复旦大学航空航天系与技术科学类

2018-2019 学年第一学期《数学分析 B》一元微积分测试

A 卷　共 16 页

课程代码：　　　　　考试形式：□ 开卷 □ 闭卷　2018 年 12 月 30 日 18:00-21:30

（本试卷答卷时间为 210 分钟，答案必须写在试卷上，做在草稿纸上无效）

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题号	1-1	1-2	1-3	2-1	2-2	3-1	3-2	
得分								
题号	4-1	4-2	4-3	4-4	5-1	5-2	5-3	5-4
得分								
题号	6-1	6-2	6-3	6-4	7-1	7-2	7-3	
得分								
题号	8-1	8-2	9-1	9-2				总分
得分								

一、计算题（每题 2 分，共 3 题，共 6 分）

1. 计算数列极限 $\lim\limits_{x \to 0} x\left[\dfrac{1}{x}\right]$

解：　考虑 $x \to 0 + 0$. $\dfrac{1}{x} \subset [n, n+1)$, 亦即 $\dfrac{1}{n+1} < n \leqslant \dfrac{1}{n}$, 此时 $\left[\dfrac{1}{x}\right] = n$.

故有 $x\left[\dfrac{1}{x}\right] \subset \left(\dfrac{n}{n+1}, 1\right)$. 按夹逼性有 $x\left[\dfrac{1}{x}\right] \to 1$.

$x \to 0 - 0$: $\dfrac{1}{x} \subset [-(n+1), -n)$, 亦即 $-\dfrac{1}{n} < n \leqslant -\dfrac{1}{n+1}$, 此时 $\left[\dfrac{1}{x}\right] = -(n+1)$.

故有 $x\left[\dfrac{1}{x}\right] \subset \left[1, \dfrac{n}{n+1}\right)$. 按夹逼性有 $x\left[\dfrac{1}{x}\right] \to 1$.

综上, 有极限为 1.

2. 计算数列极限 $\lim\limits_{x \to +\infty}\left[\left(x^3 - x^2 + \dfrac{x}{2}\right)\mathrm{e}^{\frac{1}{x}} - \sqrt{x^6 - 1}\right]$

解：

$$f(x) = \left(x^3 - x^2 + \dfrac{x}{2}\right)\mathrm{e}^{\frac{1}{x}} - \sqrt{x^6 - 1} \overset{t=\frac{1}{x}}{=} \left(\dfrac{1}{t^3} - \dfrac{1}{t^2} + \dfrac{1}{t2}\right)\mathrm{e}^t - \sqrt{\dfrac{1}{t^6} - 1}$$

$$= \dfrac{1}{t^3}\left(1 - t + \dfrac{1}{2}t^2\right)\left(1 + t + \dfrac{t^2}{2} + \dfrac{t^3}{6} + \mathrm{o}(t^3)\right) - \dfrac{1}{t^3}\left(1 - t^6\right)^{\frac{1}{2}}$$

$$= \dfrac{1}{t^3}\left[1 + (-1 + 1)t + \left(\dfrac{1}{2} - 1 + \dfrac{1}{2}\right)t^2 + \left(\dfrac{1}{2} - \dfrac{1}{2} + \dfrac{1}{6}\right)t^3 + \mathrm{o}(t^3)\right] - \dfrac{1}{t^3}\left(1 - \dfrac{1}{2}t^6 + \mathrm{o}t^6\right)$$

$$= \dfrac{1}{t^3}\left(1 + \dfrac{1}{6}t^3 + \mathrm{o}(t^3)\right) - \dfrac{1}{t^3} + \mathrm{o}(1)$$

$$= \dfrac{1}{6} + \mathrm{o}(1) \to \dfrac{1}{6}$$

3. 设 $x_1 > 0$, $x_{n+1} = \ln(1 + x_n)$, $n \in \mathbb{N}$. 证明: $\lim\limits_{n \to +\infty} x_n = 0$, $\lim\limits_{n \to +\infty} nx_n = 2$

解：　①$\delta(x) = x - \ln(1 + x)$, $\begin{cases} \delta(0) = 0 \\ \\ \delta'(x) = 1 - \dfrac{1}{1 + x} > 0, \forall x > 0 \end{cases}$

故有 $\delta(x) \uparrow\uparrow 0 \sim \mathbb{R}^+$. 由此, $\ln(1 + x) < x$. $x_{n+1} = \ln(1 + x_n) < x_n$.

而 $x_n \geqslant 0$. 故有 $x_n \downarrow 0$.

②考虑

$$\dfrac{\frac{1}{x_n}}{n} \sim \dfrac{1}{x_{n+1}} - \dfrac{1}{x_n} = \dfrac{1}{\ln(1 + x_n)} - \dfrac{1}{x_n}$$

$$= \dfrac{1}{x_n - \frac{1}{2}x_n^2 + \mathrm{o}(x_n^2)} - \dfrac{1}{x_n}$$

$$= \dfrac{1}{x_n}\left(1 + \dfrac{1}{2}x_n + \mathrm{o}(x_n)\right) - \dfrac{1}{x_n}$$

$$= \dfrac{1}{2} + \mathrm{o}(1) \to \dfrac{1}{2}$$

亦即 $nx_n \to 2$.

二、计算题（每题 4 分，共 2 题，共 8 分）

1. 计算二阶导数 $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$，函数隐式关系式为 $\mathrm{e}^{x^2+y} - x^2 y = 0$.

   解： $\mathrm{e}^{x^2+y}\left(2x + \dfrac{\mathrm{d}y}{\mathrm{d}x}\right) - 2xy - x^2\dfrac{\mathrm{d}y}{\mathrm{d}x} = 0$

   有 $\left(\mathrm{e}^{x^2+y} - x^2\right)\dfrac{\mathrm{d}y}{\mathrm{d}x} = -\mathrm{e}^{x^2+y} \cdot 2x + 2xy$

   $\Rightarrow \dfrac{\mathrm{d}y}{\mathrm{d}x}(x) = -2x\dfrac{\mathrm{e}^{x^2+y} - y}{\mathrm{e}^{x^2+y} - x^2}$

   有

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}(x) = -2\frac{\mathrm{e}^{x^2+y} - y}{\mathrm{e}^{x^2+y} - x^2}$$

$$-2x\frac{\left[\mathrm{e}^{x^2+y} \cdot \left(2x + \frac{\mathrm{d}y}{\mathrm{d}x}\right) - \frac{\mathrm{d}y}{\mathrm{d}x}\right]\left(\mathrm{e}^{x^2+y} - x^2\right) - \left(\mathrm{e}^{x^2+y} - y\right)\left[\mathrm{e}^{x^2+y} \cdot \left(2x + 2y\frac{\mathrm{d}y}{\mathrm{d}x}\right) - 2x\right]}{\left(\mathrm{e}^{x^2+y} - x^2\right)^2}$$

2. 计算二阶导数 $\dfrac{\mathrm{d}^2 y}{\mathrm{d}x^2}$，函数参数形式为 $\begin{cases} x(t) = \sqrt{1+t} \\ y(t) = \sqrt{1-t} \end{cases}$.

   解： $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{\dot{y}(t)}{\dot{x}(t)} = \dfrac{\frac{1}{2}\frac{-1}{\sqrt{1-t}}}{\frac{1}{2}\frac{1}{\sqrt{1+t}}} = -\sqrt{\dfrac{1+t}{1-t}}$

$$\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\frac{1+t}{1-t}} \cdot \frac{\mathrm{d}t}{\mathrm{d}x} = -\frac{\mathrm{d}}{\mathrm{d}t}\sqrt{\frac{1+t}{1-t}} \cdot \frac{1}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$

$$= -\frac{1}{2}\sqrt{\frac{1+t}{1-t}}\frac{1-t+(1+t)}{(1-t)^2}\frac{1}{\frac{1}{2}\frac{1}{\sqrt{1+t}}}$$

$$= -2\frac{\sqrt{1+t}}{(1-t)^2}\sqrt{\frac{1-t}{1+t}} = -2(1-t)^{-\frac{3}{2}}$$

三、定积分计算题（每题 3 分，共 2 题，共 6 分）

1. 计算 $\displaystyle\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\mathrm{d}x$

解：

$$\int_0^\pi \frac{x\sin x}{1+\cos^2 x}\mathrm{d}x = \int_0^\pi \frac{\left(x-\frac{\pi}{2}\right)\sin x}{1+\cos^2 x}\mathrm{d}x + \int_0^\pi \frac{\pi}{2}\frac{\sin x}{1+\cos^2 x}\mathrm{d}x$$

此处 $\begin{cases} x-\dfrac{\pi}{2}\text{关于}x=\dfrac{\pi}{2}\text{反对称} \\[2mm] \dfrac{\sin x}{1+\cos^2 x}\text{关于}x=\dfrac{\pi}{2}\text{对称} \end{cases}$.

$$\text{原式} = \int_0^\pi \frac{\pi}{2}\frac{\sin x}{1+\cos^2 x}\mathrm{d}x = -\int_0^\pi \frac{\pi}{2}\frac{\mathrm{d}\cos x}{1+\cos^2 x}$$

$$\overset{y=\cos x}{=\!=\!=} -\frac{\pi}{2}\int_1^{-1}\frac{\mathrm{d}y}{1+y^2} = \frac{\pi}{2}\int_{-1}^1\frac{\mathrm{d}y}{1+y^2} = \pi\int_0^1\frac{\mathrm{d}y}{1+y^2}$$

$$= \pi\arctan y\Big|_0^1 = \pi\cdot\frac{\pi}{4} = \frac{\pi^2}{4}$$

2. 计算 $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}\mathrm{d}x,\ n\in\mathbb{N}$

解：

$$\int_0^{\frac{\pi}{2}} \frac{\sin^n x}{\sin^n x + \cos^n x}\mathrm{d}x = \left(\int_0^{\frac{\pi}{4}} + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\right)\frac{\sin^n x}{\sin^n x + \cos^n x}\mathrm{d}x$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sin^n x}{\sin^n x + \cos^n x}\mathrm{d}x + \int_0^{\frac{\pi}{4}} \frac{\cos^n x}{\sin^n x + \cos^n x}\mathrm{d}x$$

$$=\!= \int_0^{\frac{\pi}{4}}\mathrm{d}x = \frac{\pi}{4}$$

利用对称性 $\displaystyle\int_0^{\frac{\pi}{4}} f(\cos x)\mathrm{d}x = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} f(\sin x)\mathrm{d}x$

Euler 积分也利用上述对称性, 进行相配.

四、不定积分计算题（每题 3 分，共 4 题，共 12 分）

1. $\displaystyle\int \frac{\mathrm{d}x}{\sqrt[3]{(x-2)(x+1)^2}}$

解：

$$I = \int \frac{\mathrm{d}(x+1)}{\sqrt[3]{(x-2)(x+1)^2}} \overset{y=x+1}{=\!=\!=} \int \frac{\mathrm{d}y}{y^{\frac{2}{3}}(y-3)^{\frac{1}{3}}} = \int y^{-1}\left(\frac{y-3}{y}\right)^{-\frac{1}{3}}\mathrm{d}y$$

令 $z = \left(\dfrac{y-3}{y}\right)^{-\frac{1}{3}}, z^3 = 1 - \dfrac{3}{y} \Rightarrow y = \dfrac{3}{1-z^3}$

$$I = \int \frac{1-z^3}{3}\frac{1}{z}\left(-\frac{z}{(1-z^3)^2}\right)(-3z^2)\mathrm{d}z = \int \frac{z}{1-z^3}\mathrm{d}z$$

$$= \int \frac{z}{(1-z)(1+z+z^2)}\mathrm{d}z = 3\int \left(\frac{\frac{1}{3}}{1-z} + \frac{\frac{2}{3}+\frac{1}{3}z}{1+z+z^2}\right)\mathrm{d}z$$

$$= \ln|1-z| + 2\int \frac{\mathrm{d}z}{1+z+z^2} + \int \frac{z}{1+z+z^2}\mathrm{d}z$$

$$= \ln|1-z| + \frac{4}{\sqrt{3}}\arctan\frac{z+\frac{1}{2}}{\frac{\sqrt{3}}{2}} + \frac{1}{2}\ln(x^2+z+1) - \frac{1}{\sqrt{3}}\arctan\frac{z+\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

2. $\displaystyle\int \frac{x^2\arcsin x}{\sqrt{1-x^2}}\mathrm{d}x$

解： 考虑 $x \in [0,1]$,

$$I \overset{t=\arcsin x}{=\!=\!=} \int \frac{\sin^2 t \cdot t}{\cos t}\cos t\,\mathrm{d}t = \int t\frac{1-\cos 2t}{2}\mathrm{d}t$$

$$= \frac{t^2}{4} - \frac{1}{2}\int t\cos 2t\,\mathrm{d}t = \frac{t^2}{4} - \frac{1}{4}\int t\,\mathrm{d}\sin 2t$$

$$= \frac{t^2}{4} - \frac{1}{4}\left(t\sin 2t + \frac{1}{2}\cos 2t\right)$$

3. $\displaystyle\int \mathrm{e}^{\sin x}\frac{x\cos^3 x-\sin x}{\cos^2 x}\mathrm{d}x$

解：

$$I_1 := \int \mathrm{e}^{\sin x}\frac{x\cos^3 x}{\cos^2 x}\mathrm{d}x = \int x\mathrm{d}\mathrm{e}^{\sin x} = \mathrm{e}^{\sin x}-\int \mathrm{e}^{\sin x}\mathrm{d}x$$

$$I_2 := \int \mathrm{e}^{\sin x}\frac{\sin x}{\cos^2 x}\mathrm{d}x = -\int \mathrm{e}^{\sin x}\frac{\mathrm{d}\cos x}{\cos^2 x} = \int \mathrm{e}^{\sin x}\mathrm{d}\frac{1}{\cos x}$$

$$= \frac{\mathrm{e}^{\sin x}}{\cos x}-\int \frac{1}{\cos x}\mathrm{e}^{\sin x}\cos x\mathrm{d}x = \frac{\mathrm{e}^{\sin x}}{\cos x}-\int \mathrm{e}^{\sin x}\mathrm{d}x$$

$$\Rightarrow I = I_1 - I_2 = \mathrm{e}^{\sin x}-\frac{\mathrm{e}^{\sin x}}{\cos x}$$

4. $\displaystyle\int \frac{\sqrt{1+\sin x}}{\cos x}\mathrm{d}x$

解：

$$I = \int \frac{\sqrt{1+\sin x}}{\cos^2 x}\cos x\mathrm{d}x \overset{y=\sin x}{=} \int \frac{\sqrt{1+y}}{1-y^2}\mathrm{d}y$$

令 $z = \sqrt{1+y},\ y = z^2-1$

$$I = \int \frac{z}{1-(z^2-1)^2}2z\mathrm{d}z = 2\int \frac{z^2}{z^2(2-z^2)}\mathrm{d}z$$

$$= \frac{1}{\sqrt{2}}\int \left(\frac{1}{\sqrt{2}+z}+\frac{1}{\sqrt{2}-z}\right)\mathrm{d}z = \frac{1}{\sqrt{2}}\ln\frac{\sqrt{2}+z}{\sqrt{2}-z}$$

五、广义积分计算题（每题 4 分，共 4 题，共 16 分）

1. 计算 $\displaystyle\int_0^{+\infty} \frac{\mathrm{d}x}{1+x^4}$

解：

$$\int \frac{\mathrm{d}x}{1+x^4} = \int \frac{\frac{1}{x^2}\mathrm{d}x}{\frac{1}{x^2}+x^2} = -\int \frac{\mathrm{d}(\frac{1}{x})}{\frac{1}{x^2}+x^2} = -\frac{1}{2}\int \frac{\mathrm{d}(\frac{1}{x}+x-x+\frac{1}{x})}{\frac{1}{x^2}+x^2}$$

$$= -\frac{1}{2}\int \frac{\mathrm{d}(\frac{1}{x}+x)}{(\frac{1}{x}+x)^2-2} + \frac{1}{2}\int \frac{\mathrm{d}(x-\frac{1}{x})}{(\frac{1}{x}-x)^2+2}$$

$$= -\frac{1}{4\sqrt{2}}\ln\left[\frac{\frac{1}{x}+x-\sqrt{2}}{\frac{1}{x}+x+\sqrt{2}}\right] + \frac{1}{2\sqrt{2}}\arctan\left(\frac{x-\frac{1}{x}}{\sqrt{2}}\right) + C$$

$$F(0+0) = -\frac{\pi}{4\sqrt{2}}, \ F(+\infty) = \frac{\pi}{4\sqrt{2}}. \ 故有 \int_0^{+\infty} \frac{\mathrm{d}x}{1+x^4} = F(+\infty) - F(0+0) = \frac{\pi}{2\sqrt{2}}$$

2. 计算 $\displaystyle\int_0^{+\infty} \frac{\mathrm{d}x}{(2x^2+1)\sqrt{1+x^2}}$

解：考虑 $\displaystyle\int \frac{\mathrm{d}x}{(2x^2+1)\sqrt{1+x^2}}$，作 $y = \frac{1}{2x^2+1}$，$x = \frac{1}{\sqrt{2}}\sqrt{\frac{1}{y}-1}$

$$\int \frac{\mathrm{d}x}{(2x^2+1)\sqrt{1+x^2}} = -\frac{1}{2}\int \frac{1}{\sqrt{\frac{1}{y^2}-1}}\frac{1}{y}\mathrm{d}y = -\frac{1}{2}\int \frac{\mathrm{d}y}{\sqrt{1-y^2}} = -\frac{1}{2}\arcsin y$$

$$\int_0^{+\infty} \frac{\mathrm{d}x}{(2x^2+1)\sqrt{1+x^2}} = -\frac{1}{2}\arcsin y\Big|_1^0 = \frac{1}{2}\frac{\pi}{2} = \frac{\pi}{4}$$

3. 计算 $\displaystyle\int_0^{+\infty} \frac{\mathrm{d}x}{\mathrm{e}^{x+1} + \mathrm{e}^{3-x}}$

解：

$$\int \frac{\mathrm{d}x}{\mathrm{e}^{x+1} + \mathrm{e}^{3-x}} = \int \frac{\mathrm{e}^x \mathrm{d}x}{\mathrm{e}^{2x+1} + \mathrm{e}^3} = \frac{1}{\mathrm{e}} \int \frac{\mathrm{d}\mathrm{e}^x}{\mathrm{e}^{2x} + \mathrm{e}^2}$$

$$\overset{y=\mathrm{e}^x}{=} \frac{1}{\mathrm{e}} \int \frac{\mathrm{d}y}{y^2 + \mathrm{e}^2} = \frac{1}{\mathrm{e}^2} \int \frac{\mathrm{d}\left(\frac{y}{\mathrm{e}}\right)}{\left(\frac{y}{\mathrm{e}}\right)^2 + 1}$$

$$= \frac{1}{\mathrm{e}^2} \arctan \frac{\mathrm{e}^x}{\mathrm{e}} =: F(x)$$

$$\int_0^{+\infty} \frac{\mathrm{d}x}{\mathrm{e}^{x+1} + \mathrm{e}^{3-x}} = F(+\infty) - F(0) = \frac{1}{\mathrm{e}^2} \frac{\pi}{2} - \frac{1}{\mathrm{e}^2} \arctan \frac{1}{\mathrm{e}}$$

4. 计算 $\displaystyle\lim_{n\to+\infty} \frac{1}{\sqrt{n}} \int_1^n \ln\left(1 + \frac{1}{\sqrt{x}}\right)\mathrm{d}x$

解：

$$\lim_{n\to+\infty} \frac{1}{\sqrt{n}} \int_1^n \ln\left(1 + \frac{1}{\sqrt{x}}\right)\mathrm{d}x \sim \frac{\ln\left(1 + \frac{1}{\sqrt{n}}\right)}{\frac{1}{2} \cdot \frac{1}{\sqrt{n}}} = \frac{\frac{1}{\sqrt{n}} + \mathrm{o}\left(\frac{1}{\sqrt{n}}\right)}{\frac{1}{2} \frac{1}{\sqrt{n}}} = 2 + \mathrm{o}(1) \to 2$$

六、常微分方程计算题（每题 5 分，共 4 题，共 20 分）

1. 求解 $\dfrac{\mathrm{d}y}{\mathrm{d}x} - \dfrac{1}{1-x^2}y = 1+x$, $y(0)=1$

   解： $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 1+x + \dfrac{1}{1-x^2}y$

   Step1: 求 $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1}{1-x^2}y$, 有 $y = C \cdot \mathrm{e}^{\displaystyle\int \frac{1}{1-x^2}\mathrm{d}x} = C \cdot \sqrt{\dfrac{1+x}{1-x}}$

   Step2: 设 $y(x) = C(x)\sqrt{\dfrac{1+x}{1-x}}$. 有 $\dfrac{\mathrm{d}C}{\mathrm{d}x}\sqrt{\dfrac{1+x}{1-x}} = 1+x$

   $\Rightarrow \dfrac{\mathrm{d}C}{\mathrm{d}x} = \sqrt{1-x^2},\ C(x) = \displaystyle\int \sqrt{1-x^2}\mathrm{d}x + C = \dfrac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right) + C$

   故有 $y(x) = \left[\dfrac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right) + C\right] \cdot \sqrt{\dfrac{1+x}{1-x}}$

   利用定解条件 $y(0)=1$, 有 $C=1$.

   亦即 $y(x) = \left[\dfrac{1}{2}\left(\arcsin x + x\sqrt{1-x^2}\right) + 1\right] \cdot \sqrt{\dfrac{1+x}{1-x}}$

2. 求解 $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{x}{\cos y} + x\tan y$

   解： $\cos y \dfrac{\mathrm{d}y}{\mathrm{d}x} = x + x\sin y = \dfrac{\mathrm{d}}{\mathrm{d}x}\sin y$

   令 $z = \sin z$, 有 $\dfrac{\mathrm{d}z}{\mathrm{d}x} = x \cdot z + x$.

   Step1: 求 $\dfrac{\mathrm{d}z}{\mathrm{d}x} = x \cdot z$, 有 $z(x) = C\mathrm{e}^{\frac{x^2}{2}}$

   Step2: 设 $z(x) = C(x)\mathrm{e}^{\frac{x^2}{2}}$, 有 $\dfrac{\mathrm{d}C}{\mathrm{d}x}\mathrm{e}^{\frac{x^2}{2}} = x$.

   $C(x) = \displaystyle\int \mathrm{e}^{-\frac{x^2}{2}}x\mathrm{d}x + C = -\int \mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}\left(-\dfrac{x^2}{2}\right) + C = -\mathrm{e}^{-\frac{x^2}{2}} + C$

   故有 $y(x) = \left[-\mathrm{e}^{-\frac{x^2}{2}} + C\right] \cdot \mathrm{e}^{\frac{x^2}{2}} = C\mathrm{e}^{\frac{x^2}{2}} - 1$

3. 求解 $y'' - 2y' + 2y = 4e^x \cos x$

解： Step1: 求解 $y'' - 2y' + 2y = 0$, $r^2 - 2r + 2 = 0$. $r_{1,2} = \dfrac{2 \pm \sqrt{-4}}{2} = 1 \pm i$

$y(x) = e^x(A\cos x + B\sin x)$

Step2: $f(x) = 4e^x \cos x = e\cos x P$

设 $y_*(x) = xe^x(E\cos x + F\sin x) = e^x[Ex\cos x + Fx\sin x]$

$\quad y_*' = e^x[Fx\cos x + E\cos x - Ex\sin x + Fx\sin x + F\sin x + Fx\cos x]$

$\quad\quad = e^x[(E+F)x\cos x + (F-E)x\sin x + F\sin x + E\cos x]$

$\quad y_*'' = e^x[(E+F)x\cos x + (E+F)\cos x - (E+F)x\sin x + (F-E)x\sin x + (F-E)x\cos x$
$\quad\quad + E\cos x - E\sin x + F\sin x + F\cos x]$

$\quad\quad = e^x[2Fx\cos x - 2Ex\sin x + 2(F-E)\sin x + 2(E+F)x\cos x]$

$y_*'' - 2y_*' + 2y_* = e^x(2F\cos x - 2E\sin x) = e^x \cdot 4\cos x$

有 $\begin{cases} F = 2 \\ E = 0 \end{cases}$, $y_*(x) = 2xe^x \sin x$

故有 $y(x) = e^x(A\cos x + B\sin x) + 2xe^x\sin x$

4. 求解 $y'' - 5y' + 6y = (12x - 7)e^{-x}$

解： Step1: 求解 $y'' - 5y' + 6y = 0$, $r^2 - 5r + 6 = 0$. $r_1 = 2$, $r_2 = 3$

$y(x) = Ae^{2x} + Be^{3x}$

Step2: $f(x) = e^{-x}(12x - 7)$, 有 $y_*(x) = e^{-x}(Ax - B)$

有

$$y_*' = e^{-x}(-Ax + A - B)$$

$$y_*'' = e^{-x}(Ax - A - A - B) = e^{-x}(Ax - 2A - B)$$

$$y'' - 5y' + 6y = e^{-x}(12Ax - 7A + 12B) = e^{-x}(12x - 7)$$

有 $\begin{cases} A = 1 \\ -7 + 12B = -7 \Rightarrow B = 0 \end{cases}$, $y_*(x) = xe^{-x}$

故有 $y(x) = Ae^{2x} + Be^{3x} + xe^{-x}$

七、应用题（每题 4 分，共 3 题，共 12 分）

1. $\lim\limits_{n\to+\infty}\displaystyle\int_n^{n+p}\dfrac{\sin x}{x}\mathrm{d}x,\ (p\in\mathbb{N}^p).$

注: 可考虑利用积分第一中值定理.

解:

$$\lim_{n\to+\infty}\int_n^{n+p}\frac{\sin x}{x}\mathrm{d}x=\lim_{n\to+\infty}\sin\xi\int_n^{n+p}\frac{\mathrm{d}x}{x}=\lim_{n\to+\infty}\sin\xi\ln\frac{n+p}{n}$$

$$=\sin\xi\ln\left(1+\frac{p}{n}\right)=\sin\xi\cdot\left(\frac{p}{n}+\mathrm{o}\left(\frac{1}{n}\right)\right)\to 0$$

2. 设 $f(x)=\begin{cases}\sin\dfrac{x}{2},x\geqslant 0\\[3mm] x\arctan x,x<0\end{cases}$ ，计算 $\displaystyle\int_0^{\pi+1}f(x-1)\mathrm{d}x$

解:

$$\int_0^{\pi+1}f(x-1)\mathrm{d}x=\int_0^{\pi+1}f(x-1)\mathrm{d}(x-1)=\int_{-1}^{\pi}f(x)\mathrm{d}x=\left(\int_{-1}^{0}+\int_0^{\pi}\right)f(x)\mathrm{d}x$$

$$=\int_{-1}^{0}x\arctan x\,\mathrm{d}x+\int_0^{\pi}\sin\frac{x}{2}\mathrm{d}x$$

式中

$$\int_{-1}^{0}x\arctan x\,\mathrm{d}x=\frac{1}{2}\int_{-1}^{0}\arctan x\,\mathrm{d}x^2=\frac{\pi}{8}-\frac{1}{2}\int_{-1}^{0}\frac{x^2+1-1}{x^2+1}\mathrm{d}x$$

$$=\frac{\pi}{4}-\frac{1}{2}$$

$$\int_0^{\pi}\sin\frac{x}{2}\mathrm{d}x=-2\cos x2|_0^{\pi}=2$$

3. 证明: 在极坐标下, 由 $0 \leqslant \alpha \leqslant \theta \leqslant \beta \leqslant \pi, 0 \leqslant r \leqslant r(\theta)$ 所表示的区域绕极轴旋转一周所成的旋成体的体积为 $V = \dfrac{2\pi}{3} \displaystyle\int_{\alpha}^{\beta} r^3(\theta) \sin\theta \, \mathrm{d}\theta$.

解: 计算 $I = \displaystyle\int_{\alpha}^{\beta} \pi y^2(\theta) x(\theta) \mathrm{d}\theta$.

其中 $\begin{cases} x(\theta) = r(\theta) \cos\theta \\[2mm] y(\theta) = r(\theta) \sin\theta \end{cases}$

$$I = \int_{\alpha}^{\beta} \pi r^2(\theta) \sin^2\theta \left[r^2(\theta) \cos\theta - r(\theta) \sin\theta \right] \mathrm{d}\theta$$

$$= \int_{\alpha}^{\beta} \pi r^2(\theta) r'(\theta) \sin^2\theta \cos\theta \, \mathrm{d}\theta - \int_{\alpha}^{\beta} \pi r^2(\theta) \sin^2\theta \, \mathrm{d}\theta$$

式中

$$\int_{\alpha}^{\beta} \pi r^2(\theta) r'(\theta) \sin^2\theta \cos\theta \, \mathrm{d}\theta = \int_{\alpha}^{\beta} \pi \sin^2\theta \cos\theta \, \mathrm{d} \frac{r^3(\theta)}{3}$$

$$= \pi \sin^2\beta \cos\beta \frac{r^3(\beta)}{3} - \pi \sin^2\alpha \cos\alpha \frac{r^3(\alpha)}{3}$$

八、计算证明题（每题 5 分，共 2 题，共 10 分）

1. $f(x) \in C^1[a, b], f(a) = 0,$ 证明 $\displaystyle\int_a^b \left|f(x)f'(x)\right| \mathrm{d}x \leqslant \frac{b-a}{2}\int_a^b \left|f'(x)\right|^2 \mathrm{d}x$

解：　令 $G(x) = \displaystyle\int_a^x \left|f'(\xi)\right| \mathrm{d}\xi,$ $\begin{cases} G'(x) = \left|f'(x)\right| \\[2mm] G(x) \geqslant \left|f(x)\right| \end{cases}$

$$\int_a^b \left|f(x)f'(x)\right| \mathrm{d}x \leqslant \int_a^b G(x)G'(x)\mathrm{d}x = \int_a^b \mathrm{d}\frac{G^2(x)}{2} = \frac{1}{2}G^2(b)$$

$$= \frac{1}{2}\left|\int_a^b \left|f'(x)\right| \mathrm{d}x\right|^2 \leqslant \frac{1}{2}\int_a^b \left|f'(x)\right|^2 \mathrm{d}x \cdot (b-a) = \frac{b-a}{2}\int_a^b \left|f'(x)\right|^2 \mathrm{d}x$$

2. 证明: $f(x) \in C^1[0, 1]$, $f'(0) \neq 0$, 则对 $\displaystyle\int_0^x f(t)\mathrm{d}t = f(\xi(x))x$ 中的 $\xi(x)$, 有 $\displaystyle\lim_{x \to 0+0} \frac{\xi(x)}{x} = \frac{1}{2}$.

解： 基于 Lagrange 中值定理:

$$\int_0^x f(t)\mathrm{d}t = \int_0^x f(t)\mathrm{d}t - \int_0^0 f(t)\mathrm{d}t = \frac{\mathrm{d}}{\mathrm{d}x}\int_0^x f(t)\mathrm{d}t(\xi) \cdot x = f(\xi(x)) \cdot x$$

当 $x \to 0$, $f(\xi(x)) = f(0) + f'(0)\xi(x) + \mathrm{o}(\xi(x))$

即 $f(\xi(x)) \cdot x = f(0)x + f'(0)x \cdot \xi(x) + \mathrm{o}(x \cdot \xi(x))$, 其中 $\mathrm{o}(x \cdot \xi(x)) = \mathrm{o}(x^2)$

另有 $-\dfrac{\mathrm{d}}{\mathrm{d}x}\displaystyle\int_0^x f(t)\mathrm{d}t = f(x) = f(0) + f'(0)x + \mathrm{o}(x) \Rightarrow \int_0^x f(t)\mathrm{d}t = f(0)x + \dfrac{f'(0)}{2}x^2 + \mathrm{o}(x^2)$

可有 $f'(0)x \cdot \xi(x) = \dfrac{f'(0)}{2}x^2 + \mathrm{o}(x^2) \Rightarrow \xi(x) = \dfrac{x}{2} + \mathrm{o}(x)$

即 $\dfrac{\xi(x)}{x} = \dfrac{1}{2} + \mathrm{o}(1) \to \dfrac{1}{2}$

九、广义积分敛散性判断题（每题 5 分，共 2 题，共 10 分）

1. 判断广义积分敛散性 $\displaystyle\int_0^{+\infty} \frac{1}{\sqrt[3]{x(x-1)^2(x-2)}}\mathrm{d}x$

解： $(1)x_* = 0, f(x) = \dfrac{C}{x^{\frac{1}{3}}}\left(1 + o(1)\right),$ 收敛.

$(2)x_* = 1, f(x) = \dfrac{C}{(x-1)^{\frac{2}{3}}}\left(1 + o(1)\right),$ 收敛.

$(3)x_* = 2, f(x) = \dfrac{C}{(x-2)^{\frac{1}{3}}}\left(1 + o(1)\right),$ 收敛.

$(4)x_* = +\infty, f(x) = \dfrac{C}{x^{\frac{4}{3}}},$ 收敛.

综上, 此广义积分收敛

2. 判断广义积分敛散性 $\displaystyle\int_0^{+\infty}\frac{\arctan x}{x^p}\mathrm{d}x$

解：　(1)$x_* = 0$, $f(x) = \dfrac{x + \mathrm{o}(x)}{x^p} = \dfrac{1}{x^{p-1}}(1 + \mathrm{o}(1))$

有 $\begin{cases} p - 1 < 1, \text{即} p < 2, \text{绝对收敛} \\[2ex] p - 1 \geqslant 1, \text{即} p \geqslant 2, \text{发散} \end{cases}$.

(2)$x_* = +\infty$, $\dfrac{\arctan x}{x^p} = \left[\dfrac{\pi}{2} + \mathrm{o}(1)\right] \cdot \dfrac{1}{x^p} = \dfrac{\pi}{2}\dfrac{1}{x^p}\left(1 + \mathrm{o}(1)\right)$

有 $\begin{cases} p > 1, \text{绝对收敛} \\[2ex] p \leqslant 1, \text{发散} \end{cases}$.

综上, 有 $\begin{cases} 1 < p < 2, \text{收敛} \\[2ex] \text{其他}, \text{发散} \end{cases}$.