectance increases w/ viewing angle



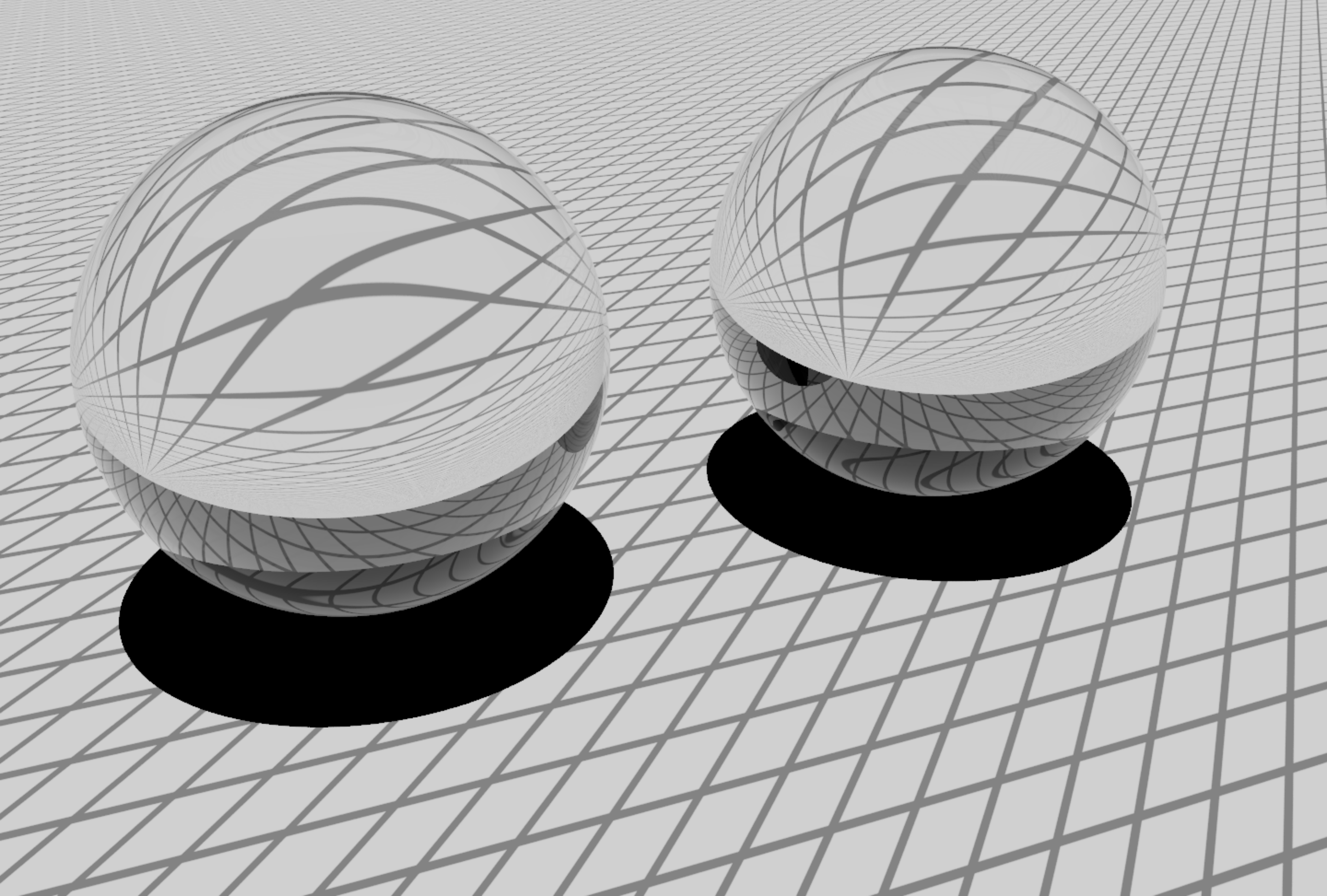
[Lafortune et al. 1997]

Snell + Fresnel: Example

Without Fresnel (fixed reflectance/transmission)



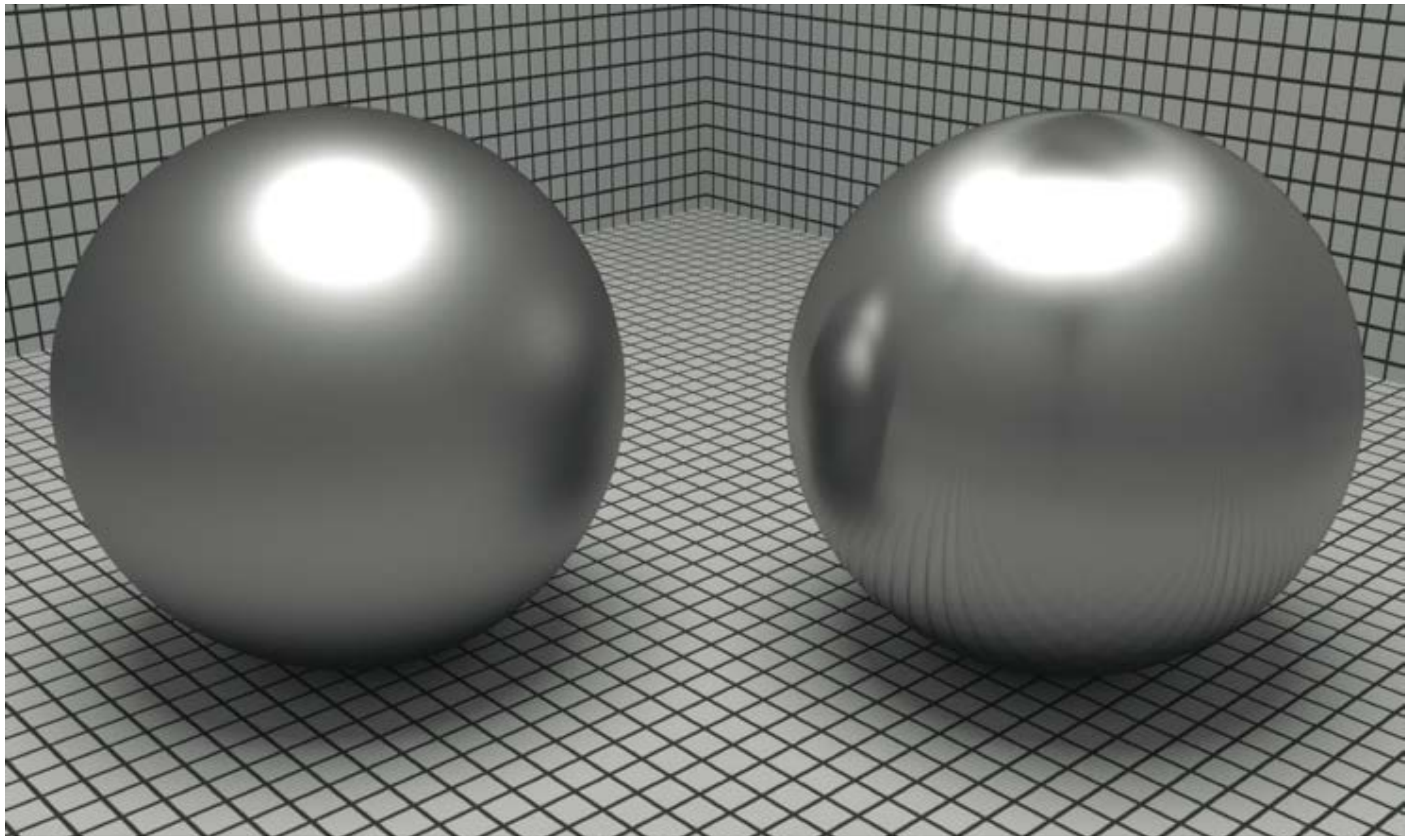
Glass with Fresnel reflection/transmission



Anisotropic reflection

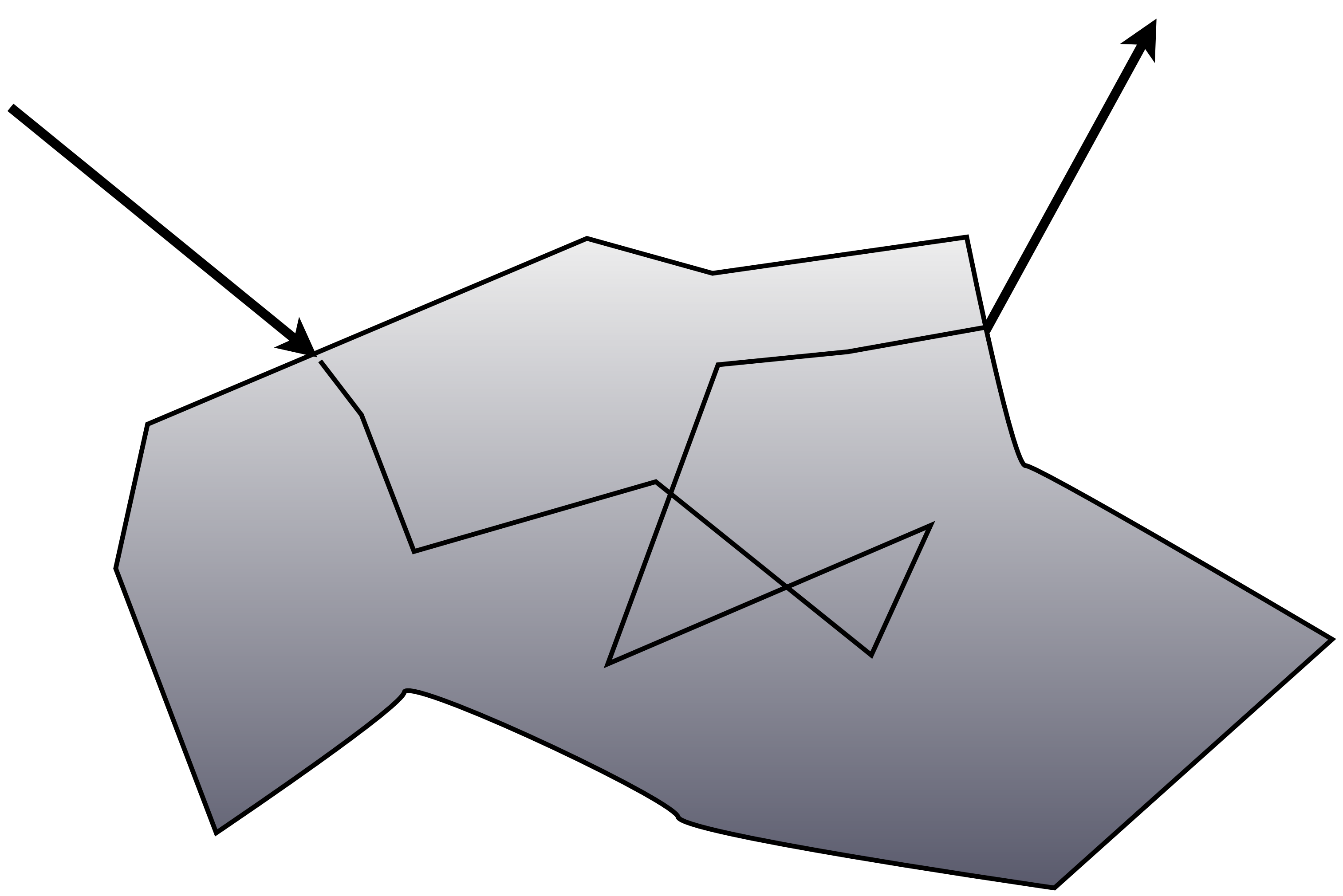
Reflection depends on azimuthal angle ϕ

Results from oriented microstructure of surface
e.g., brushed metal



Subsurface scattering

- Visual characteristics of many surfaces caused by light entering at different points than it exits
  - Violates a fundamental assumption of the BRDF



[Jensen et al 2001]

[Donner et al 2008]

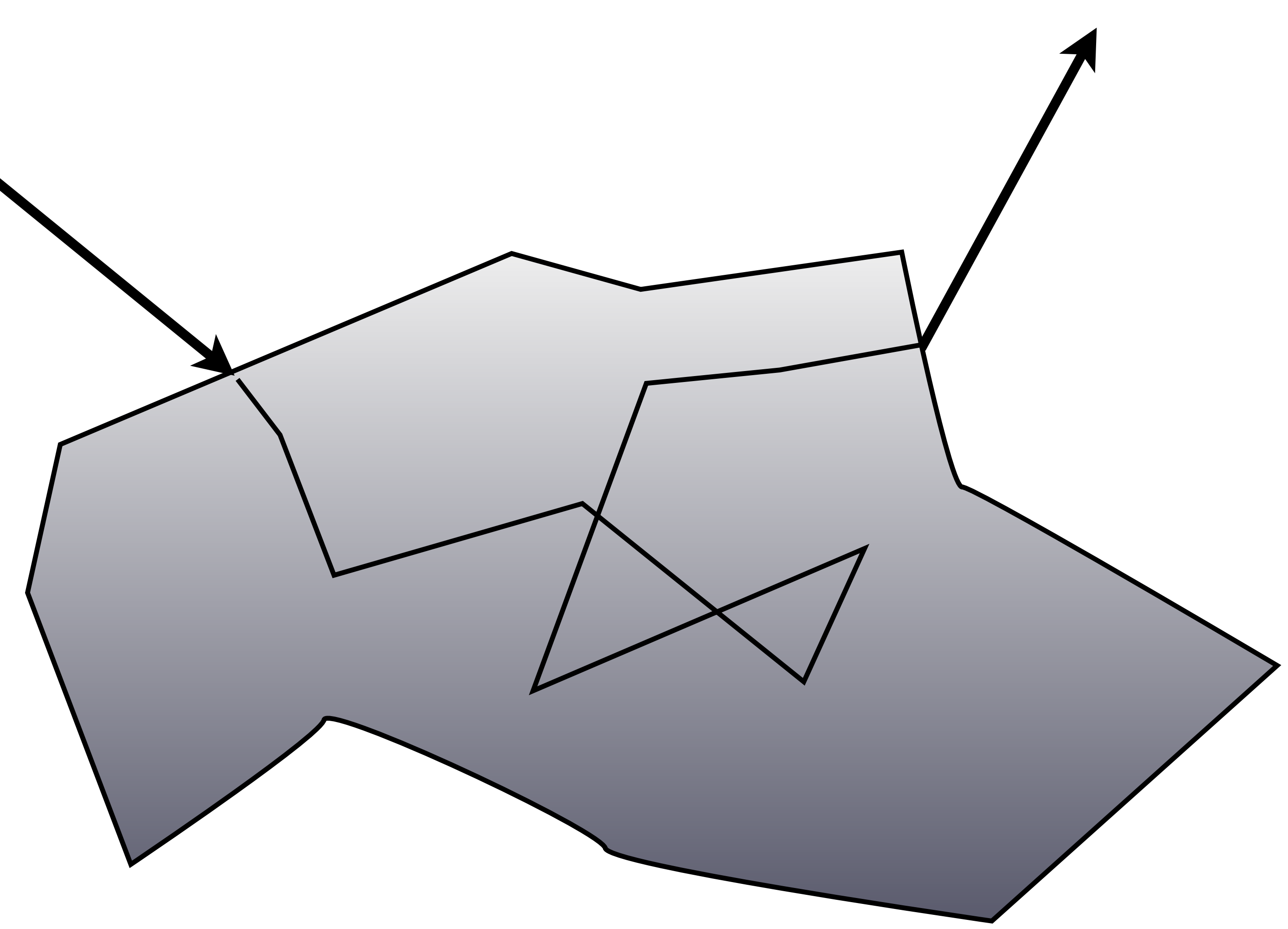
Scattering functions

- **Generalization of BRDF; describes exitant radiance at one point due to incident differential irradiance at another point:**

$$S(x_i, \omega_i, x_o, \omega_o)$$

- **Generalization of reflection equation integrates over all points on the surface and all directions(!)**

$$L(x_o, \omega_o) = \int_A \int_{H^2} S(x_i, \omega_i, x_o, \omega_o)\, L_i(x_i, \omega_i) \cos\theta_i \, \mathrm{d}\omega_i \, \mathrm{d}A$$



Translucent materials: Jade

Translucent materials: skin

Translucent materials: leaves

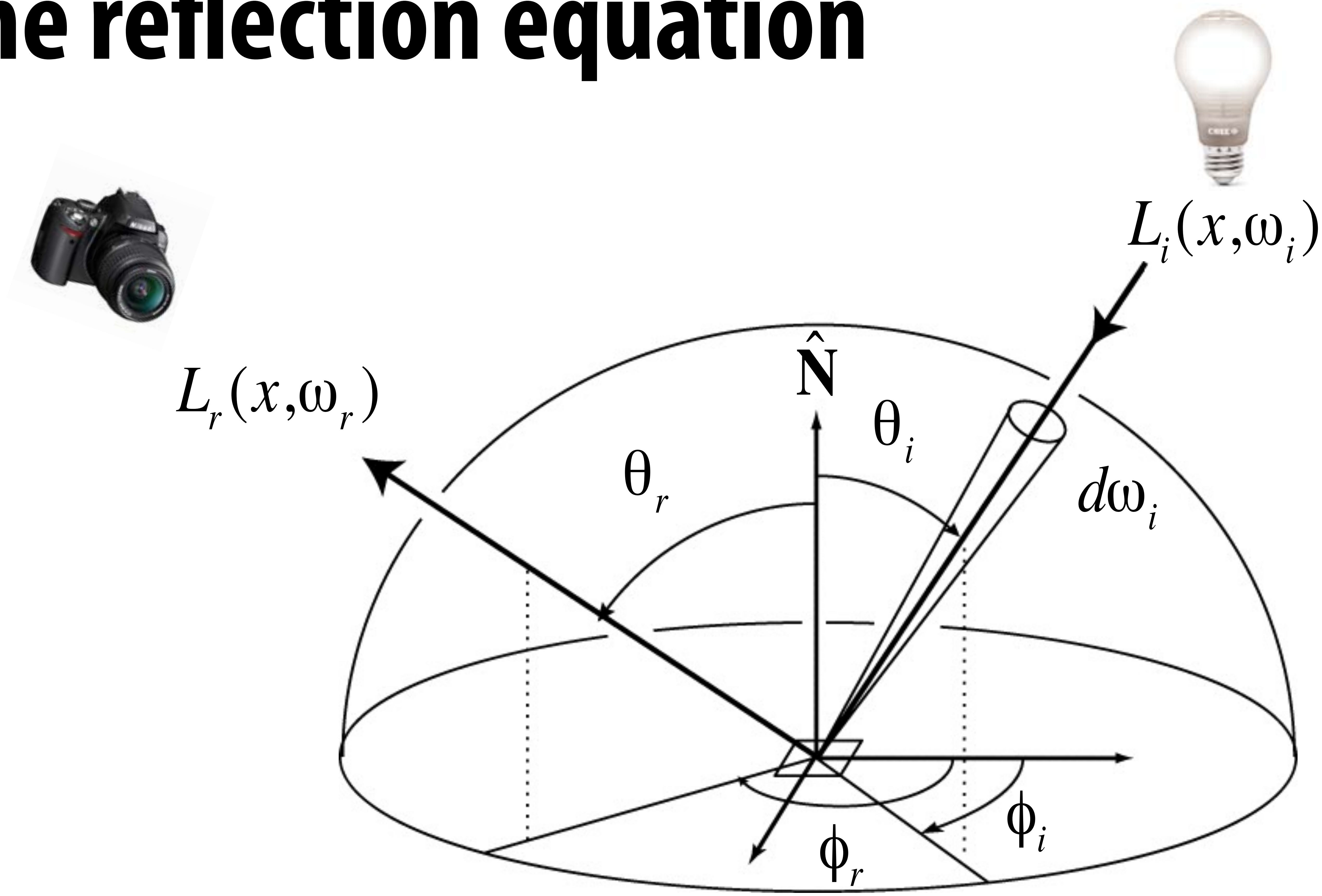
BRDF

BSSRDF

Ok, so scattering is *complicated!*

What's a (relatively simple) algorithm that can capture all this behavior?

The reflection equation



$$\mathrm{d}L_r(\omega_r) = f_r(\omega_i \to \omega_r)\,\mathrm{d}L_i(\omega_i)\,\cos\theta_i$$

$$L_r(\mathrm{p},\omega_r) = \int_{H^2} f_r(\mathrm{p},\omega_i \to \omega_r)\,L_i(\mathrm{p},\omega_i)\,\cos\theta_i\,\mathrm{d}\omega_i$$

The reflection equation

- **Key piece of overall rendering equation:**

$$L_r(\mathrm{p}, \omega_r) = \int_{H^2} f_r(\mathrm{p}, \omega_i \to \omega_r)\, L_i(\mathrm{p}, \omega_i)\, \cos\theta_i \, \mathrm{d}\omega_i$$

  - **Approximate integral via Monte Carlo integration**
  - **Generate directions ω_j sampled from some distribution $p(\omega)$**
  - **Compute the estimator**

$$\frac{1}{N} \sum_{j=1}^{N} \frac{f_r(\mathrm{p}, \omega_j \to \omega_r)\, L_i(\mathrm{p}, \omega_j)\, \cos\theta_j}{p(\omega_j)}$$

  - **To reduce variance $p(\omega)$ should match BRDF or incident radiance function**

Estimating reflected light

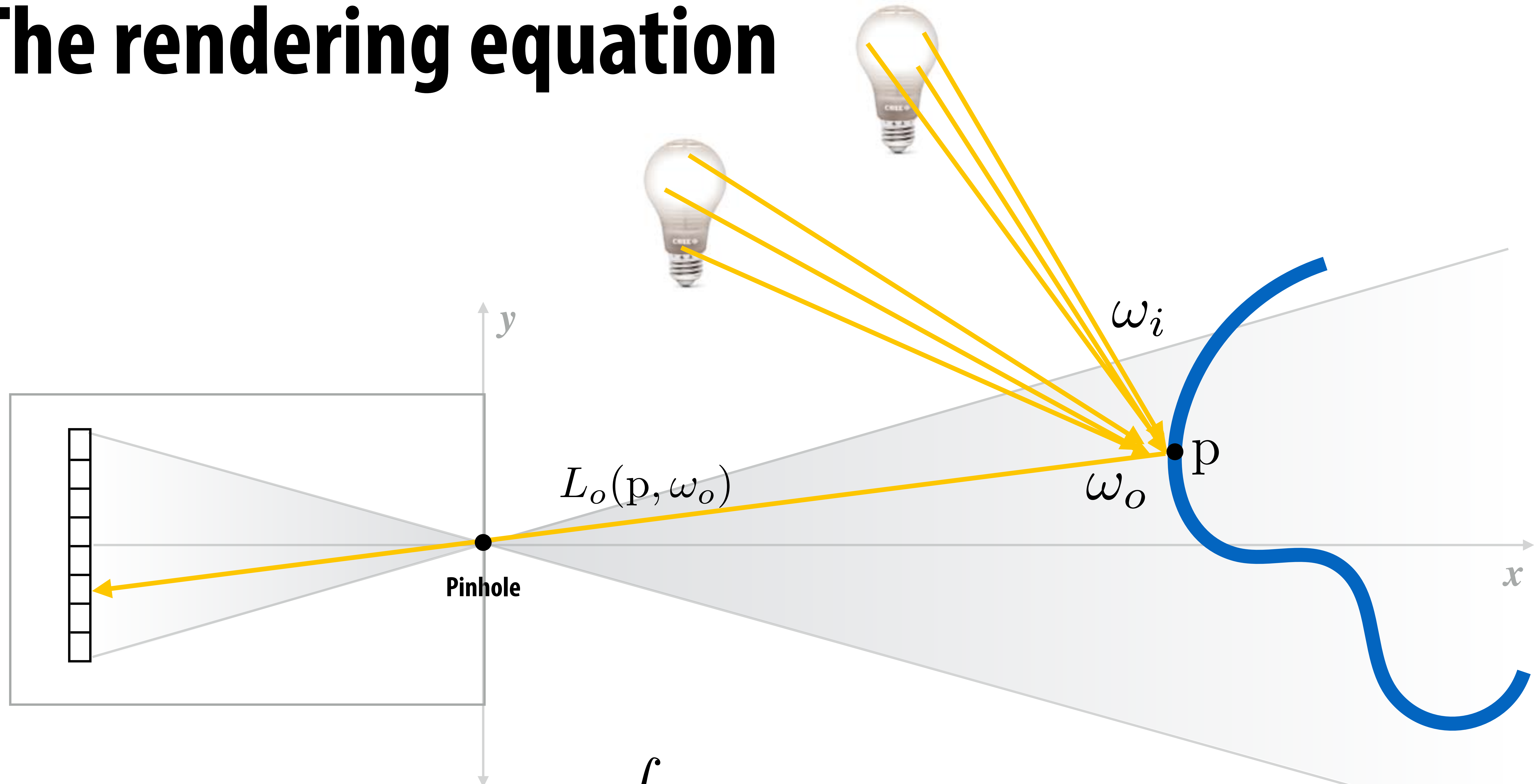
```
// Assume:
//   Ray ray hits surface at point hit_p
//   Normal of surface at hit point is hit_n

Vector3D wr = -ray.d;   // outgoing direction
Spectrum Lr = 0.;
for (int i = 0; i < N; ++i) {
    Vector3D wi;        // sample incident light from this direction
    float pdf;          // p(wi)

    generate_sample(brdf, &wi, &pdf);     // generate sample according to brdf

    Spectrum f = brdf->f(wr, wi);
    Spectrum Li = trace_ray(Ray(hit_p, wi));  // compute incoming Li
    Lr += f * Li * fabs(dot(wi, hit_n)) / pdf;
}
return Lr / N;
```

The rendering equation



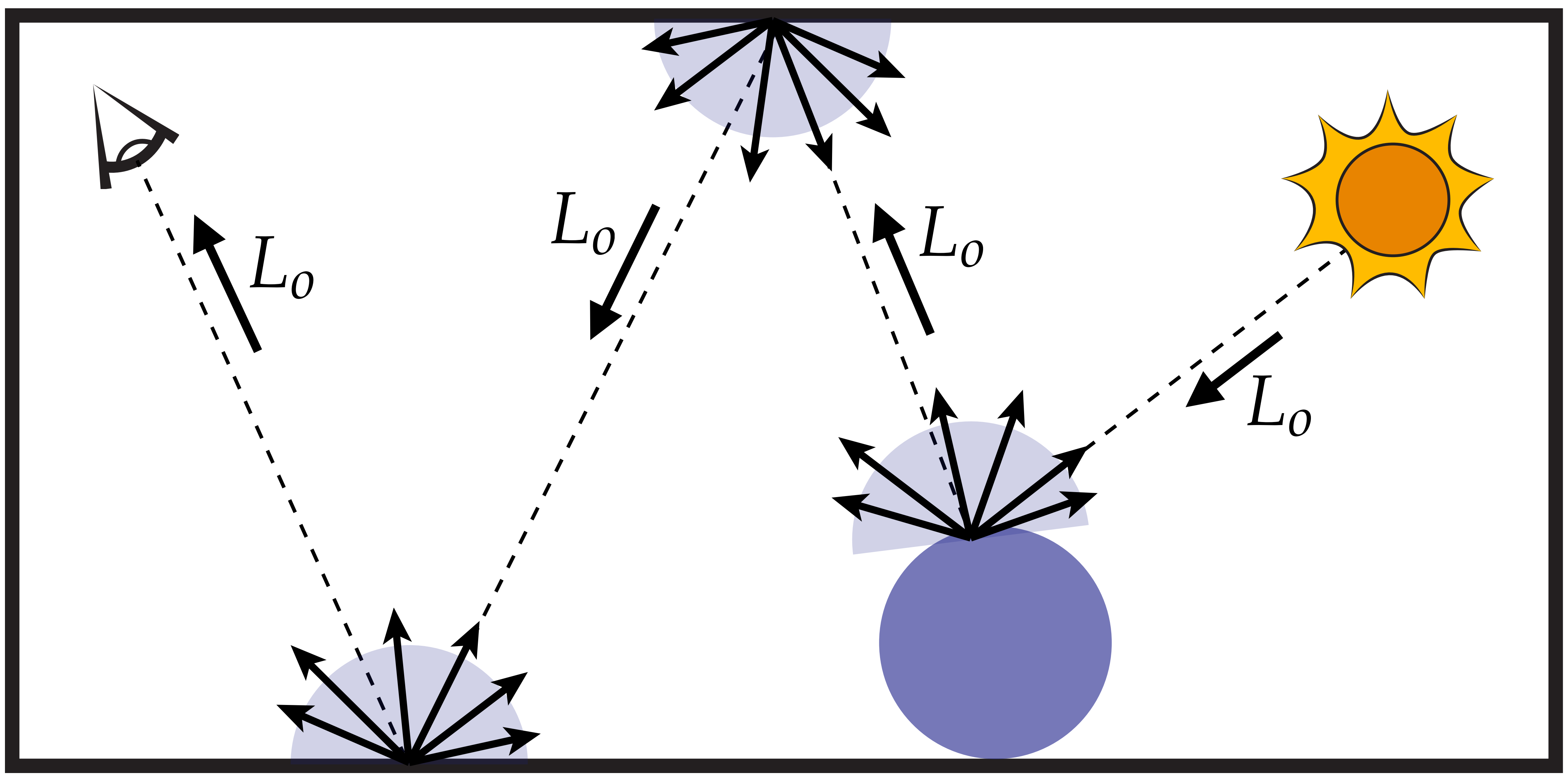
$$L_o(\mathrm{p}, \omega_o) = L_e(\mathrm{p}, \omega_o) + \int_{H^2} f_r(\mathrm{p}, \omega_i \to \omega_o)\, L_i(\mathrm{p}, \omega_i)\, \cos\theta_i \, \mathrm{d}\omega_i$$

Now that we know how to handle reflection, how do we solve the full rendering equation? Have to determine incident radiance…

Key idea in (efficient) rendering: take advantage of special knowledge to break up integration into "easier" components.

Path tracing: overview

- Partition the rendering equation into direct and indirect illumination
- Use Monte Carlo to estimate each partition separately
  - One sample for each
  - Assumption: 100s of samples per pixel
- Terminate paths with *Russian roulette*



Direct illumination + reflection + transparency

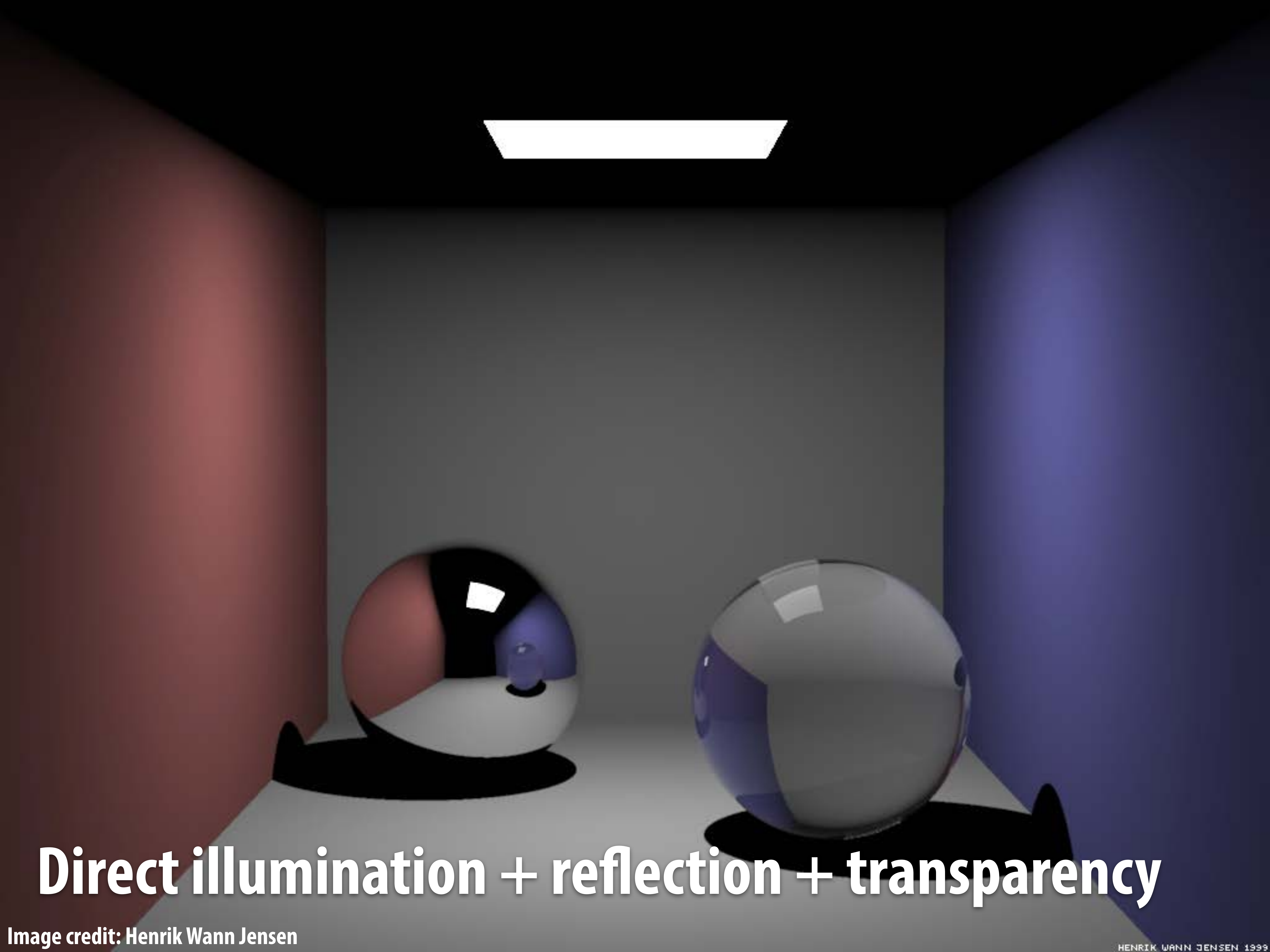
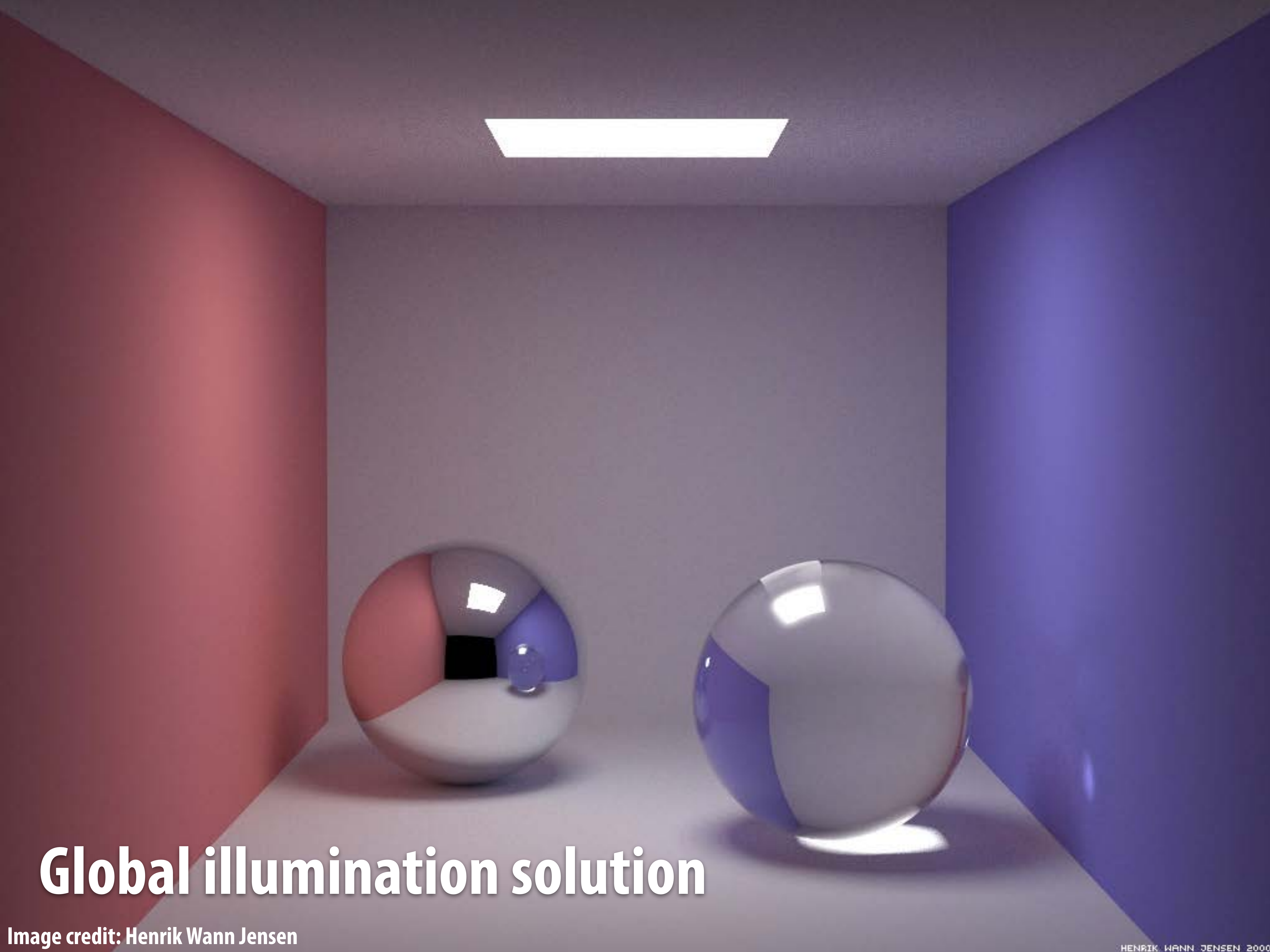


Image credit: Henrik Wann Jensen



Global illumination solution

Image credit: Henrik Wann Jensen

Review: Monte Carlo integration

- **Definite integral**
  What we seek to estimate

$$\int_a^b f(x)dx$$

- **Random variables**
  X_i **is the value of a random sample drawn from the distribution** $p(x)$
  Y_i **is also a random variable.**

$$X_i \sim p(x)$$
$$Y_i = f(X_i)$$

- **Expectation of *f***

$$E[Y_i] = E[f(X_i)] = \int_a^b f(x)\,p(x)\,\mathrm{d}x$$

- **Estimator**
  Monte Carlo estimate of $\int_a^b f(x)dx$

$$F_N = \frac{b-a}{N}\sum_{i=1}^{N} Y_i$$

Assuming samples X_i **drawn from uniform pdf. I will provide estimator for arbitrary PDFs later in lecture.**

Basic Monte Carlo estimator

$$\lim_{N\to\infty}(b-a)\frac{1}{N}\sum_{i=1}^{N}f(X_i) = \int_a^b f(x)\,dx$$

Assume uniform probability density (for now)

$$X_i \sim U(a,b)$$

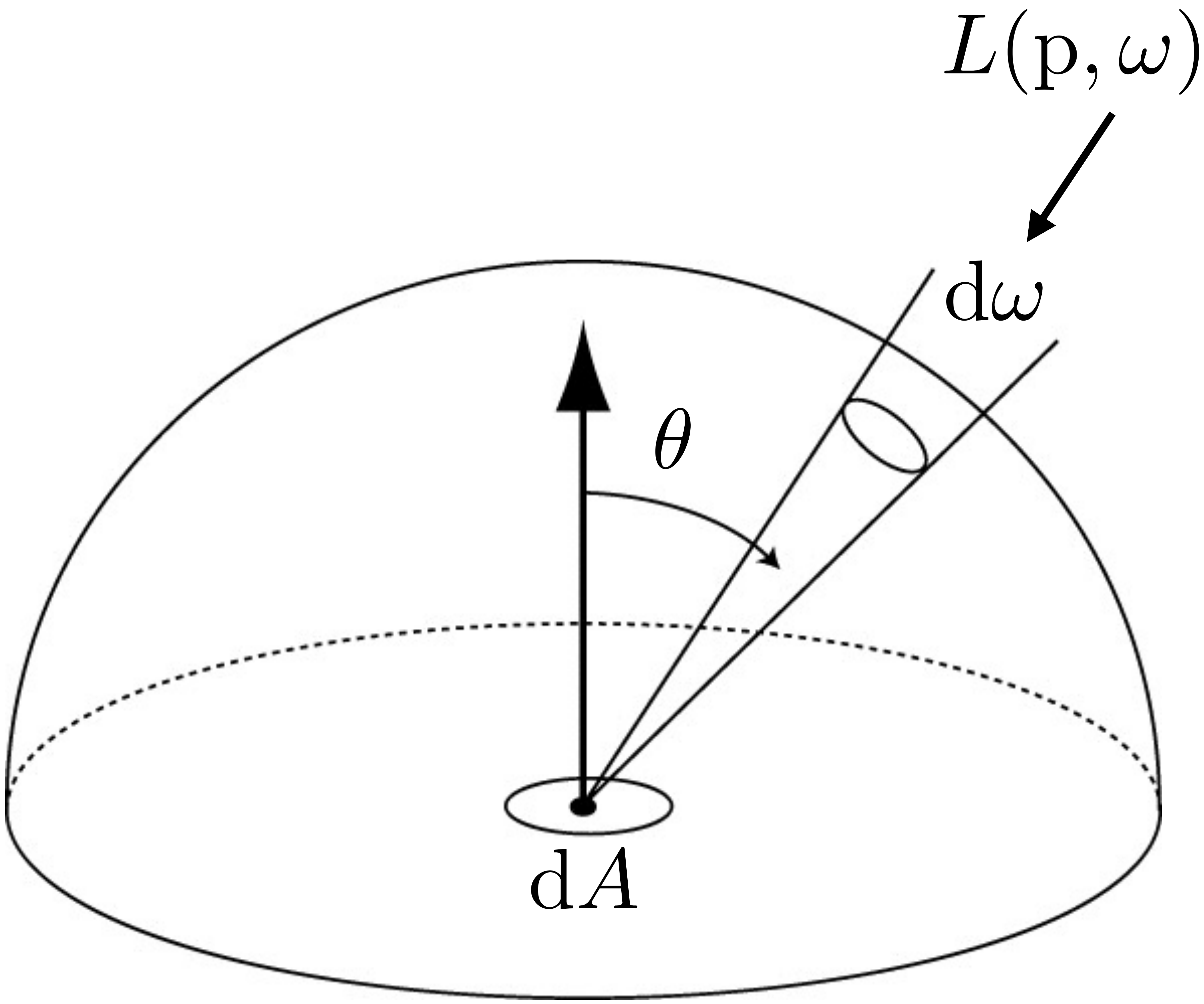
$$p(x) = \frac{1}{b-a}$$

Näive direct lighting estimate

Uniformly-sample hemisphere of directions with respect to solid angle

$$p(\omega) = \frac{1}{2\pi}$$

$$E(\mathrm{p}) = \int L(\mathrm{p}, \omega)\, \cos\theta \, \mathrm{d}\omega$$



Estimator:

$$X_i \sim p(\omega)$$

$$Y_i = f(X_i)$$

$$Y_i = L(\mathrm{p}, \omega_i) cos\, \theta_i$$

$$F_N = \frac{2\pi}{N} \sum_{i=1}^{N} Y_i$$

Näive direct lighting estimate

Uniformly-sample hemisphere of directions with respect to solid angle

$$E(\mathrm{p}) = \int L(\mathrm{p}, \omega) \cos\theta \, \mathrm{d}\omega$$

Given surface point *p*

For each of N samples:

- **Generate random direction:** ω_i
- **Compute incoming radiance arriving** L_i **at *p* from direction:** ω_i
- **Compute incident irradiance due to ray:** $dE_i = L_i cos\,\theta_i$
- **Accumulate** $\frac{2\pi}{N} dE_i$ **into estimator**

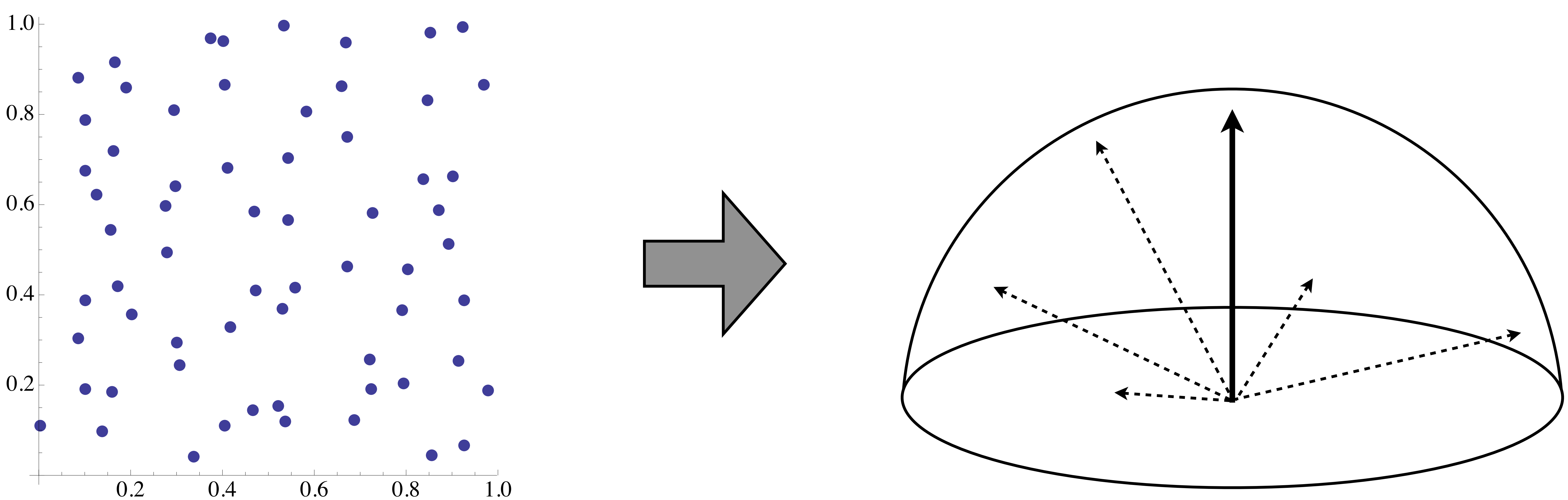
Uniform hemisphere sampling

Generate random direction on hemisphere (all directions equally likely)

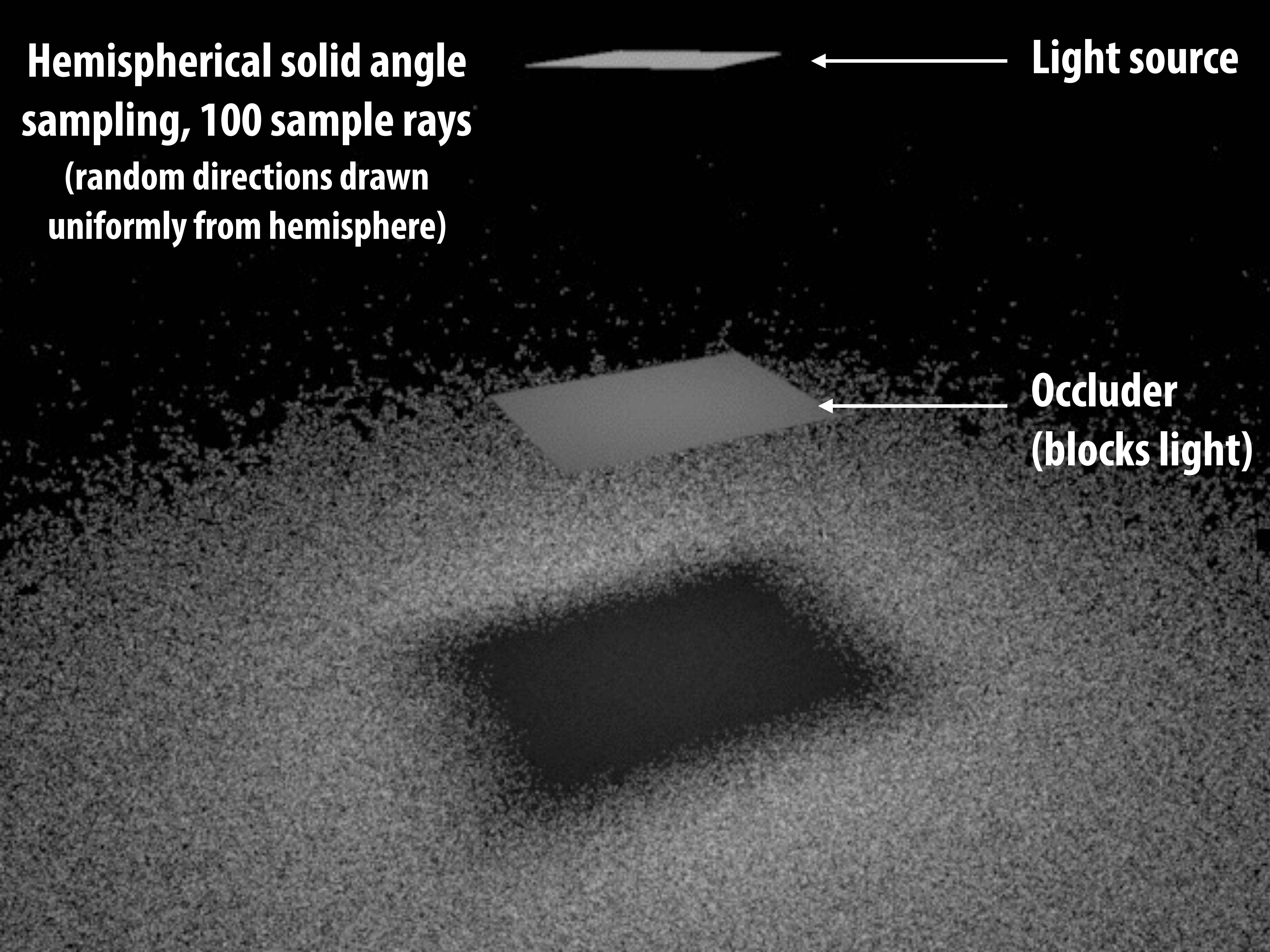
$$p(\omega) = \frac{1}{2\pi}$$

Direction computed from uniformly distributed point on 2D plane:

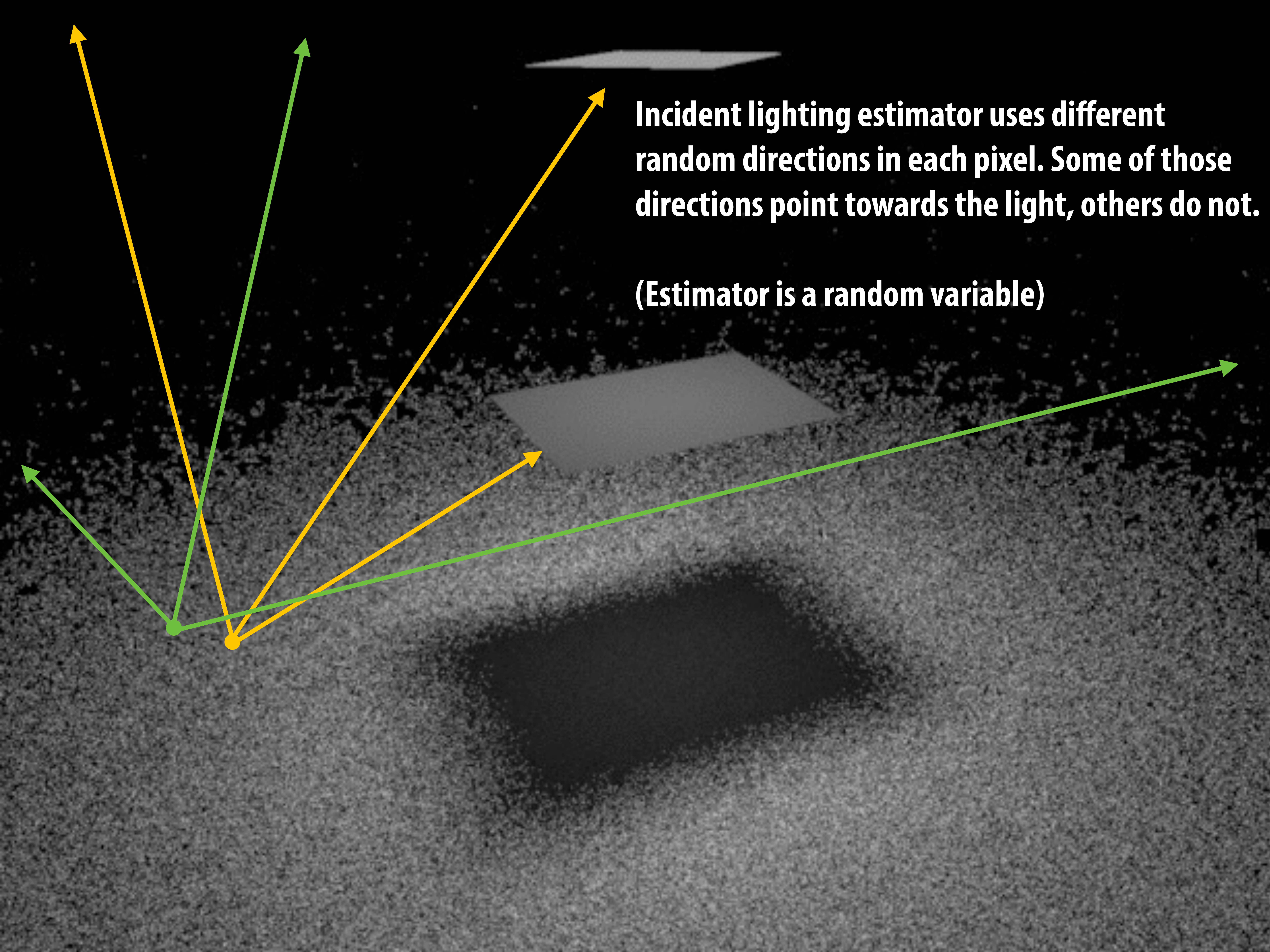
$$(\xi_1, \xi_2) = (\sqrt{1-\xi_1^2}\cos(2\pi\xi_2), \sqrt{1-\xi_1^2}\sin(2\pi\xi_2), \xi_1)$$



Try at home: derive from the inversion method [Arvo]


Hemispherical solid angle sampling, 100 sample rays
(random directions drawn uniformly from hemisphere)
Light source
Occluder
(blocks light)

Why is the image in the previous slide "noisy"?


Incident lighting estimator uses different random directions in each pixel. Some of those directions point towards the light, others do not.
(Estimator is a random variable)

Idea: don't need to integrate over entire hemisphere of directions (incoming radiance is 0 from most directions)

Only integrate over the area of the light (directions where incoming radiance is non-zero)

Direct lighting: area integral

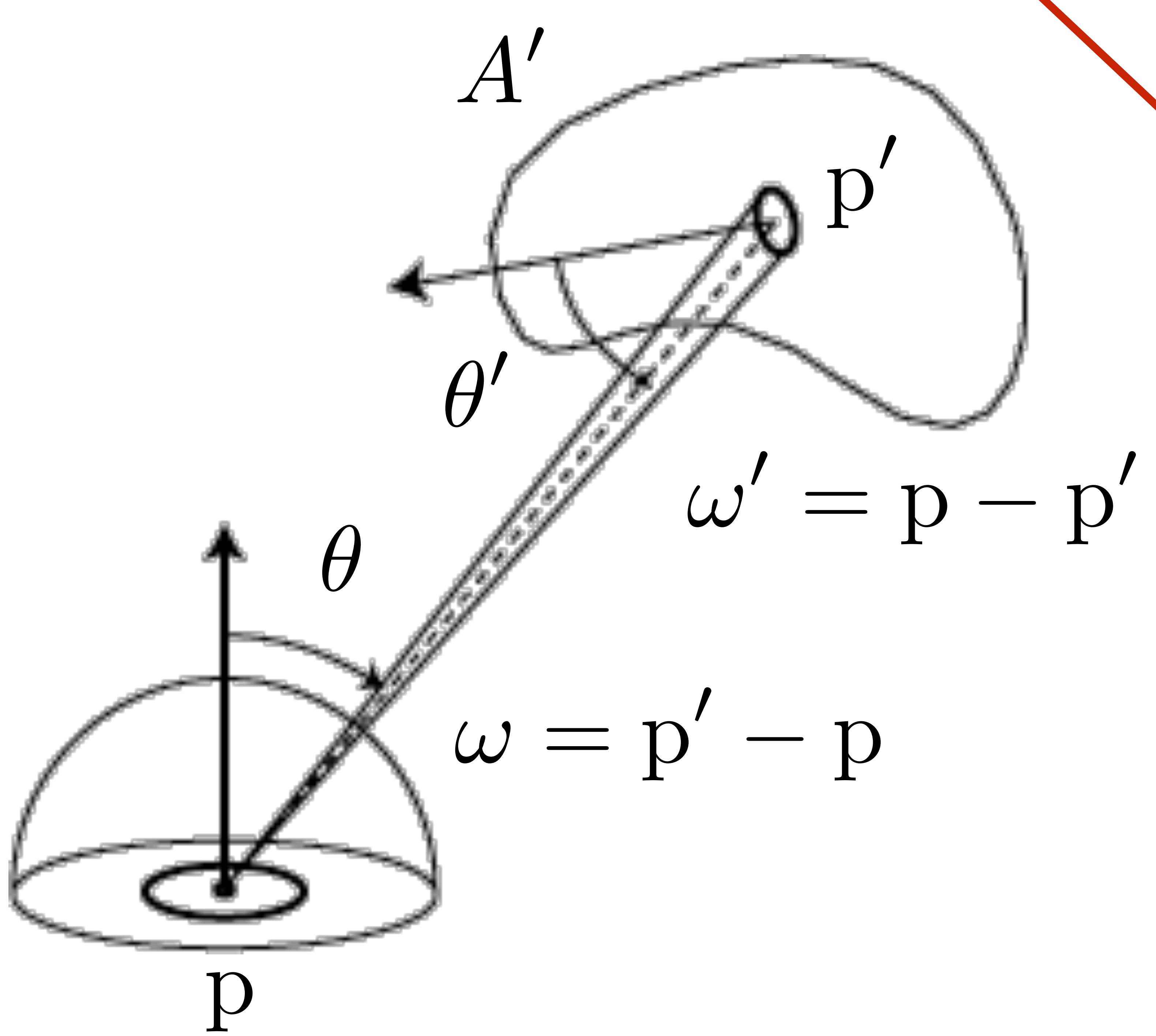
$$E(\mathrm{p}) = \int L(\mathrm{p}, \omega)\ \cos\theta \, \mathrm{d}\omega$$

Integral over directions

$$E(\mathrm{p}) = \int_{A'} L_o(\mathrm{p}', \omega')\, V(\mathrm{p}, \mathrm{p}')\, \frac{\cos\theta \cos\theta'}{|\mathrm{p} - \mathrm{p}'|^2}\, \mathrm{d}A'$$

Change of variables to integral over area of light

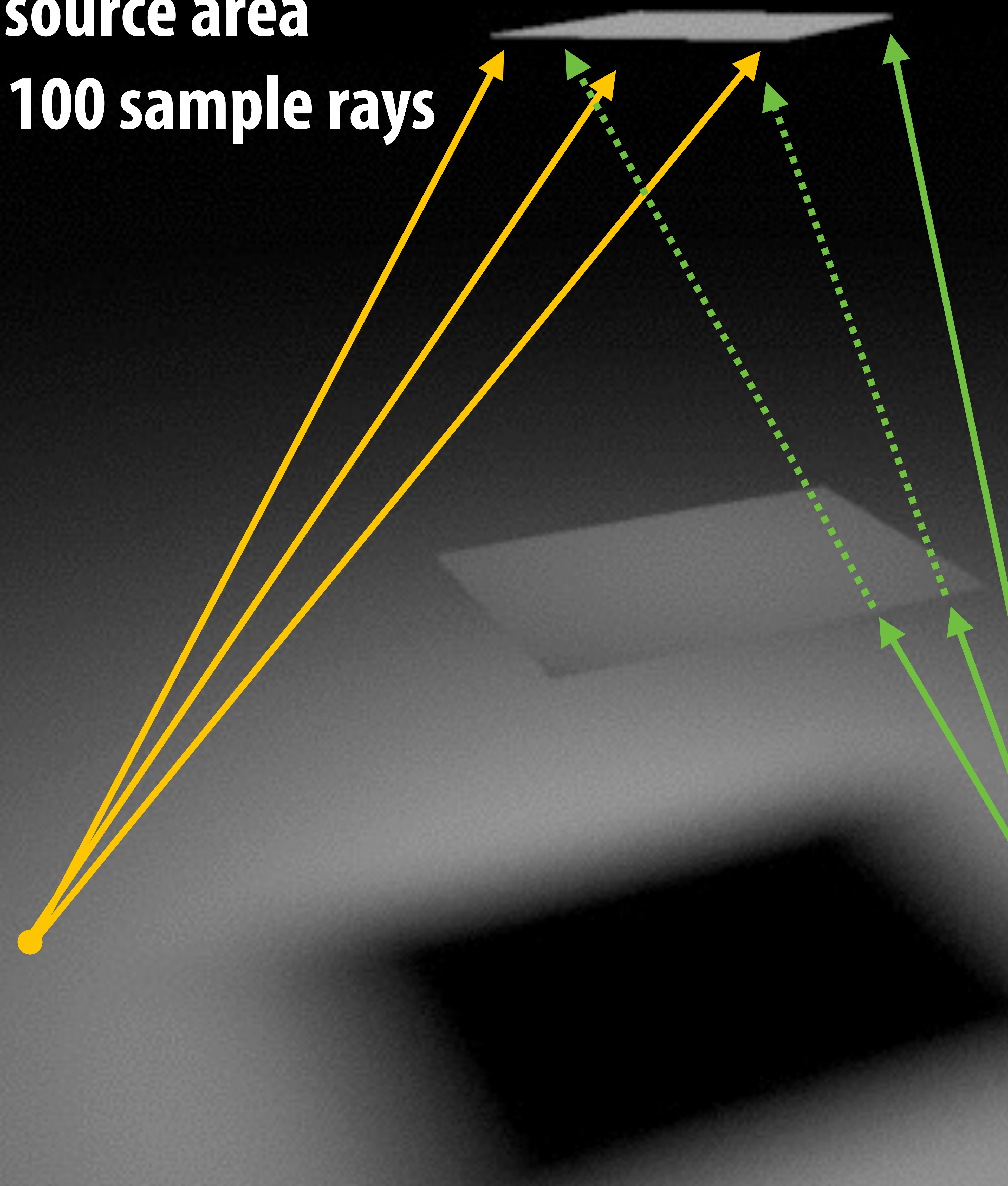
$$dw = \frac{dA}{|\mathrm{p}' - \mathrm{p}|^2} = \frac{dA' cos\,\theta}{|\mathrm{p}' - \mathrm{p}|^2}$$



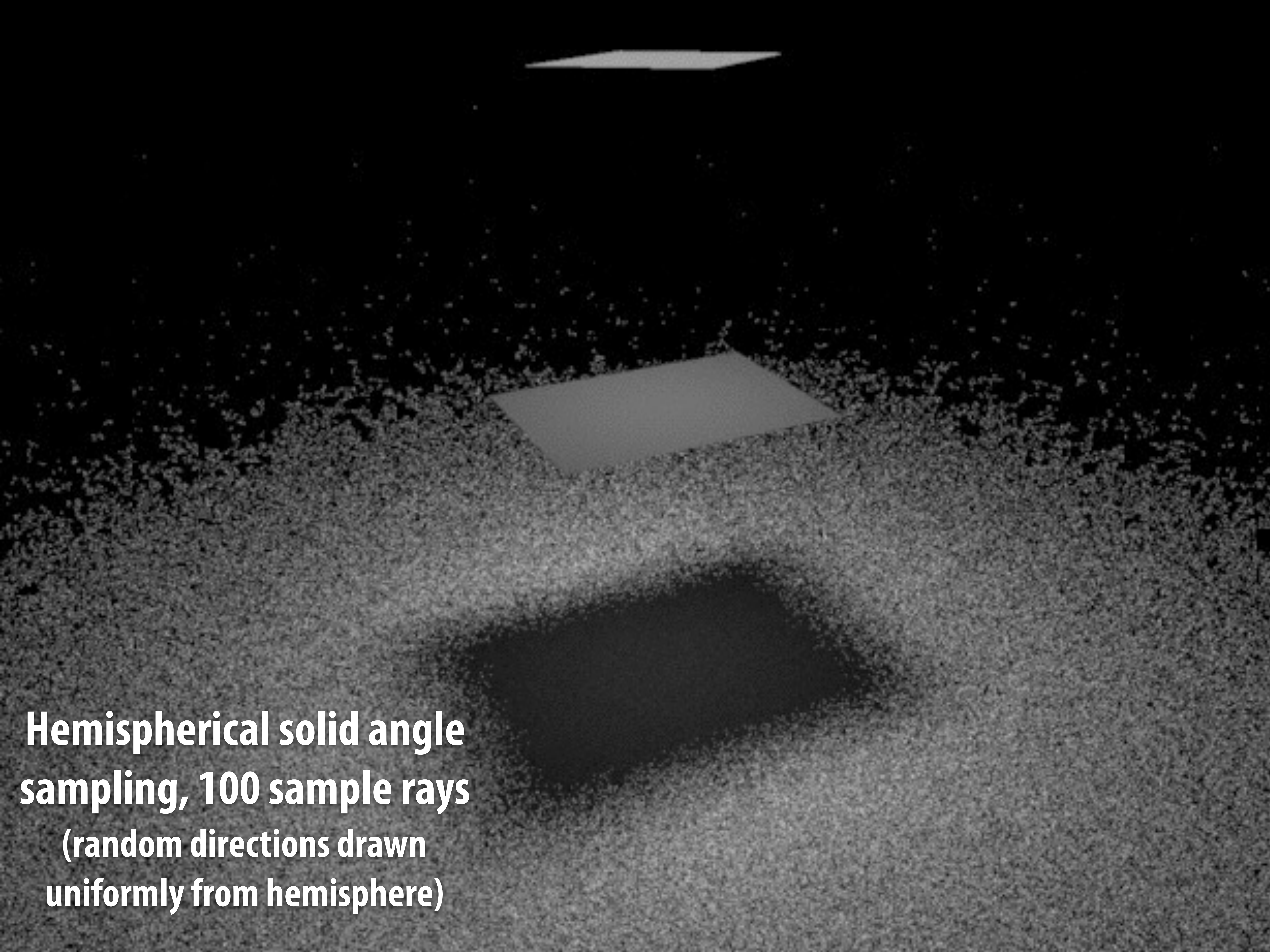
Binary visibility function:
1 if p′ is visible from p, 0 otherwise
(accounts for light occlusion)

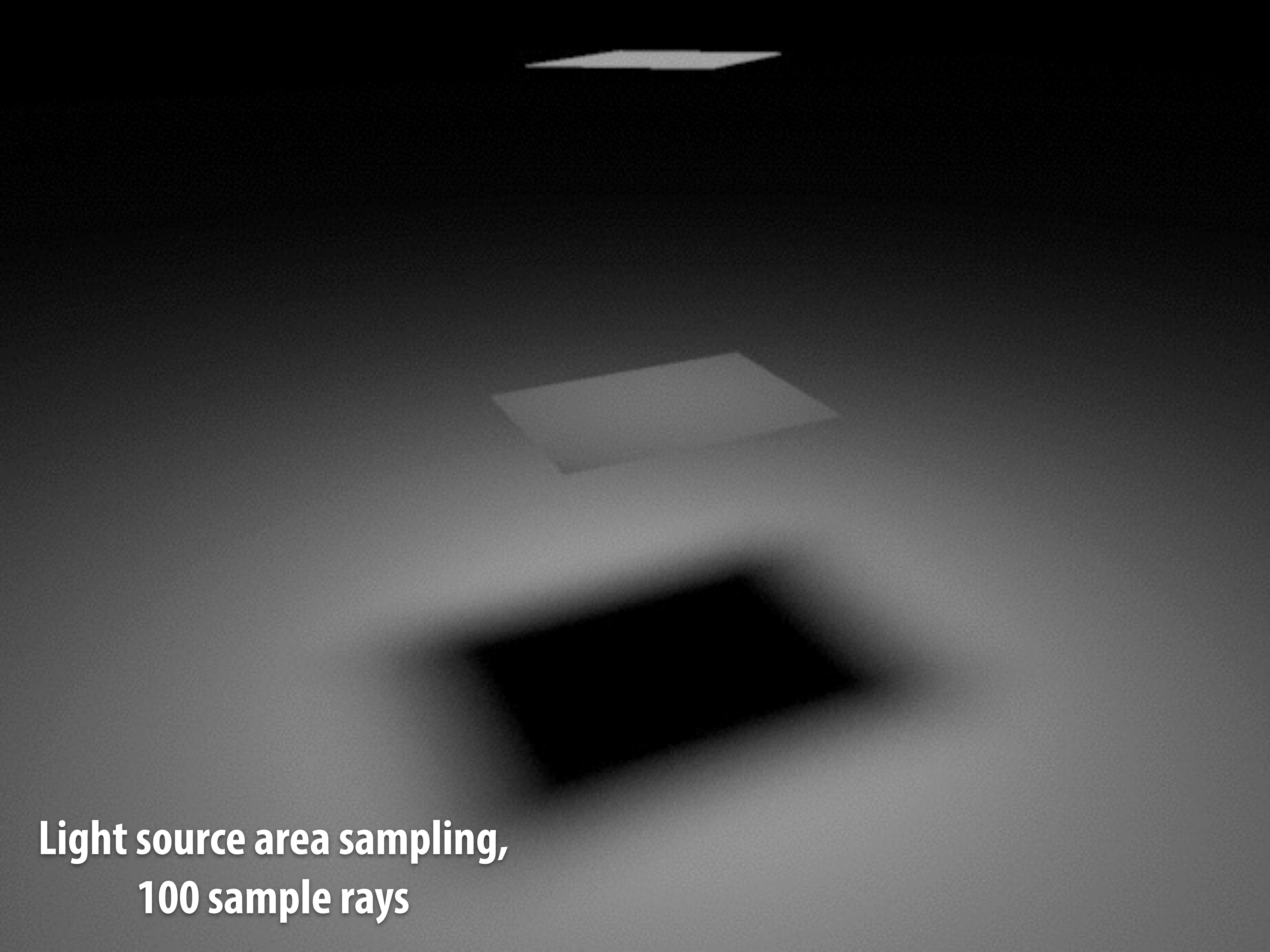
Outgoing radiance from light point p, in direction w′ towards p

Light source area sampling, 100 sample rays



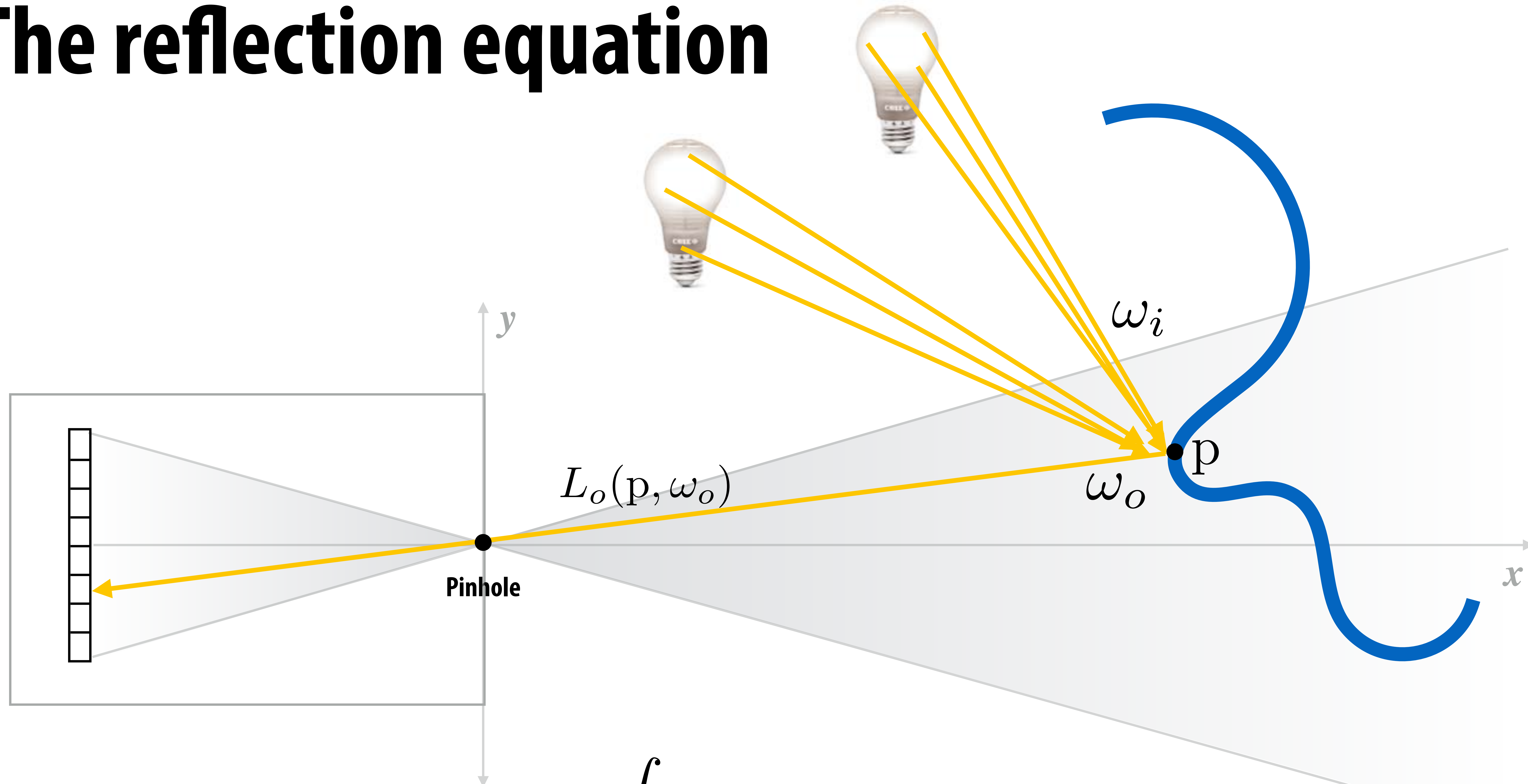
If no occlusion is present, all directions chosen in computing estimate "hit" the light source.
(Choice of direction only matters if portion of light is occluded from surface point *p*.)


Hemispherical solid angle
sampling, 100 sample rays
(random directions drawn
uniformly from hemisphere)


Light source area sampling,
100 sample rays

Estimating indirect lighting

The reflection equation

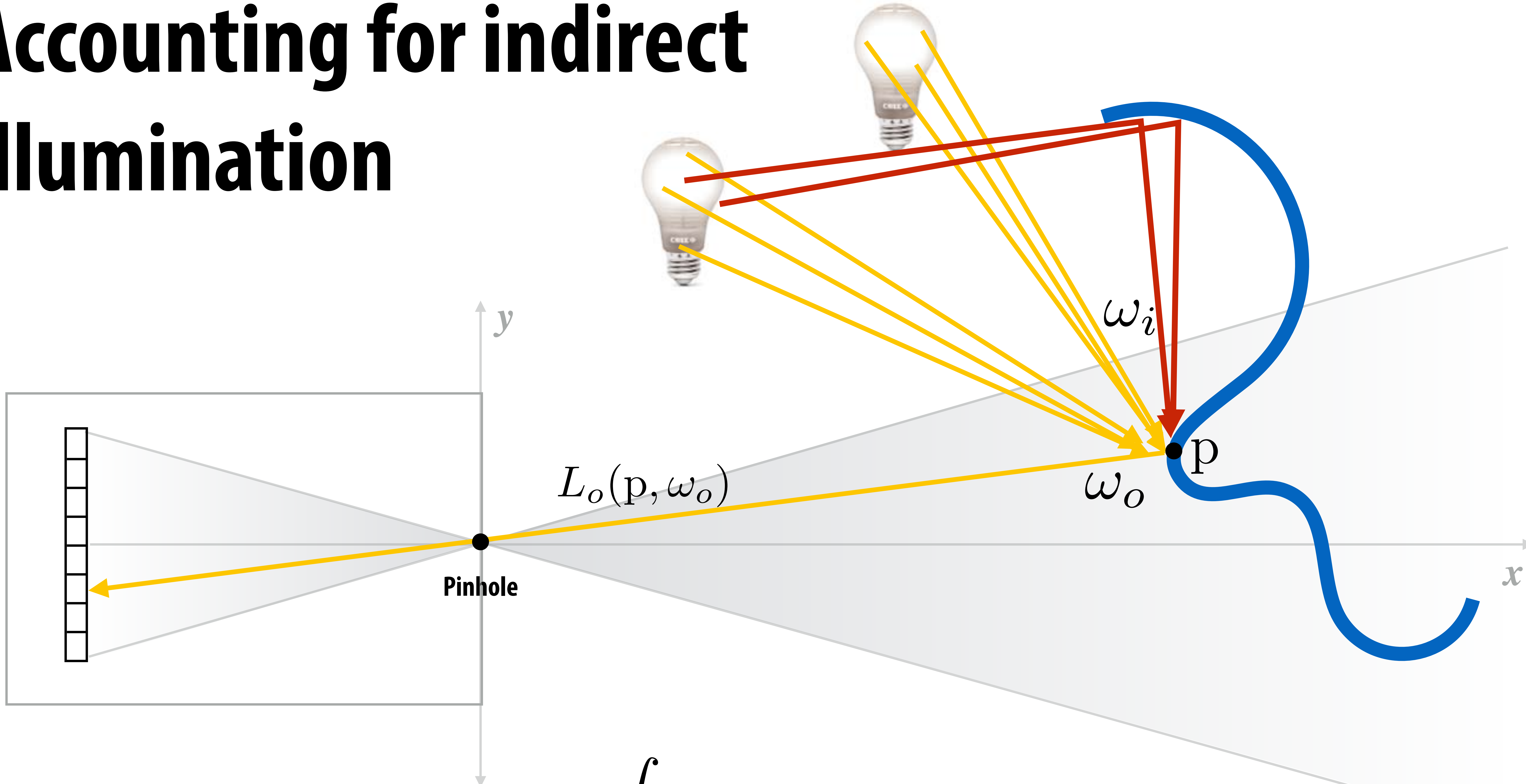


$$L_o(\mathrm{p}, \omega_o) = L_e(\mathrm{p}, \omega_o) + \int_{H^2} f_r(\mathrm{p}, \omega_i \to \omega_o)\, L_i(\mathrm{p}, \omega_i)\, \cos\theta_i \, \mathrm{d}\omega_i$$

Need to know incident radiance.

So far, have only computed incoming radiance from scene light sources.

Accounting for indirect illumination



$$L_o(\mathrm{p}, \omega_o) = L_e(\mathrm{p}, \omega_o) + \int_{H^2} f_r(\mathrm{p}, \omega_i \rightarrow \omega_o)\, L_i(\mathrm{p}, \omega_i)\, \cos\theta_i \,\mathrm{d}\omega_i$$

Incoming light energy from direction ω_i may be due to light reflected off another surface in the scene (not an emitter)

Path tracing: indirect illumination

$$\int_{H^2} f_r(\omega_i \to \omega_o)\, L_{o,i}(tr(\mathrm{p}, \omega_i), -\omega_i)\ \cos\theta_i\, \mathrm{d}\omega_i$$

- **Sample incoming direction from some distribution (e.g. proportional to BRDF):**

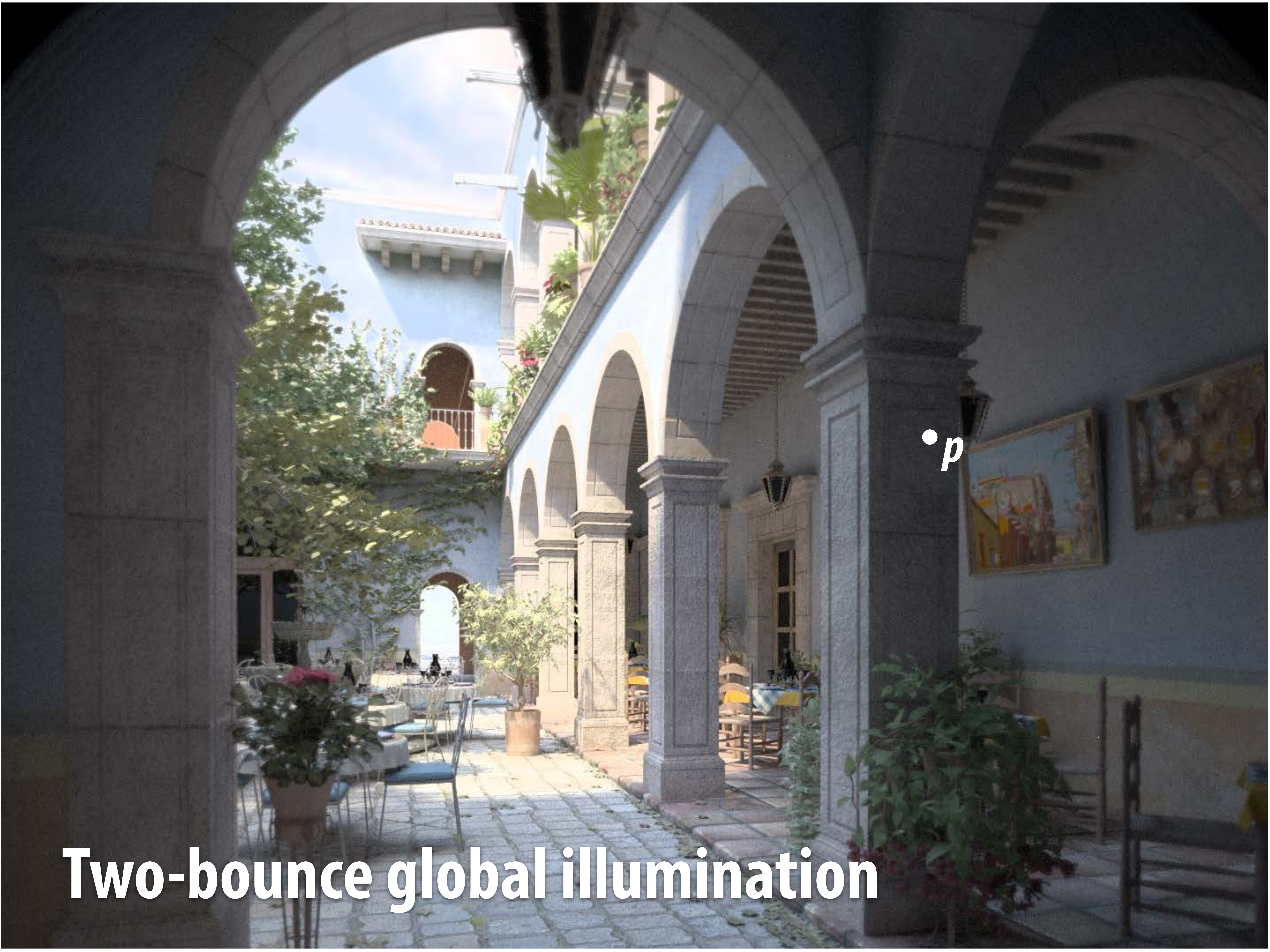
$$\omega_i \sim p(\omega)$$

- **Recursively call path tracing function to compute incident indirect radiance**
- **Monte Carlo estimator:**

$$\frac{f_r(\omega_i \to \omega_o)\, L_{o,i}(tr(\mathrm{p}, \omega_i), -\omega_i)\ \cos\theta_i}{p(\omega_i)}$$


•p
Direct illumination


p
One-bounce global illumination


p
Two-bounce global illumination

•p
Four-bounce global illumination

Eight-bounce global illumination

•p
Sixteen-bounce global illumination

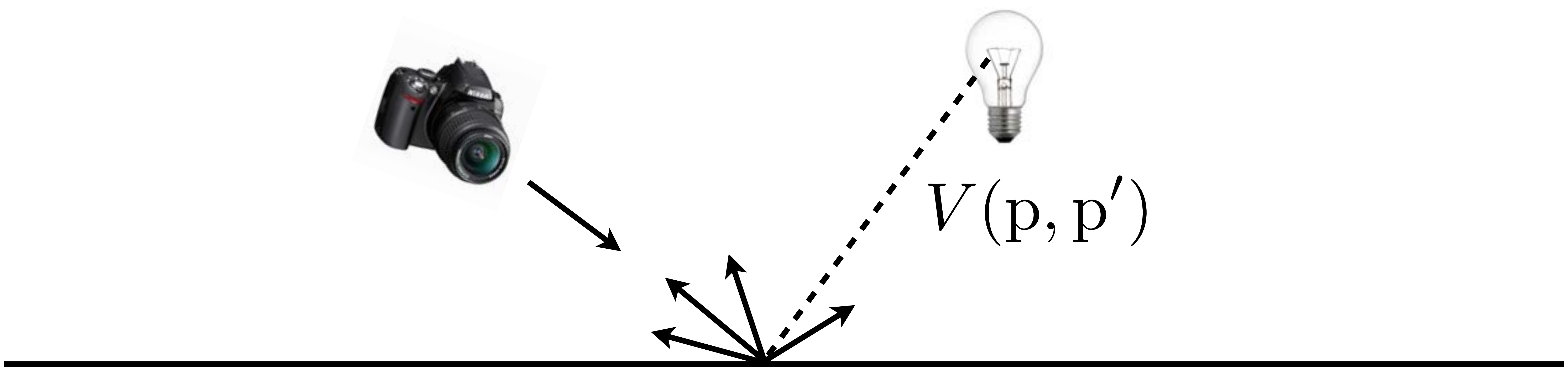
Wait a minute…

When do we *stop*?!

Russian roulette

- Idea: want to avoid spending time evaluating function for samples that make a small contribution to the final result
- Consider a low-contribution sample of the form:

$$L = \frac{f_r(\omega_i \to \omega_o)\, L_i(\omega_i)\, V(\mathrm{p}, \mathrm{p}')\, \cos\theta_i}{p(\omega_i)}$$



Russian roulette

$$L = \frac{f_r(\omega_i \to \omega_o)\, L_i(\omega_i)\, V(\mathrm{p}, \mathrm{p}')\ \cos\theta_i}{p(\omega_i)}$$

$$\downarrow$$

$$L = \left[\frac{f_r(\omega_i \to \omega_o)\, L_i(\omega_i)\ \cos\theta_i}{p(\omega_i)}\right]\ V(\mathrm{p}, \mathrm{p}')$$

- **If tentative contribution (in brackets) is small, total contribution to the image will be small regardless of** $V(\mathrm{p}, \mathrm{p}')$
- **Ignoring low-contribution samples introduces systemic error**
  - **No longer an unbiased estimator**
- **Instead, randomly discard low-contribution samples in a way that leaves estimator unbiased**

Russian roulette

- **New estimator: evaluate original estimator with probability p_{rr}, reweight. Otherwise ignore.**
- **Same expected value as original estimator:**

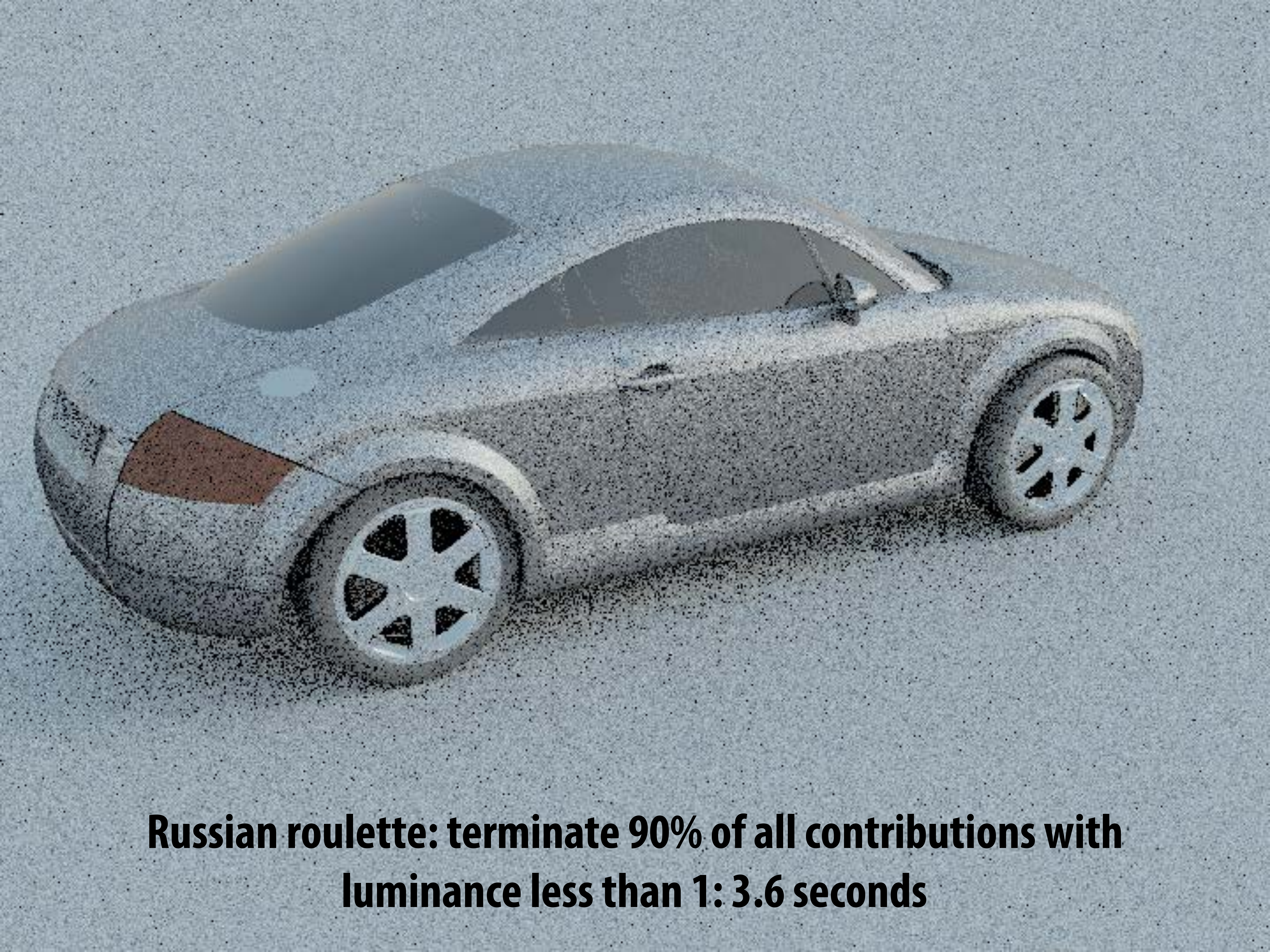
$$p_{\mathrm{rr}} E\left[\frac{X}{p_{\mathrm{rr}}}\right] + E[(1 - p_{\mathrm{rr}})0] = E[X]$$

No Russian roulette: 6.4 seconds

Russian roulette: terminate 50% of all contributions with luminance less than 0.25: 5.1 seconds

Russian roulette: terminate 50% of all contributions with luminance less than 0.5: 4.9 seconds

Russian roulette: terminate 90% of all contributions with luminance less than 0.125: 4.8 seconds



Russian roulette: terminate 90% of all contributions with luminance less than 1: 3.6 seconds

Next time:

- Variance reduction—how do we get the most out of our samples?