复旦大学航空航天系与技术科学类

2018-2019 学年第一学期《数学分析 B》一元积分学测试

A 卷　共 13 页

课程代码：　　　　　　　　考试形式：□ 开卷 □ 闭卷　2018 年 12 月 23 日 19:00-22:00

（本试卷答卷时间为 180 分钟，答案必须写在试卷上，做在草稿纸上无效）

专业：__________　学号：__________　姓名：__________　成绩：__________

题号	1-1	1-2	1-3	2-1	2-2	2-3	2-4	2-5	2-6
得分									
题号	2-7	2-8	3-1	3-2	3-3	4-1	4-2	4-3	
得分									
题号	4-4	4-5	5-1	5-2	6-1	6-2	6-3		总分
得分									

（装 订 线 内 不 要 答 题）

一、严格表述题（每题 3 分，共 3 题，共 9 分）

1. Riemann 积分的极限定义, 振幅的判定法, Riemann 判定法.

解： (i) $f(x)\in\mathscr{R}[a,b]$, 指 $\exists \lim\limits_{|\mathbb{P}|\to 0}\sum\limits_{i=1}^{N} f(\xi_i)\triangle x_i,\ \forall \xi_i\in[x_{i-1},x_i]$

(ii) $f(x)\in\mathscr{R}[a,b]\Leftrightarrow \exists \lim\limits_{|\mathbb{P}|\to 0}\sum\limits_{i=1}^{N}\omega(f;[x_{i-1},x_i])\triangle x_i=0$. 式中 $\omega(f;[x_{i-1},x_i])\triangleq \sup\limits_{[x_{i-1},x_i]} f(x)-\inf\limits_{[x_{i-1},x_i]} f(x)=\sup\limits_{\tilde{x},\hat{x}\in[x_{i-1},x_i]}|f(\tilde{x})-f(\hat{x})|$.

(iii) $f(x)\in\mathscr{R}[a,b]\Leftrightarrow \forall\lambda,\mu>0,\ \exists\delta_{\lambda\mu}>0$, 成立 $\sum\limits_{\omega(f;[x_{i-1},x_i])>\lambda}\triangle x_i<\mu,\ \forall|\mathbb{P}|<\delta_{\lambda\mu}$

2. 积分第一中值定理和积分第二中值定理.

解： (i) $\begin{cases} f(x)\in C[a,b] \\ \phi(x)\in\mathscr{R}[a,b]\text{在}[a,b]\text{保号}\end{cases}$, 有 $\int_a^b f(x)\phi(x)\mathrm{d}x=f(\xi)\int_a^b\phi(x)\mathrm{d}x,\ \exists\xi\in[a,b]$

(ii) $\begin{cases} f(x)\in\mathscr{R}[a,b] \\ \eta(x)\text{在}[a,b]\text{上单调}\end{cases}$, 有 $\int_a^b f(x)\eta(x)\mathrm{d}x=\eta(a)\int_a^c f(x)\mathrm{d}x+\eta(b)\int_c^b f(x)\mathrm{d}x,\ \exists c\in[a,b]$

3. 广义积分的 Abel-Dirichlet 判别法.

解： 考虑 $\int_a^{x_*} f(x)\eta(x)\mathrm{d}x$, 式中 $\eta(x)$ 单调. 如有 (i) $\begin{cases}\eta(x)\uparrow(\downarrow)0,\text{当}x\to x_*-0 \\ \left|\int_a^x f(\xi)\mathrm{d}\xi\right|\leqslant\mathbb{M}\in\mathbb{R}^+,\forall x\in(a,x_*)\end{cases}$ 或 (ii) $\begin{cases}\left|\eta(x)\right|\leqslant\mathbb{M}\in\mathbb{R}^+,\forall x\in(a,x_*) \\ \exists\int_a^{x_*} f(x)\mathrm{d}x\in\mathbb{R}\end{cases}$

则有 $\exists\int_a^{x_*} f(x)\eta(x)\mathrm{d}x\in\mathbb{R}$

二、不定积分计算题（每题 3 分，共 8 题，共 24 分）

1. $\int \frac{\mathrm{d}x}{1+x^4}$

解：

$$\int \frac{\mathrm{d}x}{1+x^4} = \int \frac{\frac{1}{x^2}\mathrm{d}x}{\frac{1}{x^2}+x^2} = -\int \frac{\mathrm{d}(\frac{1}{x})}{\frac{1}{x^2}+x^2} = -\frac{1}{2}\int \frac{\mathrm{d}(\frac{1}{x}+x-x+\frac{1}{x})}{\frac{1}{x^2}+x^2}$$

$$= -\frac{1}{2}\int \frac{\mathrm{d}(\frac{1}{x}+x)}{(\frac{1}{x}+x)^2-2} + \frac{1}{2}\int \frac{\mathrm{d}(x-\frac{1}{x})}{(\frac{1}{x}-x)^2+2}$$

$$= -\frac{1}{2\sqrt{2}}\int \frac{\mathrm{d}(\frac{\frac{1}{x}+x}{\sqrt{2}})}{(\frac{\frac{1}{x}+x}{\sqrt{2}})^2-1} + \frac{1}{2\sqrt{2}}\int \frac{\mathrm{d}(\frac{x-\frac{1}{x}}{\sqrt{2}})}{(\frac{\frac{1}{x}-x}{\sqrt{2}})^2+1}$$

$$= -\frac{1}{\sqrt{2}}\ln\left[\frac{\frac{\frac{1}{x}+x}{\sqrt{2}}-1}{\frac{\frac{1}{x}+x}{\sqrt{2}}+1}\right] + \frac{1}{2\sqrt{2}}\arctan\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) + C$$

2. $\int \frac{x\arctan x}{(1+x^2)^2}\mathrm{d}x$

解：

$$\int \frac{x\arctan x}{(1+x^2)^2}\mathrm{d}x = \frac{1}{2}\int \frac{\arctan x}{(1+x^2)^2}\mathrm{d}(1+x^2)$$

$$= -\frac{1}{2}\int \arctan x\,\mathrm{d}\frac{1}{1+x^2}$$

$$= -\frac{1}{2}\arctan x \cdot \frac{1}{1+x^2} - \frac{1}{2}\int \frac{1}{(1+x^2)^2}\mathrm{d}x$$

式中

$$\int \frac{1}{(1+x^2)^2}\mathrm{d}x \overset{x=\tan\theta}{=} \int \frac{\sec^2\theta\,\mathrm{d}\theta}{\sec^4\theta} = \int \cos^2\theta\,\mathrm{d}\theta$$

$$= \int \frac{1+\cos 2\theta}{4}\mathrm{d}2\theta$$

$$= \frac{1}{4}(2\theta+\sin 2\theta)+C$$

$$= \frac{1}{2}\left(\arctan x + \frac{x^2}{1+x^2}\right)+C$$

（装 订 线 内 不 要 答 题）

3. $\int \ln^2(x+\sqrt{1+x^2})\mathrm{d}x$

解：

$$\int \ln^2(x+\sqrt{1+x^2})\mathrm{d}x = x\ln^2(x+\sqrt{1+x^2}) - \int 2x\ln(x+\sqrt{1+x^2})\frac{1+\frac{x}{\sqrt{1+x^2}}}{x+\sqrt{1+x^2}}\mathrm{d}x$$

$$= x\ln^2(x+\sqrt{1+x^2}) - 2\int x\frac{\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}\mathrm{d}x$$

式中 $\int x\frac{\ln(x+\sqrt{1+x^2})}{\sqrt{1+x^2}}\mathrm{d}x = \ln(x+\sqrt{1+x^2})\sqrt{1+x^2} - x + C$

代入有 $I = x\ln^2(x+\sqrt{1+x^2}) - 2\ln(x+\sqrt{1+x^2})\sqrt{1+x^2} + 2x + C$

4. $\int \frac{\mathrm{d}x}{1+x^2+x^4}$

解：

$$\int \frac{\mathrm{d}x}{1+x^2+x^4} = \int \frac{\frac{1}{x^2}\mathrm{d}x}{\frac{1}{x^2}+x^2+1} = -\int \frac{\mathrm{d}\frac{1}{x}}{\frac{1}{x^2}+x^2+1}$$

$$= -\frac{1}{2}\int \frac{\mathrm{d}(\frac{1}{x}-x+x+\frac{1}{x})}{\frac{1}{x^2}+x^2+1}$$

$$= -\frac{1}{2}\int \frac{\mathrm{d}(\frac{1}{x}-x)}{(\frac{1}{x}+x)^2+3} - \frac{1}{2}\int \frac{\mathrm{d}(\frac{1}{x}+x)}{(\frac{1}{x}+x)^2-1}$$

$$= -\frac{1}{2\sqrt{3}}\int \frac{\mathrm{d}(\frac{\frac{1}{x}-x}{\sqrt{3}})}{(\frac{\frac{1}{x}-x}{\sqrt{3}})^2+1} - \frac{1}{2}\cdot\frac{1}{2}\int\left(\frac{1}{x+\frac{1}{x}-1} - \frac{1}{x+\frac{1}{x}+1}\right)\mathrm{d}\left(x+\frac{1}{x}\right)$$

$$= -\frac{1}{2\sqrt{3}}\arctan\frac{\frac{1}{x}-x}{\sqrt{3}} - \frac{1}{4}\ln\frac{x+\frac{1}{x}-1}{x+\frac{1}{x}+1} + C$$

5. $\displaystyle\int \frac{b\sin x + a\cos x}{a\sin x + b\cos x}\mathrm{d}x$

解：令 $\begin{cases} A = \displaystyle\int \frac{\sin x\mathrm{d}x}{a\sin x + b\cos x} \\ B = \displaystyle\int \frac{\cos x\mathrm{d}x}{a\sin x + b\cos x} \end{cases} \Rightarrow \begin{cases} aA + bB = \displaystyle\int \frac{a\sin x + b\cos x}{a\sin x + b\cos x}\mathrm{d}x = x \\ aB - bA = \displaystyle\int \frac{\mathrm{d}(a\sin x + b\cos x)}{a\sin x + b\cos x} = \ln|a\sin x + b\cos x| \end{cases}$

$\Rightarrow \begin{cases} B = \dfrac{bx + a\ln|a\sin x + b\cos x|}{a^2 + b^2} \\ A = \dfrac{ax - b\ln|a\sin x + b\cos x|}{a^2 + b^2} \end{cases}$

6. $\displaystyle\int \frac{\mathrm{e}^{\arctan x}}{(1+x^2)^{\frac{3}{2}}}\mathrm{d}x$

解：

$$\int \frac{\mathrm{e}^{\arctan x}}{(1+x^2)^{\frac{3}{2}}}\mathrm{d}x \overset{x=\tan\theta}{=} \int \frac{\mathrm{e}^{\theta}}{\sec^3\theta}\mathrm{d}\theta = \int \mathrm{e}^{\theta}\cos\theta\mathrm{d}\theta = \int \cos\theta\mathrm{d}\mathrm{e}^{\theta}$$

$$= \cos\theta\cdot\mathrm{e}^{\theta} + \int \mathrm{e}^{\theta}\sin\theta\mathrm{d}\theta$$

$$= \cos\theta\cdot\mathrm{e}^{\theta} + \int \sin\theta\mathrm{d}\mathrm{e}^{\theta}$$

$$= (\cos\theta + \sin\theta)\mathrm{e}^{\theta} - \int \mathrm{e}^{\theta}\cos\theta\mathrm{d}\theta$$

$$\Rightarrow \int \mathrm{e}^{\theta}\cos\theta\mathrm{d}\theta = \frac{1}{2}\mathrm{e}^{\theta}(\cos\theta + \sin\theta)$$

(装 订 线 内 不 要 答 题)

7. $\int \frac{1-\sqrt{x+3}}{(x+3)(1+\sqrt[3]{x+3})}\mathrm{d}x$

解:

$$\int \frac{1-\sqrt{x+3}}{(x+3)(1+\sqrt[3]{x+3})}\mathrm{d}x \overset{y=x+3}{=} \int \frac{1-\sqrt{y}}{(y)(1+\sqrt[3]{y})}\mathrm{d}y$$

$$= \int y^{-1}(1+y^{\frac{1}{3}})^{-1}\mathrm{d}y - \int y^{-\frac{1}{2}}(1+y^{\frac{1}{3}})^{-1}\mathrm{d}y$$

令 $y^{\frac{1}{3}} = z$, $y = z^3$. 有

$$\int y^{-1}(1+y^{\frac{1}{3}})^{-1}\mathrm{d}y = \int z^{-3}(1+z)^{-1}3z^2\mathrm{d}z = 3\int \frac{1}{z(1+z)}\mathrm{d}z$$

$$= 3\int (\frac{1}{z} - \frac{1}{1+z})\mathrm{d}z = 3\ln\frac{z}{1+z} + C$$

$$= 3\ln\frac{y^{\frac{1}{3}}}{1+y^{\frac{1}{3}}} + C = 3\ln\frac{(x+3)^{\frac{1}{3}}}{1+(x+3)^{\frac{1}{3}}} + C$$

$$\int y^{-\frac{1}{2}}(1+y^{\frac{1}{3}})^{-1}\mathrm{d}y = \int z^{-\frac{3}{2}}(1+z)^{-1}3z^2\mathrm{d}z = 3\int z^{\frac{1}{2}}(1+z)^{-1}\mathrm{d}z$$

$$= 3\int \frac{\sqrt{z}}{1+z}\mathrm{d}z \overset{w=\sqrt{z}}{=} 3\int \frac{w}{1+w^2}2w\mathrm{d}w = 6\int \frac{w^2+1-1}{1+w^2}\mathrm{d}w$$

$$= 6(w - \arctan w) + C = 6(\sqrt{z} - \arctan\sqrt{z}) + C, z = (x+3)^{\frac{1}{3}}$$

8. $\int \sin x \ln(\sin x)\mathrm{d}x$

解:

$$\int \sin x\ln(\sin x)\mathrm{d}x = -\int \ln(\sin x)\mathrm{d}\cos x = -\cos x\cdot\ln(\sin x) + \int \frac{\cos^2 x}{\sin x}\mathrm{d}x$$

式中

$$\int \frac{\cos^2 x}{\sin x}\mathrm{d}x = \int \frac{1-\sin^2 x}{\sin x}\mathrm{d}x = \int \frac{-\mathrm{d}\cos x}{\sin^2 x} - \int \sin x\mathrm{d}x$$

$$= -\int \frac{\mathrm{d}\cos x}{1-\cos^2 x} + \cos x$$

$$= -\frac{1}{2}\int \left(\frac{1}{1+\cos x} + \frac{1}{1-\cos x}\right)\mathrm{d}\cos x + \cos x$$

$$= -\frac{1}{2}\ln\frac{1+\cos x}{1-\cos x} + \cos x + C$$

三、定积分计算题（每题 4 分，共 3 题，共 12 分）

1. $\displaystyle\int_{-2}^{2} x\ln(1+\mathrm{e}^x)\mathrm{d}x$

解：

$$\begin{aligned}\int_{-2}^{2} x\ln(1+\mathrm{e}^x)\mathrm{d}x &= \int_{-2}^{2} x\ln[\mathrm{e}^{\frac{x}{2}}(\mathrm{e}^{-\frac{x}{2}}+\mathrm{e}^{\frac{x}{2}})]\mathrm{d}x \\ &= \int_{-2}^{2} \frac{x^2}{2}\mathrm{d}x + \int_{-2}^{2} x\ln(\mathrm{e}^{-\frac{x}{2}}+\mathrm{e}^{\frac{x}{2}})\mathrm{d}x \\ &= 2\int_{0}^{2} \frac{x^2}{2}\mathrm{d}x = \frac{8}{3}\end{aligned}$$

2. $\displaystyle\int_{0}^{\frac{\pi}{2}} \frac{\mathrm{d}x}{1+\tan^\alpha x}, \forall\alpha\in\mathbb{R}$

解：

$$\begin{aligned}&\int_{0}^{\frac{\pi}{2}} \frac{\mathrm{d}x}{1+\tan^\alpha x}, \forall\alpha\in\mathbb{R} \\ =&\int_{0}^{\frac{\pi}{2}} \frac{\cos^\alpha x}{\sin^\alpha x+\cos^\alpha x}\mathrm{d}x \\ =&\int_{0}^{\frac{\pi}{4}} \frac{\cos^\alpha x}{\sin^\alpha x+\cos^\alpha x}\mathrm{d}x + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^\alpha x}{\sin^\alpha x+\cos^\alpha x}\mathrm{d}x \\ =&\int_{0}^{\frac{\pi}{4}} \frac{\cos^\alpha x}{\sin^\alpha x+\cos^\alpha x}\mathrm{d}x + \int_{0}^{\frac{\pi}{4}} \frac{\sin^\alpha x}{\sin^\alpha x+\cos^\alpha x}\mathrm{d}x \\ =&\frac{\pi}{4}\end{aligned}$$

（装 订 线 内 不 要 答 题）

3. 计算 $B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\mathrm{d}x,\ \forall m,n\in\mathbb{N}$

解：

$$B(m,n)=\int_0^1(1-x)^{n-1}\mathrm{d}\left(\frac{x^m}{m}\right)=\frac{n-1}{m}\int_0^1 x^m(1-x)^{n-2}\mathrm{d}x$$

$$=\frac{n-1}{m}\int_0^1(1-x)^{n-2}\mathrm{d}\left(\frac{x^{m+1}}{m+1}\right)=\frac{(n-1)(n-2)}{m(m+1)}\int_0^1 x^{m+1}(1-x)^{n-3}\mathrm{d}x$$

$$=\frac{(n-1)(n-2)}{m(m+1)}\int_0^1 x^{m-1+2}(1-x)^{n-1-2}\mathrm{d}x$$

$$=\frac{(n-1)(n-2)\cdots(n-(n-1))}{m(m+1)\cdots(m+(n-2))}\int_0^1 x^{m-1+(n-1)}(1-x)^{n-1-(n-1)}\mathrm{d}x$$

$$=\frac{(n-1)(n-2)\cdots(n-(n-1))}{m(m+1)\cdots(m+(n-2))}\int_0^1 x^{m+n-2}\mathrm{d}x$$

$$=\frac{(n-1)(n-2)\cdots(n-(n-1))}{m(m+1)\cdots(m+n-1)}$$

$$=\frac{(n-1)!\cdot(m-1)!}{(m+n-1)!}$$

四、广义积分计算题（每题 4 分，共 5 题，共 20 分）

1. 计算 $\int_0^{+\infty}\frac{x\mathrm{e}^{-x}}{(1+\mathrm{e}^{-x})^2}\mathrm{d}x$

解：

$$\int_0^{+\infty}\frac{x\mathrm{e}^{-x}}{(1+\mathrm{e}^{-x})^2}\mathrm{d}x\overset{y=\mathrm{e}^{-x}}{=}\int_1^0\frac{-\ln y\,y}{(1+y)^2}\left(-\frac{1}{y}\right)\mathrm{d}y$$

$$=-\int_0^1\frac{\ln y}{(1+y)^2}\mathrm{d}y=\int_0^1\ln y\,\mathrm{d}\frac{1}{1+y}$$

考虑

$$\int\ln y\,\mathrm{d}\frac{1}{1+y}=\frac{\ln y}{1+y}-\int\frac{1}{1+y}\cdot\frac{1}{y}\mathrm{d}y=\frac{\ln y}{1+y}-\int\left(\frac{1}{y}-\frac{1}{1+y}\right)\mathrm{d}y$$

$$=\frac{\ln y}{1+y}-\ln\frac{y}{1+y}+C=:F(y)=\frac{\ln y}{1+y}-\ln y+\ln(1+y)$$

有当 $y\to 0+0$,

$$F(y)=\ln y(1-y+o(y))-\ln y+y+o(y)=-y\ln y+o(y\ln y)\to F(0+0)=0$$

故有 $\int_0^1\ln y\,\mathrm{d}\frac{1}{1+y}=F(1)-F(0+0)=\ln 2$

2. 计算 $\int_0^a x\sqrt{\frac{x}{a-x}}\mathrm{d}x,\, a>0$

解： 计算 $I=\int x\sqrt{\frac{x}{a-x}}\mathrm{d}x$, 令 $y=\sqrt{\frac{x}{a-x}},\, x=\frac{ay^2}{1+y^2}$

$$I=\int a\left(1-\frac{1}{1+y^2}\right)\cdot y\cdot a\frac{2y}{(1+y^2)^2}\mathrm{d}y=2a^2\int\frac{y^4}{(1+y^2)^3}\mathrm{d}y$$

$$\overset{y=\tan\theta}{=} 2a^2\int\sin^4\theta\,\mathrm{d}\theta$$

考虑

$$\begin{aligned}
\int\sin^4\theta\,\mathrm{d}\theta &= \int\left(\frac{1-\cos 2\theta}{2}\right)^2\mathrm{d}\theta=\frac{1}{4}\int(1-2\cos 2\theta+\cos^2 2\theta)\mathrm{d}\theta\\
&=\frac{1}{4}\left[\theta-\sin 2\theta+\int\frac{1+\cos 4\theta}{2\times 4}\mathrm{d}(4\theta)\right]\\
&=\frac{1}{4}\left[\theta-\sin 2\theta+\frac{1}{2}\theta+\frac{1}{8}\sin 4\theta\right]\\
&=\frac{1}{4}\left[\frac{3}{2}\theta-2\sin\theta\cos\theta+\frac{1}{8}2\sin 2\theta\cos 2\theta\right]\\
&=\frac{3}{8}\theta-\frac{1}{2}\sin\theta\cos\theta+\frac{1}{8}\sin\theta\cos\theta(2\cos^2\theta-1)\\
&=\frac{3}{8}\theta-\frac{5}{8}\sin\theta\cos\theta+\frac{1}{4}\sin\theta\cos^3\theta\\
&=\frac{3}{8}\arctan y-\frac{5}{8}\frac{y}{1+y^2}+\frac{1}{4}\frac{y}{1+y^2}\frac{1}{1+y^2}\to\frac{3}{8}\cdot\frac{\pi}{2}=\frac{3\pi}{16},\, y\to+\infty
\end{aligned}$$

3. 计算 $\int_0^\pi x\ln(\sin x)\mathrm{d}x$

解：

$$\begin{aligned}
\int_0^\pi x\ln(\sin x)\mathrm{d}x &= \int_0^\pi\left(x-\frac{\pi}{2}+\frac{\pi}{2}\right)\ln(\sin x)\mathrm{d}x\\
&=\int_0^\pi\left(x-\frac{\pi}{2}\right)\ln(\sin x)\mathrm{d}x+\frac{\pi}{2}\int_0^\pi\ln(\sin x)\mathrm{d}x\\
&=\pi\int_0^{\frac{\pi}{2}}\ln(\sin x)\mathrm{d}x\\
&=\pi\left(-\frac{\pi}{2}\ln 2\right)=-\frac{\pi^2}{2}\ln 2
\end{aligned}$$

（装 订 线 内 不 要 答 题）

4. 计算 $\displaystyle\int_0^{+\infty}\frac{\ln x}{(x^2+1)(x^2+4)}\mathrm{d}x$

解： $\displaystyle\int_0^{+\infty}\frac{\ln x}{(x^2+1)(x^2+4)}\mathrm{d}x=\frac{1}{3}\int_0^{+\infty}\left(\frac{\ln x}{x^2+1}-\frac{\ln x}{x^2+4}\right)\mathrm{d}x$ 式中

$$\int_0^{+\infty}\left(\frac{\ln x}{x^2+1}\right)\mathrm{d}x=\int_0^{+\infty} lnx\,\mathrm{d}\arctan x\overset{\theta=\arctan x}{=}\int_0^{\frac{\pi}{2}}\ln\tan\theta\,\mathrm{d}\theta$$

$$=\int_0^{\frac{\pi}{2}}\ln\sin\theta\,\mathrm{d}\theta-\int_0^{\frac{\pi}{2}}\ln\cos\theta\,\mathrm{d}\theta=0$$

$$\int_0^{+\infty}\left(\frac{\ln x}{x^2+4}\right)\mathrm{d}x=\frac{1}{4}\int_0^{+\infty}\frac{\ln x}{(\frac{x}{2})^2+1}\mathrm{d}x=\frac{1}{2}\int_0^{+\infty}\frac{\ln\frac{x}{2}+\ln 2}{(\frac{x}{2})^2+1}\mathrm{d}\frac{x}{2}$$

$$=\frac{1}{2}\int_0^{+\infty}\frac{\ln x+\ln 2}{x^2+1}\mathrm{d}x=\frac{1}{2}\int_0^{+\infty}\frac{\ln x}{x^2+1}\mathrm{d}x+\frac{\ln 2}{2}\int_0^{+\infty}\frac{\mathrm{d}x}{x^2+1}$$

$$=\frac{\ln 2}{2}\arctan x\Big|_0^{+\infty}=\frac{\ln 2}{2}\cdot\frac{\pi}{2}=\frac{\pi}{4}\ln 2$$

5. 计算 $\displaystyle\lim_{n\to+\infty}\sum_{k=1}^{n}\left(1+\frac{k}{n}\right)\sin\frac{k\pi}{n^2}$

解：

$$\left(1+\frac{k}{n}\right)\sin\frac{k\pi}{n^2}=\left(1+\frac{k}{n}\right)\left(\frac{k\pi}{n^2}+\mathrm{O}\left(\frac{1}{n^2}\right)\right)=\frac{\pi}{n}\left(1+\frac{k}{n}\right)\cdot\frac{k}{n}+\mathrm{O}\left(\frac{1}{n}\right)\cdot\frac{1}{n}$$

则有

$$\lim_{n\to+\infty}\sum_{k=1}^{n}\left(1+\frac{k}{n}\right)\sin\frac{k\pi}{n^2}\to\pi\int_0^1(1+x)x\,\mathrm{d}x$$

$$=\pi\left(\frac{x^2}{2}+\frac{x^3}{3}\right)\Bigg|_0^1=\frac{5\pi}{6}$$

五、定积分应用题（每题 4 分，共 2 题，共 8 分）

1. 求心脏线 $r=a(1-\cos\theta)$ 绕极轴旋轴一周所形成的旋成体的侧面积.

解： $S=\int_0^{\pi}2\pi y(\theta)\sqrt{\dot{x}^2(\theta)+\dot{y}^2(\theta)}\mathrm{d}\theta$

式中 $\begin{cases} x(\theta)=r(\theta)\cos\theta=a(1-\cos\theta)\cos\theta=a(\cos\theta-\cos^2\theta) \\ y(\theta)=r(\theta)\sin\theta=a(1-\cos\theta)\sin\theta=a(\sin\theta-\sin\theta\cos\theta) \end{cases}$

$$\dot{x}^2(\theta)+\dot{y}^2(\theta)=a^2\left[\left(\sin\theta\cos\theta-(1-\cos\theta)\sin\theta\right)^2+\left(\sin^2\theta+(1-\cos\theta)\cos\theta\right)^2\right]$$

$$=a^2\left[\sin^2\theta+(1-\cos\theta)^2\right]=a^2\left[2-2\cos\theta\right]=4a^2\left(\sin^2\frac{\theta}{2}\right)$$

$$S=2\pi\int_0^{\pi}a(1-\cos\theta)\cdot 2a\left(\sin\frac{\theta}{2}\right)\mathrm{d}\theta$$

$$=4\pi a^2\int_0^{\pi}2\sin^3\frac{\theta}{2}\sin\theta\mathrm{d}\theta=4\pi a^2\int_0^{\pi}\sin^3\frac{\theta}{2}\cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\mathrm{d}\frac{\theta}{2}$$

$$=8\pi a^2\int_0^{\pi}\sin^4\frac{\theta}{2}\mathrm{d}\sin\frac{\theta}{2}=8\pi a^2\left.\frac{\sin^5\frac{\theta}{2}}{5}\right|_0^{\pi}=\frac{8\pi a^2}{5}$$

2. 求旋轮线 $\begin{cases} x=a(t-\sin t) \\ y=a(1-\cos t) \end{cases}$, $t\in[0,2\pi]$, 绕 x 轴一圈所围成的旋成体的体积.

解：

$$V=\int_0^{2\pi a}\pi y^2(x)\mathrm{d}x\overset{x=x(t)}{=}\int_0^{2\pi}\pi y^2(t)x'(t)\mathrm{d}t$$

$$=\pi\int_0^{2\pi}a^2(1-\cos t)^2a(t-\sin t)'\mathrm{d}t$$

$$=\pi a^3\int_0^{\pi}\left(1-3\cos t+3\cos^2 t-\cos^3 t\right)\mathrm{d}t$$

$$=\pi a^3\left(2\pi+0+3\int_0^{2\pi}\frac{\cos 2t+1}{2}\mathrm{d}t-\int_0^{2\pi}(1-\sin^2 t)\mathrm{d}t\right)$$

$$=\pi a^3\left(2\pi+\frac{3}{2}\cdot 2\pi\right)=5\pi^2a^3$$

（装 订 线 内 不 要 答 题）

六、广义积分敛散性判断题（每题 9 分，共 3 题，共 27 分）

1. 判断积分敛散性 $\int_0^{+\infty}\left[\ln\left(1+\frac{1}{x}\right)-\frac{1}{1+x}\right]\mathrm{d}x$

解： (i)$x_*=0$:

$$f(x):=\ln\left(1+\frac{1}{x}\right)-\frac{1}{1+x}=\ln(1+x)-\ln x-\frac{1}{1+x}$$
$$=\mathrm{o}(1)-(1+\mathrm{o}(1))-\ln x$$
$$=-1+\mathrm{o}(1)-\ln x$$

$\int_0^1 \ln x\mathrm{d}x$ 收敛, 故 $\int_0^1 f(x)\mathrm{d}x$ 收敛.

(ii)$x_*=+\infty$:

$$f(x):=\ln\left(1+\frac{1}{x}\right)-\frac{1}{x}\frac{1}{1+\frac{1}{x}}=\frac{1}{x}-\frac{1}{2}\cdot\frac{1}{x^2}+\mathrm{o}\left(\frac{1}{x^2}\right)-\frac{1}{x}\left(1-\frac{1}{x}+\mathrm{o}\left(\frac{1}{x}\right)\right)$$
$$=\frac{1}{2x^2}+\mathrm{o}\left(\frac{1}{x^2}\right)=\frac{1}{2x^2}\left(1+\mathrm{o}(1)\right)$$

故 $\int_1^{+\infty} f(x)\mathrm{d}x$ 收敛.

综上, 此广义积分收敛.

2. 判断积分敛散性 $\displaystyle\int_3^{+\infty} \frac{\ln(\ln x)}{\ln x} \sin x \, dx$

解： (i) 绝对收敛性:

$$\left|\frac{\ln(\ln x)}{\ln x} \sin x\right| \leqslant \frac{\ln(\ln x)}{\ln x} \leqslant \frac{C \cdot \ln^{\mu} x}{\ln x} \leqslant \frac{C}{\ln^{1-\mu} x}, \text{取} \mu \in (0,1)$$

考虑 $\dfrac{\ln(\ln x)}{\ln x} \geqslant \dfrac{C}{\ln x} \geqslant \dfrac{C}{x^{\mu}}$, 可取 $\mu = \dfrac{1}{2}$, 故 $\displaystyle\int_3^{+\infty} \frac{\ln(\ln x)}{\ln x} dx$ 发散.

(ii) 自身收敛性:

$$\phi(x) = \frac{\ln(\ln x)}{\ln x}, \quad \phi'(x) = \frac{\frac{1}{\ln x} \cdot \frac{1}{x} \ln x - \ln(\ln x) \cdot \frac{1}{x}}{\ln^2 x} = \frac{1 - \ln(\ln x)}{x \ln^2 x} < 0, \text{当} x >> 1$$

且有 $\phi(x) \sim \dfrac{\frac{1}{\ln x} \cdot \frac{1}{x}}{\frac{1}{x}} = \dfrac{1}{\ln x} \to 0$, 当 $x \to +\infty$.

故有 $\phi(x) \downarrow 0$. 按 Abel-Dirichlet 判别法, 有积分自身收敛.

(iii) 绝对发散性:

$$\left|\frac{\ln(\ln x)}{\ln x} \sin x\right| \geqslant \frac{\ln(\ln x)}{\ln x} \sin^2 x = \frac{\ln(\ln x)}{\ln x} \frac{1 - \cos 2x}{2}$$

式中 $\begin{cases} \displaystyle\int_3^{+\infty} \frac{\ln(\ln x)}{\ln x} dx & \text{发散} \\ \displaystyle\int_3^{+\infty} \frac{\ln(\ln x)}{\ln x} \cos 2x \, dx & \text{收敛} \end{cases}$

故有绝对发散性.

3. 判断积分敛散性 $\int_0^{+\infty}\frac{x^p\sin x}{1+x^q}\mathrm{d}x,\quad p,q\in\mathbb{R}$

解： (i)$x_*=0$:

$$f(x):=\frac{x^p\sin x}{1+x^q}=\frac{x^px(1+\mathrm{o}(1))}{1+x^q}=\frac{x^{p+1}(1+\mathrm{o}(1))}{1+x^q}$$

(a) 当 $q\geqslant 0$, $f(x)=\frac{1}{x^{-(p+1)}}(1+\mathrm{o}(1))$, 有 $\begin{cases}-(p+1)<1,\text{即}p>-2,\text{收敛}\\ -(p+1)\geqslant 1,\text{即}p\leqslant -2,\text{发散}\end{cases}$

(b) 当 $q<0$, $f(x)=\frac{x^px(1+\mathrm{o}(1))}{x^q(1+x^{-q})}=\frac{1}{x^{q-p-1}}(1+\mathrm{o}(1))$, 有 $\begin{cases}q-p-1<1,\text{即}q<p+2,\text{收敛}\\ q-p-1\geqslant 1,\text{即}q\geqslant p+2,\text{发散}\end{cases}$

(ii)$x_*=+\infty$:

$$f(x):=\frac{x^p\sin x}{1+x^q}=\frac{\sin x}{x^{-p}}\frac{1}{1+x^q}$$

(a) 当 $q=0$, 有 $\begin{cases}-p>1,\text{即}p<-1,\text{绝对收敛}\\ 0<-p\leqslant 1,\text{即}-1\leqslant p<0,\text{条件收敛}\\ -p\geqslant 0,\text{即}p\leqslant 0,\text{发散}\end{cases}$

(b) 当 $q>0$, $f(x)=\frac{\sin x}{x^{q-p}}\frac{1}{1+(\frac{1}{x})^{1+q}}=\frac{\sin x}{x^{q-p}}\left(1-\frac{1}{x^q}+\mathrm{O}\left(\frac{1}{x^{2q}}\right)\right)$,

有 $2q-p>1$ 时, $\begin{cases}q-p>1,\text{绝对收敛}\\ 0<q-p\leqslant 1,\text{条件收敛}\\ q-p\geqslant 0,\text{发散}\end{cases}$; $2q-p\leqslant 1$ 时, $\begin{cases}q-p>0,\text{发散}\\ q-p\leqslant 0,\text{发散}\end{cases}$

(c) 当 $q<0$, $f(x)=\frac{\sin x}{x^{-p}}\left(1-\frac{1}{x^{-q}}+\mathrm{O}\left(\frac{1}{x^{-2q}}\right)\right)$

当 $-p-q>1$, 有 $\begin{cases}-p>1,\text{绝对收敛}\\ 0<-p\leqslant 1,\text{条件收敛}\\ -p\geqslant 0,\text{发散}\end{cases}$; 当 $-p-q\leqslant 1$, 有 $\begin{cases}-p>0,\text{发散}\\ -p\leqslant 0,\text{发散}\end{cases}$

（装 订 线 内 不 要 答 题）