# The Identity Problem in the special affine group of $\mathbb{Z}^2$

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June 2023

# An old decidability problem

Markov (1940s): is (semigroup) **Membership Problem** decidable?

**Input:** Set of square matrices $\mathcal{G} = \{A_1, \ldots, A_K\}$, target matrix $T$.
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Bell, Potapov (2000s) : undecidable in $\mathsf{SL}(4, \mathbb{Z})$.

**Known results.**

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* There exist groups where Membership Problem is undecidable but Identity Problem is decidable.

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Group law: $(A, \boldsymbol{a})(B, \boldsymbol{b}) = (AB, A\boldsymbol{b} + \boldsymbol{a})$.

# Main result

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If $A_s^{-1} \notin \langle A_1, \ldots, A_K \rangle$, then

$$(A_i, \boldsymbol{a}_i) \cdots (A_s, \boldsymbol{a}_s) \cdots (A_{i'}, \boldsymbol{a}_{i'}) \neq (I, \boldsymbol{0}).$$

So we can delete $(A_s, \boldsymbol{a}_s)$ from $\mathcal{G}$.

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We can perform Step 1 iteratively, until $A_s^{-1} \in \langle A_1, \ldots, A_K \rangle$ for all $s$.
So the semigroup $\langle A_1, \ldots, A_K \rangle$ becomes a group.

# Step 2: dichotomy on the matrix part

$$\mathcal{G} = \{(A_1, \boldsymbol{a}_1), \ldots, (A_K, \boldsymbol{a}_K)\}.$$

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**Theorem (Tits alternative)**

*Let $H$ be a subgroup of $\mathrm{SL}(n, \mathbb{Z})$. Then*

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Furthermore, the two cases can be distinguished in PTIME (Beals 1999).

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**Proof idea:**

Since $\langle A_1, \ldots, A_K \rangle$ is a group containing $A$ and $B$, it also contains some $Y$ such that $AYB = I$. In particular $\langle \mathcal{G} \rangle$ contains some elements $(A, \boldsymbol{a}), (Y, \boldsymbol{y}), (B, \boldsymbol{b})$ such that $(A, \boldsymbol{a})(Y, \boldsymbol{y})(B, \boldsymbol{b}) = (I, \boldsymbol{x})$ for some $\boldsymbol{x} \in \mathbb{Z}^2$.

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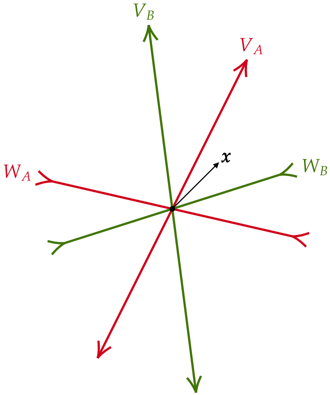
But $\boldsymbol{x}$ might not be $\mathbf{0}$. So we need to "pump" the word $(A, \boldsymbol{a})(Y, \boldsymbol{y})(B, \boldsymbol{b})$ to change $\boldsymbol{x}$.

# Step 3: getting 0 in the vector part

Suppose $A$ has eigenspaces $V_A, W_A$, and $B$ has eigenspaces $V_B, W_B$, since $A$ and $B$ are not simultaneously triangularizable, we can suppose $V_A, W_A, V_B, W_B$ pairwise distinct.



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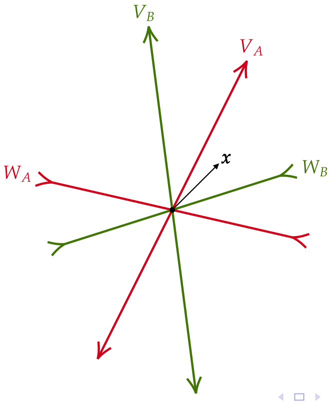

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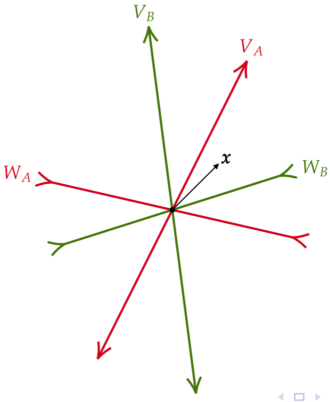

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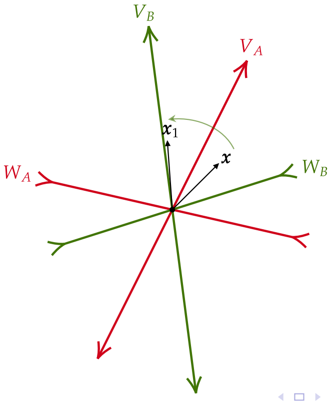
When $m \to \infty$, the vector $\boldsymbol{x}_1$ tends towards $V_B$.



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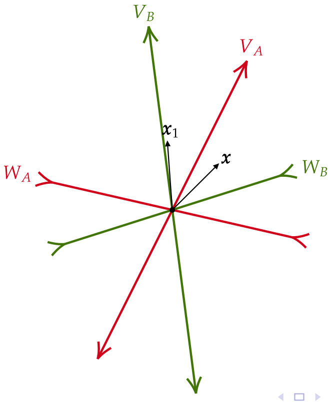

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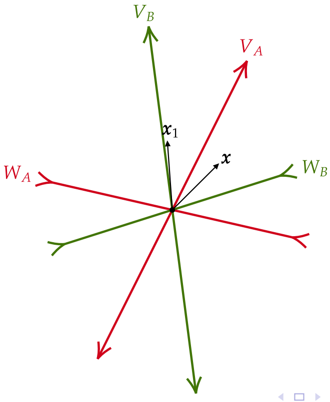

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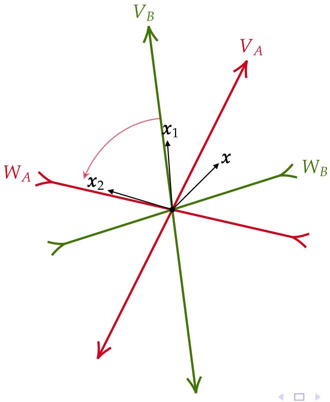
When $m \to \infty$, the vector $\boldsymbol{x}_2$ tends towards $W_A$.



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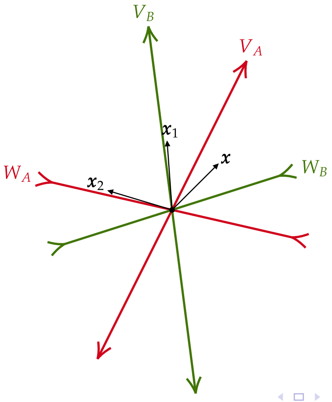

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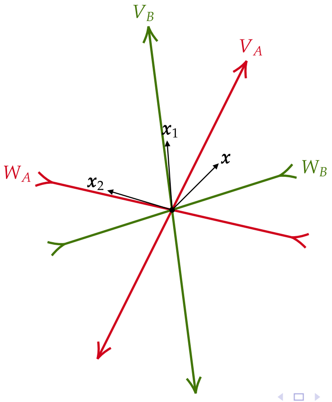
$$(A, \boldsymbol{a})(Y_2, \boldsymbol{y}_2)(B, \boldsymbol{b})^m \underbrace{(Y_2, \boldsymbol{y}_2)(A, \boldsymbol{a}) \cdots (Y_2, \boldsymbol{y}_2)(A, \boldsymbol{a})}_{m \text{ times}}(B, \boldsymbol{b}) = (I, \boldsymbol{x}_3) \in \langle \mathcal{G} \rangle$$



## Step 3: getting 0 in the vector part

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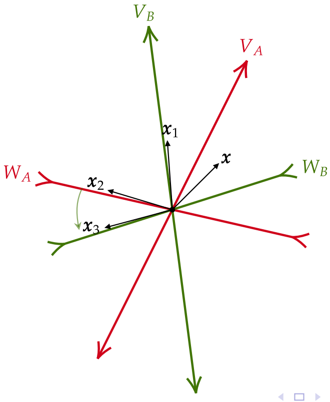
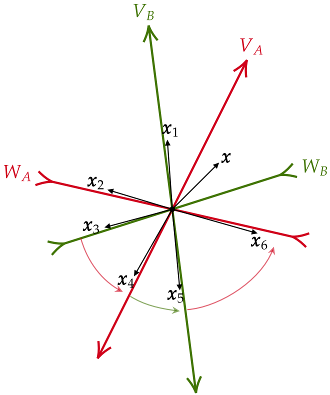
When $m \to \infty$, the vector $\boldsymbol{x}_3$ tends towards $W_B$.



# Step 3: getting 0 in the vector part

Continue like this, we obtain $(I, \boldsymbol{x}_1), (I, \boldsymbol{x}_2), \dots, (I, \boldsymbol{x}_6) \in \langle \mathcal{G} \rangle$.



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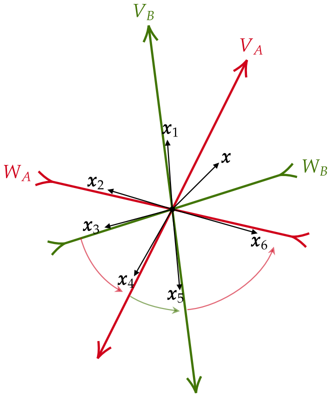
There exist **positive** integers $n_1, \ldots, n_6$ such that $n_1\boldsymbol{x}_1 + \cdots n_6\boldsymbol{x}_6 = \boldsymbol{0}$.



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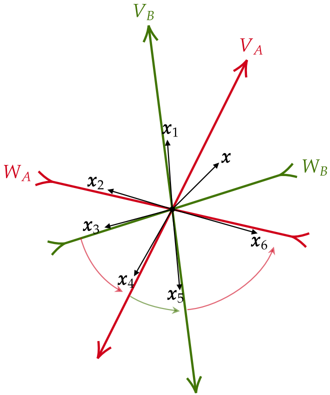
Therefore

$$(I, \mathbf{0}) = (I, \boldsymbol{x}_1)^{n_1} (I, \boldsymbol{x}_2)^{n_2} \cdots (I, \boldsymbol{x}_6)^{n_6} \in \langle \mathcal{G} \rangle.$$



# Step 4: second case of dichotomy

$$\mathcal{G} = \{(A_1, \boldsymbol{a}_1), \ldots, (A_K, \boldsymbol{a}_K)\}.$$

We have proved the first case of the dichotomy:

**Proposition**

*Suppose* $\langle A_1, \ldots, A_K \rangle \leq \mathrm{SL}(2, \mathbb{Z})$ *is a group containing two matrices* $A, B$ *that are not simultaneously triangularizable, then* $(I, \mathbf{0}) \in \langle \mathcal{G} \rangle$.

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We can also prove the second case of the dichotomy:

**Proposition**

*Suppose* $\langle A_1, \ldots, A_K \rangle$ *is a group containing a finite-index subgroup that is isomorphic to* $\mathbb{Z}$ *or* $\{I\}$*. Then it is decidable in PTIME whether or not* $(I, \mathbf{0}) \in \langle \mathcal{G} \rangle$.

# Identity Problem in $\mathsf{SA}(2, \mathbb{Z})$: recap

Let $\mathcal{G} = \{(A_1, \boldsymbol{a}_1), \ldots, (A_K, \boldsymbol{a}_K)\}$, we want to decide if $(I, \mathbf{0}) \in \langle \mathcal{G} \rangle$.

We defined $H = \langle A_1, \ldots, A_K \rangle$.

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Step 1: narrowing down to the case where $H$ is a group is done in NP.
Step 2: distinguishing dichotomy is in PTIME.
Step 3: first dichotomy case, always true.
Step 4: second dichotomy case, complexity is PTIME.

In total, complexity is in NP.

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**Open Problem**

*Is Membership Problem in* $\mathrm{SA}(2, \mathbb{Z})$ *decidable?*