Semigroup intersection problems in the Heisenberg groups

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An old decidability problem

Post Correspondence Problem (1946). The following is undecidable:

Input: A set of pairs of words $\mathcal{G} = \{(v_1, w_1), \ldots, (v_K, w_K)\}$ over the alphabet $\{a, b\}$.

Output: Can we find a sequence (possibly with repetition) $(v_{i_1}, w_{i_1}), (v_{i_2}, w_{i_2}), \ldots, (v_{i_m}, w_{i_m}) \in \mathcal{G}$ such that the concatenations:

$$v_{i_1} v_{i_2} \cdots v_{i_m} = w_{i_1} w_{i_2} \cdots w_{i_m}?$$

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Markov (1940s): is the following decidable?

Input: Two sets of matrices $\mathcal{G} = \{A_1, \ldots, A_M\}$, $\mathcal{H} = \{B_1, \ldots, B_N\}$.

Output: Can we find there two sequences $A_{i_1}, A_{i_2}, \ldots, A_{i_m} \in \mathcal{G}$ and $B_{j_1}, B_{j_2}, \ldots, B_{j_n} \in \mathcal{H}$, such that

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Reformulation as Semigroup Intersection Emptiness

Intersection Emptiness Problem: is the following decidable?

Input: m finite sets of elements $\mathcal{G}_1, \mathcal{G}_2, \ldots, \mathcal{G}_m$ in an ambient (semi)group S.

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Undecidability results (for $m = 2$):

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Open problem: decidability for $S = \mathbb{Z}^{2\times 2}$.

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Example ($S = \mathbb{Z}^d$)

Let $\mathcal{G}_1 = \{a_1, \dots, a_n\}, \mathcal{G}_2 = \{b_1, \dots, b_m\} \subset \mathbb{Z}^d$. Then $\langle \mathcal{G}_1 \rangle \cap \langle \mathcal{G}_2 \rangle \neq \emptyset$ iff

$$\ell_1 a_1 + \cdots + \ell_n a_n = k_1 b_1 + \cdots + k_m b_m$$

for some $(\ell_1, \dots, \ell_n) \in \mathbb{N}^n \setminus \{0\}, (k_1, \dots, k_m) \in \mathbb{N}^m \setminus \{0\}$.

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What about simple non-abelian groups?

Our result:

- **Decidable** when S is *2-step nilpotent*.
- **PTIME** when S is *torsion-free 2-step nilpotent*.
  For example, when $S = \mathrm{H}_n(\mathbb{K})^m$.

2-step nilpotent groups

Definition (2-step nilpotent groups)

A group S is called *2-step nilpotent* if the its quotient by its center is abelian.

Motivation: These are the simplest non-abelian groups!

Example (Heisenberg Groups)

Let $\mathbb{K}$ be a number field.
The *Heisenberg Group* $\mathrm{H}_3(\mathbb{K})$ is 2-step nilpotent:

$$\mathrm{H}_3(\mathbb{K}) := \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \middle| * \in \mathbb{K} \right\}.$$

Lie algebra

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Definition ($\mathfrak{u}(3)$)

Define $\mathfrak{u}(3)$ to be the $\mathbb{K}$-linear space of 3 by 3 upper triangular matrices with *zeros* on the diagonal. It is naturally a $\mathbb{Q}$-linear space.

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$$\log : \mathrm{H}_3(\mathbb{K}) \to \mathfrak{u}(3), \quad A \mapsto (A - I) - \frac{1}{2}(A - I)^2$$

and

$$\exp : \mathfrak{u}(3) \to \mathrm{H}_3(\mathbb{K}), \quad X \mapsto I + X + \frac{1}{2}X^2$$

are inverse of one another. In particular, $\log I = 0$ and $\exp(0) = I$.

Baker-Campbell-Hausdorff formula

$$\log(AB) = \log A + \log B + \frac{1}{2}[\log A, \log B].$$

where $[X, Y] := XY - YX$ is the **Lie bracket**.

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$$\log(C_1 C_2 \cdots C_k) = \sum_{i=1}^{k} \log C_i + \frac{1}{2} \sum_{1 \leq i < j \leq k} [\log C_i, \log C_j].$$

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Given a set $\mathcal{G} = \{A_1, \ldots, A_M\}$ and a word $w = A_{i_1} A_{i_2} \cdots A_{i_m} \in \mathcal{G}^*$. For $i = 1, \ldots, M$, denote

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- **"Inversion Number"** $\delta_{ij}(w) := \delta_{ij}^{+}(w) - \delta_{ij}^{-}(w)$.

Example

If $w = A_1 A_1 A_2 A_1$ then

$$\ell_1(w) = 3, \ell_2(w) = 1, \delta_{AB}(w) = 2 - 1 = 1.$$

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Then (1) becomes

$$\log w = \sum_{i=1}^{M} \ell_i(w) \cdot \log A_i + \frac{1}{2} \sum_{1 \le i < j \le M} \delta_{ij}(w) \cdot [\log A_i, \log A_j].$$

Back to Semigroup Intersection

Let $\mathcal{G} = \{A_1, \ldots, A_M\}, \mathcal{G}' = \{A'_1, \ldots, A'_N\}$.

$\langle \mathcal{G} \rangle \cap \langle \mathcal{G}' \rangle \neq \emptyset \iff$
we can find words $w = A_{i_1} A_{i_2} \cdots A_{i_m} \in \mathcal{G}^*, w' = A'_{j_1} A'_{j_2} \cdots A_{j_m} \in \mathcal{G}'^*$
that $\log w = \log w'$.

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This is equivalent to solving the *word equation*

$$\sum_{i=1}^{M} \ell_i(w) \cdot \log A_i + \frac{1}{2} \sum_{1 \leq i < j \leq M} \delta_{ij}(w) \cdot [\log A_i, \log A_j]$$

$$= \sum_{i=1}^{N} \ell'_i(w') \cdot \log A'_i + \frac{1}{2} \sum_{1 \leq i < j \leq N} \delta'_{ij}(w') \cdot [\log A'_i, \log A'_j].$$

for $w \in \mathcal{G}^*, w' \in \mathcal{G}'^*$.

From word equation to linear equation

We want to find $w \in \mathcal{G}^*, w' \in \mathcal{G}'^*$ that satisfy

$$\sum_{i=1}^{M} \ell_i(w) \cdot \log A_i + \frac{1}{2} \sum_{1 \leq i < j \leq M} \delta_{ij}(w) \cdot [\log A_i, \log A_j]$$
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Proposition

Equation (2) has solution $w \in \mathcal{G}^*, w' \in \mathcal{G}'^*$ if and only if the following relaxed equation has solution $s_i, s'_i \in \mathbb{N}, d_{ij}, d'_{ij} \in \mathbb{Z}$.

$$\sum_{i=1}^{M} s_i \cdot \log A_i + \frac{1}{2} \sum_{1 \leq i < j \leq M} d_{ij} \cdot [\log A_i, \log A_j]$$
$$= \sum_{i=1}^{N} s'_i \cdot \log A'_i + \frac{1}{2} \sum_{1 \leq i < j \leq N} d'_{ij} \cdot [\log A'_i, \log A'_j]. \quad (3)$$

Finding the word w

Main difficulty: given $s_i \in \mathbb{N}, d_{ij} \in \mathbb{Z}$, how to find $w \in \mathcal{G}^*$ with $\ell_i(w) = s_i, \delta_{ij}(w) = d_{ij}$ for all i, j?

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Example for words over two letters:

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Let $\mathcal{G} = \{A, B\}$, can we find $w \in \mathcal{G}^*$ with

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- When $w = AABBBAB$, then $\delta_{AB}(w) = 6$.

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- When $w = AABBBAB$, then $\delta_{AB}(w) = 6$.
- When $w = AABBBBA$, then $\delta_{AB}(w) = 4$. **Found it!**

2 letter case

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No! Because $14 > 3 \times 4$.

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Example

Let $\mathcal{G} = \{A, B\}$, we can find $w \in \mathcal{G}^*$ with

$$\ell_A(w) = 3n, \ell_B(w) = 4n, \delta_{AB}(w) = 14n.$$

for some large n.

Because $|14n| < |3n \times 4n|$ when n large.

In general:
We can find words w with $\ell_i(w) = s_i n, \ell_j(w) = s_j n$ such that $\delta_{ij}(w)$ covers all possible values between $-s_i s_j n^2$ and $s_i s_j n^2$. (Subject to oddity constraints.)

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Linear equation is homogeneous: it suffices to find $w \in \mathcal{G}^*$ and $n \in \mathbb{N}$ such that $\ell_i(w) = s_i n, \delta_{ij}(w) = d_{ij} n$ for all i, j.

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By taking large enough n, we can suppose

$$-(s_i n)(s_j n) \leq d_{ij} n \leq (s_i n)(s_j n).$$

Problem solved!

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Linear equation is homogeneous: it suffices to find $w \in \mathcal{G}^*$ and $n \in \mathbb{N}$ such that $\ell_i(w) = s_i n, \delta_{ij}(w) = d_{ij} n$ for all i, j.

By taking large enough n, we can suppose

$$-(s_i n)(s_j n) \leq d_{ij} n \leq (s_i n)(s_j n).$$

Problem solved!

The case with more letters is more complicated, but the idea is similar.

From Semigroup Intersection to linear equation

Proposition

We have $\langle \mathcal{G} \rangle \cap \langle \mathcal{G}' \rangle \neq \emptyset$ if and only if the following relaxed equation has non-zero solution $\ell_i, \ell'_i \in \mathbb{N}, \delta_{ij}, \delta'_{ij} \in \mathbb{Z}$.

$$\sum_{i=1}^{M} \ell_i \cdot \log A_i + \frac{1}{2} \sum_{1 \leq i < j \leq M} \delta_{ij} \cdot [\log A_i, \log A_j] = \sum_{i=1}^{N} \ell'_i \cdot \log A'_i + \frac{1}{2} \sum_{1 \leq i < j \leq N} \delta'_{ij} \cdot [\log A'_i, \log A'_j]. \quad (4)$$

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This is a *homogeneous* linear Diophantine equation. So it is solvable in **PTIME**.

Generalizations

Theorem (Semigroup Intersection Emptiness)

Let S be a 2-step nilpotent group. Given finite sets $\mathcal{G}_1, \ldots, \mathcal{G}_m \subset S$, it is decidable whether

$$\langle \mathcal{G}_1 \rangle \cap \cdots \cap \langle \mathcal{G}_m \rangle = \emptyset.$$

Furthermore the decision procedure is PTIME if S is torsion-free.

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Generalization using similar idea:

Theorem (Orbit Intersection)

Given finite sets $\mathcal{G}, \mathcal{H} \subset \mathsf{H}_3(\mathbb{Q})$ and elements $T, S \in \mathsf{H}_3(\mathbb{Q})$, it is decidable whether

$$T \cdot \langle \mathcal{G} \rangle \cap S \cdot \langle \mathcal{H} \rangle = \emptyset.$$

However, Orbit Intersection in $\mathsf{H}_3(\mathbb{Q})^{10000}$ is undecidable.

Open problem: Semigroup Intersection Emptiness for higher-order nilpotent groups?

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Theorem (Tit's alternative for semigroups)

Every matrix group G is either virtually nilpotent or it contains a free monoid over two generators.

If G contains a free monoid over two generators, then Semigroup Intersection is undecidable in G^2.

If G is nilpotent, then G^2 is also nilpotent.