Chapter 2
Mean-Field Theory and the Gaussian Approximation

The Wilsonian renormalization group (RG) was invented in order to study the effect of strong fluctuations and the mutual coupling between different degrees of freedom in the vicinity of continuous phase transitions. Before embarking on the theory of the RG, let us in this chapter describe two less sophisticated methods of dealing with this problem, namely the mean-field approximation and the Gaussian approximation. Within the mean-field approximation, fluctuations of the order parameter are completely neglected and the interactions between different degrees of freedom are taken into account in some simple average way. The Gaussian approximation is in some sense the leading fluctuation correction to the mean-field approximation. Although these methods are very general and can also be used to study quantum mechanical many-body systems[1], for our purpose it is sufficient to introduce these methods using the nearest-neighbor Ising model in D dimensions as an example. The Ising model is defined in terms of the following classical Hamiltonian,

$$H = -\frac{1}{2} \sum_{ij=1}^{N} J_{ij} s_i s_j - h \sum_{i=1}^{N} s_i \ . \tag{2.1}$$

Here, i and j label the sites $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$ of a D-dimensional hypercubic lattice with N sites, the variables $s_i = \pm 1$ correspond to the two possible states of the z-components of spins localized at the lattice sites, the J_{ij} denote the exchange interaction between spins localized at sites $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$, and h is the Zeeman energy associated with an external magnetic field in the z-direction. The above model is called classical because it does not involve any noncommuting operators. By simply rotating the magnetic field in the x-direction we obtain a quantum Ising model, as discussed in Exercise 2.1. The quantum mechanical origin of magnetism is hidden in the exchange energies J_{ij}. Due to the exponential decay of localized wave functions, it is often sufficient to assume that the J_{ij} are nonzero only if the sites $\boldsymbol{r}_i$ and $\boldsymbol{r}_j$

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[1] For example, in quantum many-body systems the *self-consistent Hartree–Fock approximation* can be viewed as a variant of the mean-field approximation, while the Gaussian approximation is usually called *random phase approximation*.

are nearest neighbors on the lattice. Denoting the nonzero value of J_{ij} for nearest neighbors by J, we obtain from Eq. (2.1) the nearest-neighbor Ising model

$$H = -J \sum_{\langle ij \rangle} s_i s_j - h \sum_{i=1}^{N} s_i \ . \tag{2.2}$$

Here, $\langle ij \rangle$ denotes the summation over all distinct pairs of nearest neighbors. In order to obtain thermodynamic observables, we have to calculate the partition function,

$$\mathcal{Z}(T, h) = \sum_{\{s_i\}} e^{-\beta H} \equiv \sum_{s_1 = \pm 1} \sum_{s_2 = \pm 1} \cdots \sum_{s_N = \pm 1} \exp \left[\beta J \sum_{\langle ij \rangle} s_i s_j + \beta h \sum_i s_i \right],$$
$$\tag{2.3}$$

where we have introduced the notation $\beta = 1/T$ for the inverse temperature.[2] While in one dimension it is quite simple to carry out this summation (see Exercise 1.1), the corresponding calculation in $D = 2$ is much more difficult and until now the exact $\mathcal{Z}(T, h)$ for $h \neq 0$ is not known. For $h = 0$ the partition function of the two-dimensional Ising model was first calculated by Onsager (1944), who also presented an exact expression for the spontaneous magnetization at a conference in 1949. A proof for his result was given in 1952 by C. N. Yang. See the textbooks by Wannier (1966), by Huang (1987), or by Mattis (2006) for pedagogical descriptions of the exact solution for $h = 0$. In dimensions $D = 3$ there are no exact results available, so one has to rely on approximations. The simplest is the mean-field approximation discussed in the following section.

2.1 Mean-Field Theory

2.1.1 Landau Function and Free Energy

Mean-field theory is based on the assumption that the fluctuations around the average value of the order parameter are so small that they can be neglected. Let us therefore assume that the system has a finite magnetization,[3]

$$m = \langle s_i \rangle \equiv \frac{\sum_{\{s_j\}} e^{-H/T} s_i}{\sum_{\{s_j\}} e^{-H/T}} \ , \tag{2.4}$$

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[2] The inverse temperature $\beta = 1/T$ should not be confused with the order-parameter exponent β. Because these notations are both standard we shall adopt them here; usually the meaning of the symbol β is clear from the context.

[3] For lattice models it is convenient to divide the total magnetic moment by the number N of lattice sites and not by the total volume. For simplicity we use here the same symbol as for the magnetization in Eq. (1.3), which has units of inverse volume.

where we have used the fact that by translational invariance the thermal expectation values $\langle s_i \rangle$ are independent of the site label i. Writing $s_i = m + \delta s_i$ with the fluctuation $\delta s_i = s_i - m$, we have

$$s_i s_j = m^2 + m(\delta s_i + \delta s_j) + \delta s_i \delta s_j = -m^2 + m(s_i + s_j) + \delta s_i \delta s_j . \tag{2.5}$$

We now assume that the fluctuations are small so that the last term $\delta s_i \delta s_j$ which is quadratic in the fluctuations can be neglected. Within this approximation, the Ising Hamiltonian (2.1) is replaced by the mean-field Hamiltonian

$$H_{\mathrm{MF}} = \frac{m^2}{2} \sum_{ij} J_{ij} - \sum_i \left(h + \sum_j J_{ij} m \right) s_i$$

$$= N \frac{zJ}{2} m^2 - \sum_i (h + zJm) s_i , \tag{2.6}$$

where the second line is valid for the nearest-neighbor interactions, and $z = 2D$ is the number of nearest neighbors (coordination number) of a given site of a D-dimensional hypercubic lattice. As the spins in the mean-field Hamiltonian (2.6) are decoupled, the partition function factorizes into a product of N independent terms which are just the partition functions of single spins in an effective magnetic field $h_{\mathrm{eff}} = h + zJm$,

$$\mathcal{Z}_{\mathrm{MF}}(T, h) = e^{-\beta N z J m^2/2} \sum_{\{s_i\}} e^{\beta(h + zJm) \sum_i s_i}$$

$$= e^{-\beta N z J m^2/2} \prod_i \left[e^{\beta(h + zJm)} + e^{-\beta(h + zJm)} \right]$$

$$= e^{-\beta N z J m^2/2} \left[2 \cosh[\beta(h + zJm)] \right]^N . \tag{2.7}$$

Writing this as

$$\mathcal{Z}_{\mathrm{MF}}(T, h) = e^{-\beta N \mathcal{L}_{\mathrm{MF}}(T, h; m)} , \tag{2.8}$$

we obtain in mean-field approximation

$$\mathcal{L}_{\mathrm{MF}}(T, h; m) = \frac{zJ}{2} m^2 - T \ln \left[2 \cosh[\beta(h + zJm)] \right] . \tag{2.9}$$

As will be explained in more detail below, the function $\mathcal{L}_{\mathrm{MF}}(T, h; m)$ is an example for a Landau function, which describes the probability distribution of the order parameter: the probability density of observing for the order parameter the value m is proportional to $\exp[-\beta N \mathcal{L}_{\mathrm{MF}}(T, h; m)]$. The so far unspecified parameter m is now determined from the condition that the physical value m_0 of the order parameter

maximizes its probability distribution, corresponding to the minimum of the Landau function,

$$\left.\frac{\partial \mathcal{L}_{MF}(T, h; m)}{\partial m}\right|_{m_0} = 0 \ . \tag{2.10}$$

From Eq. (2.9) we then find that the magnetization m_0 in mean-field approximation satisfies the self-consistency condition

$$\boxed{m_0 = \tanh[\beta(h + zJm_0)] \ ,} \tag{2.11}$$

which defines $m_0 = m_0(T, h)$ as a function of T and h. The mean-field result for the free energy per site is thus

$$f_{MF}(T, h) = \mathcal{L}_{MF}(T, h; m_0(T, h)) \ . \tag{2.12}$$

The mean-field self-consistency equation (2.11) can easily be solved graphically. As shown in Fig. 2.1, for $h \neq 0$ the global minimum of $\mathcal{L}_{MF}(T, h; m)$ occurs always at a finite $m_0 \neq 0$. On the other hand, for $h = 0$ the existence of nontrivial solutions with $m_0 \neq 0$ depends on the temperature. In the low-temperature regime $T < zJ$ there are two nontrivial solutions with $m_0 \neq 0$, while at high temperatures $T > zJ$ our self-consistency equation (2.11) has only the trivial solution $m_0 = 0$, see Fig. 2.2. In D dimensions the mean-field estimate for the critical temperature is therefore

$$T_c = zJ = 2DJ \ . \tag{2.13}$$

For $D = 1$ this is certainly wrong, because we know from the exact solution (see Exercise 1.1) that $T_c = 0$ in one dimension. In two dimensions the exact critical temperature of the nearest-neighbor Ising model satisfies $\sinh(2J/T_c) = 1$ (Onsager 1944), which yields $T_c \approx 2.269J$ and is significantly lower than the mean-field prediction of $4J$. As a general rule, in lower dimensions fluctuations are more important and tend to disorder the system or at least reduce the critical temperature.

2.1.2 Thermodynamic Critical Exponents

For temperatures close to T_c and small $\beta|h|$ the value m_0 of the magnetization at the minimum of $\mathcal{L}_{MF}(T, h; m)$ is small. We may therefore approximate the Landau function (2.9) by expanding the right-hand side of Eq. (2.9) up to fourth order in m and linear order in h. Using

$$\ln[2\cosh x] = \ln 2 + \frac{x^2}{2} - \frac{x^4}{12} + \mathcal{O}(x^6) \ , \tag{2.14}$$
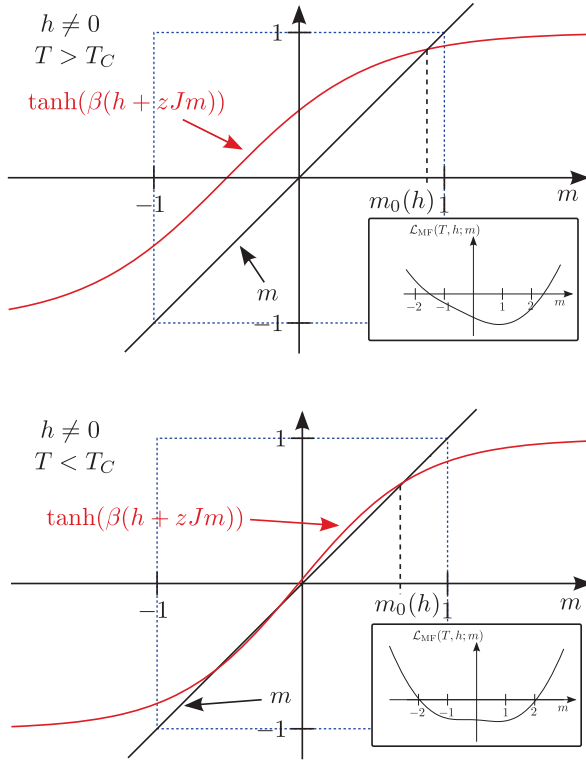
Fig. 2.1 Graphical solution of the mean-field self-consistency equation (2.11) for $h > 0$. The inset shows the behavior of the corresponding Landau function $\mathcal{L}_{\text{MF}}(T, h; m)$ defined in Eq. (2.9). For $T > T_c$ or h sufficiently large the Landau function exhibits only one minimum at finite $m_0 > 0$. For $T < T_c$ and h sufficiently small, however, there is another local minimum at negative m, but the global minimum of $\mathcal{L}_{\text{MF}}(T, h; m)$ is still at $m_0 > 0$

we obtain from Eq. (2.9),

$$\mathcal{L}_{\text{MF}}(T, h; m) = f + \frac{r}{2}m^2 + \frac{u}{4!}m^4 - hm + \dots \,, \tag{2.15}$$

with

$$f = -T \ln 2 \,, \tag{2.16a}$$

$$r = \frac{zJ}{T}\left(T - zJ\right) \approx T - T_c \,, \tag{2.16b}$$

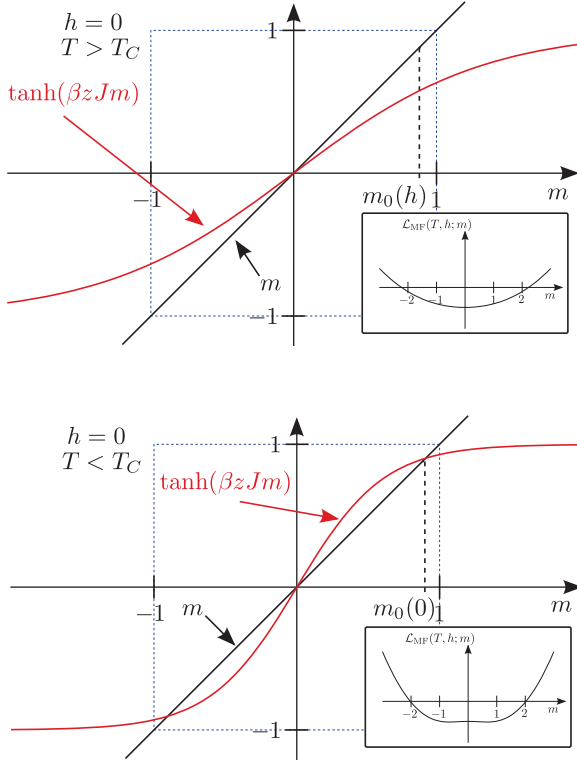$$u = 2T\left(\frac{zJ}{T}\right)^4 \approx 2T_c \,, \tag{2.16c}$$

Fig. 2.2 Graphical solution of the mean-field self-consistency equation (2.11) for $h = 0$. The upper figure shows the typical behavior in the disordered phase $T > T_c$, while the lower figure represents the ordered phase $T < T_c$. The behavior of the Landau function is shown in the insets: while for $T > T_c$ it has a global minimum at $m_0 = 0$, it develops for $T < T_c$ two degenerate minima at $\pm|m_0| \neq 0$

where the approximations are valid close to the critical temperature, where $|T - T_c| \ll T_c$ and $zJ/T \approx 1$. Obviously, the sign of the coefficient r changes at $T = T_c$, so that for $h = 0$ the global minimum of $\mathcal{L}_{\mathrm{MF}}(T, h; m)$ for $T > T_c$ evolves into a local maximum for $T < T_c$, and two new minima emerge at finite values of m, as shown in Fig. 2.2. The crucial point is now that for a small reduced temperature $t = (T - T_c)/T_c$ the value of m at the minima of $\mathcal{L}_{\mathrm{MF}}(T, h; m)$ is small compared with unity, so that our expansion (2.15) in powers of m is justified a posteriori. Taking the derivative of Eq. (2.15) with respect to m, it is easy to see that Eq. (2.11) simplifies to

$$\left.\frac{\partial \mathcal{L}_{\mathrm{MF}}(T, h; m)}{\partial m}\right|_{m_0} = rm_0 + \frac{u}{6}m_0^3 - h = 0 . \tag{2.17}$$

The behavior of thermodynamic observables close to T_c is now easily obtained:

(a) *Spontaneous magnetization:* Setting $h = 0$ in Eq. (2.17) and solving for m_0, we obtain for $T \leq T_c$ (with $r \leq 0$),

$$m_0 = \sqrt{\frac{-6r}{u}} \propto (-t)^{1/2} . \tag{2.18}$$

Comparing this with the definition (1.6) of the critical exponent β, we conclude that the mean-field approximation predicts for the Ising universality class $\beta = 1/2$, independently of the dimension D.

(b) *Zero-field susceptibility:* To obtain the mean-field result for the susceptibility exponent γ, we note that for small but finite h and $T \geq T_c$ we may neglect the terms of order m_0^3 in Eq. (2.17), so that $m_0(h) \propto h/r$, and hence the zero-field susceptibility behaves for $t \to 0$ as

$$\chi = \left. \frac{\partial m_0(h)}{\partial h} \right|_{h=0} \propto \frac{1}{r} \propto \frac{1}{T - T_c} . \tag{2.19}$$

It is a simple exercise to show that $\chi \propto |T - T_c|^{-1}$ also holds for $T < T_c$. The susceptibility exponent is therefore $\gamma = 1$ within mean-field approximation.

(c) *Critical isotherm:* The equation of state at the critical point can be obtained by setting $r = 0$ in Eq. (2.17), implying

$$m_0(h) \propto \left(\frac{h}{u} \right)^{1/3} , \tag{2.20}$$

and hence the mean-field result $\delta = 3$.

(d) *Specific heat:* The specific heat C per lattice site can be obtained from the thermodynamic relation

$$C = -T \frac{\partial^2 f_{\mathrm{MF}}(T, h)}{\partial T^2} , \tag{2.21}$$

where the mean-field free energy per site $f_{\mathrm{MF}}(T, h)$ is given in Eq. (2.12). Setting $h = 0$, we find from Eq. (2.17) for $T > T_c$ that $f_{\mathrm{MF}}(t, 0) = f$ because $m_0 = 0$ in this case. On the other hand, for $T < T_c$ we may substitute Eq. (2.18), so that

$$f_{\mathrm{MF}}(t, 0) = f - \frac{3}{2} \frac{r^2}{u} , \quad T < T_c . \tag{2.22}$$

Setting $r \approx T - T_c$ and taking two derivatives with respect to T, we obtain

$$C \approx -T_c \frac{\partial^2 f}{\partial T^2} , \qquad T > T_c , \tag{2.23a}$$

$$\approx -T_c \frac{\partial^2 f}{\partial T^2} + 3\frac{T_c}{u} , \quad T < T_c . \tag{2.23b}$$

Note that according to Eq. (2.16c) $u \approx 2T_c$ so that $3T_c/u \approx 3/2$. We conclude that within the mean-field approximation the specific heat is discontinuous at T_c, so that $C \propto |t|^0$, implying $\alpha = 0$. Note that the mean-field results are consistent with the scaling relations (1.23) and (1.24), $2 - \alpha = 2\beta + \gamma = \beta(\delta + 1)$.

In order to obtain the exponents ν and η, we need to calculate the correlation function $G(\boldsymbol{r})$, which we shall do in Sect. 2.3 within the so-called Gaussian approximation. However, a comparison of the mean-field results $\alpha = 0$, $\beta = 1/2$, $\gamma = 1$, and $\delta = 3$ with the correct values given in Table 1.1 at the end of Sect. 1.2 shows that the mean-field approximation is not suitable to obtain quantitatively accurate results, in particular in the physically accessible dimensions $D = 2$ and $D = 3$.

2.2 Ginzburg–Landau Theory

Our ultimate goal will be to develop a systematic method for taking into account the fluctuations neglected in mean-field theory, even if the fluctuations are strong and qualitatively change the mean-field results. As a first step to achieve this ambitious goal, we shall in this chapter derive from the microscopic Ising Hamiltonian (2.1) an effective (classical) field theory whose fields $\varphi(\boldsymbol{r})$ represent suitably defined spatial averages of the fluctuating magnetization over sufficiently large domains such that $\varphi(\boldsymbol{r})$ varies only slowly on the scale of the lattice spacing. The concept of an effective field theory representing *coarse-grained* fluctuations averaged over larger and larger length scales lies at the heart of the Wilsonian RG idea. It turns out that in the vicinity of the critical point the form of the effective coarse-grained action $S[\varphi]$ is constrained by symmetry and can be written down on the basis of phenomenological considerations. Such a strategy has been adopted by Ginzburg and Landau (1950) to develop a phenomenological theory of superconductivity which is particularly useful to treat spatial inhomogeneities.

2.2.1 Exact Effective Field Theory

To derive the effective order-parameter field theory for the Ising model, let us write the partition function of the Hamiltonian (2.1) in compact matrix form,

$$\mathcal{Z} = \sum_{\{s_i\}} \exp\left[\frac{\beta}{2} \sum_{ij} J_{ij} s_i s_j + \beta h \sum_i s_i\right] = \sum_{\{s_i\}} \exp\left[\frac{1}{2}\boldsymbol{s}^T \tilde{\boldsymbol{J}} \boldsymbol{s} + \tilde{\boldsymbol{h}}^T \boldsymbol{s}\right] ,$$

$$\tag{2.24}$$

where s and $\tilde{\mathbf{h}}$ are N-dimensional column vectors with $[s]_i = s_i$ and $[\tilde{\mathbf{h}}]_i = \beta h$, and $\tilde{\mathbf{J}}$ is an $N \times N$-matrix with matrix elements $[\tilde{\mathbf{J}}]_{ij} = \tilde{J}_{ij} = \beta J_{ij}$. We now use the following mathematical identity for N-dimensional Gaussian integrals, valid for any positive symmetric matrix $\mathbf{A}$,

$$\left(\prod_{i=1}^{N} \int_{-\infty}^{\infty} \frac{dx_i}{\sqrt{2\pi}} \right) e^{-\frac{1}{2}x^T \mathbf{A}x + x^T s} = [\det \mathbf{A}]^{-1/2} e^{\frac{1}{2}s^T \mathbf{A}^{-1}s} , \qquad (2.25)$$

to write the Ising interaction in the following form (we identify $\tilde{\mathbf{J}} = \mathbf{A}^{-1}$)

$$e^{\frac{1}{2}s^T \tilde{\mathbf{J}}s} = \frac{\int \mathcal{D}[x] \exp\left[-\frac{1}{2}x^T \tilde{\mathbf{J}}^{-1}x + x^T s\right]}{\int \mathcal{D}[x] \exp\left[-\frac{1}{2}x^T \tilde{\mathbf{J}}^{-1}x\right]} , \qquad (2.26)$$

where we have introduced the notation

$$\int \mathcal{D}[x] \equiv \prod_{i=1}^{N} \int_{-\infty}^{\infty} \frac{dx_i}{\sqrt{2\pi}} . \qquad (2.27)$$

In fact, the identity (2.26) can be easily proven without using the formula (2.25) by redefining the integration variables in the numerator via the shift $x = x' + \tilde{\mathbf{J}}s$ and completing the squares,

$$-\frac{1}{2}(x' + \tilde{\mathbf{J}}s)^T \tilde{\mathbf{J}}^{-1}(x' + \tilde{\mathbf{J}}s) + (x' + \tilde{\mathbf{J}}s)^T s = -\frac{1}{2}x'^T \tilde{\mathbf{J}}^{-1}x' + \frac{1}{2}s^T \tilde{\mathbf{J}}s . \qquad (2.28)$$

In a sense, Eq. (2.26) amounts to reading the Gaussian integration formula from right to left, introducing an auxiliary integration to write the right-hand side in terms of a Gaussian integral. Analogous transformations turn out to be very useful to introduce suitable collective degrees of freedom in quantum mechanical many-body systems; in this context transformations of the type (2.26) are called Hubbard–Stratonovich transformations (Hubbard 1959, Stratonovich 1957, Kopietz 1997). The Ising partition function (2.24) can now be written as

$$\mathcal{Z} = \frac{\int \mathcal{D}[x] \exp\left[-\frac{1}{2}x^T \tilde{\mathbf{J}}^{-1}x\right] \sum_{\{s_i\}} \exp\left[(\tilde{\mathbf{h}} + x)^T s\right]}{\int \mathcal{D}[x] \exp\left[-\frac{1}{2}x^T \tilde{\mathbf{J}}^{-1}x\right]} . \qquad (2.29)$$

For a given configuration of the Hubbard–Stratonovich field x, the spin summation in the numerator factorizes again in a product of independent terms, each describing the partition function of a single spin in a site-dependent magnetic field $h + x_i/\beta$. Therefore, this spin summation can easily be carried out,

$$\sum_{\{s_i\}} \exp\left[(\tilde{\mathbf{h}} + \mathbf{x})^T \mathbf{s}\right] = \prod_{i=1}^{N}\left[\sum_{s_i=\pm 1} e^{(\beta h + x_i)s_i}\right] = \prod_{i=1}^{N}\left[2\cosh(\beta h + x_i)\right]$$

$$= \exp\left[\sum_{i=1}^{N} \ln\left[2\cosh(\beta h + x_i)\right]\right] . \tag{2.30}$$

The partition function (2.29) can thus be written as

$$\mathcal{Z} = \frac{\int \mathcal{D}[x] e^{-\tilde{S}[x]}}{\int \mathcal{D}[x] \exp\left[-\frac{1}{2}\mathbf{x}^T \tilde{\mathbf{J}}^{-1}\mathbf{x}\right]} = \frac{1}{\sqrt{\det \tilde{\mathbf{J}}}} \int \mathcal{D}[x] e^{-\tilde{S}[x]} , \tag{2.31}$$

where

$$\tilde{S}[x] = \frac{1}{2}\mathbf{x}^T \tilde{\mathbf{J}}^{-1}\mathbf{x} - \sum_{i=1}^{N} \ln\left[2\cosh(\beta h + x_i)\right] . \tag{2.32}$$

To understand the physical meaning of the variables $\mathbf{x}$, we calculate the expectation value of its i-th component

$$\langle x_i \rangle_{\tilde{S}} \equiv \frac{\int \mathcal{D}[x] e^{-\tilde{S}[x]} x_i}{\int \mathcal{D}[x] e^{-\tilde{S}[x]}} . \tag{2.33}$$

Introducing an auxiliary N-component column vector $\mathbf{y} = (y_1, \ldots, y_N)^T$ to write

$$x_i = \lim_{y \to 0} \frac{\partial}{\partial y_i} e^{\mathbf{x}^T \mathbf{y}} , \tag{2.34}$$

and performing the Gaussian integration using Eq. (2.26), we have the following chain of identities,

$$\langle x_i \rangle_{\tilde{S}} = \lim_{y \to 0} \frac{\partial}{\partial y_i} \frac{\int \mathcal{D}[x] \exp\left[-\frac{1}{2}\mathbf{x}^T \tilde{\mathbf{J}}^{-1}\mathbf{x}\right] \sum_{\{s_i\}} \exp\left[(\tilde{\mathbf{h}} + \mathbf{x})^T \mathbf{s} + \mathbf{x}^T \mathbf{y}\right]}{\int \mathcal{D}[x] e^{-\tilde{S}[x]}}$$

$$= \lim_{y \to 0} \frac{\partial}{\partial y_i} \frac{\sum_{\{s_i\}} \exp\left[\frac{1}{2}(\mathbf{s} + \mathbf{y})^T \tilde{\mathbf{J}}(\mathbf{s} + \mathbf{y}) + \tilde{\mathbf{h}}^T \mathbf{s}\right]}{\sum_{\{s_i\}} \exp\left[\frac{1}{2}\mathbf{s}^T \tilde{\mathbf{J}}\mathbf{s} + \tilde{\mathbf{h}}^T \mathbf{s}\right]}$$

$$= \frac{\sum_{\{s_i\}} e^{-\beta H}[\tilde{\mathbf{J}}\mathbf{s}]_i}{\sum_{\{s_i\}} e^{-\beta H}} = \langle[\tilde{\mathbf{J}}\mathbf{s}]_i\rangle , \tag{2.35}$$

or in vector notation,

$$\langle \mathbf{x} \rangle_{\tilde{S}} = \tilde{\mathbf{J}}\langle \mathbf{s} \rangle . \tag{2.36}$$

In order to introduce variables φ_i whose expectation value can be identified with the average magnetization $m = \langle s_i \rangle$, we simply define

$$\varphi = \tilde{\mathbf{J}}^{-1} \mathbf{x} \, , \tag{2.37}$$

so that the expectation value of the i-th component of φ is

$$\langle \varphi_i \rangle_{\tilde{S}} = [\tilde{\mathbf{J}}^{-1} \langle \mathbf{x} \rangle_{\tilde{S}}]_i = \langle s_i \rangle = m \, . \tag{2.38}$$

Substituting $\varphi = \tilde{\mathbf{J}}^{-1} \mathbf{x}$ in Eqs. (2.31) and (2.32), we finally obtain the following exact representation of the partition function,

$$\boxed{\mathcal{Z} = \frac{\int \mathcal{D}[\varphi] e^{-S[\varphi]}}{\int \mathcal{D}[\varphi] \exp\left[-\frac{1}{2}\varphi^T \tilde{\mathbf{J}} \varphi\right]} = \sqrt{\det \tilde{\mathbf{J}}} \int \mathcal{D}[\varphi] e^{-S[\varphi]} \, ,} \tag{2.39}$$

where the effective action $S[\varphi] \equiv \tilde{S}[\mathbf{x} \to \tilde{\mathbf{J}}\varphi]$ is given by

$$\boxed{S[\varphi] = \frac{\beta}{2} \sum_{ij}^{N} J_{ij} \varphi_i \varphi_j - \sum_{i=1}^{N} \ln\left[2\cosh\left[\beta\left(h + \sum_{j=1}^{N} J_{ij}\varphi_j\right)\right]\right] \, .} \tag{2.40}$$

Equation (2.39) expresses the partition function of the Ising model in terms of an N-dimensional integral over variables φ_i whose expectation value is simply the magnetization per site. The integration variables φ_i can therefore be interpreted as the fluctuating magnetization. The infinite-dimensional integral obtained from Eq. (2.39) in the limit $N \to \infty$ is an example of a *functional integral*. The resulting effective action $S[\varphi]$ then defines a (classical) effective field theory for the order-parameter fluctuations of the Ising model. We shall refer to the integration variables φ_i as the *fields* of our effective field theory.

It should be noted that Eq. (2.38) does not precisely generalize to higher order correlation functions. If we proceed as above and use the identity

$$x_{i_1} x_{i_2} \dots x_{i_n} = \lim_{y \to 0} \frac{\partial}{\partial y_{i_1}} \frac{\partial}{\partial y_{i_2}} \dots \frac{\partial}{\partial y_{i_n}} e^{x^T y} \tag{2.41}$$

to carry out the same transformations as in Eq. (2.35), we arrive at

$$\langle \varphi_{i_1} \varphi_{i_2} \dots \varphi_{i_n} \rangle_S = \langle s_{i_1} s_{i_2} \dots s_{i_n} \rangle + \big\{ \text{averages involving at most} \tag{2.42}$$
$$n-2 \text{ of the spins } s_{i_1}, \dots, s_{i_n} \big\} \, ,$$

where the lower order correlation functions involving $n - 2$ and less spin variables arise from the repeated differentiation of the y-dependent terms in the exponent of the second line of Eq. (2.35). Fortunately, for short-range J_{ij} these additional terms

can be neglected as long as the distances between the external points $i_1, \ldots, i_n$ are large compared with the range of J_{ij}. For example, for $n = 2$ we have

$$\langle \varphi_i \varphi_j \rangle_S = \langle s_i s_j \rangle - [\tilde{\mathbf{J}}^{-1}]_{ij} \sim \langle s_i s_j \rangle \; . \tag{2.43}$$

We thus conclude that we may also use the effective action $S[\boldsymbol{\varphi}]$ given in Eq. (2.40) to calculate the long distance behavior of correlation functions involving two and more spins.

2.2.2 Truncated Effective Action: φ^4-Theory

The effective action given in Eqs. (2.39) and (2.40) is formally exact but very complicated. In order to make progress, let us assume that the integral (2.39) is dominated by configurations where the integration variables φ_i are in some sense small, so that we may expand the second term in the effective action (2.40) in powers of the φ_i, truncating the expansion at fourth order. Keeping in mind that physically φ_i represents the fluctuating magnetization, we expect that this truncation can only be good in the vicinity of the critical point, where the magnetization is small. Whether or not this truncation is sufficient to obtain quantitatively correct results for the critical exponents is a rather subtle question which will be answered with the help of the RG.[4] With the expansion (2.14) we obtain

$$S[\boldsymbol{\varphi}] = -N \ln 2 + \frac{\beta}{2} \sum_{ij} J_{ij} \varphi_i \varphi_j - \frac{\beta^2}{2} \sum_i \Big[h + \sum_j J_{ij} \varphi_j \Big]^2$$

$$+ \frac{\beta^4}{12} \sum_i \Big[h + \sum_j J_{ij} \varphi_j \Big]^4 + \mathcal{O}(\varphi_i^6) \; . \tag{2.44}$$

Since our lattice model has discrete translational invariance, we may simplify Eq. (2.44) by Fourier transforming the variables φ_i to wave vector space, defining

$$\varphi_i = \frac{1}{\sqrt{N}} \sum_k e^{i \mathbf{k} \cdot \mathbf{r}_i} \varphi_k \; , \tag{2.45}$$

where the wave vector sum is over the first Brillouin zone of the lattice, which may be chosen as $0 \leq k_\mu < 2\pi/a$, where a is the lattice spacing and $\mu = 1, \ldots, D$ labels the components. Imposing for convenience periodic boundary conditions, the

---

[4] It turns out that the quartic truncation of the expansion of Eq. (2.40) can be formally justified close to four dimensions. In the physically most interesting dimension $D = 3$ the term involving six powers of the φ_i cannot be neglected. However the truncated φ^4-theory is in the same universality class as the original theory and the effect of all couplings neglected can be absorbed by a redefinition of the remaining couplings.

wave vectors are quantized as $k_\mu = 2\pi n_\mu / L$, with $n_\mu = 0, 1, \ldots, N_\mu - 1$, where N_μ is the number of lattice sites in direction μ such that $\prod_{\mu=1}^{D} N_\mu = N$ is the total number of lattice sites. Using the identity

$$\frac{1}{N} \sum_i e^{i(k-k')\cdot r_i} = \delta_{k,k'} , \tag{2.46}$$

we obtain for the Fourier transform of the terms on the right-hand side of Eq. (2.44),

$$\frac{\beta}{2} \sum_{ij} J_{ij} \varphi_i \varphi_j = \frac{\beta}{2} \sum_k J_k \varphi_{-k} \varphi_k , \tag{2.47}$$

$$\frac{\beta^2}{2} \sum_i \left[\sum_j J_{ij} \varphi_j\right]^2 = \frac{\beta^2}{2} \sum_k J_{-k} J_k \varphi_{-k} \varphi_k , \tag{2.48}$$

$$\frac{\beta^4}{12} \sum_i \left[\sum_j J_{ij} \varphi_j\right]^4 = \frac{\beta^4}{12N} \sum_{k_1,k_2,k_3,k_4} \delta_{k_1+k_2+k_3+k_4,0}$$
$$\times J_{k_1} J_{k_2} J_{k_3} J_{k_4} \varphi_{k_1} \varphi_{k_2} \varphi_{k_3} \varphi_{k_4} , \tag{2.49}$$

where J_k is the Fourier transform of the exchange couplings $J_{ij} \equiv J(r_i - r_j)$,

$$J_k = \sum_i e^{-ik\cdot r_i} J(r_i) . \tag{2.50}$$

Because φ_i and J_{ij} are real and $J(-r) = J(r)$, we have $\varphi_{-k} = \varphi_k^*$ and $J_{-k} = J_k$. In Fourier space Eq. (2.44) thus reduces to

$$S[\varphi] = -N \ln 2 - \beta^2 J_{k=0} h \sqrt{N} \varphi_{k=0} + \frac{\beta}{2} \sum_k J_k (1 - \beta J_k) \varphi_{-k} \varphi_k$$
$$+ \frac{\beta^4}{12N} \sum_{k_1,k_2,k_3,k_4} \delta_{k_1+k_2+k_3+k_4,0} J_{k_1} J_{k_2} J_{k_3} J_{k_4} \varphi_{k_1} \varphi_{k_2} \varphi_{k_3} \varphi_{k_4}$$
$$+ \mathcal{O}(\varphi_i^6, h^2, h\varphi_i^3) . \tag{2.51}$$

Finally, we anticipate that sufficiently close to the critical point only long-wavelength fluctuations (corresponding to small wave vectors) are important. We therefore expand the function J_k appearing in Eq. (2.51) in powers of k. From Eq. (2.50) it is easy to show that for nearest-neighbor interactions on a D-dimensional hypercubic lattice with coordination number $z = 2D$,

$$J_k = J[z - k^2 a^2] + \mathcal{O}(k^4) = T_c \left[1 - \frac{k^2 a^2}{z}\right] + \mathcal{O}(k^4) , \tag{2.52}$$

where $T_c = zJ$ is the mean-field result for the critical temperature, see Eq. (2.13). The coefficient of the quadratic term in Eq. (2.51) can then be written as

$$\beta J_{\boldsymbol{k}}(1 - \beta J_{\boldsymbol{k}}) = a^2(r_0 + c_0 \boldsymbol{k}^2) + \mathcal{O}(k^4) , \tag{2.53}$$

where we have assumed that $|T - T_c| \ll T_c$ and the constants r_0 and c_0 are defined by

$$r_0 = \frac{T - T_c}{a^2 T_c} , \tag{2.54}$$

$$c_0 = \frac{1}{z} = \frac{1}{2D} . \tag{2.55}$$

In the limit of infinite volume $V = Na^D \to \infty$, the discrete set of allowed wave vectors merges into a continuum, so that the momentum sums can be replaced by integrations according to the following prescription,

$$\frac{1}{V} \sum_{\boldsymbol{k}} \to \int \frac{d^D k}{(2\pi)^D} \equiv \int_{\boldsymbol{k}} . \tag{2.56}$$

It is then convenient to normalize the fields differently, introducing a new (continuum) field $\varphi(\boldsymbol{k})$ via[5]

$$\varphi(\boldsymbol{k}) = a\sqrt{V}\varphi_{\boldsymbol{k}} . \tag{2.57}$$

Defining

$$f_0 = -\frac{N}{V} \ln 2 = -a^{-D} \ln 2 \tag{2.58}$$

and the (dimensionful) coupling constants

$$u_0 = 2a^{D-4}(\beta J_{\boldsymbol{k}=0})^4 \approx 2a^{D-4} , \tag{2.59}$$

$$h_0 = \frac{\beta^2 J_{\boldsymbol{k}=0} h}{a^{1+D/2}} \approx \frac{\beta h}{a^{1+D/2}} , \tag{2.60}$$

where the approximations are again valid for $|T - T_c| \ll T_c$, we find that our lattice action $S[\varphi]$ in Eq. (2.51) reduces to

---

[5] The different normalizations of lattice and continuum fields are represented by different positions of the momentum labels: while in the lattice normalization the momentum label $\boldsymbol{k}$ is attached to the dimensionless field $\varphi_{\boldsymbol{k}}$ as a subscript, the label of the continuum field $\varphi(\boldsymbol{k})$ is written in brackets after the field symbol.

$$S_{\Lambda_0}[\varphi] = V f_0 - h_0 \varphi(\mathbf{k} = 0) + \frac{1}{2} \int_{\mathbf{k}} [r_0 + c_0 k^2] \varphi(-\mathbf{k}) \varphi(\mathbf{k})$$
$$+ \frac{u_0}{4!} \int_{\mathbf{k}_1} \int_{\mathbf{k}_2} \int_{\mathbf{k}_3} \int_{\mathbf{k}_4} (2\pi)^D \delta(\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 + \mathbf{k}_4) \varphi(\mathbf{k}_1) \varphi(\mathbf{k}_2) \varphi(\mathbf{k}_3) \varphi(\mathbf{k}_4) ,$$

(2.61)

where it is understood that the momentum integrations in Eq. (2.61) have an implicit *ultraviolet cutoff* $|\mathbf{k}| < \Lambda_0 \ll a^{-1}$ which takes into account that in deriving Eq. (2.61) we have expanded $J_{\mathbf{k}}$ for small wave vectors. The functional $S_{\Lambda_0}[\varphi]$ is called the *Ginzburg–Landau–Wilson action* and describes the long-wavelength order-parameter fluctuations of the D-dimensional Ising model, in the sense that the integration over the fields $\varphi(\mathbf{k})$ yields the contribution of the associated long-wavelength fluctuations to the partition function. The contribution of the neglected short-wavelength fluctuations to the partition function can be taken into account implicitly by simply redefining the field-independent part f_0 and the coupling constants r_0, c_0, and u_0 of our effective action (2.61). Moreover, also the prefactor $\sqrt{\det \tilde{\mathbf{J}}} = e^{\frac{1}{2} \text{Tr} \ln \tilde{\mathbf{J}}}$ of Eq. (2.39) can be absorbed into a redefinition of the field-independent constant, $f_0 - \frac{1}{2V} \text{Tr} \ln \tilde{\mathbf{J}} \to f_0$. Actually, for periodic boundary conditions the eigenvalues of the matrix $\tilde{\mathbf{J}}$ are simply given by $J_{\mathbf{k}}/T \approx J_{\mathbf{k}=0}/T_c = 1$, so that at long wavelengths and to leading order in $T - T_c$ we may approximate the factor $\sqrt{\det \tilde{\mathbf{J}}}$ by unity. The functional integral (2.39) representing the partition function of the Ising model can thus be written as

$$\mathcal{Z} = \int \mathcal{D}[\varphi] e^{-S_{\Lambda_0}[\varphi]} .$$

(2.62)

It is sometimes more convenient to consider the effective action $S_{\Lambda_0}[\varphi]$ in real space. Defining the real-space Fourier transforms of the continuum fields $\varphi(\mathbf{k})$ via

$$\varphi(\mathbf{r}) = \int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} \varphi(\mathbf{k}) ,$$

(2.63)

and using the identity

$$\int_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} = \delta(\mathbf{r}) ,$$

(2.64)

we obtain from Eq. (2.61),

$$S_{\Lambda_0}[\varphi] = \int d^D r \left[f_0 + \frac{r_0}{2} \varphi^2(\mathbf{r}) + \frac{c_0}{2} [\nabla \varphi(\mathbf{r})]^2 + \frac{u_0}{4!} \varphi^4(\mathbf{r}) - h_0 \varphi(\mathbf{r}) \right] . \quad (2.65)$$

For obvious reasons the classical field theory defined by this expression is called φ^4-theory. In the first part of this book, we shall use this field theory to illustrate the

main concepts of the RG. Strictly speaking, the identity (2.64) is not quite correct if we keep in mind that we should impose an implicit ultraviolet cutoff Λ_0 on the $\mathbf{k}$-integration. Roughly, the continuum field $\varphi(\mathbf{r})$ corresponds to the coarse-grained magnetization, which is averaged over spatial regions with volume of order Λ_0^{-D} containing $(\Lambda_0 a)^{-D}$ spins.

   The quantity $e^{-S_{\Lambda_0}[\varphi]}$ is proportional to the probability density for observing an order-parameter distribution specified by the field configuration $\varphi(\mathbf{r})$. According to Eq. (2.62), the partition function is given by the average of $e^{-S_{\Lambda_0}[\varphi]}$ over all possible order-parameter configurations. With this probabilistic interpretation of the effective action (2.65), we can now give a more satisfactory interpretation of the mean-field Landau function $\mathcal{L}_{\mathrm{MF}}(T, h; m)$ defined in Eqs. (2.8) and (2.9). Let us therefore evaluate the functional integral (2.62) in saddle point approximation, where the entire integral is estimated by the integrand at a single constant value $\bar{\varphi}_0$ of the field which minimizes the action $S_{\Lambda_0}[\varphi]$. Setting $\varphi(\mathbf{r}) \to \bar{\varphi}$ in Eq. (2.62) we obtain

$$\mathcal{Z} \approx \int_{-\infty}^{\infty} \frac{d\bar{\varphi}}{\sqrt{2\pi}} e^{-S_{\Lambda_0}[\bar{\varphi}]} \, , \tag{2.66}$$

with

$$S_{\Lambda_0}[\bar{\varphi}] = V\left[f_0 + \frac{r_0}{2}\bar{\varphi}^2 + \frac{u_0}{4!}\bar{\varphi}^4 - h_0\bar{\varphi} \right] . \tag{2.67}$$

Note that in Eq. (2.66) we still integrate over all configurations of the homogeneous components $\bar{\varphi}$ of the order-parameter field; the quantity $e^{-S_{\Lambda_0}[\bar{\varphi}]}$ is therefore proportional to the probability of observing for the homogeneous component of the order parameter the value $\bar{\varphi}$. Requiring consistency of Eq. (2.67) with our previous definitions (2.8) and (2.15) of the Landau function, we should identify

$$S_{\Lambda_0}[\bar{\varphi}] = \beta N \mathcal{L}_{\mathrm{MF}}(T, h; m) = \beta N \left[f + \frac{r}{2}m^2 + \frac{u}{4!}m^4 - hm \right] , \tag{2.68}$$

which is indeed the case if

$$\bar{\varphi} = a^{1-D/2}m \, , \tag{2.69}$$

keeping in mind the definitions of f, r, and u in Eqs. (2.16a), (2.16b), and (2.16c) on the one hand, and of f_0, r_0, and u_0 in Eqs. (2.58), (2.54), and (2.59) on the other hand. For $V \to \infty$ the value of the one-dimensional integral (2.66) is essentially determined by the saddle point of the integrand, corresponding to the most probable value of $\bar{\varphi}$. The physical value $\bar{\varphi}_0$ of the order-parameter field is therefore fixed by the saddle point condition,

$$\left. \frac{\partial S_{\Lambda_0}[\bar{\varphi}]}{\partial \bar{\varphi}} \right|_{\bar{\varphi}_0} = r_0\bar{\varphi}_0 + \frac{u_0}{6}\bar{\varphi}_0^3 - h_0 = 0 \, , \tag{2.70}$$

which is equivalent to the mean-field self-consistency equation (2.17).

In summary, the mean-field Landau function defined in Eq. (2.8) can be recovered from the functional integral representation (2.62) of the partition function by ignoring spatial fluctuations of the order parameter; the corresponding saddle point equation is equivalent to the mean-field self-consistency equation (2.11). Because of its probabilistic interpretation the Landau function cannot be defined within the framework of thermodynamics, which only makes statements about averages. The concept of a Landau function is further illustrated in Exercise 2.2, where the Landau functions of noninteracting bosons are defined and calculated.

2.3 The Gaussian Approximation

While the mean-field approximation amounts to evaluating the functional integral (2.62) in saddle point approximation, the Gaussian approximation retains quadratic fluctuations around the saddle point and thus includes the lowest-order correction to the mean-field approximation in an expansion in fluctuations around the saddle point. In the field-theory language, the Gaussian approximation corresponds to describing fluctuations in terms of a free field theory, where the fluctuations with different momenta and frequencies are independent. In condensed matter physics, the Gaussian approximation is closely related to the so-called random phase approximation. It turns out that quite generally the Gaussian approximation is only valid if the dimensionality D of the system is larger than a certain upper critical dimension D_{up}, which we have already encountered in the context of the scaling hypothesis in Sect. 1.3, see the discussion after Eq. (1.32). Because for the Ising universality class $D_{\mathrm{up}} = 4$, the Gaussian approximation is not sufficient to describe the critical behavior of Ising magnets in experimentally accessible dimensions. Nevertheless, in order to motivate the RG it is instructive to see how and why the Gaussian approximation breaks down for $D < D_{\mathrm{up}}$. For simplicity, we set $h = 0$ in this section.

2.3.1 Gaussian Effective Action

To derive the Gaussian approximation, we go back to our Ginzburg–Landau–Wilson action $S_{A_0}[\varphi]$ defined in Eq. (2.65). Let us decompose the field $\varphi(\boldsymbol{r})$ describing the coarse-grained fluctuating magnetization as follows,

$$\varphi(\boldsymbol{r}) = \bar{\varphi}_0 + \delta\varphi(\boldsymbol{r}) , \qquad (2.71)$$

or in wave vector space

$$\varphi(\boldsymbol{k}) = (2\pi)^D \delta(\boldsymbol{k})\bar{\varphi}_0 + \delta\varphi(\boldsymbol{k}) . \qquad (2.72)$$

Here, $\bar{\varphi}_0$ is the mean-field value of the order parameter satisfying the saddle point equation (2.70) and $\delta\varphi(\boldsymbol{r})$ describes inhomogeneous fluctuations around the saddle

point. Substituting Eq. (2.71) into the action (2.65) and retaining all terms up to quadratic order in the fluctuations, we obtain in momentum space

$$
\begin{aligned}
S_{\Lambda_0}[\bar{\varphi}_0 + \delta\varphi] \approx{} & V\left[f_0 + \frac{r_0}{2}\bar{\varphi}_0^2 + \frac{u_0}{4!}\bar{\varphi}_0^4\right] \\
& + \left[r_0\bar{\varphi}_0 + \frac{u_0}{6}\bar{\varphi}_0^3\right]\delta\varphi(\boldsymbol{k} = 0) \\
& + \frac{1}{2}\int_{\boldsymbol{k}}\left[r_0 + \frac{u_0}{2}\bar{\varphi}_0^2 + c_0\boldsymbol{k}^2\right]\delta\varphi(-\boldsymbol{k})\delta\varphi(\boldsymbol{k})\,.
\end{aligned}
\tag{2.73}
$$

This is the *Gaussian approximation* for the Ginzburg–Landau–Wilson action. To further simplify Eq. (2.73), we note that the second line on the right-hand side of Eq. (2.73) vanishes because the coefficient of $\delta\varphi(\boldsymbol{k} = 0)$ satisfies the saddle point condition (2.70). The first line on the right-hand side of Eq. (2.73) can be identified with the mean-field free energy. Explicitly substituting for $\bar{\varphi}_0$ in Eq. (2.73) the saddle point value,

$$
\bar{\varphi}_0 = \begin{cases} 0 & \text{for}\quad r_0 > 0\,, \\ \sqrt{-6r_0/u_0} & \text{for}\quad r_0 < 0\,, \end{cases}
\tag{2.74}
$$

we obtain for the effective action in Gaussian approximation for $T > T_c$, where $\bar{\varphi}_0 = 0$ and $\delta\varphi = \varphi$,

$$
S_{\Lambda_0}[\varphi] = Vf_0 + \frac{1}{2}\int_{\boldsymbol{k}}\left[r_0 + c_0\boldsymbol{k}^2\right]\varphi(-\boldsymbol{k})\varphi(\boldsymbol{k})\,,\quad T > T_c\,.
\tag{2.75}
$$

On the other hand, for $T < T_c$, where $r_0 < 0$, we have

$$
\frac{r_0}{2}\bar{\varphi}_0^2 + \frac{u_0}{4!}\bar{\varphi}_0^4 = -\frac{3}{2}\frac{r_0^2}{u_0}\,,
\tag{2.76}
$$

$$
r_0 + \frac{u_0}{2}\bar{\varphi}_0^2 = -2r_0\,,
\tag{2.77}
$$

and hence

$$
S_{\Lambda_0}[\varphi] = V\left[f_0 - \frac{3}{2}\frac{r_0^2}{u_0}\right] + \frac{1}{2}\int_{\boldsymbol{k}}\left[-2r_0 + c_0\boldsymbol{k}^2\right]\delta\varphi(-\boldsymbol{k})\delta\varphi(\boldsymbol{k})\,,\quad T < T_c\,.
\tag{2.78}
$$

With the help of the Gaussian effective action given in Eqs. (2.75) and (2.78), we may now estimate the effect of order-parameter fluctuations on the mean-field results for the thermodynamic critical exponents. Moreover, because the Gaussian approximation takes the spatial inhomogeneity of the order-parameter fluctuations into account, we may also calculate the critical exponents ν and η associated with the order-parameter correlation function.

2.3.2 Gaussian Corrections to the Specific Heat Exponent

Because in the thermodynamic limit both the Gaussian approximation and the mean-field approximation predict the same contribution from the homogeneous fluctuations represented by $\bar{\varphi}$ to the free energy[6], Gaussian fluctuations do not modify the mean-field predictions for the exponents β, γ, and δ, which are related to the homogeneous order-parameter fluctuations. Within the Gaussian approximation, we therefore still obtain $\beta = 1/2$, $\gamma = 1$, and $\delta = 3$. On the other hand, the Gaussian approximation for the specific heat exponent α is different from the mean-field prediction $\alpha = 0$, because the fluctuations with finite wave vectors give a nontrivial contribution Δf to the free energy per lattice site, which according to Eqs. (2.75) and (2.78) can be written as

$$e^{-\beta N \Delta f} = \int \mathcal{D}[\varphi] \exp\left[-\frac{c_0}{2} \int_k (\xi^{-2} + k^2) \delta\varphi(-k)\delta\varphi(k)\right]. \qquad (2.79)$$

Here, we have introduced the length ξ via

$$\frac{c_0}{\xi^2} = \begin{cases} r_0 & \text{for } T > T_c \,, \\ -2r_0 & \text{for } T < T_c \,. \end{cases} \qquad (2.80)$$

In Sect. 2.3.3 we shall identify the length ξ with the order-parameter correlation length introduced in Sect. 1.2. Recall that for $T > T_c$ we may write $\delta\varphi(k) = \varphi(k)$ because in this case $\bar{\varphi}_0 = 0$. To evaluate the Gaussian integral in Eq. (2.79), it is convenient to discretize the integral $\int_k$ in the exponent with the help of the underlying lattice and use the associated lattice normalization of the fields, which according to Eq. (2.57) amounts to the substitution $\delta\varphi(k) = a\sqrt{V}\delta\varphi_k$. Then we obtain from Eq. (2.79),

$$e^{-\beta N \Delta f} = \int_\infty^\infty \frac{d\delta\varphi_0}{\sqrt{2\pi}} \left[\prod_{k,\, k\cdot\hat{n}>0} \int_\infty^\infty \frac{d\mathrm{Re}\delta\varphi_k}{\sqrt{2\pi}} \int_\infty^\infty \frac{d\mathrm{Im}\delta\varphi_k}{\sqrt{2\pi}}\right]$$
$$\times \exp\left[-\frac{c_0 a^2}{2} \sum_k (\xi^{-2} + k^2) |\delta\varphi_k|^2\right], \qquad (2.81)$$

where the product in the square braces is restricted to those wave vectors k whose projection $k \cdot \hat{n}$ onto an arbitrary direction $\hat{n}$ is positive. This restriction is necessary in order to avoid double counting of the finite k-fluctuations, which are represented by a real field φ_i whose Fourier components satisfy $\varphi_{-k} = \varphi_k^*$. Since the integrations

---

[6] Recall that the mean-field approximation for the free energy is $f_{\mathrm{MF}}(t, 0) = f_0$ for $T > T_c$, and $f_{\mathrm{MF}}(t, 0) = f_0 - \frac{3}{2}\frac{r^2}{u}$ for $T < T_c$, see Eq. (2.22). Taking the different normalization of $S_{A_0}[\varphi]$ into account, these expressions correspond to the field-independent terms on the right-hand sides of Eqs. (2.75) and (2.78).

in Eq. (2.81) treat the real and the imaginary parts of $\delta\varphi_k$ as independent variables, we have to correct for this double counting by integrating only over half of the possible values of $\boldsymbol{k}$. The functional integral is now reduced to a product of one-dimensional Gaussian integrals, which are of course a special case of our general Gaussian integration formula (2.25) for $N = 1$,

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} \, e^{-\frac{A}{2}x^2} = A^{-1/2} = e^{-\frac{1}{2}\ln A} \; . \tag{2.82}$$

We thus obtain from Eq. (2.81) for the correction from Gaussian fluctuations to the free energy per lattice site,

$$\Delta f = \frac{T}{2N} \sum_k \ln\left[c_0 a^2 (\xi^{-2} + k^2)\right] , \tag{2.83}$$

where it is understood that the sum is regularized via an ultraviolet cutoff $|\boldsymbol{k}| < \Lambda_0$. The corresponding correction to the specific heat per lattice site is

$$\Delta C = -T \frac{\partial^2 \Delta f}{\partial T^2} = -\frac{T}{2N} \frac{\partial^2}{\partial T^2}\left\{ T \sum_k \ln\left[c_0 a^2 (\xi^{-2} + k^2)\right] \right\}$$

$$= -\frac{T}{2N} \frac{\partial}{\partial T}\left\{ \sum_k \ln\left[c_0 a^2 (\xi^{-2} + k^2)\right] + T\left[\frac{\partial}{\partial T} \frac{1}{\xi^2}\right] \sum_k \frac{1}{\xi^{-2} + k^2} \right\}$$

$$= \frac{T^2}{2}\left[\frac{\partial}{\partial T} \frac{1}{\xi^2}\right]^2 \frac{1}{N} \sum_k \frac{1}{[\xi^{-2} + k^2]^2} - T\left[\frac{\partial}{\partial T} \frac{1}{\xi^2}\right] \frac{1}{N} \sum_k \frac{1}{\xi^{-2} + k^2} \; . \tag{2.84}$$

In the thermodynamic limit, the lattice sums can be converted into integrals using Eq. (2.56) so that for $n = 1, 2$

$$\lim_{N\to\infty} \frac{1}{N} \sum_k \frac{1}{[\xi^{-2} + k^2]^n} = a^D \int_k \frac{1}{[\xi^{-2} + k^2]^n}$$

$$= K_D a^D \int_0^{\Lambda_0} dk \frac{k^{D-1}}{[\xi^{-2} + k^2]^n} = K_D a^D \xi^{2n-D} \int_0^{\Lambda_0 \xi} dx \frac{x^{D-1}}{[1 + x^2]^n} \; . \tag{2.85}$$

Here, the numerical constant K_D is defined by

$$K_D = \frac{\Omega_D}{(2\pi)^D} = \frac{1}{2^{D-1}\pi^{D/2}\Gamma(D/2)} \; , \tag{2.86}$$

where Ω_D is the surface area of the D-dimensional unit sphere. Using the fact that according to Eqs. (2.54) and (2.80) the derivative of the square of the inverse correlation length with respect to temperature is given by

$$\frac{\partial}{\partial T} \frac{1}{\xi^2} = \begin{cases} c_0^{-1} \frac{\partial r_0}{\partial T} , & \text{for } T > T_c \\ -2c_0^{-1} \frac{\partial r_0}{\partial T} , & \text{for } T < T_c \end{cases} = \frac{A_t}{c_0 a^2 T_c} , \tag{2.87}$$

where

$$A_t = \begin{cases} 1 , & \text{for } T > T_c , \\ -2 , & \text{for } T < T_c , \end{cases} \tag{2.88}$$

we obtain from Eq. (2.84) for $|t| \equiv |T - T_c|/T_c \ll 1$,

$$\Delta C = \frac{K_D}{2} \frac{A_t^2}{c_0^2} \left(\frac{\xi}{a}\right)^{4-D} \int_0^{\Lambda_0 \xi} dx \frac{x^{D-1}}{[1+x^2]^2}$$
$$- K_D \frac{A_t}{c_0} \left(\frac{a}{\xi}\right)^{D-2} \int_0^{\Lambda_0 \xi} dx \frac{x^{D-1}}{[1+x^2]} . \tag{2.89}$$

Keeping in mind that close to the critical point the dimensionless parameter $\Lambda_0 \xi$ is large compared with unity, it is easy to see that the second term on the right-hand side of Eq. (2.89) can be neglected in comparison with the first one, because the dimensionless integral is for $D > 2$ proportional to $(\Lambda_0 \xi)^{D-2}$, and has for $D < 2$ a finite limit for $\Lambda_0 \xi \to \infty$. On the other hand, the behavior of the first term on the right-hand side of Eq. (2.89) depends on whether D is larger or smaller than the upper critical dimension $D_{up} = 4$. For $D > D_{up}$ the integral depends on the ultraviolet cutoff and is proportional to $(\Lambda_0 \xi)^{D-4}$, so that $\Delta C \propto (\xi/a)^{4-D}(\Lambda_0 \xi)^{D-4} = (\Lambda_0 a)^{D-4}$ which gives rise to a finite correction to the specific heat. Hence, for $D > 4$ long-wavelength Gaussian fluctuations merely modify the size of the jump discontinuity in the specific heat, but do not qualitatively modify the mean-field result $\alpha = 0$. On the other hand, for $D < 4$ the first integral on the right-hand side of Eq. (2.89) remains finite for $\Lambda_0 \xi \to \infty$,

$$I_D = \int_0^\infty dx \frac{x^{D-1}}{[1+x^2]^2} = \frac{(D-2)\pi}{4 \sin(\frac{(D-2)\pi}{2})} , \tag{2.90}$$

so that to leading order for large $\xi \propto |r_0|^{-1/2} \propto |t|^{-1/2}$ the contribution from Gaussian fluctuations to the specific heat is

$$\Delta C = \frac{K_D}{2} I_D \frac{A_t^2}{c_0^2} \left(\frac{\xi}{a}\right)^{4-D} \propto |t|^{-(2-D/2)} . \tag{2.91}$$

This is more singular than the jump discontinuity predicted by mean-field theory and implies $\alpha = 2 - D/2$. We thus conclude that in Gaussian approximation the specific heat exponent is given by

$$\alpha = \begin{cases} 2 - \frac{D}{2} & \text{for } D < D_{\text{up}} = 4 \ , \\ 0 & \text{for } D \geq D_{\text{up}} \ . \end{cases} \tag{2.92}$$

It should be noted that within the Gaussian model considered here the scaling relations $2 - \alpha = 2\beta + \gamma = \beta(\delta + 1)$ [see Eqs. (1.23) and (1.24)] are violated for $D < 4$. This failure of the Gaussian approximation will be further discussed in Sect. 2.3.4.

2.3.3 Correlation Function

Next, we calculate the order-parameter correlation function $G(r_i - r_j)$ of the Ising model and the associated critical exponents ν and η in Gaussian approximation. We define $G(r_i - r_j)$ via the following thermal average,

$$G(r_i - r_j) = a^{2-D} \langle \delta s_i \delta s_j \rangle \equiv a^{2-D} \frac{\sum_{\{s_i\}} e^{-\beta H} \delta s_i \delta s_j}{\sum_{\{s_i\}} e^{-\beta H}} \ , \tag{2.93}$$

where $\delta s_i = s_i - \langle s_i \rangle = s_i - m$ is the deviation of the microscopic spin from its average. Using the identity (2.43) we can express $G(r_i - r_j)$ for distances large compared with the range of the J_{ij} as a functional average over all configurations of the fluctuating order-parameter field,

$$G(r_i - r_j) = a^{2-D} \langle \delta \varphi_i \delta \varphi_j \rangle_S \equiv a^{2-D} \frac{\int \mathcal{D}[\varphi] e^{-S[\varphi]} \delta \varphi_i \delta \varphi_j}{\int \mathcal{D}[\varphi] e^{-S[\varphi]}} \ . \tag{2.94}$$

The prefactor a^{2-D} is introduced such that $G(r_i - r_j)$ can also be written as

$$G(r_i - r_j) = \langle \delta \varphi(r_i) \delta \varphi(r_j) \rangle_S \ , \tag{2.95}$$

where the continuum field $\varphi(r)$ is related to the real-space lattice field φ_i via

$$\varphi(r_i) = a^{1-D/2} \varphi_i \ , \tag{2.96}$$

which follows from the definitions (2.63), (2.45), and (2.57), see also Eq. (2.69). Expanding $\delta \varphi_i$ and $\delta \varphi_j$ in terms of their Fourier components $\delta \varphi_k$ defined analogously to Eq. (2.45), we obtain

$$G(r_i - r_j) = \frac{a^2}{V} \sum_{k_1, k_2} e^{i(k_1 \cdot r_i + k_2 \cdot r_j)} \langle \delta \varphi_{k_1} \delta \varphi_{k_2} \rangle_S$$

$$= \frac{1}{V} \sum_{k_1} e^{i k_1 \cdot (r_i - r_j)} G(k_1) \ , \tag{2.97}$$

where

$$G(k_1) = a^2 \langle \delta\varphi_{k_1} \delta\varphi_{-k_1} \rangle_S = a^2 \langle |\delta\varphi_{k_1}|^2 \rangle_S \tag{2.98}$$

is the Fourier transform of $G(r_i - r_j)$. In the second line of Eq. (2.97) we have used the fact that by translational invariance the functional average is only nonzero if the total wave vector $k_1 + k_2$ vanishes,

$$\langle \delta\varphi_{k_1} \delta\varphi_{k_2} \rangle_S = \delta_{k_1, -k_2} \langle \delta\varphi_{k_1} \delta\varphi_{-k_1} \rangle_S . \tag{2.99}$$

In the infinite volume limit, it is more convenient to work with the continuum fields $\varphi(k) = a\sqrt{V}\varphi_k$ introduced in Eq. (2.57); then the relations (2.98) and (2.99) can be written in the compact form,[7]

$$\langle \delta\varphi(k_1) \delta\varphi(k_2) \rangle_S = (2\pi)^D \delta(k_1 + k_2) G(k_1) . \tag{2.100}$$

Let us now evaluate the functional average in Eq. (2.98) within the Gaussian approximation where the long-wavelength effective action $S_{\Lambda_0}[\varphi]$ is given by Eqs. (2.75) and (2.78). Similar to Eq. (2.81), we write the Gaussian effective action for finite V as

$$S_{\Lambda_0}[\varphi] \approx S_{\Lambda_0}[\bar\varphi_0] + \frac{c_0 a^2}{2} \sum_k (\xi^{-2} + k^2) |\delta\varphi_k|^2$$

$$= c_0 a^2 \left(\xi^{-2} + k_1^2 \right) |\delta\varphi_{k_1}|^2 + \text{terms independent of } \delta\varphi_{k_1} , \tag{2.101}$$

where in the second line we have used the fact that the terms $k = k_1$ and $k = -k_1$ give the same contribution to the sum. Using the fact that within the Gaussian approximation fluctuations with different wave vectors are decoupled, the evaluation of the functional average in Eq. (2.98) is now reduced to simple one-dimensional Gaussian integrations. Renaming again $k_1 \to k$, we see that the Gaussian approximation $G_0(k)$ for the Fourier transform of the correlation function $G(k)$ is for $k \neq 0$,

$$G_0(k) = a^2 \frac{\int_{-\infty}^{\infty} \frac{d\mathrm{Re}\delta\varphi_k}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d\mathrm{Im}\delta\varphi_k}{\sqrt{2\pi}} |\delta\varphi_k|^2 \exp\left[-c_0 a^2(\xi^{-2} + k^2) |\delta\varphi_k|^2\right]}{\int_{-\infty}^{\infty} \frac{d\mathrm{Re}\delta\varphi_k}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{d\mathrm{Im}\delta\varphi_k}{\sqrt{2\pi}} \exp\left[-c_0 a^2(\xi^{-2} + k^2) |\delta\varphi_k|^2\right]}$$

$$= \frac{\int_0^{\infty} d\rho \, \rho^3 \, e^{-c_0(\xi^{-2}+k^2)\rho^2}}{\int_0^{\infty} d\rho \, \rho \, e^{-c_0(\xi^{-2}+k^2)\rho^2}}$$

$$= \frac{1}{c_0(\xi^{-2} + k^2)} = \frac{1}{A_t r_0 + c_0 k^2} , \tag{2.102}$$

---

[7] In a finite volume V the singular expression $(2\pi)^D \delta(k = 0)$ should be regularized with the volume of the system, $(2\pi)^D \delta(k = 0) \to V$.

where A_t is defined in Eq. (2.88) and in the second line we have introduced $\rho = a\,|\delta\varphi_k|$ as a new integration variable. In particular, at the critical point, where $r_0 = 0$, we obtain

$$G_0(k) = \frac{1}{c_0 k^2}\,,\quad T = T_c\,. \tag{2.103}$$

Comparing this with the definition of the correlation function exponent η in Eq. (1.14), we conclude that in Gaussian approximation

$$\boxed{\eta = 0\,.} \tag{2.104}$$

Next, let us calculate the correlation length exponent ν and justify our identification of the length ξ defined in Eq. (2.80) with the correlation length. Therefore, we calculate the correlation function $G(r)$ in real space and compare the result with Eq. (1.11). For $V \to \infty$ the Fourier transformation of the correlation function in Gaussian approximation can be written as

$$G_0(r) = \int_k e^{ik\cdot r} G_0(k) = \frac{1}{c_0}\int_k \frac{e^{ik\cdot r}}{\xi^{-2} + k^2}\,. \tag{2.105}$$

Although there is an implicit ultraviolet cutoff Λ_0 hidden in our notation, for $\min\{|r|, \xi\} \gg \Lambda_0^{-1}$ the integral is dominated by wave vectors much smaller than Λ_0, so that we may formally move the cutoff to infinity. To evaluate the D-dimensional Fourier transform in Eq. (2.105), let us denote by $k_\parallel$ the component of k parallel to the direction of r, and collect the other components of k into a $(D-1)$-dimensional vector $k_\perp$. Then the integration over $k_\parallel$ can be performed using the theorem of residues. Introducing $(D-1)$-dimensional spherical coordinates in $k_\perp$-space we obtain

$$G_0(r) = \frac{K_{D-1}}{2c_0}\int_0^\infty dk_\perp k_\perp^{D-2}\,\frac{\exp\left[-\sqrt{\xi^{-2} + k_\perp^2}\,|r|\right]}{\sqrt{\xi^{-2} + k_\perp^2}}\,. \tag{2.106}$$

Although for general D the integration cannot be performed analytically, the leading asymptotic behavior for $|r| \ll \xi$ and for $|r| \gg \xi$ is easily extracted. In the regime $|r| \ll \xi$, the exponential factor cuts off the $k_\perp$-integration at $k_\perp \approx 1/|r|$. Since for $|r| \ll \xi$ this cutoff is large compared with $1/\xi$, we may neglect ξ^{-2} as compared with $k_\perp^2$ in the integrand.[8] The integral can then be expressed in terms of the Gamma-function $\Gamma(D-2)$ and we obtain

---

[8] For $D \le 2$ this approximation is not justified because it generates an artificial infrared divergence. In this case a more careful evaluation of the integral (2.106) is necessary. Because for $D \le 2$ our simple Gaussian approximation is not justified anyway, we focus here on $D > 2$.

$$G_0(\boldsymbol{r}) \sim \frac{K_{D-1}}{2c_0} \frac{\Gamma(D-2)}{|\boldsymbol{r}|^{D-2}} , \quad \text{for } |\boldsymbol{r}| \ll \xi . \tag{2.107}$$

In particular, at the critical point where $\xi = \infty$, this expression is valid for all $\boldsymbol{r}$. A comparison with Eq. (1.13) confirms again $\eta = 0$.

In the opposite limit $|\boldsymbol{r}| \gg \xi$ the integration in Eq. (2.105) is dominated by the regime $k_\perp \lesssim (\xi|\boldsymbol{r}|)^{-1/2} \ll \xi^{-1}$. Then we may ignore the $k_\perp^2$-term as compared with ξ^{-2} in the denominator of Eq. (2.106) and approximate in the exponent,

$$\sqrt{\xi^{-2} + k_\perp^2}\,|\boldsymbol{r}| \approx \frac{|\boldsymbol{r}|}{\xi} + \frac{\xi|\boldsymbol{r}|k_\perp^2}{2} , \tag{2.108}$$

so that

$$G_0(\boldsymbol{r}) \sim \frac{K_{D-1}}{2c_0} \xi e^{-|\boldsymbol{r}|/\xi} \int_0^\infty dk_\perp k_\perp^{D-2} e^{-\frac{\xi|\boldsymbol{r}|k_\perp^2}{2}} . \tag{2.109}$$

The Gaussian integration is easily carried out and we finally obtain

$$G_0(\boldsymbol{r}) \sim \frac{K_{D-1}}{2c_0} 2^{\frac{D-3}{2}} \Gamma\!\left(\frac{D-1}{2}\right) \frac{e^{-|\boldsymbol{r}|/\xi}}{\sqrt{\xi^{D-3}|\boldsymbol{r}|^{D-1}}} , \quad \text{for } |\boldsymbol{r}| \gg \xi . \tag{2.110}$$

This has the same form as postulated in Eq. (1.11), justifying the identification of ξ in Eq. (2.80) with the order-parameter correlation length. Hence, within Gaussian approximation we obtain

$$\xi = \sqrt{\frac{c_0}{|A_t r_0|}} \propto |t|^{-1/2} , \tag{2.111}$$

implying for the correlation length exponent in Gaussian approximation

$$\boxed{\nu = 1/2 .} \tag{2.112}$$

Note that for $D < D_{\text{up}} = 4$, where the Gaussian approximation yields $\alpha = 2 - D/2$ for the specific heat exponent [see Eq. (2.92)], the Gaussian results $\eta = 0$ and $\nu = 1/2$ are also consistent with the hyper-scaling relations (1.29) and (1.32) connecting ν and η with the thermodynamic critical exponents, $\alpha = 2 - D\nu = 2 - D/2$ and $\gamma = (2 - \eta)\nu = (2 - 0) \times \frac{1}{2} = 1$.

2.3.4 Failure of the Gaussian Approximation in $D < 4$

The Gaussian approximation amounts to the quadratic truncation (2.73) of the Ginzburg–Landau–Wilson action $S[\boldsymbol{\varphi}]$ defined in Eq. (2.40). The important question is now whether the Gaussian approximation is sufficient to calculate the critical

exponents or not. As already mentioned, the answer depends crucially on the dimensionality of the system: only for $D > D_{up} = 4$ the critical exponents obtained within the Gaussian approximation are correct, while for $D < D_{up}$ the Gaussian approximation is not sufficient. To understand the special role of the upper critical dimension $D_{up} = 4$ for the Ising model, let us attempt to go beyond the Gaussian approximation by retaining the quartic term in the expansion of $S[\varphi]$ in powers of the field. As shown in Sect. 2.2.2, for small wave vectors our effective action then reduces to the φ^4-theory $S_{\Lambda_0}[\varphi]$ defined via Eq. (2.61) or (2.65). At the first sight it seems that if we assume that the coupling constant u_0 associated with the quartic term in Eqs. (2.61) and (2.65) is arbitrarily small, then we may calculate the corrections to the Gaussian approximation perturbatively in powers of u_0. However, this strategy fails in $D < 4$ in the vicinity of the critical point, because the effective dimensionless parameter which is relevant for the perturbative expansion is

$$\bar{u}_0 = \frac{u_0 \xi^{4-D}}{c_0^2} = 2(2D)^2 \left(\frac{\xi}{a}\right)^{4-D} , \qquad (2.113)$$

where we have used Eq. (2.59) to approximate $u_0 \approx 2a^{D-4}$. The fact that the perturbative expansion of the interaction in the effective action (2.61) is controlled by the dimensionless coupling $\bar{u}_0$ can be made manifest by introducing dimensionless wave vectors

$$\bar{k} = k\xi , \qquad (2.114)$$

and dimensionless fields

$$\bar{\varphi}(\bar{k}) = \sqrt{\frac{c_0}{\xi^{2+D}}} \varphi(\bar{k}/\xi) . \qquad (2.115)$$

Assuming for simplicity $T > T_c$ so that $r_0 = c_0/\xi^2$, it is easy to show that in terms of these dimensionless variables our effective Ginzburg–Landau–Wilson action (2.61) takes the form

$$
\begin{aligned}
S_{\Lambda_0}[\bar{\varphi}] = V f_0 &+ \frac{1}{2} \int_{\bar{k}} [1 + \bar{k}^2] \bar{\varphi}(-\bar{k}) \bar{\varphi}(\bar{k}) \\
&+ \frac{\bar{u}_0}{4!} \int_{\bar{k}_1} \int_{\bar{k}_2} \int_{\bar{k}_3} \int_{\bar{k}_4} (2\pi)^D \delta(\bar{k}_1 + \bar{k}_2 + \bar{k}_3 + \bar{k}_4) \bar{\varphi}(\bar{k}_1) \bar{\varphi}(\bar{k}_2) \bar{\varphi}(\bar{k}_3) \bar{\varphi}(\bar{k}_4) .
\end{aligned}
$$
$$(2.116)$$

Because the coefficient of the quadratic fluctuations in this expression is fixed to unity for small wave vectors, the coefficient $\bar{u}_0$ in front of the quartic part directly gives the relative strength of the quartic interaction between the fluctuations as compared with the free field theory described by the Gaussian part. The crucial point is now that close to the critical point the correlation length $\xi \propto |t|^{-\nu}$ diverges,

so that below four dimensions the effective dimensionless coupling $\bar{u}_0$ defined in
Eq. (2.113) becomes arbitrarily large for $T \to T_c$, no matter how small the bare
coupling constant u_0 is chosen. As a consequence, for $D < 4$ any attempt to cal-
culate perturbatively corrections to the Gaussian approximation is bound to fail for
temperatures sufficiently close to T_c. Because the critical exponents are defined in
terms of the asymptotic behavior of physical observables for $T \to T_c$, they cannot
be obtained perturbatively for $D < D_{\mathrm{up}} = 4$. Obviously, this problem can only be
solved with the help of nonperturbative methods, where all orders in the relevant
dimensionless coupling $\bar{u}_0$ are taken into account. The most powerful and general
method available to deal with this problem is the RG.

In the ordered phase where $\langle \varphi(\mathbf{r}) \rangle = \bar{\varphi}_0$ is finite the dimensionless interaction $\bar{u}_0$
in Eq. (2.113) has a nice interpretation in terms of the relative importance of order-
parameter fluctuations, which is measured by the following dimensionless ratio,

$$Q = \frac{\int^\xi d^D r \, \langle \delta \varphi(\mathbf{r}) \delta \varphi(\mathbf{r}=0) \rangle_S}{\int^\xi d^D r \, \bar{\varphi}_0^2} = \frac{\int^\xi d^D r \, G_0(\mathbf{r})}{\xi^D \bar{\varphi}_0^2} . \tag{2.117}$$

Here, the integrals $\int^\xi d^D r$ extend over a cube with volume ξ^D enclosing the ori-
gin, the mean-field value $\bar{\varphi}_0$ of the order parameter is given in Eq. (2.74), and the
correlation function in the numerator is evaluated in Gaussian approximation. Intu-
itively, the quantity Q sets the strength of order-parameter fluctuations in relation
to the average order parameter, taking into account that only within a volume of
linear extension ξ the fluctuations are correlated. If Q is small compared with unity,
then fluctuations are relatively weak and it is reasonable to expect that a mean-field
description of the thermodynamics and the Gaussian approximation for the correla-
tion function are sufficient. Taking into account that for $|\mathbf{r}| \gtrsim \xi$ the function $G_0(\mathbf{r})$
decays exponentially, the value of the integral in the numerator of Eq. (2.117) is not
changed if we extend the integration regime over the entire volume of the system,
so that we arrive at the estimate,

$$\int^\xi d^D r \, G_0(\mathbf{r}) \approx \int d^D r \, G_0(\mathbf{r}) = G_0(\mathbf{k}=0) = \frac{\xi^2}{c_0} . \tag{2.118}$$

But according to Eqs. (2.74) and (2.80) for $T < T_c$ we may write

$$\bar{\varphi}_0^2 = \frac{-6 r_0}{u_0} = \frac{3 c_0}{\xi^2 u_0} , \tag{2.119}$$

so that we finally obtain

$$Q = \frac{\xi^{4-D} u_0}{3 c_0^2} = \frac{\bar{u}_0}{3} . \tag{2.120}$$

Hence, our dimensionless coupling constant $\bar{u}_0$ defined in Eq. (2.113) is proportional to the ratio Q which measures the relative importance of fluctuations. Obviously, for $Q \ll 1$ fluctuations are weak and it is allowed to treat the interaction term in Eq. (2.116) perturbatively. On the other hand, in the regime $Q \gg 1$ we are dealing with a strongly interacting system, so that mean-field theory and the Gaussian approximation are not sufficient. The condition $Q \ll 1$ is called the *Ginzburg criterion* (Ginzburg 1960, Amit 1974).

Exercises

2.1 Mean-field Analysis of the Ising Model in a Transverse Field

Below a few Kelvin, the magnetic properties of the rare-earth insulator lithium holmium fluoride (LiHoF$_4$) are well described by the Ising model in a transverse field (also known as the quantum Ising model). Its Hamiltonian reads

$$\hat{H} = -J \sum_{\langle i,j \rangle} \sigma_i^z \sigma_j^z - \Gamma \sum_i \sigma_i^x \ .$$

Here, $J > 0$ is the exchange coupling for nearest-neighbor spins, $\Gamma \geq 0$ denotes the strength of the transverse field and σ_i^z and σ_i^x are Pauli matrices which measure the x- and z-components of the spins residing on the lattice sites of a hypercubic lattice. Let us denote the eigenstates of the σ_i^z by $| \uparrow \rangle_i$ and $| \downarrow \rangle_i$ (with eigenvalues $s_i = \pm 1$).

(a) Show that for $\Gamma = 0$ and in the above basis spanned by the eigenstates of the σ_i^z, the Hamiltonian $\hat{H}$ reduces to the Hamiltonian of the familiar classical Ising model.

(b) Introduce $g \equiv \Gamma/zJ$ (where $z = 2D$ is the coordination number) to tune a quantum phase transition at $T = 0$. Show that for $g = 0$ ($\Gamma = 0$) and for $g \to \infty$ ($J \to 0$) you get two qualitatively different ground states. These states are called the ferromagnetic and the paramagnetic ground states. Why?

(c) To derive the mean-field Hamiltonian $\hat{H}_{MF}$ of the quantum Ising model, write $\sigma_i^z = m^z + \delta\sigma_i^z$ with the fluctuation matrix $\delta\sigma_i^z = \sigma_i^z - m^z$ and

$$m^z = \langle \sigma_i^z \rangle = \frac{\text{Tr}\,[\sigma_i^z e^{-\beta\hat{H}}]}{\text{Tr}\,[e^{-\beta\hat{H}}]} \ .$$

Expand the term $\sigma_i^z \sigma_j^z$ in $\hat{H}$ to linear order in $\delta\sigma_i^z$ and neglect terms quadratic in the fluctuation matrices.

(d) Calculate the eigenvalues of the single site mean-field Hamiltonian and use these eigenvalues to evaluate the partition function $\mathcal{Z}_{MF}$ and the Landau func-

tion $\mathcal{L}_{MF}$. Minimize the Landau function with respect to the magnetization m^z
to obtain a self-consistency condition for $m_0^z = m_0^z(T, \Gamma)$.

(e) Following the procedure outlined in Sect. 2.1.1, solve the mean-field consistency equation graphically. Analyze in particular regions in parameter space where $m^z \neq 0$ and draw the phase boundary (in the T-g-plane) separating this ferromagnetic phase from the paramagnetic phase with $m^z = 0$. What kind of phase transitions can you identify?

(f) For $T = 0$ calculate m^z explicitly and sketch m^z as a function of g. Determine the mean-field value for the critical exponent β_g defined by $m^z \propto (g_c - g)^{\beta_g}$.

(g) Calculate also $m^x = \langle \sigma_i^x \rangle$ and for $T = 0$ sketch m^x as a function of g.

2.2 Landau Functions for the Free Bose Gas

Consider N noninteracting bosons in a D-dimensional harmonic trap with Hamiltonian $\hat{H} = \sum_{\mathbf{m}} E_{\mathbf{m}} b_{\mathbf{m}}^{\dagger} b_{\mathbf{m}}$, where $E_{\mathbf{m}} = \hbar\omega(m_1 + \cdots + m_D)$, and $m_i = 0, 1, 2, \ldots$.
The partition function in the canonical ensemble is given by $Z_N = \mathrm{Tr}_N e^{-\beta\hat{H}}$,
where the trace runs over the Hilbert space with fixed particle number N. It can be expressed as $Z_N = \sum_{n=0}^{\infty} Z_N(n)$, where $Z_N(n) = \mathrm{Tr}_N(\delta_{n, b_0^{\dagger} b_0} e^{-\beta\hat{H}})$ has a fixed occupation of the single particle ground state. We define the dimensionless Landau function for Bose–Einstein condensation via $Z_N(n) = e^{-N\mathcal{L}_N^{\mathrm{BEC}}(n/N)}$. Provided $\mathcal{L}_N^{\mathrm{BEC}}(q)$ is smooth and has an absolute minimum at $q \neq 0$ in the thermodynamic limit, the single particle ground state will be macroscopically occupied.

(a) Using $\mathrm{Tr}_N \hat{A} = \int_0^{2\pi} \frac{d\theta}{2\pi} \mathrm{Tr}\left[e^{i(N-\hat{N})\theta} \hat{A}\right]$, where $\hat{N} = \sum_{\mathbf{m}} b_{\mathbf{m}}^{\dagger} b_{\mathbf{m}}$ and the trace on the right-hand side runs over the entire Hilbert space containing any number of particles (this is the so-called Fock space), show that the dimensionless Landau function $\mathcal{L}_N^{\mathrm{BEC}}(q)$ is given by

$$\mathcal{L}_N^{\mathrm{BEC}}(q) = -\frac{1}{N} \ln \left[\int_0^{2\pi} \frac{d\theta}{2\pi} e^{i\theta N(1-q) - \sum_{\mathbf{m}\neq 0} \ln\left(1 - e^{-(\epsilon_{\mathbf{m}} + i\theta)}\right)} \right].$$

Here, $\epsilon_{\mathbf{m}} \equiv E_{\mathbf{m}}/T$. What is the physical interpretation of $\mathcal{L}_N^{\mathrm{BEC}}(q)$?

(b) Show that alternatively, we can write $Z_N = \int d^2\phi \, e^{-N\mathcal{L}_N^{\mathrm{SSB}}(\phi)}$, where $d^2\phi = d[\mathrm{Re}\phi] \, d[\mathrm{Im}\phi]$ and the Landau function for spontaneous symmetry breaking is given by

$$\mathcal{L}_N^{\mathrm{SSB}}(\phi) = |\phi|^2 - \frac{1}{N} \ln \left[\frac{N}{\pi} \sum_{n=0}^{N} \frac{(N|\phi|^2)^n}{n!} e^{-N\mathcal{L}_N^{\mathrm{BEC}}(n/N)} \right].$$

Hint: Carry out the partial trace over $\mathbf{m} = 0$ by using coherent states $|z\rangle = e^{zb_0^{\dagger}} |0\rangle$, i.e., eigenstates of b_0, $b_0|z\rangle = z|z\rangle$. Recall that $\mathrm{Tr}_{\mathbf{m}=0}\hat{A} = \int \frac{d^2z}{\pi} e^{-|z|^2}$

$\langle z|\hat{A}|z\rangle$ (see e.g., Shankar 1994, Chap. 21). You will also need the overlap $\langle n|z\rangle = \frac{z^n}{\sqrt{n!}}$ with a state of fixed boson number.

(c) Show the recursion relation $Z_N = \frac{1}{N}\sum_{k=1}^{N} Z_{N-k}Z_1(k)$, where $Z_1(k) = \sum_{\mathbf{m}} e^{-k\epsilon_{\mathbf{m}}} = [1 - e^{-k/\tau}]^{-D}$, and the dimensionless temperature is $\tau = T/(\hbar\omega)$. *Hint:* Show for the generating function $\mathcal{Z}(u) \equiv \sum_N Z_N u^N = \prod_{\mathbf{m}}(1 - ue^{-\epsilon_m})^{-1}$, by expressing it as a trace over Fock space. Take derivatives on both sides of this equality and express the right-hand side again in terms of $\mathcal{Z}(u)$.

(d) Show that $Z_N(n) = Z_{N-n} - Z_{N-n-1}$.
Hint: Derive a relation between the generating functions $\mathcal{Z}(u)$ and $h(u) := \sum_N Z_N(n)u^N$.

(e) Plot $\mathcal{L}_N^{\mathrm{BEC}}(q)$ and $\mathcal{L}_N^{\mathrm{SSB}}(\phi)$ for $N = 10$ and $D = 3$ as a function of $\sqrt{q}$ and ϕ, respectively, evaluating the expressions derived above numerically for different temperatures τ. What is the temperature at which the Landau functions start to develop nontrivial minima? More details and a discussion of $N \to \infty$ can be found in Sinner et al. (2006).

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Introduction to the Functional Renormalization Group