复旦大学力学与工程科学系

2009～2010学年第二学期期末考试试卷－－参考答案

□A卷　　　□B卷

课程名称：数学分析（II）　　　　课程代码：MATH120009.09

开课院系：力学与工程科学系　　　　考试形式：开卷/闭卷/课程论文

姓名：__________　学号：__________　专业：__________

题 号	1/(1)	1/(2)	2/(1)	2/(2)	2/(3)	3/(1)	3/(2)	3/(3)	4/(1)	4/(2)
得 分										
题 号	5/(1)	5/(2)	6	7/(1)	7/(2)	7/(3)	7/(4)	8		总 分
得 分										

Problem 1 (*Fourier* 级数基本概念) *现有有限区间上定义的分段函数：*

$$f(x) \triangleq \begin{cases} 0 & x \in [0,1) \\ x^2 & x \in [1,2] \\ 0 & x \in (2,3] \end{cases}$$

1. (10%) *按点收敛的概念，将 $f(x)$, $x \in [0,3]$ 表示成 Fourier 级数的形式。要求写出具体的表达形式，但相关系数仅需给出具体计算式。*

2. (05%) *示意性绘出 Fourier 级数对应的和函数（极限函数）的图像。*

$$(1) f(x) \triangleq= \begin{cases} 0 & x \in [0,1) \\ x^2 & x \in [1,2] \\ 0 & x \in (2,3) \end{cases}$$

$$\frac{f(x+0)+f(x-0)}{2} = \frac{a_0}{2} + \sum_{n=1}^{+\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

$$\text{此处}\quad a_n = \frac{1}{l}\int_a^{a+2l} f(\xi)\cos\frac{n\pi\xi}{l}d\xi = \frac{1}{3/2}\int_0^3 f(\xi)\cos\frac{n\pi\xi}{l}d\xi \quad n \in \mathbb{N}\cup\{0\}$$

$$b_n = \frac{1}{l}\int_a^{a+2l} f(\xi)\sin\frac{n\pi\xi}{l}d\xi = \frac{1}{3/2}\int_0^3 f(\xi)\sin\frac{n\pi\xi}{l}d\xi \quad n \in \mathbb{N}$$

(2)*图像为上图所示*

Problem 2 (二次曲面综合性研究) *现已知以下二次曲面*

$$ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz = 1$$

为 $\mathbb{R}^3$ *中的椭球面。*

1. (10%) *基于条件极值的处理方法，证明此椭球面最大轴长* l *为如下关于其方程之最大实根（具体处理可考虑结合矩阵特征问题基本处理方法）*

$$\begin{vmatrix} a-\frac{1}{l^2} & d & e \\ d & b-\frac{1}{l^2} & f \\ e & f & c-\frac{1}{l^2} \end{vmatrix} = 0$$

2. (10%) *基于二次型有关理论，重新获得上述结论（主要基于对称阵的正交相似对角化）。*

3. (10%) *计算上述椭球面所围的体积（需要有一般体积分的计算过程）。*

本题为条件极值问题，按$Lagrange$乘子法处理

$(1)L(x,y,z,\lambda)\triangleq= x^2+y^2+z^2+\lambda(ax^2+by^2+cz^2+2dxy+2exz+2fyz-1)$

确定临界点的方程：

$$\begin{cases}\frac{\partial L}{\partial x}=2[x+\lambda(ax+by+cz)]=0---(1)\\ \frac{\partial L}{\partial y}=2[y+\lambda(by+dx+fz)]=0---(2)\\ \frac{\partial L}{\partial z}=2[z+\lambda(cz+ex+fy)]=0---(3)\\ \frac{\partial L}{\partial \lambda}=ax^2+by^2+cz^2+2dxy+2exz+2fyz-1=0---(4)\end{cases}$$

由(1)-(4)得：

$x^2+y^2+Z^2+\lambda[ax^2+by^2+cz^2+2dxy+2exz+2fyz]=0$

$x^2+y^2+z^2+\lambda=0$　即：$x^2+y^2+z^2=-\lambda$

由(1)(2)(3)

$$\begin{bmatrix}1+\lambda a & \lambda d & \lambda e\\ \lambda d & 1+\lambda b & \lambda f\\ \lambda e & \lambda f & 1+\lambda c\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=0\in\mathbb{R}^3$$

由于$[x,y,z]^T\neq 0\in\mathbb{R}^3$故$\det\begin{bmatrix}1+\lambda a & \lambda d & \lambda e\\ \lambda d & 1+\lambda b & \lambda f\\ \lambda e & \lambda f & 1+\lambda c\end{bmatrix}=0\in\mathbb{R}$

由于$\lambda\neq 0$　$\det\begin{bmatrix}\frac{1}{\lambda}+a & d & e\\ d & \frac{1}{\lambda}+b & f\\ e & f & \frac{1}{\lambda}+c\end{bmatrix}=0$

又由于$x^2+y^2+z^2=-\lambda=l^2$(此处l为最大轴长)

故$\det\begin{bmatrix}a-\frac{1}{l^2} & d & e\\ d & b-\frac{1}{l^2} & f\\ e & f & c-\frac{1}{l^2}\end{bmatrix}=0$

(2)对于所研究的二次曲面

$ax^2+by^2+cz^2+2dxy+2exz+2fyz=1$

即为：$[x,y,z]\begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}=1$

令$A=\begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}\in Sym\quad$ 故$\exists Q\in Orth,\ s.t.$

$Q^TAQ=\Lambda\quad$ 则 $\quad[x,y,z]Q\Lambda Q^T\begin{bmatrix} x \\ y \\ z \end{bmatrix}=1$

令$\begin{bmatrix} \zeta \\ \eta \\ \xi \end{bmatrix}=Q^T\begin{bmatrix} x \\ y \\ z \end{bmatrix}\quad$ 则有$[\zeta,\eta,\xi]\Lambda\begin{bmatrix} \zeta \\ \eta \\ \xi \end{bmatrix}=1$

设$\Lambda=\begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}\quad$ 此处$\lambda_1,\lambda_2,\lambda_3>0$

故$\lambda_1\zeta^2+\lambda_2\eta^2+\lambda_3\xi^2=1$

$\frac{\zeta^2}{(\frac{1}{\sqrt{\lambda_1}})^2}+\frac{\eta^2}{(\frac{1}{\sqrt{\lambda_2}})^2}+\frac{\xi^2}{(\frac{1}{\sqrt{\lambda_3}})^2}=1$

设$\frac{1}{\sqrt{\lambda_1}}=l$(最大轴长)

由$\lambda_1,\lambda_2,\lambda_3$的确定方程为$\det\begin{bmatrix} a-\lambda & d & e \\ d & b-\lambda & f \\ e & f & c-\lambda \end{bmatrix}=0$

故有$\det\begin{bmatrix} a-\frac{1}{l^2} & d & e \\ d & b-\frac{1}{l^2} & f \\ e & f & c-\frac{1}{l^2} \end{bmatrix}=0$

(3)按(2)中的分析，可做C^p微分同胚$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = Q \begin{bmatrix} \zeta \\ \eta \\ \xi \end{bmatrix}$

故$\displaystyle\int_{\gamma_{xyz}} d\tau = \int_{\gamma_{\zeta\eta\xi}} \left| \det \begin{bmatrix} x \\ y \\ z \end{bmatrix} (\zeta,\eta,\xi) \right| d\tau$

$\displaystyle= \int_{\gamma_{\zeta\eta\xi}} |\det Q|(\zeta,\eta,\xi) d\tau = \int_{\gamma_{\zeta\eta\xi}} d\tau$

再作椭球坐标系变换$\begin{cases} \zeta = \frac{1}{\sqrt{\lambda_1}} r \sin\theta\cos\phi \\ \eta = \frac{1}{\sqrt{\lambda_2}} r\sin\theta\sin\phi \\ \xi = \frac{1}{\sqrt{\lambda_3}} r\cos\theta \end{cases}$

故：$D \begin{bmatrix} \zeta \\ \eta \\ \xi \end{bmatrix} (r,\theta,\phi) = \dfrac{1}{\sqrt{\lambda_1\lambda_2\lambda_3}} r^2 \sin\theta$

故：$\displaystyle I = \int_{\gamma_{\zeta\eta\xi}} d\tau = \int_{\gamma_{r\theta\phi}} \frac{1}{\sqrt{\lambda_1\lambda_2\lambda_3}} r^2\sin\theta d\tau$

$\displaystyle= \frac{1}{\sqrt{\lambda_1\lambda_2\lambda_3}} \int_{\gamma_{r\theta\phi}} r^2 \sin\theta d\tau = \frac{1}{\sqrt{\lambda_1\lambda_2\lambda_3}} \cdot 2\pi \int_{\mathscr{D}_{r\theta}} r^2\sin\theta d\sigma$

$\displaystyle= \frac{2\pi}{\sqrt{\lambda_1\lambda_2\lambda_3}} \int_0^\pi \sin\theta d\theta \cdot \int_0^1 r^2 dr = \frac{2\pi}{\sqrt{\lambda_1\lambda_2\lambda_3}} \frac{1}{3} \cdot 2 = \frac{4\pi}{3\sqrt{\lambda_1\lambda_2\lambda_3}}$

即：$|\gamma_{xyz}| = \dfrac{4\pi}{3} \dfrac{1}{\sqrt{\lambda_1\lambda_2\lambda_3}} = \dfrac{4\pi}{3} \dfrac{1}{\sqrt{\begin{vmatrix} a & d & e \\ d & b & f \\ e & f & c \end{vmatrix}}}$

Problem 3 (隐映照定理有关应用) *设有*$\mathbb{R}^3$ *中集合*

$$\Sigma = \{[x,y,z]^T \mid f(x,y,z) = 0\}$$

如有$(x_0,y_0,z_0) \in \Sigma$，*且*$\frac{\partial f}{\partial z}(x_0,y_0,z_0) \neq 0$，*则按隐映照定理可局部定义曲面：*

$$\overset{\sigma}{\Sigma}(x,y) :\, B_\delta((x_0,y_0)) \ni \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \overset{\sigma}{\Sigma}(x,y) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} (x,y) \triangleq \begin{bmatrix} x \\ y \\ z(x,y) \end{bmatrix} (x,y) \in \mathbb{R}^3$$

1. (10%) *证明上述曲面面积可一般表示为*

$$|\overset{\sigma}{\Sigma}| = \int_{B_\delta((x_0,y_0))} \frac{|gradf|_{\mathbb{R}^3}}{|\frac{\partial f}{\partial z}|}(x,y)\, d\sigma$$

2. (10%) 设有

$$\Sigma = \{[x,y,z]^T \mid f(x,y,z) = e^{-xy} + 2z - e^z = 0\}$$

则请检验，在 $(0,0,0) \in \Sigma$ 符合隐映照定理有关条件，并写出按上述公式计算 $|\overset{\sigma}{\Sigma}|$ 的一种近似表示形式（无需进行完整计算）。

3. (10%) 另设有定义在整个 Σ 上的向量场

$$\mathbf{a}(x,y,z) = x\mathbf{i} - y\mathbf{j} + (x^2+y^2)\mathbf{k}$$

试完整计算通量形式的曲面积分

$$\int_{B_\delta((0,0))} \mathbf{a}(x,y,z) \cdot \mathbf{n}\, d\sigma$$

此处 $\mathbf{n}$ 指向同 $\mathbf{k}$ 成锐角。

(1)由按隐映照定理论述的局部意义下定义的曲面：

$$\sum^{\sigma}(x,y):B_0((x_0,y_0))\ni\begin{bmatrix}x\\y\end{bmatrix}\longmapsto\sum^{\sigma}(x,y)=\begin{bmatrix}x\\y\\z\end{bmatrix}(x,y)\triangleq\begin{bmatrix}x\\y\\z(x,y)\end{bmatrix}\in\mathbb{R}^3$$

由$|\sum^{\sigma}|=\int_{B_\delta(\begin{bmatrix}x_0\\y_0\end{bmatrix})}\left|\frac{\partial\sum^{\sigma}}{\partial x}\times\frac{\partial\sum^{\sigma}}{\partial y}\right|(x,y)d\sigma$

$$(\frac{\partial\sum^{\sigma}}{\partial x}\times\frac{\partial\sum^{\sigma}}{\partial y})(x,y)=\begin{vmatrix}i&j&k\\1&0&\frac{\partial z}{\partial x}(x,y)\\0&1&\frac{\partial z}{\partial y}(x,y)\end{vmatrix}=\begin{bmatrix}-\frac{\partial z}{\partial x}(x,y)\\-\frac{\partial z}{\partial y}(x,y)\\1\end{bmatrix}$$

再由隐映照定理

$f(x,y,z)=0$　即：$f(\begin{bmatrix}x\\y\end{bmatrix},z)=0\in\mathbb{R}$

$$D_{\begin{bmatrix}x\\y\end{bmatrix}}f(\begin{bmatrix}x\\y\end{bmatrix},z)+D_zf(\begin{bmatrix}x\\y\end{bmatrix},z)\cdot Dz(x,y)=0\in\mathbb{R}^{1\times 2}$$

即：$[\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}](x,y,z)+\frac{\partial f}{\partial z}(x,y,z)[\frac{\partial z}{\partial x},\frac{\partial z}{\partial y}](x,y)=0$

$$\begin{cases}\frac{\partial z}{\partial x}(x,y)=-\frac{\partial f/\partial x(x,y,z)}{\partial f/\partial z(x,y,z)}\\\frac{\partial z}{\partial y}(x,y)=-\frac{\partial f/\partial y(x,y,z)}{\partial f/\partial z(x,y,z)}\end{cases}\quad\forall(x,y)\in B_\delta(\begin{bmatrix}x_0\\y_0\end{bmatrix})\text{此处}z=z(x,y)$$

故$|\sum^{\sigma}|=\int_{B_\delta(\begin{bmatrix}x_0\\y_0\end{bmatrix})}\sqrt{1+(\frac{\partial z}{\partial x})^2+(\frac{\partial z}{\partial y})^2}(x,y)d\sigma$

$$=\int_{B_\delta(\begin{bmatrix}x_0\\y_0\end{bmatrix})}\frac{\sqrt{(\frac{\partial f^2}{\partial^2 z})(x,y,z(x,y))+(\frac{\partial f}{\partial x})^2(x,y,z(x,y))+(\frac{\partial f}{\partial y})^2(x,y,z(x,y))}}{|\frac{\partial f}{\partial z}|(x,y,z(x,y))}d\sigma$$

$$=\int_{B_\delta(\begin{bmatrix}x_0\\y_0\end{bmatrix})}\frac{|\nabla f|(x,y,z(x,y))}{|\frac{\partial f}{\partial z}|(x,y,z(x,y))}d\sigma$$

(2)对于$f(x,y,z)=e^{-xy}+2z-e^z=0$

$f(0,0,0)=1+0-1=0$

$\dfrac{\partial f}{\partial z}(0,0,0)=[2-e^z]_{z=0}=1\neq 0$ 故符合隐映照定理条件

由$\nabla f(x,y,z)=\begin{bmatrix}\frac{\partial f}{\partial x}\\ \frac{\partial f}{\partial y}\\ \frac{\partial f}{\partial z}\end{bmatrix}(x,y,z)=\begin{bmatrix}-ye^{-xy}\\ -xe^{-xy}\\ 2-e^z\end{bmatrix}(x,y,z)$

$$\frac{|\nabla f|(x,y,z(x,y))}{|\frac{\partial f}{\partial z}|(x,y,z(x,y))}=\frac{\sqrt{(x^2+y^2)e^{-2xy}+(2-e^z)^2}(x,y,z(x,y))}{|2-e^z|(x,y,z(x,y))}$$

$=\sqrt{1+\dfrac{(x^2+y^2)e^{-2xy}}{(2-e^z)^2}}$ 此处$z=z(x,y)$

由于$z_0=0$ 故$z(x_0,y_0)=z(0,0)=0$

故在$B_\delta(\begin{bmatrix}0\\0\end{bmatrix})$上$z(x,y)$近似于$0\in\mathbb{R}$,考虑到$z(x,y)$在$B_\delta(\begin{bmatrix}0\\0\end{bmatrix})$上的连续性

故$(2-e^z)^{-2}=[2-(1+z+o(z))]^{-2}=[1-z+o(z)]^{-2}$

$=1-2(-z+o(z))+o(-z+o(z))$

$=1+2z+o(z)\quad as\quad z\to 0$

故$(2-e^{z(x,y)})^{-2}\doteq 1+2z(x,y)\quad \forall (x,y)\in B_\delta(\begin{bmatrix}0\\0\end{bmatrix})\subset\mathbb{R}^2$

故$|\overset{\sigma}{\sum}|\doteq\displaystyle\int_{B_\delta(\begin{bmatrix}0\\0\end{bmatrix})}\sqrt{1+(x^2+y^2)e^{-2xy}\cdot(1+2z(x,y))}d\sigma$

又由$e^{-xy}+2z=e^z=1+z+o(z)$

$z_{(x,y)}\doteq 1-e^{-xy}$

故$1+(x^2+y^2)e^{-2xy}(1+2z(x,y))\doteq 1+(x^2+y^2)e^{-2xy}\cdot(3-2e^{-xy})$

$$(3)\int_{\sum^\sigma} a(x,y,z)\cdot n d\sigma = \int_{B_\delta(\begin{bmatrix}0\\0\end{bmatrix})} a(x,y,z(x,y))\cdot(\frac{\partial\sum^\sigma}{\partial x}\times\frac{\partial\sum^\sigma}{\partial y})(x,y)d\sigma$$

$$由 a(x,y,z(x,y))\cdot(\frac{\partial\sum^\sigma}{\partial x}\times\frac{\partial\sum^\sigma}{\partial y})(x,y)$$

$$= [x,-y,x^2+y^2]\cdot\begin{bmatrix}-\frac{\partial z}{\partial x}(x,y)\\-\frac{\partial z}{\partial y}(x,y)\\1\end{bmatrix}$$

$$= -x\frac{\partial z}{\partial x}(x,y)+y\frac{\partial z}{\partial y}(x,y)+x^2+y^2 = RHS$$

$$此处\begin{cases}\frac{\partial z}{\partial x}(x,y)=\frac{ye^{xy}}{2-e^z}\\\frac{\partial z}{\partial y}(x,y)=\frac{xe^{xy}}{2-e^z}\end{cases}$$

$$RHS = \frac{-xye^{-xy}}{2-e^z}+\frac{xye^{-xy}}{2-e^z}+x^2+y^2 = x^2+y^2$$

$$故\int_{\sum^\sigma} a(x,y,z)\cdot n d\sigma = \int_{B_\delta(\begin{bmatrix}0\\0\end{bmatrix})}(x^2+y^2)d\sigma$$

$$= \int_{D_{r\theta}} r^2\cdot r d\sigma = \int_0^\delta r^3 dr\cdot(2\pi) = 2\pi\frac{\delta^4}{4} = \frac{\pi\delta^4}{2}$$

Problem 4 (微分同胚应用－积分换元公式) *考虑体上积分*

$$\int_{\mathscr{D}_{xyz}} x^2y^2z\, d\tau$$

此处，$\mathscr{D}_{xyz}$*为由以下曲面族*

$$\begin{cases}z=\frac{x^2+y^2}{a}\\z=\frac{x^2+y^2}{b}\end{cases},\ 0<a<b;\quad\begin{cases}xy=c\\xy=d\end{cases},\ 0<c<d;\quad\begin{cases}y=\alpha x\\y=\beta x\end{cases},\ 0<\alpha<\beta$$

所围成的体积。

1. (10%) *考虑映照*

$$\Phi(x,y,z):\ \mathscr{D}_{xyz}\ni\begin{bmatrix}x\\y\\z\end{bmatrix}\mapsto\Phi(x,y,z)=\begin{bmatrix}\xi\\\eta\\\zeta\end{bmatrix}(x,y,z)\triangleq\begin{bmatrix}\frac{x^2+y^2}{z}\\xy\\\frac{y}{x}\end{bmatrix}$$

   以此建立微分同胚，需说明相应的开集以及确定为微分同胚的所有条件。

2. (10%) *完成上述积分计算。*

(1) 由映照 $\Phi(x,y,z):\mathscr{D}_{xyz}\ni\begin{bmatrix}x\\y\\z\end{bmatrix}\longmapsto\Phi(x,y,z)=\begin{bmatrix}\xi\\\eta\\\zeta\end{bmatrix}(x,y,z)\triangleq\begin{bmatrix}\frac{x^2+y^2}{z}\\xy\\\frac{y}{x}\end{bmatrix}$

$D\Phi(x,y,z)\triangleq\begin{bmatrix}\frac{\partial\xi}{\partial x}&\frac{\partial\xi}{\partial y}&\frac{\partial\xi}{\partial z}\\\frac{\partial\eta}{\partial x}&\frac{\partial\eta}{\partial y}&\frac{\partial\eta}{\partial z}\\\frac{\partial\zeta}{\partial x}&\frac{\partial\zeta}{\partial y}&\frac{\partial\zeta}{\partial z}\end{bmatrix}(x,y,z)$

$\begin{bmatrix}\frac{2x}{z}&\frac{2y}{z}&-\frac{x^2+y^2}{z^2}\\y&x&0\\-\frac{y}{x^2}&\frac{1}{x}&0\end{bmatrix}$

故 $\det D\Phi(x,y,z)=-\dfrac{x^2+y^2}{z^2}\cdot\dfrac{2y}{x}=-2\dfrac{y(x^2+y^2)}{xz^2}\neq 0$

易见不同 $(x_0,y_0)\in\mathscr{D}_{xy}$ 对应不同的 $\dfrac{y}{x}=\dfrac{y_0}{x_0},xy=x_0y_0$ 曲线交点故 $\Phi(x,y,z)$ 在包含 $\mathscr{D}_{xyz}$ 的大开集上为单射，

且 $\det D\Phi(x,y,z)\neq 0$ 故 $\Phi(x,y,z)$ 实现 $\mathscr{D}_{xyz}$ 与 $\mathscr{D}_{\xi\eta\zeta}$ 之间的微分同胚

$$(2)\int_{\mathscr{D}_{xyz}} d\tau=\int_{\mathscr{D}_{\xi\eta\zeta}}\left|\det\frac{D(x,y,z)}{D(\xi\eta\zeta)}\right|(\xi,\eta,\zeta)d\tau$$

$$由于\det\frac{D(x,y,z)}{D(\xi,\eta,\zeta)}(\xi,\eta,\zeta)=\frac{1}{\det\frac{D(\xi,\eta,\zeta)}{D(x,y,z)}(x,y,z)}$$

$$=\frac{1}{-\frac{2\cdot y(x^2+y^2)}{xz^2}}=-\frac{xz^2}{2y(x^2+y^2)}$$

$$由于\begin{cases}\xi=\frac{x^2+y^2}{z}\\ \eta=xy\\ \zeta=\frac{y}{x}\end{cases}$$

$$\det\frac{D(x,y,z)}{D(\xi,\eta,\zeta)}(\xi,\eta,\zeta)$$

$$=-\frac{1}{2}\cdot\frac{1}{\zeta}\frac{1}{\xi}z$$

$$=-\frac{1}{2\xi\zeta}\cdot\frac{x^2+y^2}{\xi}=-\frac{1}{2\xi\zeta}\frac{\eta/\zeta+\eta\zeta}{\xi}$$

$$=-\frac{1}{2}\frac{1}{\xi^2}\eta(\frac{\frac{1}{\zeta}+\zeta}{\zeta})$$

$$=-\frac{1}{2}\frac{1}{\xi^2}\eta(1+\frac{1}{\zeta^2})$$

$$故\int_{\mathscr{D}_{xyz}} d\tau=\int_{\mathscr{D}_{\xi\eta\zeta}}\frac{1}{2}\frac{1}{\xi^2}\eta(1+\frac{1}{\zeta^2})d\tau$$

$$=\frac{1}{2}\int_a^b\xi^{-2}d\xi\cdot\int_c^d\eta d\eta\cdot\int_\alpha^\beta(1+\zeta^{-2})d\zeta$$

$$=\frac{1}{2}\left[\frac{\frac{1}{b}-\frac{1}{a}}{-1}\right]\frac{d^2-c^2}{2}\left[\beta-\alpha+\frac{\frac{1}{\beta}-\frac{1}{\alpha}}{-1}\right]$$

$$=\frac{1}{4}\left(\frac{1}{a}-\frac{1}{b}\right)(d^2-c^2)\left(\beta-\alpha+\frac{1}{\alpha}-\frac{1}{\beta}\right)$$

Problem 5 (正项级数判别法的核心思想－“比较观点”) *此处所言比较观点包含二个要素，参照级数及比较方式。现设参照级数为$\sum b_n=\sum\frac{1}{n(\log n)^p}$，比较方式为*

1. *如有 $\frac{a_n}{a_{n+1}}\geq\frac{b_n}{b_{n+1}}\, as\, n\geq N_\epsilon$，则当$\sum b_n$收敛，有$\sum a_n$收敛；*
2. *如有 $\frac{a_n}{a_{n+1}}\leq\frac{b_n}{b_{n+1}}\, as\, n\geq N_\epsilon$，则当$\sum b_n$发散，有$\sum a_n$发散。*

则有判别结论，如有：

$$\frac{a_n}{a_{n+1}}=1+\frac{1}{n}+\frac{p}{n\log n}+o(\frac{1}{n\log n})$$

1. *当$p>1$，则$\sum a_n$收敛，*
2. *当$p<1$，则$\sum a_n$发散，*

1. *(10%) 证明上述有关比较方式的第2条*

2. (10%) 证明上述有关判别结论的第1条

(1)*需证明对于正项级数，有结论*

如果有$\dfrac{a_n}{a_{n+1}} \leqslant \dfrac{b_n}{b_{n+1}} \quad as \quad n \geqslant N_\epsilon$,则当$\sum b_n$发散，有$\sum a_n$发散。

由$\dfrac{a_n}{a_{n+1}} \leqslant \dfrac{b_n}{b_{n+1}}, \forall n \geqslant N_\epsilon$　即：$\dfrac{a_{n+1}}{a_n} \geqslant \dfrac{b_{n+1}}{b_n}, \forall n \geqslant N_\epsilon$

$\dfrac{a_n}{a_{n-1}} \cdots \dfrac{a_{N_\epsilon+1}}{a_{N_\epsilon}} \geqslant \dfrac{b_n}{b_{n-1}} \cdots \dfrac{b_{N_\epsilon+1}}{b_{N_\epsilon}}, \forall n \geqslant N_\epsilon$

即：$\dfrac{a_n}{a_{N_\epsilon}} \geqslant \dfrac{b_n}{b_{N_\epsilon}}, \forall n \geqslant N_\epsilon$　即：$a_n \geqslant \dfrac{a_{N_\epsilon}}{b_{N_\epsilon}} b_n, \forall n \geqslant N_\epsilon$

故$\sum b_n$发散，则有$\sum a_n$发散(均趋于正无穷)

(2)需证明结论:如有

$$\frac{a_n}{a_{n+1}}=1+\frac{1}{n}+\frac{p}{n\log n}+o(\frac{1}{n\log n})$$

当$p>1$,则$\sum a_n$发散

取$b_n=\dfrac{1}{n(\log n)^p}$

考虑:$\dfrac{b_n}{b_{n+1}}=\dfrac{\frac{1}{n(\log n)^p}}{\frac{1}{(n+1)(\log n+1)^p}}=\dfrac{n+1}{n}(\dfrac{\log n+1}{\log n})^p=(1+\dfrac{1}{n})(\dfrac{\log\frac{n+1}{n}+\log n}{\log n})^p$

$$=(1+\frac{1}{n})(1+\frac{\log\frac{n+1}{n}}{\log n})^p=(1+\frac{1}{n})(1+\frac{\log(1+\frac{1}{n})}{\log n})^p$$

考虑到$\log(1+x)=x+o(x)\quad as\,x\to 0$

故$\log(1+\dfrac{1}{n})=\dfrac{1}{n}+o(\dfrac{1}{n})$

则$RHS=(1+\dfrac{1}{n})(1+\dfrac{\frac{1}{n}+o(\frac{1}{n})}{\log n})^p$

$$=(1+\frac{1}{n})[1+p(\frac{1}{n\log n}+o(\frac{1}{n\log n}))+o(\frac{1}{n\log n}+o(\frac{1}{n\log n}))]$$

$$=(1+\frac{1}{n})[1+\frac{p}{n\log n}+o(\frac{1}{n\log n})]=1+\frac{1}{n}+\frac{p}{n\log n}+\frac{p}{n\cdot n\log n}+o(\frac{1}{n\log n})$$

$$=1+\frac{1}{n}+\frac{p}{n\log n}+o(\frac{1}{n\log n})$$

即:$\dfrac{b_n}{b_{n+1}}=1+\dfrac{1}{n}+\dfrac{p}{n\log n}+o(\dfrac{1}{n\log n})$

考虑到$o(\dfrac{1}{n\log n})$,则有$|o(\dfrac{1}{n\log n})|<\epsilon\dfrac{1}{n\log n}\,as\,n\geqslant N_\epsilon$

故有$\dfrac{b_n}{b_{n+1}}\leqslant 1+\dfrac{1}{n}+\dfrac{p}{n\log n}+\dfrac{\epsilon}{n\log n}\quad\forall n\geqslant N_\epsilon$

如有:$\dfrac{a_n}{a_{n+1}}=1+\dfrac{1}{n}+\dfrac{q}{n\log n}+o(\dfrac{1}{n\log n})$

则有:$\dfrac{a_n}{a_{n+1}}\geqslant 1+\dfrac{1}{n}+\dfrac{q}{n\log n}-\dfrac{\epsilon}{n\log n}\,\forall n\geqslant N_\epsilon$

$\Longrightarrow\dfrac{a_n}{a_{n+1}}\geqslant\dfrac{b_n}{b_{n+1}},\forall n\geqslant N_\epsilon$

如果:$q-\epsilon\geqslant p+\epsilon$

当$q>1$自然可选取$p>1$满足条件$q-\epsilon\geqslant p+\epsilon$

Problem 6 (幂级数基本应用) (10%) 为计算和式$\sum_{n=1}^{+\infty}\frac{(-1)^n}{n}$,可考虑幂级数$\sum_{n=1}^{+\infty}\frac{x^n}{n}$ 试获

得上述幂级数和函数的表示式。

$$\text{解：基于}\sum_{n=1}^{+\infty}\frac{x^n}{n}\text{计算}\sum_{n=1}^{+\infty}\frac{(-1)^n}{n}$$

$$\text{令}S(n)=\sum_{n=1}^{+\infty}\frac{x^n}{n}\,,\,x\in[-1,1)$$

$$\text{由}S(n)=x+\sum_{n=2}^{+\infty}\frac{x^n}{n}$$

$$\text{考虑}\sum_{n=2}^{+\infty}(\frac{x^n}{n})'=\sum_{n=2}^{+\infty}x^{n-1}\,,\,x\in(-1,1)$$

$$=\sum_{n=1}^{+\infty}x^n=1+\sum_{n=1}^{+\infty}x^n-1=\frac{1}{1-x}-1$$

$$\text{故有}\sum_{n=2}^{+\infty}\frac{x^n}{n}=\int_0^x(\frac{1}{1-\xi}-1)d\xi\,,\,x\in(-1,1)$$

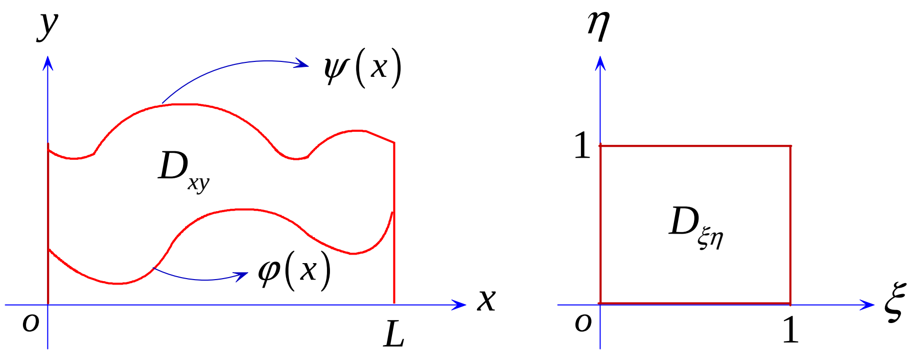
$$=-\ln(1-x)-x$$

$$\text{即得：}\sum_{n=2}^{+\infty}\frac{x^n}{n}+x=-\ln(1-x)=\sum_{n=1}^{+\infty}\frac{x^n}{n}\,,\,x\in(-1,1)$$

$$\text{而}\sum_{n=1}^{+\infty}\frac{x^n}{n}\text{收敛域为}[-1,1),\lim_{x\to-1+0}-\ln(1+x)=-\ln 2$$

$$\text{故有：}\sum_{n=1}^{+\infty}\frac{x^n}{n}=-\ln(1-x)\,,\,\forall x\in[-1,1)$$

$$\text{有}S(n)=-\ln(1-x)\,,\,\forall x\in[-1,1)\Rightarrow\sum_{n=1}^{+\infty}\frac{(-1)^n}{n}=S(-1)=-\ln 2$$



Problem 7 (微分同胚应用—变换“不规则区域”至“规则区域”) *如上图所示，对于平面上不规则区域$\mathscr{D}_{xy}$，可考虑如下映照*

$$\Phi(\xi,\eta):\mathscr{D}_{\xi\eta}\ni\begin{bmatrix}\xi\\ \eta\end{bmatrix}\mapsto\Phi(\xi,\eta)=\begin{bmatrix}x\\ y\end{bmatrix}(\xi,\eta)\triangleq\begin{bmatrix}L\xi\\ \phi(L\xi)+\eta[\psi(L\xi)-\phi(L\xi)]\end{bmatrix}$$

此处

$$\mathscr{D}_{\xi\eta}\triangleq\{\xi\in(0,1),\,\eta\in(0,1)\}$$

1. (10%) 利用微分同胚的充分性定理，证明上述 $\Phi(\xi,\eta)$ 实现 $\mathscr{D}_{\xi\eta}$ 同 $\mathscr{D}_{xy}=\Phi(\mathscr{D}_{\xi\eta})$ 间的微分同胚，需明确获得 $D\Phi(\xi,\eta)$ 。

2. (10%) 基于上述微分同胚，可得定义在 $\mathscr{D}_{xy}$ 上的局部基 $\{\mathbf{g}_\xi,\mathbf{g}_\eta\}(\xi,\eta)$

$$D\Phi(\xi,\eta)=:[\mathbf{g}_\xi,\mathbf{g}_\eta](\xi,\eta)$$

试导出 $\partial\mathbf{g}_\xi/\partial\eta(\xi,\eta)$ 在典则基 $\{\mathbf{i},\mathbf{j}\}$ 及局部基 $\{\mathbf{g}_\xi,\mathbf{g}_\eta\}(\xi,\eta)$ 下的表达式。

3. (10%) 如在 $\mathscr{D}_{\xi\eta}$ 上定义曲线

$$\mathbf{C}_{\xi\eta}(\lambda):\,[\alpha,\beta]\ni\lambda\mapsto\mathbf{C}_{\xi\eta}(\lambda)=\begin{bmatrix}\xi\\ \eta\end{bmatrix}(\lambda)$$

则其对应于 $\mathscr{D}_{xy}$ 的曲线为

$$\mathbf{C}_{xy}(\lambda)\triangleq\Phi\circ\mathbf{C}_{\xi\eta}(\lambda)$$

试推导 $\mathbf{C}_{xy}(\lambda)$ 弧长的积分表达式，需给出以参数 λ 为积分变量的被积函数的计算式。

4. (10%) 如有感兴趣的函数 $f(x,y)$ 建立在 $\mathscr{D}_{xy}$ 上，且满足偏微分方程：

$$\frac{\partial^2 f}{\partial x^2}(x,y)+\frac{\partial^2 f}{\partial y\partial x}(x,y)+\cdots=0$$

则基于微分同胚，可将定义在 $\mathscr{D}_{xy}$ 上的 $f(x,y)$ 转化至定义在 $\mathscr{D}_{\xi\eta}$ 上的

$$\hat{f}(\xi,\eta)\triangleq f(x(\xi,\eta),y(\xi,\eta))$$

具体说明如何获得 $\frac{\partial^2 f}{\partial y\partial x}(x,y)$ 经由 $\hat{f}(\xi,\eta)$ 及其偏导数的表达式。

$$(1)\Phi(\xi,\eta)=\begin{bmatrix}x\\ y\end{bmatrix}(\xi,\eta)\triangleq\begin{bmatrix}L\xi\\ \phi(L\xi)+\eta[\psi(L\xi)-\phi(L\xi)]\end{bmatrix}$$

显然$\Phi(\xi,\eta)$为$\mathscr{D}_{\xi\eta}$上的单射

$$\text{而}D\Phi(\xi,\eta)=\begin{bmatrix}\frac{\partial x}{\partial\xi} & \frac{\partial x}{\partial\eta}\\ \frac{\partial y}{\partial\xi} & \frac{\partial y}{\partial\eta}\end{bmatrix}(\xi,\eta)$$

$$=\begin{bmatrix}L & 0\\ \dot{\phi}L+\eta[\dot{\psi}-\dot{\phi}]L & \psi-\phi\end{bmatrix}$$

$$\Longrightarrow\det D\Phi(\xi,\eta)=L[\psi(L\xi)-\phi(L\xi)]\neq 0\,,\forall\xi\in\mathscr{D}_{\xi\eta}$$

故$\Phi(\xi,\eta)$实现$\mathscr{D}_{\xi,\eta}$同$\Phi\mathscr{D}_{\xi,\eta}$间的微分同胚。

$(2) D\Phi(\xi,\eta) =: [g_\xi, g_\eta](\xi,\eta)$

即：$g_\xi(\xi,\eta) = L\begin{bmatrix} 1 \\ \dot{\phi}+\eta(\dot{\psi}-\dot{\phi}) \end{bmatrix} = L\begin{bmatrix} 1 \\ \dot{\phi}(L\xi)+\eta[\dot{\psi}(L\xi)-\dot{\phi}(L\xi)] \end{bmatrix}$

$\Longrightarrow \dfrac{\partial g_\xi}{\partial \eta}(\xi,\eta) = L\begin{bmatrix} 0 \\ \dot{\psi}(L\xi)-\dot{\phi}(L\xi) \end{bmatrix} = [i_1, i_2]\begin{bmatrix} 0 \\ L[\dot{\psi}(L\xi)-\dot{\phi}(L\xi)] \end{bmatrix}$

又由$[g_\xi, g_\eta](\xi,\eta) = D\Phi(\xi,\eta) = [i_1,i_2]D\Phi(\xi,\eta)$

$\Longrightarrow \dfrac{\partial g_\xi}{\partial \eta}(\xi,\eta) = [g_\xi, g_\eta](\xi,\eta)(D\Phi)^{-1}(\xi,\eta)\begin{bmatrix} 0 \\ L[\dot{\psi}(L\xi)-\dot{\phi}(L\xi)] \end{bmatrix}$

(3) 由$C_{xy}(\lambda) \triangleq \Phi \cdot C_{\xi\eta}(\lambda)$

故$I = \displaystyle\int_{C_{xy}(\lambda)} dl = \int_\alpha^\beta \left|\frac{dC_{xy}}{d\lambda}(\lambda)\right| d\lambda$

由于$\dfrac{dC_{xy}}{d\lambda}(\lambda) = DC_{xy}(\lambda) = D\Phi(C_{\xi\eta}(\lambda))DC_{\xi\eta}(\lambda)$

$= D\Phi(C_{\xi\eta}(\lambda))\dfrac{dC_{\xi\eta}}{d\lambda}(\lambda)$

$\Longrightarrow \left|\dfrac{dC_{xy}}{d\lambda}(\lambda)\right|^2 = \left(\dfrac{dC_{\xi\eta}}{d\lambda}\right)^T(\lambda)(D\Phi)^T(C_{\xi\eta}(\lambda))(D\Phi)(C_{\xi\eta}(\lambda))\dfrac{dC_{\xi\eta}}{d\lambda}(\lambda)$

$\Longrightarrow I = \displaystyle\int_\alpha^\beta \left\{\left(\frac{dC_{\xi\eta}}{d\lambda}\right)^T(\lambda)(D\Phi)^T(D\Phi)(C_{\xi\eta}(\lambda))\frac{dC_{\xi\eta}}{d\lambda}(\lambda)\right\}^{\frac{1}{2}} d\lambda$

此处$\dfrac{dC_{\xi\eta}}{d\lambda}(\lambda) = \begin{bmatrix} \dot{\xi}(\lambda) \\ \dot{\eta}(\lambda) \end{bmatrix}$

$(D\Phi)^T D\Phi(\xi,\eta) = \begin{bmatrix} L & \dot{\phi}L+\eta[\dot{\psi}-\dot{\phi}]L \\ 0 & \psi-\phi \end{bmatrix} \cdot \begin{bmatrix} L & 0 \\ \dot{\phi}L+\eta[\dot{\psi}-\dot{\phi}]L & \psi-\phi \end{bmatrix}$

$= \begin{bmatrix} L^2+L^2(\dot{\phi}+\eta[\dot{\psi}-\dot{\phi}])^2 & (\dot{\phi}L+\eta[\dot{\psi}-\dot{\phi}]L)(\psi-\phi) \\ (\dot{\phi}L+\eta[\dot{\psi}-\dot{\phi}]L)(\psi-\phi) & (\psi-\phi)^2 \end{bmatrix}(\xi,\eta)$

(4) 由$\hat{f}(\xi,\eta) \triangleq f(x(\xi,\eta),y(\xi,\eta))$

$$f(x,y) = \hat{f}(\xi(x,y),\eta(x,y))$$

$$\Longrightarrow \frac{\partial f}{\partial x}(x,y) = \frac{\partial \hat{f}}{\partial \xi}(\xi,\eta)\frac{\partial \xi}{\partial x}(x,y) + \frac{\partial \hat{f}}{\partial \eta}\frac{\partial \eta}{\partial x}(x,y)$$

由$\begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{bmatrix}(x,y) = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}^{-1}(\xi,\eta) = (D\Phi)^{-1}(\xi,\eta)$

由$D\Phi(\xi,\eta) = \begin{bmatrix} L & 0 \\ \dot{\phi}L + \eta[\dot{\psi} - \dot{\phi}]L & \psi - \phi \end{bmatrix}$

$$\Longrightarrow (D\Phi)^{-1}(\xi,\eta) = \frac{\begin{bmatrix} \psi - \phi & -\dot{\phi}L - \eta(\dot{\psi} - \dot{\phi})L \\ 0 & L \end{bmatrix}^T}{L(\psi - \phi)}$$

$$= \frac{1}{L(\psi-\phi)}\begin{bmatrix} \psi - \phi & 0 \\ -\dot{\phi}L - \eta(\dot{\psi} - \dot{\phi})L & L \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{L} & 0 \\ \frac{\dot{\phi}+\eta(\dot{\psi}-\dot{\phi})}{\phi-\psi} & \frac{1}{\psi-\phi} \end{bmatrix}(\xi,\eta)$$

可得$\begin{cases} \frac{\partial \xi}{\partial x}(x,y) = \frac{1}{L} \\ \frac{\partial \xi}{\partial y}(x,y) = 0 \end{cases}$ $\begin{cases} \frac{\partial \eta}{\partial x}(x,y) = \frac{\dot{\phi}+\eta(\dot{\psi}-\dot{\phi})}{\phi-\psi}(\xi,\eta) \\ \frac{\partial \eta}{\partial y}(x,y) = \frac{1}{\psi-\phi}(\xi,\eta) \end{cases}$

进一步

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = [\frac{\partial^2 \hat{f}}{\partial \xi^2}(\xi,\eta)\frac{\partial \xi}{\partial y}(x,y) + \frac{\partial^2 \hat{f}}{\partial \eta \partial \xi}(\xi,\eta)\frac{\partial \eta}{\partial y}(x,y)]\frac{\partial \xi}{\partial x}(x,y)$$

$$+ \frac{\partial \hat{f}}{\partial \xi}(x,y)[\frac{\partial}{\partial \xi}(\frac{\partial \xi}{\partial x})(\xi,\eta)\frac{\partial \xi}{\partial y}(x,y) + \frac{\partial}{\partial \eta}(\frac{\partial \xi}{\partial x})(\xi,\eta)\frac{\partial \eta}{\partial y}(x,y)]$$

$$+ [\frac{\partial^2 \hat{f}}{\partial \eta^2}(\xi,\eta)\frac{\partial \eta}{\partial y}(x,y) + \frac{\partial^2 \hat{f}}{\partial \eta \partial \xi}(\xi,\eta)\frac{\partial \xi}{\partial y}(x,y)]\frac{\partial \eta}{\partial x}(x,y)$$

$$+ \frac{\partial \hat{f}}{\partial \eta}(x,y)[\frac{\partial}{\partial \xi}(\frac{\partial \eta}{\partial x})(\xi,\eta)\frac{\partial \xi}{\partial y}(x,y) + \frac{\partial}{\partial \eta}(\frac{\partial \eta}{\partial x})(\xi,\eta)\frac{\partial \eta}{\partial y}(x,y)]$$

Problem 8 (Gauss-Ostrigradskii公式应用—浮力定理) (10%)推导被二种不同流体浸没的

物体所受的浮力，设上层和下层流体的密度分别为 ρ_1 *和* ρ_2 。

$$
\begin{aligned}
&由Gauss公式\\
&\int_{\partial\gamma}\phi(x,y,z)nd\sigma=\int_{\gamma}grad\phi(x,y,z)d\tau\\
&由F=\int_{\Sigma_1}(p_a+\rho_1gz)nd\sigma+\int_{\Sigma_2}(p_a+\rho_1gH+\rho_2g(z-H))nd\sigma\\
&=\int_{\Sigma_1+\Sigma_c}(p_a+\rho_1gz)nd\sigma+\int_{\Sigma_2+\Sigma_c}[p_a+\rho_1gH+\rho_2g(z-H)]nd\sigma=RHS\\
&此处\int_{\Sigma_c}(p_a+\rho_1gz)nd\sigma=\int_{\Sigma_c}(p_a+\rho_1gH)\vec{k}d\sigma\\
&\int_{\Sigma_c}[p_a+\rho_1gH+\rho_2g(z-H)]nd\sigma=\int_{\Sigma_c}(p_a+\rho_1gH)(-\vec{k})d\sigma\\
&RHS=\int_{\gamma_1}\rho_1g\vec{k}d\tau+\int_{\gamma_2}\rho_2g\vec{k}d\tau\\
&=[\rho_1g|\gamma_1|+\rho_2g|\gamma_2|]\vec{k}
\end{aligned}
$$

注：尽量详细地给出推理和运算步骤，给分上侧重正确的思想和方法

注：本卷各题计分主要考虑为批阅方便，试题设计力求体现对微积分基本思想及方法的理解与应用