*Advanced Computer Graphics*

# *Rendering Equation*

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# *Outline*

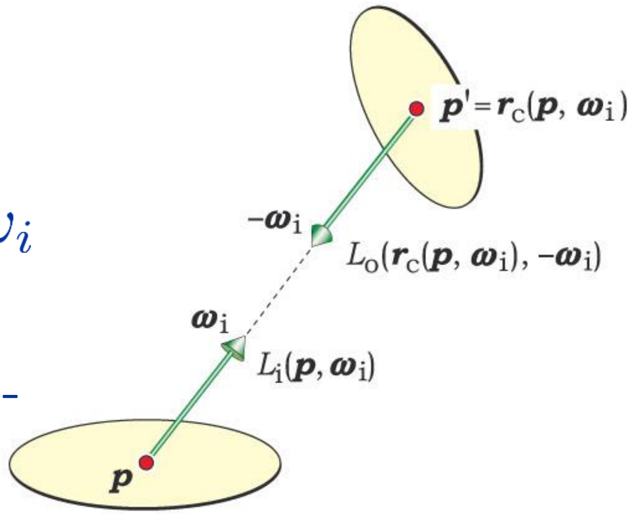
- rendering equation
- Monte Carlo integration
- sampling of random variables

# Reflection and Rendering Equation

- reflection equation at point p for reflective surfaces
  - $L_o(p,\omega_o) = \int_{2\pi^+} f_r(p,\omega_i,\omega_o) L_i(p,\omega_i) \cos\theta_i \mathrm{d}\omega_i$
  - incident radiance - weighted with the BRDF - is integrated over the hemisphere to compute the outgoing radiance
  - expresses energy balance between surfaces
  - outgoing radiance from a surface can be incident to another surface
- rendering equation at point p for reflective surfaces
  - $L_o(p,\omega_o) = L_e(p,\omega_o) + \int_{2\pi^+} f_r(p,\omega_i,\omega_o) L_i(p,\omega_i) \cos\theta_i \mathrm{d}\omega_i$
  - adds emissive surfaces to the reflection equation
  - exitant radiance is the sum of emitted and reflected radiance
  - expresses the steady state of radiance in a scene including light sources

# Ray-Casting Operator

- in general, the incoming radiance is not only determined by light sources, but also by outgoing radiance of reflective surfaces
- incident radiance $L_i(p, \omega_i)$ can be computed by tracing a ray from p into direction $\omega_i$
- ray-casting operator $r_c(p, \omega_i)$
  - nearest hit point from p into direction $\omega_i$
  - $L_i(p, \omega_i) = L_o(r_c(p, \omega_i), -\omega_i)$
  - if no surface is hit, radiance from a back-ground or light source can be returned

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# *Rendering Equation with Ray-Casting Operator*

- using the ray-casting operator,
  $L_o(p, \omega_o) = L_e(p, \omega_o) + \int_{2\pi^+} f_r(p, \omega_i, \omega_o) L_i(p, \omega_i) \cos\theta_i \mathrm{d}\omega_i$
  can be rewritten as
  $L_o(p, \omega_o) = L_e(p, \omega_o) + \int_{2\pi^+} f_r(p, \omega_i, \omega_o) L_o(r_c(p, \omega_i), -\omega_i) \cos\theta_i \mathrm{d}\omega_i$
- goal: computation of the outgoing radiance $L_o(p, \omega_o)$ at all points p into all directions $\omega_o$
  - towards the camera to compute the image
  - towards other surface points to account for indirect illumination

# Forms of the Rendering Equation

- hemisphere form

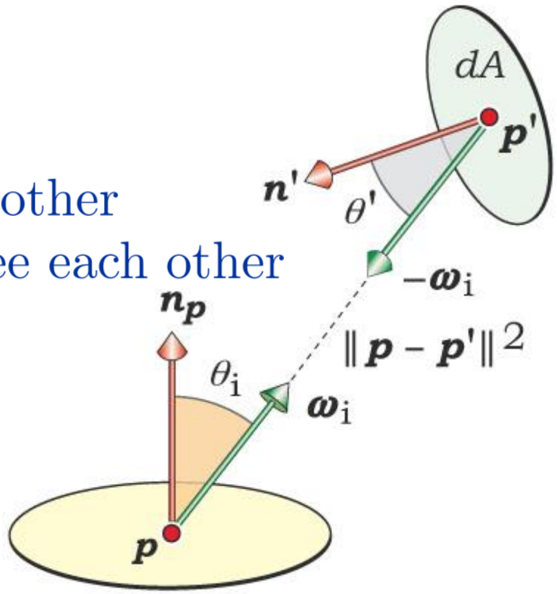
$$L_o(p,\omega_o) = L_e(p,\omega_o) + \int_{2\pi^+} f_r(p,\omega_i,\omega_o) L_o(r_c(p,\omega_i),-\omega_i)\cos\theta_i \mathrm{d}\omega_i$$

- area form
  - p is a sample point on a surface dA
  - visibility function

$$\forall(p,p') : V(p,p') = \begin{cases} 1 & \text{if p and p' see each other} \\ 0 & \text{if p and p' do not see each other} \end{cases}$$

  - solid angle vs. area $\mathrm{d}\omega_i = \frac{\cos\theta' \mathrm{d}A}{\|p'-p\|^2}$
  - $\cos\theta' = n' \cdot -\omega_i$

$$L_o(p,\omega_o) = L_e(p,\omega_o) +$$
$$\int_A f_r(p,\omega_i,\omega_o) L_o(p',-\omega_i) \frac{\cos\theta_i \cos\theta'}{\|p'-p\|^2} V(p,p') \mathrm{d}A$$

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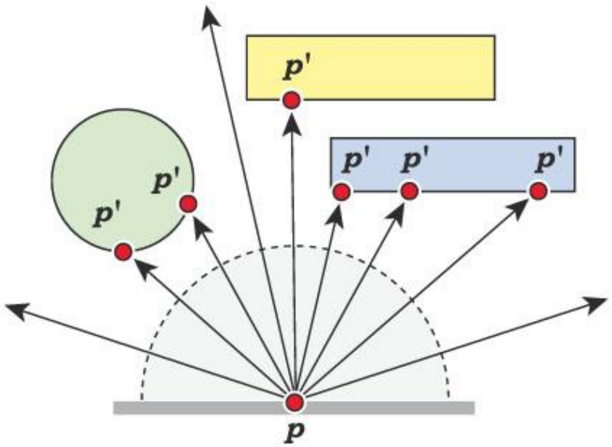
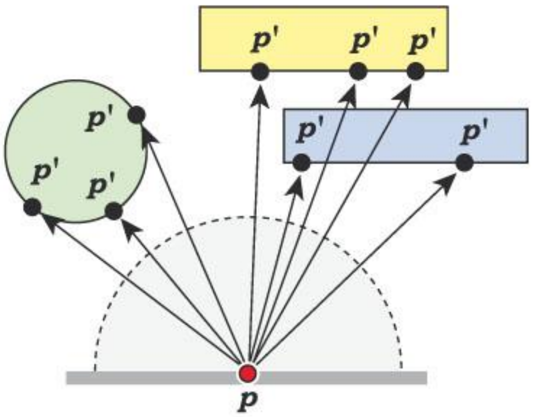
# Forms of the Rendering Equation

- the area form works with a visibility term
  - useful for direct illumination from area lights
- the hemisphere form works with the ray-casting operator
  - useful for indirect illumination




hemisphere form




area form

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# Solving the Rendering Equation

- recursively cast rays into the scene
- maximum recursion depth due to absorption of light
- for point lights, directional lights, perfect reflection and transmission, the integrals reduce to simple sums
  - radiance from only a few directions contributes to the outgoing radiance
- for area lights and indirect illumination, i. e. diffuse-diffuse light transport, Monte Carlo techniques are used to numerically evaluate the multi-dimensional integrals

# *Outline*

- rendering equation
- Monte Carlo integration
- sampling of random variables

# Introduction

- approximately evaluate the integral
  $\int_{2\pi^+} f_r(p, \omega_i, \omega_o) L_o(r_c(p, \omega_i), -\omega_i) \cos\theta_i \mathrm{d}\omega_i$ by
  - randomly sampling the hemisphere
  - tracing rays into the sample directions
  - computing the incoming radiance from the sample directions
- challenge
  - approximate the integral as exact as possible
  - trace as few rays as possible
  - trace relevant rays
    - for diffuse surfaces, rays in normal direction are more relevant than rays perpendicular to the normal
    - for specular surfaces, rays in reflection direction are relevant
    - rays to light sources are relevant

# Properties

- benefits
  - processes only evaluations of the integrand at arbitrary points in the domain
  - works for a large variety of integrands, e.g., it handles discontinuities
  - appropriate for integrals of arbitrary dimensions
- drawbacks
  - using n samples, the scheme converges to the correct result with O ($n^{½}$), i.e. to half the error, 4n samples are required
  - errors are perceived as noise, i.e. pixels are arbitrarily too bright or dark
  - evaluation of the integrand at a point is expensive

# Continuous Random Variables

- continuous random variables $X$ infinite number of possible values (in contrast to discrete random variable)
- canonical uniform random variable $0 \leq \xi < 1$
  - samples from arbitrary distributions can be computed from $\xi$
- probability density function (PDF) $p(x)$
  - the probability of a random variable taking certain value ranges
  - $Pr(x_0 \leq X \leq x_1) = \int_{x_0}^{x_1} p(x)\mathrm{d}x$ The probability, that the random variable has a certain exact value, is 0.
  - $p(x) \geq 0 \quad \forall x \in [a, b]$
  - $\int_a^b p(x)\mathrm{d}x = 1$ The probability, that the random variable is in the specified domain, is 1.
- cumulative distribution function (CDF) $P(x)$
  - describes the probability of a random variable to be less or equal to x
  - $Pr(X \leq x) = P(x) \quad Pr(x_0 \leq X \leq x_1) = P(x_1) - P(x_0)$
  - $0 \leq P(x) \leq 1$

# Expected Value

- motivation: expected value of an estimator function is equal to the integral in the rendering equation
- expected value $E_p[f(x)]$ of a function $f(x)$ is defined as the weighted average value of the function over a domain $D$
- $E_p[f(x)] = \int_D f(x)\ p(x)\ \mathrm{d}x$ with $\int_D p(x)\ \mathrm{d}x = 1$

  processes an infinite number of samples x according to a PDF p(x)
- properties
  - $E[af(x)] = aE[f(x)]$
  - $E\left[\sum_i f(X_i)\right] = \sum_i E[f(X_i)]$ for independent random variables $X_i$
- example for uniform $p$

  $E_p[cos(x)] = \int_0^\pi cos(x)\ \frac{1}{\pi}\ \mathrm{d}x = \frac{1}{\pi}(-\sin\pi + \sin 0) = 0$

# Variance

- motivation: quantifies the error of a Monte Carlo algorithm
- variance $V$ of a function is the expected deviation of the function from its expected value $V[f(x)] = E\left[(f(x) - E[f(x)])^2\right]$
- properties
  - $V[af(x)] = a^2 V[f(x)]$
  - $V[f(x)] = E\left[(f(x))^2\right] - E[f(x)]^2$
  - $\sum_i V[f(X_i)] = V\left[\sum_i f(X_i)\right]$ for independent random variables $X_i$



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# Monte Carlo Estimator
# Uniform Random Variables

- motivation: approximation of the integral in the rendering equation
- goal: computation of $\int_a^b f(x)\mathrm{d}x$
- uniformly distributed random variables $X_i \in [a, b]$
- probability density function $p(x) = \frac{1}{b-a}$ constant and integration to one
- Monte Carlo estimator $F_N = \frac{b-a}{N} \sum_{i=1}^{N} f(X_i)$
- expected value of $F_N$ is equal to the integral $\int_a^b f(x)\mathrm{d}x$
  - $E[F_N] = \int_a^b f(x)\mathrm{d}x$
- variance $V = \frac{1}{N-1} \sum_{i=1}^{N} [f(X_i) - E[F_N]]^2$
  - convergence rate of $O(\sqrt{N})$
  - independent from the dimensionality
    $\Rightarrow$ appropriate for high-dimensional integrals

# Monte Carlo Estimator
# Uniform Random Variables

- $$E[F_N] = E\left[\frac{b-a}{N}\sum_{i=1}^{N} f(X_i)\right]$$
$$= \frac{b-a}{N}\sum_{i=1}^{N} E[f(X_i)]$$
$$= \frac{b-a}{N}\sum_{i=1}^{N} \int_a^b f(x)p(x)\mathrm{d}x$$
$$= \frac{b-a}{N}\sum_{i=1}^{N} \int_a^b f(x)\frac{1}{b-a}\mathrm{d}x$$
$$= \frac{1}{N}\sum_{i=1}^{N} \int_a^b f(x)\mathrm{d}x$$
$$= \int_a^b f(x)\mathrm{d}x$$

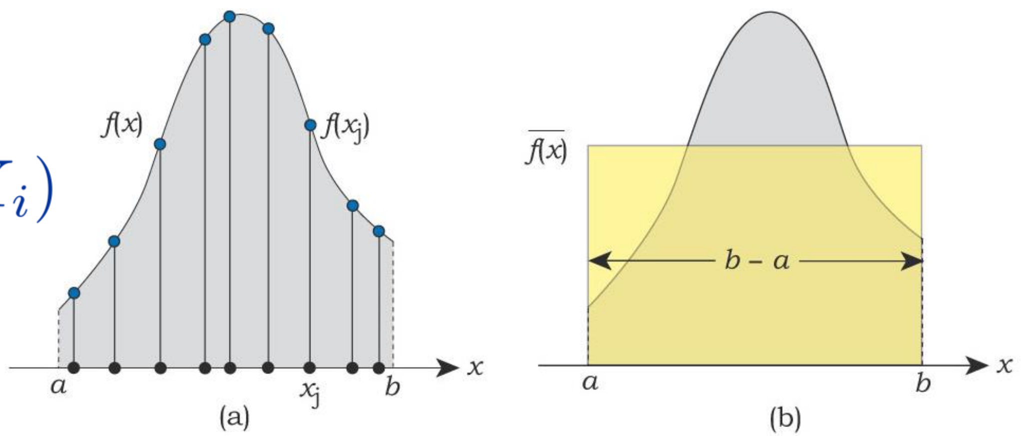
# Examples - Uniform Random Variables

- integral $\int_0^1 5x^4 \mathrm{d}x = 1$
- estimator $F_N = \frac{1}{N} \sum_{i=1}^{N} 5X_i^4$
- for an increasing number of uniformly distributed random variables $X_i$, the estimator converges to one
- $F_N = \frac{b-a}{N} \sum_{i=1}^{N} f(X_i)$

  $F_N = (b-a) \frac{1}{N} \sum_{i=1}^{N} f(X_i)$

  $F_N = (b-a) \overline{f(x)}$

  $E[F_N] = \int_a^b f(x) \mathrm{d}x$




uniformly distributed random samples

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# Monte Carlo Estimator
# Non-uniform Random Variables

- Monte Carlo estimator $F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)}$ $p(X_i) \neq 0$

- $$\begin{aligned} E[F_N] &= E\left[\frac{1}{N} \sum_{i=1}^{N} \frac{f(X_i)}{p(X_i)}\right] \\ &= \frac{1}{N} \sum_{i=1}^{N} \int_a^b \frac{f(x)}{p(x)} p(x) \mathrm{d}x \\ &= \frac{1}{N} \sum_{i=1}^{N} \int_a^b f(x) \mathrm{d}x \\ &= \int_a^b f(x) \mathrm{d}x \end{aligned}$$

# Monte Carlo Estimator
# Multiple Dimensions

- samples $X_i$ are multidimensional
- e.g. $\int_{x_0}^{x_1} \int_{y_0}^{y_1} \int_{z_0}^{z_1} f(x,y,z) \mathrm{d}x \mathrm{d}y \mathrm{d}z$
- uniformly distributed random samples

  $(x_0, y_0, z_0) \leq X_i = (x_i, y_i, z_i) \leq (x_1, y_1, z_1)$
- probability density function $p(X_i) = \frac{1}{x_1 - x_0} \frac{1}{y_1 - y_0} \frac{1}{z_1 - z_0}$
- Monte Carlo estimator

  $F_N = \frac{(x_1 - x_0)(y_1 - y_0)(z_1 - z_0)}{N} \sum_{i=1}^{N} f(X_i)$
- N can be arbitrary, N is independent from the dimensionality



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# Monte Carlo Estimator
# Integration over a Hemisphere

- approximate computation of the irradiance at a point

$$E_i(p) = \int_{2\pi^+} L_i(p,\omega)\cos\theta d\omega$$
$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}} L_i(p,\theta,\phi)\cos\theta\sin\theta \mathrm{d}\theta\mathrm{d}\phi$$

- estimator

$$F_N = \frac{1}{N}\sum_{i=1}^{N}\frac{f(X_i)}{p(X_i)} = \frac{1}{N}\sum_{i=1}^{N}\frac{L_i(p,\theta,\phi)\cos\theta\sin\theta}{p(\theta,\phi)}$$

- probability distribution
  - should be similar to the shape of the integrand
  - as incident radiance is weighted with cos $\theta$, it is appropriate to generate more samples close to the top of the hemisphere
  - $p(\omega) \propto \cos\theta$

# Monte Carlo Estimator
# Integration over a Hemisphere

- probability distribution (cont.)

$$\int_{2\pi^+} c\, p(\omega) \mathrm{d}\omega = 1$$

$$\int_0^{2\pi} \int_0^{\frac{\pi}{2}} c \; \cos\theta \sin\theta \; \mathrm{d}\theta \; \mathrm{d}\phi = 1$$

$$c \, \frac{2\pi}{1+1} = 1$$

$$c = \frac{1}{\pi}$$

$$p(\theta, \phi) = \frac{\cos\theta \sin\theta}{\pi}$$

- estimator

$$F_N = \frac{1}{N} \sum_{i=1}^{N} \frac{L_i(p,\theta,\phi) \cos\theta \sin\theta}{p(\theta,\phi)}$$

$$= \frac{\pi}{N} \sum_{i=1}^{N} L_i(p, \theta, \phi)$$

# Monte Carlo Integration
## Steps

- choose an appropriate probability density function
- generate random samples according to the PDF
- evaluate the function for all samples
- average the weighted function values

# Monte Carlo Estimator
# Error

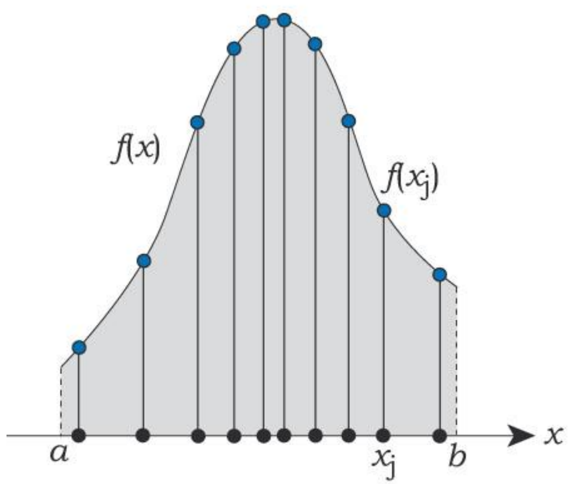
- variance

  $V = \frac{1}{N} \int_a^b \left( \frac{f(x)}{p(x)} - F_N \right)^2 p(x) \mathrm{d}x$
- estimator

  $V_N = \frac{1}{N-1} \sum_{i=1}^{N} \left[ f(X_i) - F_N \right]^2$
- for increasing N
  - the variance decreases with O(N)
  - the standard deviation decreases with O(N½)
- variance is perceived as noise

# Monte Carlo Estimator
# Variance Reduction / Error Reduction

- importance sampling
  - motivation: contributions of larger function values are more important
  - PDF should be similar to the shape of the function
  - optimal PDF $p(x) = \frac{f(x)}{\int f(x)\mathrm{d}x}$
  - e.g., if incident radiance is weighted with cos $\theta$, the PDF should choose more samples close to the normal direction
- stratified sampling
  - domain subdivision into strata does not increase the variance
- multi-jittered sampling
  - alternative to random samples for, e.g., uniform sampling of area lights

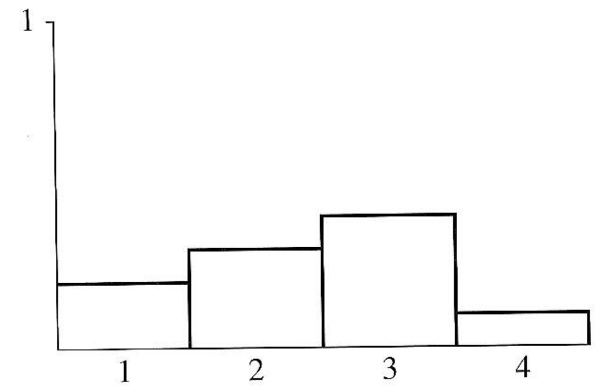


# *Outline*

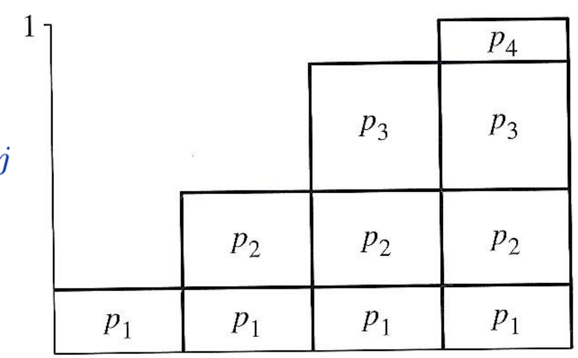
- rendering equation
- Monte Carlo integration
- sampling of random variables
  - inversion method
  - rejection method
  - transforming between distributions
  - 2D sampling
  - examples

# Inversion Method

- mapping of a uniform random variable to a goal distribution
- discrete example
  - four outcomes with probabilities $p_1, p_2, p_3, p_4$ and $\sum_i p_i = 1$




  - computation of the cumulative distribution function $P(i) = \sum_{j=1}^{i} p_j$

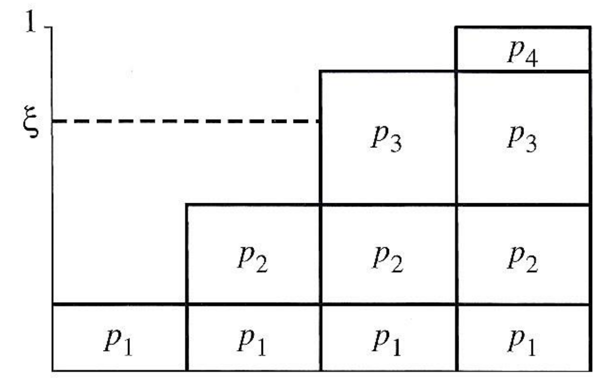





[Pharr, Humphreys]

# Inversion Method

- discrete example cont.
  - take a uniform random variable $\xi$
  - $P^{-1}(\xi)$ has the desired distribution
- continuous case
  - $P$ and $P^{-1}$ are continuous functions
  - start with the desired PDF $p(x)$
  - compute $P(x) = \int_0^x p(x')\mathrm{d}x'$
  - compute the inverse $P^{-1}(x)$
  - obtain a uniformly distributed variable
  - compute $X_i = P^{-1}(\xi)$ which adheres to $p(x)$

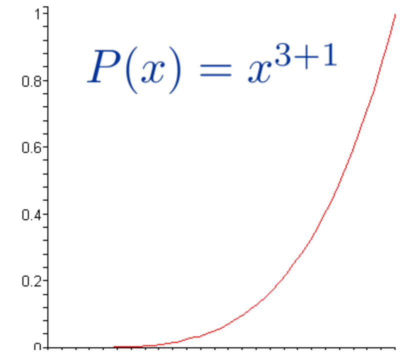


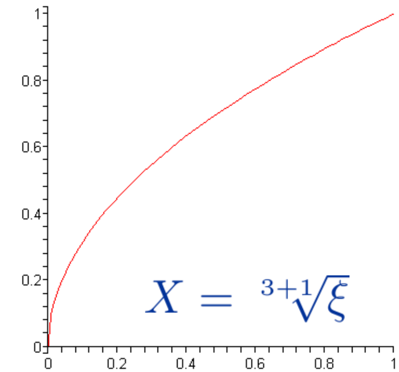



[Pharr, Humphreys]

# Inversion Method
# Example 1

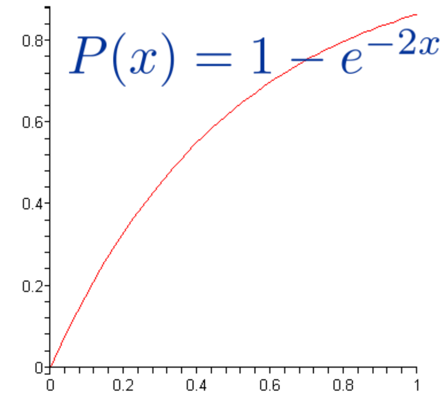
- power distribution $p(x) \propto x^n$
  - e.g., for sampling the Blinn microfacet model
- computation of the PDF
  - $\int_0^1 c\, x^n \mathrm{d}x = 1 \Rightarrow c \left.\frac{x^{n+1}}{n+1}\right|_0^1 = 1 \Rightarrow c = n+1$
- PDF $p(x) = (n+1)x^n$
- CDF $P(x) = \int_0^x p(x')\mathrm{d}x' = x^{n+1}$
- inverse of the CDF $P^{-1}(x) = \sqrt[n+1]{x}$
- sample generation
  - generate uniform random samples $0 \leq \xi \leq 1$
  - $X = \sqrt[n+1]{\xi}$ are samples from the power distribution $p(x) = (n+1)x^n$

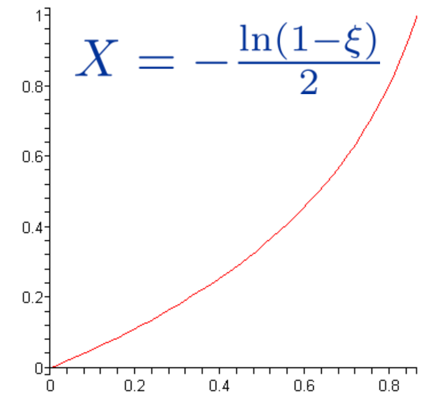


# Inversion Method
# Example 2

- exponential distribution $p(x) \propto e^{-ax}$
  - e.g., for considering participating media
- computation of the PDF
  - $\int_0^\infty c\ e^{-ax} \mathrm{d}x = -\frac{c}{a}\ e^{-ax}\big|_0^\infty = \frac{c}{a} = 1$
- PDF $p(x) = a\ e^{-ax}$
- CDF $P(x) = \int_0^x p(x')\mathrm{d}x' = 1 - e^{-ax}$
- inverse of the CDF $P^{-1}(x) = -\frac{\ln(1-x)}{a}$
- sample generation
  - generate uniform random samples $0 \le \xi \le 1$
  - $X = -\frac{\ln(1-\xi)}{a}$ are samples from the power distribution $p(x) = a\ e^{-ax}$

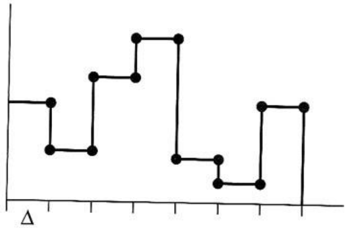


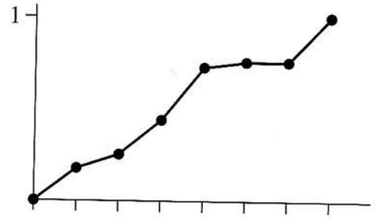
# Inversion Method
# Example 3

- piecewise-constant distribution
  - e.g., for environment lighting

$$f(x) = \begin{cases} v_0 & x_0 \leq x < x_1 \\ v_1 & x_1 \leq x < x_2 \\ \vdots & \vdots \end{cases} \qquad x_i = \Delta \cdot i \qquad \Delta = \frac{1}{N}$$




$p(x)$




$P(x)$

- PDF $p(x) = \frac{1}{c} f(x)$
  with $c = \int_0^1 f(x)\mathrm{d}x = \sum_{i=0}^{N-1} \Delta \cdot v_i = \sum_{i=0}^{N-1} \frac{v_i}{N}$



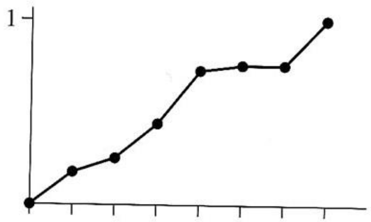
[Pharr, Humphreys]

# Inversion Method
# Example 3

- CDF

$$P(x_0) = 0$$
$$P(x_1) = \int_{x_0}^{x_1} p(x)\mathrm{d}x = \Delta \cdot \frac{v_0}{c} = \frac{v_0}{Nc} = P(x_0) + \frac{v_0}{Nc}$$
$$P(x_2) = \int_{x_0}^{x_2} p(x)\mathrm{d}x = \int_{x_0}^{x_1} p(x)\mathrm{d}x + \int_{x_1}^{x_2} p(x)\mathrm{d}x = P(x_1) + \frac{v_1}{Nc}$$
$$P(x_i) = P(x_{i-1}) + \frac{v_{i-1}}{Nc}$$

- CDF is linear between $x_i$ and $x_{i+1}$ with slope $\frac{v_i}{c}$
- sample generation
  - generate uniform random samples $0 \leq \xi \leq 1$
  - compute $x_i$ with $P(x_i) \leq \xi$ and $\xi < P(x_{i+1})$
  - compute $d$ with $P(x_i) + d(P(x_{i+1}) - P(x_i)) = \xi$
  - $X = x_i + d(x_{i+1} - x_i) = x_i + \frac{d}{N}$ are samples from $p(x) = \frac{1}{c} f(x)$



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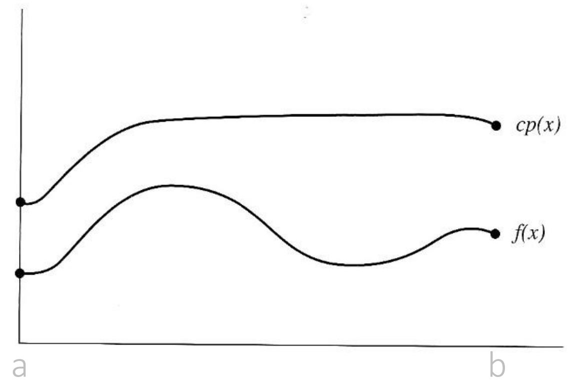
  - inversion method
  - rejection method
  - transforming between distributions
  - 2D sampling
  - examples

# Rejection Method

- draws samples according to a function $f(x)$
  - dart-throwing approach
  - works with a PDF $p(x)$ and a scalar $c$ with $f(x) < c \cdot p(x)$
- properties
  - $f(x)$ is not necessarily a PDF
  - PDF, CDF and inverse CDF do not have to be computed
  - simple to implement
  - useful for debugging purposes
- sample generation
  - generate a uniform random sample $0 \leq \xi < 1$
  - generate a sample $X$ according to $p(x)$
  - accept $X$ if $\xi \cdot c \cdot p(X) \leq f(X)$

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# *Transforming Between Distributions*

- computation of a resulting PDF, when a function is applied to samples from an arbitrary distribution
  - random variables $X_i$ are drawn from $p_x(x)$
  - bijective transformation (one-to-one mapping) $Y_i = y(X_i)$
  - How does the distribution $p_y(y)$ look like?
- $Pr\{Y \leq y(x)\} = Pr\{X \leq x\}$

  $P_y(y) = P_y(y(x)) = P_x(x)$

  $p_y(y) = \frac{p_x(x)}{|y'(x)|}$
- example $p_x(x) = 2x \quad 0 \leq x \leq 1$
  - $y(x) = \sin x \qquad x(y) = \arcsin y$
  - $y'(x) = \cos x$
  - $p_y(y) = \frac{p_x(x)}{|\cos x|} = \frac{2 \arcsin y}{\sqrt{1-y^2}}$

# *Transforming Between Distributions*

- multiple dimensions
  - $X_i$ is an n-dimensional random variable
  - $Y_i = T(X_i)$ is a bijective transformation
- transformation of the PDF
  - $p_y(y) = \frac{p_x(x)}{|J_T(x)|} \quad J_T(x) = \begin{pmatrix} \frac{\partial T_1}{\partial x_1} & \cdots & \frac{\partial T_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial T_n}{\partial x_1} & \cdots & \frac{\partial T_n}{\partial x_n} \end{pmatrix}$
- example (polar coordinates)
  - samples $(r, \theta)$ with density $p(r, \theta)$
  - corresponding density $p(x, y)$ with $x = r\cos\theta$ and $y = r\sin\theta$
  - $J_T(x) = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix} \quad |J_T(x)| = r(\cos^2\theta + \sin^2\theta) = r$
  - $p(x, y) = \frac{1}{r} p(r, \theta) \quad p(r, \theta) = r \cdot p(x, y)$

# Transforming Between Distributions

- example (spherical coordinates)
  - $x = r \sin\theta \cos\phi$
  - $y = r \sin\theta \sin\phi$
  - $z = r \cos\theta$
  - $p(r, \theta, \phi) = r^2 \sin\theta \cdot p(x, y, z)$
- example (solid angle)
  - $Pr\{\omega \in \Omega\} = \int_\Omega p(\omega) \mathrm{d}\omega$
  - $\mathrm{d}\omega = \sin\theta \ \mathrm{d}\theta \ \mathrm{d}\phi$
  - $p(\theta, \phi) = \sin\theta \cdot p(\omega)$

# *Outline*

- rendering equation
- Monte Carlo integration
- sampling of random variables
  - inversion method
  - rejection method
  - transforming between distributions
  - 2D sampling
  - examples

# Concept

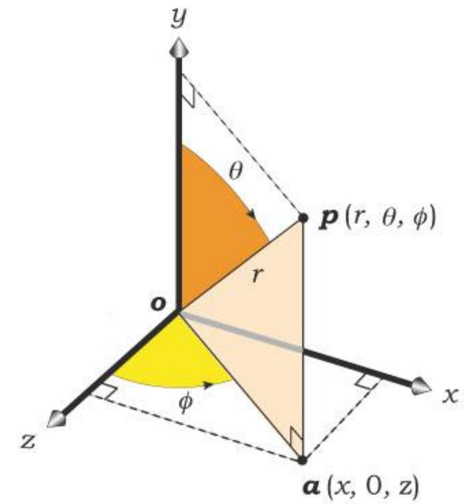
- generation of samples from a 2D joint density function $p(x,y)$
- general case
  - compute the marginal density function $p_x(x) = \int p(x,y)\mathrm{d}y$
  - compute the conditional density function $p_y(y|x) = \frac{p(x,y)}{p_x(x)}$
  - generate a sample $X$ according to $p_x(x)$
  - generate a sample $Y$ according to $p_y(y|X) = \frac{p(x,y)}{p_x(X)}$
- marginal density function
  - integral of $p(x,y)$ for a particular $x$ over all $y$-values
- conditional density function
  - density function for $y$ given a particular $x$

# Outline

- rendering equation
- Monte Carlo integration
- sampling of random variables
  - inversion method
  - rejection method
  - transforming between distributions
  - 2D sampling
  - examples

# Uniform Sampling of a Hemisphere

- PDF is constant with respect to a solid angle $p(\omega) = c$
- $\int_{2\pi^+} p(\omega)\mathrm{d}\omega = 1 \;\Rightarrow\; c\int_{2\pi^+} \mathrm{d}\omega = 1 \;\Rightarrow\; c = \frac{1}{2\pi}$
- $p(\omega) = \frac{1}{2\pi} \;\Rightarrow\; p(\theta, \phi) = \frac{\sin\theta}{2\pi}$
- marginal density function
  - $p_\theta(\theta) = \int_0^{2\pi} p(\theta, \phi)\mathrm{d}\phi = \int_0^{2\pi} \frac{\sin\theta}{2\pi}\mathrm{d}\phi = \sin\theta$
- conditional density for $\phi$
  - $p_\phi(\phi|\theta) = \frac{p(\theta,\phi)}{p_\theta(\theta)} = \frac{1}{2\pi}$
- inversion method
  - $P_\theta(\theta) = \int_0^\theta \sin\theta' \mathrm{d}\theta' = -\cos\theta + 1$
  - $P_\phi(\phi|\theta) = \int_0^\phi \frac{1}{2\pi}\mathrm{d}\phi' = \frac{\phi}{2\pi}$





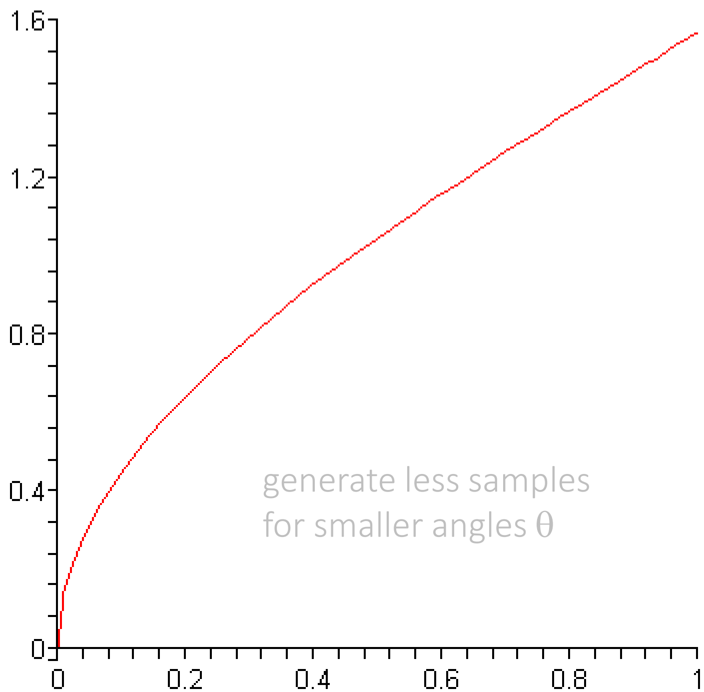

[Suffern]

# Uniform Sampling of a Hemisphere

- inversion method cont.
  - inverse functions of the cumulative distribution functions
  - $\theta = \arccos(1 - \xi_1)$
  - $\phi = 2\pi\xi_2$
  - generating uniformly sampled random values $\xi_1$ and $\xi_2$
  - applying the inverse CDFs to obtain $\theta$ and $\phi$
- conversion to Cartesian space
  - $x = \sin\theta\cos\phi = \cos(2\pi\xi_2)\sqrt{1 - (1 - \xi_1)^2}$
  - $y = \sin\theta\sin\phi = \sin(2\pi\xi_2)\sqrt{1 - (1 - \xi_1)^2}$
  - $z = \cos\theta = 1 - \xi_1$
- $(x, y, z)^T$ is a normalized direction

# Uniform Sampling of a Hemisphere

- illustration for $\theta$



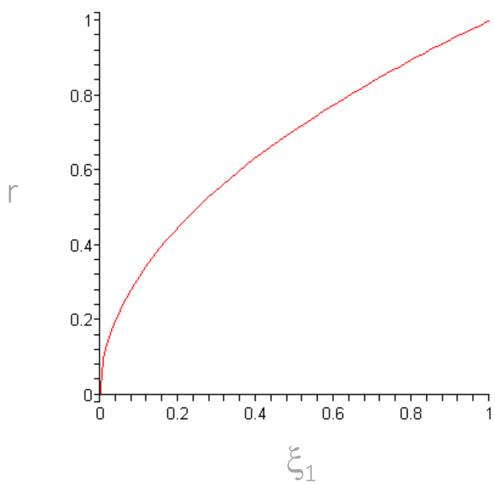

$$\theta = \arccos(1 - \xi_1)$$

# *Uniform Sampling of a Unit Disk*

- PDF is constant with respect to area $p(x,y) = \frac{1}{\pi}$
- $p(r,\theta) = r \cdot p(x,y) \;\Rightarrow \frac{r}{\pi}$
- marginal density function
  - $p_r(r) = \int_0^{2\pi} p(r,\theta)\mathrm{d}\theta = 2r$
- conditional density
  - $p_\theta(\theta|r) = \frac{p(r,\theta)}{p_r(r)} = \frac{1}{2\pi}$
- inversion method
  - $P_r(r) = \int_0^r 2r'\mathrm{d}r' = r^2$
  - $P_\theta(\theta|r) = \int_0^{2\pi} \frac{1}{2\pi}\mathrm{d}\theta' = \frac{\theta}{2\pi}$

# Uniform Sampling of a Unit Disk

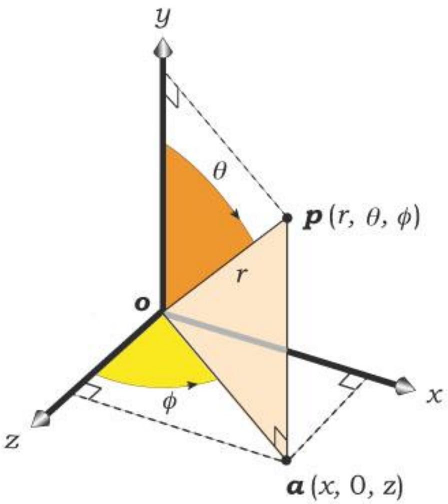
- inversion method cont.
  - inverse functions of the cumulative distribution functions
  - $r = \sqrt{\xi_1}$
  - $\theta = 2\pi\xi_2$
  - generating uniformly sampled random values $\xi_1$ and $\xi_2$
  - applying the inverse CDFs to obtain $r$ and $\theta$




generate less samples
for smaller radii

# Uniform Sampling of a Cosine-Weighted Hemisphere

- PDF is proportional to $\cos\theta$ $\quad p(\omega) \propto \cos\theta$
  - $\int_{2\pi^+} c\, p(\omega)\ \mathrm{d}\omega = 1 = \int_0^{2\pi} \int_0^{\frac{\pi}{2}} c\ \cos\theta \sin\theta\ \mathrm{d}\theta\ \mathrm{d}\phi = c\ 2\pi \int_0^{\frac{\pi}{2}} \cos\theta\sin\theta\ \mathrm{d}\theta$
  - $p(\theta,\phi) = \frac{1}{\pi}\cos\theta\sin\theta$ $\qquad = c\ 2\pi\frac{1}{2} = 1$
- marginal density function
  - $p_\theta(\theta) = \int_0^{2\pi} p(\theta,\phi)\mathrm{d}\phi = \int_0^{2\pi}\frac{1}{\pi}\cos\theta\sin\theta\mathrm{d}\phi = 2\cos\theta\sin\theta$
- conditional density for $\phi$
  - $p_\phi(\phi|\theta) = \frac{p(\theta,\phi)}{p_\theta(\theta)} = \frac{1}{2\pi}$
- inversion method
  - $P_\theta(\theta) = \int_0^\theta 2\cos\theta'\sin\theta'\mathrm{d}\theta' = 2\left[-\frac{\cos^2\theta'}{2}\right]_0^\theta$
    $= 2\left(-\frac{\cos^2\theta}{2} + \frac{1}{2}\right) = \sin^2\theta$
  - $P_\phi(\phi|\theta) = \int_0^\phi \frac{1}{2\pi}\mathrm{d}\phi' = \frac{\phi}{2\pi}$






[Suffern]

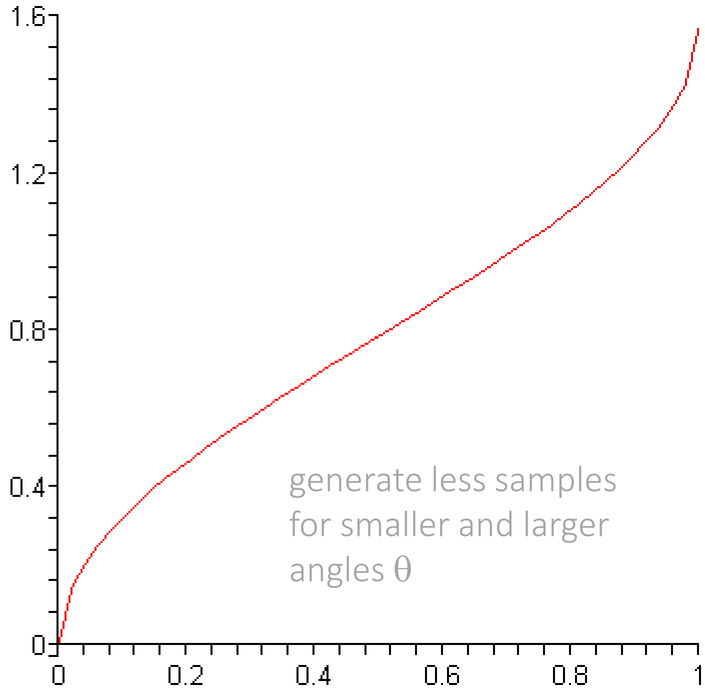
# Uniform Sampling of a Cosine-Weighted Hemisphere

- inversion method cont.
  - inverse functions of the cumulative distribution functions
  - $\theta = \arcsin(\sqrt{\xi_1})$
  - $\phi = 2\pi\xi_2$
  - generating uniformly sampled random values $\xi_1$ and $\xi_2$
  - applying the inverse CDFs to obtain $\theta$ and $\phi$
- conversion to Cartesian space
  - $x = \sin\theta\cos\phi = \cos(2\pi\xi_2)\sqrt{\xi_1}$
  - $y = \sin\theta\sin\phi = \sin(2\pi\xi_2)\sqrt{\xi_1}$
  - $z = \cos\theta = \sqrt{1-\xi_1}$
- $(x, y, z)^T$ is a normalized direction

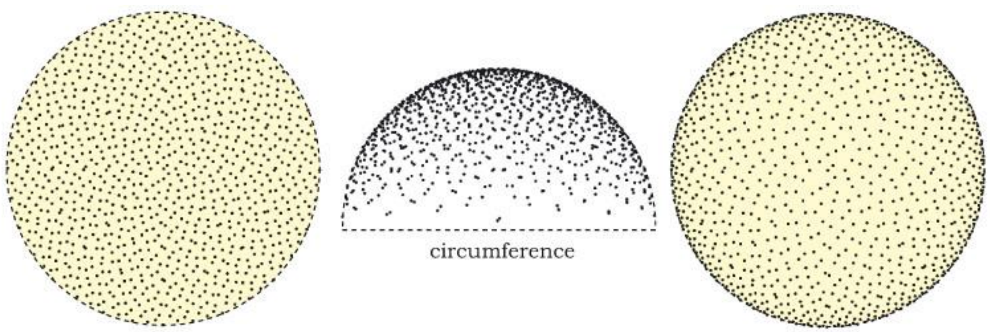
x- y- values uniformly sample a unit disk, i. e., cosine-weighted samples of the hemisphere can also be obtained by uniformly sampling a unit sphere and projecting the samples onto the hemisphere

# *Uniform Sampling of a Cosine-Weighted Hemisphere*

- illustration for $\theta$




$$\theta = \arcsin(\xi_1)$$




cosine-weighted hemisphere (top view, side view)

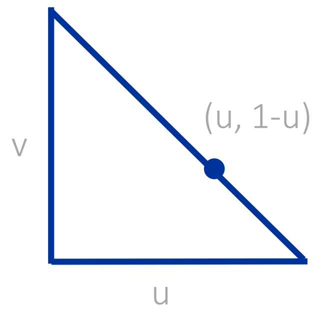
uniform hemisphere (top view)



[Suffern]

# Uniform Sampling of a Triangle

- sampling an isosceles right triangle of area 0.5
  - u, v can be interpreted as Barycentric coordinates
  - can be used to generate samples for arbitrary triangles




- $p(u,v) = 2$
- marginal density function
  - $p_u(u) = \int_0^{1-u} p(u,v)\ \mathrm{d}v = 2\int_0^{1-u} \mathrm{d}v = 2(1-u)$
- conditional density
  - $p_v(v|u) = \frac{p(u,v)}{p_u(u)} = \frac{1}{1-u}$
- inversion method
  - $P_u(u) = \int_0^u 2 - 2u' \mathrm{d}u' = 2u - u^2$
  - $P_v(v|u) = \int_0^v \frac{1}{1-u} \mathrm{d}v' = \frac{v}{1-u}$



[Pharr, Humphreys]

# *Uniform Sampling of a Triangle*

- inversion method cont.
  - inverse functions of the cumulative distribution functions
  - $u = 1 - \sqrt{\xi_1}$
  - $v = \xi_2 \sqrt{\xi_1}$
  - generating uniformly sampled random values $\xi_1$ and $\xi_2$
  - applying the inverse CDFs to obtain $u$ and $v$

# Piecewise-Constant 2D Distribution

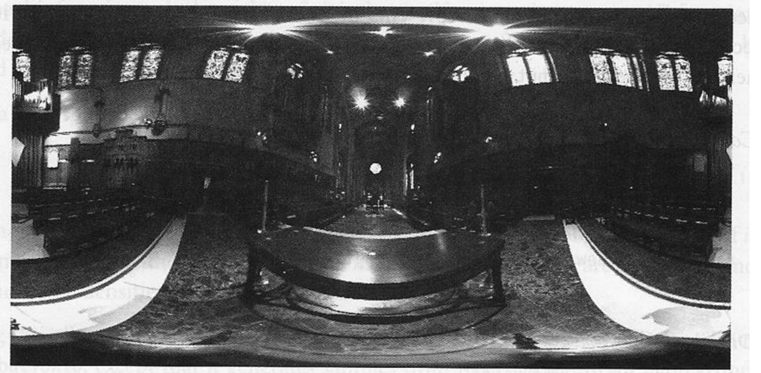
- $n_u \times n_v$ samples defined over $(u, v) \in [0, 1]^2$
  - e.g., an environment map
- $f(u, v)$ is defined by a set of $n_u \times n_v$ values $f[u_i, v_i]$
  - $u_i \in [0, \ldots, n_u - 1] \quad v_i \in [0, \ldots, n_v - 1]$
  - $f[u_i, v_i]$ is the value of $f(u, v)$ in the range $\left[\frac{i}{n_u}, \frac{i+1}{n_u}\right) \times \left[\frac{j}{n_v}, \frac{j+1}{n_v}\right)$
  - $f(u, v) = f[u_i, v_i]$ with $\tilde{u} = \lfloor n_u u \rfloor$ and $\tilde{v} = \lfloor n_v v \rfloor$
- integral over the domain
  - $I_f = \int \int f(u, v) \,\mathrm{d}u \,\mathrm{d}v = \frac{1}{n_u n_v} \sum_i \sum_j f[u_i, v_j]$
- PDF
  - $p(u, v) = \frac{1}{I_f} f(u, v) = \frac{1}{I_f} f[\tilde{u}, \tilde{v}]$

# Piecewise-Constant 2D Distribution

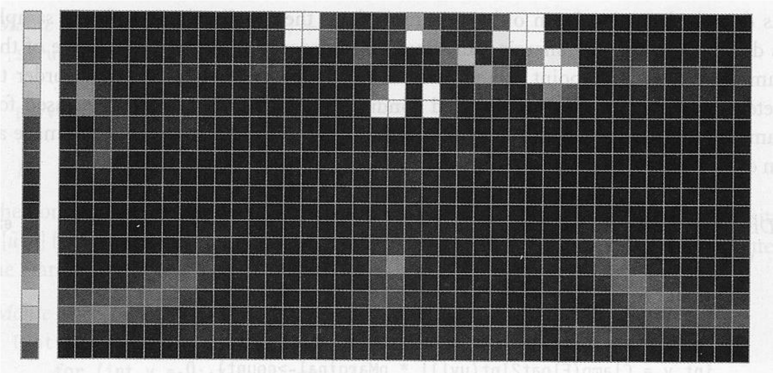
- marginal density function
  - $p_v(v) = \int p(u,v)\ \mathrm{d}u = \frac{1}{I_f}\frac{1}{n_u}\sum_i f[u_i, \tilde{v}]$
  - piecewise-constant 1D function
  - defined by $n_v$ values $p_v[\tilde{v}]$
- conditional density
  - $p_u(u|v) = \frac{p(u,v)}{p_v(v)} = \frac{1}{I_f}\frac{f[\tilde{u},\tilde{v}]}{p[\tilde{v}]}$
  - piecewise-constant 1D function
- sample generation
  - see example 3 of the inversion method

# Piecewise-Constant 2D Distribution

- environment map



- low-resolution of the marginal density function and the conditional distributions for rows



Paul Debevec, Grace Cathedral

- first, a row is selected according to the marginal density function
- then, a column is selected from the row's 1D conditional distribution



[Pharr, Humphreys]