

复旦大学力学与工程科学系

2010 ~ 2011 学年第二学期期末考试试卷——参考答案

□A 卷

□B 卷

课程名称：数学分析（II）

课程代码：MATH120009.09

开课院系：力学与工程科学系

考试形式：开卷/闭卷/课程论文

姓名：_____ 学号：_____ 专业：_____

题号	1/(1)	1/(2)	2/(1)	2/(2)	3/(1)	3/(2)	4/(1)	4/(2)	4/(3)	5/(1)
得分										
题号	5/(2)	5/(3)	5/(4)	6/(1)	6/(2)	6/(3)	7/(1)	7/(2)	8/(1)	8/(2)
得分										
题号	8/(3)	8/(4)	8/(5)							总分
得分										

Problem 1 (多维函数极限的基本概念) 有二维函数：

$$f(x, y) \triangleq \begin{cases} xy \cdot \frac{x^2 - y^2}{x^2 + y^2} & \text{as } (x, y) \neq (0, 0) \\ 0 & \text{as } (x, y) = (0, 0) \end{cases}$$

- (10%) 研究 $f(x, y)$ 在 $(0, 0)$ 点的可微性。
- (10%) 计算 $\frac{\partial^2 f}{\partial x \partial y}(0, 0)$ 以及 $\frac{\partial^2 f}{\partial y \partial x}(0, 0)$ 。如不存在，则说明理由。

解答：

- (1/1)

$$f(x, y) = \begin{cases} xy \frac{x^2 - y^2}{x^2 + y^2} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

对于可微性

$$\begin{cases} \frac{\partial f}{\partial x}(0, 0) = \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x - 0} = \lim_{x \rightarrow 0} \frac{0 - 0}{x} = 0 \\ \frac{\partial f}{\partial y}(0, 0) = \lim_{y \rightarrow 0} \frac{f(0, y) - f(0, 0)}{y - 0} = \lim_{y \rightarrow 0} \frac{0 - 0}{y} = 0 \end{cases}$$

估计

$$\left| \frac{f(x, y) - f(0, 0) - \left[\frac{\partial f}{\partial x}(0, 0)x + \frac{\partial f}{\partial y}(0, 0)y \right]}{\sqrt{x^2 + y^2}} \right| = \frac{|f(x, y)|}{\sqrt{x^2 + y^2}} = \left| \frac{x^2 - y^2}{x^2 + y^2} \frac{xy}{\sqrt{x^2 + y^2}} \right|$$

$$= \left| \frac{x^2 - y^2}{x^2 + y^2} \right| \frac{|x|}{\sqrt{x^2 + y^2}} |y| \leq |y| \leq \sqrt{x^2 + y^2}$$

故有：

$$f(x, y) = f(0, 0) + \left[\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right](0, 0) \begin{bmatrix} x \\ y \end{bmatrix} + o(\sqrt{x^2 + y^2})$$

亦即 $f(x, y)$ 在 $(0, 0)$ 点可微。

2. (1/2)

$$\frac{\partial f}{\partial x}(x, y) = y \frac{x^2 - y^2}{x^2 + y^2} + xy \frac{2x(x^2 + y^2) - (x^2 - y^2)2x}{(x^2 + y^2)^2} = y \frac{x^2 - y^2}{x^2 + y^2} + xy \frac{4xy^2}{(x^2 + y^2)^2} \quad (x, y) \neq (0, 0)$$

$$\frac{\partial f}{\partial y}(x, y) = x \frac{x^2 - y^2}{x^2 + y^2} + xy \frac{-2y(x^2 + y^2) - (x^2 - y^2)2y}{(x^2 + y^2)^2} = x \frac{x^2 - y^2}{x^2 + y^2} - xy \frac{4yx^2}{(x^2 + y^2)^2} \quad (x, y) \neq (0, 0)$$

$$\frac{\partial^2 f}{\partial x \partial y}(0, 0) \triangleq \lim_{x \rightarrow 0} \frac{\frac{\partial f}{\partial y}(x, 0) - \frac{\partial f}{\partial y}(0, 0)}{x - 0} = \lim_{x \rightarrow 0} \frac{x - 0}{x} = 1$$

$$\frac{\partial^2 f}{\partial y \partial x}(0, 0) \triangleq \lim_{y \rightarrow 0} \frac{\frac{\partial f}{\partial x}(0, y) - \frac{\partial f}{\partial x}(0, 0)}{y - 0} = \lim_{y \rightarrow 0} \frac{-y}{y} = -1$$

Problem 2 (*Fourier* 级数基本概念) 现有区间 $[0, 3]$ 上的函数：

$$f(x) \triangleq \begin{cases} 0 & x \in [0, 1) \\ x^2 & x \in [1, 2] \\ 0 & x \in (2, 3] \end{cases}$$

1. (10%) 按点收敛的意义，将 $f(x), x \in [0, 3]$ 表示成 *Fourier* 正弦级数的形式。要求写出具体的表达形式，其中相关系数仅需给出具体计算式。

2. (10%) 示意性绘出 *Fourier* 级数对应的和函数（极限函数）的图像。

解答：

1. (2/1)

$$f(x) = \begin{cases} 0 & x \in [0, 1) \\ x^2 & x \in [1, 2] \\ 0 & x \in (2, 3] \end{cases}$$

首先进行奇延拓，即

$$\hat{f}(x) = \begin{cases} f(x) & x \in [0, 3] \\ -f(x) & x \in [-3, 0] \end{cases}$$

对奇函数 $\hat{f}(x)$ 作以 6 为周期的延拓, 再做其 Fourier 级数展开。

$[-3, 3]$ 上有:

$$\frac{\hat{f}(x+0) + \hat{f}(x-0)}{2} = \frac{a_0}{2} + \sum_{n=1}^{+\infty} (a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l})$$

$$\begin{cases} a_n = \frac{1}{l} \int_{-l}^l \hat{f}(x) \cos \frac{n\pi x}{l} dx = \frac{1}{3} \int_{-3}^3 \hat{f}(x) \cos \frac{n\pi x}{3} dx & n \in \mathbb{N} \cup \{0\} \\ b_n = \frac{1}{l} \int_{-l}^l \hat{f}(x) \sin \frac{n\pi x}{l} dx = \frac{2}{3} \int_0^3 \hat{f}(x) \sin \frac{n\pi x}{3} dx = \frac{2}{3} \int_1^2 x^2 \sin \frac{n\pi x}{3} dx \end{cases}$$

2. (2/2) 略

Problem 3 (基于一般原理构造数项级数收敛性的判别法) 一般数项级数的收敛性判别法, 主要基于级数的 *Cauchy* 收敛原理并结合 *Abel* 估计。

1. (10%) 证明 *Leibnize* 判别法: 对于交叉级数

$$\sum (-1)^n a_n \quad (a_n > 0)$$

当 $\{a_n\}$ 单调趋于零, 则此级数收敛。

2. (10%) 基于 *Leibnize* 判别法, 证明: 对正项级数 $\sum a_n \quad (a_n > 0)$, 如有:

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \frac{\mu}{n \ln n} + o\left(\frac{1}{n \ln n}\right), \quad \text{as } n \rightarrow +\infty$$

则有: 当 $\mu > 0$ 时, 交叉级数 $\sum (-1)^n a_n$ 收敛。

解答:

1. (3/1) 考虑 *Abel* 估计

$$\left| \sum_{k=n+1}^{n+p} a_k b_k \right| \leq \max_{n+1 \leq j \leq n+p} \left| \sum_{k=n+1}^j b_k \right| (|a_{n+1}| + 2|a_{n+p}|)$$

此处要求 $\{a_k\}_{k=n+1}^{n+p}$ 单调

对于交叉级数 $\sum (-1)^n a_n \quad (a_n > 0)$, 可取 $b_n = (-1)^n$, 故 $|\sum_{k=1}^n b_k| = |\sum_{k=1}^n (-1)^k| \leq 1$ 而 $\{a_n\}$ 单调趋于零。故 $\sum (-1)^n a_n$ 收敛。

注:

$$\left| \sum_{k=n+1}^{n+p} (-1)^k a_k \right| \leq |a_{n+1}| + 2|a_{n+p}| < \epsilon \quad \text{as } n \geq N_\epsilon$$

2. (3/2) 基于 Leibniz 判别法, 需证 $\{a_n\}$ 单调趋于零。

由

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \frac{\mu}{n \ln n} + o\left(\frac{1}{n \ln n}\right) \quad \text{as } n \rightarrow +\infty$$

由

$$\frac{|o(\frac{1}{n \ln n})|}{\frac{1}{n \ln n}} \rightarrow 0$$

故有

$$\left| o\left(\frac{1}{n \ln n}\right) \right| < \epsilon \frac{1}{n \ln n} \quad n > \hat{N}_\epsilon$$

故有:

$$\frac{a_n}{a_{n+1}} = 1 + \frac{1}{n} + \frac{\mu}{n \ln n} + o\left(\frac{1}{n \ln n}\right) > 1 + \frac{1}{n} + \frac{\mu - \epsilon}{n \ln n} > 1 + \frac{1}{n} > 1 \quad n \geq \hat{N}_\epsilon$$

此处需 $0 < \epsilon < \mu$

故有: $a_n > a_{n+1} \quad n \geq N_\epsilon$

即有: $\{a_n\}_{n \geq N_\epsilon}$ 单调下降。

另考虑到, 取 $b_n = \frac{1}{n(\ln n)^p} \quad (p > 0)$ 有

$$\frac{b_n}{b_{n+1}} = 1 + \frac{1}{n} + \frac{p}{n \ln n} + o\left(\frac{1}{n \ln n}\right) < 1 + \frac{1}{n} + \frac{p + \epsilon}{n \ln n} \quad n \geq \hat{N}_\epsilon$$

注:

$$\begin{aligned} \frac{b_n}{b_{n+1}} &= \frac{\frac{1}{n(\ln n)^p}}{\frac{1}{(n+1)(\ln(n+1))^p}} = \frac{n+1}{n} \left(\frac{\ln(n+1)}{\ln n} \right)^p = \left(1 + \frac{1}{n}\right) \left[\frac{\ln n + \ln(1 + \frac{1}{n})}{\ln n} \right]^p \\ &= \left(1 + \frac{1}{n}\right) \left[1 + \frac{1}{n \ln n} + o\left(\frac{1}{n \ln n}\right) \right]^p = \left(1 + \frac{1}{n}\right) \left[1 + \frac{p}{n \ln n} + o\left(\frac{1}{n \ln n}\right) \right] = 1 + \frac{1}{n} + \frac{p}{n \ln n} + o\left(\frac{1}{n \ln n}\right) \end{aligned}$$

且

$$\frac{a_n}{a_{n+1}} > 1 + \frac{1}{n} + \frac{\mu - \epsilon}{n \ln n} \quad n \geq \hat{N}_\epsilon$$

取 $\epsilon > 0$ 满足 $\mu - \epsilon \geq p + \epsilon$, 且 $N_\epsilon = \max\{\tilde{N}, \hat{N}\}$ 有: $\frac{a_n}{a_{n+1}} > \frac{b_n}{b_{n+1}} \quad n \geq N_\epsilon$

故有 $a_n \leq \frac{a_{N_\epsilon}}{b_{N_\epsilon}} b_n \quad \forall n > N_\epsilon$ 注: 由

$$\begin{aligned} \frac{a_n}{a_{n+1}} &> \frac{b_n}{b_{n+1}} \quad \forall n \geq N_\epsilon \\ \Rightarrow \frac{a_{n+1}}{a_n} &< \frac{b_{n+1}}{b_n} \quad \forall n \geq N_\epsilon \\ \Rightarrow \frac{a_n}{a_{n-1}} \cdots \frac{a_{N_\epsilon+1}}{a_{N_\epsilon}} &< \frac{b_n}{b_{n-1}} \cdots \frac{b_{N_\epsilon+1}}{b_{N_\epsilon}} \quad \forall n > N_\epsilon \\ \Rightarrow \frac{a_n}{a_{N_\epsilon}} &< \frac{b_n}{b_{N_\epsilon}} \quad \forall n > N_\epsilon \end{aligned}$$

$$\text{即: } a_n < \frac{a_{N_\epsilon}}{b_{N_\epsilon}} b_n \quad \forall n > N_\epsilon$$

$$\text{而: } b_n = \frac{1}{n(\ln n)^p} \rightarrow 0 \quad \text{as } n \rightarrow +\infty \text{ 故有 } a_n \rightarrow 0$$

Problem 4 (实践幂级数基本性质) 有函数

$$f(x) = \sum_{n=1}^{+\infty} \frac{x^n}{n^2 \cdot \ln(1+n)}$$

1. (10%) 证明: $f(x) \in C[-1, 1]$
2. (10%) 证明: $\exists f'(x) \in C[-1, 1]$
3. (10%) 证明: $\lim_{x \rightarrow 1-0} f'(x) = +\infty$ 。

解答:

1. (4/1)

$$f(x) = \sum_{n=1}^{+\infty} \frac{x^n}{n^2 \ln(1+n)}$$

考虑

$$\frac{\frac{1}{(n+1)^2 \ln(2+n)}}{\frac{1}{n^2 \ln(1+n)}} |x| = \frac{n^2}{(n+1)^2} \frac{\ln(1+n)}{\ln(2+n)} |x| \rightarrow |x| \quad \text{as } n \rightarrow +\infty$$

故收敛半径为 1

在考虑:

$$f(1) = \sum_{n=1}^{+\infty} \frac{1}{n^2 \ln(1+n)} \quad \frac{1}{n^2 \ln(1+n)} \leq \frac{1}{n^2} \quad \text{as } n \gg 1$$

故 $\sum_{n=1}^{+\infty} \frac{1}{n^2 \ln(1+n)}$ 收敛。

$$f(-1) = \sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2 \ln(1+n)}$$

按 Leibnize 判别法有 $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2 \ln(1+n)}$ 收敛。综上 $f(x) = \sum_{n=1}^{+\infty} \frac{x^n}{n^2 \ln(1+n)}$ 的收敛域为 $[-1, 1]$

据幂级数的内闭一致收敛性有 $f(x) \in C[-1, 1]$

2. (4/2) 已有

$$f(x) = \frac{x}{\ln 2} + \sum_{n=2}^{+\infty} \frac{x^n}{n^2 \ln(1+n)} \quad x \in [-1, 1]$$

考虑逐项求导级数

$$\frac{1}{\ln 2} + \sum_{n=2}^{+\infty} \frac{nx^{n-1}}{n^2 \ln(1+n)} = \frac{1}{\ln 2} + \sum_{n=2}^{+\infty} \frac{x^{n-1}}{n \ln(1+n)} = \frac{1}{\ln 2} + \sum_{n=1}^{+\infty} \frac{x^n}{(n+1) \ln(2+n)}$$

收敛半径为 1, 且在 $x = -1$ 处按 Leibnize 判别法级数收敛。

考虑 $x = 1$ 处的收敛性

$$\sum_{n=1}^{+\infty} \frac{1}{(n+1) \ln(2+n)} = \sum_{n=2}^{+\infty} \frac{1}{n \ln(1+n)}$$

考虑

$$\begin{aligned} \frac{1}{n \ln(1+n)} &= \frac{1}{n \ln n (1 + \frac{1}{n})} = \frac{1}{n [\ln n + \ln(1 + \frac{1}{n})]} = \frac{1}{n \ln n \left[1 + \frac{\ln(1 + \frac{1}{n})}{\ln n} \right]} \\ &= \frac{1}{n \ln n \left[1 + \frac{\frac{1}{n} + o(\frac{1}{n})}{\ln n} \right]} = \frac{1}{n \ln n \left[1 + \frac{1}{n \ln n} + o(\frac{1}{n \ln n}) \right]} \\ &= \frac{1}{n \ln n} \left[1 - \frac{1}{n \ln n} + o(\frac{1}{n \ln n}) \right] = \frac{1}{n \ln n} + o(\frac{1}{n \ln n}) \end{aligned}$$

而 $\sum \frac{1}{n \ln n}$ 发散, 故 $\sum_{n=2}^{+\infty} \frac{1}{n \ln(1+n)}$ 发散。

综上, 逐项求导级数

$$\frac{1}{\ln 2} + \sum_{n=1}^{+\infty} \frac{x^n}{(n+1) \ln(2+n)}$$

的收敛域为 $[-1, 1)$

故有:

$$f'(x) = \frac{1}{\ln 2} + \sum_{n=1}^{+\infty} \frac{x^n}{(n+1) \ln(2+n)} \quad x \in [-1, 1)$$

故有

$$\exists f'(x) \in C[-1, 1)$$

3. (4/3) 由

$$f'(1-0) \triangleq \lim_{x \rightarrow 1-0} f'(x) = \lim_{x \rightarrow 1-0} \left[\frac{1}{\ln 2} + \sum_{n=1}^{+\infty} \frac{x^n}{(n+1) \ln(2+n)} \right]$$

由 $f'(x) \in C[-1, 1)$, 且在 $(0, 1)$ 上大于零, 故严格单调上升。

另有:

$$\frac{1}{\ln 2} + \sum_{n=1}^{+\infty} \frac{1}{(n+1) \ln(2+n)} = +\infty$$

采用反证法，即设 $\lim_{x \rightarrow 1-0} f'(x) = M \in \mathbb{R}^+$ 。则有

$$S_N(x) \triangleq \frac{1}{\ln 2} + \sum_{n=1}^N \frac{x^n}{(n+1)\ln(2+n)} \leq M, \quad \forall x \in [0, 1), \quad \forall N \in \mathbb{N}$$

进一步有

$$\lim_{x \rightarrow 1-0} S_N(x) = \lim_{x \rightarrow 1-0} \left[\frac{1}{\ln 2} + \sum_{n=1}^N \frac{x^n}{(n+1)\ln(2+n)} \right] = \frac{1}{\ln 2} + \sum_{n=1}^N \frac{1}{(n+1)\ln(2+n)} \leq M, \quad \forall N \in \mathbb{N}$$

故有

$$\frac{1}{\ln 2} + \sum_{n=1}^{+\infty} \frac{1}{(n+1)\ln(2+n)} \leq M$$

故得矛盾。

Problem 5 (实践隐映照定理) 可基于隐映照定理认识 \mathbb{R}^4 中的集合

$$\Sigma = \left\{ [x, y, z, w]^T \in \mathbb{R}^4 \mid \text{满足: } \begin{cases} yz + zx + xy - w = 0 \\ x^2 + y^2 + z^2 - 1 = 0 \end{cases} \right\}$$

1. (10%) 按隐映照定理，说明： Σ 在局部可看作 \mathbb{R}^4 中 2 维曲面

$$\overset{\sigma}{\Sigma}(x, y) : \mathbb{R}^2 \supset B_\sigma((x_0, y_0)) \ni \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \Sigma(x, y) \triangleq \begin{bmatrix} x \\ y \\ z(x, y) \\ w(x, y) \end{bmatrix} \in \mathbb{R}^4$$

需指明， $(x_0, y_0) \in \mathbb{R}^2$ 的适用范围。

2. (10%) 考虑定义在局部曲面 $\overset{\sigma}{\Sigma}$ 上函数 $\theta(x, y, z, w)$ ，显见此函数等价于函数

$$\hat{\theta}(x, y) : B_\sigma((x_0, y_0)) \ni \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \hat{\theta}(x, y) \triangleq \theta(x, y, z(x, y), w(x, y))$$

推导 $\hat{\theta}(x, y)$ 之临界点所满足的方程。

3. (10%) 可按 Lagrange 乘子法，确定函数 $\theta(x, y, z, w)$ 在曲面 $\overset{\sigma}{\Sigma}$ 上之临界点所满足的方程。证明：按 Lagrange 乘子法确定的 $\theta(x, y, z, w)$ 之临界点方程同 $\hat{\theta}(x, y)$ 之临界点方程一致。

4. (10%) \mathbb{R}^4 中 2 维曲面 $\overset{\sigma}{\Sigma}$ 可视作“抽象体”，其体积 $|\overset{\sigma}{\Sigma}|$ 定义为

$$|\overset{\sigma}{\Sigma}| \triangleq \int_{B_\sigma((x_0, y_0))} \det D \begin{bmatrix} z \\ w \end{bmatrix} (x, y) d\tau$$

计算 $(x_0, y_0) = (0, 0)$ 时，局部“抽象体”的体积（此时 σ 为确定的正数）。

解答

1. (5/1) 考虑

$$\Sigma \triangleq \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4 \left| \begin{cases} yz + zx + xy - w = 0 \\ x^2 + y^2 + z^2 - 1 = 0 \end{cases} \right. \right\}$$

如果

$$[x_0, y_0, z_0, w_0]^T \in \Sigma$$

使得

$$D \begin{bmatrix} z \\ w \end{bmatrix} \Sigma \left(\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ w_0 \end{bmatrix} \right) = \begin{bmatrix} x+y & -1 \\ 2z & 0 \end{bmatrix} \bigg| \begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ w_0 \end{bmatrix} = \begin{bmatrix} x_0 + y_0 & -1 \\ 2z_0 & 0 \end{bmatrix}$$

故有

$$\det D \begin{bmatrix} z \\ w \end{bmatrix} \Sigma \left(\begin{bmatrix} x_0 \\ y_0 \\ z_0 \\ w_0 \end{bmatrix} \right) = 2z_0$$

故对于隐映照定理要求 $z_0 \neq 0$

而

$$x_0^2 + y_0^2 + z_0^2 - 1 = 0 \Rightarrow x_0^2 + y_0^2 = 1 - z_0^2 \geq 0 \Rightarrow 0 < z_0^2 \leq 1$$

即:

$$z_0 \in (-1, 1) - \{0\}$$

而 (x_0, y_0) 满足:

$$x_0^2 + y_0^2 = 1 - z_0^2 \in [0, 1)$$

2. (5/2) 进一步由隐映照定理:

$$\Sigma \left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} z \\ w \end{bmatrix} (x, y) \right) = 0 \in \mathbb{R}^2$$

$$\Rightarrow D \begin{bmatrix} x \\ y \end{bmatrix} \left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} z \\ w \end{bmatrix} (x, y) \right) + D \begin{bmatrix} z \\ w \end{bmatrix} \Sigma \left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} z \\ w \end{bmatrix} (x, y) \right) D \begin{bmatrix} z \\ w \end{bmatrix} (x, y) = 0 \in \mathbb{R}^{2 \times 2}$$

即:

$$\begin{bmatrix} y+z & z+x \\ 2x & 2y \end{bmatrix} + \begin{bmatrix} x+y & -1 \\ 2z & 0 \end{bmatrix} \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} (x, y) = 0 \in \mathbb{R}^{2 \times 2}$$

$$\Rightarrow \begin{bmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} \end{bmatrix} (x, y) = - \begin{bmatrix} x+y & -1 \\ 2z & 0 \end{bmatrix}^{-1} \begin{bmatrix} y+z & z+x \\ 2x & 2y \end{bmatrix}$$

此处 $z = z(x, y)$

对于

$$\hat{\theta}(x, y) \triangleq \theta(x, y, z(x, y), w(x, y)) = \theta \left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} z \\ w \end{bmatrix} (x, y) \right) \in \mathbb{R}$$

临界点方程为

$$D\hat{\theta} \left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = D \begin{bmatrix} x \\ y \end{bmatrix} \theta + D \begin{bmatrix} z \\ w \end{bmatrix} \theta D \begin{bmatrix} z \\ w \end{bmatrix} (x, y) = 0 \in \mathbb{R}^{2 \times 2}$$

即:

$$\begin{bmatrix} \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{bmatrix} \left(\begin{bmatrix} x_* \\ y_* \end{bmatrix}, \begin{bmatrix} z \\ w \end{bmatrix} (x_*, y_*) \right) + \begin{bmatrix} \frac{\partial \theta}{\partial z} & \frac{\partial \theta}{\partial w} \end{bmatrix} \left(\begin{bmatrix} x_* \\ y_* \end{bmatrix}, \begin{bmatrix} z \\ w \end{bmatrix} (x_*, y_*) \right) D \begin{bmatrix} z \\ w \end{bmatrix} (x_*, y_*) = 0 \in \mathbb{R}^{1 \times 2}$$

此处

$$D \begin{bmatrix} z \\ w \end{bmatrix} (x_*, y_*) = - \begin{bmatrix} x_* + y_* & -1 \\ 2z_* & 0 \end{bmatrix}^{-1} \begin{bmatrix} y_* + z_* & z_* + x_* \\ 2x_* & 2y_* \end{bmatrix}$$

$$= - \left(D \begin{bmatrix} z \\ w \end{bmatrix} \Sigma \right) D \begin{bmatrix} x \\ y \end{bmatrix} \Sigma \left| \begin{bmatrix} x_* \\ y_* \\ \begin{bmatrix} z \\ w \end{bmatrix} (x_*, y_*) \end{bmatrix} \right.$$

3. (5/3) Lagrange 乘子法。构造 Lagrange 函数。

$$L\left(\begin{bmatrix} x \\ y \end{bmatrix}, \begin{bmatrix} z \\ w \end{bmatrix}, \begin{bmatrix} \lambda \\ \mu \end{bmatrix}\right) = \theta(x, y, z, w) + \begin{bmatrix} \lambda & \mu \end{bmatrix}^T \Sigma(x, y, z, w)$$

$$\Rightarrow \begin{cases} D\begin{bmatrix} x \\ y \end{bmatrix} L = D\begin{bmatrix} x \\ y \end{bmatrix} \theta + \begin{bmatrix} \lambda & \mu \end{bmatrix}^T D\begin{bmatrix} x \\ y \end{bmatrix} \Sigma \in \mathbb{R}^{1 \times 2} \\ D\begin{bmatrix} z \\ w \end{bmatrix} L = D\begin{bmatrix} z \\ w \end{bmatrix} \theta + \begin{bmatrix} \lambda & \mu \end{bmatrix}^T D\begin{bmatrix} z \\ w \end{bmatrix} \Sigma \in \mathbb{R}^{1 \times 2} \\ D\begin{bmatrix} \lambda \\ \mu \end{bmatrix} L = \Sigma^T \in \mathbb{R}^{1 \times 2} \end{cases}$$

由第二式

$$\begin{bmatrix} \lambda & \mu \end{bmatrix}^T = -D\begin{bmatrix} z \\ w \end{bmatrix} \theta (D\begin{bmatrix} z \\ w \end{bmatrix} \Sigma)^{-1}$$

代入第一式

$$D\begin{bmatrix} x \\ y \end{bmatrix} \theta - D\begin{bmatrix} z \\ w \end{bmatrix} \theta (D\begin{bmatrix} z \\ w \end{bmatrix} \Sigma)^{-1} D\begin{bmatrix} x \\ y \end{bmatrix} \Sigma = 0 \in \mathbb{R}^{1 \times 2}$$

4. (5/4)

$$\begin{aligned}
\left| \sum^{\sigma} \right| &\triangleq \int_{B_{\sigma} \left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \right)} \det D \begin{bmatrix} z \\ w \end{bmatrix} (x, y) d\tau \\
&= \int_{B_{\sigma} \left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \right)} \frac{\det \begin{bmatrix} y+z & z+x \\ 2x & 2y \end{bmatrix}}{2z} d\tau \\
&= \int_{B_{\sigma} \left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} \right)} \frac{(y+z)y - (z+x)x}{z} d\sigma \\
\text{此处 } \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} &= 0 \in \mathbb{R}^2 \quad z = \sqrt{1 - (x^2 + y^2)} \\
&= \int_{B_{\sigma}(0,0)} \frac{(y^2 - x^2) + z(y - x)}{z} d\sigma \\
&= \int_{B_{\sigma}(0,0)} \frac{y^2 - x^2}{\sqrt{1 - (x^2 + y^2)}} d\sigma + \int_{B_{\sigma}(0,0)} (y - x) d\sigma \\
\text{根据对称性 } \int_{B_{\sigma}(0,0)} y d\sigma &= \int_{B_{\sigma}(0,0)} x d\sigma = 0 \\
\int_{B_{\sigma}(0,0)} \frac{y^2}{\sqrt{1 - (x^2 + y^2)}} d\sigma &= \int_{B_{\sigma}(0,0)} \frac{x^2}{\sqrt{1 - (x^2 + y^2)}} d\sigma = 0 \\
\text{故 } \left| \sum^{\sigma} \right| &= 0
\end{aligned}$$

Problem 6 (实践 \mathbb{R}^m 中体积积分换元公式) 考虑体上积分

$$\int_{\mathcal{D}_{xyz}} y^4 d\tau$$

此处, \mathcal{D}_{xyz} 为由以下曲面族

$$\left\{ \begin{array}{l} x = a \cdot z^2 \\ x = b \cdot z^2 \end{array} \right., (z > 0, 0 < a < b); \quad \left\{ \begin{array}{l} x = \alpha \cdot y \\ x = \beta \cdot y \end{array} \right., (0 < \alpha < \beta); \quad x = h (h > 0)$$

所围成的体积。

1. (10%) 首先, 需说明 \mathcal{D}_{xyz} 为 \mathbb{R}^3 中 Jordan 可测集, 亦即有界且其边界为零测集。为此, 证明如下结论: 定义在闭方块上的 Monge 型曲面为零测集:

$$\Sigma(x, y) : [a, b] \times [c, d] \ni \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \Sigma(x, y) \triangleq \begin{bmatrix} x \\ y \\ f(x, y) \end{bmatrix}$$

此处 $f(x, y) \in \mathcal{R}([a, b] \times [c, d])$ 。

2. (10%) 考虑映照

$$\Phi(x, y, z) : \mathcal{D}_{xyz} \ni \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \Phi(x, y, z) = \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} (x, y, z) \triangleq \begin{bmatrix} \frac{x}{z^2} \\ \frac{x}{y} \\ x \end{bmatrix}$$

以此建立微分同胚，需说明相应的开集以及确定为微分同胚的所有条件。

3. (10%) 完成上述积分计算。需严格检验积分换元公式的有关要求。

解答

1. (6/1) 定义在比方块上的 Monge 型曲面

$$\Sigma(x, y) : [a, b] \times [c, d] \ni \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \Sigma(x, y) \triangleq \begin{bmatrix} x \\ y \\ f(x, y) \end{bmatrix}$$

此处

$$f(x, y) \in \mathcal{R}[a, b] \times [c, d]$$

则有

$$\omega(f(x, y), P) \triangleq \sum_{i=1}^{N_x} \sum_{j=1}^{N_y} \sup_{\substack{\begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}, \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix} \\ \in [x_{i-1}, x_i] \times [y_{j-1}, y_j]}} |f(\tilde{x}, \tilde{y}) - f(\hat{x}, \hat{y})| \Delta x_i \Delta y_j < \epsilon$$

$$as \quad |P| \triangleq \max_{1 \leq i \leq N_x} \max_{1 \leq j \leq N_y} |\Delta x_i \Delta y_j| < \delta_\epsilon$$

故以 $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$ 为底面，以 $f(\xi_i, \eta_j)$ 为中心，以 $\sup |f(\tilde{x}, \tilde{y}) - f(\hat{x}, \hat{y})|$ 为半高度作闭方块 I_{ij}

则

$$\Sigma \subset \bigcup_{i=1}^{N_x} \bigcup_{j=1}^{N_y} I_{ij}$$

且

$$\sum_{i=1}^{N_x} \sum_{j=1}^{N_y} |I_{ij}| = 2\omega(f(x, y), P) < \epsilon \quad \forall |P| < \delta_\epsilon$$

故得证 Σ 为零测集（Jordan 意义）。

2. (6/2)

$$\Phi(x, y, z) \triangleq \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} (x, y, z) = \begin{bmatrix} \frac{x}{z^2} \\ \frac{x}{y} \\ x \end{bmatrix}$$

$$D\Phi(x, y, z) \triangleq \frac{D(\xi, \eta, \zeta)}{D(x, y, z)} = \begin{bmatrix} \frac{1}{z^2} & 0 & -\frac{2x}{z^3} \\ \frac{1}{y} & -\frac{x}{y^2} & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$\det D\Phi(x, y, z) = -2\frac{x^2}{y^2 z^3} \neq 0 \quad \text{as} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \overset{\circ}{\mathcal{D}}_{xyz}$$

此处

$$\overset{\circ}{\mathcal{D}}_{xyz} \triangleq \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \left| \begin{array}{l} \frac{x}{z^2} \in (a, b) \\ \frac{x}{y} \in (\alpha, \beta) \\ x \in (0, h) \end{array} \right. \right\}$$

易见 $\Phi(x, y, z)$ 在 $\overset{\circ}{\mathcal{D}}_{xyz}$ 上为单射

$$\begin{aligned} \mathcal{D}_{\xi\eta\zeta} &\triangleq \Phi(\mathcal{D}_{xyz}) = \left\{ \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \in \mathbb{R}^3 \left| \begin{array}{l} \xi \in [a, b] \\ \eta \in [\alpha, \beta] \\ \zeta \in [0, h] \end{array} \right. \right\} \\ \overset{\circ}{\mathcal{D}}_{\xi\eta\zeta} &\triangleq \left\{ \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \in \mathbb{R}^3 \left| \begin{array}{l} \xi \in (a, b) \\ \eta \in (\alpha, \beta) \\ \zeta \in (0, h) \end{array} \right. \right\} = \Phi(\overset{\circ}{\mathcal{D}}_{xyz}) \end{aligned}$$

有

$$\Phi(x, y, z) \in C^1(\overset{\circ}{\mathcal{D}}_{xyz}, \overset{\circ}{\mathcal{D}}_{\xi\eta\zeta})$$

3. (6/3)

$$\begin{aligned} \int_{\mathcal{D}_{xyz}} y^4 d\tau &= \int_{\overset{\circ}{\mathcal{D}}_{xyz}} y^4 d\tau = \int_{\overset{\circ}{\mathcal{D}}_{\xi\eta\zeta}} y^4(\xi, \eta, \zeta) \left| \det \frac{D(x, y, z)}{D(\xi, \eta, \zeta)} \right| (\xi, \eta, \zeta) d\tau \\ &= \int_{\overset{\circ}{\mathcal{D}}_{\xi\eta\zeta}} y^4(\xi, \eta, \zeta) \left(\frac{y^2 z^3}{2x^2} \right) (\xi, \eta, \zeta) d\tau = \int_{\mathcal{D}_{\xi\eta\zeta}} \frac{1}{2} \xi^{-\frac{3}{2}} \eta^{-6} \zeta^{\frac{11}{2}} d\tau \\ &= \frac{1}{2} \int_a^b \xi^{-\frac{3}{2}} d\xi \int_{\alpha}^{\beta} \eta^{-6} d\eta \int_0^h \zeta^{\frac{11}{2}} d\zeta = \frac{1}{2} \frac{b^{-\frac{3}{2}} - a^{-\frac{3}{2}}}{-\frac{1}{2}} \frac{\beta^{-5} - \alpha^{-5}}{-5} h^{\frac{13}{2}} \end{aligned}$$

Problem 7 (实践 Stokes 公式) 有做功形式的曲线积分 (第二类曲线积分)

$$\oint_{C_1 \cup C_2} (y^2 + z^2)dx + (x^2 + z^2)dy + (x^2 + y^2)dz \equiv \oint_{C_1 \cup C_2} [(y^2 + z^2)\mathbf{i} + (x^2 + z^2)\mathbf{j} + (x^2 + y^2)\mathbf{k}] \cdot \tau dl$$

其中 C_1 和 C_2 分别为 z 轴上半球面 $(x-R)^2 + y^2 + z^2 = R^2$ ($z > 0$) 同柱面 $(x-r_1)^2 + y^2 = r_1^2$ 和 $(x-r_2)^2 + y^2 = r_2^2$ 的截线, 此处 $0 < r_1 < r_2 < R$; 从 z 轴正向望去, 沿 C_1 和 C_2 曲线的走向分别为逆时针和顺时针方向。

1. (10%) 利用 Stokes 公式计算上述积分。
2. (10%) 对于上述沿曲线 C_1 的积分, 给出一种线积分的直接计算方案; 需给出 C_1 的向量值映照表达式以及参数域上的积分计算式, 无需具体计算。

解答

1. (7/1)

$$\begin{aligned} \Sigma(x, y) : \mathcal{D}_{xy} \ni \begin{bmatrix} x \\ y \end{bmatrix} &\mapsto \Sigma(x, y) \triangleq \begin{bmatrix} x \\ y \\ \sqrt{R^2 - (x-R)^2 - y^2} \end{bmatrix} \\ \frac{\partial \Sigma}{\partial x} \times \frac{\partial \Sigma}{\partial y}(x, y) &= \begin{vmatrix} i & j & k \\ 1 & 0 & -\frac{(x-R)}{\sqrt{R^2 - (x-R)^2 - y^2}} \\ 0 & 1 & -\frac{y}{\sqrt{R^2 - (x-R)^2 - y^2}} \end{vmatrix} = \begin{bmatrix} \frac{x-R}{\sqrt{R^2 - (x-R)^2 - y^2}} & \frac{y}{\sqrt{R^2 - (x-R)^2 - y^2}} & 1 \end{bmatrix} \\ \nabla \times a(x, y, z) &= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 + z^2 & x^2 + z^2 & x^2 + y^2 \end{vmatrix} = \begin{bmatrix} 2(y-z) \\ 2(z-x) \\ 2(x-y) \end{bmatrix} \\ I &= \oint_{C_1 \cup C_2} [(y^2 + z^2)i + (x^2 + z^2)j + (x^2 + y^2)k] \cdot \tau dl \\ &= \int_{\mathcal{D}_{xy}} 2[y-z, z-x, x-y] \begin{bmatrix} \frac{x-R}{z} \\ \frac{y}{z} \\ 1 \end{bmatrix} d\sigma \\ &= 2 \int_{\mathcal{D}_{xy}} \frac{(y-z)(x-R) + (z-x)y}{z} + (x-y) d\sigma \\ &= 2 \int_{\mathcal{D}_{xy}} (-R\frac{y}{z} + R) d\sigma = 2R|\mathcal{D}_{xy}| = 4\pi R(r_2^2 - r_1^2) \end{aligned}$$

2. (7/2) 对于沿曲线 C_1 的积分, 作

$$\begin{cases} x = r_1 + r_1 \cos \theta \\ y = r_1 \sin \theta \end{cases}$$

而

$$z(\theta) = \sqrt{R^2 - (x - R)^2 - y^2(\theta)}$$

即:

$$C_1(\theta) : [0, 2\pi] \ni \theta \mapsto C_1(\theta) = \begin{bmatrix} r_1 + r_1 \cos \theta \\ r_1 \sin \theta \\ \sqrt{R^2 - (r_1 + r_1 \cos \theta - R)^2 - (r_1 \cos \theta)^2} \end{bmatrix}$$

$$\Rightarrow \frac{dC_1}{d\theta}(\theta) = \begin{bmatrix} -r_1 \sin \theta \\ r_1 \cos \theta \\ \frac{2(r_1 + r_1 \cos \theta - R)r_1 \sin \theta - 2r_1 \sin \theta \cos \theta}{\sqrt{R^2 - (r_1 + r_1 \cos \theta - R)^2 - (r_1 \sin \theta)^2}} \end{bmatrix}$$

$$\begin{aligned} & \int_{C_1} [(y^2 + z^2)i + (x^2 + z^2)j + (x^2 + y^2)k] \cdot \tau dl \\ &= \int_0^{2\pi} [y^2 + z^2, x^2 + z^2, x^2 + y^2](\theta) \frac{dC_1}{d\theta}(\theta) d\theta \end{aligned}$$

Problem 8 (实践微分同胚——将“三维轴对称柱体”变换为“规则矩体”) \mathbb{R}^3 中区域

$$\mathcal{D}_{xyz} \triangleq \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \mid 0 < x^2 + y^2 < R^2(z), z \in (0, H), \text{ 且将 } xz \text{ 平面的上半平面剖开} \right\}$$

为沿轴向管径具有轴对称变形的“柱体”。现考虑，将此不规则的柱体 \mathcal{D}_{xyz} 通过向量值映照化成规则的矩体。以下研究向量值映照:

$$X(\xi, \eta, \zeta) : \mathcal{D}_{\xi\eta\zeta} \ni \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \mapsto X(\xi, \eta, \zeta) \triangleq \begin{bmatrix} x(\xi, \eta, \zeta) \\ y(\xi, \eta, \zeta) \\ z(\xi, \eta, \zeta) \end{bmatrix} = \begin{bmatrix} \xi \cdot R(\zeta) \cos \eta \\ \xi \cdot R(\zeta) \sin \eta \\ \zeta \end{bmatrix}$$

此处，参数区域

$$\mathcal{D}_{\xi\eta\zeta} \triangleq \left\{ \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \mid \xi \in (0, 1), \eta \in (0, 2\pi), \zeta \in (0, H) \right\}$$

1. (10%) 基于微分同胚的有关充分性定理，证明： $X(\xi, \eta, \zeta)$ 为 $\mathcal{D}_{\xi\eta\zeta}$ 同 \mathcal{D}_{xyz} 间的微分同胚。籍此， $\{\xi, \eta, \zeta\}$ 成为物理区域 \mathcal{D}_{xyz} 的曲线坐标系。注：此处需求出向量值映照 $X(\xi, \eta, \zeta)$ 的 Jacobian 矩阵 $DX(\xi, \eta, \zeta) \in \mathbb{R}^{3 \times 3}$ 。

2. (10%) 给出三维轴对称柱体 \mathcal{D}_{xyz} 的积分表达式（需给出参数域及参数域上被积函数）。

3. (10%) 给出三维轴对称柱体之侧面积积分表达式（需给出参数域及参数域上被积函数）。

4. (10%) 计算

$$\int_{\partial \mathcal{D}_{xyz}} [-\sin \eta \mathbf{i} + \cos \eta \mathbf{j} + R(\zeta) \mathbf{k}] \cdot \mathbf{n} d\sigma$$

此处 $\partial \mathcal{D}_{xyz}$ 表示三维轴对称柱体的侧面， \mathbf{n} 指向向外。

5. (10%) 设有函数 $f(x, y, z) \in C^2(\mathcal{D}_{xyz}; \mathbb{R})$ ，则有：

$$f(x, y, z) = f(x(\xi, \eta, \zeta), y(\xi, \eta, \zeta)) =: \hat{f}(\xi, \eta, \zeta) = \hat{f}(\xi(x, y, z), \eta(x, y, z), \zeta(x, y, z))$$

试将： $\frac{\partial^2 f}{\partial y \partial x}(x, y, z)$ 由 $\hat{f}(\xi, \eta, \zeta)$ 以及 $X(\xi, \eta, \zeta) \in \mathbb{R}^3$ 等关于 ξ, η, ζ 的函数来表示，仅需说明必要的步骤，无需给出具体形式。注：本问题基于逆映照定理。

解答

1. (8/1)

$$X(\xi, \eta, \zeta) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} (\xi, \eta, \zeta) = \begin{bmatrix} \xi R(\zeta) \cos \eta \\ \xi R(\zeta) \sin \eta \\ \zeta \end{bmatrix}$$

$$DX(\xi, \eta, \zeta) \triangleq \frac{D(x, y, z)}{D(\xi, \eta, \zeta)}(\xi, \eta, \zeta) = \begin{bmatrix} R(\zeta) \cos \eta & -\xi R(\zeta) \sin \eta & \xi \dot{R}(\zeta) \cos \eta \\ R(\zeta) \sin \eta & \xi R(\zeta) \cos \eta & \xi \dot{R}(\zeta) \sin \eta \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det DX(\xi, \eta, \zeta) = \xi R(\zeta) \neq 0 \quad \forall \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} \in \mathcal{D}_{\xi\eta\zeta}$$

对于单射根据映照几何意义易证。

2. (8/2)

$$\begin{aligned} |\mathcal{D}_{xyz}| &= \int_{\mathcal{D}_{xyz}} d\tau = \int_{\mathcal{D}_{\xi\eta\zeta}} \left| \det \frac{D(x, y, z)}{D(\xi, \eta, \zeta)}(\xi, \eta, \zeta) \right| d\tau \\ &= \int_{\mathcal{D}_{\xi\eta\zeta}} \xi R(\zeta) d\tau = 2\pi \int_0^1 \xi d\xi \cdot \int_0^H R(\zeta) d\zeta = \pi \int_0^H R(\zeta) d\zeta \end{aligned}$$

3. (8/3)

$$\Sigma(\eta, \zeta, \xi) : \mathcal{D}_{\eta\zeta} \ni \begin{bmatrix} \eta \\ \zeta \end{bmatrix} \mapsto \Sigma(\eta, \zeta) = X(1, \eta, \zeta) = \begin{bmatrix} R(\zeta) \cos \eta \\ R(\zeta) \sin \eta \\ \zeta \end{bmatrix} \in \mathbb{R}^3$$

$$\Rightarrow \frac{\partial \Sigma}{\partial \eta} \times \frac{\partial \Sigma}{\partial \zeta}(\eta, \zeta) = \begin{vmatrix} i & j & k \\ -R(\zeta) \sin \eta & R(\zeta) \cos \eta & 0 \\ \dot{R} \cos \eta & \dot{R}(\zeta) \sin \eta & 1 \end{vmatrix} = \begin{bmatrix} R(\zeta) \cos \eta \\ R(\zeta) \sin \eta \\ -R(\zeta) \dot{R}(\zeta) \end{bmatrix} \in \mathbb{R}^3$$

故

$$\begin{aligned} |\Sigma| &= \int_{\Sigma} d\sigma = \int_{\mathcal{D}_{\eta\zeta}} \left| \frac{\partial \Sigma}{\partial \eta} \times \frac{\partial \Sigma}{\partial \zeta} \right|_{\mathbb{R}^3}(\eta, \zeta) d\sigma = \int_{\mathcal{D}_{\eta\zeta}} \sqrt{R^2(\zeta) + R^2(\zeta) \dot{R}^2(\zeta)} d\sigma \\ &= 2\pi \int_0^H R(\zeta) \sqrt{1 + \dot{R}^2(\zeta)} d\zeta \end{aligned}$$

4. (8/4)

$$\begin{aligned} & \int_{\partial \mathcal{D}_{xyz}} [-\sin \eta i + \cos \eta j + R(\zeta) k] \cdot n d\sigma \\ &= \int_{\mathcal{D}_{\eta\zeta}} [-\sin \eta, \cos \eta, R(\zeta)] \cdot \left(\frac{\partial \Sigma}{\partial \eta} \times \frac{\partial \Sigma}{\partial \zeta} \right)(\eta, \zeta) d\sigma \\ &= \int_{\mathcal{D}_{\eta\zeta}} R(\zeta) [-\sin \eta, \cos \eta, R(\zeta)] \cdot \begin{bmatrix} \cos \eta \\ \sin \eta \\ -\dot{R}(\zeta) \end{bmatrix} d\sigma \\ &= \int_{\mathcal{D}_{\eta\zeta}} R(\zeta) R(\zeta) \dot{R}(\zeta) d\sigma \\ &= 2\pi \int_0^H R^2(\zeta) \dot{R}(\zeta) d\zeta = 2\pi \int_0^H R^2(\zeta) dR(\zeta) \\ &= 2\pi \frac{R^3(H) - R^2(0)}{3} \end{aligned}$$

5. (8/5) 已有:

$$DX(\xi, \eta, \zeta) = \frac{D(x, y, z)}{D(\xi, \eta, \zeta)}(\xi, \eta, \zeta) = \begin{bmatrix} R(\zeta) \cos \eta & -\xi R(\zeta) \sin \eta & \xi \dot{R}(\zeta) \cos \eta \\ R(\zeta) \sin \eta & \xi R(\zeta) \cos \eta & \xi \dot{R}(\zeta) \sin \eta \\ 0 & 0 & 1 \end{bmatrix}$$

由

$$\frac{\partial f}{\partial x}(x, y, z) = \frac{\partial \hat{f}}{\partial \xi}(\xi, \eta, \zeta) \frac{\partial \xi}{\partial x}(x, y, z) + \frac{\partial \hat{f}}{\partial \eta}(\xi, \eta, \zeta) \frac{\partial \eta}{\partial x}(x, y, z) + \frac{\partial \hat{f}}{\partial \zeta}(\xi, \eta, \zeta) \frac{\partial \zeta}{\partial x}(x, y, z)$$

由逆映照定理

$$\frac{D(\xi, \eta, \zeta)}{D(x, y, z)}(x, y, z) = \left(\frac{D(x, y, z)}{D(\xi, \eta, \zeta)} \right)^{-1}(\xi, \eta, \zeta) \quad (*)$$

故

$$\frac{\partial \xi}{\partial x}(x, y, z), \frac{\partial \eta}{\partial x}(x, y, z), \frac{\partial \zeta}{\partial x}(x, y, z)$$

等, 均为 (ξ, η, ζ) 的函数, 可记

$$\frac{\partial \xi}{\partial x}(x, y, z) = \overline{\frac{\partial \xi}{\partial x}}(\xi(x, y, z), \eta(x, y, z), \zeta(x, y, z))$$

由此

$$\frac{\partial^2 f}{\partial y \partial x}(x, y, z) = \left[\frac{\partial^2 \hat{f}}{\partial \xi \partial \xi} \frac{\partial \xi}{\partial y} + \frac{\partial^2 \hat{f}}{\partial \eta \partial \xi} \frac{\partial \eta}{\partial y} + \frac{\partial^2 \hat{f}}{\partial \zeta \partial \xi} \frac{\partial \zeta}{\partial y} \right] \cdot \frac{\partial \xi}{\partial x} + \frac{\partial \hat{f}}{\partial \xi} \left[\frac{\partial}{\partial \xi} \left(\overline{\frac{\partial \xi}{\partial x}} \right) \frac{\partial \xi}{\partial y} + \frac{\partial}{\partial \eta} \left(\overline{\frac{\partial \xi}{\partial x}} \right) \frac{\partial \eta}{\partial y} + \frac{\partial}{\partial \zeta} \left(\overline{\frac{\partial \xi}{\partial x}} \right) \frac{\partial \zeta}{\partial y} \right] + \dots$$

而 $\frac{\partial \xi}{\partial y}, \frac{\partial \eta}{\partial y}, \frac{\partial \zeta}{\partial y}$ 等可再由 (*) 式获得。