Termination of linear loops under commutative updates

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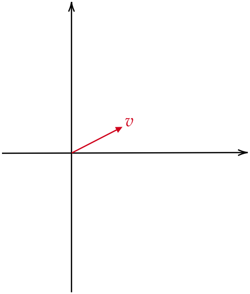

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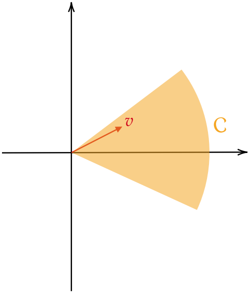

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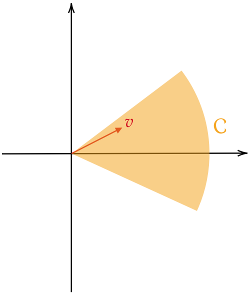

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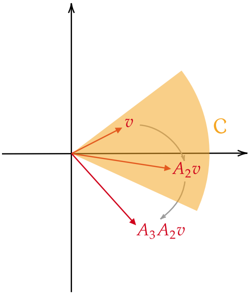

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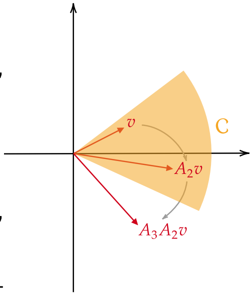

As a decidability problem:

Input: $v \in \mathbb{R}^d$, polyhedral cone $\mathcal{C} \subseteq \mathbb{R}^d$, matrices $S = \{A_1, \ldots, A_n\}$.

Output: does there exist $M \in \langle S \rangle$ such that $Mv \notin \mathcal{C}$?
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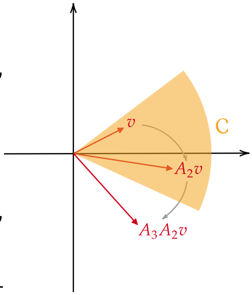

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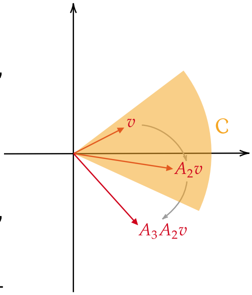

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Special case where $\mathrm{card}(S) = 1$: open, subsumes hard problems in Diophantine approximation. (Worrell, Ouaknine 2014)

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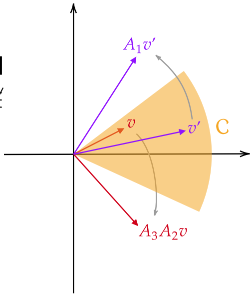

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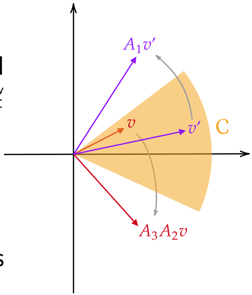

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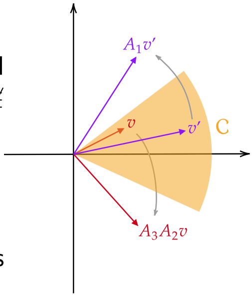

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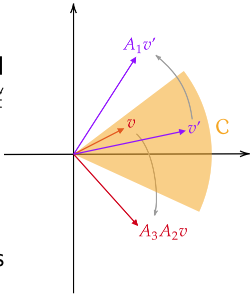

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Motivation: verify termination of linear programs.

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Let's consider the complement of the previous problem:

Definition

Termination of linear loops is the following decision problem.
Input: a closed polyhedral cone $\mathcal{C} \subseteq \mathbb{R}^d \setminus \{0^d\}$ generated by rational vectors, a set of matrices $S \subseteq \mathrm{GL}(d, \mathbb{Q})$.
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In other words, let $S = \{A_1, \ldots, A_n\}$ be a set of pairwise commuting matrices. It is decidable whether there exists v, such that

$$A_1^{k_1} A_2^{k_2} \cdots A_n^{k_n} v \in \mathcal{C} \quad \text{for all } k_1, \ldots, k_n \in \mathbb{N}.$$

Proof idea. Step 1: dual problem

Suppose $\mathcal{C}$ is defined by

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\iff & {A_n^\top}^{k_n} \cdots {A_1^\top}^{k_1} c_i \text{ are in some closed } \textbf{halfspace } \mathcal{H} := \{x \mid v^\top x \geq 0\} \\
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Let $S^\top := \{A_1^\top, \ldots, A_n^\top\}$. Denote by $\langle c_1, \ldots, c_m \rangle$ the cone generated by $c_1, \ldots, c_m$. It suffices to decide whether the orbit $\langle S^\top \rangle \cdot \langle c_1, \ldots, c_m \rangle$ lies in a closed halfspace.

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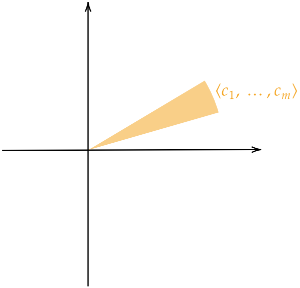

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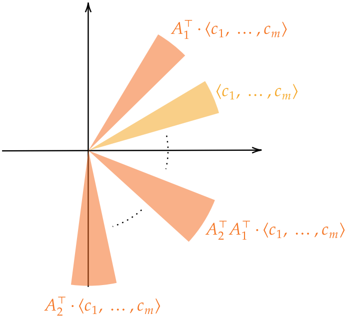

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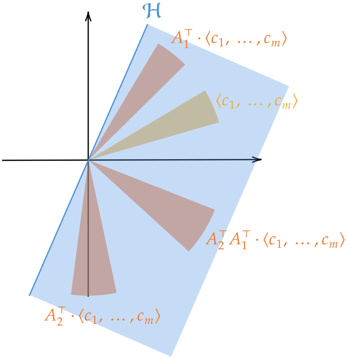

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Instead of deciding whether $\langle S^\top \rangle \cdot \langle c_1, \ldots, c_m \rangle$ lies in a closed halfspace, we first decide whether it is **salient**.

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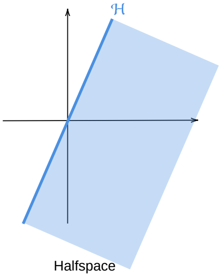
A set $\mathcal{O} \subseteq \mathbb{R}^d$ is called *salient* if the cone it generates is salient.



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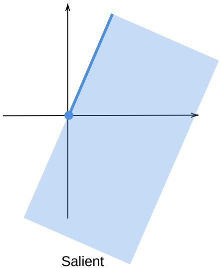

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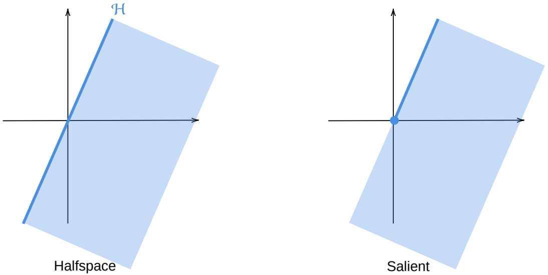

Lemma

Suppose we have a procedure that decides whether $\langle S^\top \rangle \cdot \langle c_1, \ldots, c_m \rangle$ is salient, then we can decide whether it is contained in a closed halfspace.

Step 4: from salient cone to positive polynomials

Now it suffices to whether $\langle S^\top \rangle \cdot \langle c_1, \ldots, c_m \rangle$ is salient. That is, whether there exist $x \neq 0^d$ such that both x and $-x$ are in the cone generated by $\langle S^\top \rangle \cdot \langle c_1, \ldots, c_m \rangle$.

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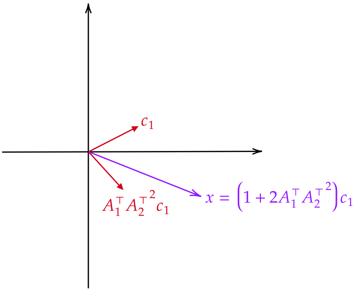

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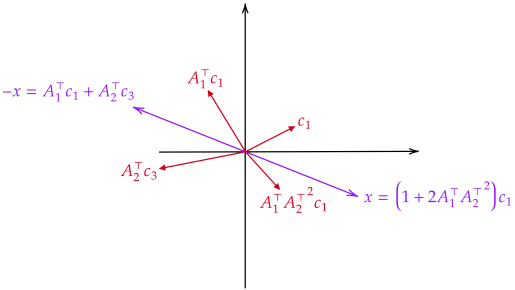

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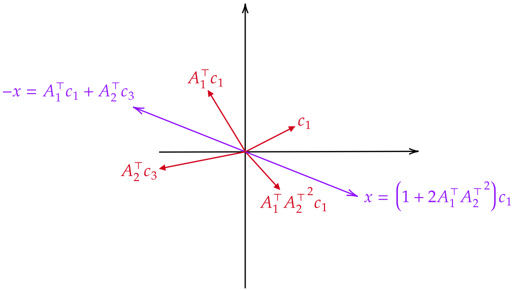

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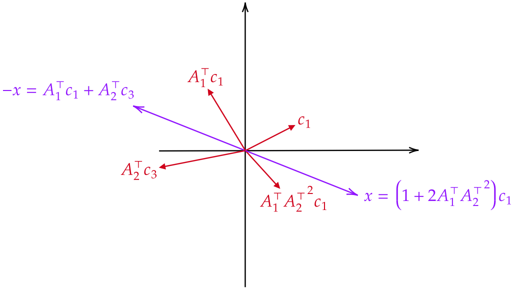

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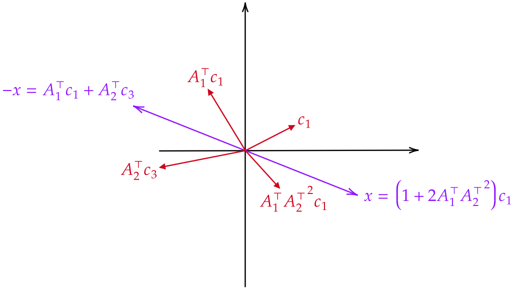

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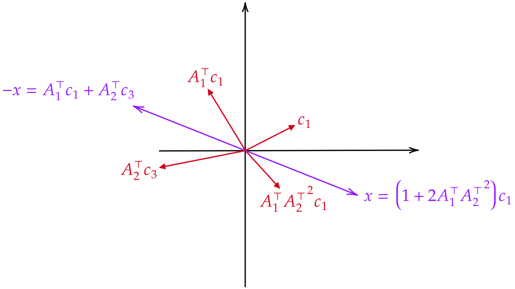

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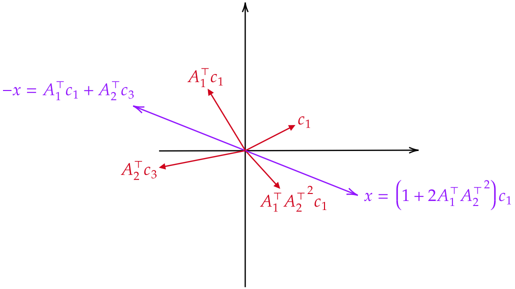

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Proposition

The orbit $\langle S^\top \rangle \cdot \langle c_1, \ldots, c_m \rangle$ is not salient if and only if there exist "positive polynomials" $f_1, \ldots, f_m \in \mathbb{R}_{\geq 0}[X_1, \ldots, X_n]$, not all zero, such that $0^d = f_1(A_1^\top, \ldots, A_n^\top) \cdot c_1 + \cdots + f_m(A_1^\top, \ldots, A_n^\top) \cdot c_m$.

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Let $\mathcal{M}$ be the $\mathbb{R}[X_1, \dots, X_n]$-submodule of $\mathbb{R}[X_1, \dots, X_n]^m$ consisting of all tuples $(f_1, \dots, f_m)$ such that

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The orbit $\langle S^\top \rangle \cdot \langle c_1, \ldots, c_m \rangle$ is not salient if and only if

$$\mathcal{M} \cap (\mathbb{R}_{\geq 0}[X_1, \ldots, X_n])^m \neq \{0^m\}.$$

Step 5: positive polynomial in a module

Let $\mathcal{M}$ be the $\mathbb{R}[X_1, \ldots, X_n]$-submodule of $\mathbb{R}[X_1, \ldots, X_n]^m$ consisting of all tuples $(f_1, \ldots, f_m)$ such that

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Proof idea: the characteristic polynomials of $A_1^\top, \ldots, A_n^\top$ are in $\mathcal{M}$. The module $\mathbb{R}[X_1, \ldots, X_n]^m$ becomes finite dimensional $\mathbb{R}$-linear space after quotient by these characteristic polynomials, the rest is linear algebra.

Step 6: local-global principle by Einsiedler et al.

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$$0 = f_1 \cdot (2X^2 - 1) + f_2 \cdot (X + 2). \tag{1}$$

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If $\mathcal{M} \cap (\mathbb{R}_{\geq 0}[X_1, \ldots, X_n]^*)^m$ is empty, such "certificate" always exists!

Theorem (Einsiedler, Mouat, Tuncel (2003))

Let $\mathcal{M}$ be an $\mathbb{R}[X_1, \ldots, X_n]$-submodule of $\mathbb{R}[X_1, \ldots, X_n]^m$. Then there exists $\boldsymbol{f} \in \mathcal{M} \cap (\mathbb{R}_{\geq 0}[X_1, \ldots, X_n]^*)^m$ if and only if:

1. *For every $r \in \mathbb{R}^n_{>0}$, there exists $\boldsymbol{f}_r \in \mathcal{M}$ such that $\boldsymbol{f}_r(r) \in \mathbb{R}^m_{>0}$.*
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Condition 1 can be checked using the first order theory of the reals.

Condition 2 only needs to be checked for a finite number of v (consider the Newton polytopes of a Gröbner basis of $\mathcal{M}$).

Corollary

Given a finite set of generators for the $\mathbb{R}[X_1, \ldots, X_n]$-submodule $\mathcal{M}$ of $\mathbb{R}[X_1, \ldots, X_n]^m$, it is decidable whether $\mathcal{M} \cap (\mathbb{R}_{\geq 0}[X_1, \ldots, X_n]^*)^m$ is empty.

Conclusion

Termination of linear loops with commuting matrices $\longleftrightarrow$ whether a submodule $\mathcal{M}$ of $\mathbb{R}[X_1, \ldots, X_n]^m$ contains a "positive" element.

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What about non-commuting matrices?
Let $\mathbb{R}\langle X_1, \ldots, X_n \rangle$ denote the ring of non-commutative polynomials.

Open Problem

Given the generators of a left submodule $\mathcal{M}$ of $\mathbb{R}\langle X_1, \ldots, X_n \rangle^m$, can we decide whether $\mathcal{M}$ contains an element with only positive coefficients?

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Open Problem (Interesting special cases)

- Given $f \in \mathbb{R}\langle X_1, \ldots, X_n \rangle$, decide if there exists $g \neq 0$ such that $g \cdot f$ has only positive coefficients?
- Let G be a 2-step nilpotent group, decide if a left ideal of $\mathbb{R}[G]$ contains an element of $\mathbb{R}_{\geq 0}[G]^*$.