

复旦大学航空航天系与技术科学类

2018-2019 学年第一学期《数学分析 B》一元微分学一般教学测试

A 卷 共 11 页

课程代码:

考试形式: ☐ 开卷 ☐ 闭卷 2018 年 11 月 02 日 7:30-9:40

(本试卷答卷时间为 130 分钟, 答案必须写在试卷上, 做在草稿纸上无效)

专业: _____ 学号: _____ 姓名: _____ 成绩: _____

题号	1-1	1-2	1-3	2-1	2-2	2-3	2-4	3-1	3-2	3-3
得分										
题号	3-4	3-5	3-6	3-7	3-8	3-9	3-10	4-1	4-2	4-3
得分										
题号	5-1	5-2	5-3						总分	百分
得分										

一、严格表述题 (每题 5 分, 共 3 题, 共 15 分)

1. 叙述: 函数极限的集聚刻画、序列刻画、振幅刻画.

解: ①集聚刻画: $\forall \varepsilon > 0, \exists \delta_\varepsilon > 0$, 成立 $f(x) \in B_\varepsilon(y_0), \forall x \in \overset{\circ}{B}_{\delta_\varepsilon}(x_0) \cap \mathcal{D}_x$.

②序列刻画: $\forall \{x_n\} \subset \mathcal{D}_x \setminus \{x_0\}, x_n \rightarrow x_0 \in \overline{\mathbb{R}}$, 有 $f(x_n) \rightarrow y_0 \in \overline{\mathbb{R}}$

③振幅刻画: $\forall \varepsilon > 0, \exists \delta_\varepsilon > 0$, 成立 $|f(\tilde{x}_n) - f(\hat{x}_n)| < \varepsilon, \forall \tilde{x}_n, \hat{x}_n \in \overset{\circ}{B}_{\delta_\varepsilon}(x_0) \cap \mathcal{D}_x$

2. 叙述: 反函数的存在性 (连续性) 定理与可导性定理.

解: ①反函数存在性定理: 设 $f(x)$ 在 $[a, b]$ 上 \uparrow , 则有 $f([a, b]) = [f(a), f(b)] \Leftrightarrow f(x) \in C[a, b]$

②进一步设有 $\exists \frac{df}{dx}(x_0) \neq 0$, 则有 $\exists \frac{df^{-1}}{dy}(y_0) = \frac{1}{\frac{df}{dx}(x_0)}, y_0 = f(x_0)$.

3. 叙述: 有界函数上、下极限的定义.

解: ①上极限: $\overline{\lim} x_n \triangleq \lim_{n \rightarrow +\infty} \sup_{k \geq n} x_k = \inf_n \sup_{k \geq n} x_k$

②下极限: $\underline{\lim} x_n \triangleq \lim_{n \rightarrow +\infty} \inf_{k \geq n} x_k = \sup_n \inf_{k \geq n} x_k$

二、判断简答题（判断下列命题是否正确，如果是正确的，请回答“是”，并予以简要证明；如果是错误的，请回答“否”，并举反例。答“是或否”2分，简要证明或举反例3分）（每题5分，共4题，共20分）

1. ①函数在一点单侧连续，则其在该点单侧可导.

②函数在一点单侧可导，则其在该点单侧连续.

解：①否. 考虑 $f(x) = \begin{cases} x \sin \frac{1}{x}, & x > 0 \\ 0, & x = 0 \end{cases}$. 有 $\exists f(0+0) = \lim_{x \rightarrow 0+0} x \sin \frac{1}{x} = 0$.

而 $f'_+(0) \triangleq \lim_{x \rightarrow 0+0} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0+0} \frac{x \sin \frac{1}{x}}{x} = \lim_{x \rightarrow 0+0} \sin \frac{1}{x}$ 不存在.

故不单侧可导.

②是. 设有 $\exists f'_+(0) = \lim_{x \rightarrow 0+0} \frac{f(x) - f(0)}{x} \in \mathbb{R}$, 则有

$f(x) = f(0) + f'_+(0)x + x \cdot o(1) \rightarrow f(0)$, 当 $x \rightarrow 0+0$.

亦即有单侧连续性.

2. ①设函数 $f(x), g(x)$ 在 (a, b) 上一致连续, 则有 $(f \cdot g)(x)$ 在 (a, b) 上一致连续.

②设函数 $f(x), g(x)$ 在 $(a, +\infty)$ 上一致连续, 则有 $(f \cdot g)(x)$ 在 $(a, +\infty)$ 上一致连续.

解：①是. 此处设 (a, b) 有界, $f(x), g(x)$ 在 (a, b) 上一致连续, 即

$$\begin{cases} \exists f(a+0), g(a+0) \in \mathbb{R} \\ \exists f(b-0), g(b-0) \in \mathbb{R} \end{cases}, \text{ 故有 } \begin{cases} \exists (f \cdot g)(a+0) = f(a+0) \cdot g(a+0) \\ (f \cdot g)(b-0) = f(b-0) \cdot g(b-0) \end{cases}.$$

由此, $(f \cdot g)(x)$ 在 (a, b) 上一致连续.

②否. 反例 $f(x) = g(x) = x$, 有 x^2 在 $(a, +\infty)$ 上不一致连续.

$$\text{考虑 } \begin{cases} \tilde{x}_n = \sqrt{x+\lambda} \\ \hat{x}_n = \sqrt{n} \end{cases}, \text{ s.t. } \begin{cases} \tilde{x}_n - \hat{x}_n = \sqrt{x+\lambda} - \sqrt{n} = \sqrt{n} \left(1 + \frac{1}{2} \frac{\lambda}{n} + o\left(\frac{1}{n}\right) \right) - \sqrt{n} \\ \tilde{x}_n^2 - \hat{x}_n^2 = \lambda \end{cases}$$

3. $f(x)$ 在 $[a, b]$ 上单调下降, 如有 $f([a, b]) = [f(b), f(a)]$, 则有 $f(x) \in C[a, b]$.

解: 是. 由于 $f(x)$ 单调下降, 则有

$$\begin{cases} \exists f(x_0 - 0) = \inf_{[a, x_0)} f(x) \\ \exists f(x_0 + 0) = \sup_{(x_0, b]} f(x) \end{cases}$$

设 $f(x)$ 在 x_0 点不连续, 则有 $f(x_0 - 0), f(x_0), f(x_0 + 0)$ 至少有两点分离.

由此, $f([a, b]) \subset [f(b), f(a)]$ 为真包含.

4. 考虑 $E \subset \mathbb{R}$ (可以有界或者无界) 上任意的二个序列 $\{\tilde{x}_n\}, \{\hat{x}_n\}$, 当 $\exists \lim_{n \rightarrow +\infty} |\tilde{x}_n - \hat{x}_n| = 0$ 时, 有 $\exists \lim_{n \rightarrow +\infty} |f(\tilde{x}_n) - f(\hat{x}_n)| = 0$, 则有 $f(x)$ 在 E 上一致连续.

解: 是. 采用反证法. 假设 $f(x)$ 在 E 上不一致连续, 则有

$$\exists \varepsilon_* > 0, \forall \delta > 0, \exists \tilde{x}_\delta, \hat{x}_\delta \in E, \text{ s.t. } \begin{cases} |\tilde{x}_\delta - \hat{x}_\delta| < \delta \\ |f(\tilde{x}_\delta) - f(\hat{x}_\delta)| \geq \varepsilon_* \end{cases}$$

$$\text{可取 } \delta_n = \frac{1}{n}, \text{ 则有 } \exists \{\tilde{x}_n\}, \{\hat{x}_n\} \subset E, \text{ s.t. } \begin{cases} |\tilde{x}_n - \hat{x}_n| < \frac{1}{n} \\ |f(\tilde{x}_n) - f(\hat{x}_n)| \geq \varepsilon_* \end{cases}$$

此时 $\exists \lim_{n \rightarrow +\infty} (\tilde{x}_n - \hat{x}_n) = 0$, 而 $f(\tilde{x}_n) - f(\hat{x}_n) \not\rightarrow 0$.

故与题设矛盾.

三、计算证明题（每题 10 分，共 6 题，共 60 分）

1. 计算 $\lim_{x \rightarrow 0} \frac{\cos(\sin x) - \cos x}{x^4}$

解： 分子部分需要展开至 $o(x^4)$

$$\begin{aligned}\cos(\sin x) &= 1 - \frac{\sin^2 x}{2!} + \frac{\sin^4 x}{4!} + o(\sin^5 x) \\&= 1 - \frac{1}{2!} \left(x - \frac{x^3}{3!} + o(x^4) \right)^2 + \frac{1}{4!} (x + o(x^4))^4 + o(x^5) \\&= 1 - \frac{1}{2}x^2 + \left(\frac{1}{3!} + \frac{1}{4!} \right) x^4 + o(x^4) \\ \cos x &= 1 - \frac{1}{2}x^2 + \frac{1}{4!}x^4 + o(x^4)\end{aligned}$$

故有 $\cos(\sin x) - \cos x = \frac{1}{3!}x^4 + o(x^4)$

2.0ex] 故有原极限为 $\frac{1}{3!} = \frac{1}{6}$.

2. 在 $x = 0$ 处 (相应的邻域), 展开 $\ln(x + \sqrt{x^2 + 1})$ 至 $o(x^3)$

解： $f(x) = \ln(x + \sqrt{x^2 + 1})$, 有

$$f'(x) = \frac{1}{x + \sqrt{x^2 + 1}} \cdot \left(1 + \frac{x}{\sqrt{x^2 + 1}} \right) = \frac{1}{\sqrt{x^2 + 1}} = (1 + x^2)^{\frac{1}{2}} = 1 + \frac{1}{2}x^2 + o(x^3).$$

有 $f(x) = f(0) + x - \frac{1}{6}x^3 + o(x^4) = x - \frac{1}{6}x^3 + o(x^4)$.

3. 计算 $\lim_{n \rightarrow +\infty} n^2(\sqrt[n]{a} - \sqrt[n+1]{a}), a > 0$

解:

$$\begin{aligned}
 n^2(\sqrt[n]{a} - \sqrt[n+1]{a}) &= n^2(e^{\frac{1}{n} \ln a} - e^{\frac{1}{n+1} \ln a}) \\
 &= n^2 \cdot \left[1 + \frac{1}{n} \ln a + \frac{1}{2n^2} \ln^2 a + o\left(\frac{1}{n^2}\right) - 1 - \frac{1}{n+1} \ln a - \frac{1}{2(n+1)} \ln^2 a + o\left(\frac{1}{n^2}\right) \right] \\
 &= n^2 \cdot \left[\left(\frac{1}{n} - \frac{1}{n+1}\right) \ln a + \frac{1}{2} \left(\frac{1}{n^2} - \frac{1}{(n+1)^2}\right) \ln^2 a + o\left(\frac{1}{n^2}\right) \right] \\
 &= n^2 \cdot \left[\frac{1}{n} \left(1 - \frac{1}{1 + \frac{1}{n}}\right) \ln a + \frac{1}{2n^2} \left(1 - \frac{1}{(1 + \frac{1}{n})^2}\right) \ln^2 a + o\left(\frac{1}{n^2}\right) \right] \\
 &= n^2 \cdot \left[\frac{1}{n} \left(\frac{1}{n} + o\left(\frac{1}{n}\right)\right) \ln a + \frac{1}{2n^2} o(1) \ln^2 a + o\left(\frac{1}{n^2}\right) \right] = \ln a + o(1) \rightarrow \ln a
 \end{aligned}$$

4. 计算函数 $y = \left[\frac{\arcsin(\sin^2 x)}{\arccos(\cos^2 x)} \right]^{\arctan^2 x}$ 的一阶导数.

解: $y(x) = e^{\arctan^2 x \ln \frac{\arcsin(\sin^2 x)}{\arccos(\cos^2 x)}}$

$$\begin{aligned}
 \frac{dy}{dx}(x) &= y(x) \frac{d}{dx} \left[\arctan^2 x \ln \frac{\arcsin(\sin^2 x)}{\arccos(\cos^2 x)} \right] \\
 &= y(x) \left[\frac{d}{dx} \arctan^2 x \cdot \ln \frac{\arcsin(\sin^2 x)}{\arccos(\cos^2 x)} + \arctan^2 x \frac{d}{dx} \ln \frac{\arcsin(\sin^2 x)}{\arccos(\cos^2 x)} \right]
 \end{aligned}$$

式中 $\frac{d}{dx} \arctan^2 x = 2 \arctan x \cdot \frac{1}{1+x^2}$

$$\begin{aligned}
 \frac{d}{dx} \ln \frac{\arcsin(\sin^2 x)}{\arccos(\cos^2 x)} &= \frac{1}{\arcsin(\sin^2 x)} \frac{d}{dx} \arcsin(\sin^2 x) - \frac{1}{\arccos(\cos^2 x)} \frac{d}{dx} \arccos(\cos^2 x) \\
 &= \frac{1}{\arcsin(\sin^2 x)} \cdot \frac{\sin 2x}{\sqrt{1 - \sin^4 x}} - \frac{1}{\arccos(\cos^2 x)} \frac{\sin 2x}{\sqrt{1 - \cos^4 x}}
 \end{aligned}$$

5. 计算 $f(x) = \begin{cases} x \left| \sin \frac{\pi}{x} \right|, & x \neq 0 \\ 0, & x = 0 \end{cases}$ 在零点的导数, 并说明: 零点的任意邻域都有不可导点.

解: ①易见 $f(x)$ 在 0 点连续.

② $\lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0} \left| \sin \frac{\pi}{x} \right|$, 极限不存在, 亦即 $f(x)$ 在 $x = 0$ 点不可导.

③考虑 $|\sin y|$, $y = \frac{\pi}{x} = 2n\pi + \pi \Rightarrow x_n = \frac{1}{2n+1}$.

$$\text{考虑} \begin{cases} f'_+(x_n \pm 0) = \lim_{x \rightarrow x_n+0} \frac{x \left| \sin \frac{\pi}{x} \right|}{x - x_n} = \lim_{x \rightarrow x_n+0} \frac{x \sin \frac{\pi}{x}}{x - x_n} = +\infty \\ f'_-(x_n \pm 0) = \lim_{x \rightarrow x_n-0} \frac{x \left| \sin \frac{\pi}{x} \right|}{x - x_n} = \lim_{x \rightarrow x_n-0} \frac{-x \sin \frac{\pi}{x}}{x - x_n} = -\infty \end{cases}$$

亦即 $f(x)$ 在 $x_n = \frac{1}{2n+1} (n \in \mathbb{N})$ 点均不可导.

6. 计算 $\lim_{n \rightarrow +\infty} \frac{n}{a^{n+1}} \left(a + \frac{a^2}{2} + \frac{a^3}{3} + \cdots + \frac{a^n}{n} \right) \quad (a > 1)$

解: 考虑到 $\frac{a^{n+1}}{n} \rightarrow +\infty$, 当 $n \rightarrow +\infty, a > 1$.

故考虑 Stolz 定理

$$\frac{a + \frac{a^2}{2} + \frac{a^3}{3} + \cdots + \frac{a^n}{n}}{a^{n+1}n} \sim \frac{\frac{a^{n+1}}{n+1}}{\frac{a^{n+2}}{n+1} - \frac{a^{n+1}}{n}} = \frac{n}{a \cdot n - (n+1)} = \frac{1}{a - (1 + \frac{1}{n})} \rightarrow \frac{1}{a-1}$$

7. ①证明: 当 $x \geq 0$ 时有 $\sqrt{x+1} - \sqrt{x} = \frac{1}{2\sqrt{x+\theta(x)}}$, 式中 $\theta(x) \in (0, 1)$.

②证明: $\exists \lim_{x \rightarrow 0+0} = \frac{1}{4}, \exists \lim_{x \rightarrow +\infty} = \frac{1}{2}$.

解: ①利用有限增量公式/(现为 Lagrange 中值定理)

$$\sqrt{x+1} - \sqrt{x} = \frac{d}{dy} \sqrt{y} \Big|_{y=\xi} \cdot 1 = \frac{1}{2} \cdot \frac{1}{\sqrt{\xi}} = \frac{1}{2} \frac{1}{\sqrt{x+\theta(x)}}$$

②当 $x \rightarrow 0+0$

$$\begin{cases} \sqrt{x+1} - \sqrt{x} = 1 - \sqrt{x} \frac{1}{2}x + o(x) \rightarrow 1 \\ \frac{1}{2} \cdot \frac{1}{\sqrt{x+\theta(x)}} \rightarrow \frac{1}{2} \frac{1}{\lim_{x \rightarrow 0+0} \theta(x)} \end{cases} \Rightarrow \lim_{x \rightarrow 0+0} \theta(x) = \frac{1}{4}.$$

当 $x \rightarrow +\infty$,

$$\begin{aligned} \sqrt{x+1} - \sqrt{x} &= \sqrt{x} \left[\left(1 + \frac{1}{x}\right)^{\frac{1}{2}} - 1 \right] \\ &= \sqrt{x} \left[1 + \frac{1}{2} \cdot \frac{1}{x} - \frac{1}{8} \cdot \frac{1}{x^{\frac{3}{2}}} + o\left(\frac{1}{x^{\frac{3}{2}}}\right) - 1 \right] \\ &= \frac{1}{2} \frac{1}{\sqrt{x}} - \frac{1}{8} \frac{1}{x^{\frac{3}{2}}} + o\left(\frac{1}{x^{\frac{3}{2}}}\right) \end{aligned}$$

另有

$$\begin{aligned} \frac{1}{2\sqrt{x+\theta(x)}} &= \frac{1}{2\sqrt{x}} \cdot \left(1 + \frac{\theta(x)}{x}\right)^{-\frac{1}{2}} \\ &= \frac{1}{2\sqrt{x}} \cdot \left(1 - \frac{1}{2} \cdot \frac{\theta(x)}{x} + o\left(\frac{1}{x}\right)\right) \\ &= \frac{1}{2\sqrt{x}} - \frac{1}{4} \frac{\theta(x)}{x^{\frac{3}{2}}} + o\left(\frac{1}{x^{\frac{3}{2}}}\right) \end{aligned}$$

故有: $-\frac{1}{8} \cdot \frac{1}{x^{\frac{3}{2}}} = -\frac{1}{4} \frac{\theta(x)}{x^{\frac{3}{2}}} + o\left(\frac{1}{x^{\frac{3}{2}}}\right) \Rightarrow -\frac{1}{8} = -\frac{1}{4} \theta(x) + o(1) \Rightarrow \exists \lim_{x \rightarrow +\infty} \theta(x) = \frac{1}{2}$

8. 设 $|x_1| \leq 2, x_{n+1} = \sqrt{4-x_n} (n \in \mathbb{N})$, 求: $\lim_{n \rightarrow +\infty} x_n$

解: ①分析有界性: 显然 $\{x_n\}$ 有界且非负.

②计算上下极限:

$$\begin{cases} \overline{x_{n+1}} = \sqrt{4 - \underline{x_n}} = \sqrt{4 - \underline{x_n}} \\ \underline{x_{n+1}} = \sqrt{4 - \overline{x_n}} = \sqrt{4 - \overline{x_n}} \end{cases}, \text{ 故有 } \overline{x_n} = \underline{x_n} = \frac{-1 + \sqrt{17}}{2}.$$

9. 设 $f(x) \in C[a, b]$, 在 (a, b) 上可导, 且 $f(0) = f(1) = 0$, $f\left(\frac{1}{2}\right) = 1$,

①证明: 存在 $\xi \in \left(\frac{1}{2}, 1\right)$, $f(\xi) = \xi$

②证明: 对 $\forall \lambda \in \mathbb{R}$, $\exists \eta \in (0, \xi)$, s.t. $f'(\eta) - \lambda[f(\eta) - \eta] = 1$

解: ①作 $\varphi(x) = f(x) - x$, s.t. $\begin{cases} \varphi(\frac{1}{2}) = f(\frac{1}{2}) - \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2} \\ \varphi(1) = f(1) - 1 = -1 \end{cases}$.

按 Rolle 定理, 有 $\exists \xi \in (\frac{1}{2}, 1)$, $\varphi(\xi) = 0$, 亦即 $f(\xi) = \xi$.

②考虑到 $f'(\eta) - \lambda f(\eta) = \frac{\frac{d}{dx}(e^{-\lambda\eta} f(\eta))}{e^{-\lambda\eta}}$ 以及 $e^{-\lambda\eta}(1 - \lambda\eta) = \frac{d}{dx}(\eta e^{-\lambda\eta})$.

故作 $\varphi(x) = e^{-\lambda x} f(x) - e^{-\lambda x} x = e^{-\lambda x}(f(x) - x)$, s.t. $\varphi(0) = 0$, $\varphi(\xi) = 0$.

按 Rolle 定理, 有 $\exists \eta \in (0, \xi)$, $\varphi'(\eta) = 0$, 即得证.

10. ①设 $\varphi(x) \in C[0, +\infty)$, $\varphi(0) = 0$, $\exists \lim_{x \rightarrow +\infty} \varphi(x) = +\infty$, 研究 $\exists \lim_{x \rightarrow +\infty} \varphi(x) \sin x$ 的存在性.

②研究: $\cos(\varphi(x) \sin x)$ 在 \mathbb{R}^+ 上的一致连续性.

解: ① 考虑 $x_n = 2n\pi + \delta_n$, 有

$$\begin{aligned} \varphi(x_n) \sin x_n &= \varphi(2n\pi + \delta_n) \sin(+\delta_n) \\ &= \varphi(2n\pi + \delta_n)(\delta_n + o(\delta_n)) \\ &= \lambda + o(1), \forall \lambda \in \mathbb{R}^+. \end{aligned}$$

求解 $\varphi(2n\pi + \delta_n) \cdot \delta_n = \lambda$.

考虑 $\varphi(2n\pi + n) = \frac{\lambda}{x}$. 按介值定理 $\exists \delta_n \in (0, +\infty)$, s.t. $\varphi(2n\pi + \delta_n) = \frac{\lambda}{\delta_n}$.

且有 $\delta_n = \frac{\lambda}{\varphi(2n\pi - \frac{\pi}{2} + \delta_n)} \rightarrow 0$, 当 $n \rightarrow +\infty$.

② 基于结论①, $\lambda, \mu \in \mathbb{R}^+$, 可有 $\begin{cases} \tilde{x}_n = 2n\pi + \tilde{\delta}_n, \varphi(\tilde{x}_n) \sin \tilde{x}_n \rightarrow \tilde{\lambda} \in \mathbb{R}^+ \\ \hat{x}_n = 2n\pi + \hat{\delta}_n, \varphi(\hat{x}_n) \sin \hat{x}_n \rightarrow \hat{\lambda} \in \mathbb{R}^+ \end{cases}$

s.t. $\begin{cases} \cos(\varphi(\tilde{x}_n) \sin \tilde{x}_n) \rightarrow \cos \tilde{\lambda} \\ \cos(\varphi(\hat{x}_n) \sin \hat{x}_n) \rightarrow \cos \hat{\lambda} \end{cases}$, 且 $\tilde{\delta}_n - \hat{\delta}_n \rightarrow 0$.

故有 $\cos(\varphi(x) \sin x)$ 在 \mathbb{R}^+ 上不一致连续.

四、应用题（每题 10 分，共 1 题，共 10 分）

1. 数列的上下极限.

① 设 $\{x_n\}$ 有界, 则有结论: 存在子列分别集聚至上、下极限. 证明: 集聚至上极限的结论.

② 证明: 对于所有的收敛之列的极限都不小于下极限、不大于上极限.

③ 证明: 设 $\exists \lim_{n \rightarrow +\infty} (x_{n+1} + \lambda x_n) = l \in \mathbb{R}, 0 < \lambda < 1$, 分析 $\exists \lim_{n \rightarrow +\infty} x_n$ 的存在性, 如果极限存在则求出极限值.

解: ① 证明 $\exists x_{n_k} \rightarrow \overline{\lim} x_n \triangleq \lim_{n \rightarrow +\infty} \sup_{k \geq n} x_k$

$\exists x_{n_{m_l}} \in \left(\overline{\lim} x_n - \frac{1}{m}, \overline{\lim} x_n + \frac{1}{m} \right)$, 故有 $x_{n_{m_l}} \rightarrow \overline{\lim} x_n$, 当 $l \rightarrow +\infty$, 当 $m \rightarrow +\infty$.

式中 $n_{1_l} < n_{2_l} < \cdots < n_{m_l} < \cdots$.

② 设 $x_{n_k} \rightarrow x_*$, 当 $k \rightarrow +\infty$.

$$\inf_{j \geq n_k} x_j \leq x_{n_k} \leq \sup_{j \geq n_k} x_j, \forall k \in \mathbb{N}$$

$$\Rightarrow \underline{\lim} x_n = \lim_{k \rightarrow +\infty} \inf_{j \geq n_k} x_j \leq \lim_{k \rightarrow +\infty} x_{n_k} \leq \lim_{k \rightarrow +\infty} \sup_{j \geq n_k} x_j = \overline{\lim} x_n$$

③ 设 $\exists \lim(\lambda x_{n+1} + x_n) = l \in \mathbb{R}, \lambda > 1$, 则有

$$|\lambda x_{n+1}| \leq |x_n| + (|l| + \varepsilon), n > N_\varepsilon \Rightarrow |x_{n+1}| \leq \frac{1}{\lambda} |x_n| + \frac{|l| + \varepsilon}{|\lambda|}, \forall n > N_\varepsilon. \text{ 则 } \{x_n\} \text{ 有界.}$$

$y_n := \lambda x_{n+1} + x_n$, 计算上下极限, 有:

$$\underline{y}_n = \underline{\lambda x_{n+1} + x_n} \sim \begin{cases} \geq \underline{\lambda x_{n+1} + x_n} \\ \leq \begin{cases} \underline{\lambda x_{n+1} + \overline{x_n}} \\ \underline{\lambda \overline{x_{n+1}} + \underline{x_n}} \end{cases} \leq \lambda \overline{x_{n+1}} + \overline{x_n} \end{cases}.$$

$$\overline{y}_n = \overline{\lambda x_{n+1} + x_n} \sim \begin{cases} \leq \lambda \overline{x_{n+1}} + \overline{x_n} \\ \geq \begin{cases} \lambda \overline{x_{n+1}} + \underline{x_n} \\ \lambda \underline{x_{n+1}} + \overline{x_n} \end{cases} \geq \lambda \underline{x_{n+1}} + \underline{x_n} \end{cases}.$$

$$\text{式中 } \underline{y}_n = \overline{y}_n = l, \text{ 且有 } \begin{cases} (1 + \lambda) \underline{x_n} = l \\ (1 + \lambda) \overline{x_n} = l \end{cases}, \text{ 亦即: } \underline{x_n} = \overline{x_n} = \frac{l}{1 + \lambda}.$$

五、应用题（每题 9 分，共 1 题，共 9 分）

1. Stolz 定理与 Bernoulli-L'Hospital 法则的通识性结构.

①阐述 Stolz 定理与 Bernoulli-L'Hospital 法则的一般形式.

②设定数列差比的极限、函数导数之比的形式为有限值, 给出 Stolz 定理与 Bernoulli-L'Hospital 法则分析中所建立的关系式.

③就 $\frac{0}{0}$ -型与 $\frac{*}{\infty}$ -型, 获得最终的结论.

解: ①Stolz 定理: $\frac{y_n}{x_n} \sim \frac{y_{n+1} - y_n}{x_{n+1} - x_n} \rightarrow l \in \mathbb{R} \cup \{\pm\infty\}$.

式中 $\begin{cases} y_n \rightarrow 0 \\ x_n \downarrow 0 \end{cases} \quad (\frac{0}{0} \text{ 型}); \text{ 或 } x_n \uparrow +\infty (\frac{*}{\infty} \text{ 型})$

Bernoulli-L'Hospital 法则: $\frac{\psi(x)}{\varphi(x)} \sim \frac{\psi'(x)}{\varphi'(x)} \rightarrow l \in \mathbb{R} \cup \{\pm\infty\}$

式中 $\begin{cases} \psi(x) \rightarrow 0 \\ \varphi(x) \rightarrow 0 \end{cases}, \text{ 当 } x \rightarrow x_0 \in \overline{\mathbb{R}};$

或 $\varphi(x) \rightarrow +\infty$, 当 $x \rightarrow x_0 \in \overline{\mathbb{R}}$, 要求 $\exists \psi'(x), \psi'(x) \in \mathbb{R}, \forall x \in \mathring{B}_{\delta_\varepsilon}(x_0)$

②设有 $\exists \lim_{n \rightarrow +\infty} \frac{y_{n+1} - y_n}{x_{n+1} - x_n} = l \in \mathbb{R}$, 则有 $y_{n+1} - y_n = l(x_{n+1} - x_n) + \delta_n(x_{n+1} - x_n), \delta_n \rightarrow 0$.

故有 $\begin{cases} y_{m+1} - y_m = l x_{m+1} - x_m + \delta_m(x_{m+1} - x_m) \\ y_m - y_{m-1} = l x_m - x_{m-1} + \delta_{m-1}(x_m - x_{m-1}) \\ \dots \\ y_{n+2} - y_{n+1} = l x_{n+2} - x_{n+1} + \delta_{n+1}(x_{n+2} - x_{n+1}) \\ y_{n+1} - y_n = l(x_{n+1} - x_n) + \delta_n(x_{n+1} - x_n) \end{cases}$

$\Rightarrow y_{m+1} - y_n = l(x_{m+1} - x_n) + \delta_m(x_{m+1} - x_m) + \dots + \delta_n(x_{n+1} - x_n).$

$\Rightarrow \frac{y_{m+1}}{x_{m+1}} - l = \frac{y_n}{x_{m+1}} - l \frac{x_n}{x_{m+1}} + \frac{\delta_m(x_{m+1} - x_m) + \dots + \delta_n(x_{n+1} - x_n)}{x_{m+1}},$

式中 $\left| \frac{\delta_m(x_{m+1} - x_m) + \dots + \delta_n(x_{n+1} - x_n)}{x_{m+1}} \right| < \varepsilon \cdot \left(1 + \frac{|x_n|}{|x_{n+1}|} \right), \forall n, m > N_\varepsilon.$

考虑 $\tilde{x}, \hat{x} \in \mathring{B}_{\delta_\varepsilon}(x_0)$, 应用 Cauchy 中值定理:

$\frac{\psi(\tilde{x}) - \psi(\hat{x})}{\varphi(\tilde{x}) - \varphi(\hat{x})} = \frac{\psi'(\xi)}{\varphi'(\xi)} \rightarrow l \in \mathbb{R}$

$\Rightarrow \frac{\psi(\tilde{x})}{\varphi(\tilde{x})} - l = \frac{\psi(\hat{x})}{\varphi(\hat{x})} + \left(1 - \frac{\psi(\hat{x})}{\varphi(\hat{x})} \right) \frac{\psi'(\xi)}{\varphi'(\xi)} - l = \frac{\psi(\tilde{x})}{\varphi(\tilde{x})} - \frac{\psi(\tilde{x})}{\varphi(\tilde{x})} \frac{\psi'(\xi)}{\varphi'(\xi)} + \frac{\psi'(\xi)}{\varphi'(\xi)} - l$

式中 $\left| \frac{\psi'(\xi)}{\varphi'(\xi)} - l \right| < \varepsilon, \forall \tilde{x}, \hat{x} \in \mathring{B}_{\delta_\varepsilon}(x_0).$

当 $\psi(x) \rightarrow \infty$, 对于确定的 \hat{x} , 可有 $\left| \frac{\psi(\tilde{x})}{\varphi(\tilde{x})} \right|, \left| \frac{\psi(\hat{x})}{\varphi(\hat{x})} \right| < \varepsilon$.

$$\textcircled{3} \text{ 当 } \begin{cases} y_n \rightarrow 0 \\ x_n \downarrow 0 \end{cases}, \text{ 令 } y_n \rightarrow +\infty, \text{ 有 } \left| \frac{y_{n+1}}{x_{n+1}} - l \right| \leq \varepsilon$$

$$(i) \text{ 当 } \begin{cases} \psi(\tilde{x}) \rightarrow 0 \\ \varphi(\tilde{x}) \rightarrow 0 \end{cases}, \text{ 令 } \tilde{x} \rightarrow x_0 \in \overline{\mathbb{R}}, \text{ 有 } \left| \frac{\psi(\hat{x})}{\varphi(\hat{x})} \right| \leq \varepsilon, \forall \hat{x} \in \mathring{B}_{\delta_\varepsilon}(x_0), \text{ 当 } x \rightarrow x_0 \in \mathbb{R}.$$

$$(ii) x_m \uparrow +\infty, \text{ 对确定的 } n \text{ 可有 } \left| \frac{y_n}{x_{m+1}} \right| < \varepsilon, \left| \frac{x_n}{x_{m+1}} \right| < \varepsilon$$