Least Squares

1. problem statement.

1.1 The system is described by a set of functions $\{f_i(x)\}$ $i=1:n$.

$z_{1:n}$ are n noisy measurements about an unknown state x.

Goal: estimate the state x which best explains our measurements $z_{1:n}$.

By Best explain: $f_i(x) = \hat{z}_i$ will be our predicted measurements A good state estimate should make $\hat{z}_{1:n}$ as close to $z_{1:n}$ as possible.

1.2 Error function.

$$\vec{e}_i(x) = z_i - f_i(x) \qquad \text{error}_i(x) = \underbrace{e_i(x)^T \Omega_i e_i(x)}_{\text{A weighted sum}}.$$

Assume error is zero-mean

And it is Gaussian

1.3 As an optimization

$$x^* = \operatorname*{argmin}_x \sum_i e_i^T(x) \Omega_i e_i(x).$$

2. Solution

2.1 We make strong assumptions:

① A good initial guess is available

②. Error function $\vec{e_i}(x)$ is smoothy.

③ we linearize error term at current guess, solve the minimization problem

by taking derivative and set to zero.

④ After we obtain a new state value x. We iterate above process.

2.2. linearization error.

$$e_i(x+\Delta x) \simeq \underbrace{e_i(x)}_{e_i} + J_i(x)\Delta x \qquad J_f(x) = \begin{pmatrix} \frac{\partial f_1(x)}{\partial x_1} & \frac{\partial f_1(x)}{\partial x_2} & \cdots & \frac{\partial f_1(x)}{\partial x_n} \\ \frac{\partial f_2(x)}{\partial x_1} & & & \\ \vdots & & & \\ \frac{\partial f_n(x)}{\partial x_1} & & \cdots & \frac{\partial f_n(x)}{\partial x_n} \end{pmatrix}$$

$$\Rightarrow \text{error}_i(x+\Delta x) = e_i^T(x+\Delta x)\Omega_i e_i(x+\Delta x)$$
$$\simeq (e_i + J_i\Delta x)^T \Omega_i (e_i + J_i \Delta x)$$
$$= e_i^T \Omega e_i + e_i^T \Omega_i J_i \Delta x + \Delta x^T J_i^T \Omega_i e_i + \Delta x^T J_i^T \Omega_i J_i \Delta x.$$

now we solve the problem as finding Δx to minimize $\text{error}_i(x+\Delta x)$.

now

$$e_i(x+\Delta x) = \underline{e_i^T \Omega_i e_i} + \overline{2 e_i^T \Omega_i J_i} \Delta x + \Delta x^T \underline{J_i^T \Omega_i J_i} \Delta x$$

$$= c_i + 2 b_i^T \Delta x + \Delta x^T H_i \Delta x$$

$\Rightarrow$ Global error:

$$F(x+\Delta x) \simeq \sum_i c_i + 2\left(\sum_i b_i^T\right)\Delta x + \Delta x^T \left(\sum H_i\right)\Delta x$$

$$= c + 2 b^T \Delta x + \Delta x^T H \Delta x.$$

$b^T = \sum_i e_i^T \Omega_i J_i$

$H = \sum_i J_i^T \Omega J_i$

$$\frac{\partial F(x+\Delta x)}{\partial \Delta x} = (H+H^T)\Delta x + 2b = 2b + 2H\Delta x \overset{!}{=} 0$$

$\Leftrightarrow$ solving $H\Delta x = -b$

$\Rightarrow \Delta x^* = -H^{-1} b$

2.3. Gauss-Newton Solution Summary.

① Initial x

② $b^T = \sum_i e_i^T \Omega_i J_i$ $\quad H = \sum_i J_i^T \Omega_i J_i$.

③ $\Delta x^* = -H^{-1} b \rightarrow$ inversion is costy. many other algorithms for this step.

④ $x \leftarrow x + \Delta x^*$. repeat.

3. Odometry Calibration

3.1 problem: we have odometry measurements u_i. but they are noisy.

Assume we have ground truth u_i^*.

Calibrate: change system parameters. s.t. u_i and u_i^* are as close as possible.

3.2. Assume a function f, it takes biased x and return unbiased x'.

$$u_i' = f_i(x) = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{pmatrix} u_i.$$ Thus we need to estimate these parameters

$\Rightarrow$ state vector

$\vec{x} = (x_{11} \cdots x_{33})^T$

error function: $e_i(x) = u_i^* - \begin{pmatrix} x_{11} & \cdots & x_{13} \\ \vdots & & \vdots \\ x_{31} & \cdots & x_{33} \end{pmatrix} u_i$

$\Rightarrow$ Jacobian

$$J_t = \frac{\partial e_i(x)}{\partial x} = -\begin{pmatrix} u_{i,x} & u_{i,y} & u_{i,o} & & & & & & \\ & & & u_{i,x} & u_{i,y} & u_{i,o} & & & \\ & & & & & & u_{i,x} & u_{i,y} & u_{i,o} \end{pmatrix}$$

note $$\frac{\partial(u^{*}_{i,x} - x_{11}u_{i,x} - x_{12}u_{i,y} - x_{13}u_{i,o})}{\partial x_{11}} = -u_{i,x}.$$

Graph SLAM

1. Idea.

use graph to represent the problem

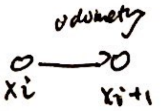
every node is pose of a robot.

edge between node is a spatial constraints

Task: build the graph and find a node configuration that minimize the error introduced by the constraints.

2. error function

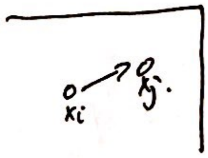
2.1 odometry edge.



transformation

$x_i^{-1} x_{i+1}$

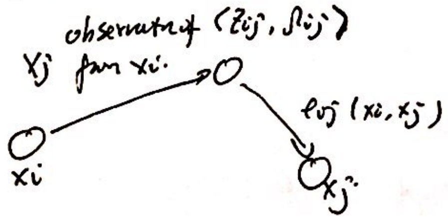
2.2. measurement edge.



transformation $x_i^{-1} x_j$

how node i sees node j.

2.3. pose graph.



optimization problem

$$x^* = \arg\min_x \sum_{ij} e_{ij}^T \Omega_{ij} e_{ij}$$

$$e_{ij}(x_i, x_j) = t2v\big(\underbrace{Z_{ij}^{-1}}_{\substack{\text{measurement.} \\ \text{(has seen)}}} \underbrace{(x_i^{-1} x_j)}_{\substack{\text{should see} \\ \text{based on configuration}}} \big)$$

$e_{ij} = 0$ if $z_{ij} = x_i^{-1} x_j$

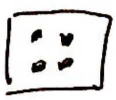
2.4. now we need to linearize the system using least squares.

$e_{ij}(x + \Delta x) \simeq e_{ij}(x) + J_{ij} \Delta x$ $\quad J_{ij} = \dfrac{\partial e_{ij}(x)}{\partial x}$ $\quad$ since $e_{ij}(x) = e_{ij}(x_i, x_j)$

$$J_{ij} = \left(0 \cdots 0 \ \frac{\partial e}{\partial x_i} \ 0 \cdots 0 \ \frac{\partial e}{\partial x_j} \ 0 \cdots 0 \right)$$

$\Rightarrow b_{ij} = J_{ij}^T \Omega_{ij} e_{ij}$

$H_{ij} = J_{ij}^T \Omega_{ij} J_{ij}$



Finally b will be dense, H will be somewhat sparse

3. Solving the system

3.1. $\Delta x^T = (\Delta x_1^T \cdots \Delta x_n^T)$

$b^T = (b_1^T \cdots b_n^T)$

$$H = \begin{pmatrix} H^{11} & H^{12} & H^{1n} \\ & \ddots & \\ H^{n1} & & H^{nn} \end{pmatrix}$$

$e_{ij} = t2v(Z_{ij}^{-1}(X_i^{-1} X_j))$

$A_{ij} = \frac{\partial e_{ij}}{\partial x_i}$ $\quad B_{ij} = \frac{\partial e_{ij}}{\partial x_j}$.

$b_i^T += e_{ij}^T \Omega_{ij} A_{ij}$ $\quad b_j^T += e_{ij}^T \Omega_{ij} B_{ij}$.

$H^{ii} += A_{ij}^T \Omega_{ij} A_{ij}$ $\quad H^{ij} += A_{ij}^T \Omega_{ij} B_{ij}$

$H^{ji} += B_{ij}^T \Omega_{ij} A_{ij}$ $\quad H^{jj} += B_{ij}^T \Omega_{ij} B_{ij}$

3.2 Flow:

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while (!converged)
    (H, b) = build System (x)
    Δx = solve(HΔx = -b)
    X = X + ΔX
end
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4. prior

4.1. H has not full rank. we need to fix the global reference frame.

by using prior $p(x_0)$

4.2. first pose, we specify $e(x_0) = t2v(x_0)$

4.3 fixing a variable.

construct H, suppress rows and cols of x_i. this will make x_i fixed.