Decision problems in sub-semigroups of metabelian groups

Ruiwen Dong

Saarland University

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An old decidability problem

Markov (1940s): is the following decidable?

Input: Set of square matrices $S = \{A_1, \ldots, A_K\}$, target matrix T.
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Michailova : undecidable in $F_2 \times F_2 \hookrightarrow \mathbb{Z}^{4 \times 4}$.

Membership problems

$\langle S \rangle$: the *semigroup* generated by S. $\langle S \rangle_{grp}$: the *group* generated by S.
Input: finite set $S = \{a_1, \ldots, a_K\}$ in ambient group G and target T.

Definition (Semigroup Membership)

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group types	Group Membership	Semigroup Membership
free abelian ($\mathbb{Z}^n$)	PTIME	NP-complete
2-step nilpotent	Decidable	Undecidable
polycylic	Decidable	Undecidable
metabelian	Decidable	Undecidable
free (F_2)	Decidable	Decidable
$F_2 \times F_2$	Undecidable	Undecidable

Intermediate Problems

Input: finite set $S = \{a_1, \ldots, a_K\}$ in some ambient group G.

Definition

Identity Problem: Let e be the neutral element of G. Is $e \in \langle S \rangle$?
Inverse Problem: Is $a_1^{-1} \in \langle S \rangle$?
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Let S^* be the set of words over the alphabet $S = \{a_1, \ldots, a_K\}$.

$$\pi \colon S^* \to \big(\langle S \rangle \cup \{e\}\big) \hookrightarrow G.$$

$$a_{i_1} a_{i_2} \cdots a_{i_m} \mapsto a_{i_1} \cdot a_{i_2} \cdot \cdots \cdot a_{i_m}$$

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Proposition

1. *The Identity Problem asks if* $\pi^{-1}(e) \neq \{\epsilon\}$.
2. *The Inverse Problem asks if* $\pi^{-1}(e)$ *contains a word that uses the letter* a_1.
3. *The Group Problem asks if* $\pi^{-1}(e)$ *contains a word that uses all the letters* $a_1, \ldots, a_K$.

Known results

Let $[G, H]$ denote the group generated by $\{ghg^{-1}h^{-1} \mid g \in G, h \in H\}$.

A group G is *metabelian* if $[G, G]$ is abelian, (equivalently, if there exists an abelian normal subgroup $A \trianglelefteq G$ such that G/A is also abelian).

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abelian ($[G, G] = \{e\}$)	PTIME	PTIME	NP-compl.
2-step nilpotent ($[G, [G, G]] = \{e\}$)	Decidable	Decidable	Undecidable
polycylic	Decidable (Kopytov '68)	?	Undecidable
metabelian ($[[G, G], [G, G]] = \{e\}$)	Decidable (Romanovski '74)	Decidable (D. 2023)	Undecidable
free (F_2)	Decidable	Decidable	Decidable
$F_2 \times F_2$	Undecidable (Mikhailova '66)	Undecidable (Bell 2010)	Undecidable

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An $\mathscr{A}^2$*-presentation* of a metabelian group G is $r_1, \dots, r_m \in M_n$ such that

$$G = M_n / \operatorname{ncl}(r_1, \dots, r_m).$$

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Theorem (Magnus, Baumslag)

Suppose we are given an $\mathscr{A}^2$-presentation of G. One can effectively embed G as a subgroup of $\mathcal{Y} \rtimes A$, where A is f.g. abelian and $\mathcal{Y}$ is a f.p. $\mathbb{Z}[A]$-module.

Example

From now on we will illustrate using the example $G = \mathbb{Z} \wr \mathbb{Z}^2$:

$$\mathbb{Z} \wr \mathbb{Z}^2 := \left\{ \begin{pmatrix} X_1^{a_1} X_2^{a_2} & f \\ 0 & 1 \end{pmatrix} \;\middle|\; a_1, a_2 \in \mathbb{Z}, f \in \mathbb{Z}[X_1^{\pm}, X_2^{\pm}] \right\}.$$

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Group law:

$$\begin{pmatrix} X_1^{a_1} X_2^{a_2} & f \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X_1^{a'_1} X_2^{a'_2} & f' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} X_1^{a_1+a'_1} X_2^{a_2+a'_2} & f + X_1^{a_1} X_2^{a_2} f' \\ 0 & 1 \end{pmatrix}.$$

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Let $S = \left\{ \begin{pmatrix} X_1^{a_{11}} X_2^{a_{12}} & f_1 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} X_1^{a_{K1}} X_2^{a_{K2}} & f_K \\ 0 & 1 \end{pmatrix} \right\}$.

As an example, we want to decide the Group Problem.

i.e. is there $w \in S^*$, using every letter in S, such that $\pi(w) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$?

S-graphs

As an example, take

$$S = \left\{ a_1 := \begin{pmatrix} X_1^{-2} X_2^3 & f_1 \\ 0 & 1 \end{pmatrix}, a_2 := \begin{pmatrix} X_1^2 & f_2 \\ 0 & 1 \end{pmatrix}, a_3 := \begin{pmatrix} X_2^{-2} & f_3 \\ 0 & 1 \end{pmatrix} \right\}.$$

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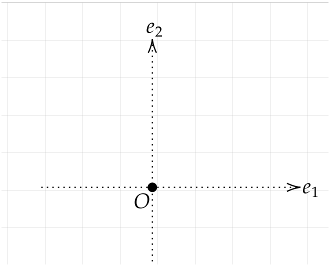

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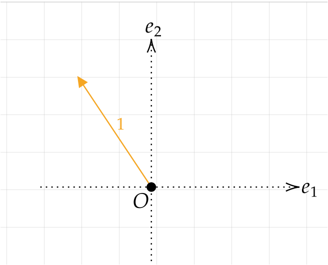

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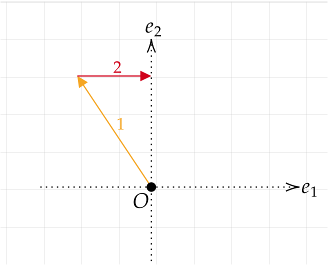

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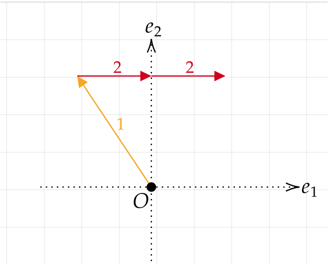

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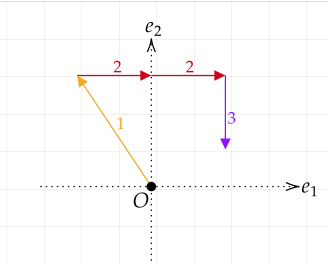

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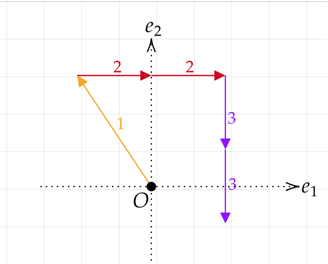

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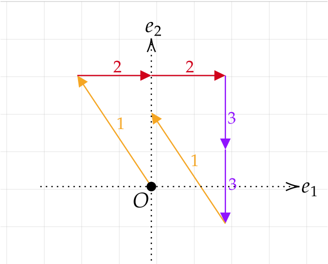

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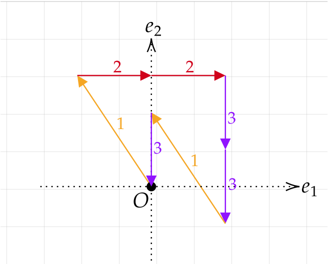

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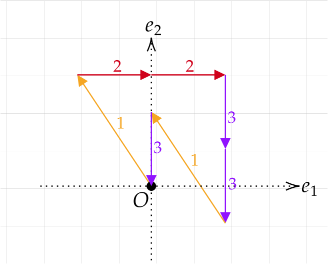

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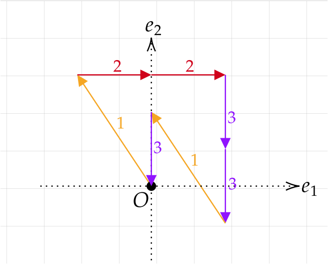

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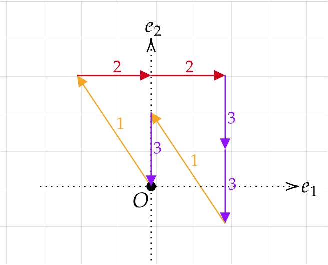

The *position polynomials* of w are p_1, p_2, p_3 in $\mathbb{N}[X_1^{\pm}, X_2^{\pm}]$, where p_i is the sum of all monomials corresponding to starting points of edges i:

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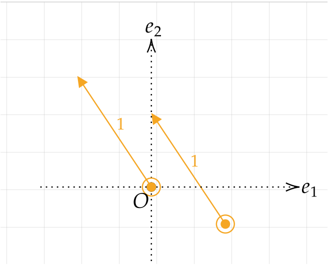

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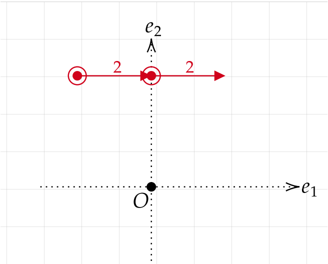

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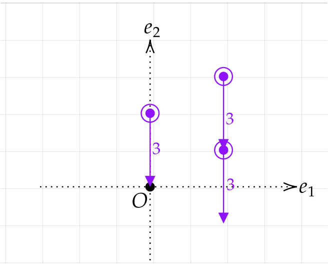

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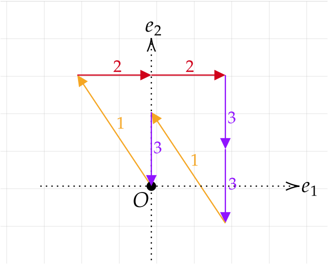

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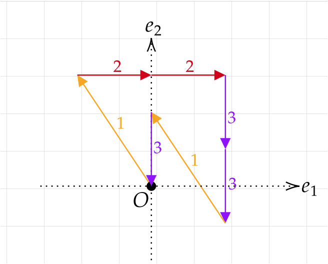

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$$\pi(w) = \begin{pmatrix} * & 0 \\ 0 & 1 \end{pmatrix} \iff f_1 \cdot p_1 + f_2 \cdot p_2 + f_3 \cdot p_3 = 0.$$

From semigroup to polynomial semiring

Let $w \in S^*$ be a word.

$$\pi(w) = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \Leftrightarrow \text{graph } \Gamma(w) \text{ is Eulerian.}$$

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$$\Gamma(w) \text{ is Eulerian}$$
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$\Updownarrow$ $\Updownarrow$

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"degree constraints"

Degree constraints: face-accessibility

Connectivity is a *global* property, but p_1, p_2, p_3 describe *local* properties.

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Definition (Face-accessibility)

Let Γ be a graph and C be the convex hull of all its vertices. Γ is called *face-accessible* if for every strict face F of C, Γ contains an edge leaving F.

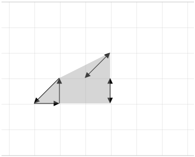

Figure: Face-accessible graph

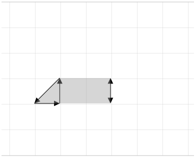

Figure: Not face-accessible

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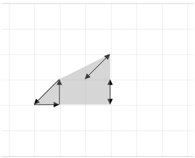

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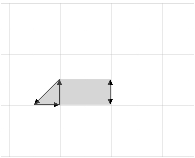

Figure: Not face-accessible

For $v \in \mathbb{R}^2 \setminus \{0\}$, define $\deg_v \left(\sum_{a \in \mathbb{Z}^2} c_a X^a\right) := \max_{c_a \neq 0}\{a \cdot v\}$.

Γ is face-accessible if and only if

$$\forall v \in \mathbb{R}^2 \setminus \{0\}, \exists i \in \{1,2,3\}, \deg_v(p_i) = \max_{1 \leq i' \leq 3}\{\deg_v(p_{i'})\} \text{ and } v \not\perp (a_{i1}, a_{i2}).$$

Degree constraints: from face-accessibility to connectivity

Proposition

Let Γ be a symmetric and face-accessible graph. Then there exist $z_1, \ldots, z_m \in \mathbb{Z}^n$, such that the union of translations $\widehat{\Gamma} := \bigcup_{i=1}^{m} (\Gamma + z_i)$ is an Eulerian graph.

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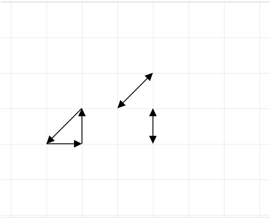

Figure: Symmetric face-accessible Γ.

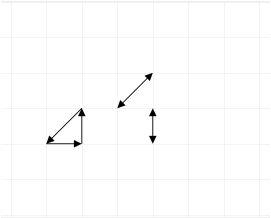

Figure: $\bigcup_{i=1}^{1} (\Gamma + z_i)$.

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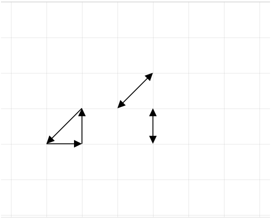

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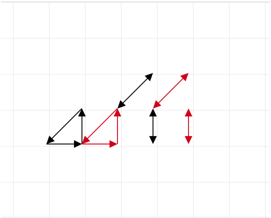

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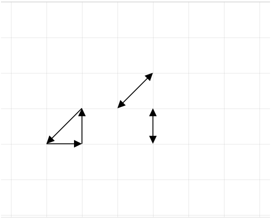

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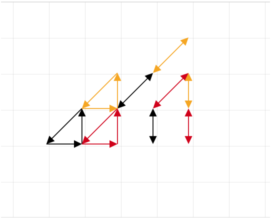

Figure: $\bigcup_{i=1}^{3} (\Gamma + z_i)$.

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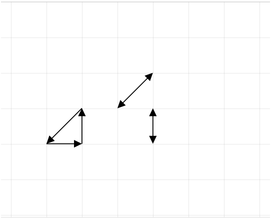

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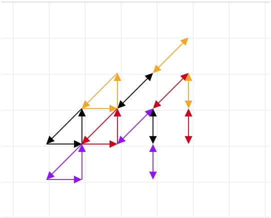

Figure: $\widehat{\Gamma} = \bigcup_{i=1}^{4} (\Gamma + z_i)$ is Eulerian.

Degree constraints: counterexample

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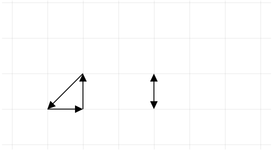

Figure: Non face-accessible Γ.

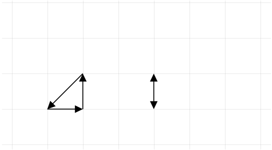

Figure: $\widehat{\Gamma}$ cannot be connected.

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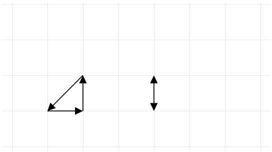

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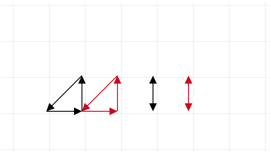

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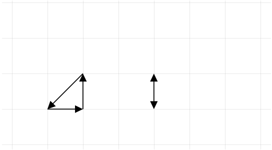

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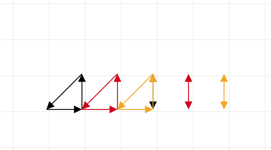

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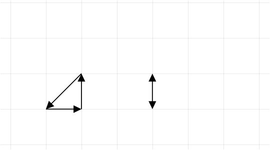

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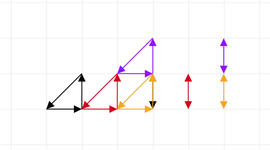

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Degree constraints: describing face-accessibility

Proposition

Let Γ be a symmetric and face-accessible graph. Then there exist $z_1, \ldots, z_m \in \mathbb{Z}^n$, such that the union of translations $\widehat{\Gamma} \coloneqq \bigcup_{i=1}^{m} (\Gamma + z_i)$ is an Eulerian graph.

If the position polynomials of Γ are p_1, p_2, p_3, then that of $\widehat{\Gamma}$ are $\widehat{p_1} \coloneqq p_1 \cdot \sum_{i=1}^{m} X^{z_i}$, $\widehat{p_2} \coloneqq p_2 \cdot \sum_{i=1}^{m} X^{z_i}$, $\widehat{p_3} \coloneqq p_3 \cdot \sum_{i=1}^{m} X^{z_i}$.

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Homogeneous linear equations satisfied by p_1, p_2, p_3 are satisfied by $\widehat{p_1}, \widehat{p_2}, \widehat{p_3}$.

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Homogeneous linear equations satisfied by p_1, p_2, p_3 are satisfied by $\widehat{p_1}, \widehat{p_2}, \widehat{p_3}$.

Recall that Γ is face-accessible if and only if

$$\forall v \in \mathbb{R}^2 \setminus \{0\}, \exists i \in \{1,2,3\}, \deg_v(p_i) = \max_{i'}\{\deg_v(p_{i'})\} \text{ and } v \not\perp (a_{i1}, a_{i2}). \tag{$*$}$$

We will call $(*)$ the "degree constraints".

From semigroup to polynomial semiring: summing up

$\langle S \rangle$ is a group

$\iff$ There exists word $w \in S^*$ containing every letter, such that $\pi(w) = e$

$\iff$ There exists an Eulerian graph Γ with position polynomials $p_1, p_2, p_3 \in \mathbb{N}[X_1^{\pm}, X_2^{\pm}]^*$ such that $f_1 p_1 + f_2 p_2 + f_3 p_3 = 0$

$\iff$ There exists a connected graph Γ with position polynomials $p_1, p_2, p_3 \in \mathbb{N}[X_1^{\pm}, X_2^{\pm}]^*$ such that $f_1 p_1 + f_2 p_2 + f_3 p_3 = 0$ and $X_1^{-2} X_2^{3} p_1 + X_1^{2} p_2 + X_2^{-2} p_3 = p_1 + p_2 + p_3$

$\iff$ There exists a face-accessible graph Γ with position polynomials $p_1, p_2, p_3 \in \mathbb{N}[X_1^{\pm}, X_2^{\pm}]^*$ such that $f_1 p_1 + f_2 p_2 + f_3 p_3 = 0$ and $X_1^{-2} X_2^{3} p_1 + X_1^{2} p_2 + X_2^{-2} p_3 = p_1 + p_2 + p_3$

$\iff$ There exist $p_1, p_2, p_3 \in \mathbb{N}[X_1^{\pm}, X_2^{\pm}]^*$ satisfying "degree constraints", such that $f_1 p_1 + f_2 p_2 + f_3 p_3 = 0$ and $(X_1^{-2} X_2^{3} - 1) p_1 + (X_1^{2} - 1) p_2 + (X_2^{-2} - 1) p_3 = 0$

Local-global principle

We need to decide whether there exist $p_1, p_2, p_3 \in \mathbb{N}[\overline{X}^{\pm}]^*$ satisfying "degree constraints" and a system of **homogeneous** linear equations.

Theorem (Einsiedler, 2003)

Let $\mathcal{M}$ be an $\mathbb{Z}[\overline{X}^{\pm}]$-submodule of $\mathbb{Z}[\overline{X}^{\pm}]^K$. Then $\mathcal{M} \cap \left(\mathbb{N}[\overline{X}^{\pm}]^*\right)^K \neq \emptyset$ if and only if the following are satisfied:

1. *For every $r \in \mathbb{R}^n_{>0}$, there exists $\boldsymbol{f}_r \in \mathcal{M}$ such that $\boldsymbol{f}_r(r) \in \mathbb{R}^K_{>0}$.*
2. *For every $v \in (\mathbb{R}^n)^*$, there exists $\boldsymbol{f}_v \in \mathcal{M}$, such that $\mathrm{in}_v\left(\boldsymbol{f}_v\right) \in \left(\mathbb{N}[\overline{X}^{\pm}]^*\right)^K$.*

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Theorem (Generalization of Einsiedler)

Let $\mathcal{M}$ be a submodule of $\mathbb{Z}[\overline{X}^{\pm}]^K$. There exists $\boldsymbol{f} \in \mathcal{M} \cap \left(\mathbb{N}[\overline{X}^{\pm}]^*\right)^K$ with <u>"degree constraints"</u> if and only if the following are satisfied:

1. *For every $r \in \mathbb{R}^n_{>0}$, there exists $\boldsymbol{f}_r \in \mathcal{M}$ such that $\boldsymbol{f}_r(r) \in \mathbb{R}^K_{>0}$.*
2. *For every $v \in (\mathbb{R}^n)^*$, there exists $\boldsymbol{f}_v \in \mathcal{M}$, such that $\operatorname{in}_v(\boldsymbol{f}_v) \in \left(\mathbb{N}[\overline{X}^{\pm}]^*\right)^K$ and satisfies some <u>"degree constraints"</u>.*

Einsiedler's proof

1. "Glue up" all $\boldsymbol{f}_v$ to obtain $\boldsymbol{f}_\infty$ satisfying $\mathrm{in}_v(\boldsymbol{f}_\infty) \in \left(\mathbb{N}[\overline{X}^{\pm}]^*\right)^K$ for all $v \in (\mathbb{R}^n)^*$. In particular, $\boldsymbol{f}_\infty(r) \in \mathbb{R}^K_{>0}$ for all $r \in \mathbb{R}^n$ except a compact set C.
2. "Glue up" all $\boldsymbol{f}_r$ to obtain $\boldsymbol{f}_C$ satisfying $\boldsymbol{f}_C(r) \in \mathbb{R}^K_{>0}$ for all $r \in C$.

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3. "Glue up" $\boldsymbol{f}_\infty$ and $\boldsymbol{f}_C$ to obtain $\boldsymbol{f}$ satisfying
   1. $\boldsymbol{f}(r) \in \mathbb{R}^K_{>0}$ for all $r \in \mathbb{R}^n_{>0}$,
   2. $\mathrm{in}_v(\boldsymbol{f})(r) \in \mathbb{R}^K_{>0}$ for all $v \in (\mathbb{R}^n)^*, r \in \mathbb{R}^n_{>0}$.

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4. Use Handelman's theorem:

Theorem (Handelman, 1985)

Let $f \in \mathbb{R}[\overline{X}^{\pm}]$ be a polynomial. There exists $g \in \mathbb{R}_{>0}[\overline{X}^{\pm}]$ such that $fg \in \mathbb{R}_{>0}[\overline{X}^{\pm}]$ if and only if f satisfies the two following conditions:

1. *For all $r \in \mathbb{R}^n_{>0}$, we have $f(r) > 0$.*
2. *For all $v \in (\mathbb{R}^n)^*$ and $r \in \mathbb{R}^n_{>0}$, we have $\text{in}_v(f)(r) > 0$.*

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Proof idea: Let F be f replacing all coefficients with their absolute value. Let S (S^+) be the subring (subsemiring) of $\mathbb{R}[\overline{X}^{\pm}, \frac{1}{F}]$ generated by $\frac{\overline{X}^a}{F}$ where $\overline{X}^a$ is a monomial of F. Consider pure states of S.

Decidability

Theorem (Generalization of Einsiedler)

Let $\mathcal{M}$ be a submodule of $\mathbb{Z}[\overline{X}^{\pm}]^K$. There exists $\boldsymbol{f} \in \mathcal{M} \cap \left(\mathbb{N}[\overline{X}^{\pm}]^*\right)^K$ with "degree constraints" if and only if the following are satisfied:

1. *For every $r \in \mathbb{R}^n_{>0}$, there exists $\boldsymbol{f}_r \in \mathcal{M}$ such that $\boldsymbol{f}_r(r) \in \mathbb{R}^K_{>0}$.*
2. *For every $v \in (\mathbb{R}^n)^*$, there exists $\boldsymbol{f}_v \in \mathcal{M}$, such that $\mathrm{in}_v(\boldsymbol{f}_v) \in \left(\mathbb{N}[\overline{X}^{\pm}]^*\right)^K$ and satisfies some "degree constraints".*

We can prove that it suffices to verify condition 2. for a countable number of v.

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3. **Procedure C** enumerates the countably many v in condition 2. and checks the condition on $\mathrm{in}_v(\mathcal{M})$.

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5. Do we have better characterization of $\pi^{-1}(e)$, other than the set of letters appearing?
6. Any structure on $\mathcal{M} \cap \left(\mathbb{N}[\overline{X}^{\pm}]^*\right)^K$?
7. Can we solve **one non-homogeneous** linear equation over $\mathbb{N}[\overline{X}^{\pm}]$?