Solving the Possible Largest Diameter Problem

SHI Xuanyu

October 25, 2023

This is a problem from the MidTerm Test of EE6605[1]:

> Given a simple graph G with N nodes and E edges. What is its possible largest diameter?

Solution:

$$D = \lceil N + 1 - \frac{3 + \sqrt{8E - 8N + 9}}{2} \rceil$$

In an undirected simple graph, the topology with the largest diameter is ideally represented by a chain ($D = N - 1$), while the topology with the smallest diameter is represented by a complete graph ($D = 1$).

To construct a graph with the maximum possible diameter given a specific number of nodes, it can be divided into three subgraphs, as shown in Fig.1.

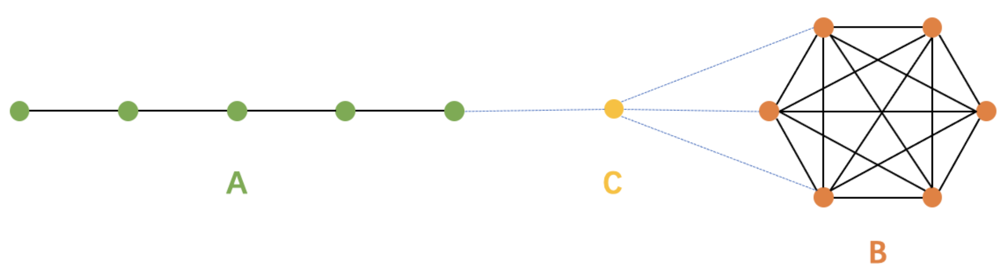

Figure 1: Topology with possible largest diameter

One of the subgraphs, A, can be a chain to increase the overall diameter of the graph, and it consists of N_A nodes with $E_A (= N_A - 1)$ edges.

[1]EE6605 Complex Networks: Modeling Dynamics and Control – City University of Hong Kong

Another subgraph, B, can be a fully-connected network that efficiently utilizes excess edges by generating the maximum number of edges using a fixed number of nodes. It has N_B nodes with $E_B = N_B(N_B - 1)/2$ edges.

The remaining subgraph, C, is a joint connecting the two subgraphs and consists of only one node ($N_C = 1$) with no edges ($E_C = 0$).

Hence, we can establish two equations:

$$N = N_A + N_B + 1 \tag{1}$$

$$\begin{aligned} E &= E_A + E_B + E_C + E_{AC} + E_{BC} \\ &= N_A + \frac{N_B(N_B - 1)}{2} + E_{BC} \end{aligned} \tag{2}$$

where E_{AC} represents the edge required to connect subgraph C and the endpoint of C, which is 1, and E_{BC} represents the number of edges that connect subgraph C and nodes of B, ranging from 1 to N_B.

Accordingly:

$$N_B = \begin{cases} \frac{3+\sqrt{8E-8N+9}}{2}, & \text{if } E_{BC} = 1 \\ \frac{1+\sqrt{8E-8N-7}}{2}, & \text{if } E_{BC} = N_B \end{cases}$$

The diameter D of this graph is:

$$\begin{aligned} D &= (N_A - 1) + D_{AtoC} + D_{CtoB} + D_B \\ &= N_A + 2 \\ &= N + 1 - N_B \\ &= [N + 1 - \frac{3 + \sqrt{8E - 8N + 9}}{2}, N + 1 - \frac{1 + \sqrt{8E - 8N - 7}}{2}] \end{aligned} \tag{3}$$

Given that the upper value has more limitations, such as it is not applicable for a chain where $N > E$ that makes $8E - 8N - 7$ negative, I have opted to use the rounded-up lower bound value in order to achieve better generalization:

$$D = \lceil N + 1 - \frac{3 + \sqrt{8E - 8N + 9}}{2} \rceil$$

Question: What if a simple Graph with $N = 1000$ and $E = 7000$?

Answer: $D = \lceil 889.9452 \rceil = 890$

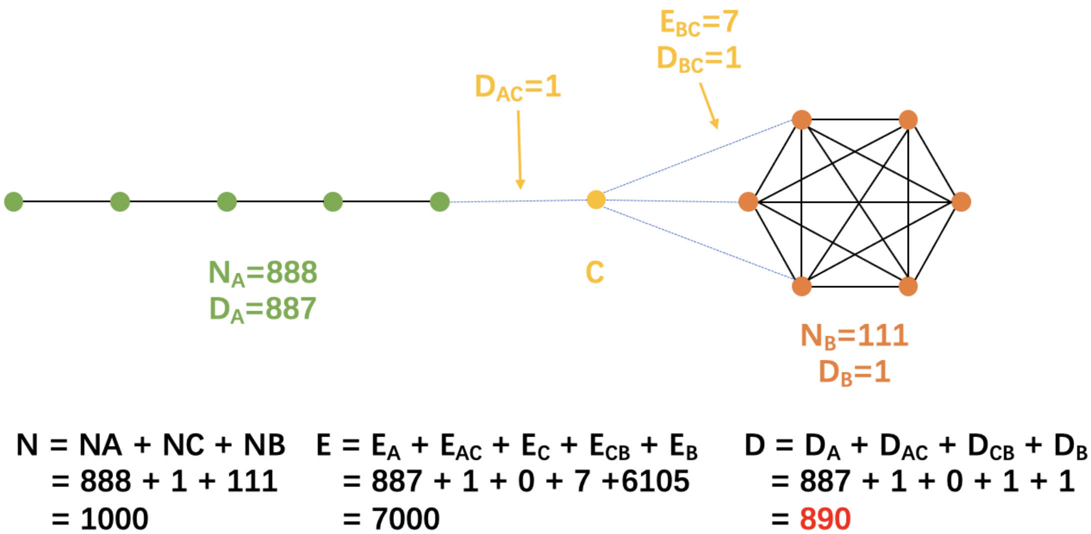

Figure 2: N=1000,E=7000

Simulation:

N	E	D_{max}
1	0	1
2	1	1
3	2	2
3	3	1
4	3	3
4	4	2
4	5	1
4	6	1