

# 复旦大学航空航天系与技术科学类

2018-2019 学年第一学期《数学分析 B》一元微分学混合教学测试

## A 卷 共 16 页

课程代码:

考试形式: ☐ 开卷 ☐ 闭卷 2018 年 11 月 02 日 13:30-17:30

(本试卷答卷时间为 240 分钟, 答案必须写在试卷上, 做在草稿纸上无效)

专业: \_\_\_\_\_ 学号: \_\_\_\_\_ 姓名: \_\_\_\_\_ 成绩: \_\_\_\_\_

题号	1-1	1-2	1-3	2-1	2-2	2-3	2-4	3-1-1	3-1-2	3-1-3
得分										
题号	3-2-1	3-2-2	3-3	3-4	3-5	3-6-1	3-6-2	3-7	3-8-1	3-8-2
得分										
题号	3-9-1	3-9-2	3-10-1	3-10-2	3-10-3	3-11-1	3-11-2	4-1	4-2	4-3
得分										
题号	5-1	5-2	5-3	6-1	6-2	6-3	6-4	6-5	总分	百分
得分										

### 一、严格表述题 (每题 5 分, 共 3 题, 共 15 分)

1. 叙述: 函数极限的集聚刻画、序列刻画、振幅刻画.

解: ①集聚刻画:  $\forall \varepsilon > 0, \exists \delta_\varepsilon > 0$ , 成立  $f(x) \in B_\varepsilon(y_0), \forall x \in \dot{B}_{\delta_\varepsilon}(x_0) \cap \mathcal{D}_x$ .

②序列刻画:  $\forall \{x_n\} \subset \mathcal{D}_x \setminus \{x_0\}, x_n \rightarrow x_0 \in \mathbb{R}$ , 有  $f(x_n) \rightarrow y_0 \in \mathbb{R}$

③振幅刻画:  $\forall \varepsilon > 0, \exists \delta_\varepsilon > 0$ , 成立  $|f(\tilde{x}_n) - f(\hat{x}_n)| < \varepsilon, \forall \tilde{x}_n, \hat{x}_n \in \dot{B}_{\delta_\varepsilon}(x_0) \cap \mathcal{D}_x$

2. 叙述: 反函数的存在性 (连续性) 定理与可导性定理.

解: ①反函数存在性定理: 设  $f(x)$  在  $[a, b]$  上  $\uparrow$ , 则有  $f([a, b]) = [f(a), f(b)] \Leftrightarrow f(x) \in C[a, b]$

②进一步设有  $\exists \frac{df}{dx}(x_0) \neq 0$ , 则有  $\exists \frac{df^{-1}}{dy}(y_0) = \frac{1}{\frac{df}{dx}(x_0)}, y_0 = f(x_0)$ .

3. 叙述: 有界函数上、下极限的定义.

解: ①上极限:  $\overline{\lim} x_n \triangleq \lim_{n \rightarrow +\infty} \sup_{k \geq n} x_k = \inf_n \sup_{k \geq n} x_k$

②下极限:  $\underline{\lim} x_n \triangleq \lim_{n \rightarrow +\infty} \inf_{k \geq n} x_k = \sup_n \inf_{k \geq n} x_k$

二、判断简答题（判断下列命题是否正确，如果是正确的，请回答“是”，并予以简要证明；如果是错误的，请回答“否”，并举反例。答“是或否”2分，简要证明或举反例3分）（每题5分，共4题，共20分）

1. ①函数在一点单侧连续，则其在该点单侧可导.

②函数在一点单侧可导，则其在该点单侧连续.

解： ①否. 考虑  $f(x) = \begin{cases} x \sin \frac{1}{x}, & x > 0 \\ 0, & x = 0 \end{cases}$ . 有  $\exists f(0+0) = \lim_{x \rightarrow 0+0} x \sin \frac{1}{x} = 0$ .

而  $f'_+(0) \triangleq \lim_{x \rightarrow 0+0} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0+0} \frac{x \sin \frac{1}{x}}{x} = \lim_{x \rightarrow 0+0} \sin \frac{1}{x}$  不存在.

故不单侧可导.

②是. 设有  $\exists f'_+(0) = \lim_{x \rightarrow 0+0} \frac{f(x) - f(0)}{x} \in \mathbb{R}$ , 则有

$f(x) = f(0) + f'_+(0)x + x \cdot o(1) \rightarrow f(0)$ , 当  $x \rightarrow 0+0$ .

亦即有单侧连续性.

2. 设函数  $f(x), g(x)$  在  $(a, +\infty)$  上一致连续, 则有  $(f+g)(x)$  在  $(a, +\infty)$  上一致连续.

解： 是. 估计

$$|(f+g)(\tilde{x}_n) - (f+g)(\hat{x}_n)| \leq |f(\tilde{x}_n) - f(\hat{x}_n)| + |g(\tilde{x}_n) - g(\hat{x}_n)| < 2\varepsilon, \forall |\tilde{x}_n - \hat{x}_n| < \{\min \delta_{\varepsilon, l}, \delta_{\varepsilon, g}\}.$$

式中  $\delta_{\varepsilon, l}, \delta_{\varepsilon, g} > 0$  为  $f(x), g(x)$  对应的自变量的限制.

3. 考虑  $E \subset \mathbb{R}$  (可以有界或者无界) 上任意的二个序列  $\{\tilde{x}_n\}, \{\hat{x}_n\}$ , 当  $\exists \lim_{x \rightarrow +\infty} |\tilde{x}_n - \hat{x}_n| = 0$  时, 有  $\exists \lim_{x \rightarrow +\infty} |f(\tilde{x}_n) - f(\hat{x}_n)| = 0$ , 则有  $f(x)$  在  $E$  上一致连续.

**解:** 是. 采用反证法. 假设  $f(x)$  在  $E$  上不一致连续, 则有

$$\exists \varepsilon_* > 0, \forall \delta > 0, \exists \tilde{x}_\delta, \hat{x}_\delta \in E, s.t. \begin{cases} |\tilde{x}_\delta - \hat{x}_\delta| < \delta \\ |f(\tilde{x}_\delta) - f(\hat{x}_\delta)| \geq \varepsilon_* \end{cases}$$

可取  $\delta_n = \frac{1}{n}$ , 则有  $\exists \{\tilde{x}_n\}, \{\hat{x}_n\} \subset E, s.t. \begin{cases} |\tilde{x}_n - \hat{x}_n| < \frac{1}{n} \\ |f(\tilde{x}_n) - f(\hat{x}_n)| \geq \varepsilon_* \end{cases}$

此时  $\exists \lim(\tilde{x}_n - \hat{x}_n) = 0$ , 而  $f(\tilde{x}_n) - f(\hat{x}_n) \not\rightarrow 0$ .

故与题设矛盾.

4. 无界区间上可导的一致连续的函数, 其导数一定有界.

**解:** 否. 考虑  $f(x) = x + \frac{1}{\sqrt{x}} \cos x^2$ . 有  $\exists \lim_{x \rightarrow +\infty} (f(x) - x) = \lim_{x \rightarrow +\infty} \frac{1}{\sqrt{x}} \cos x^2 = 0$ .

由于  $x$  在  $[1, +\infty)$  上一致连续, 故有  $f(x)$  在  $\mathbb{R}^+$  上一致连续.

$$f'(x) = 1 - \frac{1}{2x^{\frac{3}{2}}} \cos x^2 - \frac{1}{\sqrt{x}} \sin x^2 \cdot 2x = 1 - \frac{1}{2x^{\frac{3}{2}}} \cos x^2 - 2\sqrt{x} \sin x^2.$$

式中  $\frac{1}{2x^{\frac{3}{2}}} \cos x^2$  在  $[1, +\infty)$  上有界. 对于  $\sqrt{x} \sin x^2$ , 有

$\exists x_n \rightarrow +\infty, s.t. \sqrt{x_n} \sin x_n^2 \rightarrow +\infty$ , 故  $\sqrt{x} \sin x^2$  在  $[1, +\infty)$  上无界.

注: 可取  $x_n = \sqrt{2n\pi + \frac{\pi}{2}}$ .

### 三、计算证明题（每题 10 分，共 20 题，共 200 分）

1. 基于无限小分析方法，计算下列函数与数列极限. 注：可以采用其它方法.

① 计算  $\lim_{x \rightarrow 0} \frac{\ln(1 + \sin^2 x) - 6(\sqrt[3]{2 - \cos x} - 1)}{x^4}$

② 计算  $\lim_{x \rightarrow +\infty} \left[ \left( x^3 - x^2 + \frac{x}{2} \right) e^{\frac{1}{x}} - \sqrt{x^6 - 1} \right]$

③ 计算  $\lim_{n \rightarrow +\infty} n^2 (\sqrt[n]{a} - \sqrt[n+1]{a}), a > 0$

解：①

$$\begin{aligned} \ln(1 + \sin^2 x) &= \sin^2 x - \frac{\sin^4 x}{2} + o(x^5) = \left( x - \frac{x^3}{3!} + o(x^4) \right)^2 - \frac{1}{2} \left( x - \frac{x^3}{3!} + o(x^4) \right)^4 + o(x^5) \\ &= x^2 + \left( -\frac{1}{3} - \frac{1}{2} \right) x^4 + o(x^4) = x^2 - \frac{5}{6} x^4 + o(x^4). \end{aligned}$$

$$\begin{aligned} \sqrt[3]{2 - \cos x} &= (1 + 1 - \cos x)^{\frac{1}{3}} = \left( 1 + 2 \sin^2 \frac{x}{2} \right)^{\frac{1}{3}} = 1 + \frac{2}{3} \sin^2 \frac{x}{2} + \left( \frac{\frac{1}{3}}{2} \right) 4 \sin^4 \frac{x}{2} + o(x^5) \\ &= 1 + \frac{2}{3} \left( \frac{x}{2} - \frac{1}{3!} \frac{x^3}{8} + o(x^4) \right)^2 + \left( \frac{\frac{1}{3}}{2} \right) 4 \cdot \frac{x^4}{16} + o(x^4) \\ &= 1 + \frac{1}{6} x^2 - \frac{1}{24} x^4 + o(x^4) \end{aligned}$$

$$\text{故有 } \ln(1 + \sin^2 x) - 6(\sqrt[3]{2 - \cos x} - 1) = x^2 - \frac{5}{6} x^4 - x^2 + \frac{1}{4} x^4 + o(x^4) = -\frac{7}{12} x^4 + o(x^4),$$

故极限为  $-\frac{7}{12}$ .

②

$$\begin{aligned} f(x) &\stackrel{x=\frac{1}{t}}{=} \left( \frac{1}{t^3} - \frac{1}{t^2} + \frac{1}{2t} \right) e^t - \sqrt{\frac{1}{t^6} - 1} = \frac{1}{t^3} \left( 1 - t + \frac{1}{2} t^2 \right) e^t - \frac{1}{t^3} (1 - t^6)^{\frac{1}{2}} \\ &= \frac{1}{t^3} \left( 1 - t + \frac{1}{2} t^2 \right) \left( 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + o(t^3) \right) - \frac{1}{t^3} (1 + o(t^3)) \\ &= \frac{1}{t^3} \left[ 1 + \left( \frac{1}{2} - 1 + \frac{1}{2} \right) t^2 + \left( \frac{1}{2} - \frac{1}{2} + \frac{1}{6} \right) t^3 + o(t^3) \right] - \frac{1}{t^3} + o(1) \\ &= \frac{1}{6} + o(1) \rightarrow \frac{1}{6} \end{aligned}$$

③

$$\begin{aligned} n^2 (\sqrt[n]{a} - \sqrt[n+1]{a}) &= n^2 \left( e^{\frac{1}{n} \ln a} - e^{\frac{1}{n+1} \ln a} \right) \\ &= n^2 \cdot \left[ 1 + \frac{1}{n} \ln a + \frac{1}{2n^2} \ln^2 a + o\left(\frac{1}{n^2}\right) - 1 - \frac{1}{n+1} \ln a - \frac{1}{2(n+1)} \ln^2 a + o\left(\frac{1}{n^2}\right) \right] \\ &= n^2 \cdot \left[ \left( \frac{1}{n} - \frac{1}{n+1} \right) \ln a + \frac{1}{2} \left( \frac{1}{n^2} - \frac{1}{(n+1)^2} \right) \ln^2 a + o\left(\frac{1}{n^2}\right) \right] \\ &= n^2 \cdot \left[ \frac{1}{n} \left( 1 - \frac{1}{1 + \frac{1}{n}} \right) \ln a + \frac{1}{2n^2} \left( 1 - \frac{1}{(1 + \frac{1}{n})^2} \right) \ln^2 a + o\left(\frac{1}{n^2}\right) \right] \\ &= n^2 \cdot \left[ \frac{1}{n} \left( \frac{1}{n} + o\left(\frac{1}{n}\right) \right) \ln a + \frac{1}{2n^2} o(1) \ln^2 a + o\left(\frac{1}{n^2}\right) \right] = \ln a + o(1) \rightarrow \ln a \end{aligned}$$

2. ① 推导: 在  $x = 0$  处 (相应的邻域),  $\arcsin x$  至  $o(x^n)$  的展开式.

② 计算  $(\arcsin x)^2$  在  $x = 0$  处的  $n$  阶导数.

解: ①  $\arcsin x =: f(x)$ ,  $f'(x) = \frac{1}{\sqrt{1-x^2}} = (1-x^2)^{-\frac{1}{2}} = 1 + \sum_{k=1}^n \binom{-\frac{1}{2}}{k} (-1)^k x^{2k} + o(x^{2k})$ .

$\Rightarrow f(x) = x + \sum_{k=1}^n \binom{-\frac{1}{2}}{k} (-1)^k \frac{x^{2k+1}}{2k+1} + o(x^{2k+1})$ . 式中

$$\begin{aligned} \binom{-\frac{1}{2}}{k} (-1)^k \frac{1}{2k+1} &= \frac{(-\frac{1}{2})(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-(k-1))}{k!} = \frac{\frac{1}{2} \cdot (\frac{1}{2}+1) \cdots (\frac{1}{2}+(k-1))}{k!} \cdot \frac{1}{2k+1} \\ &= \frac{1 \cdot 3 \cdots (2k-1)}{2^k \cdot k!} \frac{1}{2k+1} = \frac{(2k-1)!!}{(2k)!!} \frac{1}{2k+1}. \end{aligned}$$

故有  $f^n(x) = x + \sum_{k=1}^n \frac{(2k-1)!!}{(2k)!!} \frac{1}{2k+1} x^{2k+1} + o(x^{2k+1})$ .

$$\text{② 考虑 } (\arcsin x)^2 = \left[ \sum_{k=0}^n \frac{(2k-1)!!}{(2k)!!} \frac{1}{2k+1} x^{2k+1} + o(x^{2k+1}) \right]^2$$

考虑  $(2i+1) + (2j+1) = 2n$ , 即  $i+j+1 = n$ , 则有  $2n$  阶导数为:

$$(2n)! \cdot \left[ \sum_{i+j+1=n} \frac{(2i-1)!!}{(2i)!!} \frac{1}{2i+1} \frac{(2j-1)!!}{(2j)!!} \frac{1}{2j+1} \right];$$

奇次阶导数为零.

$$3. f(x) = \begin{cases} (x-3) \arctan \frac{1}{x-3}, & x \neq 3 \\ 0, & x = 3 \end{cases}, \text{ 计算单侧变化率 } f'_+(x), f'_-(x).$$

解: ① 考虑连续性.

$$\begin{cases} f(3+0) = \lim_{x \rightarrow 3+0} (x-3) \arctan \frac{1}{x-3} = 0 \\ f(3-0) = \lim_{x \rightarrow 3-0} (x-3) \arctan \frac{1}{x-3} = 0 \end{cases}$$

则有  $f(x)$  在  $x = 3$  处连续.

$$\text{② } \begin{cases} f'_+(3) = \lim_{x \rightarrow 3+0} \frac{f(x) - f(3)}{x-3} = \lim_{x \rightarrow 3+0} \frac{(x-3) \arctan \frac{1}{x-3}}{x-3} = \frac{\pi}{2} \\ f'_-(3) = \lim_{x \rightarrow 3-0} \frac{f(x) - f(3)}{x-3} = \lim_{x \rightarrow 3-0} \frac{(x-3) \arctan \frac{1}{x-3}}{x-3} = -\frac{\pi}{2} \end{cases}.$$

$$f'_+(x) = f'_-(x) = \arctan \frac{1}{x-3} + (x-3) \frac{1}{1 + \frac{1}{(x-3)^2}} \left( -\frac{1}{(x-3)^2} \right) = \arctan \frac{1}{x-3} - \frac{x-3}{1 + (x-3)^2},$$

当  $x \neq 3$ .

4. 计算函数  $f(x) = x^{x^a} + x^{a^x} + a^{x^x}$ , ( $a > 0, x > 0$ ) 的一阶导数.

解: ①  $x^{x^a} = e^{x^a \cdot \ln x}$  有

$$\begin{aligned}\frac{d}{dx} x^{x^a} &= x^{x^a} \cdot \frac{d}{dx} (x^a \ln x) = x^{x^a} \left[ \frac{d}{dx} e^{a \ln x} \cdot \ln x + x^{a-1} \right] \\ &= x^{x^a} \left[ x^a \cdot \frac{a}{x} \cdot \ln x + x^{a-1} \right] = x^{x^a} [a x^{a-1} \ln x + x^{a-1}] = x^{x^a+a-1} (a \ln x + 1)\end{aligned}$$

②  $x^{a^x} = e^{a^x \cdot \ln x}$  有

$$\frac{d}{dx} x^{a^x} = x^{a^x} \cdot \frac{d}{dx} (a^x \ln x) = x^{a^x} \left( a^x \ln a \ln x + \frac{a^x}{x} \right)$$

③  $a^{x^x} = e^{x^x \cdot \ln a}$  有

$$\frac{d}{dx} a^{x^x} = a^{x^x} \ln a \cdot \frac{d}{dx} (x^x) = a^{x^x} \ln a \cdot \frac{d}{dx} e^{x \ln x} = a^{x^x} \ln a \cdot x^x [\ln x + 1]$$

5. 计算  $f(x) = \begin{cases} x \left| \sin \frac{\pi}{x} \right|, & x \neq 0 \\ 0, & x = 0 \end{cases}$  在零点的导数, 并说明: 零点的任意邻域都有不可导点.

解: ① 易见  $f(x)$  在 0 点连续.

②  $\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0} \left| \sin \frac{\pi}{x} \right|$ , 极限不存在, 亦即  $f(x)$  在  $x = 0$  点不可导.

③ 考虑  $|\sin y|$ ,  $y = \frac{\pi}{x} = 2n\pi + \pi \Rightarrow x_n = \frac{1}{2n+1}$ .

$$\text{考虑} \begin{cases} f'_+(x_n \pm 0) = \lim_{x \rightarrow x_n+0} \frac{x \left| \sin \frac{\pi}{x} \right|}{x - x_n} = \lim_{x \rightarrow x_n+0} \frac{x \sin \frac{\pi}{x}}{x - x_n} = +\infty \\ f'_-(x_n \pm 0) = \lim_{x \rightarrow x_n-0} \frac{x \left| \sin \frac{\pi}{x} \right|}{x - x_n} = \lim_{x \rightarrow x_n-0} \frac{-x \sin \frac{\pi}{x}}{x - x_n} = -\infty \end{cases}$$

亦即  $f(x)$  在  $x_n = \frac{1}{2n+1}$  ( $n \in \mathbb{N}$ ) 点均不可导.

6. ①计算:  $\lim_{n \rightarrow +\infty} \frac{1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}}{\ln 2\sqrt{n}}.$

②设  $x_1 = \sin x_0 \in \mathbb{R}^+, x_{n+1} = \sin x_n$ , 求:  $\lim_{n \rightarrow +\infty} \sqrt{n}x_n.$

解: ①  $\frac{1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}}{\ln 2\sqrt{n}} \sim \frac{\frac{1}{2n+1}}{\ln \sqrt{\frac{n+1}{n}}} = \frac{\frac{1}{2n+1}}{\frac{1}{2} \ln(1 + \frac{1}{n})} = \frac{\frac{1}{2n+1}}{\frac{1}{2}(\frac{1}{n} + o(\frac{1}{n}))} = \frac{\frac{1}{2+\frac{1}{n}} \cdot \frac{1}{n}}{\frac{1}{2} \ln(1 + \frac{1}{n})}.$

②  $x_{n+1} = \sin x_n \leq x_n$ , 故  $x_n \downarrow$  且有  $x_n \downarrow 0$ .

$$nx_n^2 = \frac{n}{\frac{1}{x_n^2}} \sim \frac{n+1-n}{\frac{1}{x_{n+1}^2} - \frac{1}{x_n^2}} = \frac{1}{\frac{1}{\sin^2 x_n - \frac{1}{x_n^2}}}$$

其中

$$\begin{aligned} \sin^2 x_n - \frac{1}{x_n^2} &= \frac{1}{(x_n - \frac{x_n^3}{3!} + o(x_n^4))^2} - \frac{1}{x_n^2} = \frac{1}{x_n^2} [(1 - \frac{x_n^2}{6} + o(x_n^3))^{-2} - 1] \\ &= \frac{1}{x_n^2} [\frac{x_n^2}{3} + o(x_n^2)] = \frac{1}{3} + o(1) \rightarrow \frac{1}{3} \end{aligned}$$

故有  $nx_n^2 \rightarrow 3$ , 则有  $\sqrt{n}x_n = \sqrt{3}$ .

7. 设  $|x_1| \leq 2, x_{n+1} = \sqrt{4-x_n} (n \in \mathbb{N})$ , 求:  $\lim_{n \rightarrow +\infty} x_n.$

解: ①分析有界性: 显然  $\{x_n\}$  有界且非负.

②计算上下极限:

$$\begin{cases} \overline{x_{n+1}} = \sqrt{4 - \underline{x_n}} = \sqrt{4 - \underline{x_n}} \\ \underline{x_{n+1}} = \sqrt{4 - \overline{x_n}} = \sqrt{4 - \overline{x_n}} \end{cases}, \text{ 故有 } \overline{x_n} = \underline{x_n} = \frac{-1 + \sqrt{17}}{2}.$$

8. ①设  $f(x) \in C[a, b]$ ,  $\{x_i\}_{i=1}^n \subset [a, b]$ , 证明:

$$\exists \xi \in [a, b], s.t. f(\xi) = \frac{1}{n}[f(x_1) + \cdots + f(x_n)]$$

②按上题条件, 再设有  $f(a) = f(b)$ , 证明: 对  $\forall n \in \mathbb{N}$ , 有

$$\exists \xi \in [a, b], s.t. f(\xi_n) = f\left(\xi_n + \frac{b-a}{n}\right)$$

注: ①与②都基于介值定理, 且②的获得可以考虑基于①.

$$\text{解: ①} \frac{1}{n} [f(x_1) + \cdots + f(x_n)] \sim \begin{cases} \leq \frac{1}{n} \cdot n \max\{f(x_1), \cdots, f(x_n)\} = \max\{f(x_1), \cdots, f(x_n)\} \\ \geq \frac{1}{n} \cdot n \min\{f(x_1), \cdots, f(x_n)\} = \min\{f(x_1), \cdots, f(x_n)\} \end{cases}.$$

由于  $f(x) \in C[a, b]$ , 按介值定理, 有  $\exists \xi \in [a, b], s.t. f(\xi) = \frac{1}{n}[f(x_1) + \cdots + f(x_n)]$ .

②考虑  $\varphi_n(x) = f(x) - f\left(x + \frac{b-a}{n}\right), x \in \left[a, b - \frac{b-a}{n}\right]$ .

$$\begin{cases} \varphi_n(a) = f(a) - f\left(a + \frac{b-a}{n}\right) \\ \varphi_n\left(a + \frac{b-a}{n}\right) = f\left(a + \frac{b-a}{n}\right) - f\left(a + 2 \cdot \frac{b-a}{n}\right) \\ \cdots \\ \varphi_n\left(a + (n-1)\frac{b-a}{n}\right) = f\left(a + (n-1)\frac{b-a}{n}\right) - f(b) \end{cases}$$

$$\Rightarrow \varphi_n(a) + \varphi_n\left(a + \frac{b-a}{n}\right) + \cdots + \varphi_n\left(a + (n-1)\frac{b-a}{n}\right) = f(b) - f(a) = 0.$$

$$\text{则有 } \frac{1}{n} \left[ \varphi_n(a) + \varphi_n\left(a + \frac{b-a}{n}\right) + \cdots + \varphi_n\left(a + (n-1)\frac{b-a}{n}\right) \right] = 0.$$

按①, 有  $\exists \xi_n \in \left[a, b - \frac{b-a}{n}\right], s.t. \varphi(\xi_n) = 0$ . 亦即  $f(\xi_n) = f\left(\xi_n + \frac{b-a}{n}\right)$

9. ①设  $f(x), g(x) \in C[a, b]$ , 都在  $(a, b)$  上可导, 且  $f(a) = f(b) = 0$ , 证明:

$$\exists \xi \in (a, b), \text{ s.t. } f'(\xi) + f(\xi)g'(\xi) = 0$$

②设  $f(x) \in C[a, b]$ , 在  $(a, b)$  上可导, 且  $f(a) = f(b) = 1$ , 证明:

$$\exists \xi, \eta \in (a, b), \text{ s.t. } e^{\xi-\eta}[f(\xi) + f'(\xi)] = 1$$

解: ①考虑利用 Rolle 定理.

$$f'(x) + f(x)g'(x) = \frac{-\frac{d}{dx}[e^{g(x)}f(x)]}{e^{g(x)}}.$$

$$\text{作 } \varphi(x) = e^{g(x)} \cdot f(x). \text{ s.t. } \begin{cases} \varphi(x) \in C[a, b] \\ \varphi(a) = \varphi(b) = 0 \end{cases}, \text{ 且 } \exists \varphi'(\xi) \in \mathbb{R}, \forall x \in (a, b).$$

$$\text{则有 } \exists \xi \in (a, b), \text{ s.t. } \varphi'(\xi) = e^{g(\xi)}[f'(\xi) + f(\xi)g'(\xi)] = 0 \Rightarrow f'(\xi) + f(\xi)g'(\xi) = 0.$$

$$\text{②考虑 } e^{\xi}[f'(\xi) + f(\xi)] = e^{\eta} = \frac{d}{dx}(e^x f(x)) \Big|_{x=\xi}.$$

$$\text{按 Lagrange 中值定理 } e^b f(b) - e^a f(a) = \frac{d}{dx}(e^x f(x)) \Big|_{x=\xi} \cdot (b - a).$$

另有  $e^b - e^a = e^{\eta}(b - a)$ . 结合  $f(a) = f(b) = 1$ , 则有

$$e^{\eta} = \frac{d}{dx}(e^x f(x)) \Big|_{x=\xi} = e^{\xi}[f'(\xi) + f(\xi)].$$

10. ①设  $\varphi(x) \in C[0, +\infty)$ ,  $\varphi(0) = 0$ ,  $\exists \lim_{x \rightarrow +\infty} \varphi(x) = +\infty$ , 证明: 对于  $\forall \lambda > 0$ ,  $\exists x_n \rightarrow +\infty$ ,  $s.t. \exists \varphi(x_n) \cos x_n \rightarrow \lambda$ .

②判断:  $f(x) = \sin(\varphi(x) \cos x)$  在  $\mathbb{R}^+$  上的一致连续性.

③判断:  $f(x) = \frac{x}{1+x^2 \sin^2 x}$  在  $\mathbb{R}^+$  上的一致连续性.

解: ①考虑  $x_n = 2n\pi - \frac{\pi}{2} + \delta_n$ . 有

$$\begin{aligned}\varphi(x_n) \cos x_n &= \varphi(2n\pi - \frac{\pi}{2} + \delta_n) \cos(-\frac{\pi}{2} + \delta_n) = \varphi(2n\pi - \frac{\pi}{2} + \delta_n) \sin(+\delta_n) \\ &= \varphi(2n\pi - \frac{\pi}{2} + \delta_n) \sin \delta_n = \varphi(2n\pi - \frac{\pi}{2} + \delta_n)(\delta_n + o(\delta_n)).\end{aligned}$$

求解  $\varphi(2n\pi - \frac{\pi}{2} + x) \cdot x = \lambda \in \mathbb{R}^+$ , 亦即  $\varphi(2n\pi - \frac{\pi}{2} + x) = \frac{\lambda}{x}$ ,  $\varphi(x) \rightarrow +\infty$ , 当  $x \rightarrow +\infty$ .

按介值定理, 可有  $\exists \delta_n \in (0, +\infty)$ ,  $s.t. \varphi(2n\pi - \frac{\pi}{2} + \delta_n) = \frac{\lambda}{\delta_n}$ .

且有  $\delta_n = \frac{\lambda}{\varphi(2n\pi - \frac{\pi}{2} + \delta_n)} \rightarrow 0$ , 当  $n \rightarrow +\infty$ .

因为  $2n\pi - \frac{\pi}{2} + \delta_n \rightarrow +\infty$ , 当  $n \rightarrow +\infty$ , 由此  $\varphi(2n\pi - \frac{\pi}{2} + \delta_n) \rightarrow +\infty$ .

②按结论①,  $\lambda, \mu \in \mathbb{R}^+$ , 可有  $\begin{cases} \xi_n = 2n\pi - \frac{\pi}{2} + \tilde{\delta}_n \\ \eta_n = 2n\pi - \frac{\pi}{2} + \hat{\delta}_n \end{cases}$

$s.t. \begin{cases} \varphi(\xi_n) \cos \xi_n \rightarrow \lambda \\ \varphi(\eta_n) \cos \eta_n \rightarrow \mu \end{cases}$ , 且  $\xi_n - \eta_n = \tilde{\delta}_n - \hat{\delta}_n \rightarrow 0$ . 故有  $f(x)$  在  $\mathbb{R}^+$  上不一致连续.

③考虑  $f(x) = \frac{x}{1+x^2 \sin^2 x} = \frac{1}{\frac{1}{x} + x \sin^2 x}$ .

式中  $\begin{cases} \frac{1}{x} \rightarrow 0, \text{ 当 } x \rightarrow +\infty \\ \tilde{x}_n \sin^2 \tilde{x}_n \rightarrow \lambda \in \mathbb{R}^+, \tilde{x}_n \rightarrow +\infty \\ \hat{x}_n \sin^2 \hat{x}_n \rightarrow \mu \in \mathbb{R}^+, \hat{x}_n \rightarrow +\infty \end{cases}$ , 在  $\tilde{x}_n - \hat{x}_n \rightarrow 0$ .

故有  $f(x)$  在  $\mathbb{R}^+$  上不一致收敛.

11. ①设有  $\exists f'(0) \in \mathbb{R}$ , 如果  $\exists \lim_{x \rightarrow 0} \varphi(x) = \lim_{x \rightarrow 0} \psi(x) = 0$ , 证明:

$$\lim_{x \rightarrow 0} \frac{f(\psi(x)) - f(\varphi(x))}{\psi(x) - \varphi(x)} = f'(0)$$

②设  $f(x)$  在  $B_\lambda(0)$  上存在一阶导数且连续 (连续可微),  $f'(0), \exists f''(0) \in \mathbb{R}$ , 证明:

$$\exists \lim_{x \rightarrow 0} \frac{f(x) - f(\ln(1+x))}{x^3} = \frac{1}{2}f''(0).$$

注:  $f(x)$  关于零点只能展开到二阶; 考虑利用结论①.

解: ①现有  $\begin{cases} \varphi(x) = o(1) \\ \psi(x) = o(1) \end{cases}$ , 另有  $f(x) = f(0) + f'(0)x + x \cdot o(1)$ .

$$\begin{aligned} \frac{f(\psi(x)) - f(\varphi(x))}{\psi(x) - \varphi(x)} &= \frac{f(0) + f'(0)\psi(x) + \psi(x) \cdot o(1) - [f(0) + f'(0)\varphi(x) + \varphi(x) \cdot o(1)]}{\psi(x) - \varphi(x)} \\ &= f'(0) + \frac{\psi(x) - \varphi(x)}{\psi(x) - \varphi(x)} \cdot o(1) = f'(0) + o(1) \rightarrow f'(0) \end{aligned}$$

②

$$\begin{aligned} \frac{f(x) - f(\ln(1+x))}{x^3} &\sim \frac{f'(x) - f'(\ln(1+x)) \cdot \frac{1}{1+x}}{3x^2} = \frac{f'(x) - f'(\ln(1+x))}{3x^2(1+x)} + \frac{xf'(x)}{3x^2(1+x)} \\ &= \frac{f'(x) - f'(\ln(1+x))}{x - \ln(1+x)} \cdot \frac{x - \ln(1+x)}{3x^2(1+x)} + \frac{1}{3} \frac{1}{1+x} \frac{f'(x)}{x} \\ &= \frac{f'(x) - f'(\ln(1+x))}{x - \ln(1+x)} \cdot \frac{\frac{1}{2}x^2 + o(x^2)}{3x^2(1+x)} + \frac{1}{3} \frac{1}{1+x} \frac{f'(x) - f'(0)}{x} \\ &\rightarrow f''(0) \cdot \frac{1}{6} + \frac{1}{3} \cdot f''(0) = \frac{1}{2}f''(0) \end{aligned}$$

#### 四、应用题（每题 10 分，共 11 题，共 110 分）

##### 1. 数列的上下极限.

① 设  $\{x_n\}$  有界, 则有结论: 存在子列分别集聚至上、下极限. 证明: 集聚至上极限的结论.

② 证明: 任意收敛之列的极限都不小于下极限、不大于上极限.

③ 证明: 上下极限计算的关系式

$$\underline{\lim} x_n + \underline{\lim} y_n \leq \underline{\lim} (x_n + y_n) \leq \begin{cases} \underline{\lim} x_n + \overline{\lim} y_n \\ \overline{\lim} x_n + \underline{\lim} y_n \end{cases} \leq \overline{\lim} x_n + \overline{\lim} y_n$$

解: ① 证明  $\exists x_{n_k} \rightarrow \overline{\lim} x_n \triangleq \lim_{n \rightarrow +\infty} \sup_{k \geq n} x_k$

$\exists x_{n_{m_l}} \in \left( \overline{\lim} x_n - \frac{1}{m}, \overline{\lim} x_n + \frac{1}{m} \right)$ , 故有  $x_{n_{m_l}} \rightarrow \overline{\lim} x_n$ , 当  $l \rightarrow +\infty$ , 当  $m \rightarrow +\infty$ .

式中  $n_{1_l} < n_{2_l} < \cdots < n_{m_l} < \cdots$ .

② 设  $x_{n_k} \rightarrow x_*$ , 当  $k \rightarrow +\infty$ .

$$\inf_{j \geq n_k} x_j \leq x_{n_k} \leq \sup_{j \geq n_k} x_j, \forall k \in \mathbb{N}$$

$$\Rightarrow \underline{\lim} x_n = \lim_{k \rightarrow +\infty} \inf_{j \geq n_k} x_j \leq \lim_{k \rightarrow +\infty} x_{n_k} \leq \lim_{k \rightarrow +\infty} \sup_{j \geq n_k} x_j = \overline{\lim} x_n$$

③ 考虑到关系式

$$\inf_{k \geq n} x_k + \inf_{k \geq n} y_k \stackrel{(i)}{\leq} \inf_{k \geq n} (x_k + y_k) \stackrel{(ii)}{\leq} \begin{cases} \liminf_{k \geq n} x_k + \sup_{k \geq n} y_k \\ \overline{\lim} \sup_{k \geq n} x_k + \inf_{k \geq n} y_k \end{cases} \stackrel{(iii)}{\leq} \sup_{k \geq n} x_k + \sup_{k \geq n} y_k$$

式中 (i)  $\inf_{k \geq n} x_k + \inf_{k \geq n} y_k \leq x_k + y_k, \forall k \in \mathbb{N} \Rightarrow \inf_{k \geq n} x_k + \inf_{k \geq n} y_k \leq \inf_{k \geq n} (x_k + y_k)$

(ii)  $\inf_{k \geq n} (x_k + y_k) \leq x_k + y_k \leq x_k + \sup_{k \geq n} y_k \Rightarrow \inf_{k \geq n} (x_k + y_k) \leq \inf_{k \geq n} x_k + \sup_{k \geq n} y_k$

(iii)  $x_k + y_k \leq \sup_{k \geq n} x_k + \sup_{k \geq n} y_k \Rightarrow \sup_{k \geq n} (x_k + y_k) \leq \sup_{k \geq n} x_k + \sup_{k \geq n} y_k, \forall k \in \mathbb{N}$

## 2. Stolz 定理与 Bernoulli-L'Hospital 法则的通识性结构.

①阐述 Stolz 定理与 Bernoulli-L'Hospital 法则的一般形式.

②设定数列差比的极限、函数导数之比的形式为有限值, 给出 Stolz 定理与 Bernoulli-L'Hospital 法则分析中所建立的关系式.

③就  $\frac{0}{0}$ -型与  $\frac{*}{\infty}$ -型, 获得最终的结论.

解: ①Stolz 定理:  $\frac{y_n}{x_n} \sim \frac{y_{n+1} - y_n}{x_{n+1} - x_n} \rightarrow l \in \mathbb{R} \cup \{\pm\infty\}$ .

$$\text{式中} \begin{cases} y_n \rightarrow 0 \\ x_n \downarrow 0 \end{cases} \quad \left( \frac{0}{0} \text{型}; \text{或 } x_n \uparrow +\infty \left( \frac{*}{\infty} \text{型} \right) \right)$$

Bernoulli-L'Hospital 法则:  $\frac{\psi(x)}{\varphi(x)} \sim \frac{\psi'(x)}{\varphi'(x)} \rightarrow l \in \mathbb{R} \cup \{\pm\infty\}$

$$\text{式中} \begin{cases} \psi(x) \rightarrow 0 \\ \varphi(x) \rightarrow 0 \end{cases}, \text{当 } x \rightarrow x_0 \in \bar{\mathbb{R}};$$

或  $\varphi(x) \rightarrow +\infty$ , 当  $x \rightarrow x_0 \in \bar{\mathbb{R}}$ , 要求  $\exists \psi'(x), \psi'(x) \in \mathbb{R}, \forall x \in \mathring{B}_{\delta_\varepsilon}^\pm(x_0)$

②设有  $\exists \lim_{n \rightarrow +\infty} \frac{y_{n+1} - y_n}{x_{n+1} - x_n} = l \in \mathbb{R}$ , 则有  $y_{n+1} - y_n = l(x_{n+1} - x_n) + \delta_n(x_{n+1} - x_n), \delta_n \rightarrow 0$ .

$$\text{故有} \begin{cases} y_{m+1} - y_m = l x_{m+1} - x_m + \delta_m(x_{m+1} - x_m) \\ y_m - y_{m-1} = l x_m - x_{m-1} + \delta_{m-1}(x_m - x_{m-1}) \\ \dots \\ y_{n+2} - y_{n+1} = l x_{n+2} - x_{n+1} + \delta_{n+1}(x_{n+2} - x_{n+1}) \\ y_{n+1} - y_n = l(x_{n+1} - x_n) + \delta_n(x_{n+1} - x_n) \end{cases}$$

$$\Rightarrow y_{m+1} - y_n = l(x_{m+1} - x_n) + \delta_m(x_{m+1} - x_m) + \dots + \delta_n(x_{n+1} - x_n).$$

$$\Rightarrow \frac{y_{m+1}}{x_{m+1}} - l = \frac{y_n}{x_{m+1}} - l \frac{x_n}{x_{m+1}} + \frac{\delta_m(x_{m+1} - x_m) + \dots + \delta_n(x_{n+1} - x_n)}{x_{m+1}},$$

$$\text{式中} \left| \frac{\delta_m(x_{m+1} - x_m) + \dots + \delta_n(x_{n+1} - x_n)}{x_{m+1}} \right| < \varepsilon \cdot \left( 1 + \frac{|x_n|}{|x_{n+1}|} \right), \forall n, m > N_\varepsilon.$$

考虑  $\tilde{x}, \hat{x} \in \mathring{B}_{\delta_\varepsilon}^\pm(x_0)$ , 应用 Cauchy 中值定理:

$$\frac{\psi(\tilde{x}) - \psi(\hat{x})}{\varphi(\tilde{x}) - \varphi(\hat{x})} = \frac{\psi'(\xi)}{\varphi'(\xi)} \rightarrow l \in \mathbb{R}$$

$$\Rightarrow \frac{\psi(\tilde{x})}{\varphi(\tilde{x})} - l = \frac{\psi(\hat{x})}{\varphi(\hat{x})} + \left( 1 - \frac{\psi(\hat{x})}{\varphi(\hat{x})} \right) \frac{\psi'(\xi)}{\varphi'(\xi)} - l = \frac{\psi(\tilde{x})}{\varphi(\tilde{x})} - \frac{\psi(\tilde{x})}{\varphi(\tilde{x})} \frac{\psi'(\xi)}{\varphi'(\xi)} + \frac{\psi'(\xi)}{\varphi'(\xi)} - l$$

$$\text{式中} \left| \frac{\psi'(\xi)}{\varphi'(\xi)} - l \right| < \varepsilon, \forall \tilde{x}, \hat{x} \in \mathring{B}_{\delta_\varepsilon}^\pm(x_0).$$

当  $\psi(x) \rightarrow \infty$ , 对于确定的  $\hat{x}$ , 可有  $\left| \frac{\psi(\tilde{x})}{\varphi(\tilde{x})} \right|, \left| \frac{\psi(\hat{x})}{\varphi(\hat{x})} \right| < \varepsilon$ .

$$\textcircled{3} \text{ 当 } \begin{cases} y_n \rightarrow 0 \\ x_n \downarrow 0 \end{cases}, \text{ 令 } y_n \rightarrow +\infty, \text{ 有 } \left| \frac{y_{n+1}}{x_{n+1}} - l \right| \leq \varepsilon$$

$$(i) \text{ 当 } \begin{cases} \psi(\tilde{x}) \rightarrow 0 \\ \varphi(\tilde{x}) \rightarrow 0 \end{cases}, \text{ 令 } \tilde{x} \rightarrow x_0 \in \overline{\mathbb{R}}, \text{ 有 } \left| \frac{\psi(\hat{x})}{\varphi(\hat{x})} \right| \leq \varepsilon, \forall \hat{x} \in \mathring{B}_{\delta_\varepsilon}(x_0), \text{ 当 } x \rightarrow x_0 \in \mathbb{R}.$$

$$(ii) x_m \uparrow +\infty, \text{ 对确定的 } n \text{ 可有 } \left| \frac{y_n}{x_{m+1}} \right| < \varepsilon, \left| \frac{x_n}{x_{m+1}} \right| < \varepsilon$$

### 3. 相关估计.

① 设有  $|x_{n+1}| \leq \lambda|x_n| + \mu$ ,  $0 < \lambda < 1$ ,  $\mu \in \mathbb{R}^+$ , 证明:  $\{x_n\}$  有界.

② 基于①中结果, 设  $\exists \lim_{n \rightarrow +\infty} (\lambda x_{n+1} + x_n) = l \in \mathbb{R}$ ,  $\lambda > 1$ , 证明:  $\{x_n\}$  有界并计算其极限.

③ 证明: 在  $\mathbb{R}$  上定义的一致连续函数, 满足线性增长控制:

$$|f(x)| \leq A + B|x|, A, B \in \mathbb{R}^+$$

④ 基于③中结果, 推导: 二个一致连续的函数的乘积, 依然一致连续的充分性条件.

⑤ 设有  $\exists \lim_{x \rightarrow 0} f(x) = 0$  且  $\exists \lim_{x \rightarrow 0} \frac{1}{x} \left[ f(x) - f\left(\frac{x}{\lambda}\right) \right] = 0$ ,  $\lambda > 1$ , 证明:  $\exists \lim_{x \rightarrow 0} \frac{f(x)}{x} = 0$ .

解: ①

$$\begin{aligned} |x_{n+1}| &\leq \lambda|x_n| + \mu \leq \lambda(\lambda|x_{n-1}| + \mu) + \mu = \lambda^2|x_{n-1}| + (\lambda+1)\mu \\ &\leq \lambda^2(\lambda|x_{n-2}| + \mu) + (\lambda+1)\mu = \lambda^3|x_{n-2}| + (\lambda^2 + \lambda + 1)\mu \\ &\leq \dots \\ &\leq \lambda^n|x_1| + (\lambda^{n-1} + \lambda^{n-2} + \dots + \lambda + 1)\mu \\ &< \lambda^n|x_1| + \frac{1}{1-\lambda}\mu \leq |x_1| + \frac{\mu}{1-\lambda}, \end{aligned}$$

故有  $\{x_n\}$  有界.

② 设  $\exists \lim (\lambda x_{n+1} + x_n) = l \in \mathbb{R}$ ,  $\lambda > 1$ , 则有

$$|\lambda x_{n+1}| \leq |x_n| + (|l| + \varepsilon), n > N_\varepsilon \Rightarrow |x_{n+1}| \leq \frac{1}{\lambda}|x_n| + \frac{|l| + \varepsilon}{|\lambda|}, \forall n > N_\varepsilon$$

按结论①, 则  $\{x_n\}$  有界.

$y_n := \lambda x_{n+1} + x_n$ , 计算上下极限, 有:

$$\underline{y}_n = \underline{\lambda x_{n+1} + x_n} \sim \begin{cases} \geq \underline{\lambda x_{n+1} + x_n} \\ \leq \begin{cases} \underline{\lambda x_{n+1} + \overline{x_n}} \\ \underline{\lambda \overline{x_{n+1}} + \underline{x_n}} \end{cases} \leq \lambda \overline{x_{n+1}} + \overline{x_n} \end{cases}.$$

$$\overline{y}_n = \overline{\lambda x_{n+1} + x_n} \sim \begin{cases} \leq \lambda \overline{x_{n+1}} + \overline{x_n} \\ \geq \begin{cases} \lambda \overline{x_{n+1}} + \underline{x_n} \\ \lambda \underline{x_{n+1}} + \overline{x_n} \end{cases} \geq \lambda \underline{x_{n+1}} + \underline{x_n} \end{cases}.$$

$$\text{式中 } \underline{y}_n = \overline{y}_n = l, \text{ 且有 } \begin{cases} (1+\lambda)\underline{x_n} = l \\ (1+\lambda)\overline{x_n} = l \end{cases}, \text{ 亦即: } \underline{x_n} = \overline{x_n} = \frac{l}{1+\lambda}.$$

③  $f(x)$  在  $\mathbb{R}$  上一致连续, 则有  $\forall \varepsilon > 0, \exists \delta_\varepsilon > 0$ , 成立:

$$|f(\tilde{x}) - f(\hat{x})| < \varepsilon, \forall |\tilde{x} - \hat{x}| \leq \delta_\varepsilon.$$

考虑  $0 < \delta_\varepsilon < 2\delta_\varepsilon < \dots < (n_x - 1)\delta_\varepsilon < n_x\delta_\varepsilon < x < (n_x + 1)\delta_\varepsilon$ ,

估计  $|f(n_x \delta_\varepsilon)| \leq |f(n_x \delta_\varepsilon) - f((n_x - 1)\delta_\varepsilon)| + |f((n_x - 1)\delta_\varepsilon)| < \varepsilon + |f((n_x - 1)\delta_\varepsilon)| < 2\varepsilon + |f((n_x - 2)\delta_\varepsilon)| < \cdots < n_x \varepsilon + |f(0)|$

考虑到  $n_x \delta_\varepsilon \leq x < (n_x + 1)\delta_\varepsilon \Rightarrow \leq \frac{x}{\delta_\varepsilon}$ .

则有  $|f(n_x \delta_\varepsilon)| \leq \frac{\varepsilon}{\delta_\varepsilon} x + |f(0)|$

另有  $|f(x)| \leq |f(x) - f(n_x \delta_\varepsilon)| < \varepsilon + |f(n_x \delta_\varepsilon)| < |f(0) + \varepsilon| + \frac{\varepsilon}{\delta_\varepsilon} x =: A + Bx$ .

④估计:

$$\begin{aligned} |(f \cdot g)(\tilde{x}) - (f \cdot g)(\hat{x})| &= |f(\tilde{x})g(\tilde{x}) - f(\hat{x})g(\hat{x})| \leq |f(\tilde{x})g(\tilde{x}) - f(\tilde{x})g(\hat{x})| + |f(\tilde{x})g(\hat{x}) - f(\hat{x})g(\hat{x})| \\ &\leq |f(\tilde{x}) - f(\hat{x})| \cdot |g(\hat{x})| + |g(\tilde{x}) - g(\hat{x})| \cdot |f(\hat{x})| \\ &= RHS_1 + RHS_2 \end{aligned}$$

$$\begin{aligned} RHS_1 &= |f(\tilde{x}) - f(\hat{x})| \cdot |\tilde{x}| \cdot \frac{|g(\tilde{x})|}{|\tilde{x}|} \leq |f(\tilde{x}) - f(\hat{x})| \cdot |\tilde{x}| \cdot \mathbb{M}_g (\text{设 } g(x) \text{ 一致连续}) \\ &\leq |f(\tilde{x})\tilde{x} - f(\hat{x})\hat{x} + f(\hat{x})\hat{x} - f(\hat{x})\tilde{x}| \cdot \mathbb{M}_g \leq [|f(\tilde{x})\tilde{x} - f(\hat{x})\hat{x}| + |f(\hat{x})| \cdot |\hat{x} - \tilde{x}|] \cdot \mathbb{M}_g (\text{设 } xf(x) \text{ 一致连续}) \\ &< [\varepsilon + \mathbb{M}_f \cdot \delta_\varepsilon] \mathbb{M}_g < (1 + \mathbb{M}_f) \mathbb{M}_g \cdot \varepsilon, \text{ 可令 } 0 < \delta_\varepsilon < \varepsilon \end{aligned}$$

$$RHS_2 = |g(\tilde{x}) - g(\hat{x})| \cdot |f(\hat{x})| < \varepsilon \cdot \mathbb{M}_f$$

综上, 有: 设有  $\begin{cases} x \cdot f(x) \text{ 一致连续} \\ g(x) \text{ 一致连续} \end{cases}$ , 则有  $f(x) \cdot g(x)$  一致连续.

⑤现有  $\exists \lim_{x \rightarrow 0} \frac{1}{x} \left[ f(x) - f\left(\frac{x}{\lambda}\right) \right] = 0, \lambda > 1$

则有  $|f(x)| < \left| f\left(\frac{x}{\lambda}\right) \right| + \varepsilon|x|, \dot{B}_{\delta_\varepsilon}(x_0)$ .

故有估计

$$\begin{aligned} |f(x)| &< \left| f\left(\frac{x}{\lambda}\right) \right| + \varepsilon|x| < \left| f\left(\frac{x}{\lambda^2}\right) \right| + \varepsilon \left| \frac{x}{\lambda} \right| + \varepsilon|x| = \left| f\left(\frac{x}{\lambda^2}\right) \right| + \varepsilon|x| \left(1 + \frac{1}{\lambda}\right) \\ &< \left| f\left(\frac{x}{\lambda^3}\right) \right| + \varepsilon|x| \left(1 + \frac{1}{\lambda} + \frac{1}{\lambda^2}\right) \\ &< \left| f\left(\frac{x}{\lambda^4}\right) \right| + \varepsilon|x| \left(1 + \frac{1}{\lambda} + \frac{1}{\lambda^2} + \frac{1}{\lambda^3}\right) \\ &< \cdots < \left| f\left(\frac{x}{\lambda^{n+1}}\right) \right| + \varepsilon|x| \left(1 + \frac{1}{\lambda} + \cdots + \frac{1}{\lambda^n}\right) \\ &< \left| f\left(\frac{x}{\lambda^{n+1}}\right) \right| + \varepsilon|x| \frac{1}{1 - \frac{1}{\lambda}} \end{aligned}$$

令  $n \rightarrow +\infty$ , 由于  $\exists \lim_{x \rightarrow 0} f(x) = 0$ , 则有  $|f(x)| \leq \varepsilon|x| \frac{1}{1 - \frac{1}{\lambda}} \Rightarrow \frac{|f(x)|}{|x|} \leq \varepsilon \frac{\lambda}{\lambda - 1}$

故有  $\exists \lim_{x \rightarrow 0} \frac{f(x)}{x} = 0, \forall x \in \dot{B}_{\delta_\varepsilon}(x_0)$