The Identity Problem in $\mathbb{Z} \wr \mathbb{Z}$ is decidable

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An old decidability problem

Markov (1940s): is (semigroup) **Membership Problem** decidable?

Input: Set of square matrices $\mathcal{G} = \{A_1, \ldots, A_K\}$, target matrix T.
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the Identity Problem and the Membership Problem

Known results.

$\mathrm{SL}(n, \mathbb{Z})$: the group of $n \times n$ integer matrices of determinant one.

Group	Membership Prob. $T \in \langle \mathcal{G} \rangle$?	Identity Prob. $I \in \langle \mathcal{G} \rangle$?
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"$\wr$" is the **wreath product**, important in decomposing finite automata (Krohn-Rhodes theorem), constructing symmetry groups, etc.

What is $\mathbb{Z} \wr \mathbb{Z}$? Its elements

First interpretation: as the set of matrices

$$\mathbb{Z} \wr \mathbb{Z} := \left\{ \begin{pmatrix} X^{b} & y \\ 0 & 1 \end{pmatrix} \;\middle|\; y \in \mathbb{Z}[X, X^{-1}], b \in \mathbb{Z} \right\}.$$

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Second interpretation: as a “cheap” Turing machine.

Each element of $\mathbb{Z} \wr \mathbb{Z}$ is a configuration

$\cdots$	a_{-2}	a_{-1}	a_0	a_1	a_2	$\cdots$
			$\uparrow$		$\downarrow$	

where $\cdots a_{-2}, a_{-1}, a_0, a_1, a_2 \cdots \in \mathbb{Z}$.

The arrow $\uparrow$ is placed at 0. The arrow $\downarrow$ is placed at some integer b.

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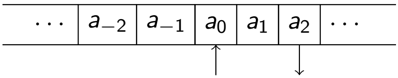

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Let $y := \cdots + a_{-1}X^{-1} + a_0 + a_1 X + a_2 X^2 + \cdots$, then the configuration represents the element $\begin{pmatrix} X^{b} & y \\ 0 & 1 \end{pmatrix}$.

What is $\mathbb{Z} \wr \mathbb{Z}$? the group law

First interpretation: as matrix multiplication

$$\begin{pmatrix} X & 2+2X \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X^2 & 3+3X+3X^2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} X^3 & 2+5X+3X^2+3X^3 \\ 0 & 1 \end{pmatrix}$$

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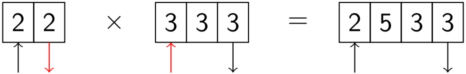
Second interpretation: align $\downarrow$ of first element and $\uparrow$ of second element, then add all cells.



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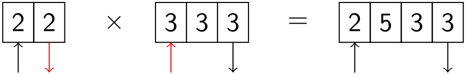
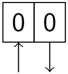
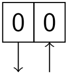
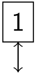


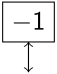
Each element is an "instruction".

$\begin{pmatrix} X & 0 \\ 0 & 1 \end{pmatrix} =$ 

$=$ "move right", $\begin{pmatrix} X^{-1} & 0 \\ 0 & 1 \end{pmatrix} =$ 

$=$ "move left".

$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} =$ 

$=$ "increase counter", $\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} =$ 

$=$ "decrease counter".

What is $\mathbb{Z} \wr \mathbb{Z}$? Membership and Identity Problem

Given a finite set of elements

$$\mathcal{G} = \left\{ \begin{pmatrix} X^{a_1} & y_1 \\ 0 & 1 \end{pmatrix}, \cdots, \begin{pmatrix} X^{a_K} & y_K \\ 0 & 1 \end{pmatrix} \right\}.$$

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Theorem (Lohrey, Steinberg, Zetzsche 2013)

Membership Problem in $\mathbb{Z} \wr \mathbb{Z}$ is undecidable.

Theorem

The Identity Problem in $\mathbb{Z} \wr \mathbb{Z}$ is decidable.

The Identity Problem in $\mathbb{Z} \wr \mathbb{Z}$: a special case

As an example, consider a set of *three* elements

$$\mathcal{G} = \left\{ \begin{pmatrix} X & y_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & y_2 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} X^{-1} & y_3 \\ 0 & 1 \end{pmatrix} \right\},$$

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Given three instructions A_1, A_2, A_3, where

- A_1 is "move right one step, change nearby counters according to y_1",
- A_2 is "don't move, change nearby counters according to y_2",
- A_3 is "move left one step, change nearby counters according to y_3".

Identity Problem: can we reach the initial configuration using A_1, A_2, A_3?

The special case: path of the run

Consider a sequence that might reach I:

$$\begin{pmatrix} X & y_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X & y_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X^{-1} & y_3 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X^{-1} & y_3 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$$

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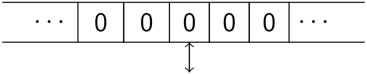

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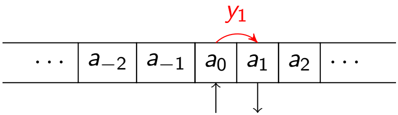

Where $y_1 = \cdots + a_{-2}X^{-2} + a_{-1}X^{-1} + a_0 + a_1 X + a_2 X^2 + \cdots$.

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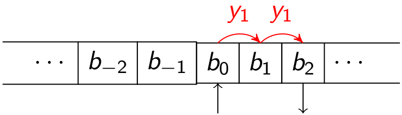

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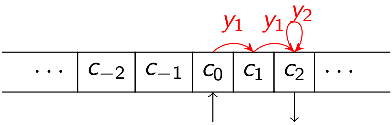

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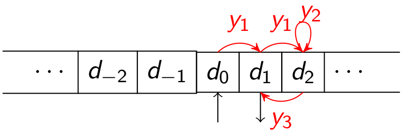

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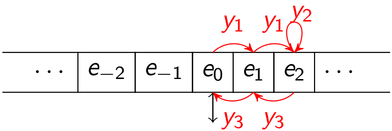

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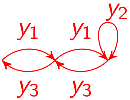
The corresponding run: 

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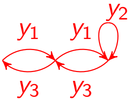
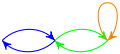
The corresponding run: 

. Decompose into tiles: 

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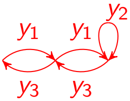
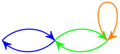
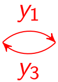
The corresponding run: 

. Decompose into tiles: 

It can be decomposed into two tiles of 

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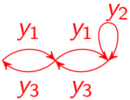
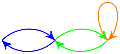
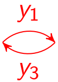
The corresponding run: 

. Decompose into tiles: 

It can be decomposed into two tiles of 

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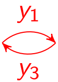
The **effect** of tile 

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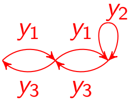
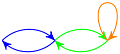
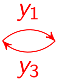
The corresponding run: 

. Decompose into tiles: 

It can be decomposed into two tiles of 

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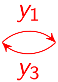
The **effect** of tile 

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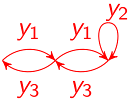
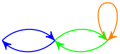
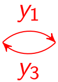
The corresponding run: 

. Decompose into tiles: 

It can be decomposed into two tiles of 

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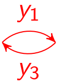
The **effect** of tile 

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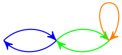
Total effect: $* = (1 + X)p_{13} + X^2 p_2$.

Runs vs Polynomials

The effect of each (cyclic) run is described by a linear combination

$$* = f_{13} p_{13} + f_2 p_2.$$

For example:



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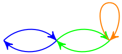
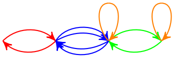
 $\longrightarrow$ $* = (1 + X)p_{13} + X^2 p_2.$

 $\longrightarrow$ $* = (X^{-1} + 2 + X)p_{13} + (X + X^2)p_2.$

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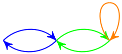

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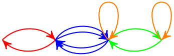

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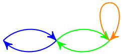

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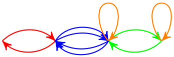

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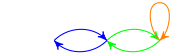

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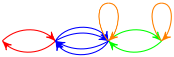

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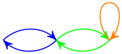

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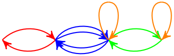

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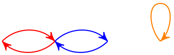

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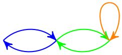

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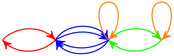

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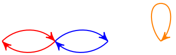

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Not connected: **degree** of $f_2 = X^2$ is too big compared to $f_{13} = X^{-1} + 1$

Identity Problem vs linear equations over $\mathbb{N}[X^{\pm}]$

Proposition

There exists a run whose effect is $* = 0$*, if and only if the equation* $0 = f_{13}p_{13} + f_2p_2$ *admits non-zero solutions* $f_{13}, f_2 \in \mathbb{N}[X^{\pm}]^*$ *satisfying* **"degree constraints"**.

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Identity Problem $\iff$ existence of run with effect $* = 0$.

This not only works for the current example (where we can only move one cell). In general:

Theorem

The Identity Problem in $\mathbb{Z} \wr \mathbb{Z}$ reduces to solving a **system** *of* **homogeneous** *linear equations over $\mathbb{N}[X^{\pm}]^*$, with additional "degree constraints".*

Local-global principle

Does $0 = f_{13} \cdot (2X^2 - 1) + f_2 \cdot (X + 2)$ have solution $f_{13}, f_2 \in \mathbb{N}[X^{\pm}]^*$?

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Theorem (Generalization of Einsiedler, Mouat, Tuncel (2003))

Let $\mathcal{M}$ be an $\mathbb{Z}[X^{\pm}]$-submodule of $\mathbb{Z}[X^{\pm}]^K$. Then there exists $\boldsymbol{f} \in \mathcal{M} \cap (\mathbb{N}[X^{\pm}]^*)^K$ satisfying "degree constraints" if and only if the following are satisfied:

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Therefore, instead of searching for solutions, we search for "certificates". This can be done using the first order theory of $\mathbb{R}$. Decidable (Tarski).

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Theorem

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