HW

1.2.1.

a) $x'' + t^2 x = 0$, $x(0) = 0$

the charact. eq: $r^2 + t^2 = 0 \Rightarrow r^2 = -t^2$

$\Rightarrow r_{1,2} = \left\langle \begin{matrix} +ti \\ -ti \end{matrix} \right.$

$e^{0t} \cos t$, $e^{0t} \sin t$ solutions

$\Rightarrow \cos(t)$, $\sin(t)$

$x(0) = 0$

$\Rightarrow x = c_1 x_1 \Rightarrow$ we have infinite solutions.

The solution space has dim $= 1$.

b) $x'' + t^2 x = 0$, $x(0) = 0$, $x'(0) = 0$

This is an equation of second order that has a single solution.

c) $x'' + t^2 x = 0$, $x(0) = 0$, $x'(0) = 0$, $x''(0) = 1$

This is an equation of second order

Let $t_0 = 0 \Rightarrow x''(0) + 0\, x(0) = 0 \Rightarrow x''(0) = 0$

which is not good, because $x''(0)$ has to be 1.

$\Rightarrow$ There is no solution

1.2.5 a) $x_p = ae^t$ $(a \in \mathbb{R})$ sol. for $x' - 2x = e^t$

$\Rightarrow x_p' = (ae^t)' = ae^t$

$\Rightarrow x_p' - 2x_p = e^t \Leftrightarrow ae^t - 2ae^t = e^t$

$e^t(a - 2a - 1) = 0$

$e^t(-a-1) = 0 \Rightarrow$

$\Rightarrow a + 1 = 0 \Rightarrow a = -1$

$\Rightarrow x_p = -e^t$

b) $x_p = be^{-t}$ $(b \in \mathbb{R})$ sol. for $x' - 2x = e^{-t}$

$\Rightarrow x_p' = (be^{-t})' = -be^{-t}$

$\Rightarrow x_p' - 2x_p = e^{-t} \Leftrightarrow -be^{-t} - 2be^{-t} = e^{-t}$

$\underbrace{e^{-t}}_{>0}(-3b-1) = 0$

$\Rightarrow -3b - 1 = 0$

$3b = -1$

$b = -\frac{1}{3}$

$\Rightarrow x_p = -\frac{1}{3}e^{-t}$

c) particular sol. for $x' - 2x = 5e^t - 3e^{-t}$

We use superposition principle.
We know that: $x' - 2x = e^t = f_1 \Rightarrow x_{p1} = -e^t$
$x' - 2x = e^{-t} = f_2 \Rightarrow x_{p2} = -\frac{1}{3}e^{-t}$

$\Rightarrow f = 5 \cdot e^t - 3e^{-t} = 5f_1 - 3f_2$

$\Rightarrow x_p = 5 \cdot x_{p1} - 3x_{p2}$

$x_p = 5 \cdot (-e^t) - 3(-\frac{1}{3}e^{-t})$

$x_p = e^{-t} - 5e^t$

d) gen. sol. of $x' - 2x = 5e^t - 3e^{-t}$

$x = x_p + x_h$ (fundamental th. of linear non-hom. diff. eq.)

We have $x_p = e^{-t} - 5e^t$

We have to find x_h, the sol. of LHDE

$x' - 2x = 0$

$r - 2 = 0 \Rightarrow r = 2 \; t \cdot e^{2t}$

$\Rightarrow x_h = c \cdot e^{2t}, \; c \in \mathbb{R}$

$\Rightarrow x = c \cdot e^{2t} + e^{-t} - 5e^t, \; c \in \mathbb{R}$

1.3.4.

M1. - the variation of the constant method

$x' - x = e^{t-1}$

$x' - x = 0$

$r - 1 = 0 \Rightarrow r = 1$

$\cdot e^t = 1 \Rightarrow x_h = c \cdot e^t$

We consider $x_p = \varphi(t) \cdot e^t$

$\Rightarrow x_p' = \varphi'(t) \cdot e^t + \varphi(t) \cdot e^t$

$x' - x = e^{t-1} \Longrightarrow \xi'(t) \cdot e^t + \xi(t) \cdot e^t - \xi(t) \cdot e^t = e^{t-1}$

$\Rightarrow \xi'(t) \cdot e^t = e^{t-1} \quad | : e^t$

$\Rightarrow \xi'(t) = e^{-1} \quad | \int$

$\Rightarrow \xi(t) = e^{-1} t + c$

$\Rightarrow x_p = \xi(t) \cdot e^t = (e^{-1} t + c) \cdot e^t =$

$= \frac{t}{e^{1-t}} + c \cdot e^t$

M_2

$x' - x = e^{t-1} \quad | \cdot e^{-t}$

$x' \cdot e^{-t} - x \cdot e^{-t} = e^{-1}$

$(x \cdot e^{-t})' = e^{-1} \Rightarrow x \cdot e^{-t} = \frac{1}{e} \cdot t + c \quad | \cdot e^t$

$\Rightarrow x = \frac{1}{e^{1-t}} \cdot t + c \cdot e^t$

1.6.1.

j) $A = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix}$

M1: we have to check if the matrix is diagonalizable

$\det(A - \lambda I_2) = \begin{vmatrix} 2-\lambda & 0 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 = 0 \Rightarrow \lambda_{1,2} = 2$

$u_1 = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow A \cdot u_1 = \lambda_1 \cdot u_1$

$\Rightarrow \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 2 \begin{pmatrix} a \\ b \end{pmatrix}$

$\Rightarrow \begin{cases} 2a = 2a \\ a + 2b = 2b \end{cases} \Rightarrow a = 0 \qquad \Rightarrow u_1 = \begin{pmatrix} 0 \\ b \end{pmatrix}$

$u_2 = \begin{pmatrix} a \\ c \end{pmatrix} \Rightarrow A \cdot u_2 = \lambda_2 \cdot u_2$

$\Rightarrow \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} a \\ c \end{pmatrix} = 2 \begin{pmatrix} a \\ c \end{pmatrix}$

$\Rightarrow \begin{cases} 2a = 2a \\ a + 2c = 2c \end{cases} \Rightarrow a = 0 \qquad \Rightarrow u_2 = \begin{pmatrix} 0 \\ c \end{pmatrix}$

$\Rightarrow$ u_1, u_2 cannot be lin. independent
$\Rightarrow$ A is not diagonalizable

M2: $\begin{cases} x' = 2x \\ y' = x + 2y \end{cases} \Rightarrow \begin{cases} x' - 2x = 0 \\ y' - 2y = x \end{cases}$

charact eq.: $r - 2 = 0 \Rightarrow r = 2 \vdash e^{2t}$

we have $x_p = 0 \Rightarrow x = x_h = c_1 \cdot e^{2t}$

$y' - 2y = x$

$\cancel{\quad}$

$\Rightarrow y' - 2y = c_1 \cdot e^{2t}$

$\Rightarrow y' \cdot e^{-2t} - 2y \cdot e^{-2t} = c_1$

$\Rightarrow (y \cdot e^{-2t})' = c_1 \quad | \int$

$\Rightarrow y \cdot e^{-2t} = c_1 \cdot t + c_2 \quad | \cdot e^{2t}$

$\Rightarrow y = c_1 \cdot t \cdot e^{2t} + c_2 \cdot e^{2t}$

$$\begin{cases} x = c_1 \cdot e^{2t} \\ y = c_1 \cdot t \cdot e^{2t} + c_2 \cdot e^{2t} \end{cases}$$

$\Rightarrow M = \begin{pmatrix} e^{2t} \\ te^{2t} + e^{2t} \end{pmatrix}$

$e^{tA} = ?$

$$\begin{cases} x' = Ax \\ x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}$$

$$\Rightarrow \begin{cases} x(0) = c_1 \cdot e^0 = c_1 = 1 \\ y(0) = c_1 \cdot 0 \cdot e^0 + c_2 \cdot e^0 = c_2 = 0 \end{cases}$$

$x_1 = \begin{pmatrix} e^{2t} \\ t \cdot e^{2t} \end{pmatrix}$ - the sol. of the ivp

$$\begin{cases} x' = Ax \\ x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$$

$$\Rightarrow \begin{cases} x(0) = c_1 \cdot e^0 = c_1 = 0 \\ y(0) = c_2 \cdot 0 \cdot e^0 + c_2 \cdot e^0 = c_2 = 1 \end{cases}$$

$\Rightarrow x_2 = \begin{pmatrix} 0 \\ e^{2t} \end{pmatrix}$ - the sol. of the IVP.

$$e^{tA} = (x_1, x_2) = \begin{pmatrix} e^{2t} & 0 \\ t \cdot e^{2t} & e^{2t} \end{pmatrix}$$

k) $A = \begin{pmatrix} 0 & 4 \\ 5 & 1 \end{pmatrix}$

M1: we have to check if the matrix is diag.

$$\det |A - \lambda I_2| = \begin{vmatrix} -\lambda & 4 \\ 5 & 1-\lambda \end{vmatrix} = \lambda^2 - \lambda - 20 = 0$$

$$\Delta = 1 + 80 = 81$$

$$\Rightarrow \lambda_{1,2} = \frac{+1 \pm 9}{2} \left\langle \begin{matrix} \lambda_1 = 5 \\ \lambda_2 = -4 \end{matrix} \right.$$

$u_1 = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow A \cdot u_1 = \lambda_1 \cdot u_1$

$$\Rightarrow \begin{pmatrix} 0 & 4 \\ 5 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = 5 \cdot \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{cases} 4b = 5a \\ 5a + b = 5b \end{cases} \Rightarrow \begin{cases} 4b = 5a \\ 4b = 5a \end{cases} \Rightarrow b = \frac{5}{4} a$$

$$\Rightarrow u_1 = \begin{pmatrix} a \\ \frac{5}{4} a \end{pmatrix}$$

$u_2 = \begin{pmatrix} c \\ d \end{pmatrix} \Rightarrow A \cdot u_2 = \lambda_2 \cdot u_2$

$$\Rightarrow \begin{pmatrix} 0 & 4 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = -4 \begin{pmatrix} c \\ d \end{pmatrix}$$

$$\begin{cases} 4d = -4c \\ 5c + d = -4d \end{cases} \Rightarrow \begin{cases} 4d = -4c \\ 5c = -5d \end{cases} \Rightarrow c = -d$$

$$\Rightarrow u_2 = \begin{pmatrix} c \\ -c \end{pmatrix}$$

Let's take $a = 1$, $c = 1$

$$\Rightarrow u_1 = \begin{pmatrix} 1 \\ \frac{5}{4} \end{pmatrix} \qquad u_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 1 \\ \frac{5}{4} & -1 \end{vmatrix} = -\frac{5}{4} - 1 = -\frac{9}{4} \neq 0$$

$\Rightarrow u_1$ and u_2 are lin. independent

$\Rightarrow A$ is diag.

We have $e^{5t}\begin{pmatrix} 1 \\ \frac{5}{4} \end{pmatrix}$, $e^{-4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$X' = AX$

$$\begin{cases} x' = 4y \\ y' = 5x + y \end{cases}$$

the gen. sol. : $\begin{cases} x = c_1 \cdot e^{5t} + c_2 \cdot e^{-4t} \\ y = c_1 \cdot \frac{5}{4} e^{5t} - c_2 \cdot e^{-4t} \end{cases}$

$$D = \begin{pmatrix} 5 & 0 \\ 0 & -4 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 1 \\ \frac{5}{4} & -1 \end{pmatrix}$$

$$\det(P) = -\frac{9}{4}$$

$${}^tP = \begin{pmatrix} 1 & \frac{5}{4} \\ 1 & -1 \end{pmatrix} \qquad P^* = \begin{pmatrix} -1 & -1 \\ -\frac{5}{4} & 1 \end{pmatrix}$$

$$P^{-1} = \frac{1}{-\frac{9}{4}} \begin{pmatrix} -1 & -1 \\ -\frac{5}{4} & 1 \end{pmatrix} = -\frac{4}{9} \begin{pmatrix} -1 & -1 \\ -\frac{5}{4} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{4}{9} & \frac{4}{9} \\ \frac{5}{9} & -\frac{4}{9} \end{pmatrix}$$

$$e^{tA} = \begin{pmatrix} 1 & 1 \\ \frac{5}{4} & -1 \end{pmatrix} \begin{pmatrix} e^{5t} & 0 \\ 0 & e^{-4t} \end{pmatrix} \begin{pmatrix} \frac{4}{9} & \frac{4}{9} \\ \frac{5}{9} & -\frac{4}{9} \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} e^{5t} & e^{-4t} \\ \frac{5}{4}e^{5t} & -e^{-4t} \end{pmatrix} \begin{pmatrix} 4 & 4 \\ 5 & -4 \end{pmatrix}$$

$$= \frac{1}{9} \begin{pmatrix} 4e^{5t} - 5e^{-4t} & 4e^{5t} - 4e^{-4t} \\ 5e^{5t} - 5e^{-4t} & 5e^{5t} + 4e^{-4t} \end{pmatrix}$$

M_2: $\begin{cases} x' = 4y \\ y' = 5x + y \end{cases}$

$y = \frac{1}{4}x'$, $y' = \frac{1}{4}x''$

$$\frac{1}{4}x'' = 5x + \frac{1}{4}x' \quad | \cdot 4$$

$$\Rightarrow x'' = 20x + x' \Rightarrow x'' - x' - 20x = 0$$

the charact. eq.: $r^2 - r - 20 = 0$

$$\Delta = 1 + 80 = 81$$

$$\Rightarrow r_{1,2} = \frac{1 \pm 9}{2} \left\langle \begin{array}{l} r = 5 \vdash e^{5t} \\ r = -4 \vdash e^{-4t} \end{array} \right.$$

$$x = c_1 \cdot e^{5t} + c_2 \cdot e^{-4t}$$

$$x' = 5c_1 \cdot e^{5t} - 4c_2 \cdot e^{-4t}$$

$$y = \frac{1}{4}x' = \frac{5}{4}c_1 \cdot e^{5t} - c_2 \cdot e^{-4t}$$

$$\Rightarrow \begin{cases} x = c_1 \cdot e^{5t} + c_2 \cdot e^{-4t} \\ y = \frac{5}{4} \cdot c_1 \cdot e^{5t} - c_2 \cdot e^{-4t} \end{cases}$$

$$M = \begin{pmatrix} e^{5t} & + e^{-4t} \\ \frac{5}{4}e^{5t} & - e^{-4t} \end{pmatrix}$$

$$\begin{cases} x' = Ax \\ x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}$$

$$\Rightarrow \begin{cases} x(0) = c_1 + c_2 = 1 \Rightarrow c_1 = 1 - c_2 \\ y(0) = \text{[illegible]} = 0 \end{cases}$$

$$\frac{5}{4}c_1 - c_2$$

$$\Rightarrow \frac{5}{4} - \frac{5}{4}c_2 - c_2 = 0$$

$$-\frac{9}{4}c_2 = -\frac{5}{4} \Rightarrow c_2 = \frac{5}{9}$$

$$c_1 = 1 - c_2 = 1 - \frac{5}{9} = \frac{4}{9}$$

$$X_1 = \begin{pmatrix} \frac{4}{9} \cdot e^{5t} + \frac{5}{9} e^{-4t} \\ \frac{5}{\cancel{4}} \cdot \frac{\cancel{4}}{9} \cdot e^{5t} - \frac{5}{9} e^{-4t} \end{pmatrix}$$

$$\begin{cases} x' = Ax \\ x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$$

$$\Rightarrow \begin{cases} x(0) = c_1 + c_2 = 0 \Rightarrow c_1 = -c_2 \\ y(0) = \frac{5}{4} c_1 - c_2 = 1 \Rightarrow -\frac{5}{4} c_2 - c_2 = 1 \end{cases}$$

$$\Rightarrow -\frac{9}{4} c_2 = 1$$

$$\Rightarrow c_2 = -\frac{4}{9}$$

$$\Rightarrow c_1 = \frac{4}{9}$$

$$X_2 = \begin{pmatrix} \frac{4}{9} \cdot e^{5t} - \frac{4}{9} \cdot e^{-4t} \\ \frac{5}{\cancel{4}} \cdot \frac{\cancel{4}}{9} \cdot e^{5t} + \frac{4}{9} e^{-4t} \end{pmatrix}$$

$$e^{tA} = (X_1, X_2) = \begin{pmatrix} \frac{4}{9} \cdot e^{5t} + \frac{5}{9} e^{-4t} & \frac{4}{9} \cdot e^{5t} - \frac{4}{9} \cdot e^{-4t} \\ \frac{5}{9} \cdot e^{5t} - \frac{5}{9} e^{-4t} & \frac{5}{9} \cdot e^{5t} + \frac{4}{9} e^{-4t} \end{pmatrix}$$

M) $A = \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix}$

M_1: we have to check if the matrix is diag.

$\det(A - xI_2) = \begin{vmatrix} -\lambda & -2 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 + 4 = 0$

$\Rightarrow \lambda_{1,2} = \pm 2i$

$u_1 = \begin{pmatrix} a \\ b \end{pmatrix} \Rightarrow A \cdot u_1 = \lambda_1 \cdot u_1$

$\Rightarrow \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = -2i \begin{pmatrix} a \\ b \end{pmatrix}$

$\begin{cases} -2b = -2ia \\ 2a = -2ib \end{cases} \Rightarrow \begin{cases} b = ia \\ a = -ib \end{cases}$

$a = -i \cdot (ia)$
$a = -i^2 a$
$a = a \ (T)$

$\Rightarrow u_1 = \begin{pmatrix} a \\ ia \end{pmatrix}$

$u_2 = \begin{pmatrix} c \\ d \end{pmatrix} \Rightarrow A \cdot u_2 = \lambda_2 \cdot u_2$

$\Rightarrow \begin{pmatrix} 0 & -2 \\ 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} c \\ d \end{pmatrix} = 2i \begin{pmatrix} c \\ d \end{pmatrix}$

$\begin{cases} -2d = 2ic \\ 2c = 2id \end{cases} \Rightarrow \begin{cases} d = -ic \\ c = id \end{cases}$

$c = i(-ic)$
$c = (-i^2) \cdot c$
$c = c \ (T)$

$\Rightarrow u_2 = \begin{pmatrix} c \\ -ic \end{pmatrix}$

Let's take $a = 1$, $c = 1$

$\Rightarrow u_1 = \begin{pmatrix} 1 \\ i \cdot 1 \end{pmatrix} = \begin{pmatrix} 1 \\ i \end{pmatrix}$, $u_2 = \begin{pmatrix} 1 \\ -i \cdot 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

~~u_1 and u_2 are lin independent~~

$$\begin{vmatrix} 1 & 1 \\ i & -i \end{vmatrix} = -i - i = -2i \neq 0$$

$\Rightarrow u_1$ and u_2 are lin. independent

$\Rightarrow A$ is diag.

We have $e^{-2it}\begin{pmatrix} 1 \\ i \end{pmatrix}$, $e^{2it}\begin{pmatrix} 1 \\ -i \end{pmatrix}$

$$x_1 = (\cos(2t) - i\sin(2t))\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \begin{pmatrix} \cos(2t) - i\sin(2t) \\ \sin(2t) + i\cos(2t) \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} \cos(2t) \\ \sin(2t) \end{pmatrix}}_{*} - i \underbrace{\begin{pmatrix} \sin(2t) \\ -\cos(2t) \end{pmatrix}}_{**}$$

so we have two lin. indep. sol. of the system

$x' = A\,(**)$ $\quad \begin{cases} x = \sin 2t \\ y = \cos 2t \end{cases}$

$\begin{cases} x' = 2y \\ y' = 2x \end{cases} \Rightarrow \begin{cases} 2\cos 2t = (-2).(-\cos 2t) \\ 2\sin 2t = 2\cos 2t \end{cases} \Rightarrow$

$\Rightarrow \begin{cases} 2\cos 2t = 2\cos 2t \\ 2\sin 2t = 2\sin 2t \end{cases}$ True for $**$

$X' = A(**)$ $\begin{cases} x = \cos 2t \\ y = \sin 2t \end{cases}$

$\begin{cases} x' = -2y \\ y' = 2x \end{cases} \Rightarrow \begin{cases} -2\sin 2t = (-2)(\sin 2t) \\ 2\cos 2t = 2\cos 2t \end{cases}$

$\Rightarrow \begin{cases} -2\sin 2t = -2\sin 2t \\ 2\cos 2t = 2\cos 2t \end{cases}$ true for $**$

$$\begin{vmatrix} \cos(2t) & \sin(2t) \\ \sin(2t) & -\cos(2t) \end{vmatrix} = -\cos^2(2t) - \sin^2(2t)$$

$$= -(\sin^2(2t) + \cos^2(2t))$$

$$= -1 \neq 0$$

$\Rightarrow$ we have proved that $*$ and $**$ are lin. independent sol.

$$M = \begin{pmatrix} \cos(2t) & \sin(2t) \\ \sin(2t) & -\cos(2t) \end{pmatrix}$$

$B = \begin{pmatrix} -2i & 0 \\ 0 & 2i \end{pmatrix}$ $P = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$ $\det(P) = -2i$

${}^tP = \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$ $P^* = \begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix}$

$$P^{-1} = -\frac{1}{2i}\begin{pmatrix} -i & -1 \\ -i & 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2}i \\ \frac{1}{2} & +\frac{1}{2}i \end{pmatrix}$$

$$e^{tA} = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \begin{pmatrix} e^{-2i} & 0 \\ 0 & e^{2i} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{1}{2} & \frac{i}{2} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-2it} & e^{2it} \\ ie^{-2it} & -ie^{2it} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & -\frac{i}{2} \\ \frac{1}{2} & \frac{i}{2} \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}(e^{-2it} + e^{2it}) & -\frac{1}{2}i(e^{-2it} + e^{2it}) \\ \frac{1}{2}i(e^{-2it} - ie^{2it}) & \frac{1}{2}(e^{-2it} + e^{2it}) \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2}\cdot 2\cos 2t & \frac{1}{2}\cdot 2 \sin 2t \\ \frac{1}{2}\, 2\cdot \sin 2t & \frac{1}{2}\cdot 2\cos 2t \end{pmatrix}$$

$$= \begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix}$$

M2: $\begin{cases} x' = -2y \\ y' = 2x \end{cases}$

$y = -\frac{1}{2}x'$, $y' = -\frac{1}{2}x''$

$-\frac{1}{2}x'' = 2x \Rightarrow 2x + \frac{1}{2}x'' = 0 \quad | \cdot 2$

$x'' + 4x = 0$

the charact. eq.: $r^2 + 4 = 0 \Rightarrow r^2 = -4$

$\Rightarrow r = \pm 2i \; t < \begin{matrix} \sin 2t \\ \cos 2t \end{matrix}$

$x = c_1 \sin 2t + c_2 \cdot \cos 2t$

$x' = 2c_1 \cos 2t + 2c_2 \cdot \sin 2t$

$$y = -\frac{1}{2}x' = -c_1 \cos 2t + c_2 \sin 2t$$

$$\Rightarrow \begin{cases} x = c_1 \sin 2t + c_2 \cos 2t \\ y = -c_1 \cos 2t + c_2 \sin 2t \end{cases}$$

$$M = \begin{pmatrix} \sin(2t) + \cos(2t) \\ -\cos(2t) + \sin(2t) \end{pmatrix}$$

$$\begin{cases} x' = Ax \\ x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \end{cases}$$

$$\Rightarrow x(0) = c_1 \cdot \sin 0 + c_2 \cdot \cos 0 \Rightarrow c_2 = 1$$

$$y(0) = -c_1 \cdot \cos 0 + c_2 \cdot \sin 0 \Rightarrow -c = 0 \Rightarrow c_1 = 0$$

$$x_1 = \begin{pmatrix} \cos 2t \\ \sin 2t \end{pmatrix}$$

$$\begin{cases} x' = Ax \\ x(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{cases}$$

$$\Rightarrow x(0) = c_1 \cdot \sin 0 + c_2 \cdot \cos 0 \Rightarrow c_2 = 0$$

$$y(0) = -c_1 \times \cos 0 + c_2 \cdot \sin 0 \Rightarrow c_1 = -1$$

$$x_2 = \begin{pmatrix} -\sin 2t \\ \cos 2t \end{pmatrix}$$

$$e^{tA} = (x_1, x_2) = \begin{pmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{pmatrix}$$

1.7.5 $e^{\alpha + i\beta} = e^{\alpha}(\cos\beta + i\sin\beta)$

For $\alpha = 0$, $\beta = t \Rightarrow e^{it} = \cos t + i\sin t$

For $\alpha = 0$, $\beta = \pi \Rightarrow e^{i\pi} = \cos\pi + i\sin\pi = -1$

For $\alpha = 0$, $\beta = \frac{\pi}{2} \Rightarrow e^{i(\frac{\pi}{2})} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$

For $\alpha = -t$, $\beta = t \Rightarrow e^{(-1+i)t} = e^{-t+it} = e^{-t}(\cos t + i\sin t)$

1.7.8. $x_p = 1 + t(1 + e^{-t})$

$= 1 + t - te^{-t}$

$\Rightarrow r_{1,2} = 0 \quad r_{3,4} = -1$

$r^2(r+1)^2 = 0$

$r^4 + 2r^3 + r^2 = 0$

$x^{IV} + 2x''' + x'' = 0$

1.7.9. $x' = k(21 - x) \Rightarrow x' + kx = k \cdot 21$

$x(0) = \eta$

We can see that $x_p = 21$

$x_p' + kx_p = k \cdot 21$, but $x_p' = 0 \Rightarrow$ True

$x' + kx = 0$

$\Rightarrow r + k = 0 \Rightarrow r = -k \vdash e^{-kt}$

$x = x_p + x_h = c \cdot e^{-kt} + 21$

$x(0) = c \cdot 1 + 21 = \eta \Rightarrow c = \eta - 21$

$x = (\eta + 21) \cdot e^{-kt} + 21$

1.7.10.

a) $x_p(t) = (at + b) \cdot e^{-2t}$

$x_p'(t) = ae^{-2t} + (at + b)(-2) \cdot e^{-2t}$

$= ae^{-2t} - 2(at + b) \cdot e^{-2t}$

$= e^{-2t}(a - 2at - 2b)$

$$x_p''(t) = -2ae^{-2t} + (-2)e^{-2t}(a-2b-2at)$$

$$= e^{-2t}(-4a+4b+4at)$$

$$x_p'' - x_p = te^{-2t} \iff e^{-2t}(-4a+4b+4at) - (at+b)e^{-2t} = te^{-2t}$$

$$e^{-2t}(-4a+4b+4at-at-b) = t\cdot e^{-2t} \quad | : e^{-2t}$$

$$(-4a+3b+3at) = t$$

$$t(3a-1) - 4a + 3b = 0$$

$$3a-1=0 \qquad -4a+3b=0$$

$$a = \frac{1}{3} \qquad -\frac{4}{3} + 3b = 0$$

$$3b = \frac{4}{3} \quad | \cdot 3$$

$$9b = 4$$

$$b = \frac{4}{9}$$

$$\Rightarrow x_p(t) = \left(\frac{1}{3}t + \frac{4}{9}\right)\cdot e^{-2t}$$

b) $x = x_p + x_h$

$$x'' - x = 0$$

$$r^2 - 1 = 0 \Rightarrow r^2 = 1 \Rightarrow r = \pm 1 \quad \begin{cases} t\,e^{-t} \\ t\,e^{t} \end{cases}$$

$$\Rightarrow x_h = c_1 e^{-t} + c_2 \cdot e^{t}, \quad c_1, c_2 \in \mathbb{R}$$

$$x = x_p + x_h = \left(\frac{1}{3}t + \frac{4}{9}\right)\cdot e^{-2t} + c_1 e^{-t} + c_2 e^{t}$$

c) $x(0) = 0 \quad x'(0) = 0$

$x(0) = \frac{4}{9} + c_1 + c_2 = 0$

$x' = (-2)\left(\frac{1}{3}t + \frac{4}{9}\right).e^{-2t} - c_1.e^{-t} + c_2 e^{t} + \frac{1}{3}e^{-2t}$

$x'(0) = (-2).\frac{4}{9} - c_1 + c_2 + \frac{1}{3}$

$$\begin{cases} \frac{4}{9} + c_1 + c_2 = 0 \\ -\frac{8}{9} + \frac{1}{3} - c_1 + c_2 = 0 \end{cases}$$

$-\frac{1}{9} + 2c_2 = 0$

$c_2 = \frac{1}{18}$

$c_1 = -\frac{1}{18} - \frac{4}{9} \Rightarrow c_1 = -\frac{9}{18} = -\frac{1}{2}$

$c_1 = -\frac{1}{2}$

$\Rightarrow x = \left(\frac{1}{3}t + \frac{4}{9}\right).e^{-2t} - \frac{1}{2}e^{-t} + \frac{1}{18}e^{t}$

1.7.12.

a) $\mathcal{L}(x+y) = (x+y)'' + 2(x+y)' + x + y$

$= x'' + 2x' + x + y'' + 2y' + y$

$= \mathcal{L}(x) + \mathcal{L}(y)$

$\mathcal{L}(kx) = (kx)'' - 2(kx)' + kx$

$= kx'' - 2kx' + kx$

$= k(x'' - 2x' + x)$

$= k.\mathcal{L}(x)$

$\Rightarrow L(x)$ is a Linear Map

$L(x) = 0 \Rightarrow x'' - 2x' + x = 0$

$\Rightarrow r^2 - 2r + 1 = 0 \Rightarrow (r-1)^2 = 0 \Rightarrow r = 1 \Rightarrow e^t, te^t$

$\Rightarrow x = c_1 e^t + c_2 t \cdot e^t \Rightarrow \dim(\operatorname{Ker} L) = 2$

b) $x'' - 2x' + x = \cos 2t$

particular sol. $x_p = a \cos 2t + b \sin 2t$

$\Rightarrow \underline{x_p'' - 2x_p' + x_p = \cos 2t}$

$x_p' = -2a \sin 2t + 2b \cos 2T$

$x_p'' = -4a \cos 2t - 4b \sin 2t$

$\Rightarrow -4a \cos 2t - 4b \sin 2t + 4a \sin 2t - 4b \cos 2t$

$+ a \cos 2t + b \sin 2t = \cos 2t$

$\Rightarrow \cos 2t(-4a - 4b + a - 1) + \sin 2t(-4b + 4a + b) = 0$

$\Rightarrow \cos 2t(-3a - 4b - 1) + \sin 2t(-3b + 4a) = 0$

$$\begin{cases} -3a - 4b - 1 = 0 & | \cdot 4 \\ 4a - 3b = 0 & | \cdot 3 \end{cases}$$

$$\Rightarrow \begin{cases} -12a - 16b - 4 = 0 \\ \underline{12a - 9b = 0} \quad (+) \end{cases}$$

$/ \; -25b = 4 \Rightarrow b = -\frac{4}{25}$

$4a + 3 \cdot \frac{4}{25} = 0 \Rightarrow 4a = -\frac{12}{25} \; | \cdot \frac{1}{4}$

$\Rightarrow a = -\frac{3}{}$

$\Rightarrow x_p = \cancel{\quad} - \frac{3}{25}\cos 2t - \frac{4}{25}\cdot \sin 2t$

$x = x_p + x_h \quad , \quad x_h = c_1 \cdot e^t + c_2 t \cdot e^t$

$\Rightarrow x = -\frac{3}{25}\cos 2t - \frac{4}{25}\cdot \sin 2t + c_1 \cdot e^t + c_2 t \cdot e^t$

c) $f_1(t) = e^{2t} \qquad f_2(t) = e^{-2t}$

$L(x) = 3f_1 + 5f_2$

From superpos. principle we have $x_p = 3x_{p1} + 5x_{p2}$

$x_{p1}'' - 2x_{p1}' + x_{p1} = e^{2t}$

$x_{p1} = e^{2t}$ can be a sol. $\Rightarrow x_{p1}' = 2e^{2t} \quad x_{p1}'' = 4e^{2t}$

$x_{p2}'' - 2x_{p2}' + x_{p2} = e^{-2t}$

we have a sol.: $x_{p2} = c e^{-2t}$

$\Rightarrow x_{p2}' = -2ce^{-2t} \qquad x_{p2}'' = -4ce^{-2t}$

$4ce^{-2t} + 4ce^{-2t} + ce^{-2t} = e^{-2t}$

$\Rightarrow 9ce^{-2t} = e^{-2t} \Rightarrow 9c = 1 \Rightarrow c = \frac{1}{9}$

$\Rightarrow x_{p2} = \frac{1}{9} e^{-2t}$

The particular sol is: $x_p = 3 \cdot e^{2t} + 5 \cdot \frac{1}{9} e^{-2t}$

1.7.19.

a) $x_p = t(a\cos 2t + b\sin 2t)$

$x_p' = a\cos 2t + b\sin 2t + t(-2a\sin 2t + 2b\cos 2t)$

$x_p'' = -2a\sin 2t + 2b\cos 2t + (-2a\sin 2t + 2b\cos 2t)$
$+ t(-4a\cos 2t - 4b\sin 2t)$

$x'' + 4x = \cos 2t$

$\Rightarrow x''_p + 4x_p = \cos t$

$\Rightarrow -2a\sin 2t + 2b\cos 2t + (-2a\sin 2t + 2b\cos 2t) +$
$+ t(-4a\cos 2t - 4b\sin 2t) + 4t(a\cos 2t + b\sin 2t) = \cos 2t$

$\Rightarrow -4a\sin 2t + 4b\cos 2t - \cancel{4t(a\cos 2t + b\sin 2t)} +$
$+ \cancel{4t(a\cos 2t + b\sin 2t)} = \cos 2t$

$\Rightarrow -4a\sin 2t + 4b\cos 2t = \cos 2t$

$\Rightarrow -4a\sin 2t + 4b\cos 2t - \cos 2t = 0$

$\Rightarrow \underbrace{-4a\sin 2t} + \cos 2t\underbrace{(4b-1)} = 0$

$-4a = 0 \qquad 4b - 1 = 0$

$a = 0 \qquad 4b = 1$

$b = \frac{1}{4}$

$x_p = t(0\cdot\cos 2t + \frac{1}{4}\sin 2t)$

$= t\cdot\frac{1}{4}\cdot\sin 2t$

b) $x = x_p + x_h$

$x'' + 4x = 0$

$r^2+4=0 \Rightarrow r^2=-4 \Rightarrow r=2i$
$r=-2i$ $\vdash \cos 2t, \sin 2t$

$\Rightarrow x_h = c_1 \cos 2t + c_2 \sin 2t$

$\Rightarrow x = t \cdot \frac{1}{4} \sin 2t + c_1 \cos 2t + c_2 \sin 2t$

c) $x'' + \frac{\nu}{m} x' + \frac{k}{m} x = f(t)$

$x'' + 4x = \cos 2t$ (*)

* models a motion without damping

$f(t) = A \cos \omega t$ $A > 0, \omega > 0$

$\Rightarrow A = 1$, $\omega t = 2t \Rightarrow \omega = 2$

$\Rightarrow \omega_0 = \sqrt{\frac{k}{m}} = \sqrt{4} = 2 \Rightarrow \frac{k}{m} = 4$

$\Rightarrow \omega_0 = \omega$

The spring oscillates with an amplitude that increases to ∞.

1.7.24. $mx'' + 25x = 12\cos(36\pi t)$

$mx'' + kx + 0x' = f(t) \Rightarrow k = 25$

The motion described is without ~~[illegible]~~ damping, but it has an external force

~~$f(t) = A \cos \omega t$~~

$f(t) = A \cos \omega t = 12 \cos(36\pi t)$

$A = 12$, $\omega = 36\pi$

$\omega_0 = \sqrt{\frac{k}{m}} = \sqrt{\frac{25}{m}} = \frac{5}{\sqrt{m}}$

$$\frac{5}{\sqrt{m}} = 36\pi$$

$$\Rightarrow \sqrt{m} = \frac{5}{36\pi} \Rightarrow m = \frac{25}{36^2\pi^2}$$

1.7.28. $\ddot{\theta} + \dot{\theta} + \theta = 0$

the charact. eq.: $r^2 + r + 1 = 0$

$\Delta = 1 - 4 = -3$

$\Rightarrow r_{1,2} = \frac{-1 \pm i\sqrt{3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ $\Big\langle$ $e^{-\frac{1}{2}t}\cos\frac{\sqrt{3}}{2}t$, $e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$

$$\Rightarrow \theta = c_1 \cdot e^{-\frac{1}{2}t} \cdot \cos\frac{\sqrt{3}}{2} + c_2 \cdot e^{-\frac{1}{2}t}\sin\frac{\sqrt{3}}{2}t$$

$$\lim_{t\to\infty} \theta(t) = \lim_{t\to\infty}\big(c_1 \cdot \underbrace{e^{-\frac{1}{2}t}}_{\to 0}\cos\frac{\sqrt{3}}{2} + c_2 \cdot \underbrace{e^{-\frac{1}{2}t}}_{\to 0} \cdot \sin\frac{\sqrt{3}}{2}t\big)$$

$$= 0$$

1.7.29. $t^2x'' + 2tx' - 2x = 0$, $t \in (0, \infty)$

a) $x(t) = t^r$

$x'(t) = rt^{r-1}$

$x''(t) = (r-1) \cdot r \cdot t^{r-2}$

$$\Rightarrow t^2(r-1) \cdot r \cdot t^{r-2} + 2t\,r\,t^{r-1} - 2t^r = 0$$

$$(r-1) \cdot r \cdot t^r + 2rt^r - 2t^r = 0$$

$$t^r(r^2 - r + 2r - 2) = 0$$

$$\Rightarrow r^2 + r - 2 = 0$$

$$\Delta = 1 + 8 = 9$$

$r_{1,2} = \frac{-1 \pm 3}{2}$ $\Big\langle$ $r_1 = 1 \Rightarrow x(t) = t$; $r_2 = -2 \Rightarrow x(t) = t^{-2}$