复旦大学力学与工程科学系

2010 ～ 2011 学年第二学期期末考试试卷——参考答案

□A 卷　　□B 卷

课程名称：数学分析（Ⅱ）　　课程代码：MATH120009.09

开课院系：力学与工程科学系　　考试形式：开卷/闭卷/课程论文

姓名：________ 学号：________ 专业：________

题 号	1/(1)	1/(2)	2/(1)	2/(2)	3/(1)	3/(2)	4/(1)	4/(2)	4/(3)	5/(1)
得 分										
题 号	5/(2)	5/(3)	5/(4)	6/(1)	6/(2)	6/(3)	7/(1)	7/(2)	8/(1)	8/(2)
得 分										
题 号	8/(3)	8/(4)	8/(5)							总 分
得 分										

Problem 1 (多维函数极限的基本概念) *有二维函数：*

$$f(x,y) \triangleq \begin{cases} xy\cdot\frac{x^2-y^2}{x^2+y^2} & as\,(x,y)\neq(0,0) \\ 0 & as\,(x,y)=(0,0) \end{cases}$$

1. (10%) *研究* $f(x,y)$ *在* $(0,0)$ *点的可微性。*

2. (10%) *计算* $\frac{\partial^2 f}{\partial x\partial y}(0,0)$ *以及* $\frac{\partial^2 f}{\partial y\partial x}(0,0)$ *。如不存在，则说明理由。*

解答：

1. (1/1)

$$f(x,y) = \begin{cases} xy\,\frac{x^2-y^2}{x^2+y^2} & (x,y)\neq(0,0) \\ 0 & (x,y)=(0,0) \end{cases}$$

对于可微性

$$\begin{cases} \frac{\partial f}{\partial x}(0,0) = \lim_{x\to0}\frac{f(x,0)-f(0,0)}{x-0} = \lim_{x\to0}\frac{0-0}{x} = 0 \\ \frac{\partial f}{\partial y}(0,0) = \lim_{y\to0}\frac{f(0,y)-f(0,0)}{x-0} = \lim_{y\to0}\frac{0-0}{y} = 0 \end{cases}$$

估计

$$\left|\frac{f(x,y)-f(0,0)-[\frac{\partial f}{\partial x}(0,0)x+\frac{\partial f}{\partial y}(0,0)y]}{\sqrt{x^2+y^2}}\right| = \frac{|f(x,y)|}{\sqrt{x^2+y^2}} = \left|\frac{x^2-y^2}{x^2+y^2}\frac{xy}{\sqrt{x^2+y^2}}\right|$$

$$=|\frac{x^2-y^2}{x^2+y^2}|\frac{|x|}{\sqrt{x^2+y^2}}|y|\leqslant|y|\leqslant\sqrt{x^2+y^2}$$

故有：

$$f(x,y)=f(0,0)+[\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}](0,0)\begin{bmatrix}x\\y\end{bmatrix}+o(\sqrt{x^2+y^2})$$

亦即 $f(x,y)$ 在 $(0,0)$ 点可微。

2. (1/2)

$$\frac{\partial f}{\partial x}(x,y)=y\frac{x^2-y^2}{x^2+y^2}+xy\frac{2x(x^2+y^2)-(x^2-y^2)2x}{(x^2+y^2)^2}=y\frac{x^2-y^2}{x^2+y^2}+xy\frac{4xy^2}{(x^2+y^2)^2}\quad(x,y)\neq(0,0)$$

$$\frac{\partial f}{\partial y}(x,y)=x\frac{x^2-y^2}{x^2+y^2}+xy\frac{-2y(x^2+y^2)-(x^2-y^2)2y}{(x^2+y^2)^2}=x\frac{x^2-y^2}{x^2+y^2}-xy\frac{4yx^2}{(x^2+y^2)^2}\quad(x,y)\neq(0,0)$$

$$\frac{\partial^2 f}{\partial x\partial y}(0,0)\triangleq\lim_{x\to0}\frac{\frac{\partial f}{\partial y}(x,0)-\frac{\partial f}{\partial y}(0,0)}{x-0}=\lim_{x\to0}\frac{x-0}{x}=1$$

$$\frac{\partial^2 f}{\partial y\partial x}(0,0)\triangleq\lim_{y\to0}\frac{\frac{\partial f}{\partial x}(0,y)-\frac{\partial f}{\partial x}(0,0)}{y-0}=\lim_{y\to0}\frac{-y}{y}=-1$$

Problem 2 (*Fourier* **级数基本概念)** *现有区间* $[0,3]$ *上的函数：*

$$f(x)\triangleq\begin{cases}0 & x\in[0,1)\\ x^2 & x\in[1,2]\\ 0 & x\in(2,3]\end{cases}$$

1. (10%) *按点收敛的意义，将* $f(x),x\in[0,3]$ *表示成 Fourier 正弦级数的形式。要求写出具体的表达形式，其中相关系数仅需给出具体计算式。*

2. (10%) *示意性绘出 Fourier 级数对应的和函数（极限函数）的图像。*

解答：

1. (2/1)

$$f(x)=\begin{cases}0 & x\in[0,1)\\ x^2 & x\in[1,2]\\ 0 & x\in(2,3]\end{cases}$$

首先进行奇延拓，即

$$\hat{f}(x)=\begin{cases}f(x) & x\in[0,3]\\ -f(x) & x\in[-3,0]\end{cases}$$

对奇函数 $\hat{f}(x)$ 作以 6 为周期的延拓，再做其 Fourier 级数展开。

$[-3,3]$ 上有：

$$\frac{\hat{f}(x+0)+\hat{f}(x-0)}{2}=\frac{a_0}{2}+\sum_{n=1}^{+\infty}(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l})$$

$$\begin{cases} a_n=\frac{1}{l}\int_{-l}^{l}\hat{f}(x)\cos\frac{n\pi x}{l}dx=\frac{1}{3}\int_{-3}^{3}\hat{f}(x)\cos\frac{n\pi x}{3}dx & n\in\mathbb{N}\cup\{0\} \\ b_n=\frac{1}{l}\int_{-l}^{l}\hat{f}(x)\sin\frac{n\pi x}{l}dx=\frac{2}{3}\int_{0}^{3}\hat{f}(x)\sin\frac{n\pi x}{3}dx=\frac{2}{3}\int_{1}^{2}x^2\sin\frac{n\pi x}{3}dx \end{cases}$$

2. (2/2) 略

Problem 3 (基于一般原理构造数项级数收敛性的判别法) *一般数项级数的收敛性判别法，主要基于级数的 Cauchy 收敛原理并结合 Abel 估计。*

1. (10%) *证明 Leibnize 判别法：对于交叉级数*

$$\sum(-1)^n a_n\,(a_n>0)$$

当 $\{a_n\}$ *单调趋于零，则此级数收敛。*

2. (10%) *基于 Leibnize 判别法，证明：对正项级数* $\sum a_n\,(a_n>0)$ *，如有：*

$$\frac{a_n}{a_{n+1}}=1+\frac{1}{n}+\frac{\mu}{n\ln n}+o(\frac{1}{n\ln n}),\quad as\quad n\to+\infty$$

则有：当 $\mu>0$ *时，交叉级数* $\sum(-1)^n a_n$ *收敛。*

解答：

1. (3/1) 考虑 Abel 估计

$$\left|\sum_{k=n+1}^{n+p}a_kb_k\right|\leqslant\max_{n+1\leqslant j\leqslant n+p}\left|\sum_{k=n+1}^{j}b_k\right|(|a_{n+1}|+2|a_{n+p}|)$$

此处要求 $\{a_k\}_{k=n+1}^{n+p}$ 单调

对于交叉级数 $\sum(-1)^n a_n\quad(a_n>0)$, 可取 $b_n=(-1)^n$, 故 $|\sum_{k=1}^n b_n|=|\sum_{k=1}^n(-1)^n|\leqslant 1$

而 $\{a_n\}$ 单调趋于零。故 $\sum(-1)^n a_n$ 收敛。

注：

$$\left|\sum_{k=n+1}^{n+p}(-1)^n a_k\right|\leqslant|a_{n+1}|+2|a_{n+p}|<\epsilon\quad as\quad n\geqslant N_\epsilon$$

2. (3/2) 基于 Leibniz 判别法，需证 $\{a_n\}$ 单调趋于零。

由

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \frac{\mu}{n\ln n} + o(\frac{1}{n\ln n}) \qquad as \quad n \to +\infty$$

由

$$\frac{|o(\frac{1}{n\ln n})|}{\frac{1}{n\ln n}} \to 0$$

故有

$$\left| o(\frac{1}{n\ln n}) \right| < \epsilon \frac{1}{n\ln n} \qquad n > \hat{N}_\epsilon$$

故有：

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \frac{\mu}{n\ln n} + o(\frac{1}{n\ln n}) > 1 + \frac{1}{n} + \frac{\mu - \epsilon}{n\ln n} > 1 + \frac{1}{n} > 1 \qquad n \geqslant \hat{N}_\epsilon$$

此处需 $0 < \epsilon < \mu$

故有：$a_n > a_{n+1} \quad n \geqslant N_\epsilon$

即有：$\{a_n\}_{n \geqslant N_\epsilon}$ 单调下降。

另考虑到，取 $b_n = \frac{1}{n(\ln n)^p} \quad (p > 0)$ 有

$$\frac{b_n}{b_{n+1}} = 1 + \frac{1}{n} + \frac{p}{n\ln n} + o(\frac{1}{n\ln n}) < 1 + \frac{1}{n} + \frac{p + \epsilon}{n\ln n} \qquad n \geqslant \hat{N}_\epsilon$$

注：

$$\frac{b_n}{b_{n+1}} = \frac{\frac{1}{n(\ln n)^p}}{\frac{1}{(n+1)(\ln(n+1))^p}} = \frac{n+1}{n}(\frac{\ln(n+1)}{\ln n})^p = (1 + \frac{1}{n}) \left[\frac{\ln n + \ln(1 + \frac{1}{n})}{\ln n} \right]^p$$

$$= (1+\frac{1}{n}) \left[1 + \frac{1}{n\ln n} + o(\frac{1}{n\ln n}) \right]^p = (1+\frac{1}{n}) \left[1 + \frac{p}{n\ln n} + o(\frac{1}{n\ln n}) \right] = 1+\frac{1}{n}+\frac{p}{n\ln n}+o(\frac{1}{n\ln n})$$

且

$$\frac{a_n}{a_{n+1}} > 1 + \frac{1}{n} + \frac{\mu - \epsilon}{n\ln n} \qquad n \geqslant \hat{N}_\epsilon$$

取 $\epsilon > 0$ 满足 $\mu - \epsilon \geqslant p + \epsilon$, 且 $N_\epsilon = \max\{\widetilde{N}, \widehat{N}\}$ 有：$\frac{a_n}{a_{n+1}} > \frac{b_n}{b_{n+1}} \quad n \geqslant N_\epsilon$

故有 $a_n \leqslant \frac{a_{N_\epsilon}}{b_{N_\epsilon}} b_n \qquad \forall n > N_\epsilon$ 注：由

$$\begin{aligned}
&\frac{a_n}{a_{n+1}} > \frac{b_n}{b_{n+1}} \quad \forall n \geqslant N_\epsilon \\
\Rightarrow &\frac{a_{n+1}}{a_n} < \frac{b_{n+1}}{b_n} \quad \forall n \geqslant N_\epsilon \\
\Rightarrow &\frac{a_n}{a_{n-1}} \cdots \frac{a_{N_\epsilon}+1}{a_{N_\epsilon}} < \frac{b_n}{b_{n-1}} \cdots \frac{b_{N_\epsilon}+1}{b_{N_\epsilon}} \quad \forall n > N_\epsilon \\
\Rightarrow &\frac{a_n}{a_{N_\epsilon}} < \frac{b_n}{b_{N_\epsilon}} \quad \forall n > N_\epsilon
\end{aligned}$$

$$\text{即：} \quad a_n < \frac{a_{N_\epsilon}}{b_{N_\epsilon}} b_n \quad \forall n > N_\epsilon$$

$$\text{而：} b_n = \frac{1}{n(\ln n)^p} \to 0 \quad as \quad n \to +\infty \text{故有} \quad a_n \to 0$$

Problem 4 (实践幂级数基本性质) *有函数*

$$f(x) = \sum_{n=1}^{+\infty} \frac{x^n}{n^2 \cdot \ln(1+n)}$$

1. (10%) *证明*：$f(x) \in C[-1,1]$

2. (10%) *证明*：$\exists f'(x) \in C[-1,1)$

3. (10%) *证明*：$\lim_{x\to 1-0} f'(x) = +\infty$ 。

解答：

1. (4/1)

$$f(x) = \sum_{n=1}^{+\infty} \frac{x^n}{n^2 \ln(1+n)}$$

考虑

$$\frac{\frac{1}{(n+1)^2 \ln(2+n)}}{\frac{1}{n^2 \ln(1+n)}} |x| = \frac{n^2}{(n+1)^2} \frac{\ln(1+n)}{\ln(2+n)} |x| \to |x| \quad as \quad n \to +\infty$$

故收敛半径为 1

在考虑：

$$f(1) = \sum_{n=1}^{+\infty} \frac{1}{n^2 \ln(1+n)} \qquad \frac{1}{n^2 \ln(1+n)} \leqslant \frac{1}{n^2} \quad as \quad n \gg 1$$

故 $\sum_{n=1}^{+\infty} \frac{1}{n^2 \ln(1+n)}$ 收敛。

$$f(-1) = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2 \ln(1+n)}$$

按 Leibnize 判别法有 $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2 \ln(1+n)}$ 收敛。综上 $f(x) = \sum_{n=1}^{+\infty} \frac{x^n}{n^2 \ln(1+n)}$ 的收敛域为 $[-1,1]$

据幂级数的内闭一致收敛性有 $f(x) \in C[-1,1]$

2. (4/2) 已有

$$f(x)=\frac{x}{\ln 2}+\sum_{n=2}^{+\infty}\frac{x^n}{n^2\ln(1+n)}\quad x\in[-1,1]$$

考虑逐项求导级数

$$\frac{1}{\ln 2}+\sum_{n=2}^{+\infty}\frac{nx^{n-1}}{n^2\ln(1+n)}=\frac{1}{\ln 2}+\sum_{n=2}^{+\infty}\frac{x^{n-1}}{n\ln(1+n)}=\frac{1}{\ln 2}+\sum_{n=1}^{+\infty}\frac{x^n}{(n+1)\ln(2+n)}$$

收敛半径为 1，且在 $x=-1$ 处按 Leibnize 判别法级数收敛。

考虑 $x=1$ 处的收敛性

$$\sum_{n=1}^{+\infty}\frac{1}{(n+1)\ln(2+n)}=\sum_{n=2}^{+\infty}\frac{1}{n\ln(1+n)}$$

考虑

$$\begin{aligned}\frac{1}{n\ln(1+n)}&=\frac{1}{n\ln n(1+\frac{1}{n})}=\frac{1}{n[\ln n+\ln(1+\frac{1}{n})]}=\frac{1}{n\ln n\left[1+\frac{\ln(1+\frac{1}{n})}{\ln n}\right]}\\&=\frac{1}{n\ln n\left[1+\frac{\frac{1}{n}+o(\frac{1}{n})}{\ln n}\right]}=\frac{1}{n\ln n\left[1+\frac{1}{n\ln n}+o(\frac{1}{n\ln n})\right]}\\&=\frac{1}{n\ln n}[1-\frac{1}{n\ln n}+o(\frac{1}{n\ln n})]=\frac{1}{n\ln n}+o(\frac{1}{n\ln n})\end{aligned}$$

而 $\sum\frac{1}{n\ln n}$ 发散，故 $\sum_{n=2}^{+\infty}\frac{1}{n\ln(1+n)}$ 发散。

综上, 逐项求导级数

$$\frac{1}{\ln 2}+\sum_{n=1}^{+\infty}\frac{x^n}{(n+1)\ln(2+n)}$$

的收敛域为 $[-1,1)$

故有：

$$f'(x)=\frac{1}{\ln 2}+\sum_{n=1}^{+\infty}\frac{x^n}{(n+1)\ln(2+n)}\qquad x\in[-1,1)$$

故有

$$\exists\quad f'(x)\in C[-1,1)$$

3. (4/3) 由

$$f'(1-0)\triangleq\lim_{x\to1-0}f'(x)=\lim_{x\to1-0}\left[\frac{1}{\ln 2}+\sum_{n=1}^{+\infty}\frac{x^n}{(n+1)\ln(2+n)}\right]$$

由 $f'(x)\in C[-1,1)$, 且在 $(0,1)$ 上大于零，故严格单调上升。

另有：

$$\frac{1}{\ln 2}+\sum_{n=1}^{+\infty}\frac{1}{(n+1)\ln(2+n)}=+\infty$$

采用反证法，即设 $\lim_{x\to 1-0} f'(x) = M \in \mathbb{R}^+$。则有

$$S_N(x) \triangleq \frac{1}{\ln 2} + \sum_{n=1}^{N} \frac{x^n}{(n+1)\ln(2+n)} \leq M, \quad \forall x \in [0,1), \quad \forall N \in \mathbb{N}$$

进一步有

$$\lim_{x\to 1-0} S_N(x) = \lim_{x\to 1-0} \left[\frac{1}{\ln 2} + \sum_{n=1}^{N} \frac{x^n}{(n+1)\ln(2+n)} \right] = \frac{1}{\ln 2} + \sum_{n=1}^{N} \frac{1}{(n+1)\ln(2+n)} \leq M, \quad \forall N \in \mathbb{N}$$

故有

$$\frac{1}{\ln 2} + \sum_{n=1}^{+\infty} \frac{1}{(n+1)\ln(2+n)} \leq M$$

故得矛盾。

Problem 5 (实践隐映照定理) *可基于隐映照定理认识 $\mathbb{R}^4$ 中的集合*

$$\Sigma = \left\{ [x,y,z,w]^T \in \mathbb{R}^4 \mid \text{满足：} \begin{cases} yz + zx + xy - w = 0 \\ x^2 + y^2 + z^2 - 1 = 0 \end{cases} \right\}$$

1. (10%) *按隐映照定理，说明：Σ 在局部可看作 $\mathbb{R}^4$ 中 2 维曲面*

$$\overset{\sigma}{\Sigma}(x,y) : \mathbb{R}^2 \supset B_\sigma((x_0,y_0)) \ni \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \Sigma(x,y) \triangleq \begin{bmatrix} x \\ y \\ z(x,y) \\ w(x,y) \end{bmatrix} \in \mathbb{R}^4$$

   需指明，$(x_0,y_0) \in \mathbb{R}^2$ 的适用范围。

2. (10%) *考虑定义在局部曲面 $\overset{\sigma}{\Sigma}$ 上函数 $\theta(x,y,z,w)$，显见此函数等价于函数*

$$\hat{\theta}(x,y) : B_\sigma((x_0,y_0)) \ni \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \hat{\theta}(x,y) \triangleq \theta(x,y,z(x,y),w(x,y))$$

   推导 $\hat{\theta}(x,y)$ 之临界点所满足的方程。

3. (10%) *可按 Lagrange 乘子法，确定函数 $\theta(x,y,z,w)$ 在曲面 $\overset{\sigma}{\Sigma}$ 上之临界点所满足的方程。证明：按 Lagrange 乘子法确定的 $\theta(x,y,z,w)$ 之临界点方程同 $\hat{\theta}(x,y)$ 之临界点方程一致。*

4. (10%) *$\mathbb{R}^4$ 中 2 维曲面 $\overset{\sigma}{\Sigma}$ 可视作"抽象体"，其体积 $|\overset{\sigma}{\Sigma}|$ 定义为*

$$|\overset{\sigma}{\Sigma}| \triangleq \int_{B_\sigma((x_0,y_0))} detD \begin{bmatrix} z \\ w \end{bmatrix} (x,y)\, d\tau$$

   计算 $(x_0,y_0) = (0,0)$ 时，局部"抽象体"的体积（此时 σ 为确定的正数）。

解答

1. (5/1) 考虑

$$\Sigma \triangleq \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4 \middle| \begin{cases} yz + zx + xy - w = 0 \\ x^2 + y^2 + z^2 - 1 = 0 \end{cases} \right\}$$

如果

$$[x_0, y_0, z_0, w_0]^T \in \Sigma$$

使得

$$D_{\begin{bmatrix} z \\ w \end{bmatrix}} \Sigma(\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ w_0 \end{bmatrix}) = \begin{bmatrix} x+y & -1 \\ 2z & 0 \end{bmatrix} \Bigg\|_{\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ w_0 \end{bmatrix}} = \begin{bmatrix} x_0+y_0 & -1 \\ 2z_0 & 0 \end{bmatrix}$$

故有

$$detD_{\begin{bmatrix} z \\ w \end{bmatrix}} \Sigma(\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ w_0 \end{bmatrix}) = 2z_0$$

故对于隐映照定理要求 $z_0 \neq 0$

而

$$x_0^2 + y_0^2 + z_0^2 - 1 = 0 \Rightarrow x_0^2 + y_0^2 = 1 - z_0^2 \geqslant 0 \Rightarrow 0 < z_0^2 \leqslant 1$$

即：

$$z_0 \in (-1, 1) - \{0\}$$

而 (x_0, y_0) 满足：

$$x_0^2 + y_0^2 = 1 - z_0^2 \in [0, 1)$$

2. (5/2) 进一步由隐映照定理：

$$\Sigma(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} z \\ w \end{bmatrix}(x, y)) = 0 \in \mathbb{R}^2$$

$$\Rightarrow D_{\begin{bmatrix} x \\ y \end{bmatrix}}(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} z \\ w \end{bmatrix}(x,y)) + D_{\begin{bmatrix} z \\ w \end{bmatrix}}\Sigma(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} z \\ w \end{bmatrix}(x,y))D\begin{bmatrix} z \\ w \end{bmatrix}(x,y) = 0 \in \mathbb{R}^{2\times 2}$$

即：

$$\begin{bmatrix} y+z & z+x \\ 2x & 2y \end{bmatrix} + \begin{bmatrix} x+y & -1 \\ 2z & 0 \end{bmatrix}\begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix}(x,y) = 0 \in \mathbb{R}^{2\times 2}$$

$$\Rightarrow \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix}(x,y) = -\begin{bmatrix} x+y & -1 \\ 2z & 0 \end{bmatrix}^{-1}\begin{bmatrix} y+z & z+x \\ 2x & 2y \end{bmatrix}$$

此处 $z = z(x,y)$

对于

$$\hat{\theta}(x,y) \triangleq \theta(x,y,z(x,y),w(x,y)) = \theta(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} z \\ w \end{bmatrix}(x,y)) \in \mathbb{R}$$

临界点方程为

$$D\hat{\theta}(\begin{bmatrix} x \\ y \end{bmatrix}) = D_{\begin{bmatrix} x \\ y \end{bmatrix}}\theta + D_{\begin{bmatrix} z \\ w \end{bmatrix}}\theta D\begin{bmatrix} z \\ w \end{bmatrix}(x,y) = 0 \in \mathbb{R}^{2\times 2}$$

即：

$$\begin{bmatrix} \frac{\partial\theta}{\partial x} & \frac{\partial\theta}{\partial y} \end{bmatrix}(\begin{bmatrix} x_* \\ y_* \end{bmatrix}, \begin{bmatrix} z \\ w \end{bmatrix}(x_*,y_*)) + \begin{bmatrix} \frac{\partial\theta}{\partial z} & \frac{\partial\theta}{\partial w} \end{bmatrix}(\begin{bmatrix} x_* \\ y_* \end{bmatrix}, \begin{bmatrix} z \\ w \end{bmatrix}(x_*,y_*))D\begin{bmatrix} z \\ w \end{bmatrix}(x_*,y_*)$$
$$= 0 \in \mathbb{R}^{1\times 2}$$

此处

$$\begin{aligned} D\begin{bmatrix} z \\ w \end{bmatrix}(x_*,y_*) = & \quad -\begin{bmatrix} x_*+y_* & -1 \\ 2z_* & 0 \end{bmatrix}^{-}\begin{bmatrix} y_*+z_* & z_*+x_* \\ 2x_* & 2y_* \end{bmatrix} \\ =& - \left(D_{\begin{bmatrix} z \\ w \end{bmatrix}}\Sigma \right) D_{\begin{bmatrix} x \\ y \end{bmatrix}}\Sigma \Bigg|_{\begin{bmatrix} x_* \\ y_* \\ \begin{bmatrix} z \\ w \end{bmatrix}(x_*,y_*) \end{bmatrix}} \end{aligned}$$

3. (5/3) Lagrange 乘子法。构造 Lagrange 函数。

$$L(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} z \\ w \end{bmatrix}, \begin{bmatrix} \lambda \\ \mu \end{bmatrix}) = \theta(x,y,z,w) + \begin{bmatrix} \lambda & \mu \end{bmatrix}^T \Sigma(x,y,z,w)$$

$$\Rightarrow \begin{cases} D_{\begin{bmatrix} x \\ y \end{bmatrix}} L = D_{\begin{bmatrix} x \\ y \end{bmatrix}} \theta + \begin{bmatrix} \lambda & \mu \end{bmatrix}^T D_{\begin{bmatrix} x \\ y \end{bmatrix}} \Sigma \in \mathbb{R}^{1\times 2} \\ D_{\begin{bmatrix} z \\ w \end{bmatrix}} L = D_{\begin{bmatrix} z \\ w \end{bmatrix}} \theta + \begin{bmatrix} \lambda & \mu \end{bmatrix}^T D_{\begin{bmatrix} z \\ w \end{bmatrix}} \Sigma \in \mathbb{R}^{1\times 2} \\ D_{\begin{bmatrix} \lambda \\ \mu \end{bmatrix}} L = \Sigma^T \in \mathbb{R}^{1\times 2} \end{cases}$$

由第二式

$$\begin{bmatrix} \lambda & \mu \end{bmatrix}^T = -D_{\begin{bmatrix} z \\ w \end{bmatrix}} \theta (D_{\begin{bmatrix} z \\ w \end{bmatrix}} \Sigma)^{-1}$$

代入第一式

$$D_{\begin{bmatrix} x \\ y \end{bmatrix}} \theta - D_{\begin{bmatrix} z \\ w \end{bmatrix}} \theta (D_{\begin{bmatrix} z \\ w \end{bmatrix}} \Sigma)^{-1} D_{\begin{bmatrix} x \\ y \end{bmatrix}} \Sigma = 0 \in \mathbb{R}^{1\times 2}$$

4. (5/4)

$$\left|\overset{\sigma}{\Sigma}\right| \triangleq \int_{B_\sigma(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix})} \det D\begin{bmatrix} z \\ w \end{bmatrix}(x,y)d\tau$$

$$= \int_{B_\sigma(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix})} \frac{\det\begin{bmatrix} y+z & z+x \\ 2x & 2y \end{bmatrix}}{2z} d\tau$$

$$= \int_{B_\sigma(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix})} \frac{(y+z)y-(z+x)x}{z} d\sigma$$

此处 $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = 0 \in \mathbb{R}^2 \quad z = \sqrt{1-(x^2+y^2)}$

$$= \int_{B_\sigma(0,0)} \frac{(y^2-x^2)+z(y-x)}{z} d\sigma$$

$$= \int_{B_\sigma(0,0)} \frac{y^2-x^2}{\sqrt{1-(x^2+y^2)}} d\sigma + \int_{B_\sigma(0,0)} (y-x) d\sigma$$

根据对称性 $\int_{B_{\sigma(0,0)}} y d\sigma = \int_{B_{\sigma(0,0)}} x d\sigma = 0$

$$\int_{B_{\sigma(0,0)}} \frac{y^2}{\sqrt{1-(x^2+y^2)}} d\sigma = \int_{B_{\sigma(0,0)}} \frac{x^2}{\sqrt{1-(x^2+y^2)}} d\sigma = 0$$

故 $\left|\overset{\sigma}{\Sigma}\right| = 0$

Problem 6 (实践 $\mathbb{R}^m$ 中体积分换元公式) *考虑体上积分*

$$\int_{\mathscr{D}_{xyz}} y^4 \, d\tau$$

此处，$\mathscr{D}_{xyz}$ 为由以下曲面族

$$\begin{cases} x = a \cdot z^2 \\ x = b \cdot z^2 \end{cases}, (z > 0,\, 0 < a < b); \quad \begin{cases} x = \alpha \cdot y \\ x = \beta \cdot y \end{cases}, (0 < \alpha < \beta); \quad x = h\,(h > 0)$$

所围成的体积。

1. *(10%) 首先，需说明 $\mathscr{D}_{xyz}$ 为 $\mathbb{R}^3$ 中 Jordan 可测集，亦即有界且其边界为零测集。为此，证明如下结论：定义在闭方块上的 Monge 型曲面为零测集：*

$$\Sigma(x,y) : [a,b]\times[c,d] \ni \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \Sigma(x,y) \triangleq \begin{bmatrix} x \\ y \\ f(x,y) \end{bmatrix}$$

此处 $f(x,y)\in\mathscr{R}([a,b]\times[c,d])$ *。*

2. (10%) *考虑映照*

$$\Phi(x,y,z):\mathscr{D}_{xyz}\ni\begin{bmatrix}x\\y\\z\end{bmatrix}\mapsto\Phi(x,y,z)=\begin{bmatrix}\xi\\\eta\\\zeta\end{bmatrix}(x,y,z)\triangleq\begin{bmatrix}\frac{x}{z^2}\\\frac{x}{y}\\x\end{bmatrix}$$

以此建立微分同胚，需说明相应的开集以及确定为微分同胚的所有条件。

3. (10%) *完成上述积分计算。需严格检验积分换元公式的有关要求。*

解答

1. (6/1) 定义在比方块上的 Monge 型曲面

$$\Sigma(x,y):[a,b]\times[c,d]\ni\begin{bmatrix}x\\y\end{bmatrix}\longmapsto\Sigma(x,y)\triangleq\begin{bmatrix}x\\y\\f(x,y)\end{bmatrix}$$

此处

$$f(x,y)\in\mathscr{R}[a,b]\times[c,d]$$

则有

$$\omega(f(x,y),P)\triangleq\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}\sup_{\begin{bmatrix}\widetilde{x}\\\widetilde{y}\end{bmatrix},\begin{bmatrix}\widehat{x}\\\widehat{y}\end{bmatrix}\in[x_{i-1},x_i]\times[y_{j-1},y_j]}|f(\widetilde{x},\widetilde{y})-f(\widehat{x},\widehat{y})|\triangle x_i\triangle y_j<\epsilon$$

$$as\quad |P|\triangleq\max_{1\leqslant i\leqslant N_x\quad 1\leqslant j\leqslant N_y}|\triangle x_i\triangle y_j|<\delta_\epsilon$$

故以 $[x_{i-1},x_i]\times[y_{j-1},y_j]$ 为底面，以 $f(\xi_i,\eta_j)$ 为中心，以 $\sup|f(\widetilde{x},\widetilde{y})-f(\widehat{x},\widehat{y})|$ 为半高度作闭方块 I_{ij}

则

$$\Sigma\subset\bigcup_{i=1}^{N_x}\bigcup_{j=1}^{N_y}I_{ij}$$

且

$$\sum_{i=1}^{N_x}\sum_{j=1}^{N_y}|I_{ij}|=2\omega(f(x,y),P)<\epsilon\quad\forall|P|<\delta_\epsilon$$

故得证 Σ 为零测集（Jordan 意义）。

2. (6/2)

$$\Phi(x,y,z) \triangleq \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}(x,y,z) = \begin{bmatrix} \frac{x}{z^2} \\ \frac{x}{y} \\ x \end{bmatrix}$$

$$D\Phi(x,y,z) \triangleq \frac{D(\xi,\eta,\zeta)}{D(x,y,z)} = \begin{bmatrix} \frac{1}{z^2} & 0 & -\frac{2x}{z^3} \\ \frac{1}{y} & -\frac{x}{y^2} & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\det D\Phi(x,y,z) = -2\frac{x^2}{y^2 z^3} \neq 0 \qquad as \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathring{\mathscr{D}}_{xyz}$$

此处

$$\mathring{\mathscr{D}}_{xyz} \triangleq \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \middle| \begin{array}{l} \frac{x}{z^2} \in (a,b) \\ \frac{x}{y} \in (\alpha,\beta) \\ x \in (0,h) \end{array} \right\}$$

易见 $\Phi(x,y,z)$ 在 $\mathring{\mathscr{D}}_{xyz}$ 上为单射

$$\mathscr{D}_{\xi\eta\zeta} \triangleq \Phi(\mathscr{D}_{xyz}) = \left\{ \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \in \mathbb{R}^3 \middle| \begin{array}{c} \xi \in [a,b] \\ \eta \in [\alpha,\beta] \\ \zeta \in [0,h] \end{array} \right\}$$

$$\mathring{\mathscr{D}}_{\xi\eta\zeta} \triangleq \left\{ \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \in \mathbb{R}^3 \middle| \begin{array}{c} \xi \in (a,b) \\ \eta \in (\alpha,\beta) \\ \zeta \in (0,h) \end{array} \right\} = \Phi(\mathring{\mathscr{D}}_{xyz})$$

有

$$\Phi(x,y,z) \in C^1(\mathring{\mathscr{D}}_{xyz}, \mathring{\mathscr{D}}_{\xi\eta\zeta})$$

3. (6/3)

$$\int_{\mathscr{D}_{xyz}} y^4 d\tau = \int_{\mathring{\mathscr{D}}_{xyz}} y^4 d\tau = \int_{\mathring{\mathscr{D}}_{\xi\eta\zeta}} y^4(\xi,\eta,\zeta) \left| \det \frac{D(x,y,z)}{D(\xi,\eta,\zeta)} \right| (\xi,\eta,\zeta) d\tau$$
$$= \int_{\mathring{\mathscr{D}}_{\xi\eta\zeta}} y^4(\xi,\eta,\zeta) (\frac{y^2 z^3}{2x^2})(\xi,\eta,\zeta) d\tau = \int_{\mathscr{D}_{\xi\eta\zeta}} \frac{1}{2} \xi^{-\frac{3}{2}} \eta^{-6} \zeta^{\frac{11}{2}} d\tau$$
$$= \frac{1}{2} \int_a^b \xi^{-\frac{3}{2}} d\xi \int_\alpha^\beta \eta^{-6} d\eta \int_0^h \zeta^{\frac{11}{2}} d\zeta = \frac{1}{2} \frac{b^{-\frac{3}{2}} - a^{-\frac{3}{2}}}{-\frac{1}{2}} \frac{\beta^{-5} - \alpha^{-5}}{-5} \frac{h^{\frac{13}{2}}}{\frac{13}{2}}$$

Problem 7 (实践 Stokes 公式) *有做功形式的曲线积分（第二类曲线积分）*

$$\oint_{C_1\bigcup C_2}(y^2+z^2)dx+(x^2+z^2)dy+(x^2+y^2)dz \equiv \oint_{C_1\bigcup C_2}[(y^2+z^2)\mathbf{i}+(x^2+z^2)\mathbf{j}+(x^2+y^2)\mathbf{k}]\cdot\tau dl$$

其中 C_1 *和* C_2 *分别为* z *轴上半球面* $(x-R)^2+y^2+z^2=R^2\,(z>0)$ *同柱面* $(x-r_1)^2+y^2=r_1^2$ *和* $(x-r_2)^2+y^2=r_2^2$ *的截线，此处* $0<r_1<r_2<R$*；从* z *轴正向望去，沿* C_1 *和* C_2 *曲线的走向分别为逆时针和顺时针方向。*

1. (10%) *利用 Stokes 公式计算上述积分。*
2. (10%) *对于上述沿曲线* C_1 *的积分，给出一种线积分的直接计算方案；需给出* C_1 *的向量值映照表达式以及参数域上的积分计算式，无需具体计算。*

解答

1. (7/1)

$$\Sigma(x,y):\mathscr{D}_{xy}\ni\begin{bmatrix}x\\y\end{bmatrix}\mapsto\Sigma(x,y)\triangleq\begin{bmatrix}x\\y\\\sqrt{R^2-(x-R)^2-y^2}\end{bmatrix}$$

$$\frac{\partial\Sigma}{\partial x}\times\frac{\partial\Sigma}{\partial y}(x,y)=\begin{vmatrix}i & j & k\\ 1 & 0 & -\frac{(x-R)}{\sqrt{R^2-(x-R)^2-y^2}}\\ 0 & 1 & -\frac{y}{\sqrt{R^2-(x-R)^2-y^2}}\end{vmatrix}=\begin{bmatrix}\frac{x-R}{\sqrt{R^2-(x-R)^2-y^2}} & \frac{y}{\sqrt{R^2-(x-R)^2-y^2}} & 1\end{bmatrix}$$

$$\nabla\times a(x,y,z)=\begin{vmatrix}i & j & k\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y^2+z^2 & x^2+z^2 & x^2+y^2\end{vmatrix}=\begin{bmatrix}2(y-z)\\2(z-x)\\2(x-y)\end{bmatrix}$$

$$\begin{aligned}I&=\oint_{C_1\cup C_2}[(y^2+z^2)i+(x^2+z^2)j+(x^2+y^2)k]\cdot\tau dl\\&=\int_{\mathscr{D}_{xy}}2[y-z,z-x,x-y]\begin{bmatrix}\frac{x-R}{z}\\\frac{y}{z}\\1\end{bmatrix}d\sigma\\&=2\int_{\mathscr{D}_{xy}}\frac{(y-z)(x-R)+(z-x)y}{z}+(x-y)d\sigma\\&=2\int_{\mathscr{D}_{xy}}(-R\frac{y}{z}+R)d\sigma=2R\,|\mathscr{D}_{xy}|=4\pi R(r_2^2-r_1^2)\end{aligned}$$

2. (7/2) 对于沿曲线 C_1 的积分，作

$$\begin{cases}x=r_1+r_1\cos\theta\\y=r_1\sin\theta\end{cases}$$

而

$$z(\theta)=\sqrt{R^2-(x-R)^2-y^2}(\theta)$$

即：

$$C_1(\theta):[0,2\pi]\ni\theta\longmapsto C_1(\theta)=\begin{bmatrix} r_1+r_1\cos\theta \\ r_1\sin\theta \\ \sqrt{R^2-(r_1+r_1\cos\theta-R)^2-(r_1\cos\theta)^2} \end{bmatrix}$$

$$\Rightarrow \frac{dC_1}{d\theta}(\theta)=\begin{bmatrix} -r_1\sin\theta \\ r_1\cos\theta \\ \frac{2(r_1+r_1\cos\theta-R)r_1\sin\theta-2r_1\sin\theta\cos\theta}{\sqrt{R^2-(r_1+r_1\cos\theta-R)^2-(r_1\sin\theta)^2}} \end{bmatrix}$$

$$\begin{aligned} &\int_{C_1}[(y^2+z^2)i+(x^2+z^2)j+(x^2+y^2)k]\cdot\tau dl \\ =&\int_0^{2\pi}[y^2+z^2,x^2+z^2,x^2+y^2](\theta)\frac{dC_1}{d\theta}(\theta)d\theta \end{aligned}$$

Problem 8 (实践微分同胚——将“三维轴对称柱体”变换为“规则矩体”) $\mathbb{R}^3$ 中区域

$$\mathscr{D}_{xyz}\triangleq\{\begin{bmatrix} x \\ y \\ z \end{bmatrix}\mid 0<x^2+y^2<R^2(z),\ z\subset(0,H),\ \text{且将 } xz \text{ 平面的上半平面剖开}\}$$

为沿轴向管径具有轴对称变形的“柱体”。现考虑，将此不规则的柱体 $\mathscr{D}_{xyz}$ 通过向量值映照化成规则的矩体。以下研究向量值映照：

$$X(\xi,\eta,\zeta):\ \mathscr{D}_{\xi\eta\zeta}\ni\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}\mapsto X(\xi,\eta,\zeta)\triangleq\begin{bmatrix} x(\xi,\eta,\zeta) \\ y(\xi,\eta,\zeta) \\ z(\xi,\eta,\zeta) \end{bmatrix}=\begin{bmatrix} \xi\cdot R(\zeta)\cos\eta \\ \xi\cdot R(\zeta)\sin\eta \\ \zeta \end{bmatrix}$$

此处，参数区域

$$\mathscr{D}_{\xi\eta\zeta}\triangleq\{\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}\mid\xi\subset(0,1),\ \eta\subset(0,2\pi),\ \zeta\subset(0,H)\}$$

1. *(10%) 基于微分同胚的有关充分性定理，证明：$X(\xi,\eta,\zeta)$ 为 $\mathscr{D}_{\xi\eta\zeta}$ 同 $\mathscr{D}_{xyz}$ 间的微分同胚。籍此，$\{\xi,\eta,\zeta\}$ 成为物理区域 $\mathscr{D}_{xyz}$ 的曲线坐标系。注：此处需求出向量值映照 $X(\xi,\eta,\zeta)$ 的 Jacobian 矩阵 $DX(\xi,\eta,\zeta)\in\mathbb{R}^{3\times3}$ 。*

2. *(10%) 给出三维轴对称柱体 $\mathscr{D}_{xyz}$ 的积分表达式（需给出参数域及参数域上被积函数）。*

3. (10%) 给出三维轴对称柱体之侧面积积分表达式（需给出参数域及参数域上被积函数）。

4. (10%) 计算

$$\int_{\partial\mathscr{D}_{xyz}}[-\sin\eta\mathbf{i}+\cos\eta\mathbf{j}+R(\zeta)\mathbf{k}]\cdot\mathbf{n}d\sigma$$

此处 $\partial\mathscr{D}_{xyz}$ 表示三维轴对称柱体的侧面，$\mathbf{n}$ 指向向外。

5. (10%) 设有函数 $f(x,y,z)\in C^2(\mathscr{D}_{xyz};\mathbb{R})$，则有：

$$f(x,y,z)=f(x(\xi,\eta,\zeta),y(\xi,\eta,\zeta))=:\hat{f}(\xi,\eta,\zeta)=\hat{f}(\xi(x,y,z),\eta(x,y,z),\zeta(x,y,z))$$

试将：$\frac{\partial^2 f}{\partial y\partial x}(x,y,z)$ 由 $\hat{f}(\xi,\eta,\zeta)$ 以及 $X(\xi,\eta,\zeta)\in\mathbb{R}^3$ 等关于 ξ,η,ζ 的函数来表示，仅需说明必要的步骤，无需给出具体形式。注：本问题基于逆映照定理。

解答

1. (8/1)

$$X(\xi,\eta,\zeta)=\begin{bmatrix}x\\y\\z\end{bmatrix}(\xi,\eta,\zeta)=\begin{bmatrix}\xi R(\zeta)\cos\eta\\\xi R(\zeta)\sin\eta\\\zeta\end{bmatrix}$$

$$DX(\xi,\eta,\zeta)\triangleq\frac{D(x,y,z)}{D(\xi,\eta,\zeta)}(\xi,\eta,\zeta)=\begin{bmatrix}R(\zeta)\cos\eta & -\xi R(\zeta)\sin\eta & \xi\dot{R}(\zeta)\cos\eta\\R(\zeta)\sin\eta & \xi R(\zeta)\cos\eta & \xi\dot{R}(\zeta)\sin\eta\\0&0&1\end{bmatrix}$$

$$\det DX(\xi,\eta,\zeta)=\xi R(\zeta)\neq 0\quad\forall\begin{bmatrix}\xi\\\eta\\\zeta\end{bmatrix}\in\mathscr{D}_{\xi\eta\zeta}$$

对于单射根据映照几何意义易证。

2. (8/2)

$$|\mathscr{D}_{xyz}|=\int_{\mathscr{D}_{xyz}}d\tau=\int_{\mathscr{D}_{\xi\eta\zeta}}\left|\det\frac{D(x,y,z)}{D(\xi,\eta,\zeta)}(\xi,\eta,\zeta)\right|d\tau$$

$$=\int_{\mathscr{D}_{\xi\eta\zeta}}\xi R(\zeta)d\tau=2\pi\int_0^1\xi d\xi\cdot\int_0^H R(\zeta)d\zeta=\pi\int_0^H R(\zeta)d\zeta$$

3. (8/3)

$$\Sigma(\eta,\zeta,\xi):\mathscr{D}_{\eta\zeta}\ni\begin{bmatrix}\eta\\\zeta\end{bmatrix}\longmapsto\Sigma(\eta,\zeta)=X(1,\eta,\zeta)=\begin{bmatrix}R(\zeta)\cos\eta\\R(\zeta)\sin\eta\\\zeta\end{bmatrix}\in\mathbb{R}^3$$

$$\Rightarrow \frac{\partial\Sigma}{\partial\eta}\times\frac{\partial\Sigma}{\partial\zeta}(\eta,\zeta)=\begin{vmatrix} i & j & k \\ -R(\zeta)\sin\eta & R(\zeta)\cos\eta & 0 \\ \dot{R}\cos\eta & \dot{R}(\zeta)\sin\eta & 1 \end{vmatrix}=\begin{bmatrix} R(\zeta)\cos\eta \\ R(\zeta)\sin\eta \\ -R(\zeta)\dot{R}(\zeta) \end{bmatrix}\in\mathbb{R}^3$$

故

$$\begin{aligned}|\Sigma|=\int_\Sigma d\sigma=\int_{\mathscr{D}_{\eta\zeta}}\left|\frac{\partial\Sigma}{\partial\eta}\times\frac{\partial\Sigma}{\partial\zeta}\right|_{\mathbb{R}^3}(\eta,\zeta)d\sigma=\int_{\mathscr{D}_{\eta\zeta}}\sqrt{R^2(\zeta)+R^2(\zeta)\dot{R}^2(\zeta)}d\sigma \\ =2\pi\int_0^H R(\zeta)\sqrt{1+\dot{R}^2(\zeta)}d\zeta\end{aligned}$$

4. (8/4)

$$\begin{aligned}&\int_{\partial\mathscr{D}_{xyz}}[-\sin\eta i+\cos\eta j+R(\zeta)k]\cdot nd\sigma \\ =&\int_{\mathscr{D}_{\eta\zeta}}[-\sin\eta,\cos\eta,R(\zeta)]\cdot(\frac{\partial\Sigma}{\eta}\times\frac{\partial\Sigma}{\partial\zeta})(\eta,\zeta)d\sigma \\ =&\int_{\mathscr{D}_{\eta\zeta}}R(\zeta)[-\sin\eta,\cos\eta,R(\zeta)]\cdot\begin{bmatrix}\cos\eta \\ \sin\eta \\ -\dot{R}(\zeta)\end{bmatrix}d\sigma \\ =&\int_{\mathscr{D}_{\eta\zeta}}R(\zeta)R(\zeta)\dot{R}(\zeta)d\sigma \\ =&2\pi\int_0^H R^2(\zeta)\dot{R}(\zeta)d\zeta=2\pi\int_0^H R^2(\zeta)dR(\zeta) \\ =&2\pi\frac{R^3(H)-R^2(0)}{3}\end{aligned}$$

5. (8/5) 已有：

$$DX(\xi,\eta,\zeta)=\frac{D(x,y,z)}{D(\xi,\eta,\zeta)}(\xi,\eta,\zeta)=\begin{bmatrix} R(\zeta)\cos\eta & -\xi R(\zeta)\sin\eta & \xi\dot{R}(\zeta)\cos\eta \\ R(\zeta)\sin\eta & \xi R(\zeta)\cos\eta & \xi\dot{R}(\zeta)\sin\eta \\ 0 & 0 & 1 \end{bmatrix}$$

由

$$\frac{\partial f}{\partial x}(x,y,z)=\frac{\partial\hat{f}}{\partial\xi}(\xi,\eta,\zeta)\frac{\partial\xi}{\partial x}(x,y,z)+\frac{\partial\hat{f}}{\partial\eta}(\xi,\eta,\zeta)\frac{\partial\eta}{\partial x}(x,y,z)+\frac{\partial\hat{f}}{\partial\zeta}(\xi,\eta,\zeta)\frac{\partial\zeta}{\partial x}(x,y,z)$$

由逆映照定理

$$\frac{D(\xi,\eta,\zeta)}{D(x,y,z)}(x,y,z)=\left(\frac{D(x,y,z)}{D(\xi,\eta,\zeta)}\right)^{-1}(\xi,\eta,\zeta) \qquad (*)$$

故

$$\frac{\partial\xi}{\partial x}(x,y,z)\,,\ \frac{\partial\eta}{\partial x}(x,y,z)\,,\ \frac{\partial\zeta}{\partial x}(x,y,z)$$

等，均为 (ξ, η, ζ) 的函数，可记

$$\frac{\partial \xi}{\partial x}(x,y,z) = \overline{\frac{\partial \xi}{\partial x}}(\xi(x,y,z), \eta(x,y,z), \zeta(x,y,z))$$

由此

$$\frac{\partial^2 f}{\partial y \partial x}(x,y,z) = [\frac{\partial^2 \hat{f}}{\partial \xi \partial \xi}\frac{\partial \xi}{\partial y}+\frac{\partial^2 \hat{f}}{\partial \eta \partial \xi}\frac{\partial \eta}{\partial y}+\frac{\partial^2 \hat{f}}{\partial \zeta \partial \xi}\frac{\partial \zeta}{\partial y}]\cdot\frac{\partial \xi}{\partial x}+\frac{\partial \hat{f}}{\partial \xi}[\frac{\partial}{\partial \xi}(\overline{\frac{\partial \xi}{\partial x}})\frac{\partial \xi}{\partial y}+\frac{\partial}{\partial \eta}(\overline{\frac{\partial \xi}{\partial x}})\frac{\partial \eta}{\partial y}+\frac{\partial}{\partial \zeta}(\overline{\frac{\partial \xi}{\partial x}})\frac{\partial \zeta}{\partial y}]+\cdots$$

而 $\frac{\partial \xi}{\partial y}$, $\frac{\partial \eta}{\partial y}$, $\frac{\partial \zeta}{\partial y}$ 等可再由 $(*)$ 式获得。