Submonoid Membership in n-dimensional lamplighter groups and S-unit equations

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ICALP 2025

[1] partially supported by ERC Advanced Grant 101097307.

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Definition (Rational Subset Membership Problem)

Input: A rational subset $R \subseteq G$ and an element $t \in G$.
Question: Does t belong to the set R?

$$\begin{aligned} \text{rational subset} &= \text{set defined by a rational expression in } G \\ &= \text{set recognized by a finite state automaton over } G \end{aligned}$$

Relation between the three membership problems

Example of a rational subset: $\{g_1\}^*\{g_2\}^* = \{g_1^n g_2^m \mid n, m \in \mathbb{N}\}$.

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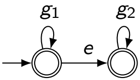

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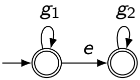

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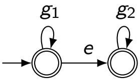

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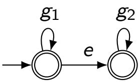

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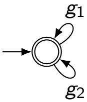

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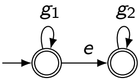

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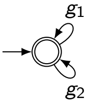

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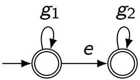

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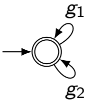

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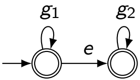

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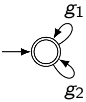

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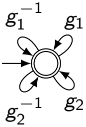

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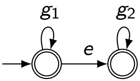

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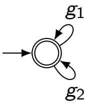

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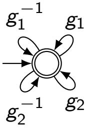

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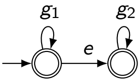

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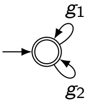

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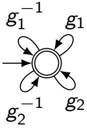

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- All three problems are decidable in $\mathbb{Z}^d$ and the free group F_2. (Benois '69)

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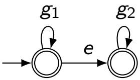

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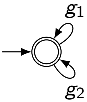

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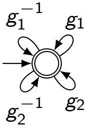

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- All three problems are undecidable in $F_2 \times F_2$. (Mikhailova '66)

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Open Problem

Is decidability of **Submonoid Membership** stable under taking finite index subgroups?

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Decidability of **Submonoid Membership** *is <u>not</u> stable under taking finite index subgroup.*

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the title of this talk:

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$$\left\{ \begin{pmatrix} X_1^{z_1} X_2^{z_2} & f \\ 0 & 1 \end{pmatrix} \;\middle|\; z_1, z_2 \in \mathbb{Z},\ \underbrace{f \in \mathbb{F}_2[X_1^{\pm}, X_2^{\pm}]}_{\text{Laurent polynomial over } \mathbb{F}_2} \right\}.$$

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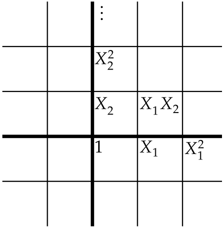

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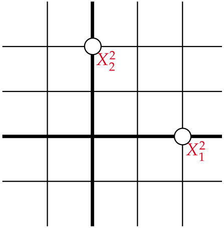

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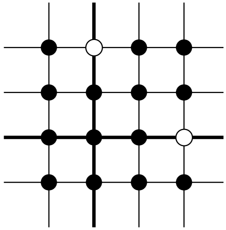

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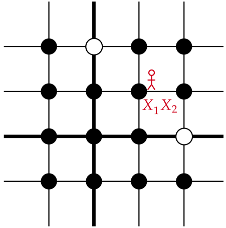

Multiplication in the 2-dimensional lamplighter group

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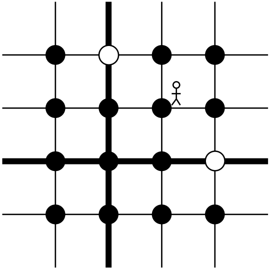
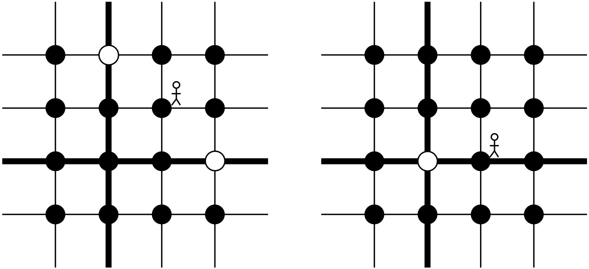


Multiplication in the 2-dimensional lamplighter group

$$\begin{pmatrix} X_1X_2 & X_1^2+X_2^2 \\ 1 & 1 \end{pmatrix} \quad \times \quad \begin{pmatrix} X_1 & 1 \\ 1 & 1 \end{pmatrix}$$



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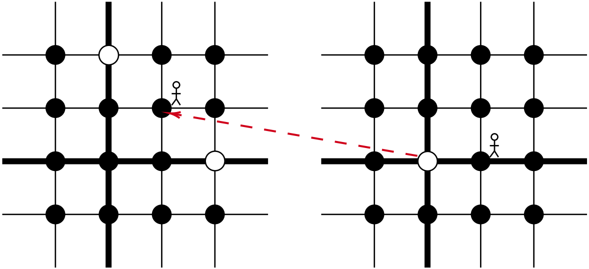

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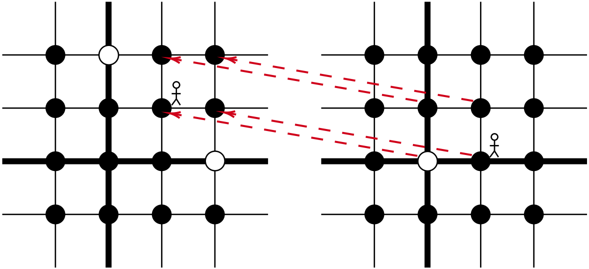

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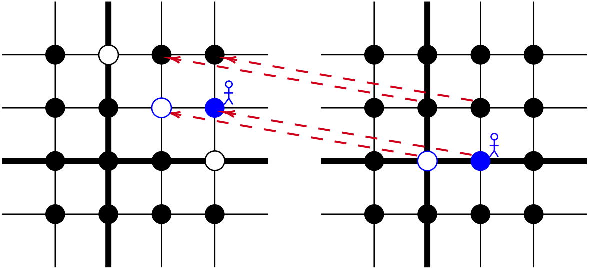
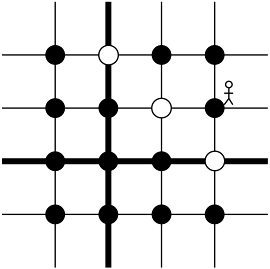
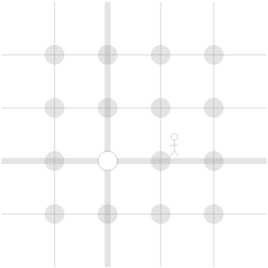


align the origin of 2nd graph to the person in 1st graph,
move the 2nd graph onto the 1st graph,
and do pointwise addition on all lamps.

Multiplication in the 2-dimensional lamplighter group

$$= \begin{pmatrix} X_1^2 X_2 & X_1^2 + X_1 X_2 + X_2^2 \\ 1 & 1 \end{pmatrix} \qquad \begin{pmatrix} X_1 & 1 \\ 1 & 1 \end{pmatrix}$$





align the origin of 2nd graph to the person in 1st graph,
move the 2nd graph onto the 1st graph,
and do pointwise addition on all lamps.

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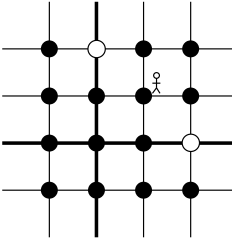
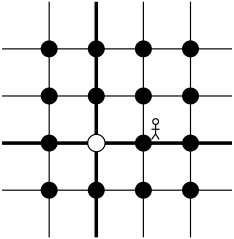


$\times$



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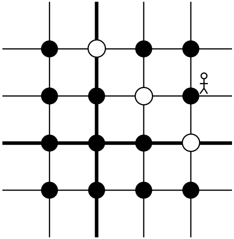

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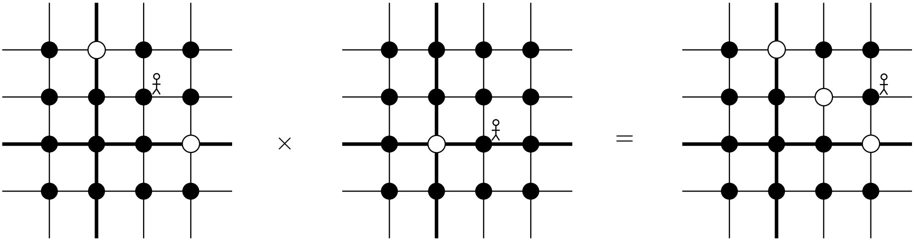

Modify the 2-dimensional lamplighter group by "wiring" all lamps in the same column together.

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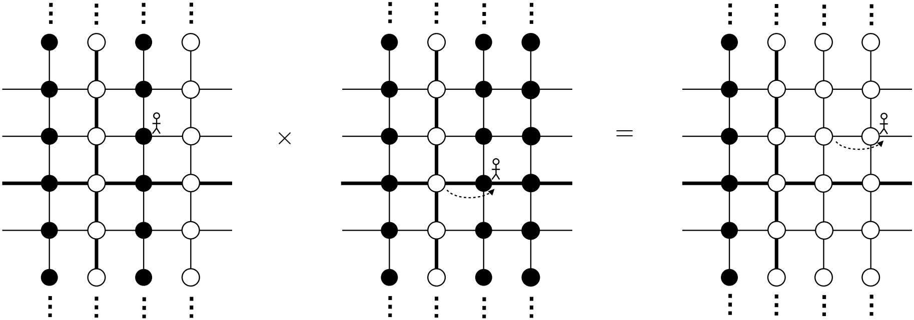

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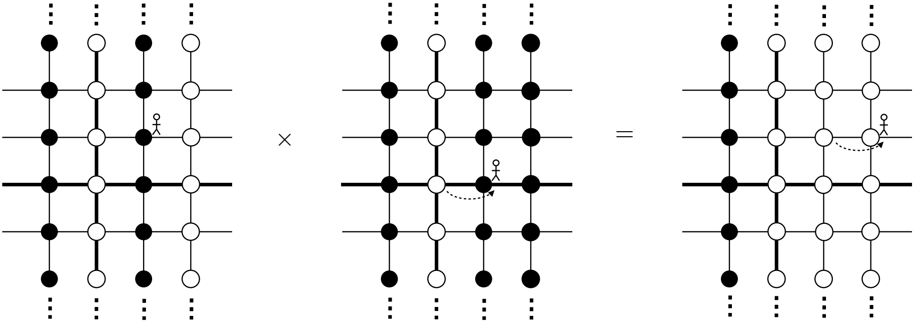

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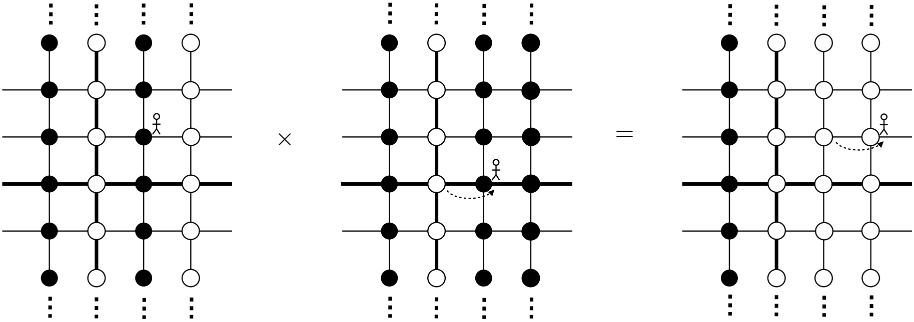

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different "wiring" patterns $\approx$ quotient by different ideals

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Deciding Submonoid Membership in lamplighter groups

Submonoid Membership in n-dimensional lamplighter groups and S-unit equations

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An **S-unit equation** is a linear equation

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Theorem (Adamczewski and Bell 2012, Derksen and Masser 2012)

The solution set of an S-unit equation over a field of characteristic $p > 0$ is effectively p-automatic.