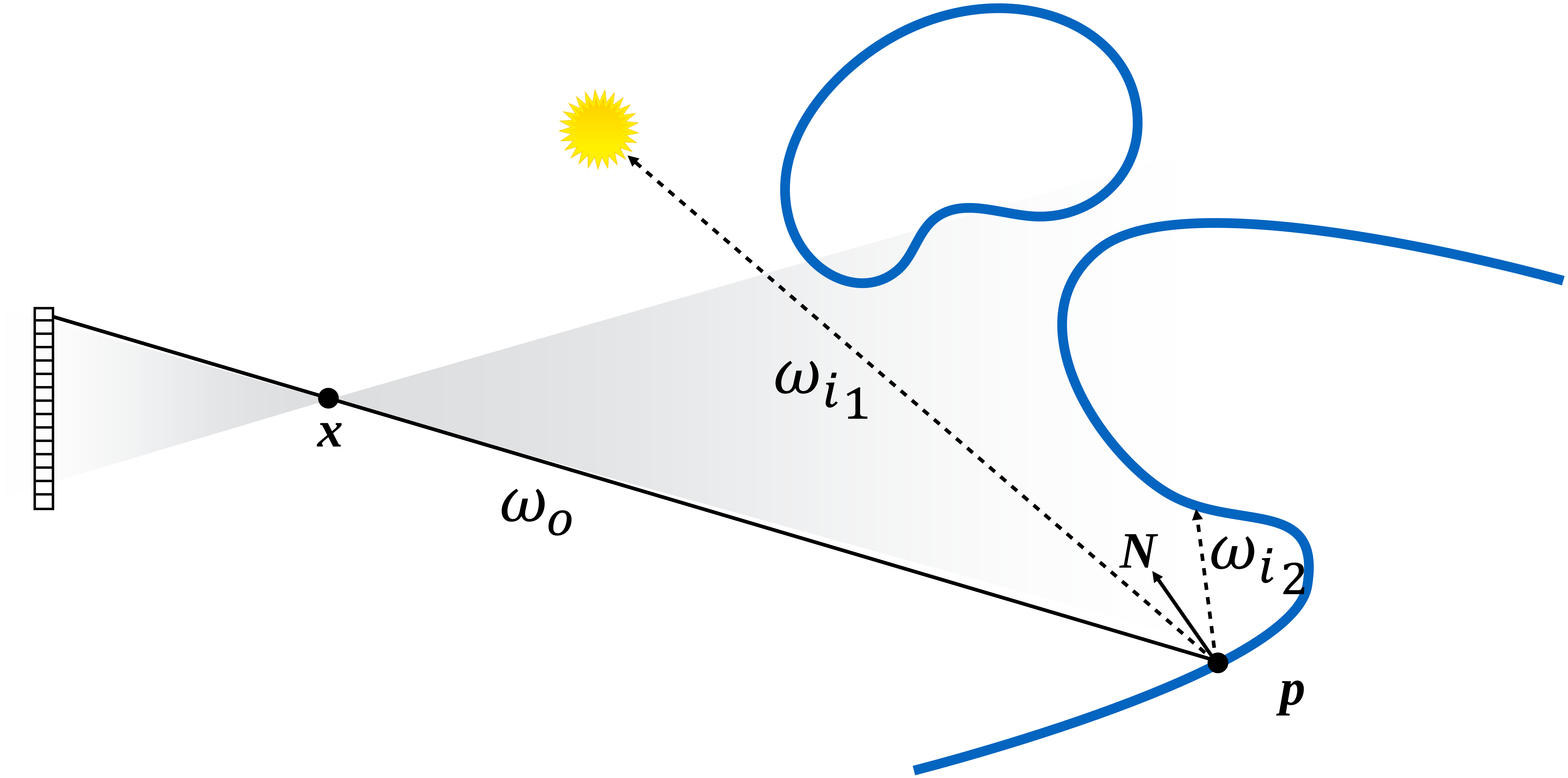
Monte Carlo Integration

Computer Graphics
CMU 15-462/15-662, Fall 2016

Talk Announcement

Jovan Popovic, Senior Principal Scientist at Adobe Research will be giving a seminar on “Character Animator” -- Monday October 24, from 3-4 in NSH 1507.

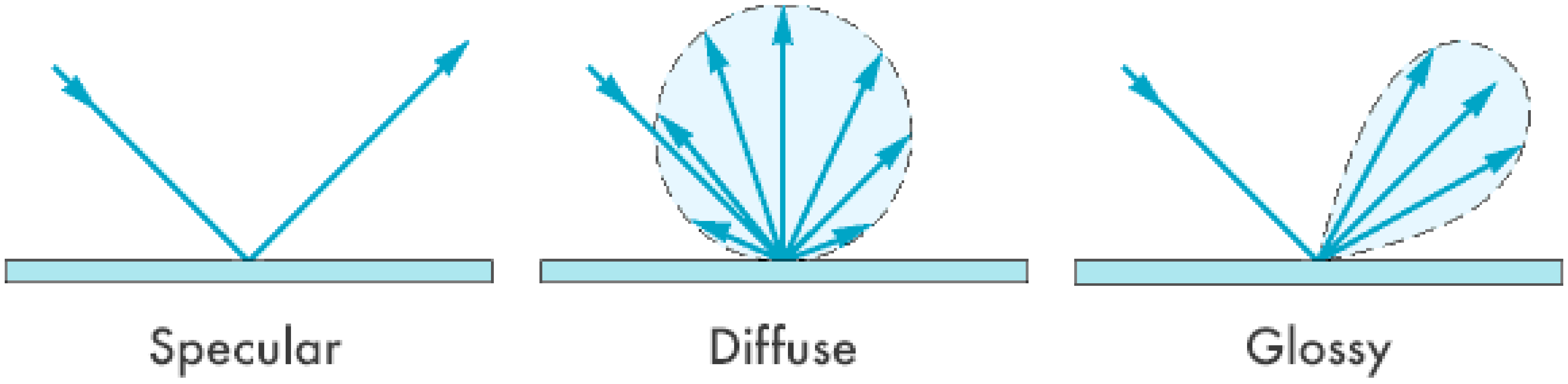
From last class…

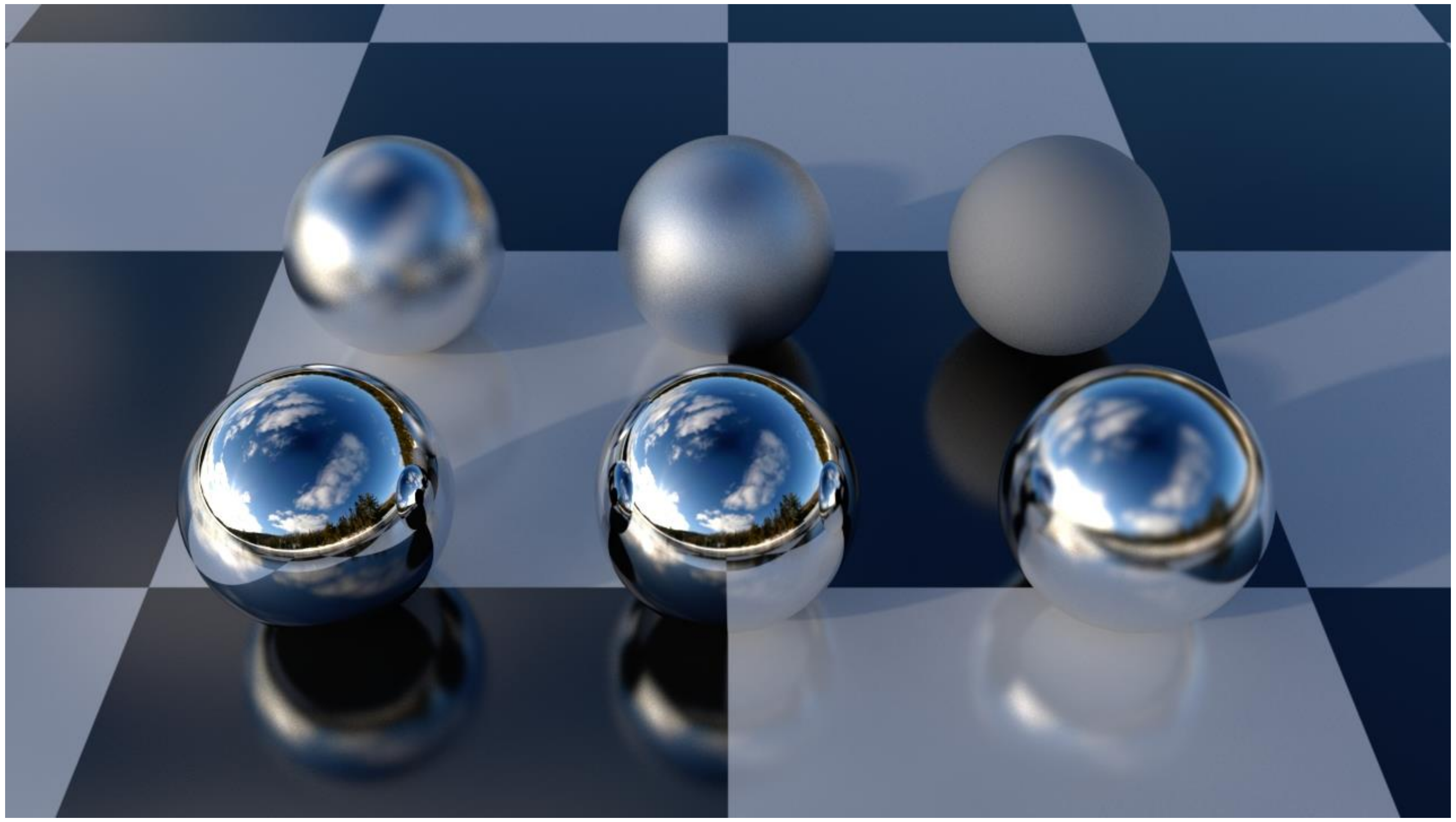


What we really want is to solve the reflection equation:

$$L_o(\mathrm{p}, \omega_o) = L_e(\mathrm{p}, \omega_o) + \int_{H^2} f_r(\mathrm{p}, \omega_i \rightarrow \omega_o)\, L_i(\mathrm{p}, \omega_i)\, \cos\theta_i \,\mathrm{d}\omega_i$$

Reflections





Review: fundamental theorem of calculus

$$\int_a^b f(x)dx = \text{ ?}$$

If f is continuous over $[a, b]$ and F is defined over $[a, b]$ as:

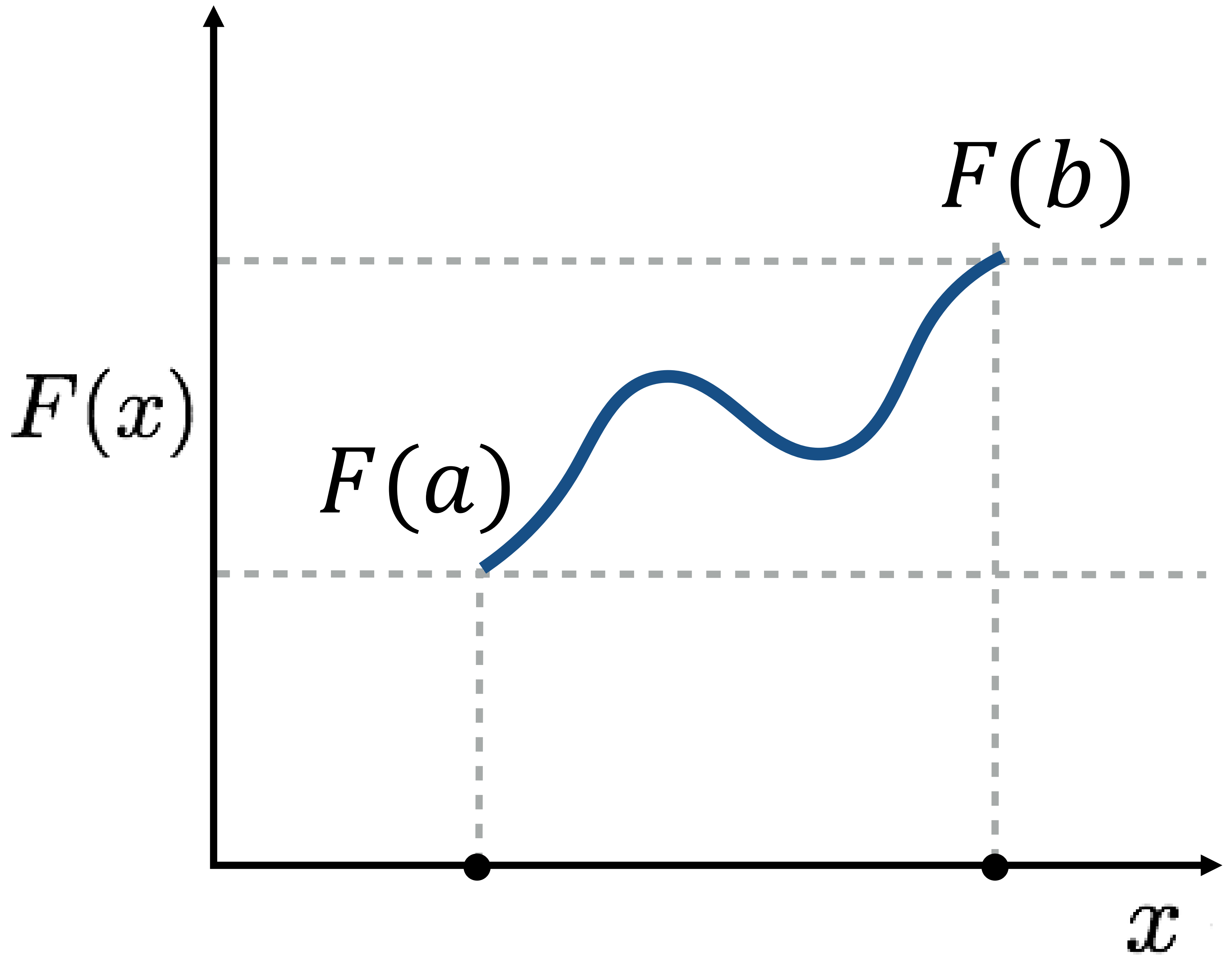
$$F(x) = \int_a^x f(t)\,dt$$

then:

$$f(x) = \frac{d}{dx}F(x)$$

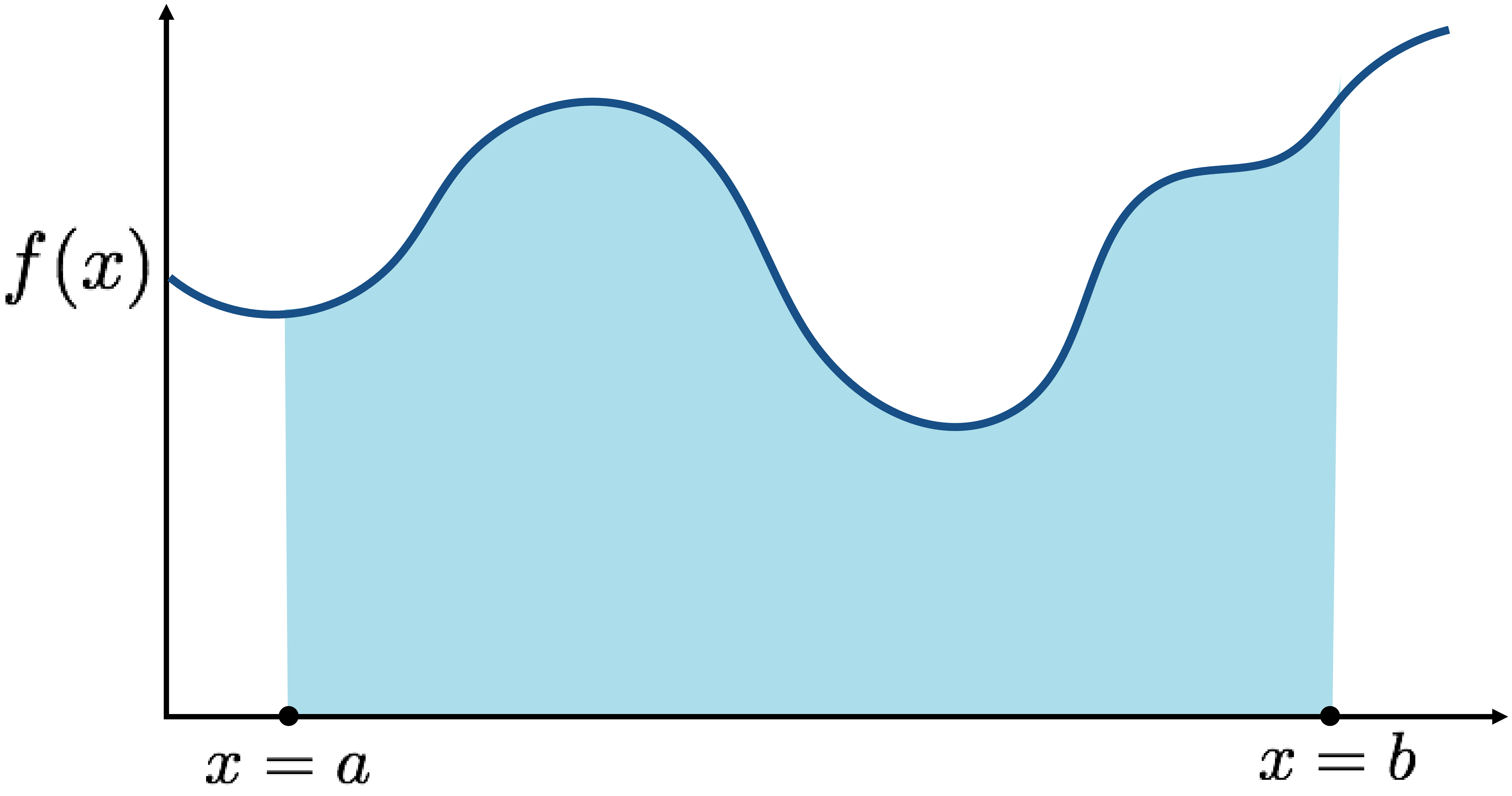
and

$$\int_a^b f(x)dx = F(b) - F(a)$$



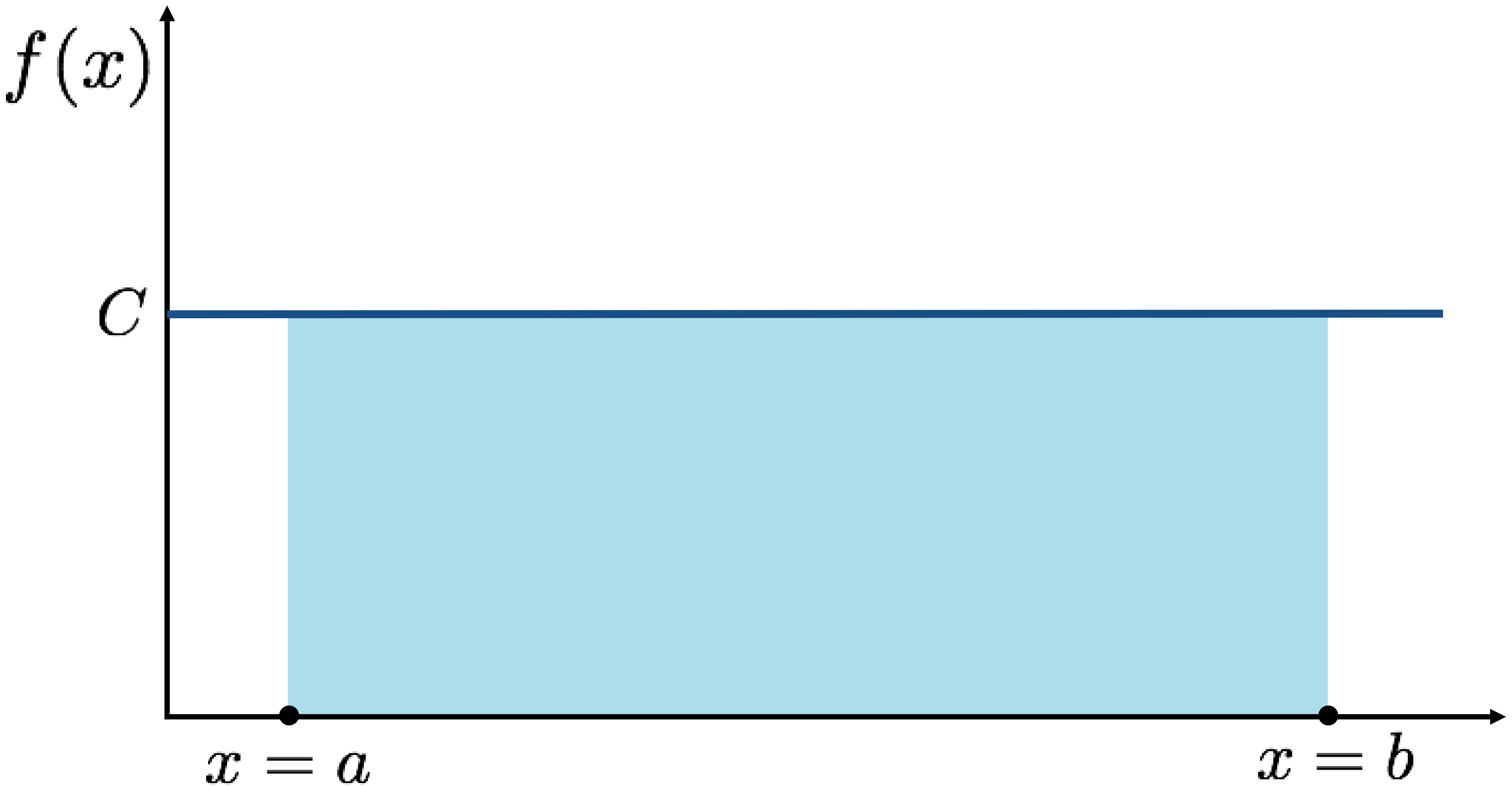
Definite integral: “area under curve”

$$\int_a^b f(x)dx$$



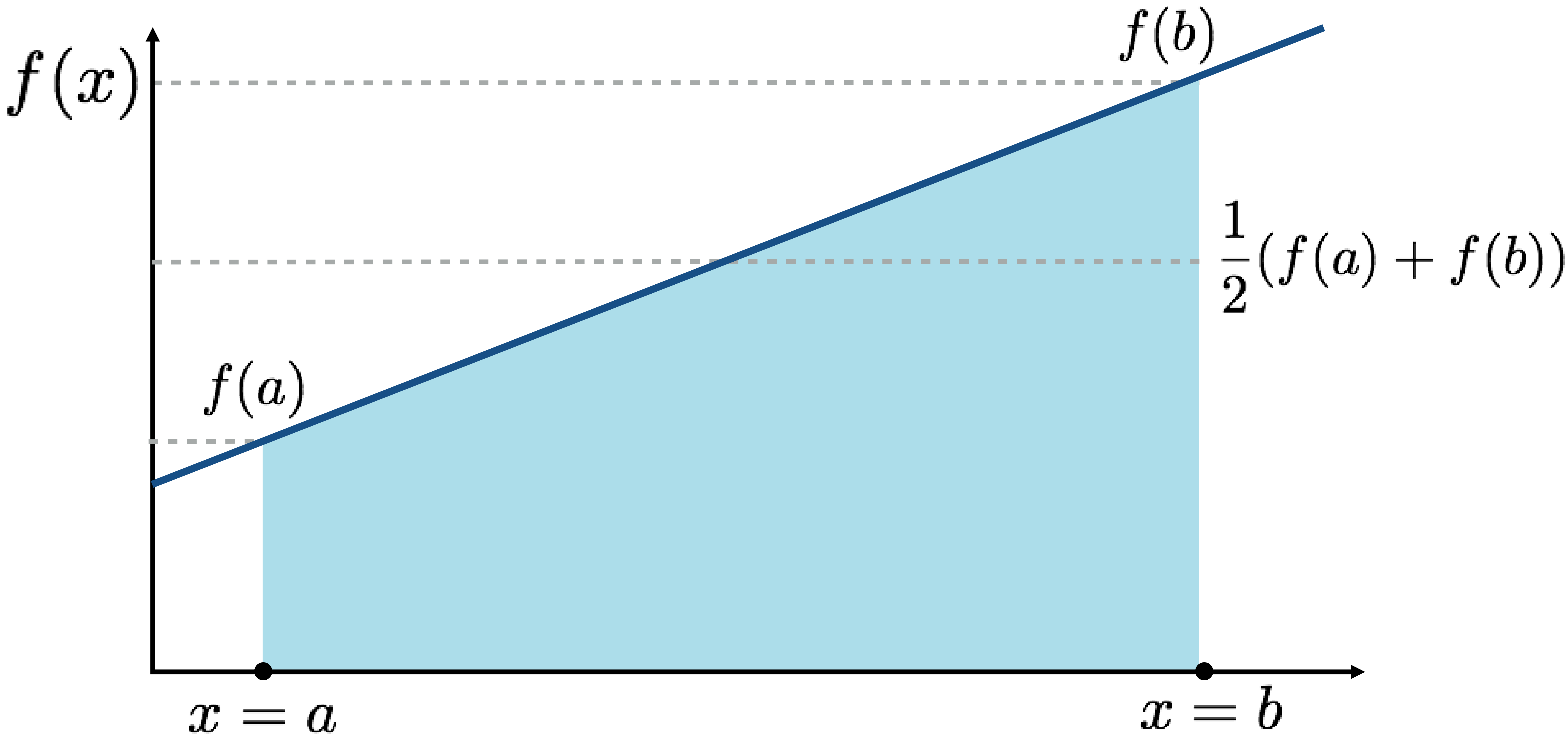
Simple case: constant function

$$\int_a^b C\,dx = (b-a)C$$



Affine function: $f(x) = cx + d$

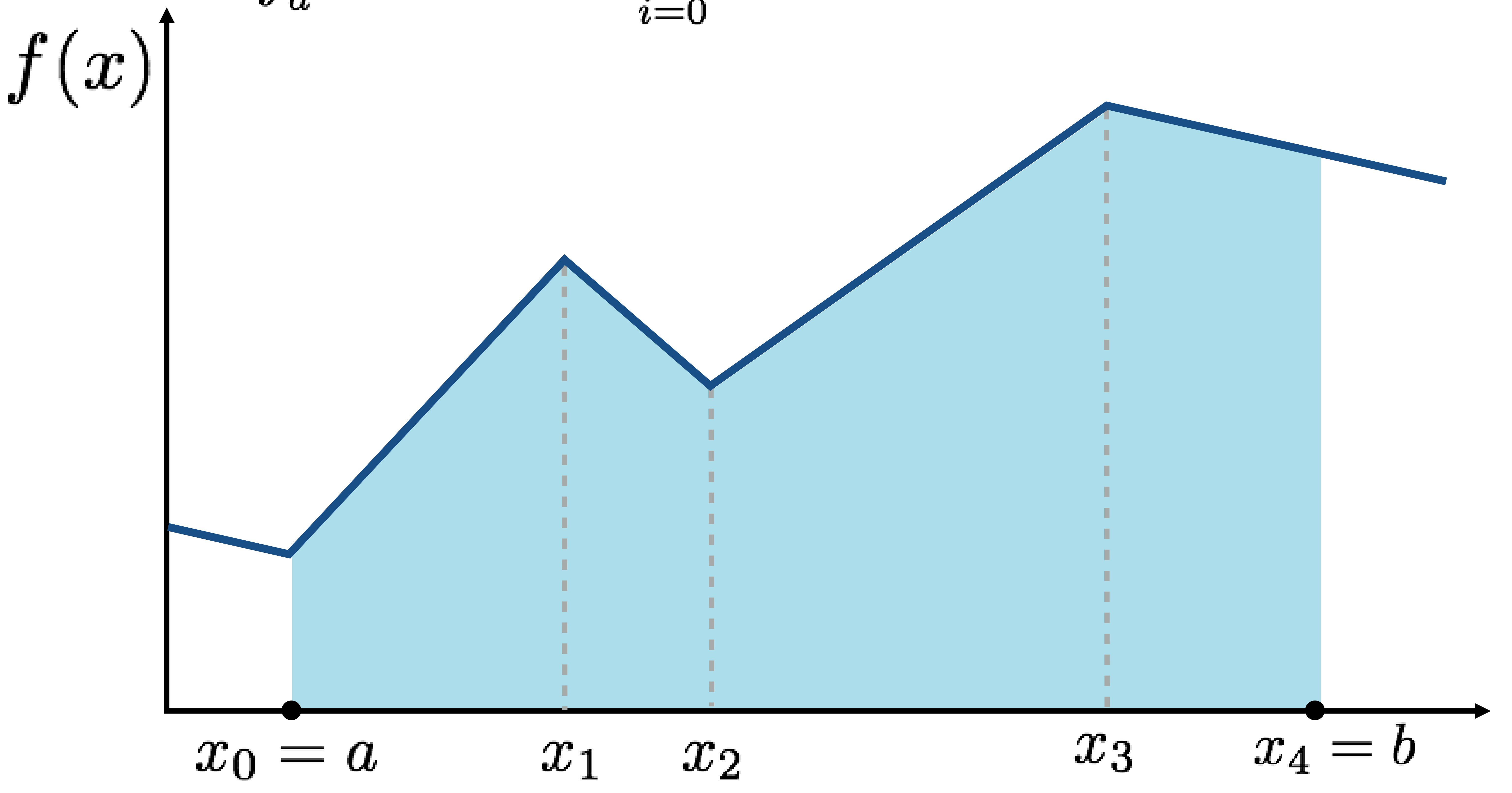
$$\int_a^b f(x)dx = \frac{1}{2}(f(a) + f(b))(b - a)$$



Piecewise affine function

Sum of integrals of individual affine components

$$\int_a^b f(x)dx = \frac{1}{2}\sum_{i=0}^{n-1}(x_{i+1} - x_i)(f(x_i) + f(x_{i+1}))$$



Piecewise affine function

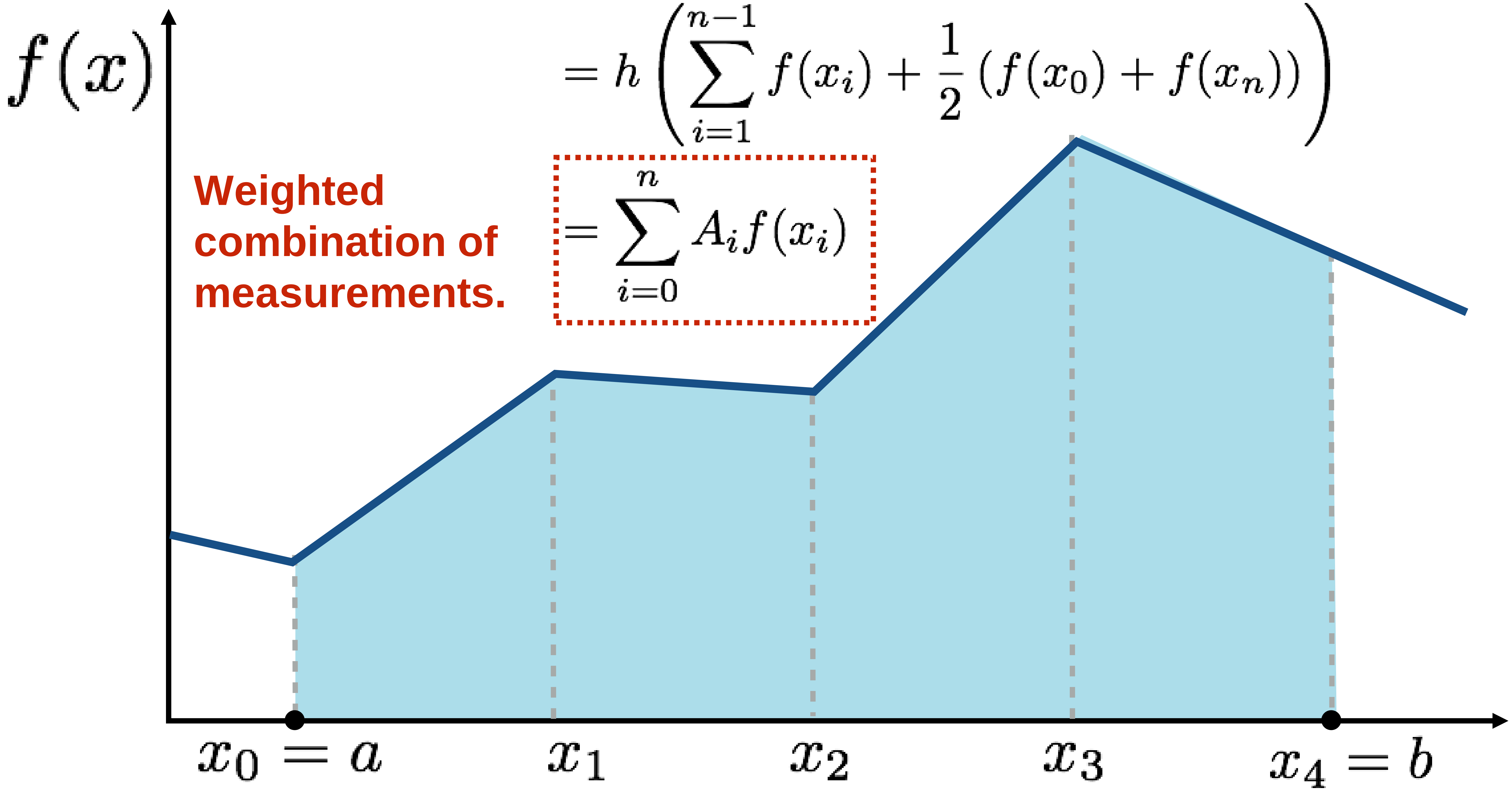
If N-1 segments are of equal length: $h = \dfrac{b-a}{n-1}$

$$\int_a^b f(x)dx = \frac{h}{2}\sum_{i=0}^{n-1}(f(x_i) + f(x_{i+1})$$

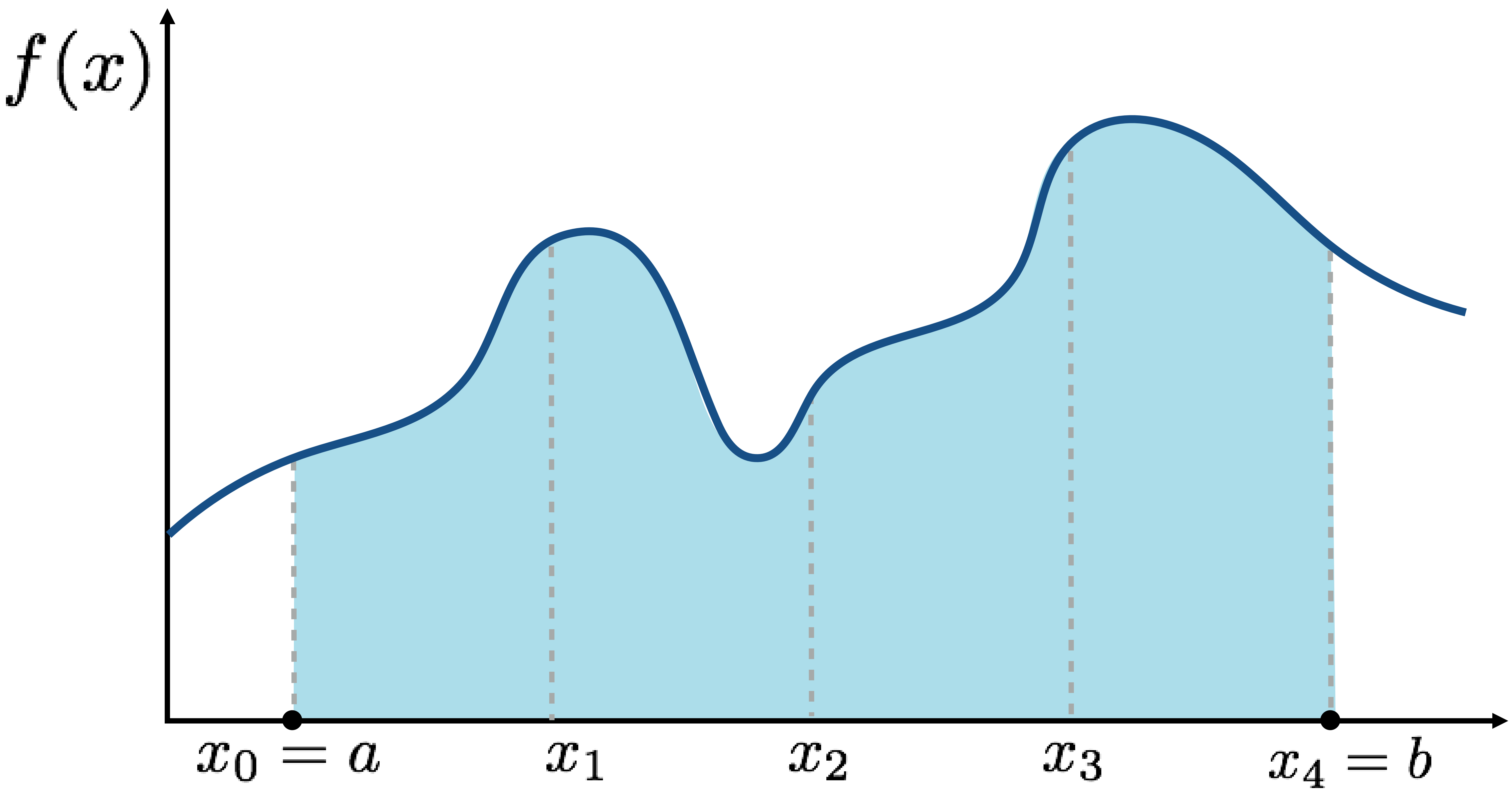
$$= h\left(\sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}\left(f(x_0) + f(x_n)\right)\right)$$

$$\boxed{= \sum_{i=0}^{n} A_i f(x_i)}$$

Weighted combination of measurements.



Arbitrary function f(x)?



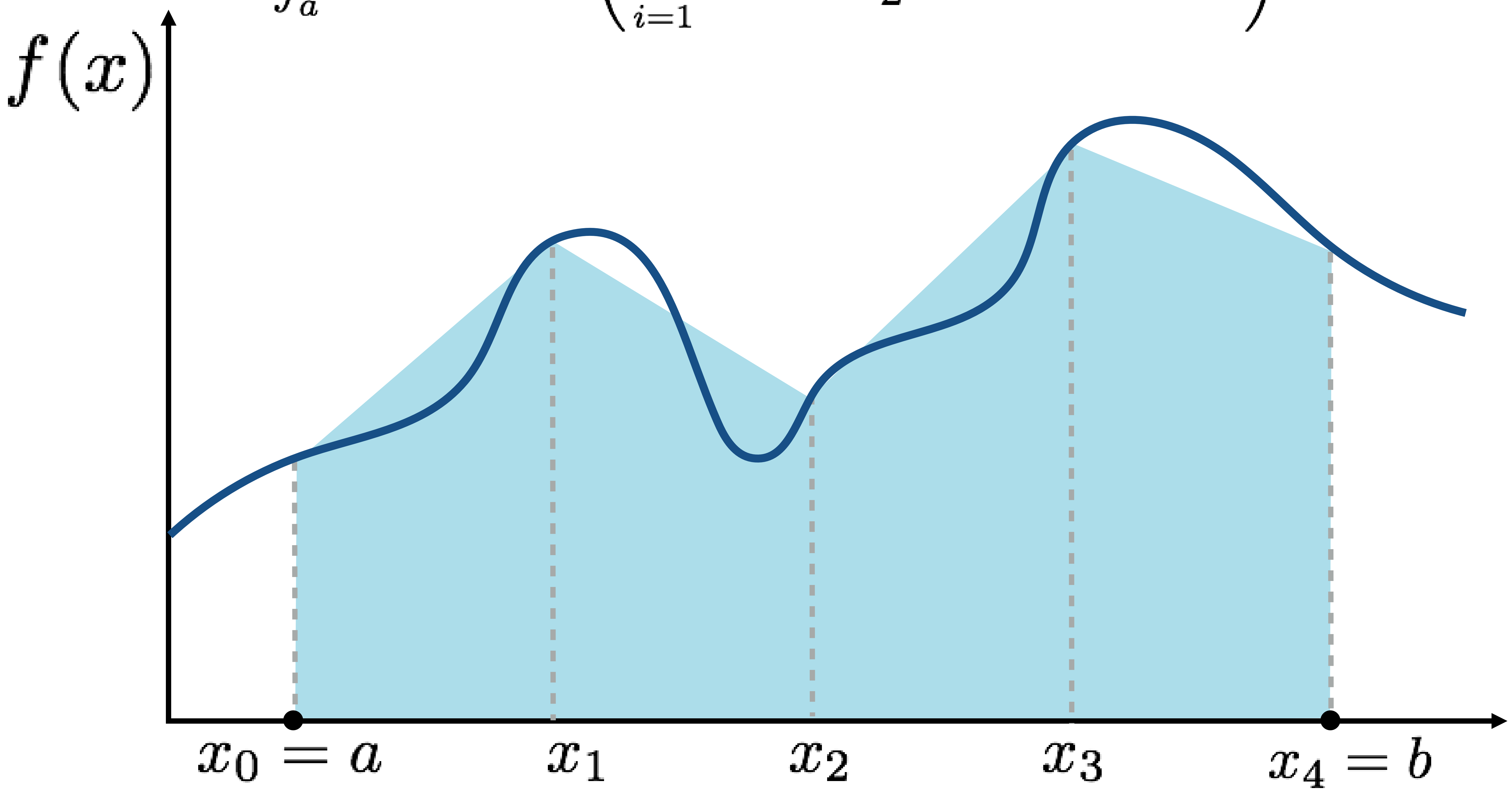
Quadrature rule: an approximation of the definite integral of a function, usually stated as a weighted sum of function values at specified points within the domain of integration.

Trapezoidal rule

<u>Approximate</u> integral of f(x) by assuming function is piecewise linear.

For equal length segments: $h = \frac{b-a}{n-1}$

$$\int_a^b f(x)dx = h\left(\sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}\left(f(x_0) + f(x_n)\right)\right)$$

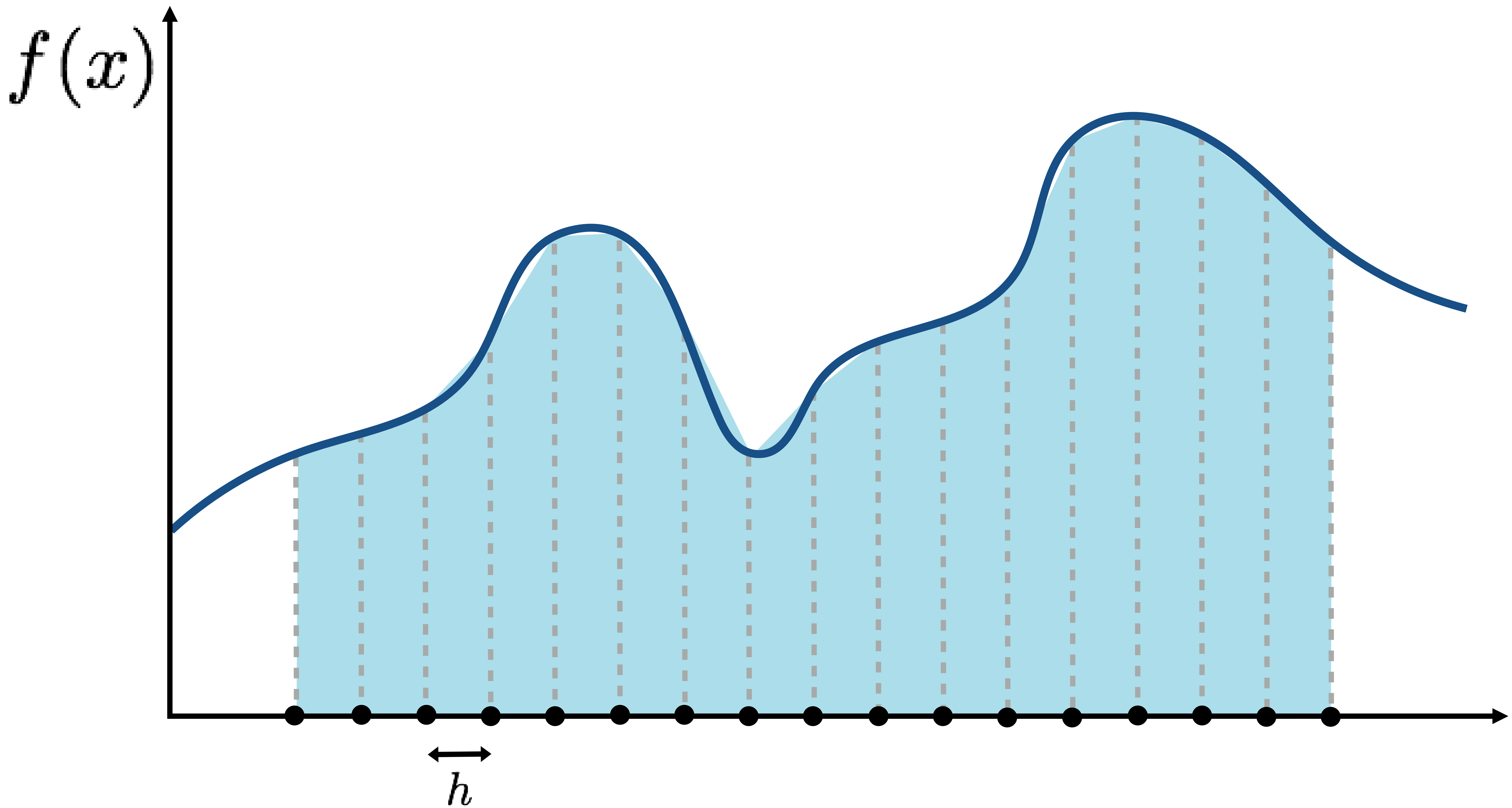


Trapezoidal rule

Consider cost and accuracy of estimate as $n \to \infty$
(or $h \to 0$ **)**

Work: $O(n)$ **Error can be shown to be:** $O(h^2) = O(\frac{1}{n^2})$

(for f(x) with continuous second derivative)



Integration in 2D

Consider integrating $f(x, y)$ using the trapezoidal rule (apply rule twice: when integrating in *x* and in *y*)

$$\int_{a_y}^{b_y} \int_{a_x}^{b_x} f(x,y)dxdy = \int_{a_y}^{b_y} \left(\boxed{O(h^2) + \sum_{i=0}^{n} A_i f(x_i, y)} \right) dy$$

First application of rule

$$= O(h^2) + \sum_{i=0}^{n} A_i \int_{a_y}^{b_y} f(x_i, y)dy$$

$$= O(h^2) + \sum_{i=0}^{n} A_i \left(\boxed{O(h^2) + \sum_{j=0}^{n} A_j f(x_i, y_j)} \right)$$

Second application

$$= O(h^2) + \sum_{i=0}^{n} \sum_{j=0}^{n} A_i A_j f(x_i, y_j)$$

Must do much more work in 2D (*n* x *n* set of measurements) to get same error bound on integral! Rendering requires computation of infinite dimensional integrals – more soon!

Monte Carlo Integration

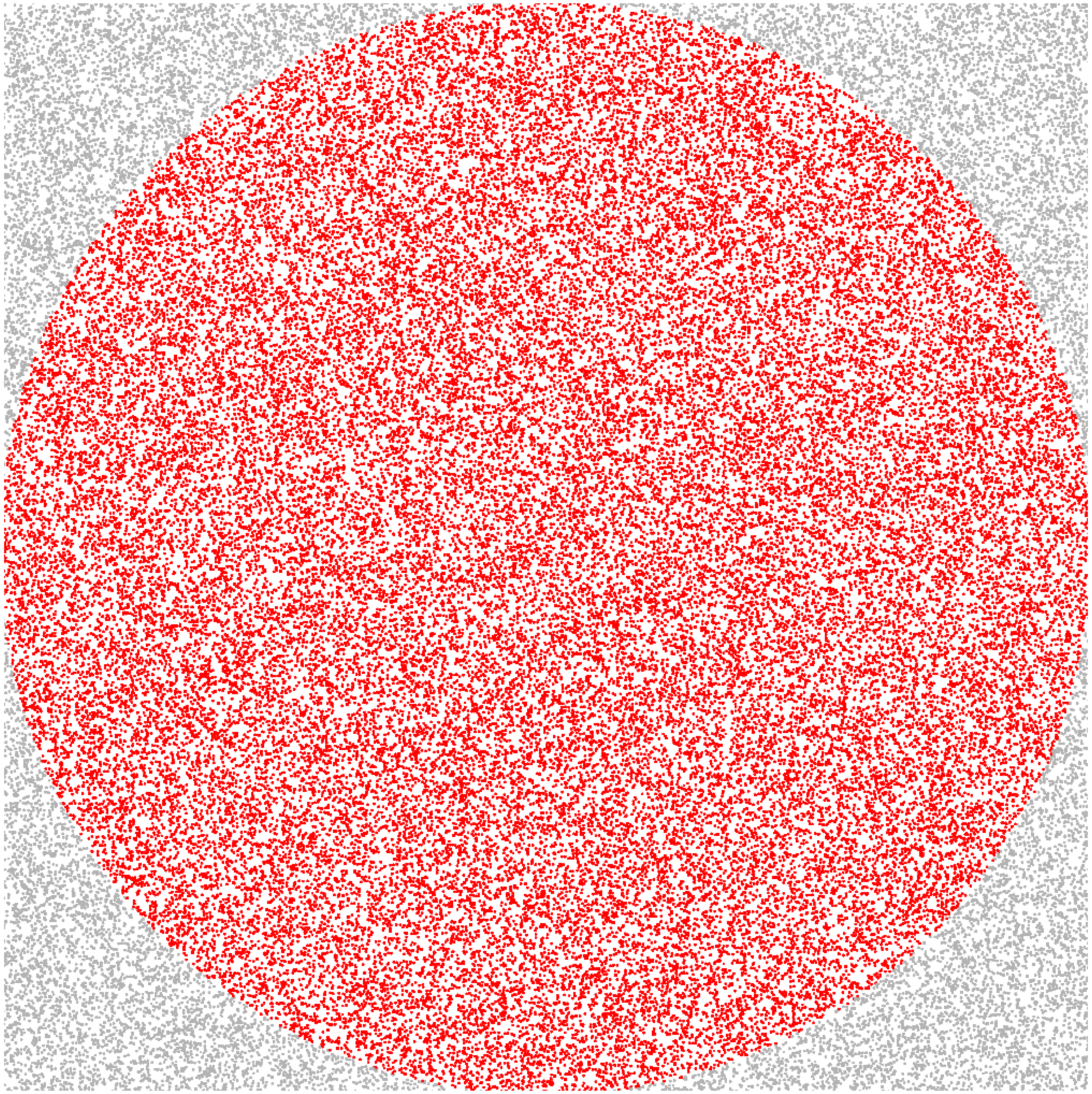
A first example…

Consider the following integral:

$$\iint f(x,y)dxdy$$

where

$$f(x,y) = \begin{cases} 1, x^2 + y^2 < 1 \\ 0, otherwise \end{cases}$$



What does this integral "mean"?

We'll come back to this…

A brief intro to probability theory…

A **random variable** x is a quantity whose value depends on a set of possible random events. A random variable can take on different values, each with an associated probability.

Random variables can be discrete or continuous

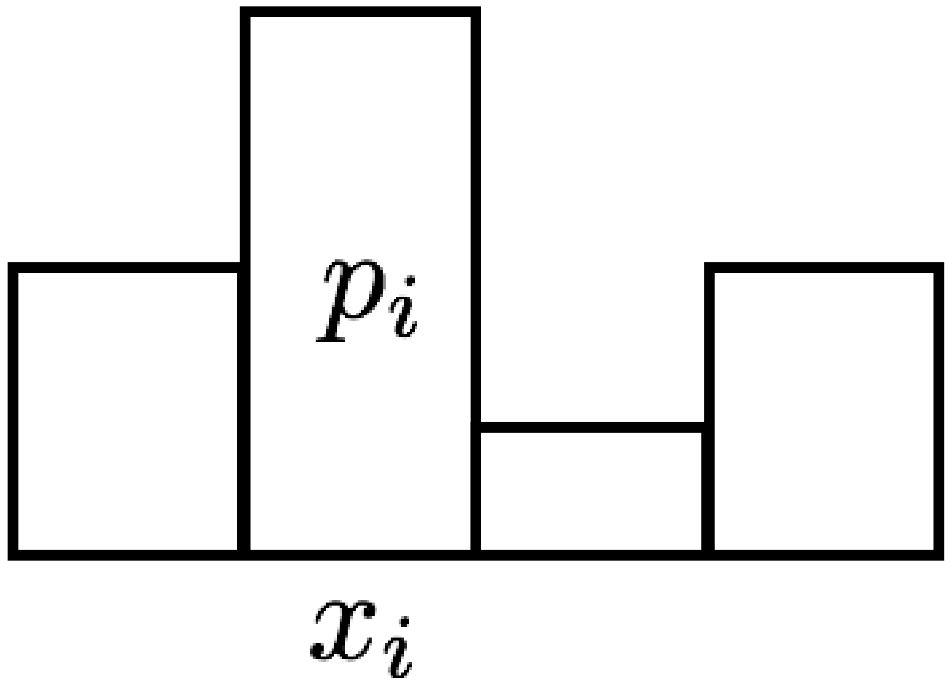
x can take on values 1, 2, 3, 4, 5 or 6 with equal probability*

x is continuous and 2-D. Probability of bulls-eye depends on who is throwing.

Discrete random variables

x can take one of n discrete values x_i, each with probability p_i

It must always be the case that



$$p_i \geq 0 \qquad \sum_{i=1}^{n} p_i = 1$$

What is the average (or **expected**) value x will take after many, many experiments?

$$E(x) = \sum_{i=1}^{n} x_i p_i \qquad E(f(x)) = \sum_{i=1}^{n} f(x_i) p_i$$

Q: What is the expected value of rolling a die?

Useful properties of Expected Value

Let x and y be independent random variables or functions of independent random variables

"expected value of sum is sum of expected values"

$$E(\boldsymbol{x}+\boldsymbol{y})=E(\boldsymbol{x})+E(\boldsymbol{y})$$

"expected value of product is product of expected values"

$$E(\boldsymbol{x}*\boldsymbol{y})=E(\boldsymbol{x})*E(\boldsymbol{y})$$

Q1: What does it mean that two random variables are independent?
Q2: What's an example of two random variables that are *correlated*?

Variance of a discrete random variable

"How far from the expected value can we expect the outcome of each instance of the experiment to be"

$$V(\boldsymbol{x}) = \sum_{\boldsymbol{i}=1}^{\boldsymbol{n}} \left(\boldsymbol{x_i} - \boldsymbol{E}(\boldsymbol{x})\right)^{\boldsymbol{2}} \ \boldsymbol{p_i} = E\left(\left(\boldsymbol{x_i} - E(\boldsymbol{x})\right)^{\boldsymbol{2}}\right)$$

$$= E\left(\boldsymbol{x}^2\right) - E(\boldsymbol{x})^2$$

Q: What are some advantages of this expression?

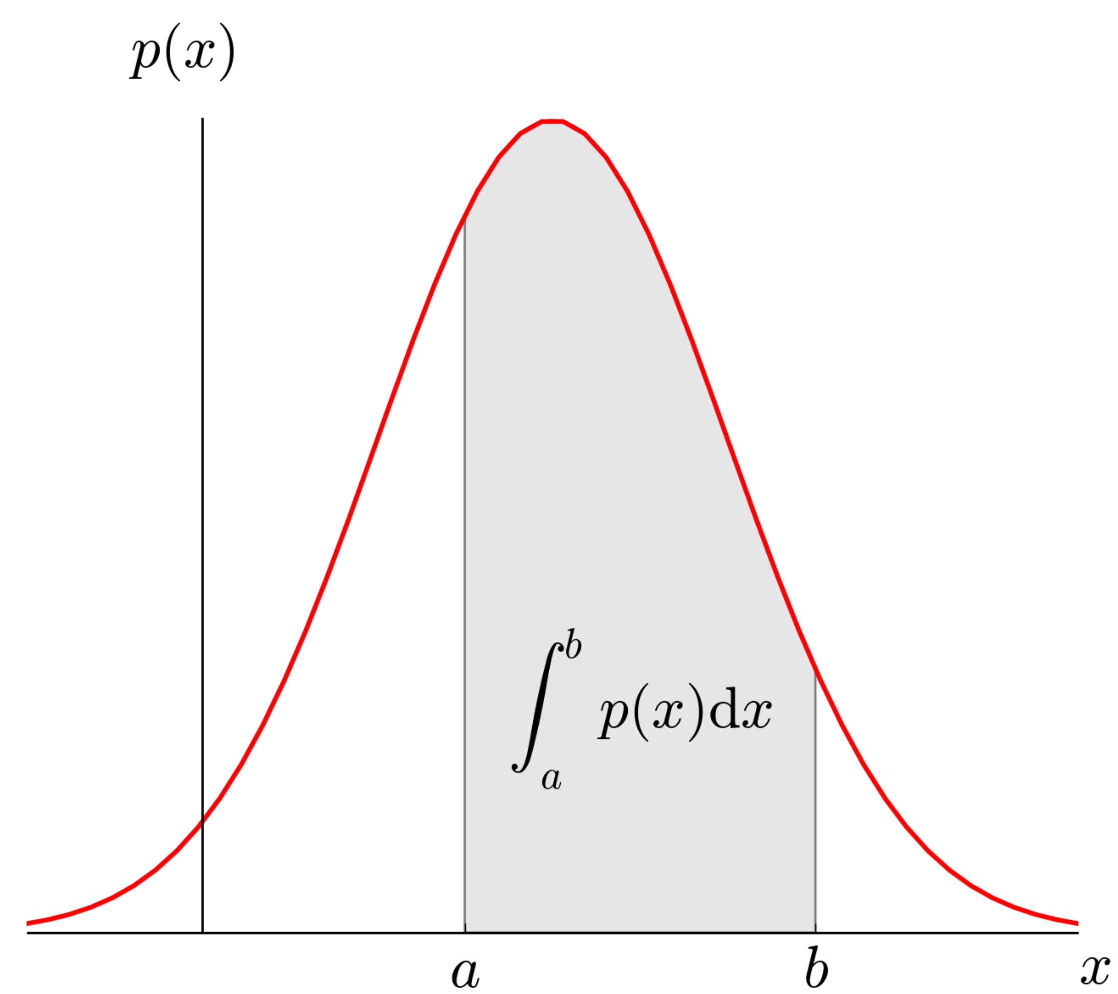
If x and y are independent random variables or functions of independent random variables:

"variance of sum is sum of variances"

$$V(\boldsymbol{x} + \boldsymbol{y}) = V(\boldsymbol{x}) + V(\boldsymbol{y})$$

Continuous random variables

x **can take infinitely many values according to a *probability density function*** $p(x)$



It must always be the case that

$$p(x) \geq 0 \qquad \int_{-\infty}^{\infty} p(x) dx = 1$$

$$p_r(a \leq x \leq b) = \int_a^b p(x) dx$$

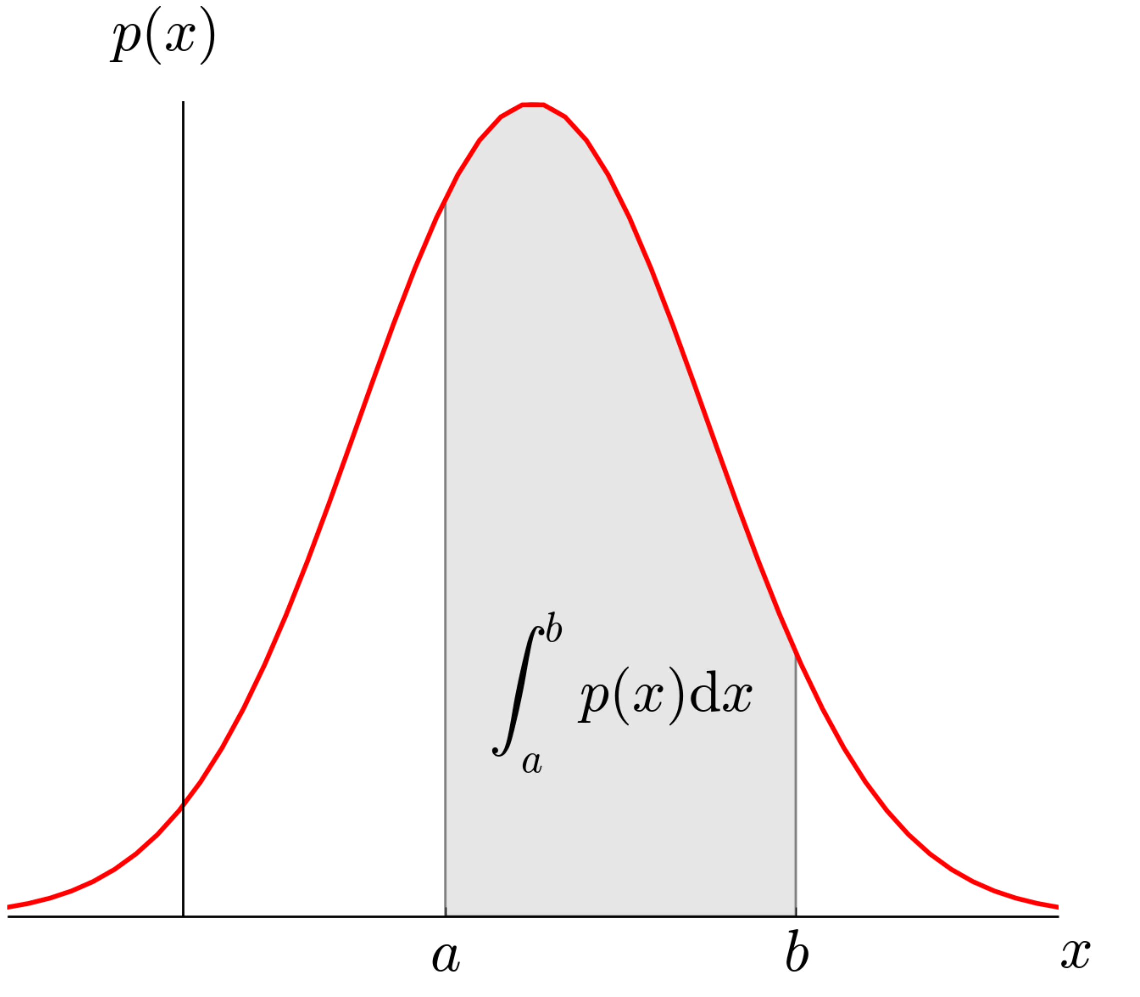
$$p_r(x = a) = 0$$

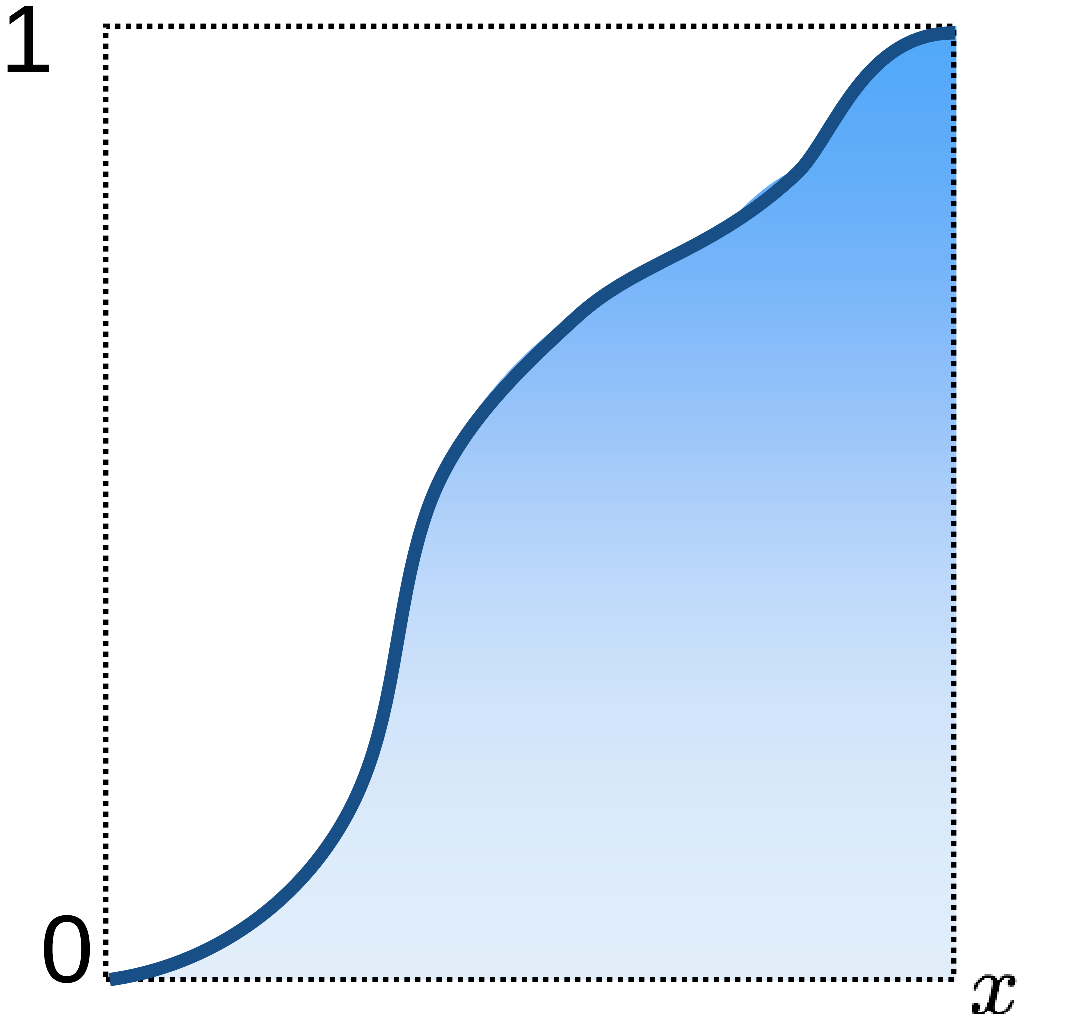
Probability of specific result for an experiment

Continuous random variables

Cumulative Distribution Function

$$CDF(b) = \int_{-\infty}^{b} p(x)dx = 1$$

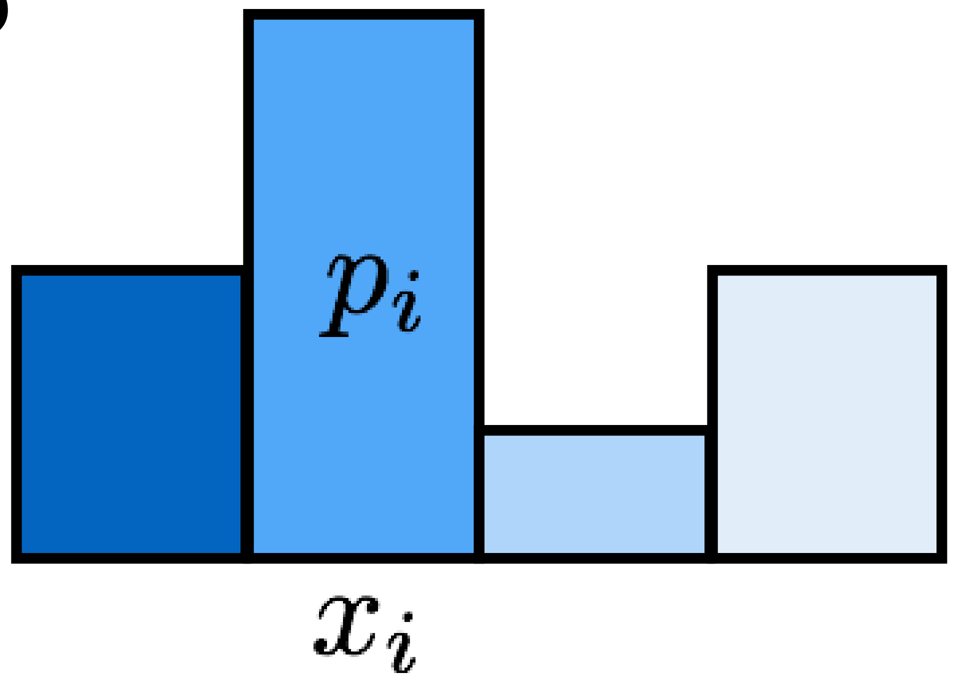




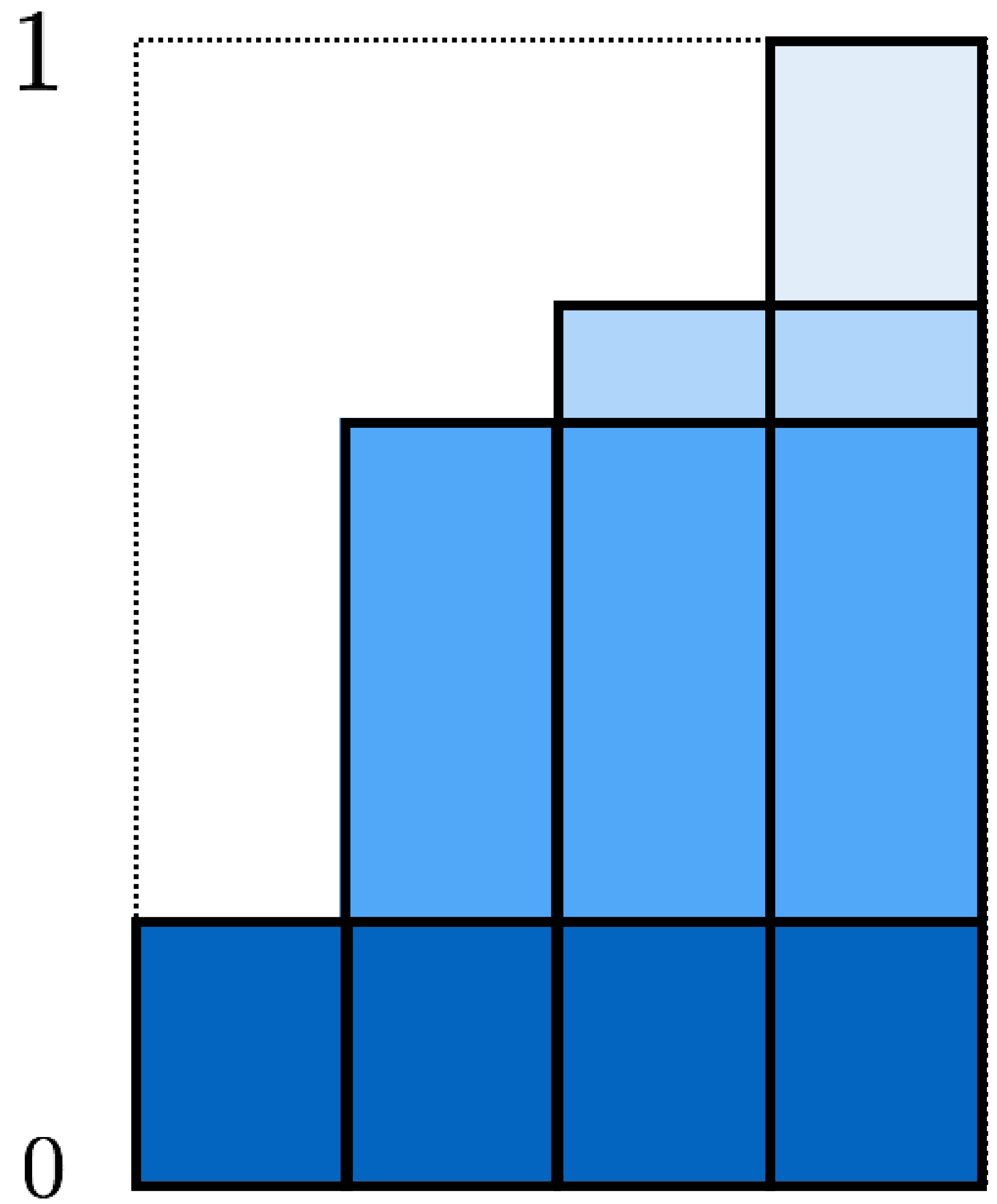
$$p_r(a \leq x \leq b) = CDF(b) - CDF(a)$$

Cumulative distribution function for discrete random variables

PDF:



CDF:



Discrete vs Continuous RVs

Concepts from discrete case carry over to continuous case. Just replace sums with integrals as needed!

$$E(\boldsymbol{f}(\boldsymbol{x})) = \sum_{\boldsymbol{i}=\mathbf{1}}^{\boldsymbol{n}} \boldsymbol{f}(\boldsymbol{x}_{\boldsymbol{i}})\boldsymbol{p}_{\boldsymbol{i}} \qquad E(\boldsymbol{x}) = \int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f}(\boldsymbol{x})\boldsymbol{p}(\boldsymbol{x})\boldsymbol{dx}$$

One more experiment

- **Assume $x_1, x_2, \ldots, x_N$ are independent random variables**
  - **e.g. repeating the same experiment over and over**
- **Let $G(\boldsymbol{X}) = \frac{1}{N}\sum_{\boldsymbol{j=1}}^{\boldsymbol{N}} g(x_j)$**
  - **e.g. average score after you throw 10 darts**

"expected value of sum is sum of expected values"

$$E\big(G(\boldsymbol{X})\big) = \boldsymbol{E}\left(\frac{1}{N}\sum_{\boldsymbol{j=1}}^{\boldsymbol{N}} g(x_j)\right) = E\big(g(x)\big) = G(\boldsymbol{X})$$

Expected value of average of N trials is the same as the expected value of 1 trial, is the same as average of N trials!

Monte Carlo Integration

- **Want to estimate the integral:** $$I = \int_{a}^{b} f(x)\, dx$$
- **Make an estimate:** $$\tilde{I}_1 = \frac{f(x_1)}{p(x_1)} = g(x_1)$$
- **Assume for now a uniform probability distribution**
  - **How do you randomly choose a point x_1 in interval [a, b]?**
  - **What is the value of $p(x_1)$?**
  - **What does estimate look like for a constant function?**

Monte Carlo Integration

- **Want to estimate the integral:** $$I = \int_{\boldsymbol{a}}^{\boldsymbol{b}} \boldsymbol{f}(\boldsymbol{x})\, \boldsymbol{dx}$$
- **Make an estimate:** $$\tilde{I}_1 = \frac{\boldsymbol{f}(\boldsymbol{x_1})}{\boldsymbol{p}(\boldsymbol{x_1})} = \boldsymbol{g}(\boldsymbol{x_1})$$
- **What is the expected value of the estimate?**

$$E(\tilde{I}_1) = \int_{\boldsymbol{a}}^{\boldsymbol{b}} \frac{\boldsymbol{f}(\boldsymbol{x})}{\boldsymbol{p}(\boldsymbol{x})} \boldsymbol{p}(\boldsymbol{x}) \boldsymbol{dx} = I = g(\boldsymbol{x_1})$$

- **So we're done…**

Monte Carlo Integration

- **Not so fast…**
  - **This is like trying to decide based on one toss if coin is fair or biased…**

- **Why is it that you expect to get better estimates by running more trials (i.e. $\tilde{I}_N$)?**
  - **expected value does not change…**

- **Look at variance of estimate after N trials:**

$$\tilde{I}_N = \sum_{i=1}^{N} g(x_i)$$

Part 2 of last experiment

- **Assume $x_1, x_2, \ldots, x_N$ are independent random variables**
  - **e.g. repeating the same experiment over and over**
- **Let $G(\boldsymbol{X}) = \frac{1}{N}\sum_{\boldsymbol{j=1}}^{\boldsymbol{N}} g(x_j)$**
  - **e.g. average score after you throw 10 darts**

"variance of sum is sum of variances (assuming independent RVs)"

$$V(G(\boldsymbol{X})) = V\left(\frac{1}{N}\sum_{\boldsymbol{j=1}}^{\boldsymbol{N}} g(x_j)\right) = \frac{1}{N}V(g(x))$$

Variance of N averaged trials decreases linearly with N as compared to variance of each trial!

Monte Carlo Integration

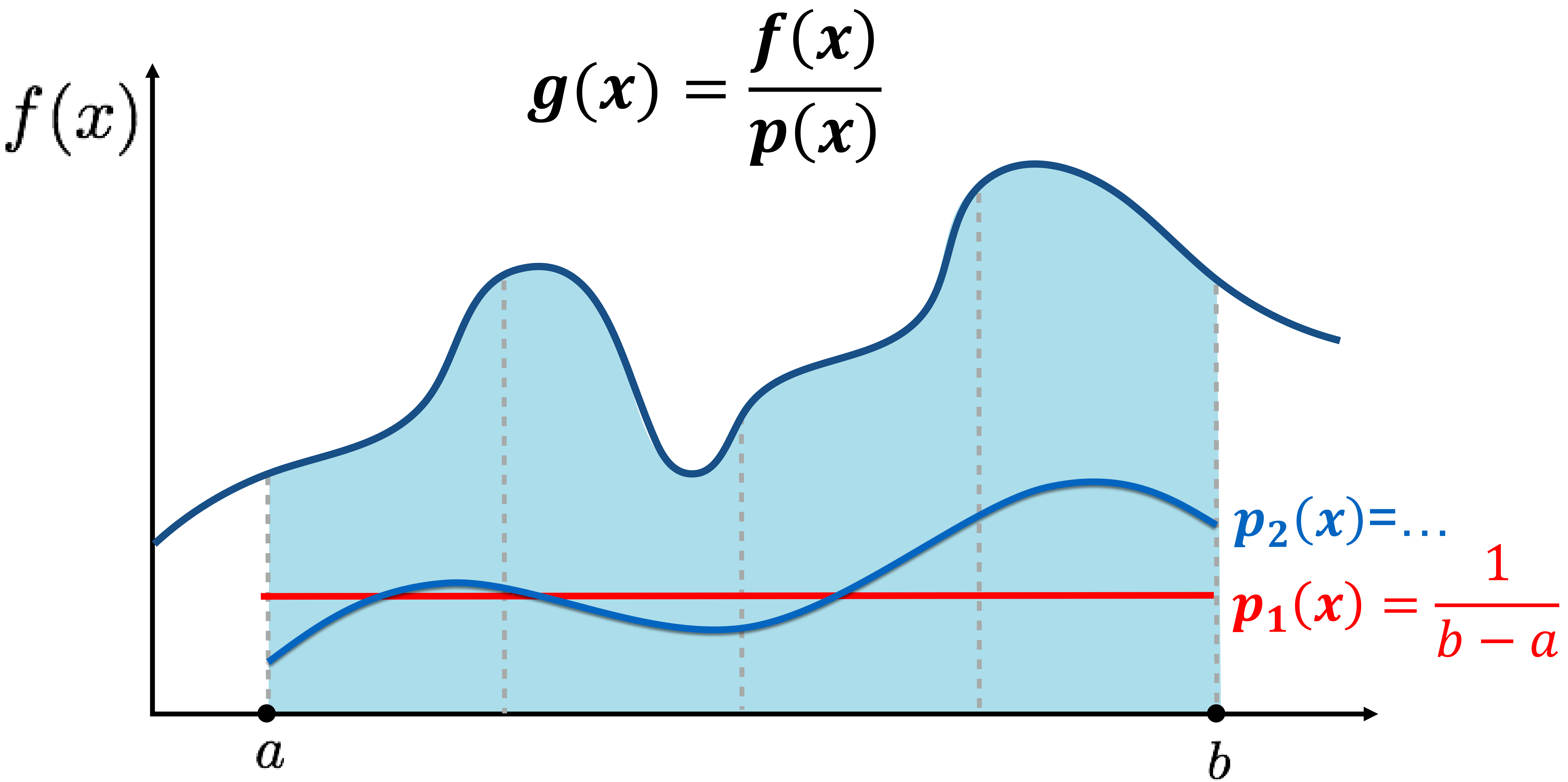
- **In 3 easy steps:**
  - **Define a probability distribution to draw samples from**
  - **Evaluate integrand**
  - **Estimate is weighted average of function samples**

$$\int_a^b f(x)dx \approx \frac{1}{N}\sum_{i=1}^{N}\frac{f(x_i)}{p(x_i)}$$

Q: how do we get the variance of the estimate to decrease?

A: Increase N, or decrease $V(g(x))$

A Variance Reduction Method



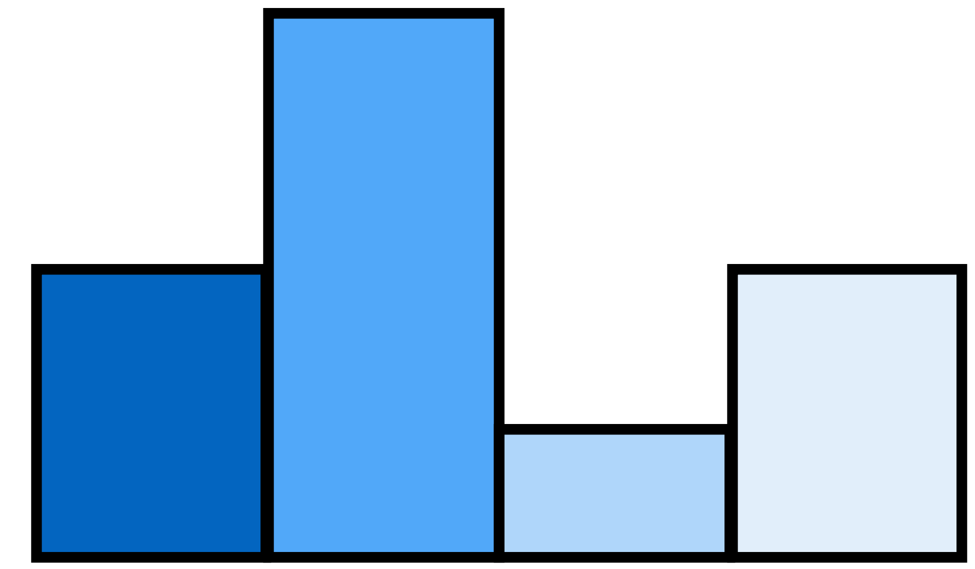
Non-uniform (biased) sampling can reduce variance of $g(x)$

If PDF is proportional to *f*, *g* will have zero variance! Only one sample needed! A free lunch?

How do we generate samples according to an arbitrary probability distribution?

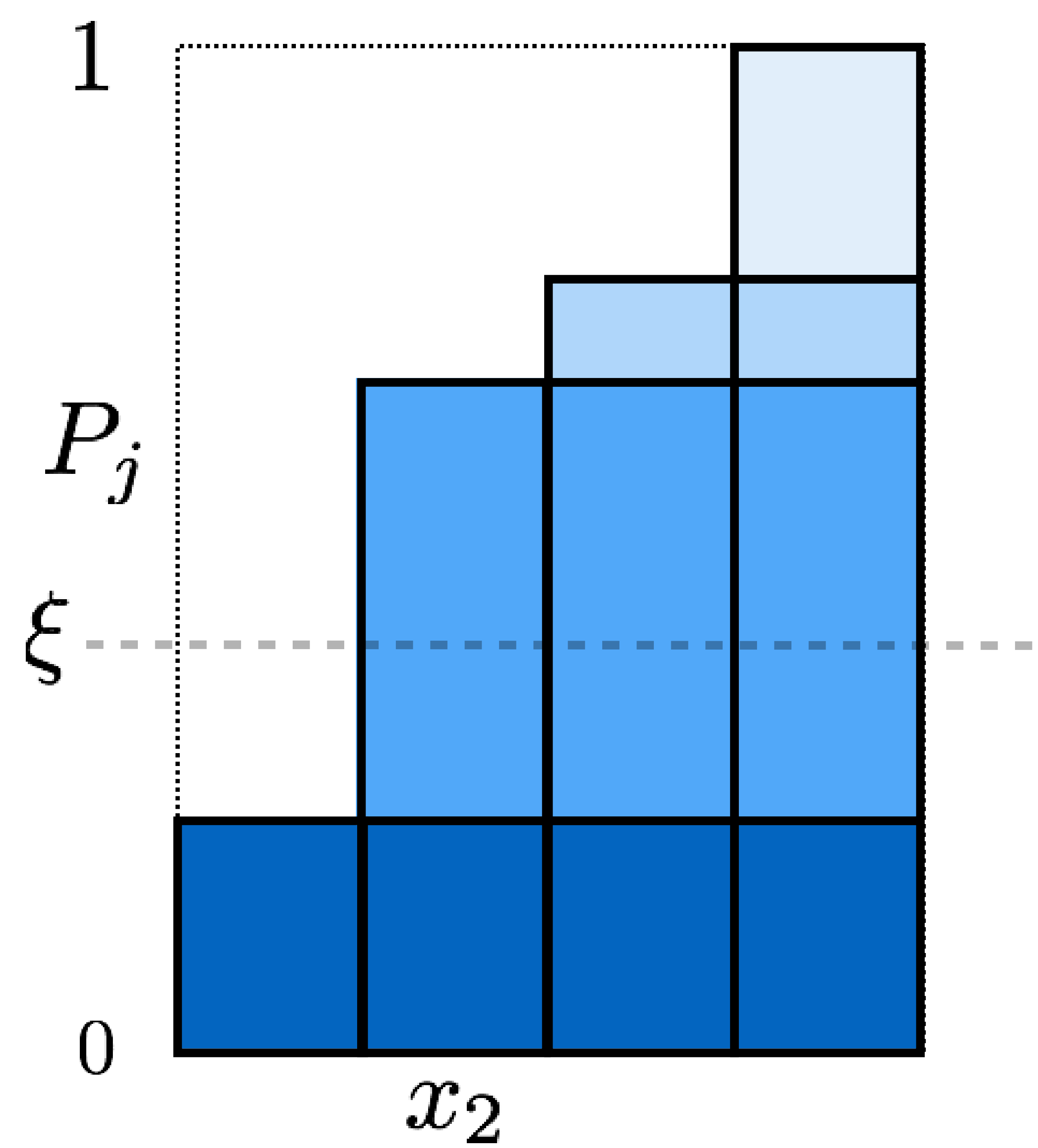
The discrete case

Consider a loaded die…



Select x_i **if** ξ **falls in its CDF bucket**

Uniform random variable $\in [0, 1)$

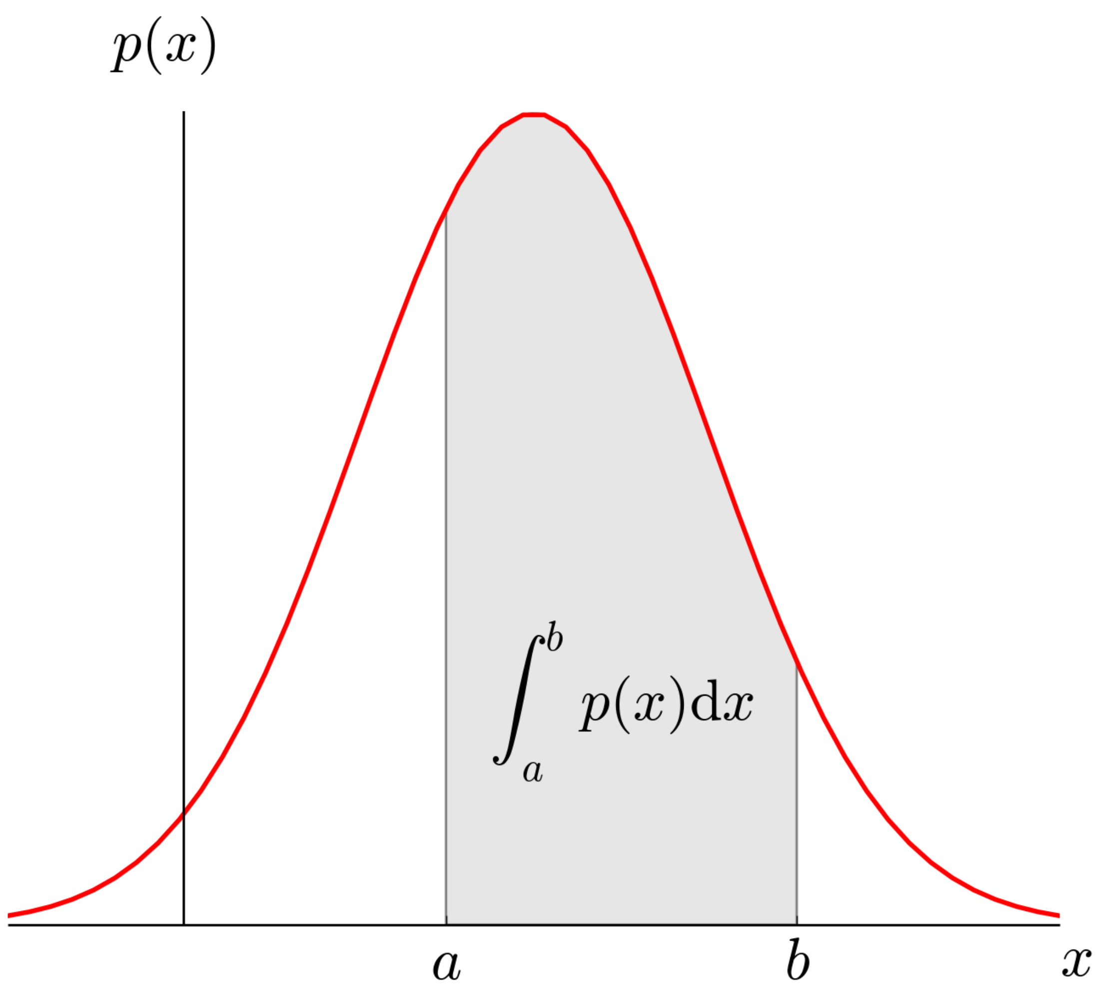


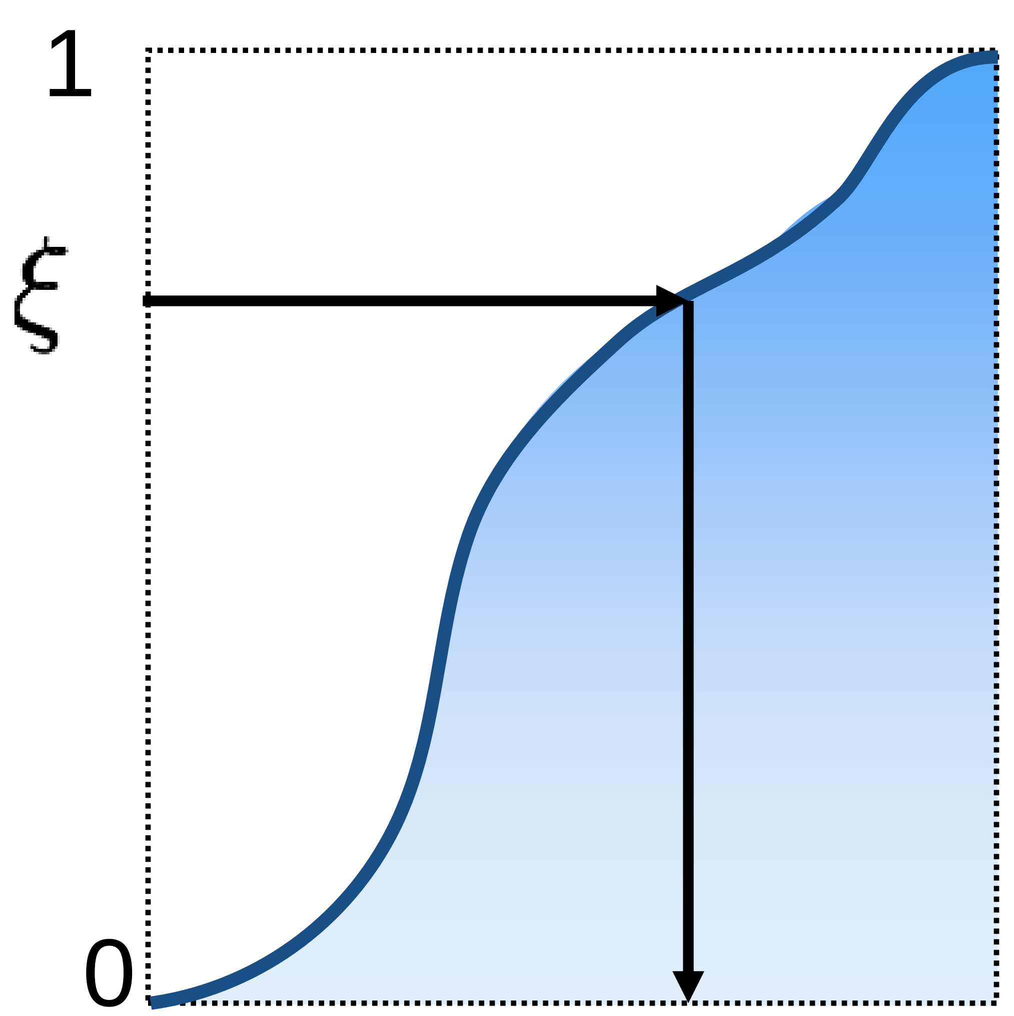
The continuous case

Exactly the same idea!

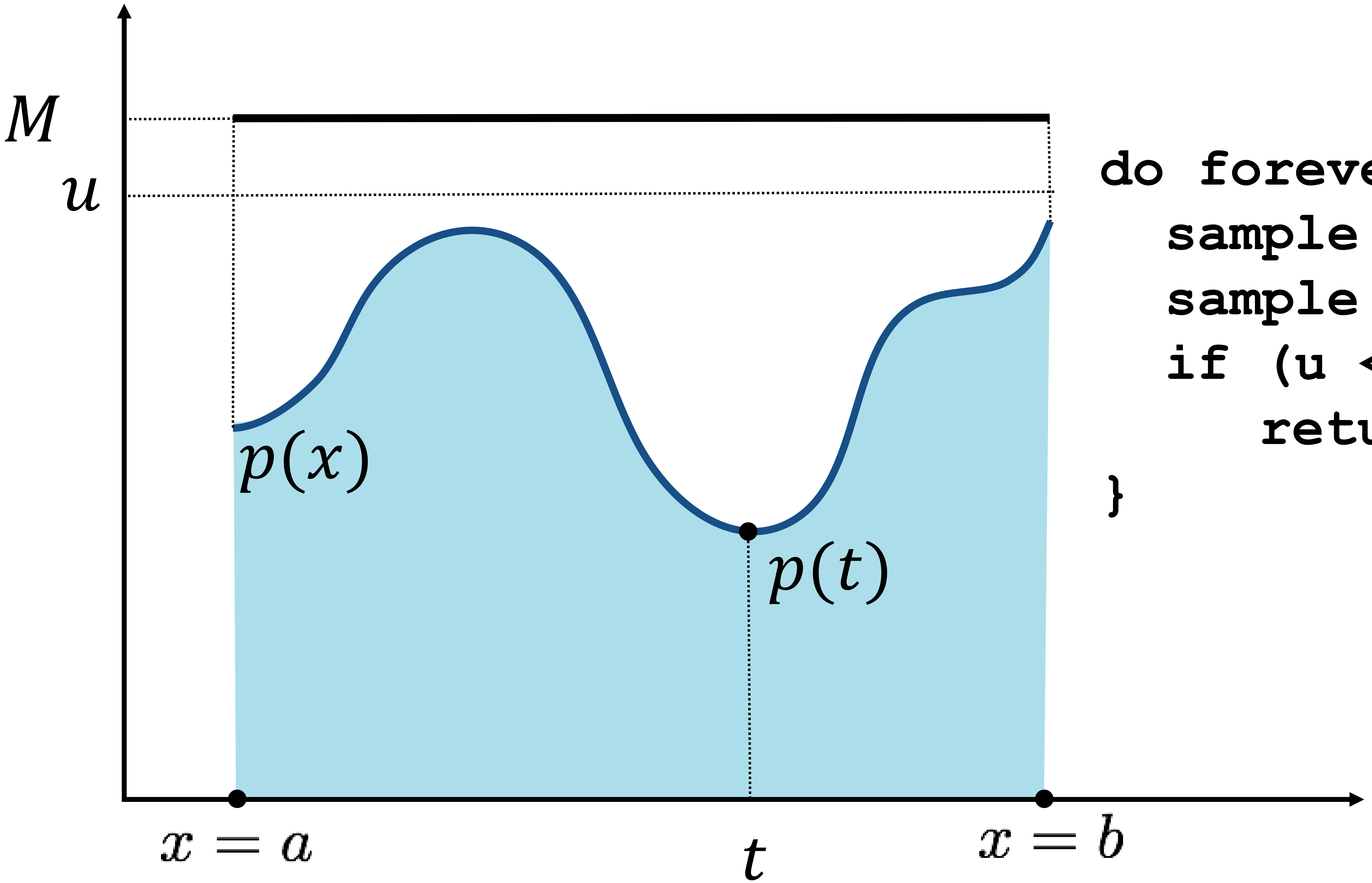
But must know the inverse of the Cumulative Distribution Function!

And if we don't?





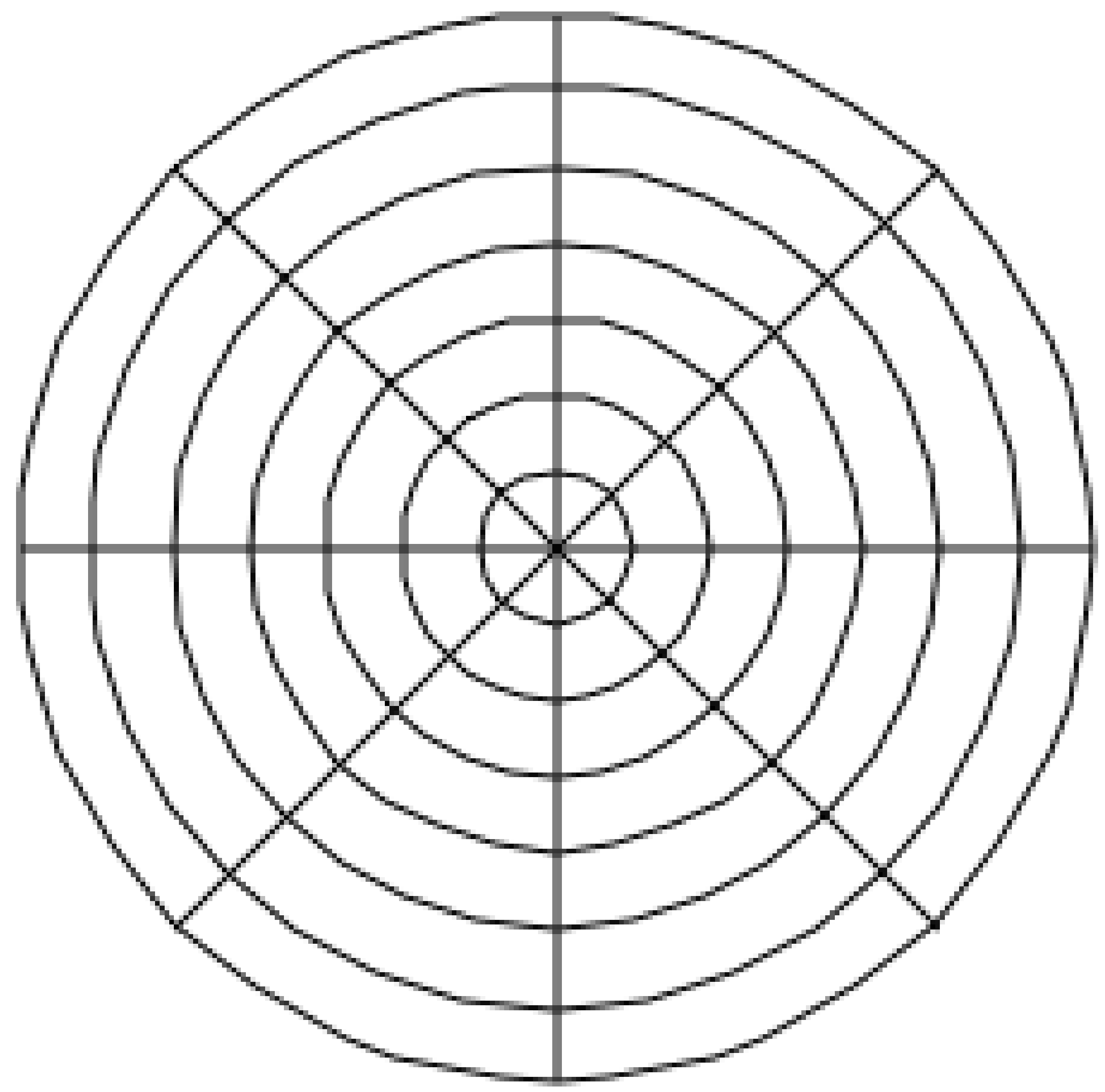
Rejection Sampling




```
do forever {
   sample u in [0, M]
   sample t in [a, b]
   if (u <= p(t))
      return t;
}
```

Uniformly sampling unit disk: a first try

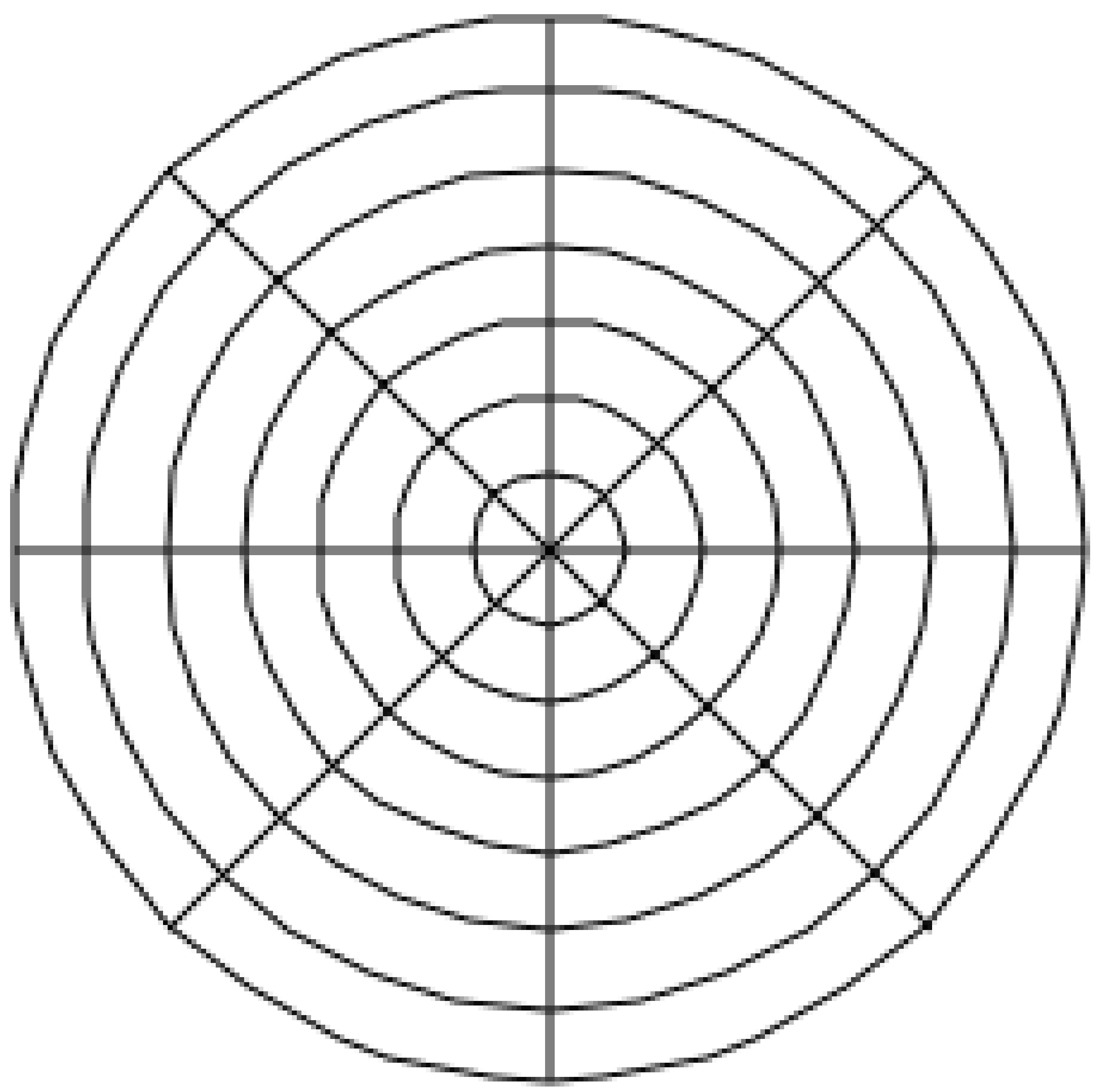
- θ = uniform random angle between 0 and 2π
- r = uniform random radius between 0 and 1
- Return point: $(r\cos\theta, r\sin\theta)$



This algorithm does not produce uniform samples on this 2d disk. Why?

Uniformly sampling unit disk

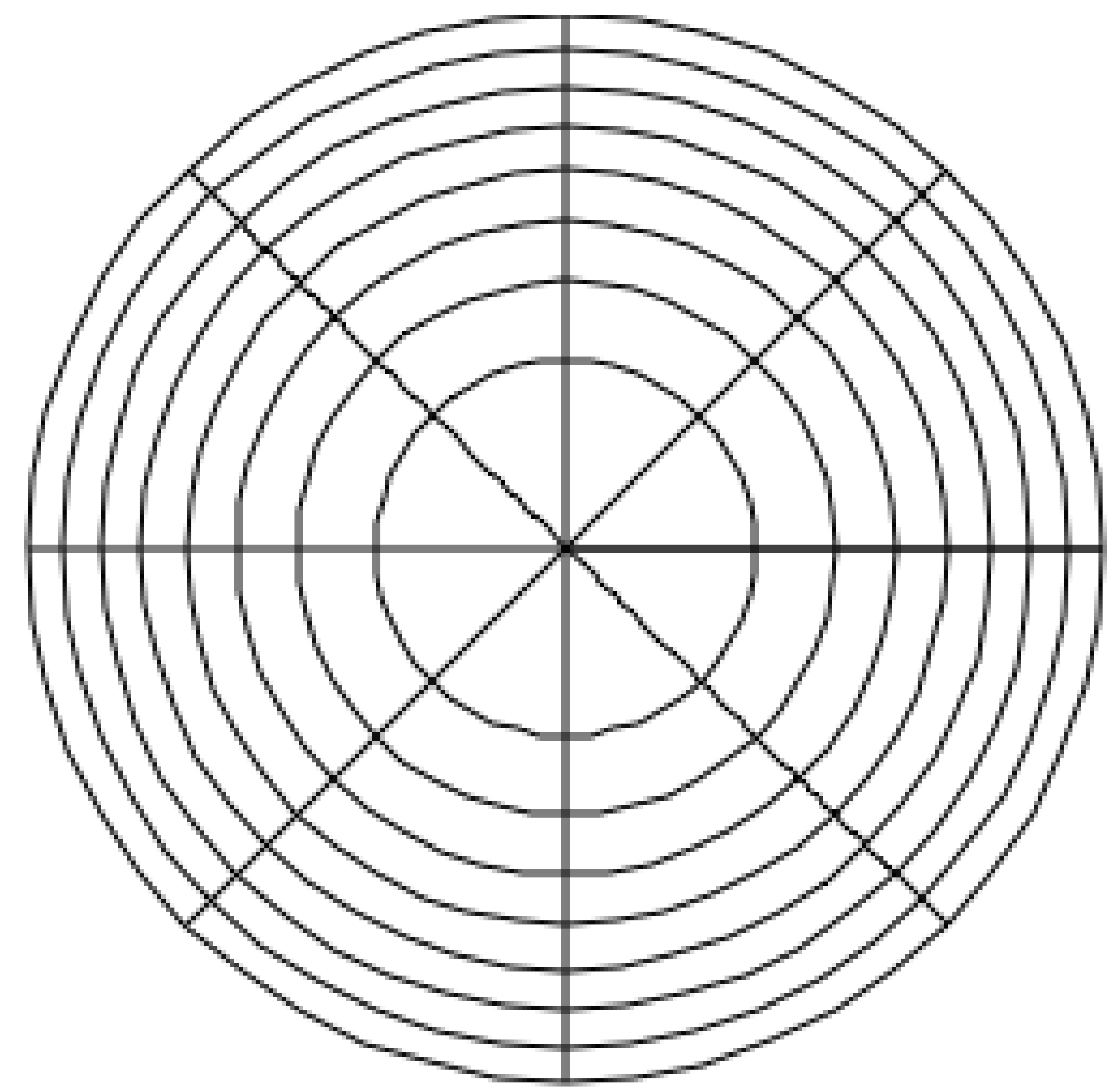
WRONG
Not Equi-areal



$$\theta = 2\pi\xi_1$$

$$r = \xi_2$$

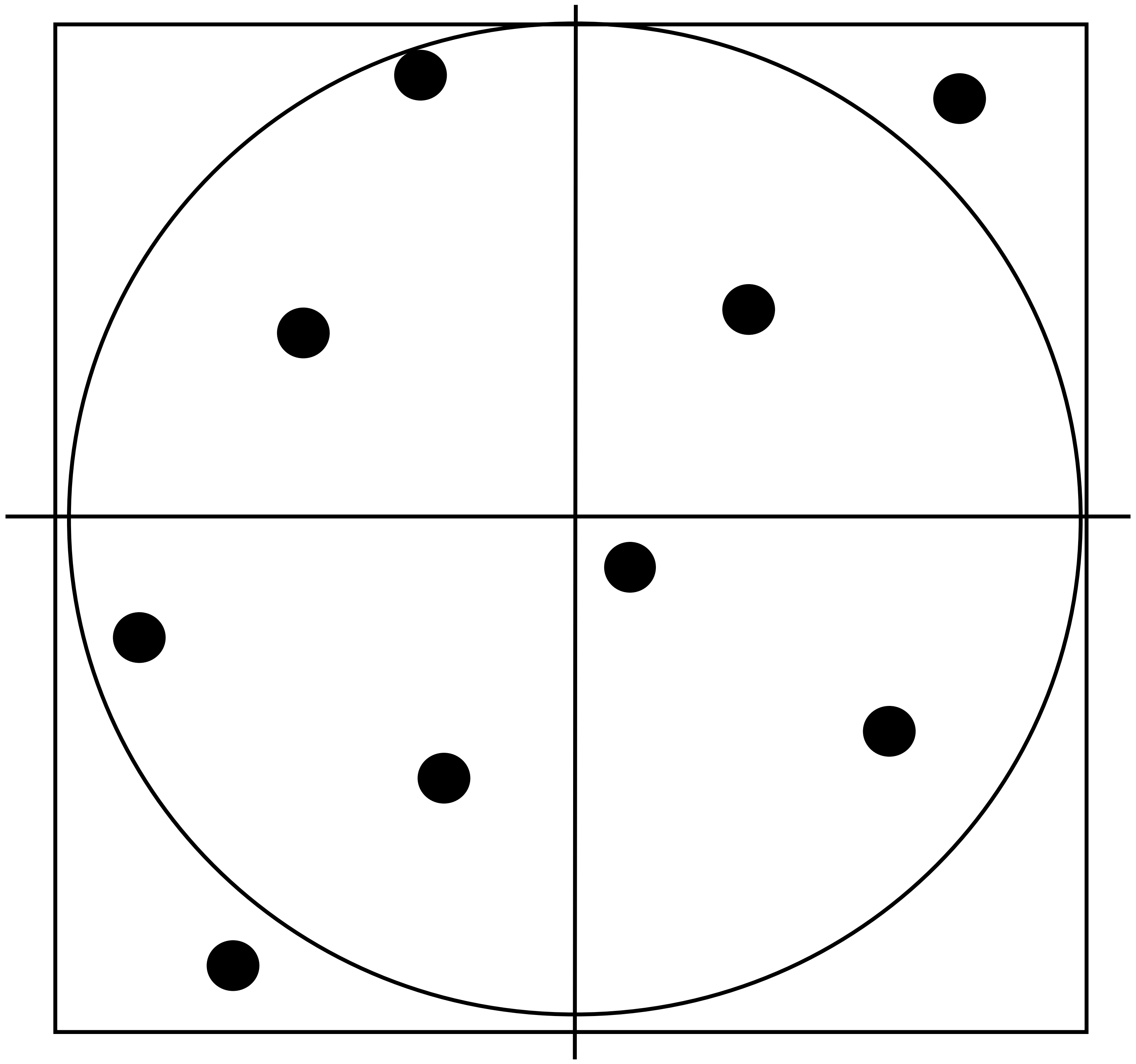
RIGHT
Equi-areal



$$\theta = 2\pi\xi_1$$

$$r = \sqrt{\xi_2}$$

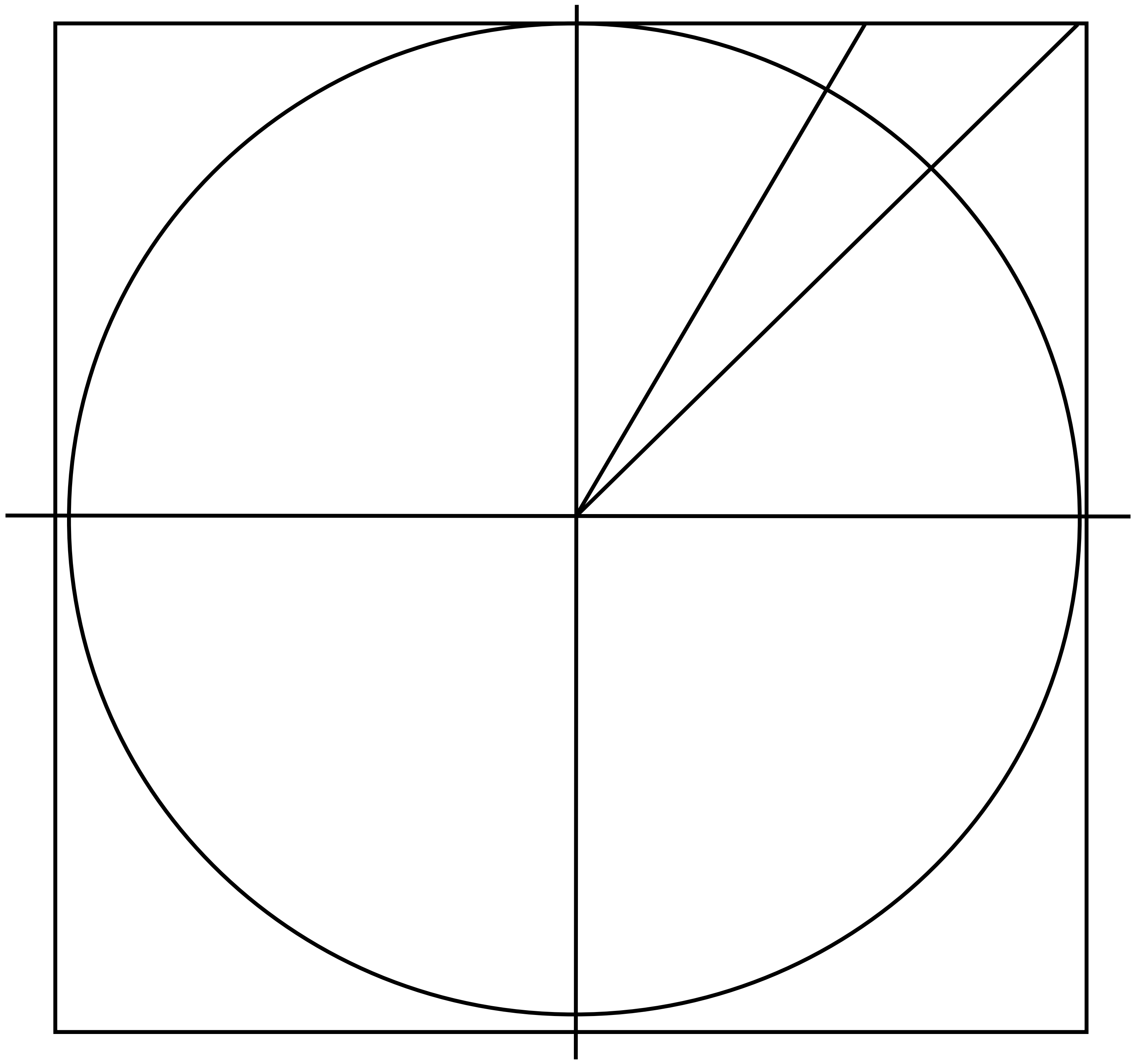
Uniform sampling of unit disk via rejection sampling



Generate random point within unit circle

```
do {
   x = 1 - 2 * rand01();
   y = 1 - 2 * rand01();
} while (x*x + y*y > 1.);
```

Sampling 2D directions



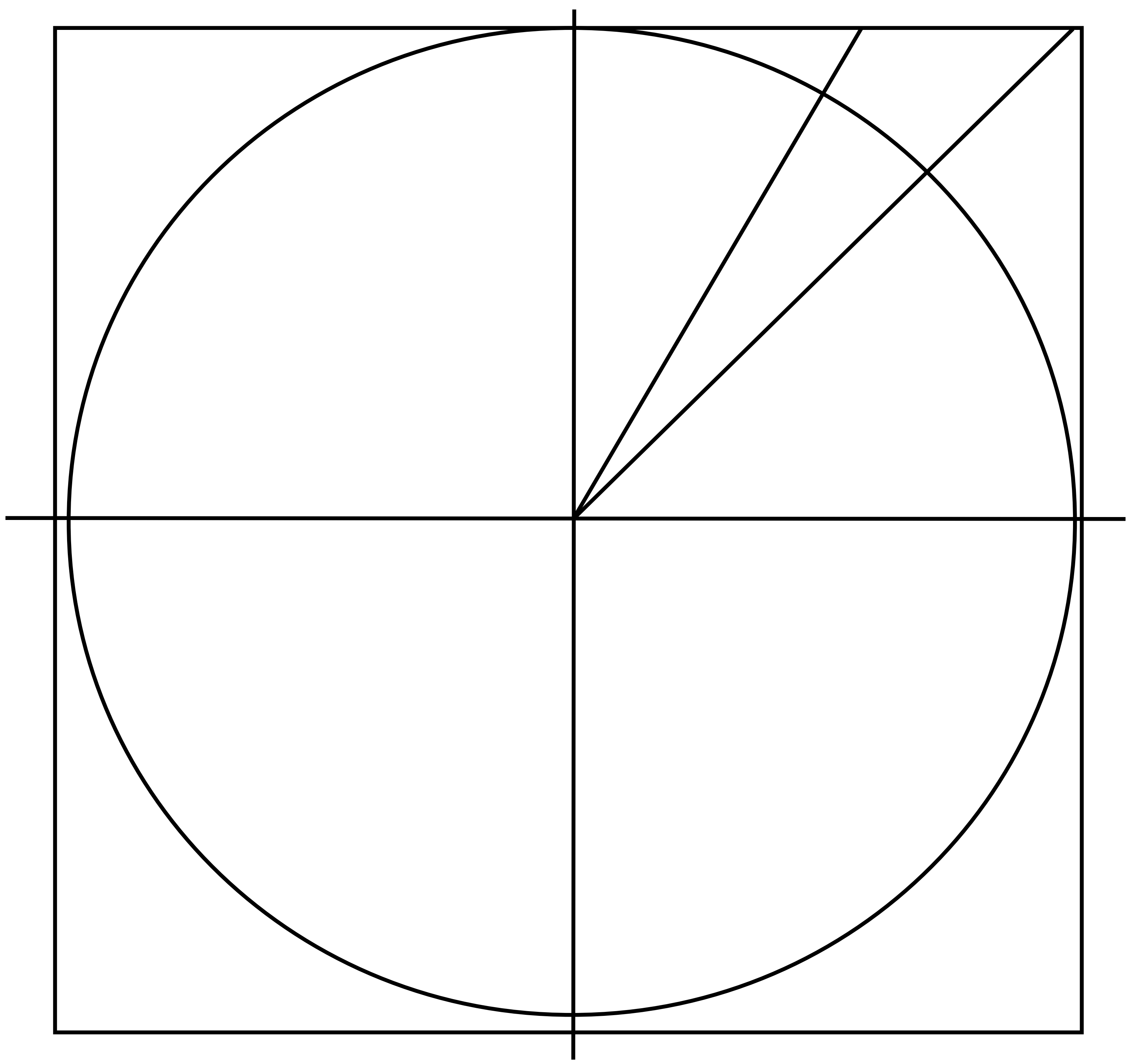
Goal: generate random directions in 2D with uniform probability

```
x = 1 - 2 * rand01();
y = 1 - 2 * rand01();

r = sqrt(x*x+y*y);
x_dir = x/r;
y_dir = y/r;
```

This algorithm is not correct. What is wrong?

Rejection sampling for 2D directions



Goal: generate random directions in 2D with uniform probability

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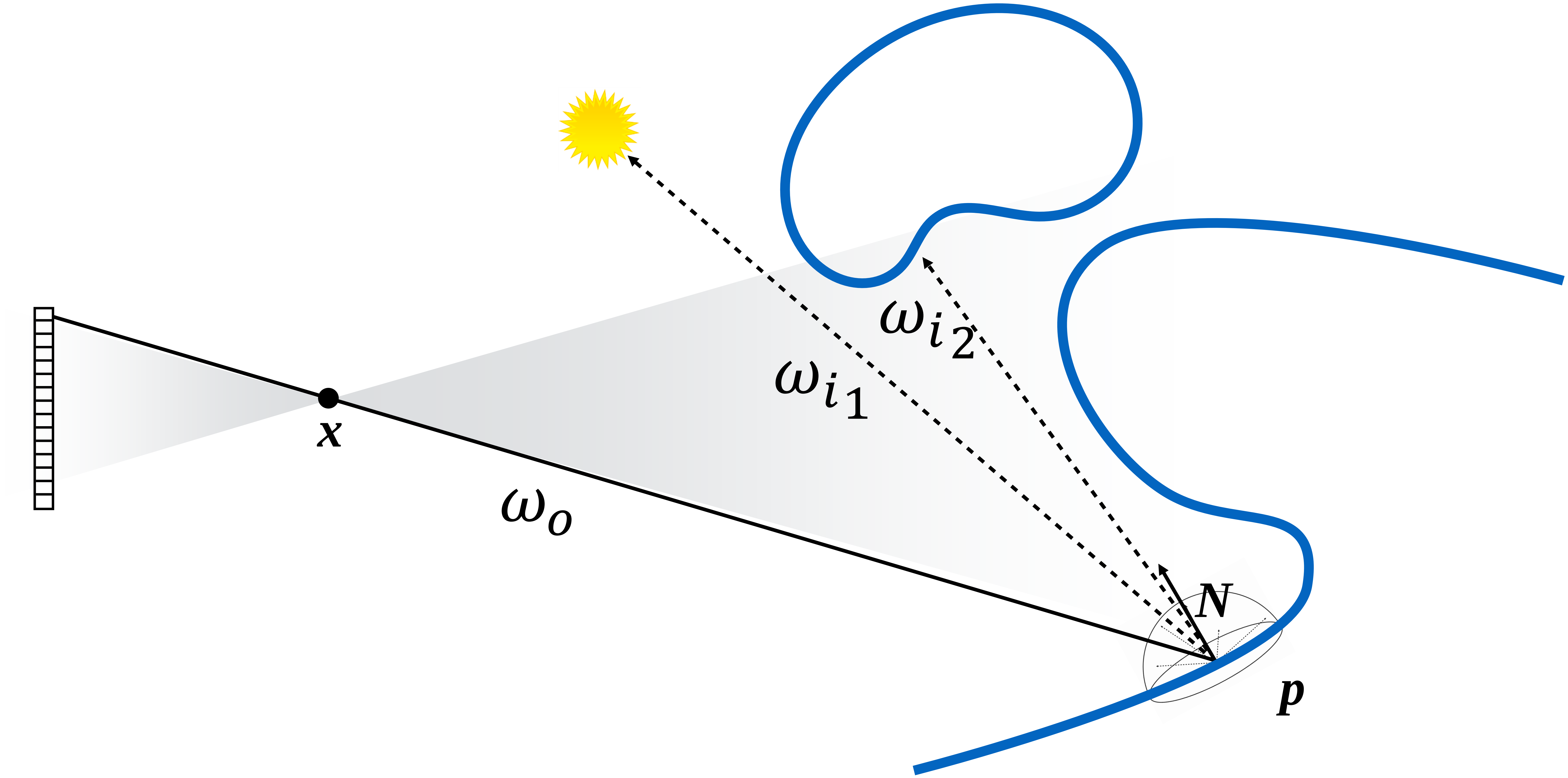
do {
   x = 1 - 2 * rand01();
   y = 1 - 2 * rand01();
} while (x*x + y*y > 1.);

r = sqrt(x*x+y*y);
x_dir = x/r;
y_dir = y/r;
```

Back to our problem: lighting estimate

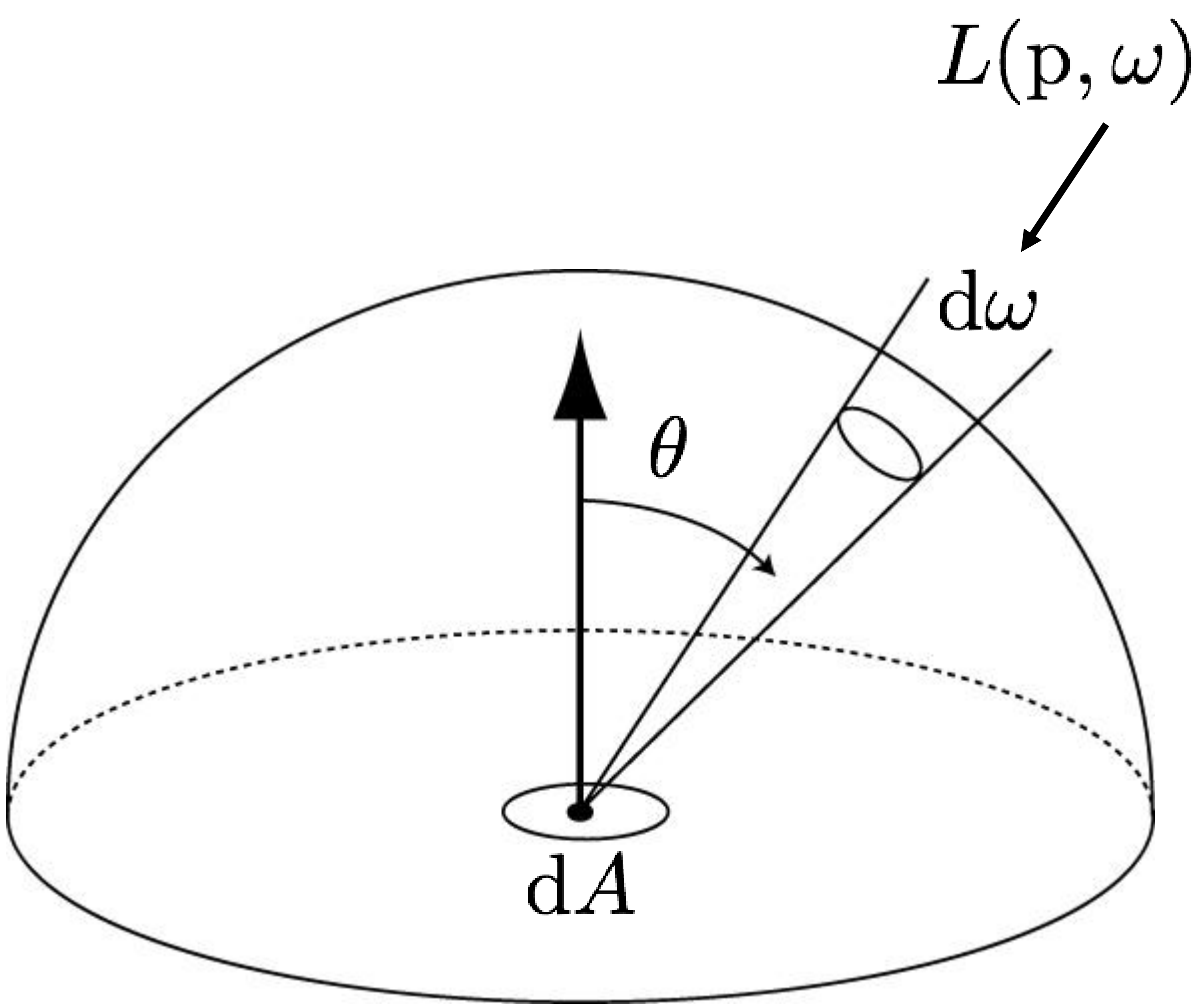


What we really want is to solve the reflection equation:

$$L_o(\mathrm{p}, \omega_o) = L_e(\mathrm{p}, \omega_o) + \int_{H^2} f_r(\mathrm{p}, \omega_i \to \omega_o)\, L_i(\mathrm{p}, \omega_i)\, \cos\theta_i \, \mathrm{d}\omega_i$$

Back to our problem: lighting estimate

Want to integrate reflection term over hemisphere
- using uniformly sampled directions



MC Estimator:

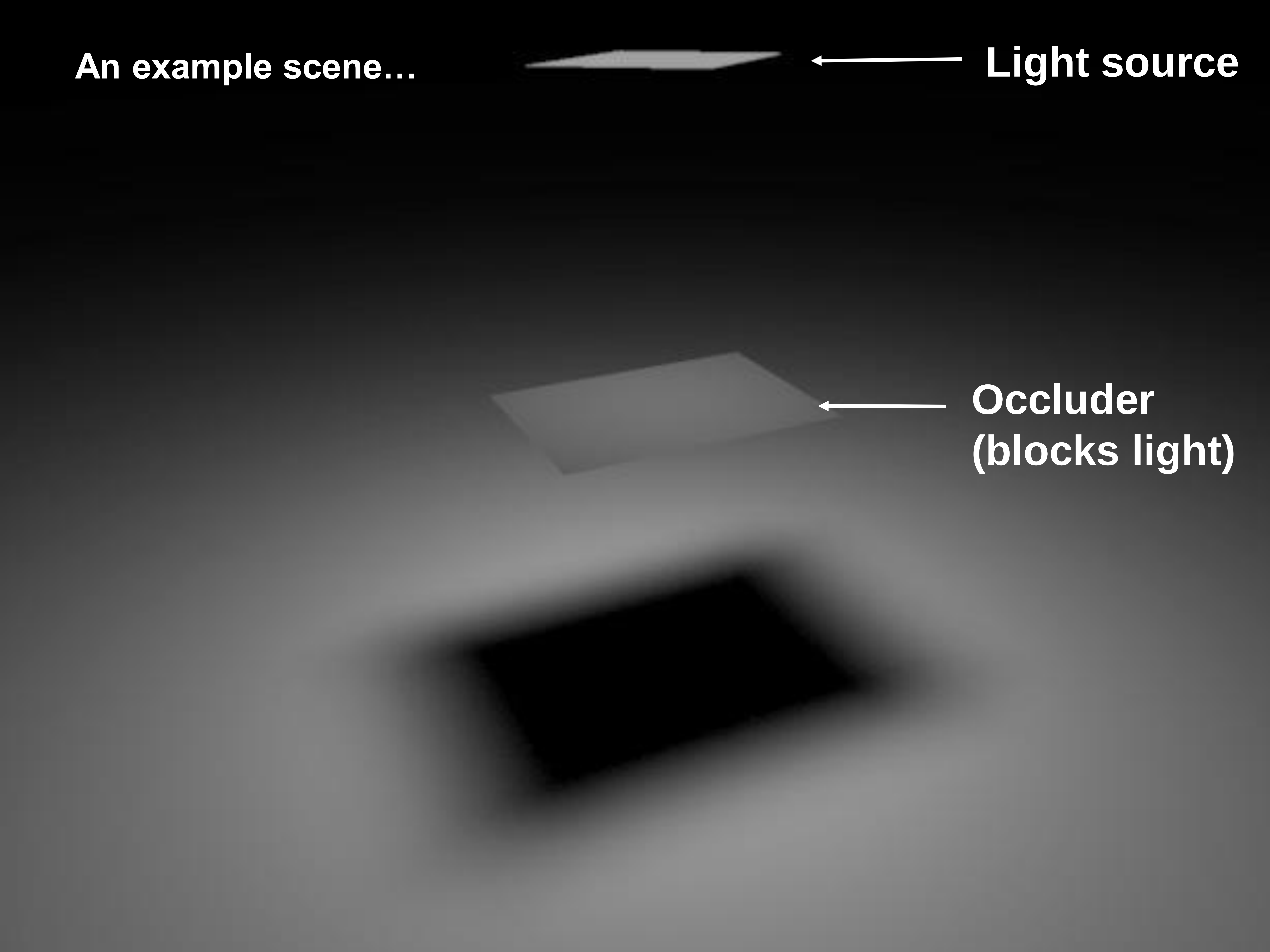
$$X_i \sim p(\omega)$$

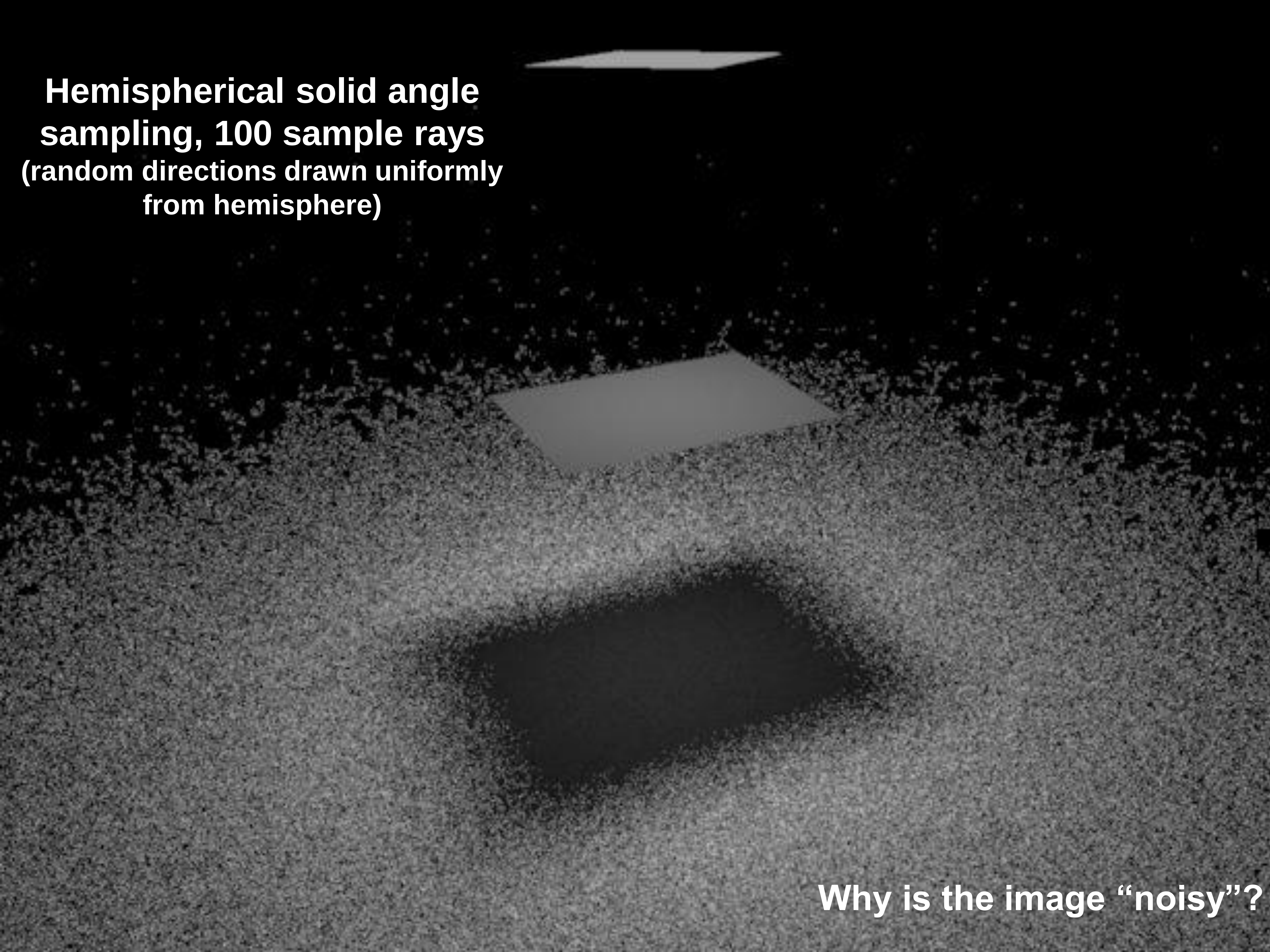
$$p(\omega) = \frac{1}{2\pi}$$

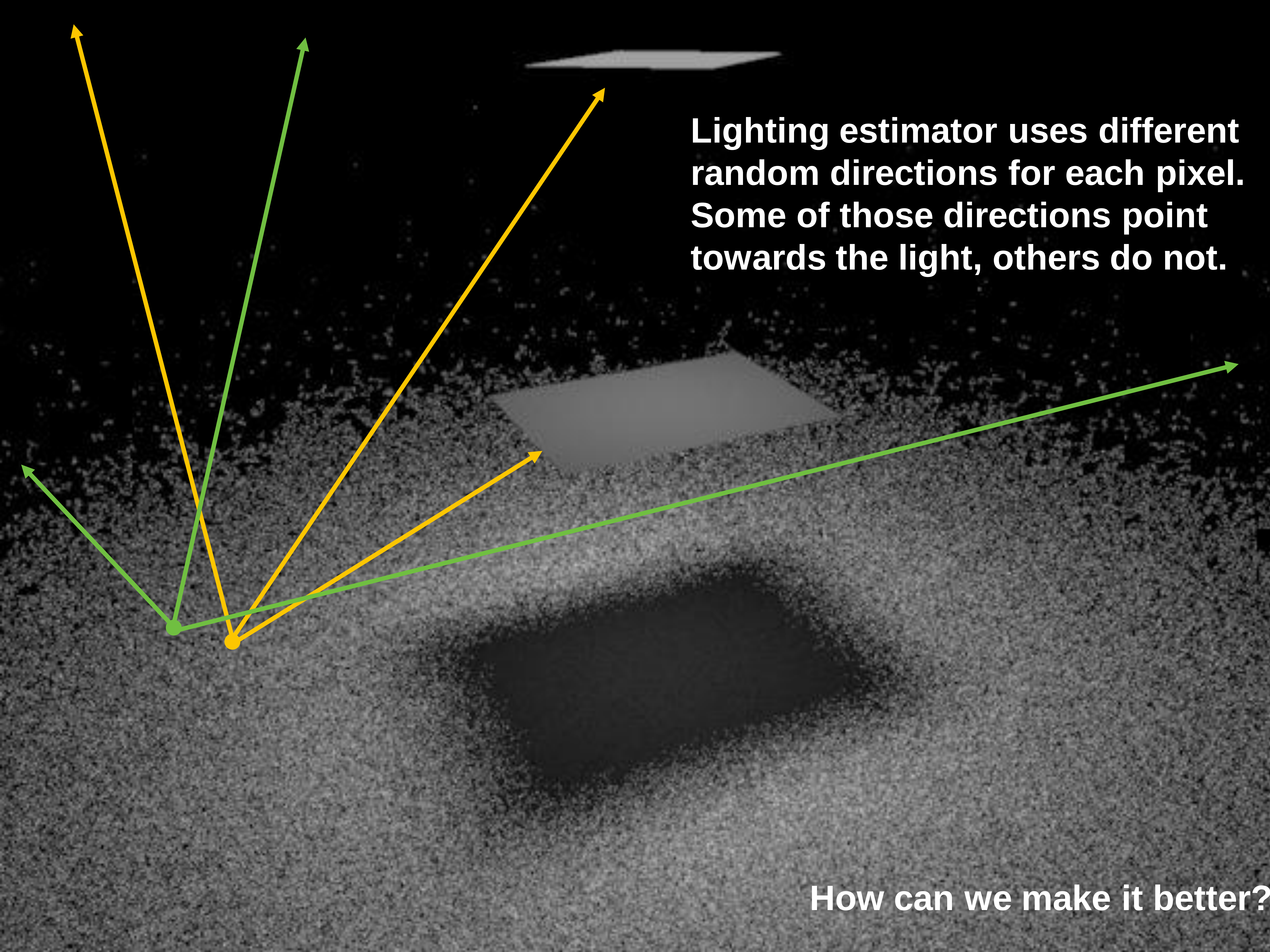
$$Y_i = f(X_i)$$

$$= f_r(\mathrm{p}, \omega_i \to \omega_o)\, L_i(\mathrm{p}, \omega_i)\, \cos\theta_i$$

$$F_N = \frac{2\pi}{N} \sum_{i=1}^{N} Y_i$$


An example scene…
Light source
Occluder
(blocks light)


Hemispherical solid angle
sampling, 100 sample rays
(random directions drawn uniformly
from hemisphere)
Why is the image “noisy”?



Lighting estimator uses different random directions for each pixel. Some of those directions point towards the light, others do not.

How can we make it better?

“Biasing” sample selection for rendering applications

- Note: “biasing” selection of random samples is different than creating a biased estimator!

- Variance reduction method, but can think of it also as importance sampling

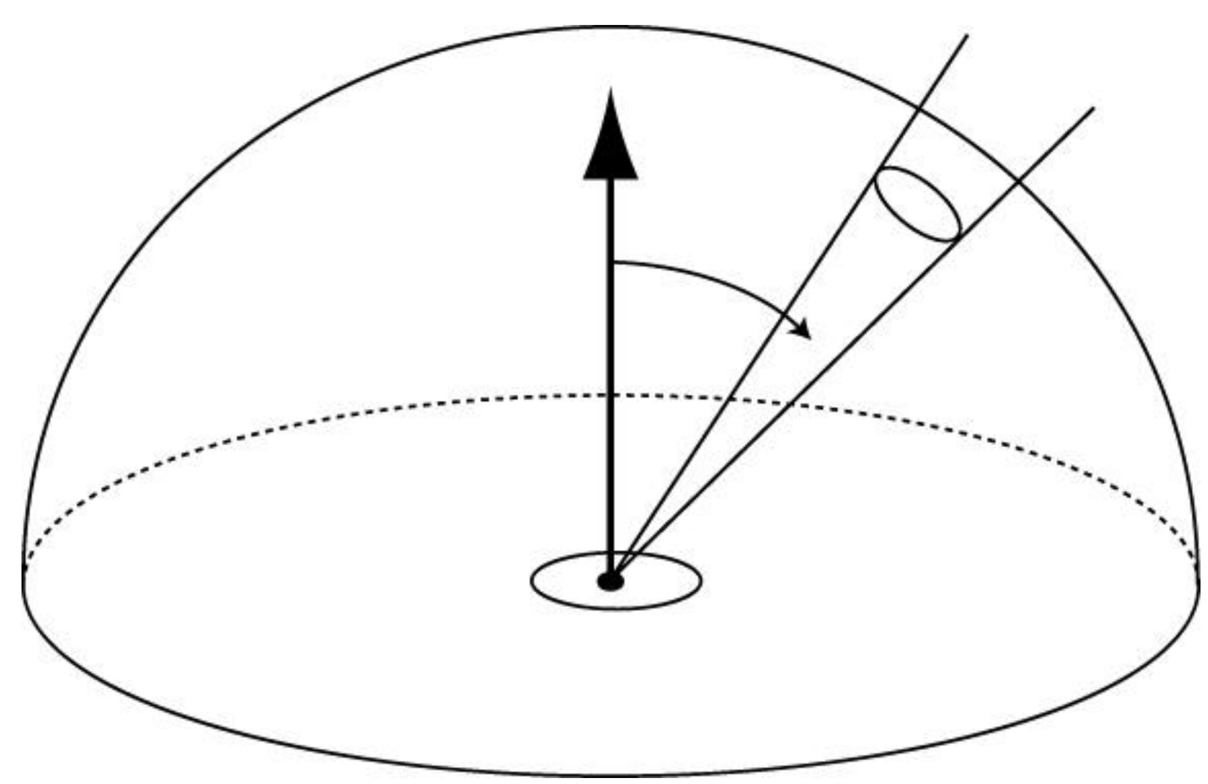
$$\int_{H^2} f_r(\mathrm{p}, \omega_i \rightarrow \omega_o)\, L_i(\mathrm{p}, \omega_i)\ \cos\theta_i \,\mathrm{d}\omega_i$$

Importance sampling example

<u>Cosine-weighted</u> hemisphere sampling in irradiance estimate:

$$f(\omega) = L_i(\omega)\cos\theta \qquad\qquad p(\omega) = \frac{\cos\theta}{\pi}$$

$$\int_\Omega f(\omega)\,\mathrm{d}\omega \approx \frac{1}{N}\sum_i^N \frac{f(\omega)}{p(\omega)} = \frac{1}{N}\sum_i^N \frac{L_i(\omega)\ \cos\theta}{\cos\theta/\pi} = \frac{\pi}{N}\sum_i^N L_i(\omega)$$



<u>Note</u>: Samples along the hemisphere must also be drawn according to this probability distribution!

Summary: Monte Carlo integration

- Estimate value of integral using random sampling
  - Why is it important that the samples are truly random?
  - Algorithm gives the correct value of integral “on average”

- Only requires function to be evaluated at random points on its domain
  - Applicable to functions with discontinuities, functions that are impossible to integrate directly

- Error of estimate independent of dimensionality of integrand
  - Faster convergence in estimating high dimensional integrals than non-randomized quadrature methods
  - Suffers from noise due to variance in estimate
    - more samples & variance reduction method can help