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1. a) i)

$$x_{n+1} - x_n = n+1 + \frac{1}{(n+1)^2} - n - \frac{1}{n^2}$$

$$= \frac{(n+1)(n+1)^2+1}{(n+1)^2} - \frac{n^3-1}{n^2}$$

$$= \frac{(n+1)(n^2+2n+1)+1}{n^2+2n+1} - \frac{n^3-1}{n^2}$$

$$= \frac{n^3+2n^2+n+n^2+2n+1+1}{n^2+2n+1} - \frac{n^3-1}{n^2}$$

$$= \frac{^{n^2)}n^3+3n^2+3n+2}{n^2+2n+1} - \frac{^{n^2+2n+1)}n^3-1}{n^2}$$

$$= \frac{\cancel{n^5}+3n^4+3n^3+2n^2-\cancel{n^5}-n^2-2n^4-2n-n^3-1}{n^4+2n^3+n^2}$$

$$= \frac{n^4+2n^3+n^2-2n-1}{n^4+2n^3+n^2} > 0$$

$\Rightarrow (x_n)_n$ is increasing

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$$(y_n) = \frac{1}{x_n} = \frac{n^2}{n^3+1}$$

$$y_{n+1} - y_n = \frac{(n+1)^2}{(n+1)^3+1} - \frac{n^2}{n^3+1}$$

$$= \frac{{}^{n^3+1)}n^2+2n+1}{n^3+3n^2+3n+2} - \frac{{}^{n^3+3n^2+3n+2)}n^2}{n^3+1}$$

$$= \frac{\cancel{n^5}+2n^4+n^3+n^2+2n+1-\cancel{n^5}-3n^4-3n^3-2n^2}{n^6+3n^5+3n^4+2n^3+n^3+3n^2+3n+2}$$

$$= \frac{-n^4-2n^3-n^2+2n+1}{n^6+3n^5+3n^4+3n^3+3n^2+3n+2} < 0$$

$\Rightarrow (y_n)$ is decreasing

$0 < y_n \leq \frac{1}{2}, \forall n \in \mathbb{N}$

$\Rightarrow (y_n)$ is bounded

Let $l = \lim_{n \to \infty} (y_n) \in \mathbb{R}$

$$\Rightarrow l = \lim_{n \to \infty} \frac{n^2}{n^3+1} = 0$$

$\Rightarrow (y_n)$ is convergent

1. a) ii)

$$\lim_{n\to\infty} \sqrt[n]{x_n} = \lim_{n\to\infty} \sqrt[n]{\frac{n^3+1}{n^2}} = \lim_{n\to\infty} \frac{(n+1)^3+1}{(n+1)^2} \cdot \frac{n^2}{n^3+1}$$

$$= \lim_{n\to\infty} \frac{x_{n+1}}{x_n}$$

$$= \lim_{n\to\infty} \frac{n^3+3n^2+3n+2}{n^2+2n+1} \cdot \frac{n^2}{n^3+1}$$

$$= \lim_{n\to\infty} \frac{n^5+3n^4+3n^3+2n^2}{n^5+2n^4+n^3+n^2+2n+1} \to 1$$

$$\lim_{n\to\infty} (1+y_n)^{n^2} = \lim_{n\to\infty} \left(1+\frac{n^2}{n^3+1}\right)^{n^2} =$$

$$= \left[\lim_{n\to\infty} \left(1+\frac{n^2}{n^3+1}\right)^{\frac{n^3+1}{n^2}}\right]^{\frac{n^2}{n^3+1}\cdot n^2} =$$

$$= e^{\lim_{n\to\infty} \frac{n^4}{n^3+1}} = e^{\infty} = \infty$$

a) iii) $\sum_{n\geq 1} y_n =$

$= \sum_{n\geq 1} \frac{n^2}{n^3+1}$, we apply Series integral Test

$$\Rightarrow \int_1^{\infty} \frac{n^2}{n^3+1}\, dn$$

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$$\int_1^{\infty} \frac{n^2}{n^3+1}\, dn$$

$$\int \frac{n^2}{n^3+1}\, dn = \frac{1}{3}\int \frac{1}{t}\, dt = \frac{1}{3}\ln|t| = \frac{1}{3}\ln|n^3+1| + C$$

$t = n^3+1$

$dt = 3n^2\, dn \Rightarrow \frac{1}{3}\, dt = n^2\, dn$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{1}{3}\ln|\underbrace{n^3+1}_{n\geq 1}|\right) =$$

$$= \lim_{n\to\infty}\left(\frac{1}{3}\ln(n^3+1)\right) =$$

$$= \frac{1}{3}\lim_{n\to\infty}(\ln(n^3+1))$$

$$= \frac{1}{3}\cdot\infty = \infty$$

$\Rightarrow \sum_{n\geq 1} y_n$ is divergent

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1. b) Let $\sum_{n\geq 1} \frac{n^{\alpha}}{a^n} = x_n$

$\alpha = 0 \Rightarrow x_n = \frac{1}{a^n}$

$n < 0 \Rightarrow x_n = n^{-a} \longrightarrow \infty \Rightarrow x_n \not\to 0$

$\Rightarrow x_n$ is not conv.

$n = 0 \Rightarrow x_n = 1 \longrightarrow 1 \Rightarrow x_n \not\to 0$

$\Rightarrow x_n$ is not conv.

$n > 0 \Rightarrow x_n = \frac{1}{a^n} \to 0$, x_n is convergent

Conclusion: the series $\sum_{n\geq 1} \frac{n^{\alpha}}{a^n}$ is not absolutely convergent.

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2. $\left(\frac{1}{n}, \frac{1}{n}\right) \to (0,0)$

$$f\left(\frac{1}{n}, \frac{1}{n}\right) = \frac{e^{\frac{1}{n} - \frac{1}{n}} - 1}{\frac{1}{n} + \frac{1}{n}} = 0 \quad (1)$$

$\left(\frac{1}{n}, 0\right) \to (0,0)$

$$f\left(\frac{1}{n}, 0\right) = \frac{e^{\frac{1}{n}} - 1}{\frac{1}{n}} \xrightarrow[n\to\infty]{} 1 \quad (2)$$

From (1) and (2) $\Rightarrow \lim\limits_{x,y\to(0,0)} f(x,y)$ exists

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3. a) For $(x,y) \in \mathbb{R}^2$,

$$\frac{\partial f}{\partial x}(x,y) = 2yx + 2x \qquad \frac{\partial f}{\partial y}(x,y) = 6y^2 + 10y + x^2$$

$$\nabla f(x,y) = (2yx + 2x,\ 6y^2 + 10y + x^2) \in \mathbb{R}^2$$

$$\frac{\partial^2 f}{\partial x^2}(x,y) = 2y + 2 \qquad \frac{\partial^2 f}{\partial y^2}(x,y) = 12y + 10$$

$$\frac{\partial^2 f}{\partial y \partial x}(x,y) = 2x = \frac{\partial^2 f}{\partial x \partial y}$$

$$Hf(x,y) = \begin{pmatrix} 2y+2 & 2x \\ 2x & 12y+10 \end{pmatrix}$$

b) For $(x,y) \in \mathbb{R}^2$,

$$\begin{cases} 2yx + 2x = 0 \Rightarrow 2x(y+1) = 0 \Rightarrow x = 0,\ y+1 = 0 \Rightarrow y = -1 \\ 6y^2 + 10y + x^2 = 0 \end{cases}$$

For $6y^2 + 10y + x^2 = 0$, we substitute x with 0

$\Rightarrow 6y^2 + 10y + 0 = 0 \Rightarrow 6y^2 + 10y = 0 \Rightarrow 2y(3y+5) = 0$

$\Rightarrow y = 0,\ 3y + 5 = 0 \Rightarrow y = -\frac{5}{3}$

For $6y^2 + 10y + x^2 = 0$, we substitute y with -1

$\Rightarrow 6 - 10 + x^2 = 0 \Rightarrow x^2 = 4 \Rightarrow x = \pm 2$

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Stationary points: $(0,0)$, $(0,-\frac{5}{3})$, $(2,-1)$, $(-2,-1)$

$$Hf(x,y) = \begin{pmatrix} 2y+2 & 2x \\ 2x & 12y+10 \end{pmatrix}$$

$$Hf(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 10 \end{pmatrix} : \Delta_2 = 20 > 0, \ \Delta_1 = 2 > 0$$

$\Rightarrow Hf(0,0)$ is pos. def. $\Rightarrow (0,0)$ is a local minimum point of f

$$Hf(0,-\tfrac{5}{3}) = \begin{pmatrix} \frac{7}{3} & 0 \\ 0 & -10 \end{pmatrix} : \Delta_2 = -\frac{70}{3} < 0 \quad \Delta_1 = \frac{7}{3} > 0$$

$\Rightarrow Hf(0,-\frac{5}{3})$ is indefinite $\Rightarrow (0,-\frac{5}{3})$ is not a local extremum point of f.

$$Hf(2,-1) = \begin{pmatrix} 0 & 4 \\ 4 & -2 \end{pmatrix} : \Delta_2 = -16 < 0$$

$\Rightarrow Hf(2,-1)$ is indefinite $\Rightarrow (2,-1)$ is not a local extremum point of f.

$$Hf(-2,-1) = \begin{pmatrix} 0 & -4 \\ -4 & -2 \end{pmatrix} : \Delta_2 = -16 < 0$$

$\Rightarrow Hf(-2,-1)$ is indefinite $\Rightarrow (-2,-1)$ is not a local extremum point of f.

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3. c) Since we have a unique local minimum point of f; $(0,0)$ is also a global minimum point for the function.

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4. a) $f:(-\infty,1] \to \mathbb{R}$, $f(x) = (1-x)e^x$

f cont.

Let $t \in (-\infty, 1]$.

$$\int_t^1 (1-x)\cdot e^x \, dx = e^x(1-x)\Big|_t^1 - \int_t^1 -e^x =$$

$u = (1-x) \quad u' = -1$

$v' = e^x \quad v = e^x$

$$= -e^t(1-t) + e^1 - e^t = -e^t(1-t+1) + e$$

$$\xrightarrow{t \to -\infty} e$$

$\Rightarrow$ f is improperly integrable on $(-\infty, 1]$ and

$$\int_{-\infty}^1 f(x)\,dx = e \quad \left(\int_{-\infty}^0 f(x)\,dx = 1 \right)$$

b) $\iint_M (y + \sqrt{x^2+1})\,dx\,dy$