复旦大学力学与工程科学系

2008～2009学年第二学期期末考试试卷答案

□A卷　　　□B卷

课程名称：数学分析（II）　　　　课程代码：MATH120009.09

开课院系：力学与工程科学系　　　　考试形式：开卷/闭卷/课程论文

姓名：＿＿＿＿＿＿　学号：＿＿＿＿＿＿　专业：＿＿＿＿＿＿

题 号	1/(1)	1/(2)	1/(3)	1/(4)	2	3/(1)	3/(2)	4/(1)	4/(2)
得 分									
题 号	5/(1)	5/(2)	6/(1)	6/(2)	7/(1)	7/(2)	8/(1)	8/(2)	总 分
得 分									

Problem 1 (向量值映照基本概念) *现有向量值映照：*

$$\Phi(x,y) : \mathbb{R}^2 \ni (x,y) \mapsto \Phi(x,y) \triangleq \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} \in \mathbb{R}^2,$$

其中：

$$f(x,y) \triangleq \begin{cases} xy \cdot \frac{x^2-y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases} \quad ; \quad g(x,y) \triangleq x \cdot \sin y \quad \forall (x,y) \in \mathbb{R}^2$$

1. (10%) *计算：* $\lim_{(x,y)\to(0,0)} \Phi(x,y)$

2. (10%) *计算：* $\frac{\partial f}{\partial x}(0,0)$，$\frac{\partial f}{\partial y}(0,0)$

3. (05%) *研究：* $\Phi(x,y) \in \mathbb{R}^2$ *在* $(0,0)$ *点的可微性（要求明确写出该点的Jacobian矩阵）*

4. (05%) *计算：*

$$\frac{\partial \Phi}{\partial \mathbf{e}} \triangleq \lim_{\lambda \to 0 \in \mathbb{R}} \frac{\Phi(\mathbf{0} + \lambda \cdot \mathbf{e}) - \Phi(\mathbf{0})}{\lambda}$$

此处： $\mathbf{e} = (1,1)/\sqrt{2} \in \mathbb{R}^2$，$\Phi(\mathbf{0}) = \Phi(0,0)$。

1/(1)：

$$\lim_{(x,y)\to 0\in\mathbb{R}^2} f(x,y) = \lim_{(x,y)\to 0\in\mathbb{R}^2} xy\frac{x^2-y^2}{x^2+y^2} = 0$$

$$\lim_{(x,y)\to 0\in\mathbb{R}^2} g(x,y) = \lim_{(x,y)\to 0\in\mathbb{R}^2} x\sin y = 0$$

$$\text{故} \lim_{(x,y)\to 0\in\mathbb{R}^2} \Phi(x,y) = \lim_{(x,y)\to 0\in\mathbb{R}^2} \begin{bmatrix} f(x,y) \\ g(x,y) \end{bmatrix} = 0 \in \mathbb{R}^2$$

1/(2)：

$$\frac{\partial f}{\partial x}(0,0) \triangleq \lim_{\Delta x\to 0} \frac{f(0+\Delta x,0)-f(0,0)}{\Delta x} = \lim_{\Delta x\to 0} \frac{0-0}{\Delta x} = 0$$
$$\frac{\partial f}{\partial y}(0,0) \triangleq \lim_{\Delta y\to 0} \frac{f(0,0+\Delta y)-f(0,0)}{\Delta y} = \lim_{\Delta y\to 0} \frac{0-0}{\Delta y} = 0$$

1/(3)： 对于$\Phi(x,y)\in\mathbb{R}^2$在$(0,0)$点的可微性，考虑$f(x,y),g(x,y)$在$(0,0)$点的可微性

显然$g(x,y)\triangleq x\sin y$在$(0,0)$点可微

$$\frac{\left|f(x,y)-f(0,0)-\left[\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right](0,0)\begin{bmatrix} x \\ y \end{bmatrix}\right|}{\sqrt{x^2+y^2}} = \frac{\left|xy\frac{x^2-y^2}{x^2+y^2}\right|}{\sqrt{x^2+y^2}} \leqslant \frac{|x|}{\sqrt{x^2+y^2}}|y| \leqslant \sqrt{x^2+y^2}$$

故$f(x,y)$在$(0,0)$点可微，由此$\Phi(x,y)$在$(0,0)$点可微。另外$\frac{\partial g}{\partial x}(0,0)=0$，$\frac{\partial g}{\partial y}(0,0)=0$

故

$$D\Phi(0,0) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{R}^{2\times 2}$$

1/(4)： 由于$\Phi(x,y)$在$(0,0)\in\mathbb{R}^2$点可微，故

$$\frac{\partial \Phi}{\partial e}(0,0) = D\Phi(0,0)e = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} e = 0$$

Problem 2 (导数计算) (10%) *求函数$u(x,y)=f(xy,y/x)$的二阶偏导数$\frac{\partial^2 u}{\partial x\partial y}(x,y)$。此处认为$f(\xi,\eta)$具有足够的光滑性。*

2：

$$
\begin{aligned}
&u(x,y)=f(xy,\frac{y}{x})\\
\text{则}&\frac{\partial u}{\partial y}(x,y)=f_1\cdot x+f_2\cdot\frac{1}{x}\\
&\frac{\partial^2 u}{\partial x\partial y}(x,y)=\Big(f_{11}\cdot y+f_{12}\cdot(-\frac{y}{x^2})\Big)x+f_1+\Big(f_{21}\cdot y+f_{22}(-\frac{y}{x^2})\Big)\frac{1}{x}+f_2(-\frac{1}{x^2})\\
&\qquad\qquad\quad=xyf_{11}-2xyf_{12}-\frac{y}{x^3}f_{22}+f_1-\frac{1}{x^2}f_2
\end{aligned}
$$

此处$f_{12}(xy,\frac{y}{x})=\frac{\partial^2 f}{\partial x\partial y}(xy,\frac{y}{x})=\frac{\partial^2 f}{\partial y\partial x}(xy,\frac{y}{x})$等。

Problem 3 (*Fourier* **级数基本概念)** *现有有限区间上定义的分段函数:*

$$
f(x)\triangleq\begin{cases}0 & x\in[0,1)\\ 1 & x\in[1,2]\\ x-2 & x\in(2,3]\end{cases}
$$

1. (10%) *按点收敛的概念，将*$f(x)$, $x\in[0,3]$*表示成Fourier 级数的形式。要求写出具体的表达形式，但相关系数仅需给出具体计算式。*

2. (05%) *示意性绘出Fourier 级数对应的和函数（极限函数）的图像。*

3/(1)：

$$
\frac{f(x+0)+f(x-0)}{2}=\frac{a_0}{2}+\sum_{n=1}^{+\infty}\Big(a_n\cos\frac{n\pi x}{l}+b_n\sin\frac{n\pi x}{l}\Big)
$$

此处$l=\frac{3}{2}$

$$
\begin{cases}a_n\triangleq\frac{1}{l}\int_0^3 f(x)\cos\frac{n\pi x}{\frac{3}{2}}dx\,,\ n\in\mathbb{N}\cup\{0\}\\ b_n\triangleq\frac{1}{l}\int_0^3 f(x)\sin\frac{n\pi x}{\frac{3}{2}}dx\,,\ n\in\mathbb{N}\end{cases}
$$

3/(2)：略

Problem 4 (条件极值问题) *在椭球面*$x^2/a^2+y^2/b^2+z^2/c^2=1$*的内积长方体中，求体积为最大的那个长方体。*

1. (10%) *按Lagrange 方法进行有约束目标函数极值点的寻找*

2. (05%) *从极值到最值的有关说明（需先计算Hasse矩阵）*

4/(1)： 利用Lagrange乘子法求条件极值

$$L(x,y,z) \triangleq xyz - \lambda(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1)$$

$$\begin{cases} \frac{\partial L}{\partial x}(x,y,z) = yz - \frac{2\lambda}{a^2}x = 0 \\ \frac{\partial L}{\partial y}(x,y,z) = xz - \frac{2\lambda}{b^2}y = 0 \\ \frac{\partial L}{\partial z}(x,y,z) = xy - \frac{2\lambda}{c^2}z = 0 \end{cases}$$ ，即为临界点所满足的条件

$$\begin{cases} 3xyz = 2\lambda(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}) = 2\lambda \\ \frac{x^2}{a^2} = \frac{y^2}{b^2} = \frac{z^2}{c^2} = \frac{xyz}{2\lambda} = \frac{1}{3} \end{cases}$$

$$\begin{cases} x = \frac{a}{\sqrt{3}} \\ y = \frac{b}{\sqrt{3}} \\ z = \frac{c}{\sqrt{3}} \end{cases}$$

$$\lambda = \frac{3}{2}xyz = \frac{3}{2}\frac{abc}{3\sqrt{3}} = \frac{abc}{2\sqrt{3}}$$

4/(2)：

$$H_L(xyz) = \begin{bmatrix} -\frac{2\lambda}{a^2} & z & y \\ z & -\frac{2\lambda}{b^2} & x \\ y & x & -\frac{2\lambda}{c^2} \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{bc}{a\sqrt{3}} & \frac{c}{\sqrt{3}} & \frac{b}{\sqrt{3}} \\ \frac{c}{\sqrt{3}} & -\frac{ac}{b\sqrt{3}} & \frac{a}{\sqrt{3}} \\ \frac{b}{\sqrt{3}} & \frac{a}{\sqrt{3}} & -\frac{ab}{c\sqrt{3}} \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} -\frac{bc}{a} & c & b \\ c & -\frac{ac}{b} & a \\ b & a & -\frac{ab}{c} \end{bmatrix}$$

对此可基于问题的实际情况，说明最大值的存在性。

Problem 5 (曲面积分计算) *设有螺旋面，其向量值映照表示为：*

$$\Sigma(u,v) : \mathscr{D}_{uv} \ni \begin{bmatrix} u \\ v \end{bmatrix} \mapsto \Sigma(u,v) \triangleq \begin{bmatrix} u\cos v \\ u\sin v \\ v \end{bmatrix} \in \mathbb{R}^3$$

此处， $\mathscr{D}_{uv} \triangleq \{[u,v]^T \in \mathbb{R}^2 \,|\, u \subset [0,a],\, v \subset [0,2\pi]\}$ 。

1. (10%) *计算：* $\int_\Sigma z^2\, d\sigma$

2. (10%) *计算：* $\int_\Sigma (y\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}) \cdot \mathbf{n}\, d\sigma \equiv \int_\Sigma y\, dydz + x\, dzdx + z^2\, dxdy$

5/(1)：

$$\int_{\Sigma} z^2 d\sigma = \int_{\mathscr{D}_{u,v}} z^2(u,v) \left| \frac{\partial \Sigma}{\partial u} \times \frac{\partial \Sigma}{\partial v} \right|_{\mathbb{R}^3} (u,v) d\sigma$$

$$\frac{\partial \Sigma}{\partial u} \times \frac{\partial \Sigma}{\partial v}(u,v) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = \begin{bmatrix} \sin v \\ -\cos v \\ u \end{bmatrix}$$

$$\int_{\Sigma} z^2 d\sigma = \int_{\mathscr{D}_{uv}} v^2 \sqrt{1+u^2} d\sigma = \int_0^{2\pi} v^2 dv \int_0^a \sqrt{1+u^2} du = \frac{8\pi^3}{3} \int_0^a \sqrt{1+u^2} du$$

$$\begin{aligned} \int \sqrt{1+u^2} du &= u\sqrt{1+u^2} - \int u \frac{u}{\sqrt{1+u^2}} du = u\sqrt{1+u^2} - \int \frac{1+u^2-1}{\sqrt{1+u^2}} du \\ &= u\sqrt{1+u^2} - \int \sqrt{1+u^2} du + \int \frac{du}{\sqrt{1+u^2}} =: I \\ 2I &= u\sqrt{1+u^2} + \int \frac{du}{\sqrt{1+u^2}} \end{aligned}$$

对于$\int \frac{du}{\sqrt{1+u^2}}$令$u(\theta) = \tan\theta$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $u(\theta)$在$(-\frac{\pi}{2}, \frac{\pi}{2})$严格单调且连续，满足$u(-\frac{\pi}{2}, \frac{\pi}{2}) = \mathbb{R}$且存在$u'(\theta) = \sec^2\theta$, $\forall \theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$$\begin{aligned} \text{故} \int \frac{du}{\sqrt{1+u^2}} &= \int \frac{\sec^2\theta d\theta}{\sec\theta} = \int \frac{d\theta}{\cos\theta} \\ &= \int \frac{d\sin\theta}{(1-\sin^2\theta)} = \int (\frac{1}{1+\sin\theta} + \frac{1}{1-\sin\theta}) d\sin\theta \\ &= \ln\frac{1+\sin\theta}{1-\sin\theta} + c = \ln\frac{1+\frac{u}{\sqrt{1+u^2}}}{1-\frac{u}{\sqrt{1+u^2}}} + c = \ln\frac{\sqrt{1+u^2}+u}{\sqrt{1+u^2}-u} + c \\ &= 2\ln(u+\sqrt{1+u^2}) + c \end{aligned}$$

$$\text{故}\, I = \frac{1}{2}u\sqrt{1+u^2} + \ln(u+\sqrt{1+u^2}) + c \text{ 为} \sqrt{1+u^2} \text{ 在 } [0,a] \text{上的原函数}$$

$$\text{故} \int_{\Sigma} z^2 d\sigma = \frac{8\pi^3}{3} \left(\frac{1}{2}a\sqrt{1+a^2} + \ln(a+\sqrt{1+a^2}) \right)$$

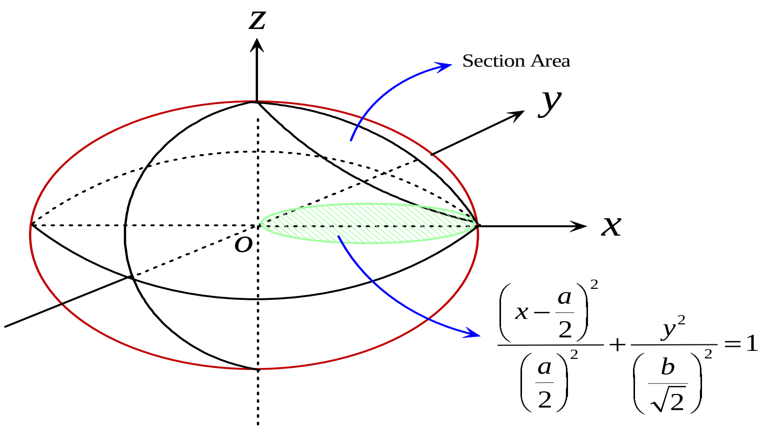
5/(2):

$$
\begin{aligned}
&\int_{\Sigma}(y\vec{i}+x\vec{j}+z^2\vec{k})\cdot\vec{n}d\sigma \\
&\quad=\int_{\mathscr{D}_{uv}}(y\vec{i}+x\vec{j}+z^2\vec{k})(u,v)\cdot\left(\frac{\partial\Sigma}{\partial u}\times\frac{\partial\Sigma}{\partial v}\right)(u,v)d\sigma \\
&\quad=\int_{\mathscr{D}_{uv}}\begin{bmatrix} u\sin v & u\cos v & v^2\end{bmatrix}\begin{bmatrix}\sin v\\ -\cos v\\ u\end{bmatrix}d\sigma \\
&\quad=\int_{\mathscr{D}_{uv}}\left[u^2(\sin^2 v-\cos^2 v)+uv^2\right]d\sigma \\
&\quad=\int_{\mathscr{D}_{uv}}(uv^2-u^2\cos 2v)d\sigma
\end{aligned}
$$

$$\int_{\mathscr{D}_{uv}}uv^2d\sigma=\int_0^a udu\int_0^{2\pi}v^2dv=\frac{8\pi^3}{3}\cdot\frac{a^2}{2}=\frac{4\pi a^2}{3}$$

$$\int_{\mathscr{D}_{uv}}u^2\cos 2vd\sigma=\int_0^a u^2du\int_0^{2\pi}\cos 2vdv=0$$

$$\text{故}\ \int_{\Sigma}(y\vec{i}+x\vec{j}+z^2\vec{k})\cdot\vec{n}d\sigma=\frac{4\pi a^2}{3}$$



Problem 6 (曲线积分计算) *如上图所示，计算曲线积分：*

$$\int_L[(y^2+z^2)\mathbf{i}+(z^2+x^2)\mathbf{j}+(x^2+y^2)\mathbf{k}]\cdot\tau dl\equiv\int_L(y^2+z^2)\,dx+(z^2+x^2)\,dy+(x^2+y^2)\,dz$$

此处曲线 L 为

$$\begin{cases}\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1\\ \frac{x}{a}+\frac{z}{c}=1\end{cases},\qquad x\ge 0,\,y\ge 0,\,z\ge 0$$

（$a>0,b>0,c>0$）的交线；积分指向面对于 x 轴为逆时针方向。消去 z，易得交线 L 在 xy 平面的投影为：

$$\frac{(x-\frac{a}{2})^2}{(\frac{a}{2})^2}+\frac{y^2}{(\frac{b}{\sqrt{2}})^2}=1$$

籍此可构造 L 的参数表示（向量值映照）。本问题可有以下两种解法：

1. (15%) *基于 $Stokes$ 公式计算上述积分。*

2. (10%) *构造 L 的向量值映照表示，并写出上述积分计算的表达式（无需完成全部计算）。*

6/(1)：

$$\begin{aligned}&\int_L [(y^2+z^2)\vec{i}+(z^2+x^2)\vec{j}+(x^2+y^2)\vec{k}]\cdot\vec{\tau}dl\\ &\quad=\int_\Sigma rot\,\vec{a}\cdot\vec{n}d\sigma\end{aligned}$$

$$rot\,\vec{a}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y^2+z^2 & z^2+x^2 & x^2+y^2\end{vmatrix}=\begin{bmatrix}2y-2z\\ 2z-2x\\ 2x-2y\end{bmatrix}=2\begin{bmatrix}y-z\\ z-x\\ x-y\end{bmatrix}$$

对于所截平面，利用Monge型表示，即为：

$$\Sigma(x,y):\mathscr{D}_{xy}\ni\begin{bmatrix}x\\ y\end{bmatrix}\longmapsto\Sigma(x,y)\triangleq\begin{bmatrix}x\\ y\\ c(1-\frac{x}{a})\end{bmatrix}$$

$$\frac{\partial\Sigma}{\partial x}\times\frac{\partial\Sigma}{\partial y}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & -\frac{c}{a}\\ 0 & 1 & 0\end{vmatrix}=\begin{bmatrix}\frac{c}{a}\\ 0\\ 1\end{bmatrix}$$

$$\begin{aligned}
\text{故}\, I &= \int_{\Sigma} rot\,\vec{a}\cdot\vec{n}d\sigma = \int_{\mathscr{D}_{xy}} 2[y-z,z-x,x-y](x,y)\cdot\begin{bmatrix}\frac{c}{a}\\ 0\\ 1\end{bmatrix}d\sigma\\
&= \frac{2c}{a}\int_{\mathscr{D}_{xy}}[y-c(1-\frac{x}{a})]d\sigma\\
&= \frac{2c^2}{a^2}\int_{\mathscr{D}_{xy}}(\frac{x}{a}-1)d\sigma\\
&= \frac{2c^2}{a^2}\left(\int_{\mathscr{D}_{xy}}\frac{x}{a}d\sigma-\pi\frac{ab}{2\sqrt{2}}\right)
\end{aligned}$$

$$\text{此处}\ \mathscr{D}_{xy}\triangleq\left\{\begin{bmatrix}x\\ y\end{bmatrix}\in\mathbb{R}^2\,\middle|\,\frac{(x-\frac{a}{2})^2}{(\frac{a}{2})^2}+\frac{y^2}{(\frac{b}{\sqrt{2}})^2}\le 1\right\}$$

$$\int_{\mathscr{D}_{xy}}\frac{x}{a}d\sigma=\frac{1}{a}\int_{\mathscr{D}_{xy}}xd\sigma$$

$$\text{作}\begin{cases}x=\frac{a}{2}+\frac{a}{2}r\cos\theta\\ y=\frac{b}{\sqrt{2}}r\sin\theta\end{cases}\quad\begin{cases}r\in[0,1]\\ \theta\in[0,2\pi]\end{cases}$$

$$\frac{D(x,y)}{D(r,\theta)}(r,\theta)=\begin{vmatrix}\frac{a}{2}\cos\theta & -\frac{a}{2}r\sin\theta\\ \frac{b}{\sqrt{2}}\sin\theta & \frac{b}{\sqrt{2}}r\cos\theta\end{vmatrix}=\frac{ab}{2\sqrt{2}}r$$

$$\text{故}\,\frac{1}{a}\int_{\mathscr{D}_{xy}}xd\sigma=\frac{1}{a}\int_{\mathscr{D}_{xy}}\left(\frac{a}{2}+\frac{a}{2}r\cos\theta\right)\frac{ab}{2\sqrt{2}}rd\sigma=\frac{ab}{4\sqrt{2}}\int_{\mathscr{D}_{xy}}rd\sigma=\frac{\pi ab}{4\sqrt{2}}$$

$$\text{综上}\quad I=\frac{2c^2}{a^2}\left(\frac{\pi ab}{4\sqrt{2}}-\frac{\pi ab}{2\sqrt{2}}\right)=-\frac{\pi bc^2}{2\sqrt{2}a}$$

6/(2)： 对于L的向量值映照

$$\begin{cases} x = \frac{a}{2} + \frac{a}{2}\cos\theta \\ y = \frac{b}{\sqrt{2}}\sin\theta \\ z = c(1 - \frac{x}{a}) = c(\frac{1}{2} - \frac{1}{2}\cos\theta) = \frac{c}{2}(1 - \cos\theta) \end{cases}$$

$$\theta \in [0, \frac{\pi}{2}]$$

由此 $\displaystyle\int_L \left[(y^2+z^2)\vec{i} + (z^2+x^2)\vec{j} + (x^2+y^2)\vec{k}\right] \cdot \vec{\tau} dl$

$$= \int_0^{2\pi} \left[(y^2+z^2)\vec{i} + (z^2+x^2)\vec{j} + (x^2+y^2)\vec{k}\right](\theta) \cdot \frac{d\vec{r}}{d\theta}(\theta) d\theta$$

此处 $\displaystyle\frac{d\vec{r}}{d\theta}(\theta) = D\vec{r}(\theta) = \begin{bmatrix} -\frac{a}{2}\sin\theta \\ \frac{b}{\sqrt{2}}\cos\theta \\ \frac{c}{2}\sin\theta \end{bmatrix} \in \mathbb{R}^3$

Problem 7 (幂级数基本性质) *已知幂级数*

$$S(x) = \sum_{n=1}^{\infty} \frac{x^n}{n \cdot 2^n}$$

1. (10%) *计算：* $S(x)$ *的收敛半径及收敛域。*
2. (10%) *计算：* $S(x)$ *的解析表达式。*

7/(1)：

$$S(x) = \sum_{n=1}^{\infty} \frac{x^n}{n \cdot 2^n}$$

则，考虑 $\displaystyle\frac{|x|^{n+1}}{(n+1)\cdot 2^{n+1}} \cdot \frac{n \cdot 2^n}{|x|^n} = |x| \cdot \frac{n}{n+1} \cdot \frac{1}{2} \longrightarrow \frac{|x|}{2} < 1$

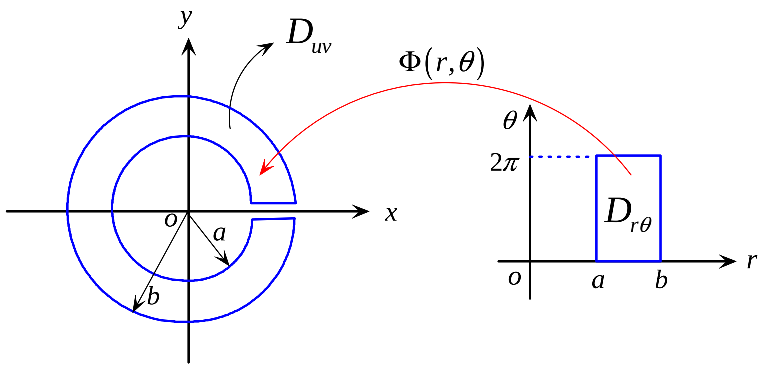
故收敛半径为2，再考虑 $\displaystyle a_n = \frac{2^n}{n \cdot 2^n} = \frac{1}{n}$

$$a_n = \frac{(-1)^n 2^n}{n \cdot 2^n} = \frac{(-1)^n}{n}$$

故 $S(x)$, $x \in [-2, 2)$为收敛域。

7/(2):

$$
\begin{aligned}
S(x) &= \sum_{n=1}^{+\infty} \frac{x^n}{n\cdot 2^n} \\
&= \frac{x}{2} + \sum_{n=2}^{+\infty} \frac{x^n}{n\cdot 2^n} \quad x\in[-2,2) \\
\text{考虑}\quad \frac{1}{2} + \sum_{n=2}^{+\infty} \frac{nx^{n-1}}{n\cdot 2^n} &= \frac{1}{2} + \sum_{n=2}^{+\infty} \frac{x^{n-1}}{2\cdot 2^{n-1}} \\
&= \frac{1}{2} + \sum_{n=1}^{+\infty} \frac{x^n}{2\cdot 2^n} \quad \text{收敛域为}(-2,2). \\
\text{故有}\quad \frac{dS}{dx}(x) &= \frac{1}{2}(1+\sum_{n=1}^{+\infty}(\frac{x}{2})^n) \quad \forall x\in(-2,2) \\
&= \frac{1}{2}\frac{1}{1-\frac{x}{2}} = \frac{1}{2-x} \\
S(x) &= S(0) + \int_0^x \frac{1}{2-\xi}d\xi \quad \forall x\in(-2,2) \\
&= -\int_0^x \frac{1}{2-\xi}d(2-\xi) = -\ln\frac{2-x}{2} = -\ln(1-\frac{x}{2}) \\
\text{即}\quad S(x) &= -\ln(1-\frac{x}{2}) \quad \forall x\in(-2,2) \\
\lim_{x\longrightarrow -2+0} -\ln(1-\frac{x}{2}) &= -\ln 2 = S(-2) \\
\text{故}\quad S(x) &= -\ln(1-\frac{x}{2}) \quad \forall x\in[-2,2)
\end{aligned}
$$



Problem 8 (区域变换) *$f(x,y)$ 在环型区域 $\mathscr{D}_{xy} \triangleq \{(x,y)\,|\,a^2\le x^2+y^2\le b^2$, 且在$x$轴正部有割缝$\}$（$a$，$b$均为正数）上有定义，且满足 Laplace 方程，即有*

$$
\frac{\partial^2 f}{\partial x^2}(x,y) + \frac{\partial^2 f}{\partial y^2}(x,y) = 0
$$

考虑极坐标变换:

$$
\Phi(r,\theta):\ \mathscr{D}_{r\theta} \ni \begin{bmatrix} r \\ \theta \end{bmatrix} \mapsto \Phi(r,\theta) \triangleq \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix} \in \mathbb{R}^2
$$

此处$\mathscr{D}_{r\theta} \triangleq \{(r,\theta)\,|\,r \subset (a,b),\, \theta \subset (0,2\pi)\}$，则实现环形区域$\mathscr{D}_{xy}$同矩形区域$\mathscr{D}_{r\theta}$间的一一对应的变换。如上图所示。

1. (10%) 计算：Φ逆映照的$Jacobian$矩阵。

2. (10%) 计算：$\hat{f}(r,\theta) \triangleq f(x(r,\theta), y(r,\theta))$所满足的方程。提示：$f(x,y) = \hat{f}(r(x,y), \theta(x,y))$。

8/(1)：

$$D\Phi(r,\theta) = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}$$

8/(2)：

$$D\Phi(r,\theta) = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} \quad detD\Phi(r,\theta) = r$$

则
$$\begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix}^{-} = \frac{1}{r}\begin{bmatrix} r\cos\theta & -\sin\theta \\ r\sin\theta & \cos\theta \end{bmatrix}^T$$

$$= \frac{1}{r}\begin{bmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\frac{\sin\theta}{r} & \frac{\cos\theta}{r} \end{bmatrix}$$

由 $f(x,y)=\hat f(r(x,y),\theta(x,y))$

$$\frac{\partial f}{\partial x}=\frac{\partial \hat f}{\partial r}\frac{\partial r}{\partial x}+\frac{\partial \hat f}{\partial \theta}\frac{\partial \theta}{\partial x}=\frac{\partial \hat f}{\partial r}\cos\theta-\frac{\partial \hat f}{\partial \theta}\frac{\sin\theta}{r}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2}&=\left(\frac{\partial^2 \hat f}{\partial r^2}\frac{\partial r}{\partial x}+\frac{\partial^2 \hat f}{\partial \theta\partial r}\frac{\partial \theta}{\partial x}\right)\cos\theta+\frac{\partial \hat f}{\partial r}(-\sin\theta)\frac{\partial \theta}{\partial x}\\
&-\left(\frac{\partial \hat f}{\partial r\partial \theta}\frac{\partial r}{\partial x}+\frac{\partial^2 \hat f}{\partial \theta^2}\frac{\partial \theta}{\partial x}\right)\frac{\sin\theta}{r}-\frac{\partial \hat f}{\partial \theta}\left(-\frac{\sin\theta}{r^2}\frac{\partial r}{\partial x}+\frac{\cos\theta}{r}\frac{\partial \theta}{\partial x}\right)\\
&=\left(\frac{\partial^2 \hat f}{\partial r^2}\cos\theta-\frac{\partial^2 \hat f}{\partial \theta\partial r}\frac{\sin\theta}{r}\right)\cos\theta+\frac{\partial \hat f}{\partial r}(-\sin\theta)\left(-\frac{\sin\theta}{r}\right)\\
&-\left(\frac{\partial^2 \hat f}{\partial r\partial \theta}\cos\theta-\frac{\partial^2 \hat f}{\partial \theta^2}\frac{\sin\theta}{r}\right)\frac{\sin\theta}{r}-\frac{\partial \hat f}{\partial \theta}\left(-\frac{\sin\theta}{r^2}\cos\theta-\frac{\cos\theta}{r}\frac{\sin\theta}{r}\right)\\
&=\frac{\partial^2 \hat f}{\partial r^2}\cos^2\theta+\frac{\partial^2 \hat f}{\partial \theta^2}\frac{\sin^2\theta}{r^2}-\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 \hat f}{\partial r\partial \theta}\\
&+\frac{\sin^2\theta}{r}\frac{\partial \hat f}{\partial r}+\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial \hat f}{\partial \theta}
\end{aligned}$$

$$\frac{\partial f}{\partial y}=\frac{\partial \hat f}{\partial r}\frac{\partial r}{\partial y}+\frac{\partial \hat f}{\partial \theta}\frac{\partial \theta}{\partial y}=\frac{\partial \hat f}{\partial r}\sin\theta+\frac{\partial \hat f}{\partial \theta}\frac{\cos\theta}{r}$$

$$\begin{aligned}
\frac{\partial^2 f}{\partial y^2}&=\left[\frac{\partial^2 \hat f}{\partial r^2}\frac{\partial r}{\partial y}+\frac{\partial^2 \hat f}{\partial \theta\partial r}\frac{\partial \theta}{\partial y}\right]\sin\theta+\frac{\partial \hat f}{\partial r}\cos\theta\frac{\partial \theta}{\partial y}\\
&+\left[\frac{\partial^2 \hat f}{\partial r\partial \theta}\frac{\partial r}{\partial y}+\frac{\partial^2 \hat f}{\partial \theta^2}\frac{\partial \theta}{\partial y}\right]\frac{\cos\theta}{r}+\frac{\partial \hat f}{\partial \theta}\left(-\frac{\cos\theta}{r^2}\frac{\partial r}{\partial y}-\frac{\sin\theta}{r}\frac{\partial \theta}{\partial y}\right)\\
&=\left[\frac{\partial^2 \hat f}{\partial r^2}\sin\theta+\frac{\partial^2 \hat f}{\partial \theta\partial r}\frac{\cos\theta}{r}\right]\sin\theta+\frac{\partial \hat f}{\partial r}\cos\theta\frac{\cos\theta}{r}\\
&+\left[\frac{\partial^2 \hat f}{\partial r\partial \theta}\sin\theta+\frac{\partial^2 \hat f}{\partial \theta^2}\frac{\cos\theta}{r}\right]\frac{\cos\theta}{r}+\frac{\partial \hat f}{\partial \theta}\left(-\frac{\cos\theta}{r^2}\sin\theta-\frac{\sin\theta}{r}\frac{\cos\theta}{r}\right)\\
&=\sin^2\theta\frac{\partial^2 \hat f}{\partial r^2}+\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 \hat f}{\partial r\partial \theta}+\frac{\cos^2\theta}{r^2}\frac{\partial^2 \hat f}{\partial \theta^2}\\
&+\frac{\cos^2\theta}{r}\frac{\partial \hat f}{\partial r}-\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial \hat f}{\partial \theta}
\end{aligned}$$

综上 $\left(\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}\right)(x,y)=\frac{\partial^2 \hat f}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 \hat f}{\partial \theta^2}+\frac{1}{r}\frac{\partial \hat f}{\partial r}$