Linear equations with monomial constraints and decision problems in abelian-by-cyclic groups

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Part I

Linear equations with extra constraints

Given a system of linear equations:

$$\begin{cases} 4y_1 + 12y_2 + 2z_1 + 3z_2 = 7 \\ 5y_1 + 17y_2 + 9z_1 + 8z_2 = 4 \\ 2y_1 + 21y_2 + 3z_1 + 4z_2 = 6 \end{cases}$$

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Decidable by a fragment of **Büchi arithmetic**.

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Undecidable (D. 2024, **first result of this talk**).

Linear equations with monomial constraints

Theorem (D. 2024)

It is undecidable whether, given system of linear equations $a_{i1}y_1 + \cdots + a_{in}y_n + b_{i1}z_1 + \cdots + b_{im}z_m = c_i$, $i = 1, \ldots, k$, *there are solutions* $y_1, \ldots, y_n \in \mathbb{Z}[X]$, $z_1, \ldots, z_n \in X^{\mathbb{N}}$.

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Lemma (Expressing squares)

Suppose $n_1, n_2, n_3 \in \mathbb{N}$. We have

$$(X-1)^3 \mid X^{n_1} + X^{n_2}(1-X) + X^{n_3} + (X-3)$$

if and only if $n_2 = n_1^2$, $n_3 = -n_1$.

Idea: $(X-1)^3 \mid f$ if and only if $f(1) = f'(1) = f''(1) = 0$.

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Note that "$(X-1)^3 \mid f$" can be expressed as "$(X-1)^3 y = f$". Therefore we can express "squaring" of integers.

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Lemma (Expressing sums)

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if and only if $n_3 = n_1 + n_2$.

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Therefore linear equations with monomial constraints can express "summing" of integers.

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Linear equations with monomial constraints can express "squaring" and "summing" of integers. Note that "product" of integers and be expressed by "squaring" and "summing": $xy = \Big((x+y)^2 - x^2 - y^2\Big)/2$.

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Therefore linear equations with monomial constraints can express any polynomial equation over integers, therefore undecidable.

Q.E.D

Part II

Let G be an (infinite) group. Let $g_1, g_2, g_3, g_4 \in G$. Consider the following problems:

Applications to group theory: Equations over groups

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Definition (Equations over groups)

Solving a system of equations over a group G is the following problem. Let $\mathcal{X} = \{x_1, \ldots, x_n\}$ be an alphabet and $\mathcal{X}^{-1} := \{x_1^{-1}, \ldots, x_n^{-1}\}$.
Input: words $w_1, \ldots, w_t$ over the alphabet $\mathcal{X} \cup \mathcal{X}^{-1} \cup G$.
Question: whether there exist $h_1, \ldots, h_n \in G$, such that each w_i evaluates to the neutral element when we replace each x_j with h_j.

Equations over groups: classic results

Theorem (Makanin 1977)

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Theorem (Romankov 1979, Duchin, Liang, Shapiro 2015)

Solving a system of equations over **free metabelian groups** *and over* **free nilpotent groups** *is undecidable.*

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Open problem: find an example of group G, **neither hyperbolic nor virtually abelian**, such that solving a system of equations is decidable.

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Most hopeful candidates: **abelian-by-cyclic** groups.

Definition

A group G is called **abelian-by-cyclic** if it admits an normal subgroup A, such that A is abelian and $G/A \cong \mathbb{Z}$.

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