Seminar Nr.1, Euler's Functions; Counting, Outcomes, Events

<u>Solutions</u>

<u>**Theory Review**</u>

Euler's Gamma Function: $\Gamma : (0,\infty) \to (0,\infty), \Gamma(a) = \int\limits_0^\infty x^{a-1}e^{-x}dx.$

1. $\Gamma(1) = 1;$
2. $\Gamma(a+1) = a\Gamma(a),\ \forall a > 0;$
3. $\Gamma(n+1) = n!\ ,\ \forall n \in \mathbb{N};$
4. $\Gamma\left(\frac{1}{2}\right) = \sqrt{2}\int\limits_0^\infty e^{-\frac{t^2}{2}}dt = \int\limits_{\mathbb{R}} e^{-t^2}dt = \sqrt{\pi}.$

Euler's Beta Function: $\beta : (0,\infty)\times(0,\infty) \to (0,\infty), \beta(a,b) = \int\limits_0^1 x^{a-1}(1-x)^{b-1}dx.$

1. $\beta(a,1) = \frac{1}{a},\ \forall a > 0;$
2. $\beta(a,b) = \beta(b,a),\ \forall a,b > 0;$
3. $\beta(a,b) = \frac{a-1}{b}\beta(a-1,b+1),\ \forall a > 1, b > 0;$
4. $\beta(a,b) = \frac{b-1}{a+b-1}\beta(a,b-1) = \frac{a-1}{a+b-1}\beta(a-1,b),\ \forall a > 1,\ b > 1;$
5. $\beta(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)},\ \forall a > 0, b > 0.$

Arrangements: $A_n^k = \frac{n!}{(n-k)!};$

Permutations: $P_n = A_n^n = n!;$

Combinations: $C_n^k = \frac{A_n^k}{P_k} = \frac{n!}{k!(n-k)!}.$

De Morgan's laws:

$$\overline{\bigcup_{i\in I} A_i} = \bigcap_{i\in I} \overline{A}_i \quad \text{and} \quad \overline{\bigcap_{i\in I} A_i} = \bigcup_{i\in I} \overline{A}_i.$$

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1. In how many ways can 10 students be seated in a classroom with
a) 15 chairs?
b) 10 chairs?

<u>**Solution**</u>:
Order *does* matter, so these are arrangements:
a) $A_{15}^{10} = 6 \cdot 7 \cdot \ldots \cdot 15;$
b) $A_{10}^{10} = P_{10} = 10!.$

2. Find the number of possible outcomes for the following events:
a) three dice are rolled;
b) two letters and three digits are randomly selected.

<u>**Solution**</u>:
All actions are *independent* of each other, so we simply multiply the corresponding numbers of outcomes for each action:
a) There are 6 possibilities for each die, so for 3 dice there are 6^3 possible outcomes;

b) There are 26 letters, 10 digits, so all together $26^2 \cdot 10^3$ possible outcomes.

3. A firm offers a choice of 10 free software packages to buyers of their new home computer. There are 25 packages available, five of which are computer games, and three of which are anti-virus programs.
a) How many selections are possible?
b) How many selections are possible, if exactly three computer games are selected?
c) How many selections are possible, if exactly three computer games and exactly two anti-virus programs are selected?

Solution:
Here, order *does not* matter, so for each individual action we use combinations.
a) Choosing 10 objects out of 25 (order does not matter): C_{25}^{10} possibilities;

b) We break the problem into subproblems:
– First, selecting exactly three computer games, i.e. choosing 3 objects out of 5 (again order does not matter). This can be done in C_5^3 ways;
– Then, choosing the rest, i.e. 7 objects out of the remaining 20 (only 20, because no more computer games can be chosen), so C_{20}^7 possibilities.
The two actions are again, independent, so we multiply the numbers.
Final answer: $C_5^3 \cdot C_{20}^7$;

c) The same type of argument as before, only now there are three independent actions, so $C_5^3 \cdot C_3^2 \cdot C_{17}^5$ possible selections.

4. A person buys n lottery tickets. For $i = \overline{1, n}$, let A_i denote the event: the i^{th} ticket is a winning one. Express the following events in terms of $A_1, ..., A_n$.
a) A: all tickets are winning;
b) B: all tickets are losing;
c) C: at least one is winning;
d) D: exactly one is winning;
e) E: exactly two are winning;
f) F: at least two are winning;
g) G: at most two are winning.

Solution:
Let's recall:
– union, $\cup$, is described with the word "or";
– intersection, $\cap$, is described with the word "and";
– opposite (contrary, complementary) event, $\overline{E}$, is described with the word "not";
– difference, $\setminus$, is described with the words "but not".

a) "All" tickets are winning, that means the first one is (event A_1) AND the second one is (event A_2) AND ... all the way to A_n. Thus, this is an intersection,

$$A = \bigcap_{i=1}^{n} A_i.$$

b) Same here, only all are now *losing* tickets, i.e. *not* winning, so

$$B = \bigcap_{i=1}^{n} \overline{A_i} = \overline{\bigcup_{i=1}^{n} A_i},$$

the last part coming from de Morgan's laws.

c) Whenever working with events (sets), always keep in mind the contrary event, as well, and choose to work with the one that is easiest. The opposite event of C would be " 0 winning tickets", which is event B, so

$$C = \overline{B} = \bigcup_{i=1}^{n} A_i.$$

d) The sole winning ticket could be either one. So, another way of describing event D would be "only the first ticket is a winning one OR only the second ticket is OR ... OR only the last one is. That is a union. Only it's not the union of events A_i, because A_i states that ticket i is winning, but it's not necessarily *the only* winning ticket.

Thus, let us denote by D_i the event: only the i^{th} ticket is a winning one, for $i = 1, \dots, n$. Then $D = \bigcup_{i=1}^{n} D_i$.

Now, all that is left to do is express D_i in terms of A_i. Let us start with D_1 and A_1. Obviously, A_1 says more than D_1. In fact, to get D_1 (only the first ticket is winning) from A_1 (the first ticket is winning), we have to "take something out", that is make sure that none of the other tickets are winning ones. This sounds like "something ... BUT NOT something else ..." Indeed, we want A_1, but not "any of the other tickets are winning". So $D_1 = A_1 \setminus \bigcup_{j=2}^{n} A_j$ and, in general,

$$D_i = A_i \setminus \left(\bigcup_{j \neq i} A_j \right) = A_i \cap \overline{\bigcup_{j \neq i} A_j} = \overline{A}_1 \cap \ldots \cap \overline{A}_{i-1} \cap A_i \cap \overline{A}_{i+1} \cap \ldots \cap \overline{A}_n.$$

Then

$$D = \bigcup_{i=1}^{n} \left(A_i \setminus \left(\bigcup_{j \neq i} A_j \right) \right)$$

e) The same type of argument leads to

$$E = \bigcup_{1 \le i < j \le n} \left((A_i \cap A_j) \setminus \left(\bigcup_{k \neq i,j} A_k \right) \right)$$

f) The contrary event of F: at most one is winning, i.e. 0 or 1 winning tickets, so $\overline{F} = B \cup D$. Thus

$$F = \overline{B \cup D} = \overline{B} \cap \overline{D} = C \cap \overline{D} = C \setminus D.$$

g) At most 2 winning tickets means 0 or 1 or 2. So

$$G = B \cup D \cup E.$$

5. Three shooters aim at a target. For $i = \overline{1,3}$, let A_i denote the event: the i^{th} shooter hits the target. Express the following events in terms of A_1, A_2 and A_3.

a) A: the target is hit;
b) B: the target is not hit;
c) C: the target is hit exactly three times;
d) D: the target is hit exactly once;
e) E: the target is hit exactly twice.

<u>**Solution**</u>:

This is the same type of problem as the previous one. We have:

a) $A = A_1 \cup A_2 \cup A_3$;

b) $B = \overline{A} = \overline{A}_1 \cap \overline{A}_2 \cap \overline{A}_3$;

c) $C = A_1 \cap A_2 \cap A_3$;

d) $D = \left(A_1 \cap \overline{A}_2 \cap \overline{A}_3\right) \cup \left(\overline{A}_1 \cap A_2 \cap \overline{A}_3\right) \cup \left(\overline{A}_1 \cap \overline{A}_2 \cap A_3\right)$;

e) $E = \left(A_1 \cap A_2 \cap \overline{A}_3\right) \cup \left(A_1 \cap \overline{A}_2 \cap A_3\right) \cup \left(\overline{A}_1 \cap A_2 \cap A_3\right)$.