The Identity Problem in virtually solvable matrix groups over algebraic numbers

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LICS 2025

[1] partially supported by the Swiss NSF Grant 200020-200400.
[2] partially supported by ERC Advanced Grant 101097307.

Identity Problem

We consider the following decision problem:

Definition (Identity Problem)

Input: A set of square matrices $S = \{A_1, \ldots, A_K\}$.
Question: Does there exist a sequence $A_{i_1}, A_{i_2}, A_{i_3}, \ldots \in S$, such that the product $A_{i_1} A_{i_2} A_{i_3} \cdots$ is equal to the identity matrix I?

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The Identity Problem has applications in:

- **theory of ordered groups** (does a group admit a left-ordering?)
- **weighted automata** (is there a word with neutral effect?)
- **computational group theory** (is a given element in a semigroup invertible?)

Identity Problem: warm up

Theorem (folklore)

The Identity Problem is decidable for 1×1 *matrices over* $\mathbb{Q}$*.*

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Let $A_1, A_2, A_3 \in \mathbb{Q}^{1\times 1}$. The Identity Problem asks whether 1 belongs to the semigroup $\langle A_1, A_2, A_3 \rangle$.

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Since A_1, A_2, A_3 commute, this is equivalent to asking whether there exist $n_1, n_2, n_3 \in \mathbb{N}$, not all zero, such that $A_1^{n_1} A_2^{n_2} A_3^{n_3} = 1$.

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This can be done using **integer programming**: For example, suppose

$$A_1 = 12 = 2^2 \times 3, \quad A_2 = \frac{27}{16} = 2^{-4} \times 3^3, \quad A_3 = 6 = 2 \times 3.$$

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Whether there exist non-trivial, non-negative integer solutions $n_1, n_2, n_3 \in \mathbb{N}$, can be decided by integer programming.

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Historical results

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meta-question: is there an "algebraic" criterion of the matrix group for decidability of the Identity Problem?

Almost-dichotomy

Theorem (Tits alternative, 1972)

Let G be a matrix group over a field (in this talk: algebraic numbers $\overline{\mathbb{Q}}$). Then

- *either G contains the free group F_2 as a subgroup,*
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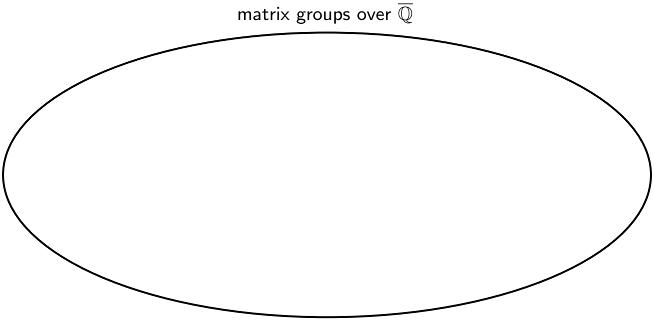

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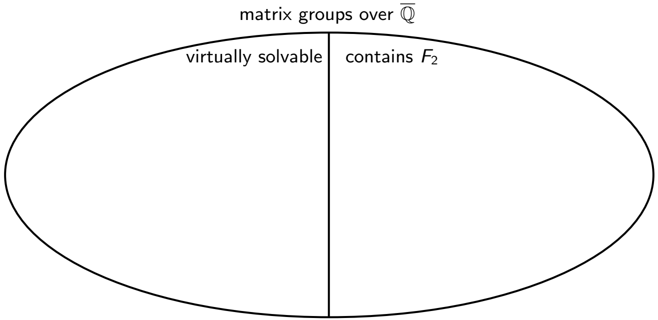

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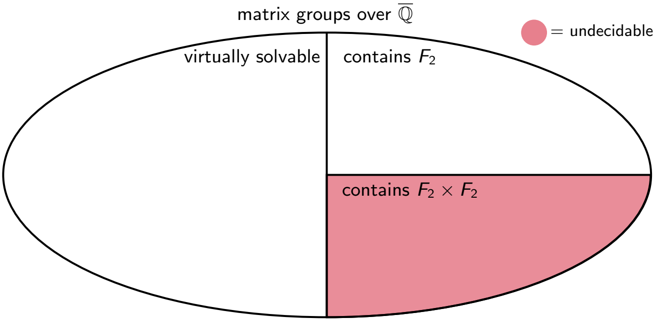

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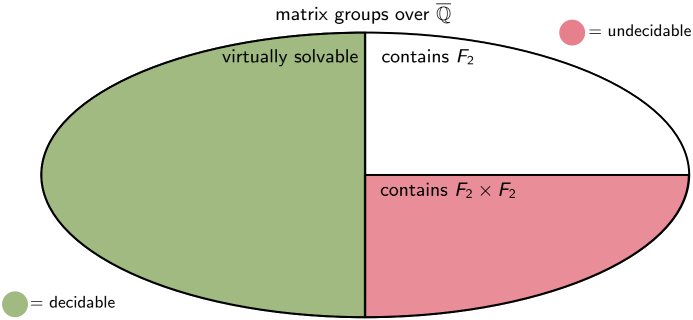

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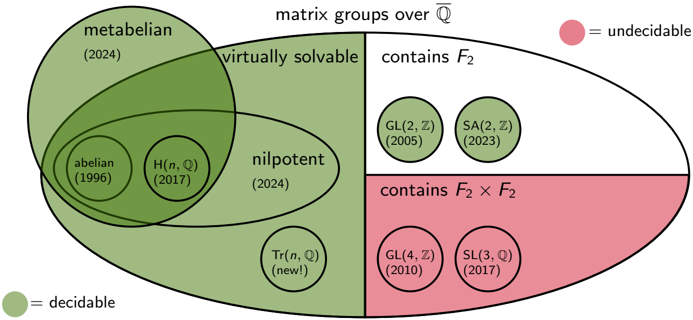

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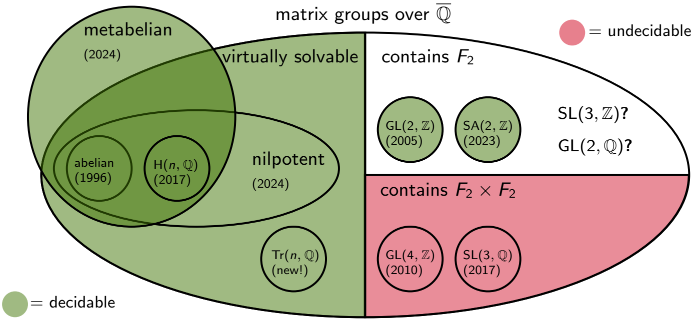

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A matrix group G over $\overline{\mathbb{Q}}$ is **virtually solvable**, if and only if it admits a finite index subgroup T that is triangularizable. (i.e. there is a matrix h such that every $hth^{-1}, t \in T$ is upper-triangular.)

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Example of a virtually solvable group: $G = \mathrm{Tr}(3, \mathbb{Q}) \times S_2$, where

$$\mathrm{Tr}(3, \mathbb{Q}) := \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \middle| a, b, c, d, e, f \in \mathbb{Q} \right\}, \qquad S_2 := \{-1, 1\}.$$

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Example of a set of elements in G:

$$A_1 = \left(\begin{pmatrix} 2 & 7 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, -1 \right), \; A_2 = \left(\begin{pmatrix} 1 & -1 & 1 \\ 0 & 2 & 5 \\ 0 & 0 & 1 \end{pmatrix}, 1 \right),$$

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Question (Identity Problem): does the semigroup $\langle A_1, A_2, A_3 \rangle$ contain the neutral element $(I, 1)$? **We now illustrate our algorithm using this example.**

Step 1: Identity Problem $\longrightarrow$ automaton over triangular matrices

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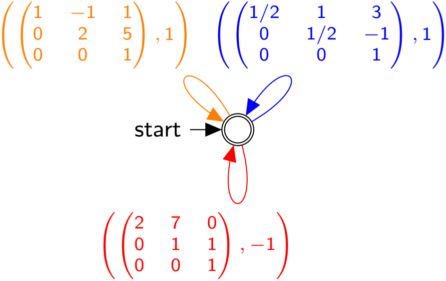

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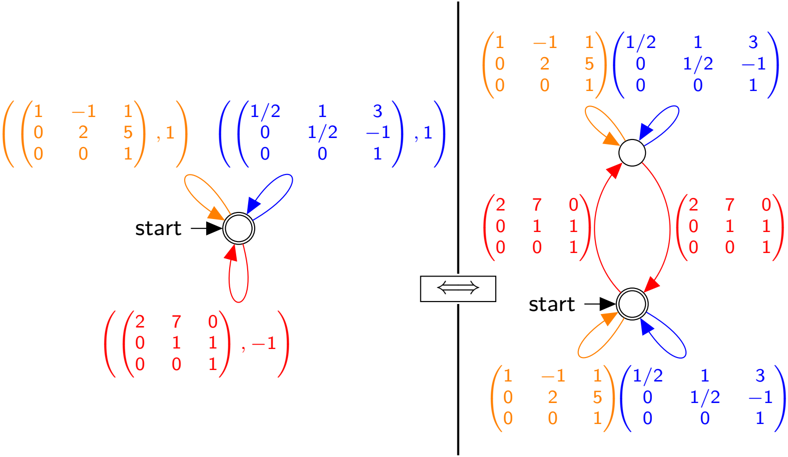

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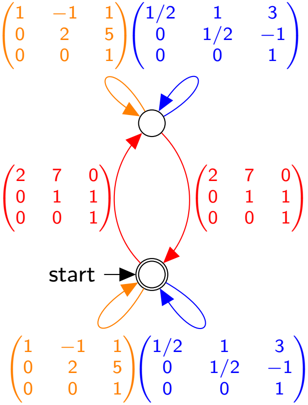

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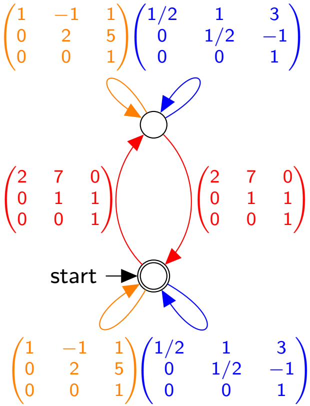

Theorem (structure theorem of subsemigroups of nilpotent groups)

Let N be a nilpotent group of finite Prüfer rank and M be a subsemigroup of N. If $M[N,N] = N$, then $M = N$.

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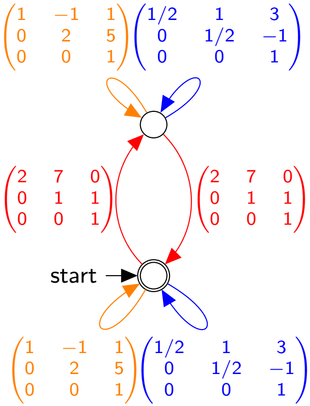

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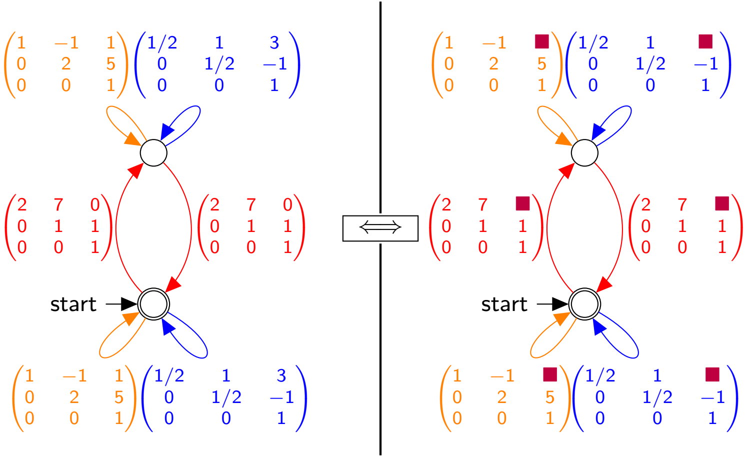

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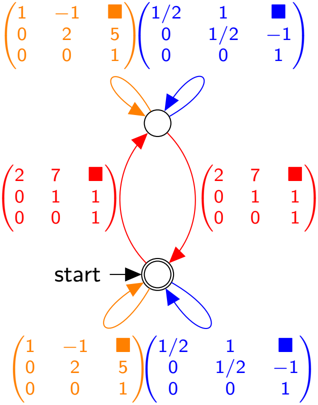
Step 3: reduce to rational subsemigroups of metabelian groups

Does the automaton admit a run, whose label product is $\begin{pmatrix} 1 & 0 & \blacksquare \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$?



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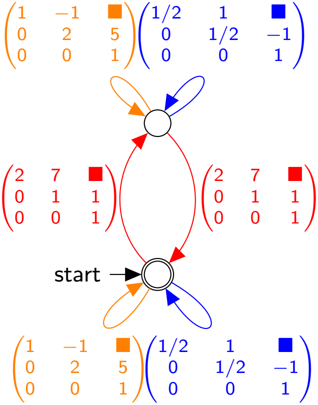

We now work in the "quotient" group

$$\mathrm{Tr}(3,\mathbb{Q})/\blacksquare := \left\{ \begin{pmatrix} a & b & \blacksquare \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \middle|\, a, b, d, e, f \in \mathbb{Q} \right\}.$$

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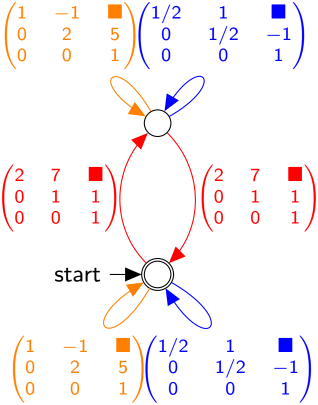

$\boxed{\Longleftrightarrow}$

The same automaton over a **metabelian** group

(A group G is called **metabelian** if it has a normal subgroup $H \trianglelefteq G$, such that both H and G/H are abelian)

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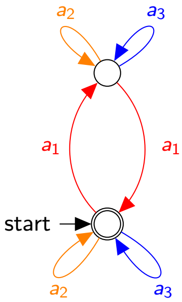

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Step 4: generalize Identity Problem in metabelian groups

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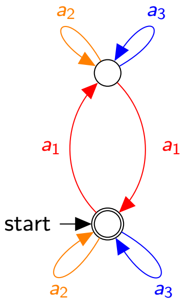

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Theorem (Dong 2024)

The Identity Problem in metabelian groups is decidable.

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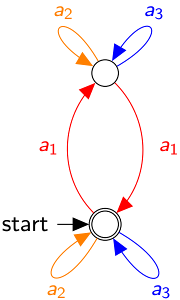

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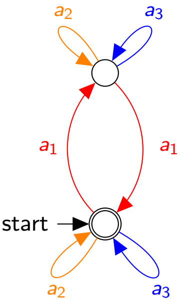

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Theorem (generalization of Dong)

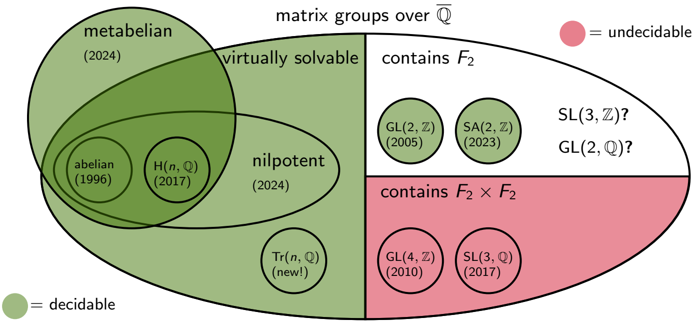
The above problem is decidable (when initial state = final state).

Conclusion and future work

To summarize our result:

Theorem (Bodart, Dong 2025)

A virtually solvable matrix group over $\overline{\mathbb{Q}}$ has decidable Identity Problem.



Decidability map of the Identity Problem

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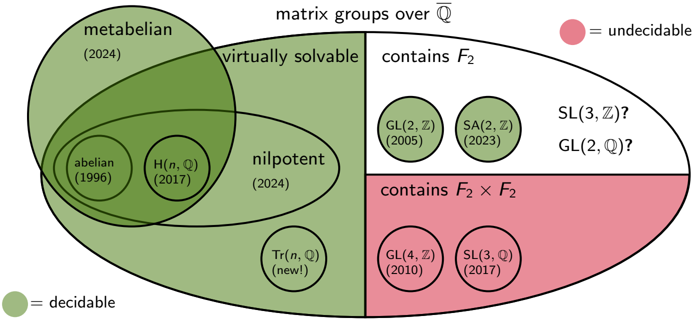

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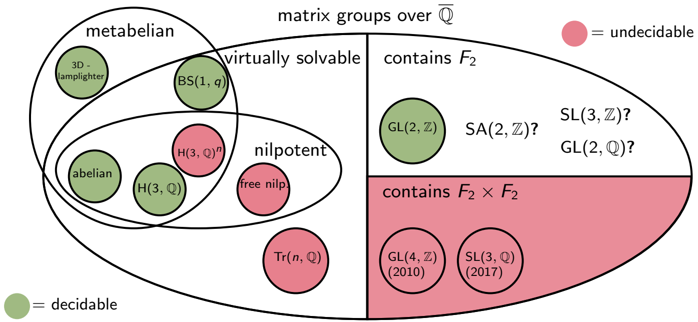

Decidability map of Semigroup Membership