一、严格表述题

1. $Cauchy$叙述：对$\forall\varepsilon>0,\exists\delta_\varepsilon>0,$使得当$0<\left|\boldsymbol{x}-\boldsymbol{x}_0\right|<\delta_\varepsilon,$都有

$\left|f\left(\boldsymbol{x}\right)-f\left(\boldsymbol{x}_0\right)\right|<\varepsilon$

$Heine$叙述：对$\forall\left\{\boldsymbol{x}_n\right\}\subset D\setminus\left\{\boldsymbol{x}_0\right\}$，满足$\boldsymbol{x}_n\to\boldsymbol{x}_0\left(n\to\infty\right)$，都有

$f\left(\boldsymbol{x}_n\right)\to f\left(\boldsymbol{x}_0\right)\left(n\to\infty\right)$

$Cauchy$收敛原理：对$\forall\varepsilon>0$，$\exists\delta_\varepsilon>0,$使得当$\boldsymbol{x}',\boldsymbol{x}''\in\mathring{B}_{\delta_\varepsilon}\left(\boldsymbol{x}_0\right)$时，都有

$\left|f\left(\boldsymbol{x}'\right)-f\left(\boldsymbol{x}''\right)\right|<\varepsilon$

2. 对向量值映照$\boldsymbol{f}\left(\boldsymbol{x}\right):\mathbb{R}^m\supset D_x\ni\boldsymbol{x}\mapsto\boldsymbol{f}\left(\boldsymbol{x}\right)\in\mathbb{R}^n,\boldsymbol{x}_0$为$D_x$的内点，有

$\boldsymbol{f}\left(\boldsymbol{x}_0+\boldsymbol{h}\right)-\boldsymbol{f}\left(\boldsymbol{x}\right)=\boldsymbol{Df}\left(\boldsymbol{x}_0\right)\left(\boldsymbol{h}\right)+o\left(\left|\boldsymbol{h}\right|_{\mathbb{R}^m}\right)\in\mathbb{R}^n$，$\boldsymbol{Df}\left(\boldsymbol{x}_0\right)\in\mathscr{L}\left(\mathbb{R}^m;\mathbb{R}^n\right),$

则称$\boldsymbol{f}\left(\boldsymbol{x}\right)$在$\boldsymbol{x}_0$点可微。

3，第二类曲线积分：设L为一条定向光滑曲线，起点为A，终点为B。在L上每一点取切向量$\boldsymbol{\tau}$,使得它与L的走向相一致。设

$\boldsymbol{f}\left(x,y,z\right)=P\left(x,y,z\right)\boldsymbol{i}+Q\left(x,y,z\right)\boldsymbol{j}+R\left(x,y,z\right)\boldsymbol{k}$定义在$L$上的向量值函数，则称

$\int_L\boldsymbol{f}\bullet\boldsymbol{\tau}dl=\int\limits_L\left(P\left(x,y,z\right)\cos\alpha+Q\left(x,y,z\right)\cos\beta+R\left(x,y,z\right)\cos\gamma\right)dl$为第二类曲线积分。

第二类曲面积分，同样，有$\boldsymbol{f}\left(x,y,z\right)=P\left(x,y,z\right)\boldsymbol{i}+Q\left(x,y,z\right)\boldsymbol{j}+R\left(x,y,z\right)\boldsymbol{k}$

为定义在定向光滑曲面Σ上的向量值函数，设曲面上单位法向量$\boldsymbol{n}=\left(\cos\alpha,\cos\beta,\cos\gamma\right),$

则$\iint\limits_\Sigma\boldsymbol{f}\bullet\boldsymbol{n}dS=\iint\limits_\Sigma\left(P\left(x,y,z\right)\cos\alpha+Q\left(x,y,z\right)\cos\beta+R\left(x,y,z\right)\cos\gamma\right)dS$为第二类曲面积分

4，若下列二条件的一个满足，则级数$\sum\limits_{n=1}^\infty a_nb_n$收敛：

$(1)\left(Abel\right)\left|a_n\right|$单调有界,$\sum\limits_{n=1}^{+\infty}b_n$收敛

$(2)\left(Dirichlet\right)\left|a_n\right|$单调趋于0,$\left|\sum\limits_{n=1}^{+\infty}b_n\right|$有界

二、简答题

1. 解：左$=\dfrac{\partial y}{\partial t}=-kn^2\sin nx\cdot e^{-kn^2t}$

$$\frac{\partial y}{\partial x}=e^{-kn^2t}n\cos nx$$

右$=k\dfrac{\partial^2y}{\partial x^2}=-e^{kn^2t}kn^2\sin nx$

则有 左=右

2. 解：$f\left(x,y\right)=\dfrac{x^3+y^3}{x^2+y}$

令$y=-x^2+\lambda x^p\left(p>0\right)$

$$\sim\begin{cases}\lambda x^p\left(1+o\left(1\right)\right) & 0<p<2\\ \left(\lambda-1\right)x^2 & p=2\\ -x^2\left(1+o\left(1\right)\right) & p>2\end{cases}$$

$\left(i\right)0<p<2\quad f\left(x,y\right)=\dfrac{x^3+\lambda^3x^{3p}\left(1+o\left(1\right)\right)}{\lambda x^p}=\dfrac{1}{\lambda}x^{3-p}+\lambda^2x^{2p}\left(1+o\left(1\right)\right)\to 0$

$\left(ii\right)p=2\qquad f\left(x,y\right)=\dfrac{x^3+\left(\lambda-1\right)^3x^6}{\lambda x^2}=\dfrac{1}{\lambda}x+\dfrac{\left(\lambda-1\right)^3}{\lambda}x^4\to 0$

$\left(iii\right)p>2\qquad f\left(x,y\right)=\dfrac{x^3-x^6\left(1+o\left(1\right)\right)}{\lambda x^3}\to\dfrac{1}{\lambda}\neq 0$

则有$f\left(x,y\right)$无极限

3，注意到积分:$\iint\limits_{\mathbb{R}^2}e^{-x^2-y^2}dxdy=\int_0^{2\pi}d\theta\int_0^{+\infty}e^{-r^2}rdr=2\pi\bullet\dfrac{1}{2}=\pi$

又:$\iint\limits_{\mathbb{R}^2}e^{-x^2-y^2}dxdy=\int_{-\infty}^{+\infty}e^{-x^2}dx\int_{-\infty}^{+\infty}e^{-y^2}dy=4\left(\int_0^{+\infty}e^{-x^2}dx\right)^2$

即$\int_0^{+\infty}e^{-x^2}dx=\dfrac{\sqrt{\pi}}{2}$

4,$\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n=e$,故收敛半径为$\dfrac{1}{e}$

$x=e^{-1}$时,$\dfrac{n^n}{n!}x^n\sim\dfrac{1}{\sqrt{2\pi n}}$,由比较判别法知发散

$x=-e^{-1}$时,$\left|\dfrac{n^n}{n!}x^n\right|\sim\dfrac{1}{\sqrt{2\pi n}}\to 0$,且$\left|\dfrac{a_{n+1}x^{n+1}}{a_nx^n}\right|=\dfrac{1}{e}\left(1+\dfrac{1}{n}\right)^n<1$,

此$Leibniz$级数收敛

三、级数题

1. 解:$\dfrac{u_{n+1}}{u_n}=\dfrac{\left(n+1\right)!}{10^{n+1}}\cdot\dfrac{10^n}{n!}=\dfrac{n+1}{10}$,

$\lim\limits_{n\to\infty}\dfrac{u_{n+1}}{u_n}=\lim\limits_{n\to\infty}\dfrac{n+1}{10}=\infty.$

则有级数发散。

2. 解：$f\left(x\right)=\dfrac{1}{x^2+4x+3}=\dfrac{1}{\left(x+1\right)\left(x+3\right)}=\dfrac{1}{2\left(1+x\right)}-\dfrac{1}{2\left(3+x\right)}$

$$=\frac{1}{4\left(1+\frac{x-1}{2}\right)}-\frac{1}{8\left(1+\frac{x-1}{4}\right)}$$

其中$\dfrac{1}{4\left(1+\frac{x-1}{2}\right)}=\dfrac{1}{4}\sum_{n=0}^{\infty}\dfrac{\left(-1\right)^n}{2^n}\left(x-1\right)^n \quad \left(-1<x<3\right)$

$$\frac{1}{8\left(1+\frac{x-1}{4}\right)}=\frac{1}{8}\sum_{n=0}^{\infty}\frac{\left(-1\right)^n}{4^n}\left(x-1\right)^n \qquad \left(-3<x<5\right)$$

则有

$$f\left(x\right)=\frac{1}{x^2+4x+3}=\sum_{n=0}^{\infty}\left(-1\right)^n\left(\frac{1}{2^{n+2}}-\frac{1}{2^{2n+3}}\right)\left(x-1\right)^n \quad \left(-1<x<3\right)$$

3, 首先对函数进行偶延拓:$\begin{cases} x, & x\in[0,\pi) \\ -x, & x\in[-\pi,0) \end{cases}$

$$a_0=\frac{2}{\pi}\int_0^{\pi}xdx=\pi, a_n=\frac{2}{\pi}\int_0^{\pi}x\cos nxdx=2\bullet\frac{\left(-1\right)^n-1}{n^2\pi}$$

故$f\left(x\right)=\dfrac{\pi}{2}-\dfrac{4}{\pi}\sum_{n=1}^{\infty}\dfrac{\cos\left(2k-1\right)x}{\left(2k-1\right)^2}$

代入$x=0$,立即有:$\sum_{n=1}^{\infty}\dfrac{1}{\left(2k-1\right)^2}=\dfrac{\pi}{2}\bullet\dfrac{\pi}{4}=\dfrac{\pi^2}{8}$

注意到:$\cos x$全部零点为$\pm\dfrac{2k-1}{2}\pi$，代入即有所求为:$\dfrac{\pi^2}{8}\bullet\dfrac{4}{\pi^2}\bullet 2=1$

四、多元函数微分题

1．　解：事物$\{x,y,z\}\in\mathbb{R}^3, s.t.\bar{\boldsymbol{F}}(x,y)=\boldsymbol{F}\left(z,\begin{bmatrix}x\\y\end{bmatrix}(z)\right)=\begin{bmatrix}x+y+z\\x^2+y^2+z^2-1\end{bmatrix}=0\in\mathbb{R}^2$

$$0=\boldsymbol{D}\bar{\boldsymbol{F}}=\boldsymbol{D}_z\boldsymbol{F}+\boldsymbol{D}_{\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]}\boldsymbol{F}\cdot\boldsymbol{D}\begin{bmatrix}x\\y\end{bmatrix}(z)\in\mathbb{R}^{2\times1}$$

$$\boldsymbol{D}\begin{bmatrix}x\\y\end{bmatrix}(z)=-\left(\boldsymbol{D}_{\left[\begin{smallmatrix}x\\y\end{smallmatrix}\right]}\boldsymbol{F}\right)^{-1}\cdot\boldsymbol{D}_z\boldsymbol{F}$$

$$\begin{bmatrix}\dfrac{\partial x}{\partial z}\\\dfrac{\partial y}{\partial z}\end{bmatrix}=-\begin{bmatrix}\dfrac{\partial F^1}{\partial x}&\dfrac{\partial F^1}{\partial y}\\\dfrac{\partial F^2}{\partial x}&\dfrac{\partial F^2}{\partial y}\end{bmatrix}^{-1}\cdot\begin{bmatrix}\dfrac{\partial F^1}{\partial z}\\\dfrac{\partial F^2}{\partial z}\end{bmatrix}$$

$$\begin{bmatrix}\dfrac{\partial x}{\partial z}\\\dfrac{\partial y}{\partial z}\end{bmatrix}=-\begin{bmatrix}1&1\\2x&2y\end{bmatrix}^{-1}\cdot\begin{bmatrix}1\\2z\end{bmatrix}$$

$$\frac{\partial x}{\partial z}=-\frac{1}{2(y-x)}\begin{vmatrix}1&1\\2z&2y\end{vmatrix}=-\frac{2(y-z)}{2(y-x)}=-\frac{y-z}{y-x}$$

$$\frac{\partial y}{\partial z}=-\frac{1}{2(y-x)}\begin{vmatrix}1&1\\2x&2z\end{vmatrix}=-\frac{2(z-x)}{2(y-x)}=-\frac{z-x}{y-x}$$

即$\dfrac{dx}{dz}=-\dfrac{y-z}{y-x}$　，　$\dfrac{dy}{dz}=-\dfrac{z-x}{y-x}$

2\. 解：自变量变换$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x+y \\ \dfrac{y}{x} \end{bmatrix}$

因变量变换$z = wx =: \Phi\left(\begin{bmatrix} x \\ y \end{bmatrix}, w\left(\begin{bmatrix} u \\ v \end{bmatrix}(x,y)\right)\right)$

则有

$$\boldsymbol{D}z(x,y) = \boldsymbol{D}_{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]}\Phi + \boldsymbol{D}_w\Phi \cdot \boldsymbol{D}w(u,v) \cdot \boldsymbol{D}\begin{bmatrix} u \\ v \end{bmatrix}(x,y)$$

$$\begin{bmatrix} z_x & z_y \end{bmatrix} = \begin{bmatrix} w & 0 \end{bmatrix} + x \cdot \begin{bmatrix} w_u & w_v \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ -\dfrac{y}{x^2} & \dfrac{1}{x} \end{bmatrix}$$

$$z_x = w + xw_u - \frac{y}{x}w_v = \frac{z}{x} + xw_u - \frac{y}{x}w_v =: \Psi\left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix}, z\right)$$

$$z_y = xw_u + w_v =: \Theta\left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} u \\ v \end{bmatrix}\right)$$

$$(i)\begin{bmatrix} z_{xx} & z_{xy} \end{bmatrix} = \boldsymbol{D}_{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]}\Psi + \boldsymbol{D}_{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]}\Psi \cdot \boldsymbol{D}\begin{bmatrix} u \\ v \end{bmatrix}(x,y) + \boldsymbol{D}_z\Psi \cdot \boldsymbol{D}z(x,y)$$

$$= \begin{bmatrix} -\dfrac{z}{x^2} + w_u + \dfrac{y}{x^2}w_v & -\dfrac{w_v}{x} \end{bmatrix}$$

$$+ \begin{bmatrix} xw_{uu} - \dfrac{y}{x}w_{uv} & xw_{uv} - \dfrac{y}{x}w_{vv} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -\dfrac{y}{x^2} & \dfrac{1}{x} \end{bmatrix}$$

$$+ \frac{1}{x}\begin{bmatrix} \dfrac{z}{x} + xw_u - \dfrac{y}{x}w_v & xw_u + w_v \end{bmatrix}$$

$$\begin{cases} z_{xx} = 2w_u + xw_{uu} - \dfrac{2y}{x}w_{uv} + \dfrac{y^2}{x^3}w_{vv} \\ z_{xy} = w_u + xw_{uu} + w_{uv} - \dfrac{y}{x}w_{uv} - \dfrac{y}{x^2}w_{vv} \end{cases}$$

$$(ii)\begin{bmatrix} z_{xy} & z_{yy} \end{bmatrix} = \boldsymbol{D}_{\left[\begin{smallmatrix} x \\ y \end{smallmatrix}\right]}\Theta + \boldsymbol{D}_{\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]}\Theta \cdot \boldsymbol{D}\begin{bmatrix} u \\ v \end{bmatrix}(x,y)$$

$$= \begin{bmatrix} w_u & 0 \end{bmatrix} + \begin{bmatrix} xw_{uu} + w_{uv} & xw_{uv} + w_{vv} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -\dfrac{y}{x^2} & \dfrac{1}{x} \end{bmatrix}$$

$$\begin{cases} z_{xy} = w_u + xw_{uu} + w_{uv} - \dfrac{y}{x}w_{uv} - \dfrac{y}{x^2}w_{vv} \\ z_{yy} = xw_{uu} + 2w_{uv} + \dfrac{1}{x}w_{vv} \end{cases}$$

代入化简即可得$\left(\dfrac{y^2}{x^3} + \dfrac{2y}{x^2} + \dfrac{1}{x}\right)w_{vv} = 0$

$\left(x+y\right)^2 w_{vv}=0$即$u^2 w_{vv}=0$

五，多元函数积分题

1，本积分为第二类曲面积分，将$Z=\sqrt{x^2+y^2}$直接向xoy面投影，立即有：

$Z_x=\frac{x}{\sqrt{x^2+y^2}},Z_y=\frac{y}{\sqrt{x^2+y^2}},$

原式为：$-\iint\limits_{D_{xy}}\frac{-x}{\sqrt{x^2+y^2}}\left[x\left[f\left(x\right)+g\left(y\right)\right]+2x-y\right]+\left[y\left[f\left(x\right)+g\left(y\right)\right]+2y+x\right]\frac{-y}{\sqrt{x^2+y^2}}$

$+\sqrt{x^2+y^2}\left[f\left(x\right)+g\left(y\right)+1\right]dxdy$

$=\iint\limits_{D_{xy}}\sqrt{x^2+y^2}dxdy=\int_0^{2\pi}d\theta\int_2^4 r\bullet rdr=\frac{112}{3}\pi$

2,容易知道，截面是一个等边三角形，边长为$\sqrt{2}$，$\boldsymbol{n}=\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$

由$Stocks$公式$:\oint\limits_{\partial S}\boldsymbol{f}\bullet\boldsymbol{\tau}dl=\iint\limits_S\left(\nabla\times\boldsymbol{a}\right)\bullet\boldsymbol{n}ds$,则原式为$:\iint\limits_S\frac{1}{\sqrt{3}}\left(-2y-2z-2z-2x-2x-2y\right)ds$

$=-\frac{4}{\sqrt{3}}\iint\limits_S\left(x+y+z\right)ds=-\frac{4}{\sqrt{3}}\iint\limits_S ds=-\frac{4}{\sqrt{3}}\bullet\frac{\sqrt{3}}{4}\bullet 2=-2$

3,引入广义极坐标$:\begin{cases}x=2r\cos\theta\\ y=3r\sin\theta\end{cases},\frac{\partial\left(r,\theta\right)}{\partial\left(x,y\right)}=6r$

曲线方程为：$r^2=3\sin 2\theta,$

由于图线具有对称性，所以仅计算第一象限部分即可。

$D=\left\{\left(r,\theta\right)\middle|\theta\in\left[0,\frac{\pi}{2}\right],r\in\left[0,\sqrt{3\sin 2\theta}\right]\right\}$

所求面积为$:2\iint\limits_D 6r\mathrm{d}r\mathrm{d}\theta=12\int_0^{\frac{\pi}{2}}\mathrm{d}\theta\int_0^{\sqrt{3\sin 2\theta}}rdr=18$

六、证明题

1. 证明：设$L(x,y,z;\lambda)=\ln x+2\ln y+3\ln z+\lambda\left(x^2+y^2+z^2-6R^2\right)$

$$(i)\begin{cases}L_x=\dfrac{1}{x}+2\lambda x=0\\ L_y=\dfrac{2}{y}+2\lambda y=0\\ L_z=\dfrac{3}{z}+2\lambda z=0\\ x^2+y^2+z^2=6R^2\end{cases}\Rightarrow\begin{cases}\lambda=-\dfrac{1}{2R^2}\\ x=R\\ y=\sqrt{2}R\\ z=\sqrt{3}R\end{cases}$$

$$(ii)\boldsymbol{H}_L=\begin{bmatrix}L_{xx}&L_{xy}&L_{xz}\\L_{yx}&L_{yy}&L_{yz}\\L_{zx}&L_{zy}&L_{zz}\end{bmatrix}=\begin{bmatrix}-\dfrac{1}{x^2}+2\lambda&0&0\\0&-\dfrac{2}{y^2}+2\lambda&0\\0&0&-\dfrac{3}{z^2}+2\lambda\end{bmatrix}=\begin{bmatrix}-\dfrac{2}{R^2}&0&0\\0&-\dfrac{2}{R^2}&0\\0&0&-\dfrac{2}{R^2}\end{bmatrix}$$

$Hesse$阵为负定矩阵

即有$\left(R,\sqrt{2}R,\sqrt{3}R\right)^T$处为严格极大值点

$$\ln x+2\ln y+3\ln z\le\ln(R)+2\ln\left(\sqrt{2}R\right)+3\ln\left(\sqrt{3}R\right)$$

$$\ln\left(xy^2z^3\right)\le\ln\left(R\cdot 2R^2\cdot 3\sqrt{3}R^3\right)=\ln\left(6\sqrt{3}R^6\right)$$

$$xy^2z^3\le 6\sqrt{3}R^6$$

$$x^2y^4z^6\le 108R^{12}=108\left(\frac{x^2+y^2+z^2}{6}\right)^6$$

令$a=x^2,b=y^2,c=z^2$

则有$ab^2c^3\le 108\left(\dfrac{a+b+c}{6}\right)^6$

命题得证

2,注意到$(0,0)$为区域内奇点，设L围成区域为Σ，构造$\Sigma_\varepsilon:4x^2+y^2\le\varepsilon^2$，边界为$L_\varepsilon$

在区域$\Sigma-\Sigma_\varepsilon$上运用格林公式：

$$\oint_{L\text{-}L_\varepsilon}\frac{(4x-y)dx+(x+y)dy}{4x^2+y^2}=\iint_{\Sigma-\Sigma_\varepsilon}\frac{-2y(4x-y)-\left(4x^2+y^2\right)-\left(4x^2+y^2\right)+(x+y)8x}{\left(4x^2+y^2\right)^2}ds=0$$

故有原式为：$\displaystyle\oint_{L_\varepsilon}\frac{(4x-y)dx+(x+y)dy}{4x^2+y^2}=\oint_{L_\varepsilon}\frac{(4x-y)dx+(x+y)dy}{\varepsilon^2}$

此时，区域无奇点，再使用格林公式：$\displaystyle\frac{1}{\varepsilon^2}\iint_{\Sigma_\varepsilon}1\text{-}(-1)\,dxdy=\frac{2}{\varepsilon^2}\pi\bullet\varepsilon\bullet\frac{\varepsilon}{2}=\pi$