S-unit equations in modules

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Based on joint work with Doron Shafrir

Workshop on Loop Invariants and Algebraic Reasoning, ICALP 2025

Consider the three following problems.

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Theorem (Jensen, Kahle and Katthän 2017)

Finding sparse polynomial in ideals over $\mathbb{Q}$ is decidable for $s = 1, 2$.

For $s \geq 3$, decidability is an open problem.

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Input: A finite set $S \subset \mathbb{K} \setminus \{0\}$, and a linear equation

$$x_1 m_1 + \cdots + x_d m_d = m_0 \qquad (*)$$

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Fact (cf. Mahler 1933, Lang 1960, Evertse 1984, Schlickewei 1990, . . .)

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Theorem (Adamczewski and Bell 2012, Derksen and Masser 2012)

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A unified approach?

The Skolem Problem

Sparse polynomial in ideals

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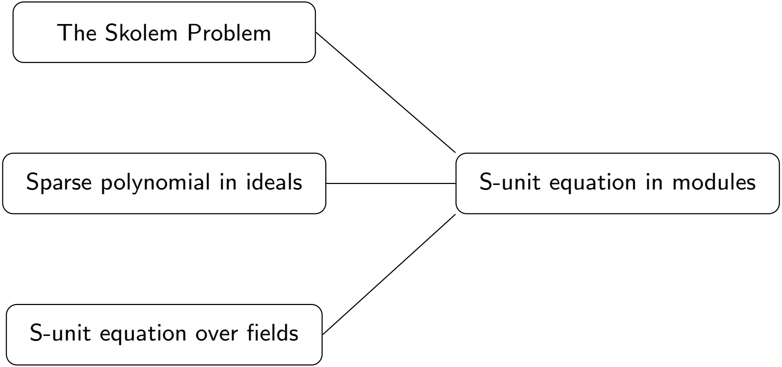

Motivation: consider the S-unit equation over $\mathbb{Q}$:

$$2^{z_1} + 2^{z_2} \cdot 3 = 6,$$

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In general, a **finitely presented module** over a ring R is defined as a quotient R^d/N, for an integer d and an R-submodule $N \subset R^d$.

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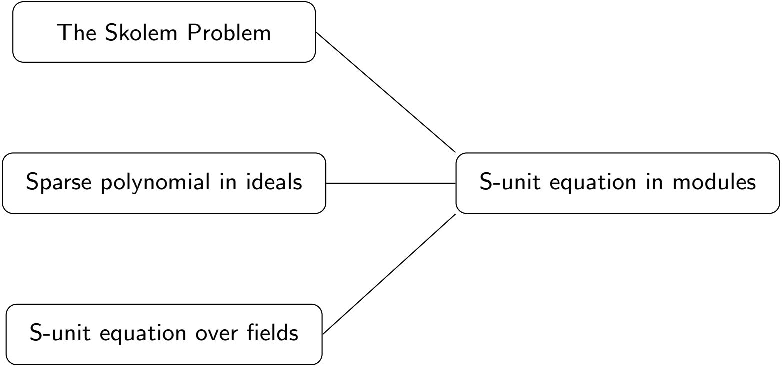
A (half-)unified approach



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Decidability

Can we decide S-unit equation in modules?

Theorem (D., SODA'25)

Solving S-unit equations in finitely presented modules is **undecidable**, *even for a fixed $\mathbb{Z}[X^{\pm}]$-module $\mathcal{M}$.*

Proof idea: embed Hilbert's tenth problem.

So what **can** we decide?

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Recall: a field $\mathbb{K}$ is of characteristic $p > 0$, if $px = 0$ for all $x \in \mathbb{K}$.

p is always prime.

T-torsion modules

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Let $T > 0$ be an integer. A module $\mathcal{M}$ is ***T*-torsion**, if $Tx = 0$ for all $x \in \mathcal{M}$.

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Let $T = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ be its prime factorization. Solving S-unit equations in T-torsion modules is Turing equivalent to finding solutions to a system of linear equations over $\mathbb{Z}$, where variables can be restricted to powers of p_i.

Example: $T = 3 \times 5 \times 7$ subsumes finding $x_1, x_2, x_3 \in \mathbb{N}$, $3^{x_1} + 5^{x_2} - 7^{x_3} = 11$.

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Difficult open problem for $k \geq 3$.

Main ingredients for our proofs

Derksen's approach: p-automata

Theorem (Derksen 2007)

The solution set $\{n \in \mathbb{N} \mid \mu(n) = 0\}$ to the Skolem Problem over a field of characteristic $p > 0$ is effectively p-automatic.

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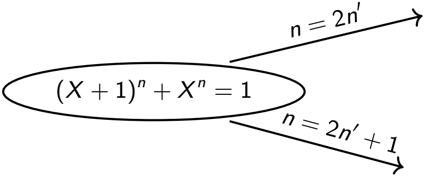

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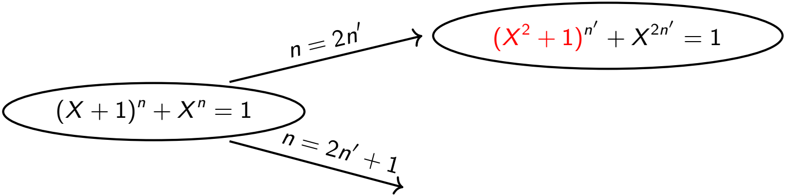

$^*(X+1)^2 = X^2 + 1$

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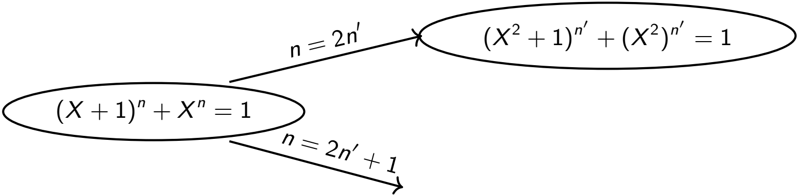

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$$(X+1)^n + X^n = 1 \xrightarrow{n = 2n'+1}$$

---

*let $X' := X^2$

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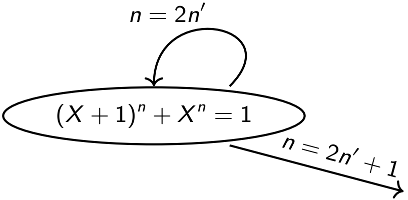

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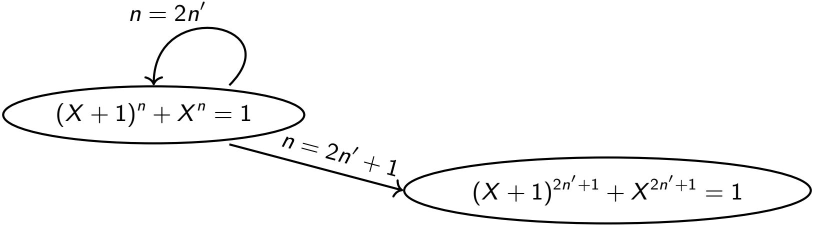

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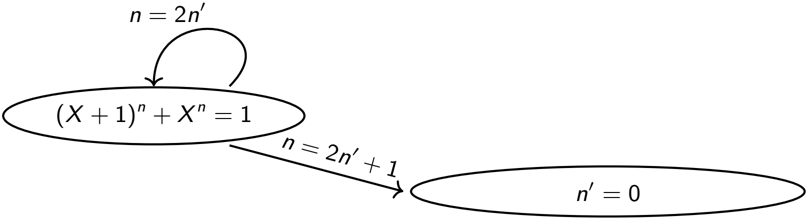

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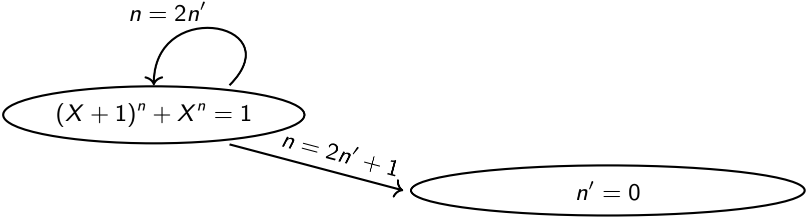

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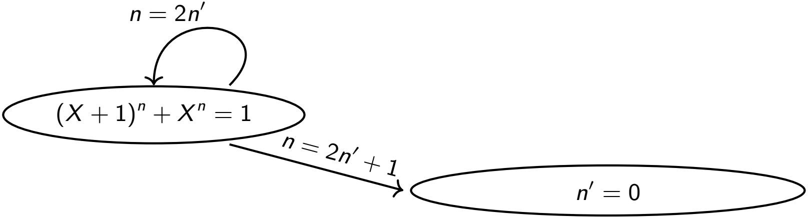

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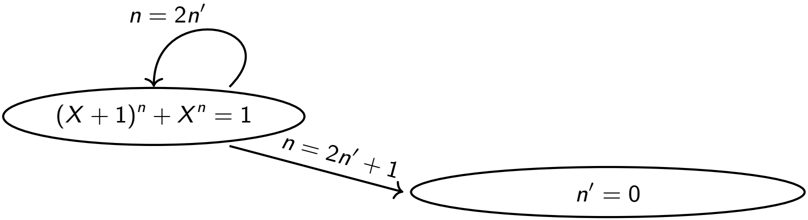

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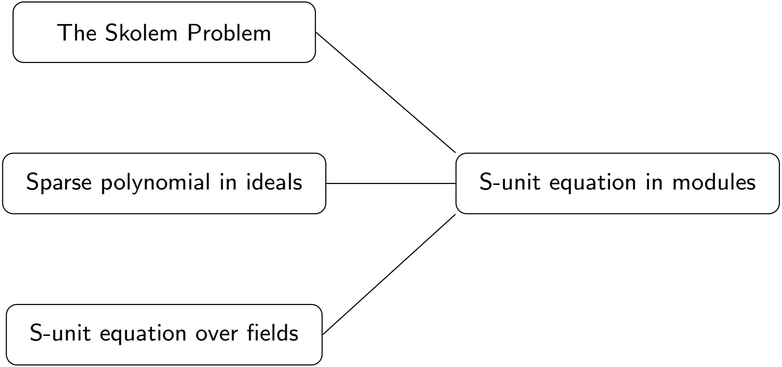
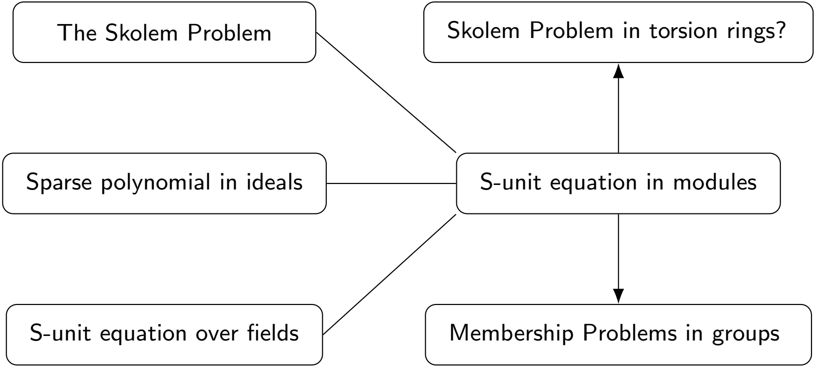
With some more work: hardness result for general T-torsion modules.

Future work and applications

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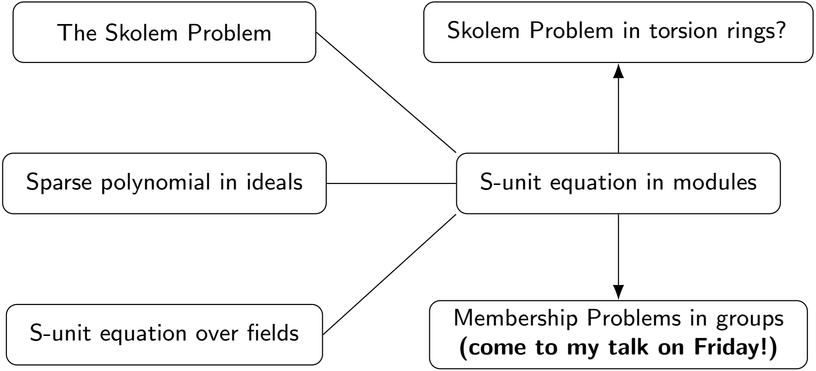
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(come to my talk on Friday!)