

Cycles of Well-Linked Sets and an Elementary Bound for the Directed Grid Theorem^{*}

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In 2015, Kawarabayashi and Kreutzer proved the directed grid theorem — the generalisation of the well-known excluded grid theorem to directed graphs — confirming a conjecture by Reed, Johnson, Robertson, Seymour, and Thomas from the mid-nineties. The theorem states the existence of a function f such that every digraph of directed tree-width $f(k)$ contains a cylindrical grid of order k as a butterfly minor, but the given function grows non-elementarily with the size of the grid minor. More precisely, it contains a tower whose height depends on the size of the grid.

In this paper we present an alternative proof of the directed grid theorem which is conceptually much simpler, more modular in its composition and also improves the upper bound for the function f to a power tower of height 22.

Our proof is inspired by the breakthrough result of Chekuri and Chuzhoy, who proved a polynomial bound for the excluded grid theorem for undirected graphs. We translate a key concept of their proof to directed graphs by introducing *cycles of well-linked sets (CWS)*, and show that any digraph of high directed tree-width contains a large CWS, which in turn contains a large cylindrical grid, improving the result due to Kawarabayashi and Kreutzer from a non-elementary to an elementary function.

^{*}The results in this manuscript were also presented in Milani's PhD thesis [Mil24].

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An immediate application of our result is that we can improve the bound for Younger’s conjecture—the directed Erdős-Pósa property—proved by Reed, Robertson, Seymour and Thomas [RRST96] from a non-elementary to an elementary function. The same improvement applies to other types of Erdős-Pósa style problems on directed graphs. To the best of our knowledge this is the first significant improvement on the bound for Younger’s conjecture since it was proved in 1996.

Since its publication in STOC 2015, the Directed Grid Theorem has found numerous applications (see for example [CLMS19, GKKK20b, JWZ23, GKW24, HRW19]), all of which directly benefit from our main result.

Finally, we believe that the theoretical tools developed in this work may find applications beyond the directed grid theorem, in a similar way as the path-of-sets-system framework due to Chekuri and Chuzhoy [CC16] did for undirected graphs (see for example [HKPS22, CC15, CN19]).

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1 Introduction

The excluded grid theorem by Robertson and Seymour is a central result in the study of graph minors and is the first major building block of their Graph Minors project [RS]. Additionally, the theorem has found a huge number of applications beyond its original scope, for instance in the theory of graph algorithms (see for example [CFK⁺15]). Based on a conjecture by Reed, Johnson, Robertson, Seymour and Thomas from the mid-nineties [JRST01b], Kawarabayashi and Kreutzer proved in 2015 [KK15] an excluded grid theorem for directed graphs, i.e. the existence of a function f such that every digraph of directed tree-width $f(k)$ contains a *cylindrical grid* of order k as a butterfly minor. In addition they proved that there is an XP algorithm that either produces a directed tree decomposition of width at most $f(k)$ or finds a cylindrical grid of order k as a butterfly minor. Campos et al. [CLMS19] improved their result from XP to FPT. The directed grid theorem has been used to prove advanced results in digraph structure theory [GKKK20b, GKKK20a, GKKK22], Erdős-Pósa/cycle packing [MMP⁺22, AKKW16, ACH⁺19, KKKX23], and matching theory [HRW19, GKW24] as well as for algorithmic results [BJCH16, EMW17].

For a certain class of problems on digraphs, the presence of a cylindrical grid minor immediately results in a positive instance. The strength of the directed grid theorem is then fully realised in providing a win-win scenario. On one hand we have low directed treewidth, which in many cases allows us to compute the problem on a subset of the vertex set of size $f(k)$. On the other we have a cylindrical grid minor. The function f is the major determinant of the efficiency of algorithms obtained through this method. Unfortunately, the original function by Kawarabayashi and Kreutzer is non-elementary, specifically it contains a tower whose height is dependent on the size of the grid. Our main contribution is an improvement on the proof of the directed grid theorem in two ways. First, by improving the bound on the function f to an elementary one, and second, by developing novel techniques which allow us to make the proof more modular and easier to understand.

More precisely, we require the directed tree-width of a digraph to be at least a power tower of height 5 for finding a *split* or *segmentation* (Theorem 5.15), a power tower of height 7 for finding a *path of well-linked sets* of width w and length ℓ (Theorem 10.9), and a power tower of height 22 if we want to obtain a cylindrical grid of order k (Theorem 1.2).

Further, our result gives better bounds for several Erdős-Pósa-like theorems for digraphs. More precisely we say that a graph H has the Erdős-Pósa property if there is a function l_H such that in any digraph D we can either find n disjoint H -butterfly minors or $l_H(n)$ vertices covering all H -butterfly minors. Amiri, Kawarabayashi, Kreutzer and Wollan [AKKW16] prove that the Erdős-Pósa property holds for strongly connected directed graphs precisely when they are minors of the cylindrical grid. Their methods relies heavily on the use of a directed grid and the functions l_H they obtained depends on the function determined in Kawarabayashi and Kreutzer's Directed Grid Theorem. Thus, our result provides new elementary functions for this result. When H is a directed cycle on two vertices C_2 , this result is equivalent to Younger's conjecture, which was proven to be true in 1996 by Reed, Robertson, Seymour and Thomas. Their proof resulted in a non-elementary function l_{C_2} and has since not been improved. Our new bound for f , together with the result in [AKKW16], gives the first (to the best of our knowledge) elementary bound for the function l_{C_2} .

Inspired by *path-of-sets system* framework due to Chekuri and Chuzhoy [CC16], which played an important role in their proof of a polynomial bound for the undirected grid theorem, we also build our proof around finding sequences of sets which are highly connected in one direction. In order to handle all the cases that appear in the directed setting, we need to consider two

types of highly connected sets, namely *well-linked* and *order-linked sets*, which in turn lead us to our definitions of *paths of well-linked sets*, *paths of order-linked sets* and *cycles of well-linked sets*. The latter three concepts naturally capture the connectivity properties provided by *fences*, *acyclic grids* and *cylindrical grids*, respectively.

In order to obtain the connectivity properties required above, we develop a framework based on the known concept of *temporal digraphs* (see, e.g. [CHMZ20, Mol20]), which also naturally models our setting where disjoint paths intersect a sequence of disjoint subgraphs in the same order. We then introduce the concept of *H-routings* for digraphs and temporal digraphs, which, on digraphs, is a weaker property than having H as an immersion or as a butterfly minor. In particular, obtaining the desired connectivity corresponds to finding temporal walks in certain temporal digraphs and constructing H -routings from these walks.

Our modular approach facilitates the transfer of the intermediate results in our proof to other settings. Well-linked sets play an important role in several results in the theory of digraphs (for example, in [RRST96, JRST01b, EMW17, KK15]) and our framework provides additional tools for obtaining such sets.

Further, by reusing the existing concept of temporal digraphs, we also make the proof of the directed grid theorem more accessible to a larger community. Indeed, one of the important steps in obtaining an acyclic grid in our proof is Lemma 6.10, whose bound is currently not polynomial. Reducing this bound is an important step towards improving the function of the directed grid theorem, and both the statement and its proof can be expressed using the language of temporal digraphs.

From an algorithmic perspective, our intermediate concepts facilitate the design of efficient algorithms for finding a directed grid, as questions regarding finding long walks in temporal digraphs, constructing H -routings, obtaining well-linked sets and constructing a cylindrical grid from a cycle of well-linked sets can all be considered independently from each other, simplifying the process of identifying bottlenecks and computational obstacles in each step of the proof.

The paper is organized as follows. In Section 2 we provide an overview of the proof. Sections 3 and 4 contain preliminary definitions which are used throughout the paper. We construct a web in a digraph of high directed treewidth in Section 5, improving the corresponding step of the proof of [KK15] from a non-elementary to an elementary bound. We introduce our framework on temporal digraphs in Section 6, where we also obtain the H -routings from which we construct our order-linked and well-linked sets. In sections 7 to 9 we introduce the concepts of paths of order-linked sets, paths of well-linked sets and cycles of well-linked sets, respectively, and show how to obtain the corresponding type of grid from each of them. Finally, in sections 10 and 11 we apply the framework developed in the sections above in order to construct a path of well-linked sets and a cycle of well-linked sets, obtaining one of our main results.

Theorem 1.1. Let w, ℓ be integers. Every digraph D with $\text{dtw}(D) \geq \text{dtw}_{1.1}(w, \ell)$ contains a cycle of well-linked sets $(\mathcal{S}, \mathcal{P})$ of width w and length ℓ .

Since we prove in Theorem 9.3 that every cycle of well-linked sets contains a cylindrical grid, the theorem above yields another of our main results.

Theorem 1.2. Every digraph D with $\text{dtw}(D) \geq \text{dtw}_{1.2}(k)$ contains a cylindrical grid of order k as a butterfly minor.

2 An overview of the proof

We provide an overview of our contribution with sketches of proofs for essential statements. We leave the numbers of all environments the same as in the full version, which follows after.

We use standard definitions for (directed) graphs without loops or multiedges (unless specifically stated otherwise), see [Section 3](#) for the formal statements. When working with a set or another structure X containing digraphs, we write $D(X)$ to mean the digraph obtained by taking the union of all digraphs in X . A set A is *ordered* when it comes equipped with an ordering \leq_A of its vertices. We denote the digraph of a path on k vertices by \mathbf{P}_k . For the *bidirected path on k vertices*, we write $\tilde{\mathbf{P}}_k := (\{u_1, u_2, \dots, u_k\}, \{(u_i, u_j) \mid 1 \leq i, j \leq k \text{ and } |i - j| = 1\})$. The *cycle on k vertices* is given by $\mathbf{C}_k := (\{u_0, u_1, \dots, u_{k-1}\}, \{(u_i, u_{i+1 \bmod k}) \mid 0 \leq i < k\})$. Finally, we write $\tilde{\mathbf{K}}_k := (\{u_1, u_2, \dots, u_k\}, \{(u_i, u_j) \mid 1 \leq i, j \leq k \text{ and } i \neq j\})$ for the *complete digraph on k vertices*.

We consider different connectivity measures for digraphs. A digraph D is said to be *strongly connected* if for every $u, v \in V$ there is a u - v -path **and** a v - u -path in D . We say D is *unilateral* [[HNC65](#)] if for every $u, v \in V$ there is a u - v -path **or** a v - u -path in D . Finally, D is *weakly-connected* if the underlying undirected graph of D is connected.

Let $A, B \subseteq V(D)$ be vertex sets in a digraph D . An *A - B -linkage* \mathcal{L} of order k is a set of k disjoint paths $\{L_1, L_2, \dots, L_k\} = \mathcal{L}$ such that $\text{start}(L_i) \subseteq A$ and $\text{end}(L_i) \subseteq B$ for all $1 \leq i \leq k$. We write $\text{start}(\mathcal{L})$ for the set $\{\text{start}(L_i) \mid L_i \in \mathcal{L}\}$ and, similarly, we write $\text{end}(\mathcal{L})$ for the set $\{\text{end}(L_i) \mid L_i \in \mathcal{L}\}$. We extend the notation for path concatenation to linkages. Given two linkages $\mathcal{P} = \{P_1, P_2, \dots, P_p\}$ and $\mathcal{Q} = \{Q_1, Q_2, \dots, Q_q\}$ such that $\text{end}(\mathcal{P}) = \text{start}(\mathcal{Q})$, we write $\mathcal{P} \cdot \mathcal{Q}$ for the linkage $\{P_a \cdot Q_b \mid P_a \in \mathcal{P}, Q_b \in \mathcal{Q} \text{ and } \text{end}(P_a) = \text{start}(Q_b)\}$. Additionally, we sometimes use a linkage \mathcal{L} as a function $\mathcal{L} : \text{start}(\mathcal{L}) \rightarrow \text{end}(\mathcal{L})$. The expression $\mathcal{L}(a) = b$ then means that \mathcal{L} contains a path starting in a and ending in b .

Let A, B be sets of vertices in a digraph D . We say that A is *well-linked to B in D* if for every $A' \subseteq A$ and every $B' \subseteq B$ with $|A'| = |B'|$ there is an A' - B' linkage of order $|A'|$ in D .

Let D be a digraph, let $H \subseteq D$ be a subgraph, and let \mathcal{L} be a linkage of order k . We say that \mathcal{L} is *minimal with respect to H* , or *H -minimal*, if for all edges $e \in \bigcup_{P \in \mathcal{L}} E(P) \setminus E(H)$ there is no $\text{start}(\mathcal{L})$ - $\text{end}(\mathcal{L})$ -linkage of order k in the graph $(\mathcal{L} \cup H) - e$. Given a linkage \mathcal{L} in a digraph D and a subgraph $H \subseteq D$, it is always possible to obtain a linkage \mathcal{L}' with same order and same endpoints as \mathcal{L} which is H -minimal by iteratively removing edges $e \in E(\mathcal{L}) \setminus E(H)$ for which a $\text{start}(\mathcal{L})$ - $\text{end}(\mathcal{L})$ linkage of order $|\mathcal{L}|$ exists avoiding e . The following is a particularly useful property of minimal linkages, and was also extensively used in the proof of [[KK15](#)].

Definition 3.5 (weak minimality). A linkage \mathcal{L} in a digraph D is *weakly k -minimal* with respect to a subgraph H of D if for every $P_1 \cdot e \cdot P_2 \in \mathcal{L}$ where $e \in E(\mathcal{L}) \setminus E(H)$ there is a $V(P_1)$ - $V(P_2)$ -separator of size at most $k - 1$ in $(\mathcal{L} \cup H) - e$.

Observation 3.6. Let H be a subgraph of a digraph D and let \mathcal{L} be a linkage which is H -minimal. Then \mathcal{L} is weakly $|\mathcal{L}|$ -minimal with respect to H .

Given a digraph D and an arc $(u, v) \in E(D)$, we say that (u, v) is *butterfly contractible* if $|N_D^{\text{in}}(v)| = 1$ or $|N_D^{\text{out}}(u)| = 1$. A digraph H is a *butterfly minor* of D if it can be obtained from a subgraph of D by contracting butterfly contractible edges.

A *cylindrical grid* of order k is a digraph consisting of k pairwise disjoint directed cycles C_1, C_2, \dots, C_k of length $2k$, together with a set of $2k$ pairwise vertex disjoint paths P_1, P_2, \dots, P_{2k} of length $k - 1$ such that

- each path P_i has exactly one vertex in common with each cycle C_j and both endpoints of P_i are in $V(C_1) \cup V(C_k)$,
- the paths P_1, P_2, \dots, P_{2k} appear on each C_i in this order, and
- for each $1 \leq i \leq 2k$, if i is odd, then the cycles C_1, C_2, \dots, C_k occur on P_i in this order and, if i is even, then the cycles occur in the reverse order C_k, C_{k-1}, \dots, C_1 .

Removing all edges ending in P_1 results in a structure called a (k, k) -fence.

Two linkages \mathcal{H} and \mathcal{V} in a digraph D build an (h, v) -web $(\mathcal{H}, \mathcal{V})$ if every path in \mathcal{V} intersects every path in \mathcal{H} . The set $\text{start}(\mathcal{V})$ is called the *top* of the web, while the set $\text{end}(\mathcal{V})$ is called the *bottom* of the web. Finally, $(\mathcal{H}, \mathcal{V})$ is *well-linked* if $\text{end}(\mathcal{V})$ is well-linked to $\text{start}(\mathcal{V})$ in D .

2.1 Cycles and paths of sets

We construct a cylindrical grid by first constructing an acyclic grid and then finding a fence inside it. To achieve this, we first construct objects which contain similar connectivity properties as these three types of grid, while not necessarily being planar.

We introduce the concepts of *r-order-linkedness* and *shifts* in order to capture the connectivity provided by acyclic grids.

Shifts and order-linkedness Let $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_m)$ be ordered sets. Let $r \in \mathbb{N}$, let A' be an ordered subset of A and let B' be an ordered subset of B such that $|A'| = |B'|$. We say that B' is an *r-shift* of A' if there is a bijection $\pi : A' \rightarrow B'$ such that

- for all $a_i \in A'$ we have that $\pi(a_i) = b_j$ implies $i \leq j$;
- there are at most r vertices $a_i \in A'$ with $\pi(a_i) \neq b_i$; and
- for all $a_i, a_j \in A'$, if $a_i \leq_A a_j$, then $\pi(a_i) \leq_B \pi(a_j)$.

Let H be a digraph, $A = (a_1, \dots, a_n), B = (b_1, \dots, b_m) \subseteq V(H)$ be ordered sets and let $r \in \mathbb{N}$. We say that A is *r-order-linked* to B in H if for every $A' \subseteq A$ and every $B' \subseteq B$ with $|A'| = |B'|$ where B' is an *r-shift* of A' witnessed by the bijection π there is an $A'-B'$ -linkage \mathcal{L} in H satisfying $\pi(a) = \mathcal{L}(a)$ for all $a \in A'$.

We can now define the concepts of *paths of well-linked sets*, which behave like fences, and of *paths of r-order-linked sets*, which behave like acyclic grids.

Definition 7.3 and 8.1 (path of *r-order-linked/well-linked* sets). A *path of r-order-linked/well-linked* sets of width w and length ℓ is a tuple $(\mathcal{S}, \mathcal{P})$ such that

1. \mathcal{S} is a sequence of $\ell + 1$ pairwise disjoint subgraphs (S_0, \dots, S_ℓ) , which are called *clusters*,
2. for every $0 \leq i \leq \ell$ there are disjoint *ordered* sets $A(S_i), B(S_i) \subseteq V(S_i)$ of size w such that $A(S_i)$ is *r-order-linked/well-linked* to $B(S_i)$ in S_i ,
3. \mathcal{P} is a sequence of ℓ pairwise disjoint linkages $(\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{\ell-1})$ such that, for every $0 \leq i < \ell$, \mathcal{P}_i is a $B(S_i)$ - $A(S_{i+1})$ -linkage of order w which is internally disjoint from S_i and S_{i+1} and disjoint from every $S \in \mathcal{S} \setminus \{S_i, S_{i+1}\}$.

Further, a *path of r-order-linked sets* $(\mathcal{S}, \mathcal{P})$ is called *uniform* if for all $0 \leq i < \ell$ and for all $b_1, b_2 \in B(S_i)$ we have that $b_1 \leq_{B(S_i)} b_2$ implies $\mathcal{P}_i(b_1) \leq_{A(S_{i+1})} \mathcal{P}_i(b_2)$. A *path of well-linked sets* is called *strict* if every vertex in S_i lies on an $A(S_i)$ - $B(S_i)$ -path.

A *cycle of well-linked sets* of width w and length ℓ is a pair $(\mathcal{S}, \mathcal{P} \cup \{\mathcal{P}_\ell\})$ where $(\mathcal{S}, \mathcal{P})$ is a path of well-linked sets of width w and length $\ell - 1$, and \mathcal{P}_ℓ is a linkage from the B -set of the last cluster to the A -set of the first cluster that is internally disjoint from $(\mathcal{S}, \mathcal{P})$.

We obtain the following connection between paths of order-linked sets and paths of well-linked sets, allowing us to focus on obtaining paths of order-linked sets when proving our main result. The construction is quite similar to the one used to obtain a fence from an acyclic grid, and the bounds we obtain are essentially the same.

Lemma 8.3. Let $w_{8.3}(w, \ell) := w(\ell + 1)$. Every path of w -order-linked sets $(\mathcal{S}, \mathcal{P})$ of width at least $w_{8.3}(w, \ell)$ and length at least ℓ contains a path of well-linked sets $(\mathcal{S}', \mathcal{P}')$ of width w and length ℓ .

Since we can construct acyclic grids, fences and cylindrical grids from the objects defined above, it suffices for our results to show that digraphs of high directed tree-width contain a large cycle of well-linked sets.

Theorem 9.3. Every cycle of well-linked sets of width $w \geq \text{w}_{9.3}(k)$ and length $\ell \geq \text{e}_{9.3}(k)$ contains a cylindrical grid of order k .

This allows us to divide our proof into roughly three main ‘‘parts’’.

The first part consists of finding a well-linked web $(\mathcal{P}, \mathcal{Q})$ from a bramble of high order where \mathcal{P} is minimal with respect to \mathcal{Q} . We emphasise that the requirement of minimality is what makes obtaining such a web very challenging.

After obtaining the well-linked web, we can obtain a similar structure where one linkage is ‘‘ordered’’ according to the other, constructing objects which are called *splits* and *segmentations* in [KK15].

Theorem 5.15. Let D be a digraph. If $\text{dtw}(D) \geq \text{t}_{5.15}(x, y, q, k)$, then D contains one of the following

- (D1) a cylindrical grid of order k as a butterfly minor,
- (D2) a (y, q) -split $(\mathcal{P}', \mathcal{Q}')$ of $(\mathcal{P}_1, \mathcal{Q}_1)$ in D , where $\text{end}(\mathcal{Q}')$ is well-linked to $\text{start}(\mathcal{Q}')$, or
- (D3) an (x, q) -segmentation $(\mathcal{P}', \mathcal{Q}')$ of $(\mathcal{P}_1, \mathcal{Q}_1)$ in D , where $\text{end}(\mathcal{P}')$ is well-linked to $\text{start}(\mathcal{P}')$.

In the second part, we construct a path of well-linked sets from the splits and segmentations obtained previously, together with a *back-linkage*, which is a linkage from the B -set of the last cluster to the A -set of the first in the path of well-linked sets.

Theorem 10.9. Every digraph D with $\text{dtw}(D) \geq \text{t}_{10.9}(w, \ell)$ contains a path of well-linked sets $(\mathcal{S} = (S_0, S_1, \dots, S_\ell), \mathcal{P})$ of width w and length ℓ such that $B(S_\ell)$ is well-linked to $A(S_0)$ in D .

In the third and final part, we obtain a cycle of well-linked sets from a path of well-linked sets with a back-linkage.

Theorem 11.22. Let w, ℓ be integers, let $(\mathcal{S} = (S_0, S_1, \dots, S_{\ell'}), \mathcal{P})$ be a strict path of well-linked sets of width w' and length ℓ' and let \mathcal{R} be a $B(S_{\ell'})$ - $A(S_0)$ linkage of order r . If $w' \geq \text{w}_{11.22}(w, \ell, r)$, $r \geq \text{r}_{11.22}(w, \ell)$ and $\ell' \geq \text{e}_{11.22}(w, \ell, r)$, then $\text{D}((\mathcal{S}, \mathcal{P}) \cup \mathcal{R})$ contains a cycle of well-linked sets of width w and length ℓ .

We then combine the statements above to produce the first of our main results.

Theorem 1.1. Let w, ℓ be integers. Every digraph D with $\text{dtw}(D) \geq \text{dtw}_{1.1}(w, \ell)$ contains a cycle of well-linked sets $(\mathcal{S}, \mathcal{P})$ of width w and length ℓ .

Proof. Let $r_1 = \text{r}_{11.22}(w, \ell)$, $w_1 = \text{w}_{11.22}(w, \ell, r) + r$ and $\ell_1 = \text{e}_{11.22}(w, \ell, r)$. By Theorem 10.9, D contains a path of well-linked sets $(\mathcal{S} = (S_0, S_1, \dots, S_{\ell_1}), \mathcal{P})$ of width $\text{w}_{11.22}(w, \ell)$ and length $\ell_1 := \text{e}_{11.22}(w, \ell)$ where $B(S_{\ell_1})$ is well-linked to $A(S_0)$ in D . Hence, there is a $B(S_{\ell_1})$ - $A(S_0)$ linkage \mathcal{R} of order r_1 in D . By Theorem 11.22, $\text{D}(\mathcal{S} \cup \mathcal{P} \cup \mathcal{R})$ contains a cycle of well-linked sets $(\mathcal{S}', \mathcal{P}')$ of width w and length ℓ . \square \square

We note that [KK22] also split their proof into three parts, which served as a base for the outline of our proof. However, the bounds they obtain for all three of their corresponding statements grow larger than any elementary function, and so we essentially need to improve upon nearly all of their proofs.

Combining Theorems 1.1 and 9.3 then yields our improvement of the Directed Grid Theorem.

Theorem 1.2. Every digraph D with $\text{dtw}(D) \geq \text{dtw}_{1,2}(k)$ contains a cylindrical grid of order k as a butterfly minor.

2.2 Obtaining a path of well-linked sets

Several steps of our proofs revolve around linkages which intersect certain subgraphs in an ordered fashion. To simplify our arguments and reasoning and to avoid repetition, we model this configuration in an abstract manner with the help of the concept of *temporal digraphs*.

A *temporal digraph* is a pair $T = (V, \mathcal{A})$ consisting of a vertex set V and sequence of arc sets $\mathcal{A} = (A_1, A_2, \dots, A_\ell)$ such that $D_t(T) := (V, A_t)$ is a digraph for all $1 \leq t \leq \ell$. We also refer to $D_t(T)$ as *layer t* of T and call t a *time step*. The *lifetime* of D is given by $\ell(D) := \ell$. A *temporal walk* of length n from v_0 to v_n in a temporal digraph T is a sequence $W := (v_0, t_0), (v_1, t_1), \dots, (v_n, t_n)$ such that $(v_i, v_{i+1}) \in A_{t_i}$ and $t_i < t_{i+1} \leq \ell(T)$ for all $0 \leq i \leq n - 1$. If such a walk exists, we say that v_0 *temporally reaches* v_n . A temporal walk is said to be a *temporal path* if no vertex appears twice in the sequence. Finally, we say that W *departs* at t_0 and *arrives* at t_n , and that $t_n - t_0$ is the *duration* of W .

Usage of temporal digraphs arise naturally in a directed setting. Consider the example given in [Figure 1](#). If we want to construct a new linkage starting and ending in a subset of the starting and endpoints of $\mathcal{P} := \{P_a, P_b, P_c\}$, then as soon as a path in our new linkage visits a vertex in Q_2 , it can no longer use vertices from Q_1 , as we only have connectivity from “left” to “right”. Further, as we want some way of ensuring that our paths are disjoint and form a linkage, we are interested in the connectivity provided by each Q_i “between” the paths in \mathcal{P} without intersecting other paths in \mathcal{P} . This intuition leads us to the following definition.

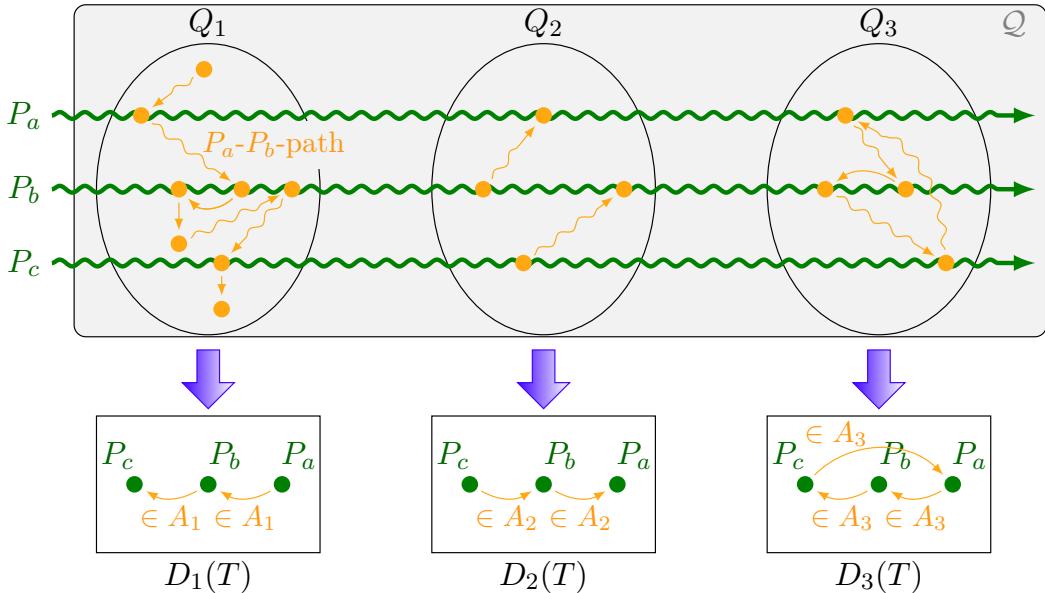


Figure 1: The layers $D_j(T)$ of the temporal graph $T := (V = \{a, b, c\}, \mathcal{A} = \{A_1, A_2, A_3\})$ constructed from the graphs Q_j displayed above as defined in [Definition 6.3](#).

Definition 6.3. Let \mathcal{P} be a linkage and let $\mathcal{Q} = \{Q_1, Q_2, \dots, Q_q\}$ be a set of pairwise disjoint digraphs such that each path $P_i \in \mathcal{P}$ can be partitioned as $P_i^1 \cdot P_i^2 \cdot \dots \cdot P_i^q = P_i$ such that $V(P_i^j) \cap V(\mathcal{Q}) \subseteq V(Q_j)$ for all $1 \leq j \leq q$. The *routing temporal digraph* (V, \mathcal{A}) of \mathcal{P} through

\mathcal{Q} , which we also refer to as $\mathcal{T}(\mathcal{P}, \mathcal{Q})$, is constructed as follows. We set $V = \mathcal{P}$ and for each $1 \leq j \leq q$ we define $A_j = \{(P_a, P_b) \mid P_a, P_b \in \mathcal{P} \text{ and there is a path from } V(P_a) \text{ to } V(P_b) \text{ inside } Q_j \text{ which is internally disjoint from } \mathcal{P}\}$. $\mathcal{T}(\mathcal{P}, \mathcal{Q})$

Routings In order to better describe the kind of connectivity that a routing temporal digraph provides between the paths of the original linkage, we define the concept of H -routings, where H is some digraph. We would like to point out to the reader that connectivity in temporal digraphs is not transitive, and hence it is not sufficient to restrict the following definition to edges.

Definition 6.4. Let H be a digraph, D be a (temporal) digraph and $S \subseteq V(D)$. An H -routing (over S) is a bijection $\varphi : V(H) \rightarrow S$ such that for each $v-u$ path P in H we can find a $\varphi(v)$ - $\varphi(u)$ (temporal) path in D which is disjoint from $S \setminus \varphi(V(P))$.

In order to apply the framework defined above, we simplify the notion of splits and segmentations to slightly more general structures we call *ordered* and *folded* webs.

Definition 10.1. Let $(\mathcal{H}, \mathcal{V})$ be an (h, v) -web. We say that $(\mathcal{H}, \mathcal{V})$ is an *ordered web* if there is an ordering of $\mathcal{V} = (V_1, V_2, \dots, V_v)$ for which each path $H \in \mathcal{H}$ can be decomposed into $H = H_1 \cdot H_2 \cdots H_v$ such that H_i intersects V_j if and only if $i = j$.

Definition 10.5. An (h, v) -web $(\mathcal{H}, \mathcal{V})$ is a *folded web* if every $V_i \in \mathcal{V}$ can be split as $V_i^a \cdot V_i^b := V_i$ such that both V_i^a and V_i^b intersect all paths of \mathcal{H} .

It is not difficult to show that a (p, q) -segmentation $(\mathcal{P}, \mathcal{Q})$ from [KK22] is also an ordered web $(\mathcal{P}, \mathcal{Q})$, and that a (p, q) -split $(\mathcal{P}, \mathcal{Q})$ is a folded ordered $(\mathcal{Q}, \mathcal{P})$ web.

Our main results related to routing temporal digraphs are about finding \mathbf{P}_k , \mathbf{C}_k and $\tilde{\mathbf{P}}_k$ -routings in different contexts. First, we show that unilateral temporal digraphs contain a walk with many vertices. The reason why we consider temporal digraphs where the layers are unilateral is that, given an ordered web $(\mathcal{H}, \mathcal{V})$, each layer of $\mathcal{T}(\mathcal{H}, \mathcal{V})$ is unilateral.

Lemma 6.10. Let $\ell_{6.10}(n, k) := 2kn \sum_{i=1}^{2kn} n^i$. Let T be a temporal digraph with n vertices where each layer is unilateral and let $S \subseteq V(T)$ be a set of size k . If $\ell(T) \geq \ell_{6.10}(n, k)$, then T contains a temporal walk W with $S \subseteq V(W)$.

We can then use the walk obtained in Lemma 6.10 in order to construct a \mathbf{P}_k -routing.

Theorem 6.12. Let $\ell_{6.12}(n, k) := \ell_{6.10}(n, k^2 - 1)$. Let T be a temporal digraph where each layer is unilateral. If $\ell(T) \geq \ell_{6.12}(n, k)$ and $n := |V(T)| \geq k^2 - 1$, then there is some set $S \subseteq V(T)$ such that T contains a \mathbf{P}_k -routing over S .

Intuitively, a \mathbf{P}_k -routing in a routing temporal digraph $\mathcal{T}(\mathcal{P}, \mathcal{Q})$ gives connectivity between the paths in \mathcal{P} which is similar to a column in an acyclic grid, which in turn is related to the concept of order-linkedness defined before. This intuition is formalised below.

Lemma 7.6. Let h, k be integers. Let T be the routing temporal digraph of some linkage \mathcal{L} through a sequence (H_1, H_2, \dots, H_h) of disjoint digraphs. Let $\mathcal{L}' \subseteq \mathcal{L}$ be a linkage of order at most k . If T contains a \mathbf{P}_k -routing on the paths $L_1, L_2, \dots, L_k \in \mathcal{L}'$, ordered according to their occurrence on the \mathbf{P}_k -routing, then A is 1-order-linked to B in $D(\mathcal{L} \cup \bigcup_{i=1}^h H_i)$, where $A = \{a_i \mid a_i \text{ is the first vertex of } L_i \text{ on } H_1\}$ and $B = \{b_i \mid b_i \text{ is the last vertex of } L_i \text{ on } H_h\}$.

It is not difficult to see that, given a folded ordered web $(\mathcal{H}, \mathcal{V})$, each layer of $\mathcal{T}(\mathcal{H}, \mathcal{V})$ is strongly-connected. Having strongly-connected layers allows us to obtain \mathbf{C}_k or $\tilde{\mathbf{P}}_k$ -routings

instead. Intuitively, because these two types of routings provide connectivity in both directions between the paths of our linkage \mathcal{P} , we are able to use them to obtain well-linked instead of order-linked sets.

Theorem 6.16. Let T be a temporal digraph such that $D_i(T)$ is strongly-connected for all $1 \leq i \leq \ell(T)$. If $\ell(T) \geq \ell_{6.16}(k)$, then for every set $S \subseteq V(T)$ with $|S| \geq s_{6.16}(k)$ there is a subset $S' \subseteq S$ with $|S'| = k$ such that D contains an H -routing over S' for some $H \in \{\mathbf{C}_k, \vec{\mathbf{P}}_k\}$.

Corollary 10.8. Let $(\mathcal{H}, \mathcal{V})$ be a folded ordered (h, v) -web. If $h \geq h_{10.7}(w)$ and $v \geq v_{10.7}(w, \ell)$, then $(\mathcal{H}, \mathcal{V})$ contains a path of well-linked sets $(\mathcal{S} = (S_0, S_1, \dots, S_\ell), \mathcal{P})$ of width w and length ℓ . Additionally, $A(S_0) \subseteq \text{start}(\mathcal{H})$ and $B(S_\ell) \subseteq \text{end}(\mathcal{H})$.

We can now use the results above to prove [Theorem 10.9](#).

Proof sketch of Theorem 10.9. By [Theorem 5.15](#), we have one of three cases. If D contains a cylindrical grid of order $w\ell$, then it contains a cycle of well-linked sets of width w and length ℓ .

If D contains a (y, q) -split $(\mathcal{V}, \mathcal{H})$, then this split is essentially a folded ordered web $(\mathcal{H}, \mathcal{V})$. By [Corollary 10.8](#), we can construct a path of well-linked paths from this split. Moreover, the beginning and the end of this path of well-linked sets coincides with $\text{start}(\mathcal{V})$ and $\text{end}(\mathcal{V})$, respectively. As $\text{end}(\mathcal{V})$ is well-linked to $\text{start}(\mathcal{V})$, the path of well-linked sets obtained satisfies the restrictions in the statement.

In the last case, D an (x, q) -segmentation $(\mathcal{H}, \mathcal{V})$, which also means that $(\mathcal{H}, \mathcal{V})$ is an ordered web. Applying [Lemma 10.4](#) to $(\mathcal{H}, \mathcal{V})$ yields a path of well-linked sets. Moreover, the beginning and the end of this path of well-linked sets coincides with $\text{start}(\mathcal{H})$ and $\text{end}(\mathcal{H})$, respectively. As $\text{end}(\mathcal{H})$ is well-linked to $\text{start}(\mathcal{H})$, the path of well-linked sets obtained satisfies the restrictions in the statement. \square \square

2.2.1 Constructing a cycle of well-linked sets

As we can find paths of well-linked sets with their final B -set being well-linked to the initial A -set, our goal is to find a linkage between these two sets that is internally disjoint from the path of well-linked sets in order to obtain a cycle of well-linked sets in the end.

Back-linkage intersecting cluster by cluster First we further analyse the ways a back-linkage can intersect a given path of well-linked sets.

Let $(\mathcal{S}, \mathcal{P})$ be a path of well-linked sets of length ℓ . A *jump* of length k over $(\mathcal{S}, \mathcal{P})$ is a path R with $\text{start}(R) \in V(S_i) \cup V(\mathcal{P}_i)$ and $\text{end}(R) \subseteq V(S_j) \cup V(\mathcal{P}_j)$ (if $j = \ell$, we require $\text{end}(R) \subseteq V(S_j)$ instead) such that $|j - i| = k$. If $i < j$, then R is a *forward jump*. If $i \geq j$ and R is internally disjoint from $(\mathcal{S}, \mathcal{P})$, then R is a *backward jump*.

Let \mathcal{R} be a partial back-linkage for $(\mathcal{S}, \mathcal{P})$. We say that \mathcal{R} intersects $(\mathcal{S}, \mathcal{P})$ *cluster by cluster* if \mathcal{R} does not contain any forward or backward jump of length greater than one over $(\mathcal{S}, \mathcal{P})$. We can show that in case we do not immediately obtain the desired cycle of well-linked sets we can find a back-linkage that intersects cluster by cluster, thus in a slightly more ordered fashion.

Lemma 11.6. Let ℓ_1, w_1, ℓ_2, w_2 be integers, let $(\mathcal{S} = (S_0, S_1, \dots, S_{\ell'}), \mathcal{P} = (\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{\ell'-1}))$ be a strict path of well-linked sets of length $\ell' \geq \ell'_{11.6}(w_1, \ell_1, \ell_2, m)$ and width $w' \geq w'_{11.6}(w_1, w_2)$ with a partial back-linkage \mathcal{R} of order at least w_2 in a digraph D such that \mathcal{R} is weakly m -minimal with respect to $(\mathcal{S}, \mathcal{P})$. Then D contains at least one of the following:

- (C1) a cycle of well-linked sets of length ℓ_1 and width w_1 , or

(C2) a path of well-linked sets of length ℓ_2 and width w_2 together with a partial back-linkage $\mathcal{R}' \subseteq \mathcal{R}$ of order w_2 intersecting it cluster-by-cluster.

Proof sketch. First we establish that if there are no long back-jumps, then we can obtain (C2). We then choose a maximal family \mathcal{J} of nested longer and longer back-jumps and consider the subpath of well-linked sets all elements of \mathcal{J} jump over.

If $|\mathcal{J}| < w_1$, then use the partial back-linkage over the whole path of well-linked sets to construct a partial back-linkage over the subpath by using linkages in the unused parts, the back-linkage and Menger's Theorem. As \mathcal{J} is chosen maximally there are no more long back-jumps over this subpath of well-linked sets, which again allows us to obtain (C2).

So assume \mathcal{J} contains w_1 jumps. We use the jumps in \mathcal{J} to construct a partial back-linkage for the subpath of well-linked sets that is completely disjoint. We do so by finding a linkage \mathcal{X}_1 from the last cluster to the start-vertices of the jumps, and a linkage \mathcal{X}_2 from the end-vertices to the first cluster. In a path of well-linked sets is always possible to find a path of well-linked sets of reduced width choosing subsets of the first A -set and the last B -set. Thus, we can find a path of well-linked sets of reduced width with the end-vertices of \mathcal{X}_2 being the first A -set and the start-vertices \mathcal{X}_1 being the last B -set, yielding the cycle of well-linked sets of length ℓ_1 and width w_1 , satisfying (C1). \square

Getting a 2-horizontal web Given a path of well-linked sets and a back-linkage intersecting it cluster by cluster, we construct a new type of web, which we call q -horizontal web, that “preserves” the cluster by cluster property of the back-linkage.

Definition 11.7 (q -horizontal web). Let $(\mathcal{H}, \mathcal{V})$ be a web. We say that $(\mathcal{H}, \mathcal{V})$ is a q -horizontal web if every path $H_i \in \mathcal{H}$ can be decomposed into paths $H_i^1 \cdot H_i^2 \cdot \dots \cdot H_i^q = H_i$ and every path $V_j \in \mathcal{V}$ can be decomposed into paths $V_j^1 \cdot V_j^2 \cdot \dots \cdot V_j^q = V_j$ such that $V_j^x \cap H_i \subseteq H_i^{q-x+1} \cup H_i^{q-x+2}$ and $V_j^x \cap H_i^{q-x+1} \neq \emptyset$ for all $1 \leq x \leq q$, where for simplicity we define H_i^{q+1} to be empty.

We can construct an ordered web $(\mathcal{R}, \mathcal{V})$ from a back-linkage \mathcal{R} and a path of well-linked sets $(\mathcal{S}, \mathcal{P})$ (Lemma 11.8). From there we find a new “horizontal” linkage \mathcal{H} which is weakly $|\mathcal{H}|$ -minimal with respect to \mathcal{V} such that $(\mathcal{H}, \mathcal{V})$ is a 2-horizontal web. Further, \mathcal{H} goes *forwards* through the path of well-linked sets $(\mathcal{S}, \mathcal{P})$, visiting the clusters of \mathcal{S} in the order given by \mathcal{S} . Clearly, $(\mathcal{S}, \mathcal{P})$ contains some forward linkage, and the construction of \mathcal{V} makes us able to construct \mathcal{H} such that it forms a web together with \mathcal{V} .

We obtain \mathcal{H} by first making it minimal with respect to \mathcal{V} . If \mathcal{H} does not intersect \mathcal{V} as required for them to form a 2-horizontal web, we can find a cycle of well-linked sets. Towards this end, we prove (in Lemma 11.10) that a path of well-linked sets that contains a forward linkage disjoint from the back-linkage also contains a cycle of well-linked sets.

Now, given a path of well-linked sets and a large forward linkage \mathcal{L} , we can construct a new path of well-linked sets $(\mathcal{S}', \mathcal{P}')$ (Lemma 11.9) and a subset $\mathcal{L}^* \subseteq \mathcal{L}$ disjoint from $(\mathcal{S}', \mathcal{P}')$ and also disjoint from the back-linkage \mathcal{R} . Applying results from Section 10 (Corollary 10.3), we obtain another path of well-linked sets $(\mathcal{S}'', \mathcal{P}'')$ going in the same direction as \mathcal{R} . With respect to $(\mathcal{S}'', \mathcal{P}'')$, the forward linkage \mathcal{L}^* now acts like a back-linkage which is disjoint from $(\mathcal{S}'', \mathcal{P}'')$, which yields a cycle of well-linked sets (Lemma 11.10).

Going back to constructing \mathcal{H} , we can argue that it must intersect \mathcal{V} enough, even when taking \mathcal{H} minimal with respect to \mathcal{V} , forming an object we call a *semi-web*.

From a semi-web $(\mathcal{H}, \mathcal{V})$ we can construct (Observation 11.12) a horizontal web $(\mathcal{H}', \mathcal{V}')$ such that \mathcal{H}' is weakly $|\mathcal{H}|$ -minimal with respect to \mathcal{V} or find large $\mathcal{H}' \subseteq \mathcal{H}$ and $\mathcal{V}' \subseteq \mathcal{V}$ which are

internally disjoint and satisfy some additional conditions. This we need for the final construction of the 2-horizontal web we are looking for unless we already find a cycle of well-linked sets during the construction ([Lemma 11.14](#)).

Lemma 11.14. Let w, ℓ, h, v be integers, let $(\mathcal{S}, \mathcal{P})$ be a strict path of well-linked sets of length $\ell' \geq \ell_{11.14}(w, m)$ and width $w' = w_{11.14}(h, w, m)$ with a back-linkage \mathcal{R} of order $r \geq r_{11.14}(h, w, v, m)$ intersecting $(\mathcal{S}, \mathcal{P})$ cluster by cluster such that \mathcal{R} is weakly m -minimal with respect to $(\mathcal{S}, \mathcal{P})$. Then, $D(\mathcal{S} \cup \mathcal{P} \cup \mathcal{R})$ contains one of the following:

- (H1) a cycle of well-linked sets of width w and length ℓ , or
- (H2) a weakly $m_{11.14}(h, w)$ -minimal 2-horizontal web $(\mathcal{H}, \mathcal{V})$ where $\mathcal{V} \subseteq \mathcal{R}$, $|\mathcal{H}| \geq h$ and $|\mathcal{V}| \geq v$.

Using the 2-horizontal web We split the 2-horizontal web $(\mathcal{H}, \mathcal{V})$ obtained above into two parts $\mathcal{H} = \mathcal{H}^1 \cdot \mathcal{H}^2$ in order to construct a new path of well-linked sets with \mathcal{H}^1 and then use \mathcal{H}^2 to complete the cycles.

To construct a path of well-linked sets $(\mathcal{S}, \mathcal{P})$ in \mathcal{H}_1 , we adapt the construction in the proof of [[KK15](#), Lemma 5.15], obtaining a split or a more suitable kind of segmentation ([Lemma 11.18](#)). This allows us to continue from the last cluster of $(\mathcal{S}, \mathcal{P})$ to \mathcal{H}^2 without intersecting $(\mathcal{S}, \mathcal{P})$ again, which in turn allows us to construct a folded ordered web or an ordered web ([Lemma 11.19](#)), allowing us to apply the results of [Section 10](#).

The tools established so far already allow us to construct a cycle of well-linked sets from a folded ordered web. So we have an ordered web $(\mathcal{H}', \mathcal{V}')$ which ends on $\text{end}(\mathcal{H}^1)$. We use the subpaths of the paths in \mathcal{H}' after their last intersection with \mathcal{V}' to construct a linkage to \mathcal{H}^2 that is disjoint from a path of order-linked sets built from the ordered web ([Lemma 11.20](#)). We use the paths in \mathcal{V}^1 to construct a back-linkage. Using the weak minimality of \mathcal{H} with respect to \mathcal{V} , we avoid intersections between \mathcal{V}^1 and the path of order-linked sets we constructed.

Lemma 11.21. Let $(\mathcal{H}, \mathcal{V})$ be a 2-horizontal web where \mathcal{H} is weakly c -minimal with respect to \mathcal{V} . If $|\mathcal{H}| \geq h_{11.21}(w, \ell)$ and $|\mathcal{V}| \geq v_{11.21}(w, \ell, c)$, then $D((\mathcal{H}, \mathcal{V}))$ contains a cycle of well-linked sets of length ℓ and width w .

From a path of well-linked sets to a cycle of well-linked sets Having gained enough insight on how to disentangle the back-linkage from the path of well-linked sets, we can now show how to obtain a cycle of well-linked sets.

Proof sketch of Theorem 11.22. Assume, without loss of generality, that \mathcal{R} is weakly r -minimal with respect to $(\mathcal{S}, \mathcal{P})$. Applying [Lemma 11.6](#) to $(\mathcal{S}, \mathcal{P})$ and \mathcal{R} yields two cases. If (C1) holds, then we obtain a cycle of well-linked sets of width w and length ℓ as desired. Otherwise, (C2) holds, and $D((\mathcal{S}, \mathcal{P}) \cup \mathcal{R})$ contains a path of well-linked sets $(\mathcal{S}', \mathcal{P}')$ of width w_1 and length ℓ_1 with a back-linkage \mathcal{R}' of order w_1 intersecting $(\mathcal{S}', \mathcal{P}')$ cluster by cluster such that $\mathcal{R}' \subseteq \mathcal{R}$. Note that \mathcal{R}' is also weakly r -minimal with respect to $(\mathcal{S}', \mathcal{P}')$.

Applying [Lemma 11.14](#) to $(\mathcal{S}', \mathcal{P}')$ and \mathcal{R}' yields two further cases. If (H1) holds, then we obtain a cycle of well-linked sets of width w and length ℓ as desired. Otherwise, (H2) holds, and we obtain a 2-horizontal (h, v) -web $(\mathcal{H}, \mathcal{V})$ such that \mathcal{H} is weakly m -minimal with respect to \mathcal{V} which, by [Lemma 11.21](#), contains a cycle of well-linked sets of width w and length ℓ . $\square \square$

3 Preliminaries

In this section we fix our notation and recall standard concepts and results from the literature used throughout the paper.

Sequences, sets, and functions. Given sequences $S_1 := (x_1, x_2, \dots, x_j)$ and $S_2 := (y_1, y_2, \dots, y_k)$, we write $S_1 \cdot S_2$ for the sequence $S_3 := (x_1, x_2, \dots, x_j, y_1, y_2, \dots, y_k)$. We say that $S_1 \cdot S_2$ is a *decomposition* of S_3 . The following is a well-known theorem about sequences of numbers due to Erdős and Szekeres.

Theorem 3.1 ([ES35]). Let $r, s \in \mathbb{N}$. Every sequence of distinct numbers of length at least $(r - 1)(s - 1) + 1$ contains a monotonically increasing subsequence of length r or a monotonically decreasing subsequence of length s .

We usually consider functions $f : A \rightarrow B$ to be *partial*, that is, the domain $\text{Dom}(f)$ is not necessarily A .

An *ordered set* is a sequence $A = (a_1, \dots, a_k)$ such that all elements of A are distinct. The order $\leq_A : A \times A$ induced by A is defined by $a_i \leq_A a_j$ for all $1 \leq i \leq j \leq k$. An *ordered subset* $A' \subseteq A$ then is just a subsequence of A , that is, the order of the elements is preserved. If we obtain an ordered set A' from a set A by fixing an order, we call A' an *ordering* of A .

Power towers and polynomials Let d be an integer and $V = \{x_1, \dots, x_k\}$ a set of variables. A *polynomial* of degree d over V is a function $p(x_1, x_2, \dots, x_k)$ of the form $p(x_1, x_2, \dots, x_k) = \sum_{i=1}^n (c_i \prod_{j=1}^k x_j^{e_{j,i}})$, where for each $1 \leq i \leq n$ and each $1 \leq j \leq k$ we have that $c_i \in \mathbb{R}$, $e_{j,i} \in \mathbb{N}$ and $\sum_{j=1}^k e_{j,i} \leq d$. We write $\text{poly}^d(V)$ for the set of all functions f for which there is a polynomial p of degree d over the variable set x_1, x_2, \dots, x_k such that $f(x_1, x_2, \dots, x_k) \in O(p(x_1, x_2, \dots, x_k))$.

We define *power towers* as follows. Given an integer h and a set of functions F over a set of variables V , we define a set of functions $2^{h \uparrow\uparrow F}$ recursively as follows. We set $2^{0 \uparrow\uparrow F} = F$ and define $2^{h \uparrow\uparrow F}$ as $\{f : \mathbb{R}^{|V|} \rightarrow \mathbb{R} \mid f \in O(2^{g(V)}) \text{, } g \in 2^{h-1 \uparrow\uparrow F}\}$ for $h > 1$. If $F = \text{poly}^d(V)$, we say that an $f \in 2^{h \uparrow\uparrow F}$ is a *power tower* of height h .

Graphs and digraphs. We denote by $E(G)$ the edge set of a graph G , directed or not, and by $V(G)$ its vertex set. We often use G for undirected and D for directed graphs.

Let D be a digraph. Given a set $X \subseteq V(D)$, we write $D - X$ for the digraph $(Y := V(D) \setminus X, E(D) \subseteq Y \times Y)$. Similarly, given a set $F \subseteq E(D)$, we write $D - F$ for the digraph $(V(D), V(A) \setminus F)$.

If $u \neq v \in V(D)$, we write $D + (v, u)$ for the digraph $(V(D), E(D) \cup \{(v, u)\})$. We also extend this operation to vertices and digraphs in the obvious way.

If D is a digraph and $v \in V(D)$, then $N_D^{\text{in}}(v) := \{u \in V \mid (u, v) \in E\}$ si the set of *in-neighbours* and $N_D^{\text{out}}(v) := \{u \in V \mid (v, u) \in E\}$ the set of *out-neighbours* of v . By $\deg_D^{\text{in}}(v) := |N_D^{\text{in}}(v)|$ we denote the *in-degree* of v and by $\deg_D^{\text{out}}(v) := |N_D^{\text{out}}(v)|$ its *out-degree*. When working with a set or another structure X containing digraphs, we write $D(X)$ to mean the digraph obtained by taking the union of all digraphs in X .

Paths and walks. A *walk* of length ℓ in a digraph D is a sequence of vertices $W := (v_0, v_1, \dots, v_\ell)$ such that $(v_i, v_{i+1}) \subseteq E(D)$, for all $0 \leq i < \ell$. We write $\text{start}(W)$ for v_0 and $\text{end}(W)$ for v_ℓ and say that W is a v_0 - v_ℓ -walk.

A walk $W := (v_0, v_1, \dots, v_\ell)$ is called a *path* if no vertex appears twice in it and it is called a *cycle* if $v_0 = v_\ell$ and $v_i \neq v_j$ for all $0 \leq i < j < \ell$.

We often identify a walk W in D with the corresponding subgraph and write $V(W)$ and $E(W)$ for the set of vertices and edges appearing on it.

Given two walks $W_1 := (x_1, x_2, \dots, x_j)$ and $W_2 := (y_1, y_2, \dots, y_k)$ with $\text{end}(W_1) = \text{start}(W_2)$, we make use of the concatenation notation for sequences and write $W_1 \cdot W_2$ for the walk $W_3 := (x_1, x_2, \dots, x_j, y_2, y_3, \dots, y_k)$. We say that $W_1 \cdot W_2$ is a decomposition of W_3 . If W_1 or W_2 is an empty sequence, then the result of $W_1 \cdot W_2$ is the other walk (or the empty sequence if both walks are empty).

Let P be a path in a digraph D and let X be a set of vertices with $V(P) \cap X \neq \emptyset$. We consider the vertices p_1, \dots, p_m of P ordered by their occurrence on P . Let i be the highest index such that $p_i \in X$ and let j be the smallest index such that $p_j \in X$. We call p_i the *last vertex of P in X* or, depending on the perspective, the last element of X on P , and p_j the *first vertex of P in X* or the first vertex of X on P .

If $v \in V(D)$ we denote by $\text{out}^*(v)$ the set of vertices reachable from v in D and by $\text{in}^*(v)$ the set of vertices from which v can be reached in D .

Special digraphs. We denote the digraph of a path on k vertices by \mathbf{P}_k . For the *bidirected path on k vertices*, we write $\vec{\mathbf{P}}_k := (\{u_1, u_2, \dots, u_k\}, \{(u_i, u_j) \mid 1 \leq i, j \leq k \text{ and } |i - j| = 1\})$. The *cycle on k vertices* is given by $\mathbf{C}_k := (\{u_0, u_1, \dots, u_{k-1}\}, \{(u_i, u_{i+1 \bmod k}) \mid 0 \leq i < k\})$. Finally, we write $\vec{\mathbf{K}}_k := (\{u_1, u_2, \dots, u_k\}, \{(u_i, u_j) \mid 1 \leq i, j \leq k \text{ and } i \neq j\})$ for the *complete digraph on k vertices*.

Connectivity. A digraph D is said to be *strongly connected* if for every $u, v \in V$ there is a u - v -path and a v - u -path in D . We say D is *unilateral* if for every $u, v \in V$ there is a u - v -path or a v - u -path in D . Finally, D is *weakly-connected* if the underlying undirected graph of D is connected.

A *feedback vertex set* of D is a set $X \subseteq V(D)$ such that $D - X$ is acyclic. Similarly, a *feedback arc set* of D is a set $F \subseteq E(D)$ such that $D - F$ is acyclic.

Linkages and separators. Let $A, B \subseteq V(D)$. An A - B -walk is a walk W that starts in A and ends in B . A set $X \subseteq V(D)$ is an A - B *separator* if there are no A - B -paths in $D - X$.

A *linkage* in D is a set \mathcal{L} of pairwise vertex disjoint paths. The *order* $|\mathcal{L}|$ of \mathcal{L} is the number of paths it contains.

An A - B -*linkage* of order k is a linkage $\mathcal{L} := \{L_1, L_2, \dots, L_k\}$ such that $\text{start}(L_i) \in A$ and $\text{end}(L_i) \in B$ for all $1 \leq i \leq k$. We write $\text{start}(\mathcal{L})$ for the set $\{\text{start}(L_i) \mid L_i \in \mathcal{L}\}$ and $\text{end}(\mathcal{L})$ for the set $\{\text{end}(L_i) \mid L_i \in \mathcal{L}\}$. We also extend the notation for path concatenation to linkages. Given linkages $\mathcal{P} = \{P_1, P_2, \dots, P_k\}$ and $\mathcal{Q} = \{Q_1, Q_2, \dots, Q_k\}$ such that $\text{end}(\mathcal{P}) = \text{start}(\mathcal{Q})$, we write $\mathcal{P} \cdot \mathcal{Q}$ for the linkage $\{P_a \cdot Q_b \mid P_a \in \mathcal{P}, Q_b \in \mathcal{Q} \text{ and } \text{end}(P_a) = \text{start}(Q_b)\}$.

It often is convenient to use a linkage \mathcal{L} as a function $\mathcal{L} : \text{start}(\mathcal{L}) \rightarrow \text{end}(\mathcal{L})$. The expression $\mathcal{L}(a) = b$ then means that \mathcal{L} contains a path starting in a and ending in b .

We frequently use the following classical result by Menger.

Theorem 3.2 (Menger's Theorem). Let D be a digraph, $A, B \subseteq V(D)$ with $|A| = |B|$. There is an A - B -linkage of size k in D if and only if every A - B separator has size at least k .

Let D be a digraph, $A, B \subseteq V(D)$, and $c \in \mathbb{N}$. An A - B -*linkage with congestion c* in D is a set \mathcal{L} of A - B -paths such that no vertex of $V(D)$ occurs in more than c distinct paths in \mathcal{L} . A linkage of congestion 1 is called *integral* and a linkage of congestion 2 is called *half-integral*.

A simple application of [Theorem 3.2](#) yields the following lemma (see e.g. [\[KK15\]](#)).

Lemma 3.3. Let D be a digraph, $A, B \subseteq V(D)$. If there is an A - B -linkage of order ck and congestion c in D , then there is an integral A - B -linkage of order k in D .

Throughout the paper we frequently work with a special kind of linkages that we define next.

Definition 3.4 (minimal linkages). Let D be a digraph, let $H \subseteq D$ be a subgraph, and let \mathcal{L} be a linkage of order k . \mathcal{L} is *minimal with respect to H* , or *H -minimal*, if for all edges $e \in \bigcup_{P \in \mathcal{L}} E(P) \setminus E(H)$ there is no $\text{start}(\mathcal{L})$ - $\text{end}(\mathcal{L})$ -linkage of order k in the graph $(\mathcal{L} \cup H) - e$.

Given a linkage \mathcal{L} in a digraph D and a subgraph $H \subseteq D$, we can always obtain a linkage \mathcal{L}' with same order and same endpoints as \mathcal{L} which is H -minimal by iteratively removing edges $e \in E(\mathcal{L}) \setminus E(H)$ for which a $\text{start}(\mathcal{L})$ - $\text{end}(\mathcal{L})$ -linkage of order $|\mathcal{L}|$ exists avoiding e .

Minimal linkages were used extensively in [\[KK15\]](#). The idea is that when constructing paths of an H -minimal linkage \mathcal{L} , we always prefer to use edges of H over edges not in $E(H)$. This implies the following property which we exploit frequently in our proofs.

Definition 3.5 (weak minimality). A linkage \mathcal{L} in a digraph D is *weakly k -minimal* with respect to a subgraph H of D if for every $P_1 \cdot e \cdot P_2 \in \mathcal{L}$ with $e \in E(\mathcal{L}) \setminus E(H)$ there is a $V(P_1)$ - $V(P_2)$ -separator of size at most $k - 1$ in $(\mathcal{L} \cup H) - e$.

Observation 3.6. Let H be a subgraph of a digraph D and let \mathcal{L} be a linkage which is H -minimal. Then \mathcal{L} is weakly $|\mathcal{L}|$ -minimal with respect to H .

Proof. Assume towards a contradiction that there is some $L \in \mathcal{L}$ and some $e \in E(L) \setminus E(H)$ such that L can be decomposed into $L_1 \cdot e \cdot L_2$ and there is no $V(L_1)$ - $V(L_2)$ separator of size less than $|\mathcal{L}|$ in $D(\mathcal{L} \cup H) - e$. By [Theorem 3.2](#), there is a $V(L_1)$ - $V(L_2)$ -linkage \mathcal{Q} of order $|\mathcal{L}|$ in $D(\mathcal{L} \cup H) - e$.

Let S be a minimum $\text{start}(\mathcal{L})$ - $\text{end}(\mathcal{L})$ separator in $D(\mathcal{L} \cup H) - e$. Because \mathcal{L} is H -minimal, we have that $|S| < |\mathcal{L}|$. Hence, S must hit every path in $\mathcal{L} \setminus \{L\}$ and it must be disjoint from L .

Since $|\mathcal{Q}| = |\mathcal{L}|$, there is some $Q \in \mathcal{Q}$ which is not hit by S . Hence, there is a $\text{start}(L)$ - $\text{end}(L)$ path in $D(\mathcal{L} \cup H) - e - S$, a contradiction to the assumption that S is a separator. Thus, \mathcal{L} is weakly $|\mathcal{L}|$ -minimal with respect to H . \square \square

We close this part by recalling the definition of well-linkedness, an important property of a central concept in our proof, the cycle-of-well-linked-sets.

Definition 3.7. Let A, B be sets of vertices in a digraph D . We say that A is *well-linked to B in D* if for every $A' \subseteq A$ and every $B' \subseteq B$ with $|A'| = |B'|$ there is an A' - B' -linkage of order $|A'|$ in D .

Minors. Given a digraph D and an arc $(v, u) \in E(D)$, we say that (v, u) is *butterfly contractible* if $\deg^{\text{out}}(v) = 1$ or $\deg^{\text{in}}(u) = 1$. The *butterfly contraction* of (v, u) is the operation which consists of removing v and u from D , then adding a new vertex vu , together with the arcs $\{(w, vu) \mid w \in \deg_D^{\text{in}}(v)\}$ and $\{(vu, w) \mid w \in \deg_D^{\text{out}}(u)\}$. Note that, by definition of digraphs, we remove duplicated arcs and loops, that is, arcs of the form (w, w) . If there is a subgraph D' of D such that we can construct another digraph H from D' by means of butterfly contractions, then we say that H is a *butterfly minor of D* , or that D contains H as a *butterfly minor*.

4 Directed Treewidth and Grids

In this section we recall directed treewidth and the dual concepts of brambles and cylindrical grids. We also define various other forms of “grids” in directed graphs that we use in the sequel.

Directed treewidth was originally introduced by Reed [Ree99] and Johnson, Robertson, Seymour, and Thomas [JRST01b], see also [JRST01a]. Adler [Adl07] showed that the original definition in [JRST01b] of directed treewidth is not closed under butterfly minors. We therefore use the variant of directed treewidth defined in [KK22], which is closed under taking butterfly minors.

An *arborescence* T is an acyclic directed graph obtained from an undirected rooted tree by orienting all edges away from the root. That is, T has a vertex r_0 , called the root of T , with the property that for every $r \in V(T)$ there is a unique directed path from r_0 to r in T . For each $r \in V(T)$ we denote the subarborescence of T induced by the set of vertices in T reachable from r by T_r . In particular, r is the root of T_r .

Definition 4.1 ([KK22, Definition 3.1]). A *directed tree decomposition* of a digraph D is a triple (T, β, γ) , where $\beta : V(T) \rightarrow 2^{V(D)}$ and $\gamma : E(T) \rightarrow 2^{V(D)}$ are functions and T is an arborescence such that

(W1) $\{\beta(t) : t \in V(T)\}$ is a partition of $V(D)$ into (possibly empty) sets and

(W2) for every $e = (s, t) \in E(T)$, there is no closed directed walk in $D - \gamma(e)$ containing a vertex in A and a vertex in B , where $A = \bigcup \beta(t) : t \in V(T_t)$ and $B = V(D) \setminus A$.

For $t \in V(T)$ we define $\Gamma(t) := \beta(t) \cup \bigcup \gamma(e) : e \sim t$, where $e \sim t$ if e is incident to t , and we define $\beta(T_t) := \bigcup \{\beta(t) : t \in V(T_t)\}$. The *width* of (T, β, γ) is the least integer w such that $|\Gamma(t)| \leq w + 1$ for all $t \in V(T)$. The *directed treewidth* of D is the least integer w such that D has a directed tree decomposition of width w . The sets $\beta(t)$ are called the bags and the sets $\gamma(e)$ are called the guards of the directed tree decomposition.

The natural dual to directed tree decompositions are objects called *brambles*. The concept of brambles was also introduced by [JRST01b]. For the same reason as before we use the variant of brambles defined in [KK15].

Definition 4.2. A *bramble* in a digraph D is a set \mathcal{B} of strongly connected subgraphs $B \subseteq D$ such that $B \cap B' \neq \emptyset$ for all $B, B' \in \mathcal{B}$.

A *cover* of \mathcal{B} is a set $X \subseteq V(D)$ of vertices such that $V(B) \cap X \neq \emptyset$ for all $B \in \mathcal{B}$. Finally, the *order* of a bramble \mathcal{B} is the minimum size of a cover for \mathcal{B} . The bramble number $bn(D)$ of D is the maximum order of a bramble in D .

We also need the following relation between brambles and directed treewidth. The following can be obtained from results due to [JRST01b] by converting brambles to havens and back, and the statement was proven formally by [KO14].

Lemma 4.3 ([KO14, Corollary 6.4.24]). There are constants c, c' such that for all digraphs D , $bn(D) \leq c \text{dtw}(D) \leq c' bn(D)$.

By combining the statement (1.1) of [JRST01b] and Lemma 6.4.20 of [KO14], we obtain the following.

Corollary 4.4 ([JRST01b] + [KO14]). Let D be a digraph. If $\text{dtw}(D) \geq 2k$, then D contains a bramble of order k .

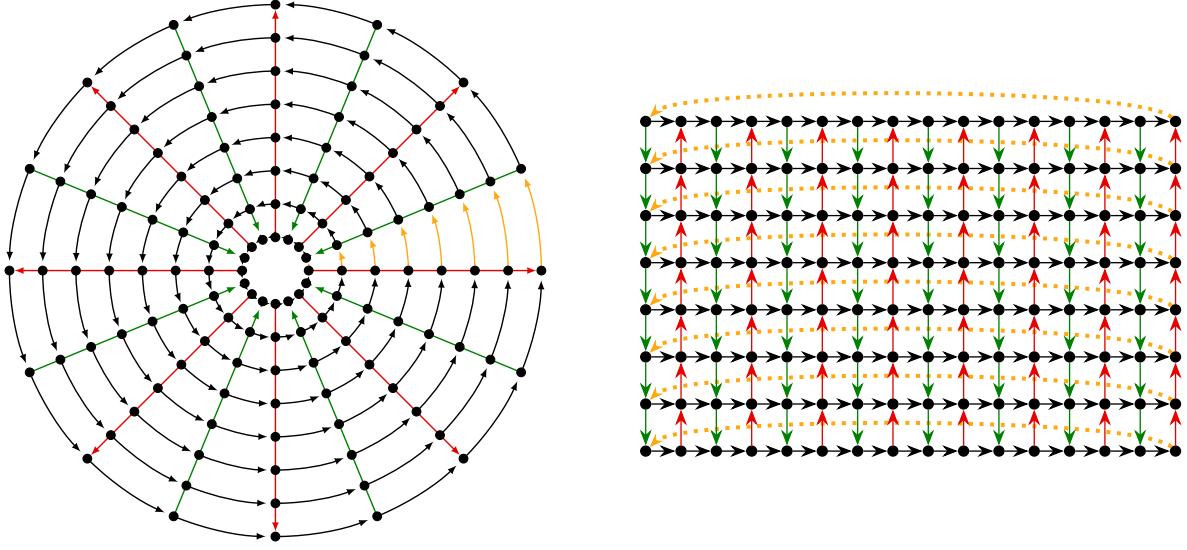


Figure 2: Cylindrical grid G_8 of order 8 drawn in two ways. The drawing on the right illustrates how a cylindrical grid is obtained from a fence. The dotted orange paths symbolise the edges e_i that close the cycles drawn solid on the left.

We now define another obstruction to directed treewidth called *cylindrical grids*. See Figure 2 for an illustration.

Definition 4.5. A *cylindrical grid* of order k is a digraph G_k consisting of k pairwise disjoint directed cycles C_1, C_2, \dots, C_k of length $2k$, together with a set of $2k$ pairwise vertex disjoint paths P_1, P_2, \dots, P_{2k} of length $k - 1$ such that

- each path P_i has exactly one vertex in common with each cycle C_j and both endpoints of P_i are in $V(C_1) \cup V(C_k)$,
- the paths P_1, P_2, \dots, P_{2k} appear on each C_i in this order, and
- for each $1 \leq i \leq 2k$, if i is odd, then the cycles C_1, C_2, \dots, C_k occur on P_i in this order and, if i is even, then the cycles occur in the reverse order C_k, C_{k-1}, \dots, C_1 .

Besides cylindrical grids several different ways of defining “directed grids” have been considered in the literature (for example [RRST96], [JRST01b], [KK15]). Two of these, called *acyclic grids* and *fences*, see Figure 3, are used at various points of our proof. Since we are interested in grids in the context of minors, we define grids by linkages instead of giving explicit vertex and edge sets.

To motivate the following definitions let us dissect a cylindrical grid $((C_1, \dots, C_k), (P_1, \dots, P_{2k}))$ as follows. An important difference between cylindrical grids and grids in undirected graphs is that cylindrical grids are *locally acyclic* in the following sense. Suppose we delete in each cycle C_i the edge e_i whose head is on the path P_1 . These edges are marked by the dotted red lines in Figure 2. The resulting digraph is acyclic and consists of two linkages: the linkage $\{P_1, \dots, P_{2k}\}$ and the linkage $\{C_1 - e_1, \dots, C_k - e_k\}$ which contains for each cycle C_i the path that remains once the edge e_i is deleted. Digraphs of this form are called *fences*. See Figure 2 for a drawing of cylindrical grids illustrating how they are constructed from a fence with additional edges closing the cycles.

Definition 4.6. A (p, q) -fence is a tuple $(\mathcal{P}, \mathcal{Q})$ such that

- $\mathcal{P} = (P_1, P_2, \dots, P_{2p})$ and $\mathcal{Q} = (Q_1, Q_2, \dots, Q_q)$ are linkages,
- for each $1 \leq i \leq 2p$ and each $1 \leq j \leq q$, the digraph $P_i \cap Q_j$ is a path (and therefore non-empty),
- for each $1 \leq j \leq q$, the paths $P_1 \cap Q_j, P_2 \cap Q_j, \dots, P_{2p} \cap Q_j$ appear in this order along Q_j , and
- for each $1 \leq i \leq 2p$, if i is odd then the paths $P_i \cap Q_1, P_i \cap Q_2, \dots, P_i \cap Q_q$ appear in this order along P_i , and if i is even instead, then the paths $P_i \cap Q_q, P_i \cap Q_{q-1}, \dots, P_i \cap Q_1$ appear in this order along P_i .

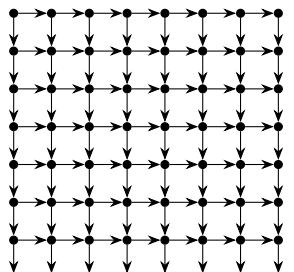
See Figure 3b for an illustration. The “horizontal” paths, or *rows*, constitute the linkage \mathcal{Q} and the *columns* form the linkage \mathcal{P} .

A useful property of a (p, q) -fence $(\mathcal{P}, \mathcal{Q})$ is that if $A \subseteq \text{start}(\mathcal{Q})$ and $B \subseteq \text{end}(\mathcal{Q})$ are sets with $|A| = |B| \leq p$ then there is an $A-B$ -linkage \mathcal{L} of order $|A|$ in the graph $\bigcup \mathcal{P} \cup \bigcup \mathcal{Q}$.

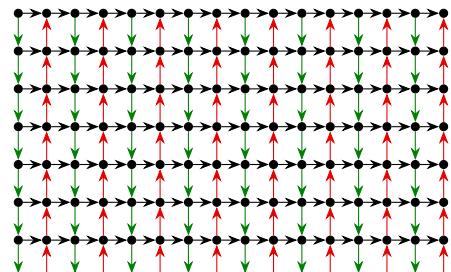
Now let us further decompose the fence constructed from the cylindrical web to obtain an even simpler form of directed grid. In a fence we can only route from “left to right” but we can route “upwards” as well as “downwards”. An even simpler form of directed grid is obtained if in a fence we remove the “upwards” paths, i.e. every second column. The resulting digraph is called an *acyclic grids*, see Figure 3a.

Definition 4.7. An *acyclic (p, q) -grid* is a pair $(\mathcal{P}, \mathcal{Q})$ such that

- $\mathcal{P} = (P_1, P_2, \dots, P_p)$ and $\mathcal{Q} = (Q_1, Q_2, \dots, Q_q)$ are linkages,
- for each $1 \leq i \leq p$ and each $1 \leq j \leq q$, the digraph $P_i \cap Q_j$ is a path (and therefore non-empty),
- for each $1 \leq j \leq q$, the paths $P_1 \cap Q_j, P_2 \cap Q_j, \dots, P_p \cap Q_j$ appear in this order along Q_j , and
- for each $1 \leq i \leq p$, the paths $P_i \cap Q_1, P_i \cap Q_2, \dots, P_i \cap Q_q$ appear in this order along P_i .



(a) An acyclic $(8, 8)$ -grid.



(b) An $(8, 8)$ -fence.

Figure 3: An acyclic grid and a fence.

Acyclic grids only allow to route from top to bottom and left to right.

The last type of grid-like structures we define is called a *web*, originally introduced by Reed et al. in [RRST96]. Webs form an important step in the proof of the directed grid theorem in [KK15].

5 Constructing splits and segmentations

The starting point for constructing our paths of well-linked sets and paths of order-linked sets are *splits* and *segmentations*, which add more structure to webs by ensuring that one linkage of the web intersects the other in an ordered fashion. We repeat below the definition of splits and segmentations from Kawarabayashi and Kreutzer [KK22].

Definition 5.1 ([KK22, Definitions 5.6 and 5.7]). Let \mathcal{P} and \mathcal{Q}^* be linkages and let $\mathcal{Q} \subseteq \mathcal{Q}^*$ be a sublinkage of order q . Let $r \geq 0$.

- (S1) An (r, q') -split of $(\mathcal{P}, \mathcal{Q})$ (with respect to \mathcal{Q}^*) is a pair $(\mathcal{P}', \mathcal{Q}')$ of linkages of order $r = |\mathcal{P}'|$ and $q' = |\mathcal{Q}'|$ with $\mathcal{Q}' \subseteq \mathcal{Q}$ such that there is a path $P \in \mathcal{P}$ and edges $e_1, \dots, e_{r-1} \in E(P) \setminus E(\mathcal{Q}^*)$ such that $P = P_1 e_1 P_2 \dots e_{r-1} P_r$ and $\mathcal{P}' := (P_1, \dots, P_r)$ and every $Q \in \mathcal{Q}'$ can be divided into subpaths Q_1, \dots, Q_r such that $Q = Q_1 e'_1 \dots e'_{r-1} Q_r$, for suitable edges $e'_1, \dots, e'_{r-1} \in E(Q)$, and $\emptyset \neq V(Q) \cap V(P_i) \subseteq V(Q_{r+1-i})$, for all $1 \leq i \leq r$.
- (S2) A subset $\mathcal{Q}' \subseteq \mathcal{Q}$ of order q' is a q' -segmentation of $P \in \mathcal{P}$ (with respect to \mathcal{Q}^*) if there are edges $e_1, \dots, e_{q'-1} \in E(P) - E(\mathcal{Q}^*)$ with $P = P_1 e_1 \dots P_{q'-1} e_{q'-1} P_{q'}$, for suitable subpaths $P_1, \dots, P_{q'}$, such that \mathcal{Q}' can be ordered as $(Q_1, \dots, Q_{q'})$ and $V(Q_i) \cap V(P) \subseteq V(P_i)$.
- (S3) An (r, q') -segmentation (with respect to \mathcal{Q}^*) is a pair $(\mathcal{P}', \mathcal{Q}')$ where \mathcal{P}' is a linkage of order r and \mathcal{Q}' is a linkage of order q' such that \mathcal{Q}' is a q' -segmentation (with respect to \mathcal{Q}^*) of every path P_i into segments $P_1^i e_1 P_2^i \dots e_{q'-1} P_{q'}^i$.
- (S4) A segmentation $(\mathcal{P}', \mathcal{Q}')$ is *ordered* if for all $P_i \in \mathcal{P}'$ the order $(Q_1, \dots, Q_{q'})$ given by the q' -segmentation of P_i is the same. We say that $(\mathcal{P}', \mathcal{Q}')$ is an (ordered) (r, q') -segmentation of $(\mathcal{P}, \mathcal{Q})$ if $\mathcal{Q}' \subseteq \mathcal{Q}$ and every path in \mathcal{P}' is a subpath of a path in \mathcal{P} .

An (r, q) -split $(\mathcal{P}, \mathcal{Q})$ or an (r, q) -segmentation $(\mathcal{P}, \mathcal{Q})$ is well-linked if $\text{end}(\mathcal{Q})$ is well-linked to $\text{start}(\mathcal{Q})$.

One can obtain splits and segmentations from webs by using the following result. We observe that $\mathbf{q}_{5.2}(p, q, x, y, c) \in 2^{2 \uparrow \uparrow \text{poly}^4(p, q, x, y, c)}$.

Lemma 5.2 ([KK15, Lemma 5.13]). Let $p, q, q', r, s, c, x, y \geq 0$ be integers such that $p \geq x$ and $q' \geq \mathbf{q}_{5.2}(p, q, x, y, c) := (pq(q+c))^{2^{(x-1)y+1}}$. If D contains a (p, q') -web $\mathcal{W} := (\mathcal{P}, \mathcal{Q})$ where \mathcal{P} is weakly c -minimal with respect to \mathcal{Q} , then D contains one of the following:

- (S1) a (y, q) -split $(\mathcal{P}', \mathcal{Q}')$ of $(\mathcal{P}, \mathcal{Q})$, or
- (S2) an (x, q) -segmentation $(\mathcal{P}', \mathcal{Q}')$ of $(\mathcal{P}, \mathcal{Q})$.

Furthermore, if \mathcal{W} is well-linked in D , then so is $(\mathcal{P}', \mathcal{Q}')$.

With a simple pigeon-hole principle argument, it is possible to construct an ordered segmentation from a segmentation. The proof provided below is based on some steps of the proof of Lemma 5.19 from [KK15].

Observation 5.3. Let $(\mathcal{P}, \mathcal{Q})$ be a (p, q) -segmentation. If $p \geq \mathbf{p}_{5.3}(k, q) := (k-1)q! + 1$, then there is $\mathcal{P}' \subseteq \mathcal{P}$ such that $(\mathcal{P}', \mathcal{Q})$ is an ordered (k, q) -segmentation.

Proof. For each $P_i \in \mathcal{P}$ there is an ordering \mathcal{Q}_i of \mathcal{Q} witnessing that \mathcal{Q} is a q -segmentation of P_i . In total, there are at most $q!$ distinct orderings \mathcal{Q}_i . Hence, by the pigeon-hole principle, there is some $\mathcal{P}' \subseteq \mathcal{P}$ of size k such that $\mathcal{Q}_i = \mathcal{Q}_j$ for all $P_i, P_j \in \mathcal{P}'$. \square \square

Since the bounds for Lemma 5.2 are already elementary, in the remainder of this section we obtain a web while making sure that all the functions that arise are elementary.

Thereby we improve upon results of [KK22], which shows that digraphs of high directed treewidth contain a large *well-linked web* or a large cylindrical grid obtaining non-elementary bounds in this step. In particular, their proof uses an *iterated Ramsey* argument and so the bounds obtained are a power tower whose height depends on h, v and k (where h, v and k are defined as in Theorem 5.4).

Theorem 5.4 ([KK22, Theorem 4.2 + Lemma 3.6 + Lemma 4.10]). Let $h, v, k \in \mathbb{N}$. There exists a function $t_{5.4} : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ such that every digraph D with $\text{dtw}(D) \geq t_{5.4}(h, v, k)$ contains a cylindrical grid of order k as a butterfly minor or a (h, v) -web $(\mathcal{H}, \mathcal{V})$ where $\text{end}(\mathcal{V}) \cup \text{start}(\mathcal{V})$ is a well-linked set in D and \mathcal{H} is \mathcal{V} -minimal.

To achieve an elementary bound, we utilise a result known as *Lovász Local Lemma*, obtaining even a polynomial bound for the step which was previously non-elementary. In total, however, the gap between directed treewidth and the size of the ordered web we can guarantee is upper bounded by a super-polynomial function.

Lemma 5.5 (Lovász Local Lemma [EL74, Spe77]). Consider a set \mathcal{E} of events such that for each $A \in \mathcal{E}$

1. $\Pr[A] \leq p < 1$, and
2. A is mutually independent of a set of all but at most d other events.

If $ep(d+1) < 1$, then with positive probability none of the events in \mathcal{E} occur.

While Lemma 5.5 above is not constructive, [MT10] provided a randomized algorithm and [CGH13] provided a deterministic algorithm for finding an assignment of the random variables which avoids all events \mathcal{E} .

The proof of [KK22] for obtaining a web works by starting with a *bramble* of high order and then showing that the existence of such a bramble implies the existence of an object called a *path-system*¹. We repeat the definition of path-system below (see Figure 4 for an illustration).

Definition 5.6 (path system [KK22]). Let G be a digraph and let $\ell, p \geq 1$. An ℓ -linked path system of order p is a sequence $\mathcal{S} := (\mathcal{P}, \mathcal{L}, \mathcal{A})$, where

- $\mathcal{A} := (A_i^{in}, A_i^{out})_{1 \leq i \leq p}$ such that $A := \bigcup_{1 \leq i \leq p} A_i^{in} \cup A_i^{out} \subseteq V(G)$ is a well-linked set of order $2\ell p$ and $|A_i^{in}| = |A_i^{out}| = \ell$, for all $1 \leq i \leq p$,
- $\mathcal{P} := (P_1, \dots, P_p)$ is a sequence of pairwise vertex disjoint paths and for all $1 \leq i \leq p$, $A_i^{in}, A_i^{out} \subseteq V(P_i)$ and all $v \in A_i^{in}$ occur on P_i before any $v' \in A_i^{out}$ and the first vertex of P_i is in A_i^{in} and the last vertex of P_i is in A_i^{out} and
- $\mathcal{L} := (L_{i,j})_{1 \leq i \neq j \leq p}$ is a sequence of linkages such that for all $1 \leq i \neq j \leq p$, $L_{i,j}$ is a linkage of order ℓ from A_i^{out} to A_j^{in} .

The system \mathcal{S} is *clean* if for all $1 \leq i \neq j \leq p$ and all $Q \in L_{i,j}$, $Q \cap P_s = \emptyset$ for all $1 \leq s \leq p$ with $s \notin \{i, j\}$.

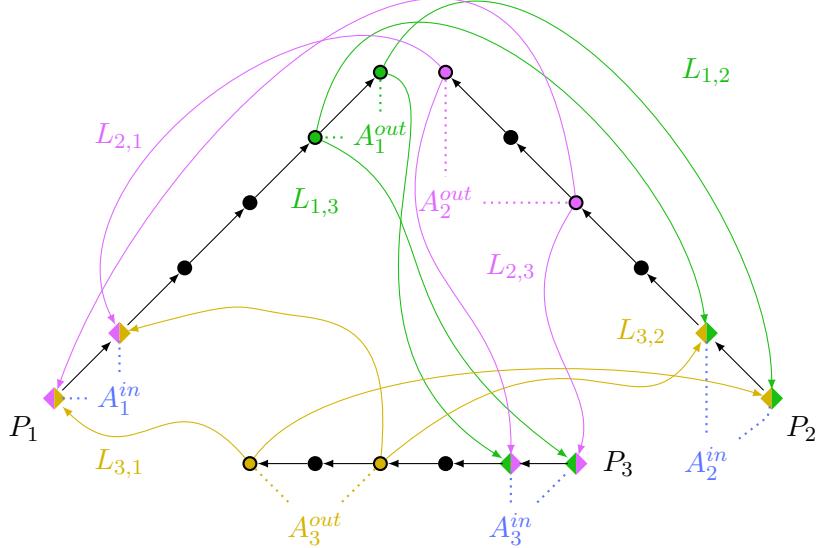


Figure 4: A clean 2-linked path-system of order 3.

We can obtain path-systems from brambles using the following lemma. We define $\text{k}_{5.7}(\ell, p) = (4\ell p)(2\ell p + 1)$ and observe that $\text{k}_{5.7}(\ell, p) \in O(\ell^2 p^2)$. $\text{k}_{5.7}$

Lemma 5.7 ([KK22, Lemma 4.6]). Let D be a digraph and let $\ell, p \geq 1$. If D contains a bramble of order $\text{k}_{5.7}(\ell, p)$, then D contains an ℓ -linked path systems \mathcal{S} of order p .

In our proof, we use a intermediate object which we call a *semi-web*. Unlike a web, we do not require from a semi-web $(\mathcal{P}, \mathcal{Q})$ that the paths in \mathcal{P} and \mathcal{Q} pairwise intersect each other. Instead, a semi-web has a degree which controls how much the linkages much intersect each other.

A (p, q) -web $(\mathcal{P}, \mathcal{Q})$ of avoidance d from [KK15] corresponds to a (p, q) -semi-web $(\mathcal{P}, \mathcal{Q})$ of degree $|\mathcal{P}| \cdot \frac{d-1}{d}$ where \mathcal{P} is minimal with respect to \mathcal{Q} in our notation. The main reason for this modification is to avoid fractional calculations when determining the bounds of our functions.

Definition 5.8. Let D be a digraph. Two linkages \mathcal{H} and \mathcal{V} in D build a $(|\mathcal{H}|, |\mathcal{V}|)$ -semi-web $(\mathcal{H}, \mathcal{V})$ of degree d if every path in \mathcal{V} intersects at least d paths in \mathcal{H} .

Finally, $(\mathcal{H}, \mathcal{V})$ is *well-linked* if $\text{end}(\mathcal{V})$ is well-linked linked to $\text{start}(\mathcal{V})$ in D .

We can obtain a web from a semi-web using Lemma 5.11 below. Towards this end, we we use observations 5.9 and 5.10, which summarize some basic properties of a linkage \mathcal{L} which is minimal with respect to another linkage \mathcal{P} .

Observation 5.9 ([KK22, Lemma 2.14]). Let D be a digraph. Let \mathcal{P}, \mathcal{L} be linkages such that \mathcal{L} is minimal with respect to \mathcal{P} . Then \mathcal{L} is minimal with respect to \mathcal{P}' for every $\mathcal{P}' \subseteq \mathcal{P}$.

Observation 5.10. Let \mathcal{P}, \mathcal{Q} be two linkages such that \mathcal{P} is minimal with respect to \mathcal{Q} . Let $P \in \mathcal{P}$. If P does not intersect any path in \mathcal{Q} , then $\mathcal{P} \setminus \{P\}$ is minimal with respect to \mathcal{Q} .

We adapt the following statement from [KK22] to our notation, fixing some small mistakes in their proof in the process. We first define

$$\text{q}_{5.11}(p', q, k) = qk(p')^k. \quad \text{q}_{5.11}$$

¹Despite the similarity in names, the path-systems we use here and the paths-of-sets systems of [CC16] are unrelated mathematical objects.

Lemma 5.11 ([KK22, Lemma 4.10]). Let $(\mathcal{P}', \mathcal{Q}')$ be a (p', q') -semi-web of degree k in a digraph D such that \mathcal{P}' is minimal with respect to \mathcal{Q}' . If $q' \geq \text{q}_{5.11}(p', q, k)$, then D contains a well-linked (p_1, q) -web $(\mathcal{P}, \mathcal{Q})$ where \mathcal{P} is minimal with respect to \mathcal{Q} , $\mathcal{P} \subseteq \mathcal{P}'$, $\mathcal{Q} \subseteq \mathcal{Q}'$ and $p_1 \geq k$.

Proof. Define a function f as $f(Q) = \{P \in \mathcal{P}' \mid V(P) \cap V(Q) = \emptyset\}$ for each $Q \in \mathcal{Q}'$. By assumption, $|f(Q)| \leq p' - k$ for each $Q \in \mathcal{Q}'$.

As there are $\binom{p'}{|f(Q)|}$ choices for each $f(Q)$, and as $\sum_{i=0}^k \binom{p'}{p'-k} = \sum_{i=0}^k \binom{p'}{k} \leq k(p')^k$, by the pigeon-hole principle there is some $\mathcal{Q} \subseteq \mathcal{Q}'$ of order q such that $X := f(Q_a) = f(Q_b)$ holds for all $Q_a, Q_b \in \mathcal{Q}$.

Let $\mathcal{P} = \mathcal{P}' \setminus X$ and let $p_1 = |\mathcal{P}|$. Note that $p_1 \geq k$. By observations 5.9 and 5.10, \mathcal{P} is minimal with respect to \mathcal{Q} . Hence, $(\mathcal{P}, \mathcal{Q})$ is a (p_1, q) -web where \mathcal{P} is minimal with respect to \mathcal{Q} , as desired. \square \square

Lemma 5.12 essentially corresponds to Lemma 4.7 from [KK22], and our proof is based on theirs. The idea is to attempt to construct a semi-web from some linkage $L_{a,b} \in \mathcal{L}$ and some $P \in \mathcal{P}$. If we do not find any semi-web, then it means that the linkages in \mathcal{L} are mostly disjoint from \mathcal{P} . We then use this observation to argue that, for each pair $P_i, P_j \in \mathcal{P}$, there are only few other paths $P_r \in \mathcal{P}$ which are “bad” for the choice P_i, P_j , that is, we cannot easily construct a clean path-system if we take P_i, P_j and P_r . This allows us to construct our “bad” events in order to apply Lovász Local Lemma.

We define

$$\begin{aligned} d'(p_2) &= 3(p_2)^2/2 - 15p_2/2 + 10, \\ \ell_{5.12}(p_2, \ell_2, d_1) &= \ell_2 + (p_2 - 2)d_1, \\ \mathbf{p}_{5.12}(q_1, p_2) &= (2e(d'(p_2) + 1)q_1 + 1)p_2. \end{aligned} \quad \begin{aligned} &\ell_{5.12} \\ &\mathbf{p}_{5.12} \end{aligned}$$

Note that $\ell_{5.12}(p_2, \ell_2, d_1) \in O(\ell_2 + d_1 p_2)$ and $\mathbf{p}_{5.12}(q_1, p_2) \in O(q_1(p_2)^3)$.

Lemma 5.12. Let d_1, p_2, ℓ_2, q_2 be integers. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathcal{A})$ be an ℓ -linked path-system of order p . If $\ell \geq \ell_{5.12}(p_2, \ell_2, d_1)$ and $p \geq \mathbf{p}_{5.12}(q_1, p_2)$, then $\mathsf{D}(\mathcal{S})$ contains one of the following

- (C1) a well-linked (ℓ, q_1) -semi-web $(\mathcal{P}_1, \mathcal{Q}_1)$ of degree d_1 where $\mathcal{P}_1 \in \mathcal{L}$, $\mathcal{Q}_1 \subseteq \mathcal{P}$ and \mathcal{P}_1 is minimal with respect to \mathcal{Q}_1 , or
- (C2) a clean ℓ_2 -linked path system $(\mathcal{P}_2, \mathcal{L}_2, \mathcal{A}_2)$ of order p_2 .

Proof. Let $d = 3(p_2)^2/2 - 15p_2/2 + 10$ and $p'_2 = \lceil 2eq_1(d + 1) \rceil + 1$.

First, if there is some $\mathcal{L}_{s,t} \in \mathcal{L}$ such that $\mathcal{L}_{s,t}$ is not minimal with respect to \mathcal{P} , then replace such a linkage with another $\mathsf{start}(\mathcal{L}_{s,t})$ - $\mathsf{end}(\mathcal{L}_{s,t})$ linkage of the same order which is minimal with respect to \mathcal{P} . This does not alter the fact that \mathcal{S} is an ℓ -linked path-system of order p . Further, if $p_2 = 1$, we can trivially obtain a clean ℓ_2 -linked path-system satisfying (C2) by taking any path in \mathcal{P} and setting $\mathcal{L} = \emptyset$. Hence, we can assume that $p_2 \geq 2$.

Define a function γ as follows. For each distinct $P_s, P_t \in \mathcal{P}$ we set (see Section 5 for an illustration)

$$\begin{aligned} \gamma(P_s, P_t) &= \{P \in \mathcal{P} \setminus \{P_s, P_t\} \mid P \text{ intersects at least } d_1 \text{ paths of } \mathcal{L}_{s,t} \text{ or} \\ &\quad P \text{ intersects at least } d_1 \text{ paths of } \mathcal{L}_{t,s}\}. \end{aligned}$$

If there is a pair of distinct $P_s, P_t \in \mathcal{P}$ such that $|\gamma(P_s, P_t)| \geq 2q_1$, then we construct our pair $(\mathcal{P}_1, \mathcal{Q}_1)$ as follows. By the pigeon-hole principle there is a choice of $\mathcal{P}_1 \in \{\mathcal{L}_{s,t}, \mathcal{L}_{t,s}\}$ and a set $\mathcal{Q}_1 \subseteq \gamma(P_s, P_t)$ of size q_1 such that every $Q \in \mathcal{Q}_1$ intersects at least d_1 paths of \mathcal{P}_1 . By assumption, \mathcal{P}_1 is minimal with respect to \mathcal{Q}_1 . Furthermore, $\mathsf{end}(\mathcal{Q}_1)$ is well-linked to $\mathsf{start}(\mathcal{Q}_1)$. Hence, $(\mathcal{P}_1, \mathcal{Q}_1)$ satisfies (C1).

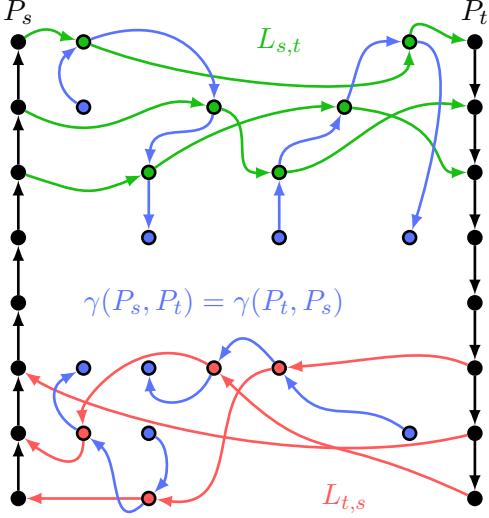


Figure 5: Illustration of the set $\gamma(P_s, P_t)$ used in the proof of Lemma 5.12, given in blue. This set consists of the paths of \mathcal{P} which intersect many paths in at least one of the linkages between P_s and P_t .

We now assume that $|\gamma(P_s, P_t)| < 2q_1$ holds for all distinct $P_s, P_t \in \mathcal{P}$. We construct a set \mathcal{P}_2 as follows.

First, distribute the elements of \mathcal{P} arbitrarily into p_2 disjoint sets $\mathcal{X}_1, \dots, \mathcal{X}_{p_2}$, each of size p'_2 . For each \mathcal{X}_i , define a random variable x_i which corresponds to sampling one element of \mathcal{X}_i from an uniform distribution.

For each three distinct $a, b, c \in \{1, 2, \dots, p_2\}$, let $A_{a,b,c}$ be the event that $x_a \in \gamma(x_b, x_c)$.

Since the event $A_{a,b,c}$ depends only on the values of x_a, x_b and x_c , we know that $A_{a,b,c}$ is independent from $A_{a',b',c'}$ if $\{a, b, c\} \cap \{a', b', c'\} = \emptyset$. Hence, $A_{a,b,c}$ is independent from all but at most $\binom{p_2}{3} - \binom{p_2-3}{3} = d$ other events.

We now bound the value of $\Pr[A_{a,b,c}]$. As there are $(p'_2)^3$ distinct choices for the tuple (x_a, x_b, x_c) and for each choice of x_b, x_c there are at most $2q_1$ choices of x_a such that $x_a \in \gamma(x_b, x_c)$, we have that $\Pr[A_{a,b,c}] \leq (2q_1(p'_2)^2)/(p'_2)^3 = 2q_1/p'_2$.

Because $p'_2 \geq e \cdot (d+1) \cdot 2q_1 + 1$, from Lemma 5.5 we know that the probability that none of the events $A_{a,b,c}$ occur is positive. That is, there is some choice of x_1, x_2, \dots, x_{p_2} such that $x_a \notin \gamma(x_b, x_c)$ for all three distinct $a, b, c \in \{1, 2, \dots, p_2\}$. We set $\mathcal{P}_2 = \{x_1, x_2, \dots, x_{p_2}\}$.

For each distinct $P_s, P_t \in \mathcal{P}_2$ define $\mathcal{L}'_{s,t} = \{L \in \mathcal{L}_{s,t} \mid \text{for all } P \in \mathcal{P}_2 \setminus \{P_s, P_t\} \text{ we have } V(P) \cap L = \emptyset\}$.

By choice of \mathcal{P}_2 , we have that $P_r \notin \gamma(P_s, P_t)$ holds for all pairwise distinct $P_s, P_t, P_r \in \mathcal{P}_2$. Hence, $|\mathcal{L}'_{s,t}| \geq |\mathcal{L}_{s,t}| - (p_2 - 2)d_1 \geq \ell_2$.

Finally, choose \mathcal{A}_2 as the elements A_i^{in}, A_i^{out} of \mathcal{A} satisfying $P_i \in \mathcal{P}_2$. Clearly, $(\mathcal{P}_2, \mathcal{L}_2, \mathcal{A}_2)$ is a clean ℓ_2 -linked path system of order p_2 , satisfying (C2). \square \square

We obtain a bidirected clique from a clean path-system by first iteratively trying to obtain disjoint paths inside \mathcal{L} pairwise connecting the paths of \mathcal{P} . If we cannot do so, then we obtain a well-linked semi-web.

As seen before in Lemma 5.11, in order to obtain a web from a semi-web $(\mathcal{P}, \mathcal{Q})$ of degree d , we require that \mathcal{Q} is much larger than \mathcal{P} . Unfortunately, it is not possible to directly use the results of [KK22] to obtain the required web, as the sizes of the linkages \mathcal{P} and \mathcal{Q} provided by their statements do not match. Instead, we need to modify the proof of [KK22, Lemma 4.8], ensuring that in each step of the iteration described above we obtain a sufficiently large gap between \mathcal{Q} and \mathcal{P} so that we can apply Lemma 5.11. Similarly, we again require a large gap between \mathcal{Q}

and \mathcal{P} when obtaining a split or segmentation from a web using Lemma 5.2. Hence, we need to “pay” the function from both Lemma 5.2 and Lemma 5.11 during each step of the iteration in our proof.

In order to avoid repetition, we only present here the part of the proof from [KK22] which we modify, which is Lemma 5.14 below. The remainder of the proof of [KK22, Lemma 4.8] is given by Lemma 5.13.

Lemma 5.13 ([KK22, proof of Lemma 4.8]). Let k be an integer and let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathcal{A})$ be a clean 1-linked path-system of order p in a digraph D such that all paths in \mathcal{L} are pairwise vertex-disjoint. If $p \geq \mathbf{p}_{5.13}(k) := 3k$, then D contains a bidirected clique of order k as a butterfly minor.

The proof of Lemma 5.14 works by iteratively constructing pairwise disjoint paths R_r . If in some step we cannot construct the desired path R_r , then we argue that some linkage $\mathcal{L}_{i,j}$ in the clean-path system must intersect many paths of some other linkage $\mathcal{L}_{s,t}$. Our modifications of the proof of [KK22, Lemma 4.8] are essentially focused on ensuring that both linkages forming the semi-web described above are large enough for us to apply lemmata 5.2 and 5.11.

We define

$$k'(k) = 2 \binom{3k}{2}, \quad \mathbf{p}_{5.14}(k) = 3k, \quad \ell_{5.14}(x, y, q, k) = (2xqk'(k))^{2^{k'(k)(y(x-1)+1)}(3x)^{k'(k)}}. \quad \begin{matrix} \mathbf{p}_{5.14} \\ \ell_{5.14} \end{matrix}$$

Observe that $\ell_{5.14}(x, y, q, k) \in 2^{2 \uparrow \uparrow \text{poly}^5(x, y, q, k)}$.

Lemma 5.14. Let x, y, q and k be integers. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathcal{A})$ be a clean ℓ' -linked path system of order p' in a digraph D . If $\ell' \geq \ell_{5.14}(x, y, q, k)$ and $p' \geq \mathbf{p}_{5.14}(k)$, then there are $\mathcal{P}_1 \subseteq \mathcal{L}_P \in \mathcal{L}$ and $\mathcal{Q}_1 \subseteq \mathcal{L}_Q \in \mathcal{L}$, where $\mathcal{L}_P \neq \mathcal{L}_Q$, such that D contains one of the following:

- (W1) a bidirected clique of order k as a butterfly minor,
- (W2) a (y, q) -split $(\mathcal{P}', \mathcal{Q}')$ of $(\mathcal{P}_1, \mathcal{Q}_1)$ where $\text{end}(\mathcal{Q}')$ is well-linked to $\text{start}(\mathcal{Q}')$, or
- (W3) an (x, q) -segmentation $(\mathcal{P}', \mathcal{Q}')$ of $(\mathcal{P}_1, \mathcal{Q}_1)$ where $\text{end}(\mathcal{P}')$ is well-linked to $\text{start}(\mathcal{P}')$.

Proof. Let $k_1 = 2 \binom{3k}{2}$. We define functions q', f, p' and q'' recursively as follows. We start by setting

$$\begin{aligned} q'(k_1) &= \mathbf{q}_{5.11}(x, q, x), & f(k_1 + 1) &= 0, f(k_1) &= q'(k_1) + 1. \\ p'(k_1) &= x, & \text{and} & & q''(k_1) &= \mathbf{q}_{5.2}(p'(k_1), q, x, y, p'(k_1)). \end{aligned}$$

For $1 \leq r < k_1$, we set

$$\begin{aligned} p'(r) &= f(r + 1) + x - 1, & q''(r) &= \mathbf{q}_{5.2}(p'(r), q, x, y, p'(r)) \\ q'(r) &= \mathbf{q}_{5.11}(p'(r), q''(r), x) & \text{and} & & f(r) &= (k_1 - r + 1)q'(r) + 1. \end{aligned}$$

By repeatedly applying the functions above, we obtain the following recursive equality, which will be used later

$$\begin{aligned} f(r) &= 1 + x(k_1 - r + 1)(x + f(r + 1) - 1)^x \\ &\quad \cdot (q(x + f(r + 1) - 1)(x + q + f(r + 1) - 1))^{2^{y(x-1)+1}}. \end{aligned}$$

Before proceeding with the proof, we give upper bounds for the functions defined above.

Claim 1. For all $0 \leq r \leq k_1 - 1$, we have

$$f(k_1 - r) \leq (2k_1 x q)^{2^{(r+1)(y(x-1)+1)}(3x)^{r+1}}$$

Proof. We prove the statement iteratively by starting at $r = 0$.

$$f(k_1) = x^{x+1}q + 1 \leq (2k_1xq)^{3x} \leq (2k_1xq)^{3x2^{y(x-1)+1}}.$$

Hence, the bounds given above hold for $r = 0$. Now assume the bounds hold for some $r - 1 \in \{0, 1, \dots, k_1 - 2\}$. We show that they also hold for r . To aid readability, we replace $y(x-1) + 1$ with w .

$$\begin{aligned} f(k_1 - r) &= x(r+1)(x + f(k_1 - r + 1) - 1)^x \\ &\quad \cdot (q(x + f(k_1 - r + 1) - 1) \\ &\quad \cdot (x + q + f(k_1 - r + 1) - 1))^{2^{y(x-1)+1}} + 1 \\ &\leq x(r+1) \left(x + (2k_1xq)^{2^{rw}(3x)^r} \right)^x \\ &\quad \cdot \left(q \left(x + (2k_1xq)^{2^{rw}(3x)^r} \right) \left(x + q + (2k_1xq)^{2^{rw}(3x)^r} \right) \right)^{2^w} \quad (\text{induction}) \\ &\leq x(r+1) \left(2(2k_1xq)^{2^{rw}(3x)^r} \right)^x \left(4q(2k_1xq)^{2^{rw+1}(3x)^r} \right)^{2^w} \quad (x + q \leq 2k_1xq) \\ &\leq \left(2(2k_1xq)^{2^{rw}(3x)^r} \right)^x \left(4k_1xq(2k_1xq)^{2^{rw+1}(3x)^r} \right)^{2^w} \quad (xk_1 \leq (xk_1)^{2^w}) \\ &= \left(2(2k_1xq)^{2^{rw}(3x)^r} \right)^x \left(2(2k_1xq)^{2^{rw+1}(3x)^r+1} \right)^{2^w} \\ &= 2^{2^w+x} (2k_1xq)^{2^w(2^{rw+1}(3x)^r+1)+2^{rw}x(3x)^r} \\ &\leq (2k_1xq)^{2^w(2^{rw+1}(3x)^r+1)+2^w+x+2^{rw}x(3x)^r} \quad (2 \leq 2k_1xq) \\ &= (2k_1xq)^{2^{w(r+1)+1}(3x)^r+2^{w+1}+x+2^{rw}x(3x)^r} \\ &\leq (2k_1xq)^{3 \cdot 2^{w(r+1)} 3^r x^{r+1}} \quad (2^{w(r+1)+1}(3x)^r, \\ &\quad 2^{rw}x(3x)^r, \\ &\quad 2^{w+1} + x \leq 2^{(r+1)w} 3^r x^{r+1}) \\ &= (2k_1xq)^{2^{(r+1)(y(x-1)+1)}(3x)^{r+1}} \quad (\text{def. of } w) \end{aligned}$$

Hence, the statement of the claim follows by induction. \square

From [Claim 1](#) we have that $f(1) \leq \ell_{5.14}(x, y, q, k)$. Let $(\mathcal{P} = (P_1, P_2, \dots, P_{3k}), \mathcal{L} = (\mathcal{L}_{i,j}), \mathcal{A} = (A_i^{\text{in}}, A_i^{\text{out}})) := \mathcal{S}$. Choose an arbitrary bijection $\sigma : [k_1] \rightarrow \{(i, j) \mid 1 \leq i, j \leq 3k, i \neq j\}$, where $[k_1] = \{1, 2, \dots, k_1\}$.

We iteratively construct linkages $\mathcal{L}_{i,j}^r$ and paths R^r , where $1 \leq r \leq k_1$, satisfying the following:

(L1) R^r is a path from A_i^{out} to A_j^{in} , where $(i, j) := \sigma(r)$, and R^r does not share any internal vertex with any path in \mathcal{P} or in any $\mathcal{L}_{\sigma(q)}^r$ where $q > r$.

(L2) $|\mathcal{L}_{\sigma(r)}^r| = f(r)$,

(L3) for all $r < q \leq k_1$ we have $|\mathcal{L}_{\sigma(q)}^r| = p'(r)$, and $\mathcal{L}_{\sigma(q)}^r$ is $\mathcal{L}_{\sigma(r)}^r$ -minimal, and

(L4) for all $1 \leq i, j \leq 3k$ where $i \neq j$ and for all $P \in \mathcal{L}_{\sigma^{-1}(i,j)}^r$ the path P has no vertex in common with any P_t for $i \neq t \neq j$.

We show that, if **(L2)** to **(L4)** hold on step $1 \leq r \leq k_1$, then **(L1)** holds on step r and if **(L2)** to **(L4)** hold on step $1 \leq r < k_1$, then **(L2)** to **(L4)** also hold on step $r + 1$.

For $r = 1$, we pick $\mathcal{L}_{\sigma(1)}^1 \subseteq \mathcal{L}_{s,t}$ arbitrarily, where $(s, t) = \sigma(1)$, so that $|\mathcal{L}_{\sigma(1)}^1| = f(1)$, satisfying **(L2)** for $r = 1$. Further, for each $1 < q \leq k_1$, we choose $\mathcal{L}_{\sigma(q)}^1$ as a $\mathcal{L}_{\sigma(1)}^1$ -minimal $\text{start}(\mathcal{L}_{\sigma(q)})$ -

$\text{end}(\mathcal{L}_{\sigma(q)})$ linkage in $D(\mathcal{L}_{\sigma(1)}^1 \cup \mathcal{L}_{\sigma(q)})$ of order $p'(1)$. This satisfies (L3) for $r = 1$. Observe that (L4) is satisfied for $r = 1$ because \mathcal{S} is a clean path-system.

Now assume that (L2) to (L4) hold for step $r \geq 1$. We construct the path R^r as follows.

First, let $(i, j) = \sigma(r)$. We consider two cases.

Case 1: There is a path $P \in \mathcal{L}_{i,j}^r$ which, for each $r < q \leq k_1$, is internally disjoint from at least $f(r+1)$ paths in $\mathcal{L}_{\sigma(q)}^r$.

We set $R^r := P$, satisfying (L1) for r . If $r = k_1$, we are done with the iteration. Otherwise, let $(s, t) := \sigma(r+1)$ and let $\mathcal{L}_{s,t}^{r+1} \subseteq \mathcal{L}_{s,t}^r$ be an $A_s^{\text{out}}\text{-}A_t^{\text{in}}$ -linkage of order $f(r+1) \leq p'(r)$ (satisfying (L2)) such that no path in $\mathcal{L}_{s,t}^{r+1}$ has an internal vertex in $V(P) \cup \bigcup_{r'=1}^r V(R^{r'})$ (towards satisfying (L1) for $r+1$). Because (L1) holds for r , we know that $\mathcal{L}_{s,t}^r$ is internally disjoint from all $R^{r'}$ with $1 \leq r' < r$. Hence, such a linkage $\mathcal{L}_{s,t}^{r+1}$ exists. Furthermore, as (L1) holds for r , we have that every path in $\mathcal{L}_{s,t}^{r+1}$ is disjoint from all $P \in \mathcal{P} \setminus \{P_s, P_t\}$ (towards satisfying (L4)).

For each $r+1 < q \leq k_1$, let $(s', t') = \sigma(q)$ and choose an $A_{s'}^{\text{out}}\text{-}A_{t'}^{\text{in}}$ -linkage $\mathcal{L}_{\sigma(q)}^{r+1}$ of order $p'(r+1)$ inside $D(\mathcal{L}_{\sigma(q)}^r \cup \mathcal{L}_{s,t}^{r+1})$ which satisfies (L4) such that every path in $\mathcal{L}_{\sigma(q)}^{r+1}$ has no inner vertex in $V(P) \cup \bigcup_{r'=1}^r V(R_{\sigma(r')})$ and which is $\mathcal{L}_{s,t}^{r+1}$ -minimal (satisfying (L3)). Thus, (L1) to (L4) hold for the step $r+1$.

Case 2: For every $P_z \in \mathcal{L}_{i,j}^r$ there is some $r < q_z \leq k_1$ for which P_z intersects least $p'(r) - f(r+1) + 1 = x$ paths in $\mathcal{L}_{\sigma(q_z)}^r$.

Let $(i', j') = \sigma(q_z)$. As $|\mathcal{L}_{i,j}^r| = f(r) = (k_1 - r - 1)q'(r) + 1$, by the pigeon-hole principle there is a $r < w \leq k_1$ and a $\mathcal{Q} \subseteq \mathcal{L}_{i,j}^r$ of order $q'(r)$ such that all paths in \mathcal{Q} intersect at least x paths in $\mathcal{L}_{\sigma(w)}^r$. Hence, $(\mathcal{L}_{\sigma(w)}^r, \mathcal{Q})$ is a $(p'(r), q'(r))$ -semi-web of degree x . Finally, as the starting points and endpoints of both \mathcal{Q} and $\mathcal{L}_{\sigma(w)}^r$ lie in the well-linked set $A_i^{\text{out}} \cup A_j^{\text{in}} \subseteq A$, we have that $(\mathcal{L}_{\sigma(w)}^r, \mathcal{Q})$ is also a well-linked semi-web.

Applying Lemma 5.11 to $(\mathcal{L}_{\sigma(w)}^r, \mathcal{Q})$ yields a well-linked $(p_2, q''(r))$ -web $(\mathcal{P}_2, \mathcal{Q}_2)$ where \mathcal{P}_2 is minimal with respect to \mathcal{Q}_2 , where $p'(r) \geq p_2 \geq x$. As $q''(r) \geq \mathbf{q}_{5.2}(p'(r), q, x, y, p'(r))$ and \mathcal{P}_2 is also p_2 -minimal with respect to \mathcal{Q}_2 , we can apply Lemma 5.2 to $(\mathcal{P}_2, \mathcal{Q}_2)$, obtaining two cases.

If Lemma 5.2(S2) holds, then we satisfy (W3). Otherwise, Lemma 5.2(S1) holds, satisfying (W2).

If Case 2 above never occurs, then we obtain a sequence of pairwise disjoint paths $\mathcal{R} := (R_1, R_2, \dots, R_{k_1})$ such that for all $1 \leq r \leq k_1$, the path R_r is an $A_i^{\text{out}}\text{-}A_j^{\text{in}}$ path which is disjoint from \mathcal{P} , where $(i, j) = \sigma(r)$. By Lemma 5.13, we obtain a bidirected clique of size k as a butterfly minor, satisfying (W1). \square \square

We conclude this section by combining the main statements proven above, yielding the following theorem which is used later on.

We define

$$\begin{aligned} \ell'(x, y, q, k) &= \ell_{5.12}(\mathbf{p}_{5.14}(2k), \ell_{5.14}(x, y, q, 2k), x), \\ \mathbf{t}_{5.15}(x, y, q, k) &= 2(\mathbf{k}_{5.7}(\ell'(x, y, q, k)), \\ &\quad \mathbf{p}_{5.12}(\mathbf{q}_{5.11}(\ell'(x, y, q, k), \\ &\quad \mathbf{q}_{5.2}(\ell'(x, y, q, k), q, x, y, \\ &\quad \ell'(x, y, q, k)), x), \mathbf{p}_{5.14}(2k))). \end{aligned} \tag{t_{5.15}}$$

Note that $\mathbf{t}_{5.15}(x, y, q, k) \in 2^{5 \uparrow \text{poly}^5(x, y, q, k)}$.

Theorem 5.15. Let D be a digraph. If $\text{dtw}(D) \geq \mathbf{t}_{5.15}(x, y, q, k)$, then D contains one of the following

- (D1) a cylindrical grid of order k as a butterfly minor,
- (D2) a (y, q) -split $(\mathcal{P}', \mathcal{Q}')$ of some pair $(\mathcal{P}_1, \mathcal{Q}_1)$ in D , where $\text{end}(\mathcal{Q}')$ is well-linked to $\text{start}(\mathcal{Q}')$, or
- (D3) an (x, q) -segmentation $(\mathcal{P}', \mathcal{Q}')$ of some pair $(\mathcal{P}_1, \mathcal{Q}_1)$ in D , where $\text{end}(\mathcal{P}')$ is well-linked to $\text{start}(\mathcal{P}')$.

Proof. Let $k_6 = 2k$, $\ell_5 = \ell_{5.14}(x, y, q, k_6)$, $p_5 = \mathbf{p}_{5.14}(k_6)$, $d_3 = x$, $\ell_2 = \ell_{5.12}(p_5, \ell_5, d_3)$, $q_4 = \mathbf{q}_{5.2}(\ell_2, q, x, y, \ell_2)$, $q_3 = \mathbf{q}_{5.11}(\ell_2, q_4, d_3)$, $p_2 = \mathbf{p}_{5.12}(q_3, p_5)$, $k_1 = \mathbf{k}_{5.7}(\ell_2, p_2)$.

Observe that $\ell_2 = \ell'(x, y, q, k) \geq x$ and that $\mathbf{t}_{5.15}(x, y, q, k) \geq 2k_1$.

By Corollary 4.4, D contains a bramble \mathcal{B}_1 of order at least k_1 . By Lemma 5.7, D contains an ℓ_2 -linked path-system \mathcal{S}_2 of order p_2 . By applying Lemma 5.12 to \mathcal{S}_2 , we obtain two cases.

Case 1: Lemma 5.12(C1) holds.

Then D contains a well-linked (ℓ_2, q_3) -semi-web $(\mathcal{P}_3, \mathcal{Q}_3)$ of degree d_3 where \mathcal{P}_3 is minimal with respect to \mathcal{Q}_3 and $\mathcal{P}_3 \in \mathcal{L}$. In particular, $\text{end}(\mathcal{P}_3)$ is well-linked to $\text{start}(\mathcal{P}_3)$ as $\text{start}(\mathcal{L}) \cup \text{end}(\mathcal{L})$ are vertices of a well-linked set, and $\text{end}(\mathcal{Q}_3)$ is well-linked to $\text{start}(\mathcal{Q}_3)$ as the starting and endpoints of the paths in \mathcal{P} are vertices of a well-linked set.

By Lemma 5.11, there is some p_4 such that $\ell_2 \geq p_4 \geq d_3$ and D contains a well-linked (p_4, q_4) -web $(\mathcal{P}_4, \mathcal{Q}_4)$ where \mathcal{P}_4 is minimal with respect to \mathcal{Q}_4 and $\mathcal{P}_4 \subseteq \mathcal{P}_3$, $\mathcal{Q}_4 \subseteq \mathcal{Q}_3$. In particular, \mathcal{P}_4 is also weakly p_4 -minimal with respect to \mathcal{Q}_4 , $\text{end}(\mathcal{P}_4)$ is also well-linked to $\text{start}(\mathcal{P}_4)$ and $\text{end}(\mathcal{Q}_4)$ is also well-linked to $\text{start}(\mathcal{Q}_4)$. Applying Lemma 5.2 to $(\mathcal{P}_4, \mathcal{Q}_4)$ yields two cases. If Lemma 5.2(S1) holds, then (D2) is satisfied. Otherwise, Lemma 5.2(S2) holds, satisfying (D3).

Case 2: Lemma 5.12(C2) holds.

That is, D contains a clean ℓ_5 -linked path-system \mathcal{S}_5 of order p_5 . Applying Lemma 5.14 to \mathcal{S}_5 yields three cases.

If Lemma 5.14(W1) holds, then D contains a bidirected clique of order k_6 as a butterfly minor. As a cylindrical grid of order k contains k_6 vertices, D also contains a cylindrical grid of order k as butterfly minor, satisfying (D1).

If Lemma 5.14(W2) holds, then we obtain a (y, q) -split, satisfying (D2).

If Lemma 5.14(W3) holds, then we obtain an (x, q) -segmentation, satisfying (D3). $\square \quad \square$

6 Temporal digraphs and routings

In our proof we are frequently faced with problems of the following form. We have already constructed two linkages, \mathcal{P} and \mathcal{Q} , say, but whereas the paths within the same linkage are disjoint by definition, a pair of paths from different linkages may intersect arbitrarily, or not at all. So the intersection pattern between the two linkages can be arbitrarily complex. The problem then is to find some kind of order within the chaos, i.e. to find a subgraph of a specific form in $\bigcup \mathcal{P} \cup \bigcup \mathcal{Q}$.

Problems of this form occur frequently in this research area and also in the proof of the directed grid theorem [KK15], where the authors had to solve the same kind of problems over and over again.

In this section we develop a framework based on temporal digraphs which allows us to rephrase these problems in a more abstract setting. This abstraction allows us to simplify many arguments and to unify proofs by isolating the core ideas common to several of these proofs. Moreover, our framework allows us to obtain much better bounds and to prove elementary bounds for results that require non-elementary bounds in [KK15].

There are several different definitions of temporal graphs and temporal walks, each useful in a different context. Here, we make use of the notation from [CHMZ20, Mol20] and adapt it to the directed setting. We first define our notion of temporal digraphs and walks within such digraphs and then discuss how they arise in our context.

Definition 6.1. A *temporal digraph* is a pair $T = (V, \mathcal{A})$ consisting of a vertex set V and sequence of arc sets $\mathcal{A} = (A_1, A_2, \dots, A_\ell)$ such that $D_t(T) := (V, A_t)$ is a digraph for all $1 \leq t \leq \ell$. We also refer to $D_t(T)$ as *layer t* of T and call t a *time step*. The *lifetime* of D is given by $\ell(D) := \ell$.

We next define paths and walks in the temporal setting. A *temporal walk* in a temporal digraph T is required to obey the “timeline” of T , i.e. the order in which edges occur on the walk must respect the time steps of T . In our setting we even need a more restrictive definition and allow a temporal walk to only use a single edge of each layer.

Definition 6.2. A *temporal walk* of length n from v_0 to v_n in a temporal digraph T is a sequence $W := (v_0, t_0), (v_1, t_1), \dots, (v_n, t_n)$ such that $(v_i, v_{i+1}) \in A_{t_i}$ and $t_i < t_{i+1} \leq \ell(T)$ for all $0 \leq i \leq n - 1$. If such a walk exists, we say that v_0 *temporally reaches* v_n . A temporal walk is said to be a *temporal path* if no vertex appears twice in the sequence. Finally, we say that W *departs* at t_0 and *arrives* at t_n , and that $t_n - t_0$ is the *duration* of W .

In our setting, temporal digraphs usually arise from a linkage \mathcal{P} intersecting pairwise disjoint digraphs Q_1, \dots, Q_q as formalised in the next definition. For this to work the individual paths of the linkage must intersect the digraphs all in the same order.

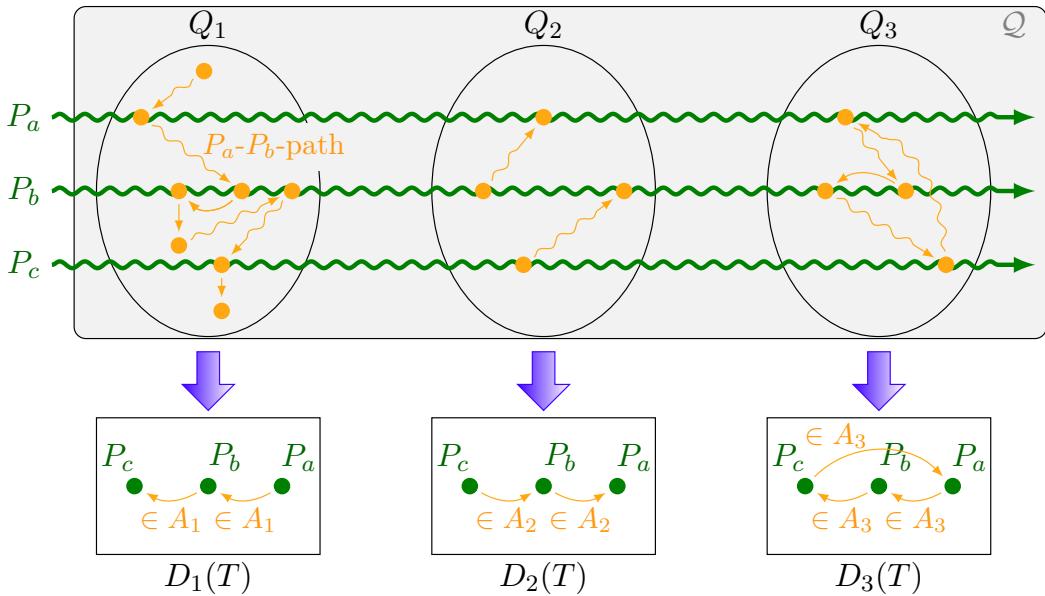


Figure 6: The layers $D_j(T)$ of the temporal graph $T := (V = \{a, b, c\}, \mathcal{A} = \{A_1, A_2, A_3\})$ constructed from the graphs Q_j displayed above as defined in Definition 6.3.

Definition 6.3. Let \mathcal{P} be a linkage and let $\mathcal{Q} = \{Q_1, Q_2, \dots, Q_q\}$ be a set of pairwise disjoint digraphs such that each path $P_i \in \mathcal{P}$ can be partitioned as $P_i^1 \cdot P_i^2 \cdot \dots \cdot P_i^q = P_i$ such that $V(P_i^j) \cap V(\mathcal{Q}) \subseteq V(Q_j)$ for all $1 \leq j \leq q$.

The *routing temporal digraph* (V, \mathcal{A}) of \mathcal{P} through \mathcal{Q} is constructed as follows. We set $V = \mathcal{P}$ and for each $1 \leq j \leq q$ we define $A_j = \{(P_a, P_b) \mid P_a, P_b \in \mathcal{P} \text{ and there is a path from } V(P_a) \text{ to } V(P_b) \text{ inside } Q_j \text{ which is internally disjoint from } \mathcal{P}\}$.

See [Figure 6](#) for an example of a temporal digraph obtained from a linkage $\mathcal{P} := \{P_a, P_b, P_c\}$ and digraphs $\{Q_1, Q_2, Q_3\}$.

In our application of routing temporal digraphs we want to translate paths or more general structures we construct in a routing temporal digraph T of a linkage \mathcal{P} through $\mathcal{Q} := \{Q_1, \dots, Q_q\}$ into corresponding subgraphs of $\bigcup \mathcal{P} \cup \bigcup \mathcal{Q}$. This is made possible by our requirement that a temporal walk in T is only allowed to use at most one edge in each layer. For, if W is a temporal walk in T and $e = (P_i, P_j) \in E(W)$ is an edge in layer A_l , say, then we can replace e by a path $L(e) \subseteq Q_l$ connecting P_i to P_j . As W contains at most one edge per layer the paths $L(e)$ and $L(e')$ for distinct edges $e \neq e'$ are pairwise disjoint. Therefore, the walk W naturally translates into a walk in $\bigcup \mathcal{P} \cup \bigcup \mathcal{Q}$.

As the example in [Figure 6](#) demonstrates, this would no longer work if temporal walks were allowed to use more than one edge per layer. For instance, the digraph Q_2 induces the edges (P_c, P_b) and (P_b, P_a) in the routing temporal digraph T_2 of \mathcal{P} through $\{Q_2\}$, but the walk (P_c, P_b, P_a) in T_2 does not correspond to any P_c - P_a -walk in $\bigcup \mathcal{P} \cup Q_2$.

6.1 H -routings

We now introduce the main tools to facilitate temporal digraphs for our purposes. Our goal is to describe connections between specific sets of vertices that involve several layers, possibly with the additional requirement that the connections avoid a given set of forbidden vertices. The concept of H -routings formalised in the next definition allows us to specify the required connections by a digraph H . We now define H -routings in digraphs and temporal digraphs.

Definition 6.4. Let H be a digraph, D be a digraph or temporal digraph, and $S \subseteq V(D)$. An H -routing (over S) is a bijection $\varphi : V(H) \rightarrow S$ such that for each v - u path P in H we can find a $\varphi(v)$ - $\varphi(u)$ -path (or temporal path, resp.) in D which is disjoint from $S \setminus \varphi(V(P))$.

Note that reachability in temporal digraphs is not transitive, as the example in [Figure 6](#) demonstrates: in the temporal digraph $T_{1,3} := (\{P_1, P_2, P_3\}, \{A_1, A_3\})$ containing only the layers A_1 and A_3 , P_a is reachable from P_c and P_b is reachable from P_a but P_b is not reachable from P_c . To get a meaningful concept of H -routings we therefore have to require that not only for every edge in H but also for every path in H there is a temporal path in T with the same start and endpoint.

To motivate our next results let us briefly consider the following statement proved by Leaf and Seymour for undirected graphs.

Lemma 6.5 ([LS15, statement 2.3]). Let $r \geq 1$ and $h \geq 3$ be integers, let G be a connected graph with $|V(G)| \geq (r+2)(2h-5) + 2$. Then, G contains one of the following

- a path with r vertices whose internal vertices have degree two in G , or
- a spanning tree T with at least h leaves.

In the undirected setting, both cases of the previous lemma can be useful in terms of connectivity they provide: a tree contains for every pair of leaves a path connecting them without intersecting any other leaves whereas a long path yields many disjoint subpaths.

In the directed setting, however, neither out-trees nor in-trees provide any connectivity whatsoever between their leaves. To get a statement that we can use in the sequel we therefore have to replace the trees used in [Lemma 6.5](#) by something else. We first need the following well-known result about acyclic digraphs, often stated in terms of chains and anti-chains in partial orderings.

Observation 6.6. Every acyclic digraph D with more than $\ell \cdot p$ vertices but no \mathbf{P}_p as a subgraph contains a set $X \subseteq V(D)$ of size ℓ such that no vertex in X can reach any other vertex in X .

Proof. For each $i \geq 0$ let $L_i := \{v \in V(D) : \text{the longest path from a source to } v \text{ in } T \text{ has length } i\}$. As, by assumption, D does not contain a path of length p , $L_i = \emptyset$ for all $i \geq p$. Furthermore, by construction, no vertex $v \in L_i$ can reach any other vertex $u \in L_i$, as otherwise the longest path from a source to u would be longer than i .

Every vertex of D lies in some L_i ; by the pigeon-hole principle, at least one L_i must contain at least ℓ vertices, which proves the claim. \square \square

In the next lemma we establish a simple base case where we are guaranteed to either find a long path or a $\tilde{\mathbf{K}}_k$ -routing.

Lemma 6.7. Let D be a strongly connected digraph. Let $s \in V(D)$ be a vertex such that $D - \{s\}$ contains at least kp strongly connected components. Then, D contains one of the following:

- (B1) a \mathbf{P}_p as a subgraph, or
- (B2) a $\tilde{\mathbf{K}}_k$ -routing over some $S \subseteq V(D)$.

Proof. We show that (B2) holds if (B1) does not hold.

Let T be the acyclic digraph of strongly connected components of $D - \{s\}$. As D has no path of length p , T also has no such path. Thus, by [Observation 6.6](#), T contains a set $X' \subseteq V(T)$ of size k such that no vertex in X' can reach any other vertex in X' in T .

Let $X \subseteq V(D)$ be a set of size $|X| = |X'|$ which contains a vertex $v_i \in V(C_i)$ for each strong component $C_i \in X'$ of $D - \{s\}$. Let $\varphi : V(\tilde{\mathbf{K}}_k) \rightarrow \{v_1, v_2, \dots, v_k\}$ be a bijection. We show that φ is a $\tilde{\mathbf{K}}_k$ -routing in D .

Let $v_i, v_j \in X$ be two distinct vertices. Let $C_t \in V(T)$ be a sink in T which is reachable from C_i and let $C_r \in V(T)$ be a source in T which can reach C_j .

Since D is strongly connected and $D - \{s\}$ is not, there is some $u_t \in V(C_t)$ and some $u_r \in V(C_r)$ such that (u_t, s) and (s, u_r) are arcs in D .

Because C_i can reach C_t and C_r can reach C_j in T , there is a v_i - u_t path $P_{i,t}$ and a u_r - v_j path $P_{r,j}$ in D such that $P_{i,t}$ and $P_{r,j}$ do not intersect any vertex in $X \setminus \{v_i, v_j\}$. Hence, $P_{i,t} \cdot (u_t, s) \cdot (s, u_r) \cdot P_{r,j}$ is a v_i - v_j path in D which is disjoint from $X \setminus \{v_i, v_j\}$.

We conclude that φ is a $\tilde{\mathbf{K}}_k$ -routing in D , and so (B2) holds, as desired. \square \square

We now prove a statement as an analogue to [Lemma 6.5](#) in the case of directed graphs.

Theorem 6.8. Let $n_{6.8}(k, p) := 2k^2p^3$. Every strongly connected digraph D with $|V(D)| \geq n_{6.8}(k, p)$ contains one of the following:

- (S1) a \mathbf{P}_p as a subgraph, or

(S2) a $\vec{\mathbf{K}}_k$ -routing over some $S \subseteq V(D)$.

Proof. We show that **(S2)** holds if **(S1)** does not hold.

We iterate from 1 to $2kp$, potentially stopping earlier. On step i , we construct a vertex sequence X_i and a digraph D_i satisfying all of the following.

(A1) $(v_1, v_2, \dots, v_i) = X_i$ and so $|X_i| \geq i$,

(A2) D_i is strongly connected component of $D_{i-1} - \{v_i\}$ (and so v_i is not on D_i),

(A3) for every $1 \leq j \leq i$, v_j lies on D_{j-1} ,

(A4) $V(D_i) \geq (2kp - i)kp^2$.

Start by setting $D_0 = D$ and X_0 as the empty sequence. Clearly, **(A1)** to **(A4)** hold for 0 (to simplify notation, we set $D_{-1} := D$ and replace $\{v_0\}$ with the empty set so that **(A2)** is well-defined for $i = 0$). On step $i \leq 2kp$, we consider the following cases.

Case 1. There is a $v \in V(D_{i-1})$ such that $D_{i-1} - v$ contains at least kp strongly connected components.

As we assume that **(S1)** does not hold, we know from Lemma 6.7 that **(S2)** holds and we are done with the construction and the proof.

Case 2. There is a $v \in V(D_{i-1})$ such that $D_{i-1} - v$ is strongly connected.

Then set $X_i := X_{i-1} \cdot (v)$ and set $D_i := D_{i-1} - v$. It is immediate from the choice of v and from our assumption over $D_{i-1} - v$ that **(A1)** to **(A4)** hold for i as they also hold for $i - 1$.

Case 3. There is a $v \in V(D_i)$ such that the largest strongly connected component C of $D_{i-1} - v$ has at least $|V(D_{i-1})| - kp^2$ many vertices.

Then set $X_i := X_{i-1} \cdot (v)$ and $D_i := C$. Note that $|V(D_{i-1})| - kp^2 \geq (2kp - i)kp^2$ as **(A4)** holds for $i - 1$. Hence, it is again immediate from the choice of v and from our assumption over C that **(A1)** to **(A4)** hold for i as they also hold for $i - 1$.

This completes the case distinction above. Now assume towards a contradiction that none of the three cases above apply and $i \leq 2kp$.

For every $v \in V(D_{i-1})$, we know that $D_{i-1} - v$ has fewer than kp strong components because *Case 1* does not apply. Further, $D_{i-1} - v$ is not strongly connected, as *Case 2* does not apply. Finally, we know that each strong component of $D_{i-1} - v$ has fewer than $|V(D_{i-1}) - kp^2|$ vertices because *Case 3* does not apply.

For each $v \in V(D_{i-1})$, let \mathcal{C}_v be the set of strong components of $D'_v := D_{i-1} - v$. Let $A_v = \{(u, w) : u, w \in V(D'_v) \text{ and there is no path from } u \text{ to } w \text{ in } D'_v\}$. Let $n_v = |V(D'_v)|$.

Now let $v \in V(D_{i-1})$ be arbitrary. For any two distinct components $C_1, C_2 \in \mathcal{C}_v$, there is no path in D'_v from any $u \in V(C_1)$ to any $w \in V(C_2)$ or vice versa. Thus, A_v contains all possible arcs from vertices in C_1 to vertices in C_2 or all possible arcs from vertices in C_2 to vertices in C_1 . Fixing some arbitrary ordering (C_1, C_2, \dots, C_c) of the elements of \mathcal{C}_v , we deduce that

$$|A_v| \geq \sum_{\substack{C_a, C_b \in \mathcal{C}_v, \\ a < b}} |V(C_a)| \cdot |V(C_b)|.$$

Let $C \in \mathcal{C}_v$ be a strong component of D'_v with the maximal number of vertices among all components in \mathcal{C}_v . The previous inequality implies that $|A_v| \geq |V(C)| \cdot \sum_{C_a \in \mathcal{C}_v \setminus \{C\}} |V(C_a)|$.

By assumption $|V(C)| \leq n_v - kp^2 - 1$, and since the strong components of D'_v form a partition of D'_v , we obtain that $\sum_{C_a \in \mathcal{C}_v \setminus \{C\}} |V(C_a)| \geq kp^2$. Note that $n_v - kp^2 - 1 \geq kp^2$ since **(A4)** holds for $i < 2kp$. Since D'_v contains fewer than kp strong components, we also obtain $|V(C)| \geq$

$(n_v - 1)/kp$. From the inequality above, we obtain

$$\begin{aligned} |A_v| &\geq |V(C)| \cdot \sum_{C_a \in \mathcal{C} \setminus \{C\}} |V(C_a)| \\ &\geq \frac{n_v - 1}{kp} \cdot \sum_{C_a \in \mathcal{C} \setminus \{C\}} |V(C_a)| \\ &\geq \frac{n_v - 1}{kp} \cdot kp^2 = (n_v - 1) \cdot p \end{aligned}$$

Hence, $\sum_{v \in V(D_{i-1})} |A_v| \geq n_v \cdot (n_v - 1) \cdot p$. Since A_v does not contain any reflexive tuples and there are $n_v(n_v - 1)$ non-reflexive tuples in the set $V(D_{i-1}) \times V(D_{i-1})$, by the pigeon-hole principle we deduce that there are $u, v \in V(D_{i-1})$ and there are $v_1, \dots, v_p \in V(D_{i-1})$ such that $(u, v) \in A_{v_j}$ for all $1 \leq j \leq p$. Thus, every path from u to w in D_{i-1} must contain each of the vertices v_1, \dots, v_p and thus be of length at least $p + 1$ (by definition, $(u, w) \notin A_u \cup A_w$), a contradiction to the assumption that **(S1)** does not hold, that is, that D does not contain a path of length p . Hence, one of the three cases must apply and we can complete the construction above.

If at any point during the construction we end up at Case 1, then, as argued above, **(S2)** is true and we are done. Otherwise, we know that **(A1)** to **(A4)** hold for $2kp$. We now show that **(S2)** holds.

Let $(v_1, v_2, \dots, v_{2kp}) := X_{2kp}$. We inductively construct disjoint sets $A_i, B_i \subseteq \{v_1, \dots, v_{2pk}\}$ and we construct for each $v_j \in B_i$ a path $P_j^+ \subseteq D_j$ from v_j to $V(D_{2kp})$ and a path $P_j^- \subseteq D_j$ from $V(D_{2kp})$ to v_j such that $(V(P_j^+) \cup V(P_j^-)) \cap A_i = \emptyset$ for all $v_j \in B_i$. Finally, $|B_i| = i$, $|A_i| \geq 2kp - 2pi$ and the elements of B_i are contained in X_i .

We start by setting $A_0 := \{v_1, \dots, v_{2kp}\}$ and $B_0 := \emptyset$, which obviously meet the requirements. On step $i < k$, let j be minimal such that $v_j \in A_{i-1}$.

Let P_j^+ be a shortest path in D_{j-1} from v_j to a vertex in D_{2kp} and let P_j^- be a shortest path from a vertex in D_{2kp} to v_j , again in D_{j-1} . As D_{j-1} is strongly connected and $D_{2kp} \subseteq D_{j-1}$ due to **(A2)**, such paths exist. Furthermore, P_j^+ and P_j^- are both of length at most p and all internal vertices of P_j^+ and P_j^- are disjoint from X_{j-1} (and so from B_{i-1}) due to **(A2)**.

We define $B_i = B_{i-1} \cup \{v_j\}$ and $A_i = A_{i-1} \setminus (V(P_j^+) \cup V(P_j^-))$. As $|V(P_j^+)|, |V(P_j^-)| < p$ and $|A_{i-1}| \geq 2kp - 2p(i+1)$, we immediately get that $|A_i| \geq 2kp - 2pi$, as required. Clearly, the other conditions are satisfied as well.

The construction stops after k steps with a set B_k such that for each $v_j \in B_k$ there are paths P_j^+, P_j^- such that P_j^+ is a v_j - $V(D_{2kp})$ path and P_j^- is a $V(D_{2kp})$ - v path and both paths are internally disjoint from B_k .

Let $\varphi : V(\vec{\mathbf{K}}_k) \rightarrow B_k$ be a bijection. We show that φ is a $\vec{\mathbf{K}}_k$ -routing in D . To see this, let $v_i, v_j \in B_k$ and let $P_{i,j}$ be a path in D_{2kp} from the end vertex of P_i^+ to the start of P_j^- . Then $P_i^+ \cup P_{i,j} \cup P_j^-$ contains a v_i - v_j -path disjoint from B_k . Hence, **(S2)** holds and this concludes the proof of the theorem. \square \square

6.2 Finding \mathbf{P}_k -routings in temporal digraphs

Of particular interest to us are H -routings in temporal digraphs where H is just a simple path \mathbf{P}_k . This is not surprising as, for example, an acyclic grid is nothing else than a “horizontal” linkage $\mathcal{Q} := (Q_1, \dots, Q_q)$ that intersects a sequence $\mathcal{P} := \{P_1, \dots, P_p\}$ of pairwise disjoint digraphs P_i in the order P_1, \dots, P_p where each P_i happens to be a simple path intersecting the paths in \mathcal{Q} in the order Q_1, \dots, Q_q .

Our next goal, therefore, is to identify properties of the individual layers of a temporal digraph T that guarantee the existence of a \mathbf{P}_k -routing in T .

Recall from [Section 3](#) that a digraph D is *unilateral* if for every pair $u, v \in V(D)$ of vertices v can reach u or u can reach v . As proved in [[HNC65](#), Theorem 3.10], a digraph D is unilateral if and only if there is a walk visiting all vertices of D .

We need a stronger characterisation of unilateral digraphs for our results.

Lemma 6.9. A digraph D is unilateral if and only if for every $S \subseteq V(D)$ there is a walk W of length at most $2kn$ in D with $S \subseteq V(W)$, where $k = |S|$ and $n = |V(D)|$.

Proof. If there is a walk W in D with $V(W) = V(D)$, then D is clearly unilateral.

Now assume D is unilateral and let $S = \{v_1, v_2, \dots, v_k\} \subseteq V(D)$. We construct for each $1 \leq i \leq k$ a walk W_i of length at most $2i \cdot |V(D)|$ such that $\{v_1, v_2, \dots, v_i\} \subseteq V(W_i)$. Start by setting W_1 as the walk containing only v_1 .

We extend the walk W_i as follows. Let u_1, u_2 be the starting point and endpoint of W_i , respectively. If there is a path P from v_{i+1} to u_1 , we construct W_{i+1} by adding P to the beginning of W_i . Similarly, if there is path P from u_2 to v_{i+1} we construct W_{i+1} by adding P to the end of W_i . Since in both cases P has length at most $|V(D)| - 1$, the length of W_{i+1} is at most $2i|V(D)| + |V(D)| \leq 2(i+1)|V(D)|$.

If none of the previous two cases apply, we know there is an arc (w_1, w_2) in W_i such that there is a path P_1 from w_1 to v_{i+1} and a path P_2 from v_{i+1} to w_2 . We construct W_{i+1} by replacing the arc (w_1, w_2) in W_i by the walk $P_1 \cdot P_2$. The walk W_{i+1} has length at most $2i|V(D)| + 2|V(D)| = 2(i+1)|V(D)|$, as desired.

Thus, the walk W_k satisfies the statement of the lemma. □

Finding long walks in unilateral digraphs is easy. The task becomes more complicated in temporal digraphs as the connectivity provided by individual layers may be very different. As we show next, one direction of the previous lemma can be retained in the temporal setting. Observe that $\ell_{6.10}(n, k) \in O(k^2 n^{2kn+2})$.

Lemma 6.10. Let $\ell_{6.10}(n, k) := 2kn \sum_{i=1}^{2kn} n^i$. Let T be a temporal digraph with n vertices where each layer is unilateral and let $S \subseteq V(T)$ be a set of size k . If $\ell(T) \geq \ell_{6.10}(n, k)$, then T contains a temporal walk W with $S \subseteq V(W)$. ℓ_{6.10}

Proof. By [Lemma 6.9](#), for each $1 \leq i \leq \ell(T)$ there is a walk W_i of length at most $2kn$ in $D_i(T)$ such that $S \subseteq V(W_i)$. Note that there are $\sum_{i=1}^{2kn} n^i$ distinct walks of length at most $2kn$ over the vertex set of T .

As $\ell(T) \geq 2kn \sum_{i=1}^{2kn} n^i$, by the pigeon-hole principle there is some walk W' which appears on at least $2kn$ different layers. Let t_1, t_2, \dots, t_{2kn} be time steps such that each $D_{t_i}(T)$ contains the walk W' . Now set $W := ((v_i, t_i) \mid 1 \leq i \leq 2kn \text{ and } v_i \text{ is the } i\text{th vertex on } W')$.

Since $V(W) = V(W')$, we also have $S \subseteq V(W)$, as desired. □

The next lemma establishes a special case where a temporal digraph is guaranteed to contain a \mathbf{P}_k -routing. Together with [Lemma 6.10](#) this implies [Theorem 6.12](#).

Lemma 6.11. Let D be a temporal digraph and let W be a temporal walk in D . If $|V(W)| \geq k^2 - 1$, then W contains a \mathbf{P}_k -routing.

Proof. If W contains a \mathbf{P}_k as a temporal subpath, then this subgraph also contains a \mathbf{P}_k -routing. So assume W does not contain any \mathbf{P}_k as a temporal subpath.

Let W' be a minimal temporal subwalk of W such that $V(W') = V(W)$. We say that a temporal subwalk R of W is a *return around u* if u appears twice on R , R starts and ends on u and all other vertices on R appear exactly once on R . As we do not have any \mathbf{P}_k as a temporal subpath of W' , we know each return contains at most $k - 1$ vertices. Furthermore, by minimality of W' , each return R around a vertex u must contain a vertex u' which only occurs on R .

Let v_1, v_2, \dots, v_a be vertices on W' which are not contained in any return and let X be the shortest subwalk of W' containing these vertices. Clearly, X is a temporal path, as otherwise we would have some return containing some v_i . Hence, $a \leq k - 1$ as there is no \mathbf{P}_k in W .

Observe that $|V(W')| \geq k^2 - 1$, that each return contains at most $k - 1$ vertices and that there are at most $k - 1$ vertices which are not contained in any return. This implies that the walk W' must contain at least $\frac{(k^2-1)-(k-1)}{k-1} = k$ distinct returns.

Each return R_i on W' contains some vertex u_i which appears exactly once on W' . Let X be a shortest temporal subwalk of W' which contains $S := \{u_1, u_2, \dots, u_k\}$. This walk contains a \mathbf{P}_k -routing as desired, as each temporal subpath of X connecting u_i to u_j is internally disjoint from $S \setminus \{u_i, \dots, u_j\}$ for every $1 \leq i < j \leq k$. \square \square

The previous two lemmas immediately imply the following result. Observe that $\ell_{6.12}(n, k) \in 2^{1 \uparrow \text{poly}^6(k, n)}$.

Theorem 6.12. Let $\ell_{6.12}(n, k) := \ell_{6.10}(n, k^2 - 1)$. Let T be a temporal digraph where each layer is unilateral. If $\ell(T) \geq \ell_{6.12}(n, k)$ and $n := |V(T)| \geq k^2 - 1$, then there is some set $S \subseteq V(T)$ such that T contains a \mathbf{P}_k -routing over S . $\ell_{6.12}$

Proof. Let $S' \subseteq V(D)$ with $|S'| = k^2 - 1$. By Lemma 6.10, T contains a temporal walk W such that $S' \subseteq V(W)$. In particular, $|V(W)| \geq k^2 - 1$. By Lemma 6.11, there is some $S \subseteq V(W)$ such that W and, hence, T contain a \mathbf{P}_k routing over S . \square \square

6.3 Finding C_k and $\tilde{\mathbf{P}}_k$ -routings in temporal digraphs

As discussed at the beginning of the previous section, \mathbf{P}_k -routings relate to the connectivity in an acyclic grid, which only allows to route from top to bottom and from left to right. If instead of an acyclic grid we consider a fence, then the fence allows us to route upwards as well as downwards as the columns alternate in direction. Two consecutive columns taken together allow to go from any row to any other row and in this way resemble a strongly connected digraph like a cycle C_k or a bioriented $\tilde{\mathbf{P}}_k$ much more than a \mathbf{P}_k . In this section we aim at finding H -routings that provide this higher level of connectivity.

We first define

$$s_{6.13}(k_1, k_2) := 6k_1(k_2)^2 - 8k_1k_2 + 2k_1 - 2(k_2)^2 + 3k_2 \quad s_{6.13}$$

and prove the following technical lemma. Note that $s_{6.13}(k_1, k_2) \in O(k_1(k_2)^2)$.

Lemma 6.13. Let T be a temporal digraph, let W be a temporal walk in T , let k_1, k_2 be integers, and let $S \subseteq V(W)$ be a set of size at least $s_{6.13}(k_1, k_2)$. Then there is some $S' \subseteq S$ such that one of the following is true:

(R1) W contains a $\tilde{\mathbf{P}}_{k_1}$ -routing over S' , or

(R2) there are (possibly arcless) walks W_1, W_a, W_b, W_c in D such that W_1 is a subwalk of W leaving and arriving at the same time steps as W , $W_a \cdot W_b \cdot W_c = W_1$, W_a and W_c are internally disjoint from S' , and W_b contains a \mathbf{P}_{k_2} -routing over S' where the first vertex of the \mathbf{P}_{k_2} is mapped to $\text{start}(W_b)$ and the last vertex of the \mathbf{P}_{k_2} is mapped to $\text{end}(W_b)$.

Proof. To simplify arithmetic steps, we define $k_3 = (k_1 - 1)(k_2 - 1)$ and $s_1 = 2(k_3 + k_2) + k_2 - 1$. Note that $(k_2 - 1)(s_1 - 1) + k_2(4k_3 + k_2) = 6k_1(k_2)^2 - 8k_1k_2 + 2k_1 - 2(k_2)^2 + 3k_2 = \text{S6.13}(k_1, k_2) \leq |S|$.

We identify in the following claim a base case for the proof, which is used several times later on.

Claim 1. Let $\widehat{W} = \widehat{W}_a \cdot \widehat{W}_b \cdot \widehat{W}_c$ be a temporal walk inside W such that $\text{start}(\widehat{W}) = \text{start}(W)$ and $\text{end}(\widehat{W}) = \text{end}(W)$ and let $\widehat{S} \subseteq V(\widehat{W}_a) \cap V(\widehat{W}_c) \cap S$ be a set such that each vertex of \widehat{S} appears exactly once on \widehat{W}_a and exactly once on \widehat{W}_c . If $|\widehat{S}| \geq k_3 + 1$, then there is some $S' \subseteq \widehat{S}$ such that **(R1)** or **(R2)** holds.

Proof. Since each vertex of \widehat{S} appears exactly once on \widehat{W}_a and exactly once on \widehat{W}_c , each of these walks induces an ordering over the vertices of \widehat{S} . By [Theorem 3.1](#), we obtain two cases.

Case 1: There is some $S' \subseteq \widehat{S}$ of size k_1 such that the vertices of S' appear on \widehat{W}_c in the reverse order compared to their order on \widehat{W}_a .

Let W_a be a shortest temporal subpath of \widehat{W}_a containing every vertex of S' and let W_c be a shortest temporal subpath of \widehat{W}_c containing every vertex of S' . Note that $\text{end}(W_a) = \text{start}(W_c)$. We show that $W_a \cdot W_c$ contains a $\tilde{\mathbf{P}}_{k_1}$ -routing over S' . Let $\{u_1, u_2, \dots, u_{k_1}\}$ be the vertices of the $\tilde{\mathbf{P}}_{k_1}$ sorting according to their occurrence on the path.

Let $u_i, u_j \in \{u_1, u_2, \dots, u_{k_1}\}$. If $i < j$, then W_a contains a u_i - u_j path avoiding $S' \setminus \{u_i, \dots, u_j\}$. If $j > i$, then W_c contains a u_i - u_j path avoiding $S' \setminus \{u_j, \dots, u_i\}$. Since both W_a and W_c are temporal paths, we have that $W_a \cdot W_c$ contains a $\tilde{\mathbf{P}}_{k_1}$ -routing over S' , satisfying **(R1)**.

Case 2: There is some $S' \subseteq \widehat{S}$ of size k_2 such that the vertices of S' appear in \widehat{W}_c in the same order as in \widehat{W}_a .

Let W_b be the shortest temporal subpath of \widehat{W}_a containing every vertex of S' . Let W_a be a temporal $\text{start}(\widehat{W})$ - $\text{start}(W_b)$ path in \widehat{W}_a and let \widehat{W}_c be a temporal $\text{end}(W_b)$ - $\text{end}(\widehat{W})$ path in \widehat{W}_c .

Since every vertex of S' appears exactly once, W_b contains a \mathbf{P}_{k_2} -routing over S' where the first vertex of the \mathbf{P}_{k_2} is mapped to $\text{start}(W_b)$ and the last vertex of the \mathbf{P}_{k_2} is mapped to $\text{end}(W_b)$. Further, W_a and W_c are internally disjoint from S' . Thus, the temporal walk $W_1 := W_a \cdot W_b \cdot W_c$ satisfies **(R2)**. \square

Let W' be a minimal temporal subwalk of W such that $S \subseteq V(W')$. We say that a temporal subwalk R of W is a *return around* a vertex $u \in S$ if u appears exactly twice on R , R starts and ends on u and all vertices of $(V(R) \cap S) \setminus \{u\}$ appear exactly once on R . Note that, by minimality of W' , each return R around a vertex u must contain a vertex $u' \in S$ whose only occurrence on W' is on R .

Let R be a return in W' such that the cardinality of $S_1 := V(R) \cap S$ is maximum. We consider two cases.

Case 1: $|S_1| \leq s_1 - 1$.

We decompose W' into $Q_1 \cdot R_1 \cdot Q_2 \cdot R_2 \cdot \dots \cdot Q_x \cdot R_x \cdot Q_{x+1} = W'$, where each Q_i is a temporal walk where no vertex of S appears twice and each R_i is a return in W' . By definition of return, such a decomposition is unique. We distinguish between two cases.

Case 1.1: $x \geq k_2$.

For each $1 \leq i \leq k_2$ let $u_i \in V(R_i) \cap S$ be a vertex which occurs exactly once on W' . Let W_a be a temporal $\text{start}(W')$ - u_1 path in W' , let W_c be a temporal u_{k_2} - $\text{end}(W')$ path in W' and let W_b be a temporal u_1 - u_{k_2} walk in W' which contains every vertex of $S' := \{u_1, u_2, \dots, u_{k_2}\}$ exactly once.

Because each vertex of S' appears exactly once, W_b contains a \mathbf{P}_{k_2} -routing over S' where $u_1 = \text{start}(W_b)$ is the first vertex of the \mathbf{P}_{k_2} and $u_{k_2} = \text{end}(W_b)$ is the last vertex of the \mathbf{P}_{k_2} . Further, W_a and W_c are temporal walks which are internally disjoint from S' . Hence, $W_1 := W_a \cdot W_b \cdot W_c$ satisfies (R2).

Case 1.2: $x < k_2$.

Since $|S| \geq (k_2 - 1)(s_1 - 1) + k_2(4k_3 + k_2)$ and $|V(R_i) \cap S| \leq |S_1| \leq s_1 - 1$ for every $1 \leq i \leq x$, there is some $1 \leq b \leq x$ such that $|V(Q_b) \cap S| \geq (|S| - (k_2 - 1)(s_1 - 1))/k_2 \geq 4k_3 + k_2$.

Let t_1 be the time step in which Q_b departs and let t_2 be the time step in which Q_b arrives. Let Q'_a be a temporal $\text{start}(W')$ - $\text{start}(Q_b)$ path in W' arriving at t_1 and let Q'_c be a temporal $\text{end}(Q_b)$ - $\text{end}(W')$ path in W' departing at t_2 . Let $S_a = V(Q'_a) \cap S$, $S_b = V(Q_b) \cap S$ and $S_c = V(Q'_c) \cap S$. We consider three cases.

Case 1.2.1: $|S_b \setminus (S_a \cup S_c)| \geq k_2$.

Let $S' \subseteq S_b \setminus (S_a \cup S_c)$ be a set of size k_2 . Let W_b be a shortest walk inside Q_b containing every vertex of S' exactly once. By construction of Q_b , this is possible. Let W_a be a temporal $\text{start}(W')$ - $\text{start}(W_b)$ walk in $Q'_a \cdot Q_b$ and let W_c be a temporal $\text{end}(W_b)$ - $\text{end}(W')$ walk in $Q_b \cdot Q'_c$.

By construction, we have that W_a and W_c are internally disjoint from S' . Further, since each vertex of S' appears exactly once, W_b contains a \mathbf{P}_{k_2} -routing over S' where the first vertex of the \mathbf{P}_{k_2} is mapped to $\text{start}(W_b)$ and the last vertex of the \mathbf{P}_{k_2} is mapped to $\text{end}(W_b)$. Hence, $W_1 := W_a \cdot W_b \cdot W_c$ satisfies (R2).

Case 1.2.2: $|(S_b \cap S_a) \setminus S_c| \geq k_3 + 1$ or $|(S_b \cap S_c) \setminus S_a| \geq k_3 + 1$.

We assume, without loss of generality, that $|(S_b \cap S_a) \setminus S_c| \geq k_3 + 1$. The other case follows analogously.

Let $S_2 = (S_b \cap S_a) \setminus S_c$. Let $\widehat{W}_c = Q_b \cdot Q'_c$. Each vertex of \widehat{S} appears exactly once on Q_a and exactly once on \widehat{W}_c . Hence, by Claim 1, there is some $S' \subseteq S_2$ such that (R1) or (R2) holds.

Case 1.2.3: The conditions of Case 1.2.1 and Case 1.2.2 do not apply.

We first show that $|S_a \cap S_c| \geq k_3 + 1$. Since $|S_b \setminus (S_a \cup S_c)| \leq k_2 - 1$ and $|S_b| \geq 4k_3 + k_2$, we have that $|S_b \cap (S_a \cup S_c)| \geq 4k_3 + 1$. Hence, $|S_b \cap S_a| \geq 2k_3 + 1$ or $|S_b \cap S_c| \geq 2k_3 + 1$.

Assume, without loss of generality, that $|S_b \cap S_a| \geq 2k_3 + 1$ holds. Because $|(S_b \cap S_a) \setminus S_c| \leq k_3$, we have that $|S_a \cap S_c| \geq k_3 + 1$, as desired.

Let $S_2 \subseteq S_a \cap S_c$ be a set of size $k_3 + 1$. Each vertex of S_2 appears exactly once on Q_a and exactly once on Q_c since Q'_a and Q_c are temporal paths. Hence, by Claim 1, there is some $S' \subseteq S_2$ such that (R1) or (R2) holds.

Case 2: $|S_1| \geq s_1$.

Let Q_a, Q_b be two temporal paths inside W' such that $Q_a \cdot R \cdot Q_b$ is a walk starting at $\text{start}(W')$ and ending at $\text{end}(W')$. Note that, as $\text{start}(R) = \text{end}(R)$, $Q_a \cdot Q_b$ is also a temporal walk. Let $S_2 \subseteq S_1$ be the vertices of S_1 which occur exactly once on $Q_a \cdot R \cdot Q_b$.

Case 2.1: $|S_2| \geq k_2$.

Let $S' \subseteq S_2$ be a set of size k_2 . Let W_b be the temporal subpath of R which contains every vertex in S' . As every internal vertex of R appears exactly once on R , such a path W_b exists. Now let W_a be a $\text{start}(W')$ - $\text{start}(W_b)$ temporal path inside W' and let W_c be an $\text{end}(W_b)$ - $\text{end}(W')$ temporal path inside W' . The temporal paths W_a and W_c are internally disjoint from S' since the only occurrence of the vertices of S' along W' is on R . Since W_b is a path containing every vertex of S' , it also contains a \mathbf{P}_{k_2} -routing φ over S' . By choice of W_b , we also have that φ maps the first vertex of the \mathbf{P}_{k_2} to $\text{start}(W_b)$ and the last vertex of the \mathbf{P}_{k_2} to $\text{end}(W_b)$. By setting $W_1 := W_a \cdot W_b \cdot W_c$, we satisfy (R2).

Case 2.2: $|S_2| < k_2$.

Because $|S_1| \geq s_1 = 2(k_3 + k_2) + k_2 - 1$, we have that $|S_1 \cap (V(Q_a) \cup V(Q_b))| = |S_1 \setminus S_2| \geq 2(k_3 + k_2)$. Hence, $|V(Q_a) \cap S_1| \geq k_3 + k_2$ or $|V(Q_b) \cap S_1| \geq k_3 + k_2$.

Without loss of generality, we assume that $|V(Q_a) \cap S_1| \geq k_3 + k_2$, as the other case follows analogously. Since every vertex of $V(Q_a) \cap S_1$ is either in $V(Q_b)$ or not, we obtain two cases.

Case 2.2.1: $|(V(Q_a) \cap S_1) \setminus V(Q_b)| \geq k_2$.

Let $S' \subseteq (V(Q_a) \cap S_1) \setminus V(Q_b)$ be a set of size k_2 and let W_b be a minimal temporal subpath of Q_a containing every vertex of S' . Let W_a be a $\text{start}(W')\text{-start}(W_b)$ temporal walk in $Q_a \cdot Q_b$ and let W_c be an $\text{end}(W_b)\text{-end}(W')$ temporal walk in $Q_a \cdot Q_b$.

Every vertex of S' appears exactly once in the temporal walk $Q_a \cdot Q_b$. Since every occurrence of S' is on W_b and W_b is a temporal path, we have that W_b contains a \mathbf{P}_{k_2} -routing over S' . Since W_b was chosen minimal, the first vertex of the \mathbf{P}_{k_2} is mapped to $\text{start}(W_b)$ and the last vertex of the \mathbf{P}_{k_2} is mapped to $\text{end}(W_b)$. Further, $W_1 = W_a \cdot W_b \cdot W_c$ and W_a and W_b are internally disjoint from S' , satisfying **(R2)**.

Case 2.2.2: $|V(Q_a) \cap V(Q_b) \cap S_1| \geq k_3 + 1$.

Let $S_3 \subseteq V(Q_a) \cap V(Q_b) \cap S_1$ be a set of size $k_3 + 1$. As Q_a and Q_b are temporal paths, we have that each vertex of S_3 appears exactly once in each of those temporal paths. Hence, by [Claim 1](#), there is some $S' \subseteq S_3$ such that **(R1)** or **(R2)** holds. \square \square

The next lemma allows us to transfer strong connectivity of each individual layer of a temporal digraph to the temporal digraph as a whole.

Lemma 6.14. Let T be a temporal digraph in which each layer is strongly connected. If $\ell(T) \geq |V(T)| - 1$, then every $u \in V(T)$ temporally reaches every $v \in V(T)$.

Proof. Let $u \in V(T)$ and let $n = |V(T)| - 1$.

For each $0 \leq i \leq n$ let R_i be the set of vertices of T which u temporally reaches in at most i time steps. Clearly $u \in R_0$ and so $|R_0| = 1$. Further, $|R_i| \leq |R_j|$ if $i \leq j$.

We show that, for every $0 \leq i < n$, if $|R_i| < |V(T)|$, then $|R_{i+1}| > |R_i|$. Let R_i be such a set and let $X = V(T) \setminus V(R_i)$. Since $D_{i+1}(T)$ is strongly connected, there is some $w \in R_i$ and some $v \in X$ such that $(w, v) \in E(D_{i+1}(T))$. By assumption, there is a temporal walk W from u to w within the first i layers in D . By extending W with the arc (w, v) , we obtain a walk from u to v within the first $i + 1$ layers. Hence, $|R_{i+1}| > |R_i|$.

Since $n \geq |V(T) - 1|$ and $|R_0| = 1$, we have that $|R_n| = n + 1$. Thus, u temporally reaches every $v \in V(T)$. \square \square

We are almost ready to prove the main result of this part which allows us to construct \mathbf{C}_k - or $\tilde{\mathbf{P}}_k$ -routings in temporal digraphs with strongly connected layers. But first we need the following lemma.

Lemma 6.15. Let D be a temporal digraph in which each layer is strongly connected, let $S \subseteq V(D)$, let $v \in V(D)$ and let $s \in S$. If $\ell(D) \geq |S| \cdot (|V(D)| - 1)$, then D contains a temporal v - s walk W with $S \subseteq V(W)$.

Proof. Let $\{s_1, \dots, s_k\} := S$ be an arbitrary ordering of S such that $s = s_k$, and let $n := |V(D)|$. We iteratively construct temporal walks W_1, W_2, \dots, W_k such that W_i is a walk from v to s_i within the first $i \cdot (n - 1)$ layers and W_i contains s_1, \dots, s_i .

Start by taking some temporal v - s_1 walk W_1 within the first $n - 1$ layers. By [Lemma 6.14](#), such a walk exists.

On step $i \geq 2$, let W' be the temporal s_{i-1} - s_i walk from layer $i \cdot (n - 1) + 1$ to layer $(i + 1) \cdot (n - 1)$ in D . By [Lemma 6.14](#), such a walk exists. Now set $W_{i+1} = W_i \cdot W'$. Since W_i arrives on $\text{end}(W_i) = \text{start}(W')$ on time step $i(n - 1)$ and W' leaves $\text{start}(W')$ on time step $i(n - 1) + 1$, we have that W_{i+1} is a temporal v - s_{i+1} walk as desired.

Thus, the walk W_k is a temporal v - s walk within the first $|S| \cdot (n - 1)$ layers which contains all vertices of S . \square \square

We define the following functions:

$$\text{s}_{6.16}(k) := \text{s}_{6.13}(k, \text{s}_{6.13}(k, (k - 1)^2 + 1)), \quad \text{s}_{6.16}$$

$$\ell_{6.16}(n, k) := \text{s}_{6.16}(k) + \text{s}_{6.13}(k, (k - 1)^2 + 1) \cdot (n - 1). \quad \ell_{6.16}$$

Observe that $\text{s}_{6.16}(k) \in O(k^{11})$ and $\ell_{6.16}(n, k) \in O(k^{11} + k^5 n)$.

We are now ready to prove the next result which guarantees an H -routing for some $H \in \{\tilde{\mathbf{P}}_k, \mathbf{C}_k\}$ in any temporal digraph of sufficiently large lifetime as long as each layer is strongly connected. Moreover, we even have some control over the vertex set of the H -routing. Note, however, that we cannot choose which of the two possible routings we obtain.

Theorem 6.16. Let T be a temporal digraph such that $D_i(T)$ is strongly connected for all $1 \leq i \leq \ell(T)$. If $\ell(T) \geq \ell_{6.16}(|V(T)|, k)$, then for every set $S \subseteq V(T)$ with $|S| \geq \text{s}_{6.16}(k)$ there is a subset $S' \subseteq S$ with $|S'| = k$ such that D contains an H -routing over S' for some $H \in \{\mathbf{C}_k, \tilde{\mathbf{P}}_k\}$.

Proof. Let $k_2 = (k - 1)^2 + 1$ and let $k_1 = \text{s}_{6.13}(k, k_2)$. Let $S_0 \subseteq S$ be a set of size $\text{s}_{6.13}(k, k_1)$. Note that $\ell(T) \geq (|S_0| + k_1) \cdot (|V(T)| - 1)$.

Let W_1 be a temporal walk of minimal length which contains all vertices of S_0 within the first $|S_0| \cdot (|V(T)| - 1)$ layers of D . By Lemma 6.15, such a walk W_1 exists.

If Lemma 6.13(R1) holds, then W_1 contains a $\tilde{\mathbf{P}}_k$ -routing over some $S' \subseteq S_0$ and we are done. Otherwise, Lemma 6.13(R2) holds. That is, there is some $S_1 \subseteq S_0$ and there are (possibly arcless) walks W_2, W_a, W_b, W_c in W_1 such that W_2 is a subwalk of W_1 departing and arriving at the same time steps as W_1 , $W_a \cdot W_b \cdot W_c = W_2$, W_a and W_c are internally disjoint from S_1 , and W_b contains a P_{k_1} -routing over S_1 where the first vertex of the P_{k_1} is mapped to $\text{start}(W_b)$ and the last vertex of the P_{k_1} is mapped to $\text{end}(W_b)$. Let φ_1 be the bijection of this P_{k_1} -routing.

Let $t_1 \leq (|S_0| \cdot (|V(T)| - 1))$ be the time step in which W_1 arrives and let W_3 be a temporal walk departing on t_1 and of duration at most $|S_1| \cdot (|V(T)| - 1)$ which visits all vertices of S_1 . By Lemma 6.15, such a walk W_3 exists.

If Lemma 6.13(R1) holds, then W_2 contains a $\tilde{\mathbf{P}}_k$ -routing over some $S' \subseteq S_1$ and we are done. Otherwise, Lemma 6.13(R2) holds. That is, there is some $S_2 \subseteq S_1$ and there are (possibly arcless) walks W_4, W_d, W_e, W_f in W_3 such that W_4 is a subwalk of W_3 departing and arriving at the same time steps as W_3 , $W_d \cdot W_e \cdot W_f = W_4$, W_d and W_f are internally disjoint from S_2 , and W_e contains a \mathbf{P}_{k_2} -routing over S_2 where the first vertex of the \mathbf{P}_{k_2} is mapped to $\text{start}(W_e)$ and the last vertex of the \mathbf{P}_{k_2} is mapped to $\text{end}(W_e)$. Let φ_2 be the bijection of this \mathbf{P}_{k_2} -routing.

By Theorem 3.1, there is some $S_3 \subseteq S_2$ of size k which satisfies one of the following two cases. Let $\varphi'_1 = \varphi_1|_{S_3}$ and $\varphi'_2 = \varphi_2|_{S_3}$.

Case 1: φ'_1 and φ'_2 induce two \mathbf{P}_k -routings over S_3 where the order of the vertices along the \mathbf{P}_k are the same. We show that φ'_1 also induces a \mathbf{C}_k -routing in D . Let u_1, u_2, \dots, u_k be the vertices of \mathbf{P}_k sorted according to their order along \mathbf{P}_k . Let $u_i, u_j \in V(\mathbf{P}_k)$.

If $i < j$, then W_b contains a $\varphi'_1(u_i)$ - $\varphi'_1(u_j)$ temporal path which is disjoint from $S_3 \setminus \{\varphi'_1(u_i), \dots, \varphi'_1(u_j)\}$.

If $i > j$, we construct the desired temporal path P' as follows. Let Q_1 be a temporal $\varphi'_1(u_i)$ - $\varphi'_1(u_k)$ walk in W_b which is disjoint from $S_3 \setminus \{\varphi'_1(u_i), \dots, \varphi'_1(u_k)\}$ and $\text{end}(Q_1) = \text{end}(W_b) = \text{start}(W_c)$. Since W_b contains a \mathbf{P}_k -routing and $\varphi'_1(u_k) = \text{end}(W_b)$, such a walk Q_1 exists.

Let Q_2 be a temporal $\varphi'_1(u_1)$ - $\varphi'_1(u_j)$ walk in W_e which is disjoint from $S_3 \setminus \{\varphi'_1(u_1), \dots, \varphi'_1(u_j)\}$ and $\text{start}(Q_2) = \text{start}(W_e) = \text{end}(W_d)$. Since W_e contains a \mathbf{P}_k -routing and $\varphi'_1(u_1) = \text{end}(W_e)$, such a walk Q_2 exists.

We now have that $Q_1 \cdot W_c \cdot W_d \cdot Q_2$ is a temporal $\varphi'_1(u_i)$ - $\varphi'_1(u_j)$ walk which is disjoint from $S_3 \setminus (\{\varphi'_1(u_i), \dots, \varphi'_1(u_k)\} \cup \{\varphi'_1(u_1), \dots, \varphi'_1(u_j)\})$ in D . Thus, $Q_1 \cdot W_c \cdot W_d \cdot Q_2$ contains the desired temporal $\varphi'_1(u_i)$ - $\varphi'_1(u_j)$ path P' . Hence, φ'_1 induces a \mathbf{C}_k -routing over $S_3 \subseteq S$ in D .

Case 2: φ'_1 and φ'_2 induce two \mathbf{P}_k -routings over S_3 where the vertices along the \mathbf{P}_k of φ'_2 are ordered in reverse compared to those of the \mathbf{P}_k of φ'_1 . We show that φ'_1 induces a $\tilde{\mathbf{P}}_k$ -routing over $S_3 \subseteq S$ in D . Let u_1, u_2, \dots, u_k be the vertices of \mathbf{P}_k sorted according to their order along the \mathbf{P}_k for φ'_1 . Let $u_i, u_j \in V(\mathbf{P}_k)$.

If $i < j$, then W_b contains a $\varphi'_1(u_i)$ - $\varphi'_1(u_j)$ temporal path which is disjoint from $S_3 \setminus \{\varphi'_1(u_i), \dots, \varphi'_1(u_j)\}$.

If $i > j$, we take a temporal $\varphi'_1(u_i)$ - $\varphi'_1(u_j)$ path P' which is disjoint from $S_3 \setminus \{\varphi'_1(u_j), \dots, \varphi'_1(u_i)\}$ in W_e . Since φ'_2 induces a \mathbf{P}_k -routing in W_e where the vertices of the \mathbf{P}_k are ordered in reverse when compared to the \mathbf{P}_k of φ'_1 , such a path P' exists. Hence, φ'_1 induces a $\tilde{\mathbf{P}}_k$ -routing over $S_3 \subseteq S$ in D . \square \square

Our next goal is to relate H -routings, for $H \in \{\mathbf{C}_k, \tilde{\mathbf{P}}_k\}$, to well-linkedness. Towards this aim, we first observe the following.

Observation 6.17. Let A and B be disjoint sets of equal cardinality and let S be a sequence containing each element of $A \uplus B$ exactly once. Then there are sequences S_1, S_2 such that $S_1 \cdot S_2 = S$ and the following holds

(P1) S_1 starts in A and ends in B or starts in B and ends in A , and

(P2) each of S_1 and S_2 contains as many elements of A as elements of B .

Proof. We assume, without loss of generality, that S starts at an element of A . The other case follows analogously by swapping A and B .

Let S_1 be the shortest prefix of S containing the same number of elements in A and B . Since the first element of S_1 lies on A , its last element must lie on B . If this were not the case, then S_1 would contain a prefix with more elements of B than elements of A , which implies that S_1 also contains a shorter prefix with as many elements of A as elements of B , a contradiction to the choice of S_1 . Hence, (P1) holds.

Let S_2 be such that $S_1 \cdot S_2 = S$. Since both S_1 and S contain as many elements of A as elements of B , we have that S_2 also contains as many elements of A as elements of B . Thus, (P2) holds. \square \square

Note that in Observation 6.17 if S does not start and end in the same, then we can always take S_1 to be S and S_2 to be empty.

The next observation is used when obtaining well-linkedness in case of a \mathbf{C}_k -routing.

Observation 6.18. Let C be a directed cycle and let $f : V(C) \rightarrow \mathbb{Z}$ be a function such that $\sum_{v \in V(C)} f(v) = 0$. Then there is some $v \in V(C)$ such that for every subpath P of C starting at v we have $\sum_{v \in V(P)} f(v) \geq 0$.

Proof. If $f(v) = 0$ for all $v \in V(C)$, then the statement is true. So assume otherwise and take a subpath P of C minimising $\sum_{v \in V(P)} f(v)$. Note that the weight of this path is negative and that P is a proper subpath of C . Let u be the vertex on C after $\text{end}(P)$. We claim that u has the desired property. Suppose not and let P' be a negative subpath of C starting in u . If $\text{end}(P') \notin V(P)$, then $P \cdot P'$ is a proper subpath of C with lower weight than P , a contradiction. Thus $\text{end}(P') \in V(P)$. As P was chosen to minimise $\sum_{v \in V(P)} f(v)$, the subpath $P \cap P'$ cannot be of positive weight and thus we obtain a contradiction to C being of total weight 0. \square \square

We are now ready to state and prove our last result on routings in temporal digraphs which shows how well-linkedness can be obtained from routing temporal digraphs T . The main idea is to use [Theorem 6.16](#) to construct a sufficiently large number of \mathbf{C}_k - or $\tilde{\mathbf{P}}_k$ -routings in T . By the pigeon-hole principle we either get enough $\tilde{\mathbf{P}}_k$ -routings or enough \mathbf{C}_k -routings. The previous two observations allow us to prove that certain sets are well-linked in either of the two cases.

Lemma 6.19. Let h be some integer, let D be a digraph, let \mathcal{L} be a linkage of order k in D and let T be the routing temporal digraph of \mathcal{L} through $\mathcal{H} := (H_1, H_2, \dots, H_h)$, where each H_i is a subgraph of D . If there is some $R \in \{\tilde{\mathbf{P}}_k, \mathbf{C}_k\}$ and there are some temporally disjoint subgraphs T_1, T_2, \dots, T_k of T such that for each $1 \leq i \leq k$ there is an R -routing φ_i over \mathcal{L} in T_i where $\varphi_i = \varphi_j$ for all $1 \leq i, j \leq k$, then $\text{start}(\mathcal{L})$ is well-linked to $\text{end}(\mathcal{L})$ in $D(\mathcal{L} \cup \mathcal{H})$.

Proof. Let $A \subseteq \text{start}(\mathcal{L})$ and $B \subseteq \text{end}(\mathcal{L})$ be sets with $n := |A| = |B|$. Let φ be the R -routing over \mathcal{L} in each T_i . We construct an A - B linkage \mathcal{W} as follows.

Let $\mathcal{L}^A = \{L \in \mathcal{L} \mid \text{start}(L) \subseteq A\}$ and let $\mathcal{L}^B = \{L \in \mathcal{L} \mid \text{end}(L) \subseteq B\}$. Let $\varphi_{A,B} = \varphi|_{\mathcal{L}^A \cup \mathcal{L}^B}$. Let $\{a_1, a_2, \dots, a_n\} := A$ and let $\{b_1, b_2, \dots, b_n\} := B$.

We start by constructing temporal walks $\mathcal{W} := \{W_1, W_2, \dots, W_n\}$ in T and by constructing sets $X_0, \dots, X_n \subseteq \mathcal{L}^A$ and $Y_0, \dots, Y_n \subseteq \mathcal{L}^B$ such that, for each $0 \leq i \leq n$, we have

$$(\mathbf{W1}) \quad |X_i| = i = |Y_i|,$$

$$(\mathbf{W2}) \quad W_i \text{ is a temporal walk in } T_i \text{ which is disjoint from } (\mathcal{L}^A \setminus X_i) \cup Y_{i-1}, \text{ and}$$

$$(\mathbf{W3}) \quad \text{start}(\mathcal{W}) = \mathcal{L}^A \text{ and } \text{end}(\mathcal{W}) = \mathcal{L}^B.$$

We consider two cases.

Case 1: $R = \mathbf{C}_k$.

Let $R' = \mathbf{C}_n$ and note that $\varphi_{A,B}$ is a \mathbf{C}_n -routing in each T_i . Partition $V(R')$ into a sequence of subpaths $\mathcal{Q} = (Q_1, Q_2, \dots, Q_x)$ of R' where each Q_i can be decomposed into $Q_i^a \cdot Q_i^b$ such that $|V(Q_i^a)| \geq 1$, $|V(Q_i^b)| \geq 1$, $\varphi_{A,B}(V(Q_i^a)) \subseteq \mathcal{L}^A$ and $\varphi_{A,B}(V(Q_i^b)) \subseteq \mathcal{L}^B$. Since $\varphi(V(R')) = \mathcal{L}^A \cup \mathcal{L}^B$, such a decomposition exists. Now define the function $f : \mathcal{Q} \rightarrow \mathbb{Z}$ with $f(Q_i = Q_i^a \cdot Q_i^b) = |\varphi(Q_i^a) \cap \mathcal{L}^A| - |\varphi(Q_i^b) \cap \mathcal{L}^B|$.

From [Observation 6.18](#) we know there is some $v \in V(R')$ for which every subpath P of R' starting at v satisfies $|\varphi_{A,B}(V(P)) \cap \mathcal{L}^A| \geq |\varphi_{A,B}(V(P)) \cap \mathcal{L}^B|$. Let Q be the subpath of R' starting at v and containing every vertex of R' , and let $\{u_1, u_2, \dots, u_n\} := V(Q)$ be an ordering of the vertices of Q according to their occurrence along Q .

We iteratively construct the desired walks W_i and sets X_i, Y_i such that, for each $0 \leq i \leq n-1$, Q can be decomposed as $Q_1^i \cdot Q_2^i \cdot Q_3^i = Q$ satisfying the following properties

$$(\mathbf{C1}) \quad \varphi_{A,B}(\text{end}(Q_1^i)) \in \mathcal{L}^A \setminus X_i,$$

$$(\mathbf{C2}) \quad \varphi_{A,B}(V(Q_2^i) \setminus \{\text{start}(Q_2^i)\}) \subseteq (\mathcal{L}^B \cup X_i) \setminus Y_i, \quad |V(Q_2^i) \cap (\mathcal{L}^B \setminus Y_i)| \geq 1, \text{ and}$$

$$(\mathbf{C3}) \quad Y_i \subseteq \varphi_{A,B}(V(Q_3^i)) \text{ and } \varphi_{A,B}(V(Q_3^i)) \subseteq Y_i \cup X_i.$$

Start by setting $X_0 := \emptyset$ and $Y_0 := \emptyset$. By choice of Q , **(C1)**, **(C2)**, **(C3)** and **(W1)** hold for $i = 0$.

On step $1 \leq i \leq n$, let $Q_1^{i-1} \cdot Q_2^{i-1} \cdot Q_3^{i-1} = Q$ be a decomposition of Q satisfying **(C1)**, **(C2)** and **(C3)** for $i-1$. Let $u_a = \text{end}(Q_1^{i-1})$ and let $u_b \in V(Q_2^{i-1}) \cap (\mathcal{L}^B \setminus Y_{i-1})$ be the last such vertex on Q_2^{i-1} . As **(C1)** and **(C2)** hold for $i-1$, $u_a \in \mathcal{L}^A \setminus X_{i-1}$ holds and such a vertex u_b exists.

Let W_i be a temporal $\varphi_{A,B}(u_a)$ - $\varphi_{A,B}(u_b)$ walk in T_i avoiding $(\mathcal{L}^A \cup \mathcal{L}^B) \setminus \varphi_{A,B}(\{u_a, \dots, u_b\})$. Since $\varphi_{A,B}$ is an R' -routing in T_i , such a walk exists. Set $X_i = X_{i-1} \cup \{\varphi_{A,B}(u_a)\}$ and $Y_i = Y_{i-1} \cup \{\varphi_{A,B}(u_b)\}$. The walk W_i satisfies **(W2)** because $Y_{i-1} \subseteq Q_3^{i-1}$ and **(C3)** holds for $i-1$.

We now show that X_i, Y_i and W_i satisfy the required properties. Clearly **(W1)** holds for i . If $i < n$, then $|\mathcal{L}^A \setminus X_i| \geq 1$. As **(C2)** and **(C3)** hold for $i-1$, we have $\mathcal{L}^A \setminus X_{i-1} \subseteq \varphi(V(Q_1^{i-1}))$.

Let Q_1^i be the shortest subpath of Q_1^{i-1} containing every vertex of $\varphi^{-1}(\mathcal{L}^A \setminus X_i)$ and let Q_3^i be the u_b -end(Q) subpath of Q . By construction, **(C1)** holds for i . Further, by choice of u_b , **(C3)** holds for i .

Let Q_2^i be the end(Q_1^i)-start(Q_3^i) subpath of Q . Since Q_1^i is a subpath of H' starting at v and ending in a vertex u'_a with $\varphi(u'_a) \in \mathcal{L}^A \setminus X_i$, we have that $Q_2^i \cdot Q_3^i$ must contain some vertex of $\varphi^{-1}(\mathcal{L}^B)$. Further, as $\varphi(V(Q_3^i)) \subseteq Y_i \cup X_i$ due to **(C3)**, we have that Q_2^i contains some vertex of $\mathcal{L}^B \setminus Y_i$. Finally, $\mathcal{L}^A \setminus X_i \subseteq V(Q_1^i)$ and so $\varphi_{A,B}(V(Q_2^i) \setminus \{\text{start}(Q_2^i)\}) \subseteq \mathcal{L}^B \cup (X_i \setminus Y_i)$. Hence, **(C2)** holds for i .

After n steps, it is immediate that **(W3)** holds by choice of W_1, W_2, \dots, W_n .

Case 2: φ is a $\tilde{\mathbf{P}}_k$ -routing.

Let $R' = \tilde{\mathbf{P}}_n$ and note that $\varphi_{A,B}$ is a $\tilde{\mathbf{P}}_n$ -routing in T_i for each i .

We iteratively construct the desired walks W_i and sets $X_i, Y_i \subseteq \mathcal{L}'$ such that, for each $0 \leq i \leq n-1$, the following statement holds

(P1) for each strongly connected component Z_i of $R' - \varphi_{A,B}^{-1}(Y_i)$ we have that the set $\varphi_{A,B}(V(Z_i))$ contains as many elements of $\mathcal{L}^A \setminus X_i$ as elements of $\mathcal{L}^B \setminus Y_i$.

Start by setting $X_0 := \emptyset$ and $Y_0 := \emptyset$. Clearly, **(W1)** and **(P1)** hold for $i=0$.

On step $1 \leq i \leq n$, let Z be a component (and hence a subpath) of $R' - \varphi_{A,B}^{-1}(X_{i-1})$ such that $\varphi_{A,B}(V(Z))$ contains at least one element of $\mathcal{L}^A \setminus X_{i-1}$ and one element of $\mathcal{L}^B \setminus Y_{i-1}$. Since **(W1)** and **(P1)** hold for $i-1$, such a component exists.

Because Z is a bidirected path, it induces a sequence over the elements of $\mathcal{L}^A \setminus X_{i-1}$ and of $\mathcal{L}^B \setminus Y_{i-1}$. Let Z' be the shortest subpath of Z satisfying $\varphi_{A,B}(V(Z')) \cap ((\mathcal{L}^A \setminus X_{i-1}) \cup (\mathcal{L}^B \setminus Y_{i-1})) = \varphi(V(Z)) \cap ((\mathcal{L}^A \setminus X_{i-1}) \cup (\mathcal{L}^B \setminus Y_{i-1}))$. By [Observation 6.17](#), there is a subpath Z'' of Z' starting at one of the endpoints of Z' such that one endpoint of Z'' is in $\mathcal{L}^A \setminus X_{i-1}$ and the other is in $\mathcal{L}^B \setminus Y_{i-1}$, and both Z'' and the rest of Z' contain as many elements of $\mathcal{L}^A \setminus X_{i-1}$ as they contain elements of $\mathcal{L}^B \setminus Y_{i-1}$. Let Z'' be the shortest such subpath of Z' .

Let $\{z_1, z_2, \dots, z_j\}$ be the vertices of Z'' sorted according to their occurrence along Z'' . Without loss of generality, we have $\varphi_{A,B}(z_1) \in \mathcal{L}^B \setminus X_{i-1}$ and $\varphi_{A,B}(z_j) \in \mathcal{L}^A \setminus Y_{i-1}$.

Let j_a be the smallest index such that $\varphi_{A,B}(z_{j_a}) \in \mathcal{L}^A \setminus X_{i-1}$. We set W_i as a temporal $\varphi_{A,B}(z_{j_a})$ - $\varphi_{A,B}(z_1)$ walk in T_i which is disjoint from $(\mathcal{L}^A \setminus X_{i-1}) \cup Y_{i-1}$. By choice of j_a and because $\varphi_{A,B}$ is an R' -routing over $\mathcal{L}^A \cup \mathcal{L}^B$ in T_i and Z is a component of $R' - \varphi_{A,B}(X_{i-1})$, such a walk W_i exists.

We set $X_i = X_{i-1} \cup \{\varphi_{A,B}(z_{j_a})\}$ and $Y_i = Y_{i-1} \cup \{\varphi_{A,B}(z_1)\}$. The vertex z_1 is an endpoint of Z'' , **(P1)** holds for $i-1$ and $\varphi_{A,B}(V(Z'') \setminus \{z_1\})$ contains one less vertex of $\mathcal{L}^B \setminus Y_{i-1}$ and one less vertex of $\mathcal{L}^A \setminus X_{i-1}$ when compared to Z . Hence, **(P1)** holds for i . Further, $|X_i| = |X_{i-1}| + 1 = |Y_{i-1}| + 1 = |Y_i|$, and so **(W1)** holds.

This completes the case distinction above and the construction of W_1, W_2, \dots, W_n . We construct an A - B linkage \mathcal{L}' as follows. For each $1 \leq i \leq n$, let $L_i^a = \text{start}(W_i)$, $L_i^b = \text{end}(W_i)$, $a_i = \text{start}(L_i^a)$ and $b_i = \text{end}(L_i^b)$. We construct a path $Q_i = Q_i^a \cdot Q_i^t \cdot Q_i^b$ such that $D(Q_i^a) \subseteq L_i^a - D(\{W_j \mid s_{i+1} \leq j \leq \ell(T)\})$, $D(Q_i^t) \subseteq D(\{H_j \mid s_i \leq j \leq s_{i+1}\})$ and $D(Q_i^b) \subseteq L_i^b - D(\{W_j \mid s_1 \leq j \leq s_i\})$.

Since (W2) holds, each arc of W_i corresponds to some path in D which is disjoint from $\{L_j^a \mid i < j \leq n\} \cup \{L_j^b \mid 1 \leq j < i\}$. Furthermore, W_i corresponds to some path Q_i^t in $D(\{W_j \mid t_i \leq j \leq t_{i+1}\})$.

We set Q_i^a as the subpath of L_i^a ending at $\text{start}(Q_i^t)$ and we set Q_i^b as the subpath of L_i^b starting at $\text{end}(Q_i^t)$. By construction, $Q_i := Q_i^a \cdot Q_i^t \cdot Q_i^b$ is an a_i - b_i path in $D(\mathcal{L}) \cup D(\mathcal{W})$ which is disjoint from all Q_j for $1 \leq j < i$. Hence, $\mathcal{Q} := \{Q_i \mid 1 \leq i \leq n\}$ is an A - B linkage as desired.

Because we can construct such an A - B linkage for any choice of A, B , we have that $\text{start}(\mathcal{L}')$ is well-linked to $\text{end}(\mathcal{L}')$ in $D(\mathcal{L} \cup \mathcal{W})$, as desired. \square \square

We can now show how to obtain well-linkedness from the routing temporal digraph of some linkage \mathcal{L} . The idea is to use [Theorem 6.16](#) to obtain many \mathbf{C}_k and $\tilde{\mathbf{P}}_k$ -routings. With the pigeon-hole principle, we get many routings which are equal. We then use observations [6.17](#) and [6.18](#) in each case to argue that certain sets are well-linked. We start by defining

$$\ell_{6.20}(k) := \mathsf{s}_{6.16}(k), \quad \text{6.20}$$

$$h_{6.20}(k) := \ell_{6.16}(\ell_{6.20}(k), k) \cdot 2k \binom{\mathsf{s}_{6.16}(k)}{k} k!. \quad \text{6.20}$$

Note that $\ell_{6.20}(k) \in O(k^{11})$ and $h_{6.20}(k) \in 2^{1 \uparrow \uparrow \text{poly}^2(k)}$. Using the pigeon-hole principle, we can combine [Theorem 6.16](#) and [Lemma 6.19](#) in order to obtain the desired statement.

Proposition 6.20. Let k be an integer, let $h \geq h_{6.20}(k)$, let D be a digraph, let \mathcal{L} be a linkage of order $\ell_{6.20}(k)$ in D and let T be the routing temporal digraph of \mathcal{L} through $\mathcal{H} := \{H_1, \dots, H_h\}$, where each H_i is a subgraph of D . If each layer $D_i(T)$ is strongly connected, then there exists some $\mathcal{L}' \subseteq \mathcal{L}$ of order k such that $\text{start}(\mathcal{L}')$ is well-linked to $\text{end}(\mathcal{L}')$ in $D(\mathcal{L} \cup \mathcal{H})$.

Proof. Let $k_1 = 2k \binom{\mathsf{s}_{6.16}(k)}{k} k!$. Define $s_1 = 1$ and for each $1 \leq i \leq k_1$ define $s_i = (i-1) \cdot \ell_{6.16}(\ell_{6.20}(k), k) + 1$. For each $1 \leq i \leq k_1$ let T_i be the temporal subgraph of T from time step s_i to $s_{i+1} - 1$. Note that $\ell(T_i) = s_{i+1} - s_i = \ell_{6.16}(\ell_{6.20}(k), k)$ and that $|\mathcal{L}| = |V(T_i)| = \mathsf{s}_{6.16}(k) = \ell_{6.20}(k)$.

By [Theorem 6.16](#) each T_i contains a \mathbf{C}_k -routing φ_i or a $\tilde{\mathbf{P}}_k$ -routing φ_i over some set $\mathcal{L}_i \subseteq \mathcal{L}$ of size k . As there are $k_1 = 2k \binom{\mathsf{s}_{6.16}(k)}{k} k!$ temporal digraphs T_i , there is some $I \subseteq \{1, \dots, k_1\}$ of size k and some $H \in \{\mathbf{C}_k, \tilde{\mathbf{P}}_k\}$ such that, for every $i, j \in I$, both T_i and T_j have an H -routing $\varphi := \varphi_i = \varphi_j$ over $\mathcal{L}' := \mathcal{L}_i = \mathcal{L}_j$. By [Lemma 6.19](#), $\text{start}(\mathcal{L}')$ is well-linked to $\text{end}(\mathcal{L}')$ in $D(\mathcal{L}') \cup \mathcal{H}$. \square \square

7 Paths of order-linked sets and acyclic grids

In the previous section we already discussed the similarities between \mathbf{P}_k -routings in routing temporal digraphs and acyclic grids. We now develop a more abstract framework in which we can model these intuitive observations. This enables us to lift certain properties of acyclic grids to a more abstract setting. The techniques and results we develop in this section play an important rôle in our proof of the directed grid theorem.

To motivate the following definitions, consider the acyclic grid illustrated in [Figure 7](#). Suppose we want to connect some vertex a_i on the left of the grid to a vertex b_j on the right. As in an acyclic grid we can never route upwards, this is possible if and only if $i \leq j$.

Let A be the ordered set containing the left-most vertices of the grid, i.e. $\{a_1, \dots, a_4\}$ in the example in [Figure 7](#), ordered from top to bottom and let B be the ordered set containing the vertices at the right, i.e. $\{b_1, \dots, b_4\}$ in the example, again ordered from top to bottom.

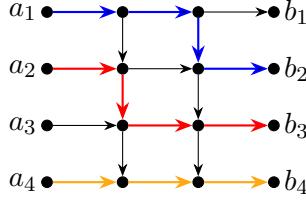


Figure 7: We illustrate r -shifts in acyclic grids. The 2-shift (b_2, b_3, b_4) of (a_1, a_2, a_4) is routable in the grid as illustrated here. However the 3-shift (b_2, b_3, b_4) of (a_1, a_2, a_3) is not routable in the grid.

Now suppose we are given a subset $A' \subseteq A$ and an equal sized subset $B' \subseteq B$. Under what conditions can we connect A' to B' by a linkage \mathcal{L} in the grid? For the same reason as before we can only connect $a_i \in A'$ to $b_j \in B'$ if $i \leq j$. Furthermore, as the grid is planar, if $\{a_{i_1}, \dots, a_{i_l}\}$ are the vertices of A' ordered by their order in A and likewise $\{b_{j_1}, \dots, b_{j_l}\}$ are the ordered vertices of B' , then we have to connect a_{i_s} to b_{j_s} , for all $1 \leq s \leq l$. This implies that $i_s \leq j_s$ for all $1 \leq s \leq l$. But even if this condition is satisfied by A' and B' , it may still not be possible to connect A' to B' . As the example in Figure 7 demonstrates, there simply may not be enough columns in the grid to route all paths downwards that connect pairs a_{i_s}, b_{j_s} with $i_s < j_s$. So we may have to restrict the number of pairs (a_{i_s}, b_{j_s}) for which we allow $i_s < j_s$.

This idea is formalised in the next definition by the concept of *r -shifts*.

Definition 7.1. Let $A = (a_1, \dots, a_n)$ and $B = (b_1, \dots, b_m)$ be ordered sets. Let $r \in \mathbb{N}$, let A' be an ordered subset of A and let B' be an ordered subset of B such that $|A'| = |B'|$. We say that B' is an *r -shift of A'* if there is a bijection $\pi : A' \rightarrow B'$ such that

1. for all $a_i \in A'$ we have that $\pi(a_i) = b_j$ implies $i \leq j$;
2. there are at most r vertices $a_i \in A'$ with $\pi(a_i) \neq b_i$; and
3. π is order preserving, that is, for all $a_i, a_j \in A'$, if $a_i \leq_A a_j$, then $\pi(a_i) \leq_B \pi(a_j)$.

In the example of Figure 7, (b_2, b_3, b_4) is a 2-shift of (a_1, a_2, a_4) but it is a 3-shift of (a_1, a_2, a_3) . We interested in pairs of equal sized ordered sets A and B which allow given a subset $A' \subseteq A$ to route A' to all possible r -shifts $B' \subseteq B$ of A' . This is formalised in the next definition. Recall from Section 3 that we may consider a linkage \mathcal{L} as a function $\mathcal{L} : \text{start}(\mathcal{L}) \rightarrow \text{end}(\mathcal{L})$ where $\mathcal{L}(a)$ is the endpoint of the path in \mathcal{L} starting at a .

Definition 7.2. Let H be a digraph, let $A = (a_1, \dots, a_n), B = (b_1, \dots, b_m) \subseteq V(H)$ be ordered sets, and let $r \in \mathbb{N}$. We say that A is *r -order-linked* to B in H if for every $A' \subseteq A$ and every $B' \subseteq B$ with $|A'| = |B'|$ where B' is an r -shift of A' witnessed by the bijection π there is an $A'-B'$ -linkage \mathcal{L} in H satisfying $\pi(a) = \mathcal{L}(a)$ for all $a \in A'$.

For (unordered) sets $A, B \subseteq V(H)$, we say that A is *r -order-linked* to B in H if there exist orderings A_1 and B_1 of A and B , respectively, such that A_1 is *r -order-linked* to B_1 in H .

To give an example, it is easily seen that in any acyclic (r, r) -grid $(\mathcal{P}, \mathcal{Q})$ the set $\text{start}(\mathcal{Q})$ is *r -order-linked* to $\text{end}(\mathcal{Q})$.

We now define a new type of structure which can be seen as an abstraction of acyclic grids.

Definition 7.3 (path of r -order-linked sets). A *path of r -order-linked sets* of width w and length ℓ is a tuple $(\mathcal{S}, \mathcal{P})$ such that

1. \mathcal{S} is a sequence of $\ell + 1$ pairwise disjoint subgraphs (S_0, \dots, S_ℓ) , which are called *clusters*,
2. for every $0 \leq i \leq \ell$ there are disjoint sets $A(S_i), B(S_i) \subseteq V(S_i)$ of size w such that $A(S_i)$ is r -order-linked to $B(S_i)$ in S_i ,
3. \mathcal{P} is a sequence of ℓ pairwise disjoint linkages $(\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{\ell-1})$ such that, for every $0 \leq i < \ell$, \mathcal{P}_i is a $B(S_i)$ - $A(S_{i+1})$ -linkage of order w which is internally disjoint from S_i and S_{i+1} and disjoint from every $S \in \mathcal{S} \setminus \{S_i, S_{i+1}\}$.

By definition, for each $1 \leq i \leq \ell$ there are orderings $\leq_{A(S_i)}$ of $A(S_i)$ and $\leq_{B(S_i)}$ of $B(S_i)$ witnessing the r -order-linkedness of $A(S_i)$ and $B(S_i)$ in S_i .

We say that $(\mathcal{S}, \mathcal{P})$ is *uniform* if for all $1 \leq i \leq \ell$ we can choose orderings $\leq_{A(S_i)}$ and $\leq_{B(S_i)}$ witnessing that $A(S_i)$ is r -order-linked to $B(S_i)$ so that for all $0 \leq i < \ell$ and all $b_1, b_2 \in B(S_i)$: if $b_1 \leq_{B(S_i)} b_2$, then $\mathcal{P}_i(b_1) \leq_{A(S_{i+1})} \mathcal{P}_i(b_2)$.

The following notation is used frequently below. Given a path of r -order-linked sets $(\mathcal{S} := (S_0, \dots, S_\ell), \mathcal{P} := (\mathcal{P}_0, \dots, \mathcal{P}_{\ell-1}))$ and indices $0 \leq i \leq j \leq \ell$ we define $(\mathcal{S}, \mathcal{P})[i, j]$ as the path of r -order-linked sets from cluster i to cluster j . That is, $(\mathcal{S}, \mathcal{P})[i, j] := ((S_i, \dots, S_j), (\mathcal{P}_i, \dots, \mathcal{P}_{j-1}))$.

To give an example, let $(\mathcal{P}, \mathcal{Q})$ be an acyclic (r^2, r) -grid, where $\mathcal{P} := (P_1^1, \dots, P_1^r, P_2^1, \dots, P_2^r, \dots, P_r^1, \dots, P_r^r)$. Then $(\mathcal{P}, \mathcal{Q})$ contains a path of r -order-linked sets as follows. The cluster S_i is obtained as the union of the columns P_i^j , for $1 \leq j \leq r$, and, for each $Q \in \mathcal{Q}$, the subpath Q^i of Q starting at the first vertex of Q on P_i^1 and ending at the last vertex of Q on P_i^r . We define $A(S_i) := \{\text{start}(Q^i) : Q \in \mathcal{Q}\}$ and $B(S_i) := \{\text{end}(Q^i) : Q \in \mathcal{Q}\}$. Finally, the linkages \mathcal{P}_i , for $0 \leq i < r$, are obtained by taking the subpaths of the rows connecting S_i to S_{i+1} in the obvious way. Then $((S_0, \dots, S_r), (\mathcal{P}_0, \dots, \mathcal{P}_{r-1}))$ is a path of r -order-linked sets.

As the example shows, we can easily obtain a path of r -order linked sets from a sufficiently large acyclic grid. Our next goal is to show that the converse is also true, albeit with bigger bounds.

We first observe the following.

Observation 7.4. Let $D = (\mathcal{S} = (S_0, S_1, \dots, S_\ell), \mathcal{P})$ be a path of 0-order-linked sets of width w . For every $0 \leq i < j \leq \ell$, every $A' \in \{A(S_i), B(S_i)\}$, and every $B' \in \{A(S_j), B(S_j)\}$ there is an A' - B' -linkage \mathcal{L} of order w in D . Furthermore, for all $i < k < j$ every path in \mathcal{L} must intersect $A(S_k)$ and $B(S_k)$.

Proof. We show the case where $A' = A(S_i)$ and $B' = B(S_j)$. The other cases follow analogously.

For each $i \leq t \leq j - 1$ construct sets A_t, B_t and a linkage \mathcal{L}_t as follows. Start by setting $A_{i-1} = A'$ and \mathcal{L}_{i-1} as the linkage containing only the vertices of A' .

On step t , let B_t be a 0-shift of A_{t-1} and let \mathcal{R}_t be an A_{t-1} - B_t -linkage of order w in S_t . Since $A(S_t)$ is 0-order-linked to $B(S_t)$, such a linkage \mathcal{R}_t exists. Let $\mathcal{R}'_t \subseteq \mathcal{P}_t$ be the set of paths with $\text{start}(\mathcal{R}'_t) = \text{end}(\mathcal{R}_t)$. Now set $A_t = \text{end}(\mathcal{R}'_t)$ and set $\mathcal{L}_t = \mathcal{L}_{t-1} \cdot \mathcal{R}_t \cdot \mathcal{R}'_t$.

It is immediate from the construction that \mathcal{L}_j is an A' - B' -linkage of order w , as desired. $\square \square$

We are now ready to show that every path of 1-order-linked sets contains an acyclic grid. Here we make use of our framework of H -routings in temporal digraphs. The idea is to use the \mathbf{P}_l -routings constructed in [Theorem 6.12](#) to obtain the columns of the grid. We start by defining

$$\begin{aligned} \mathbf{w}_{7.5}(k) &:= k^2 - 1, & \mathbf{w}_{7.5} \\ \ell_{7.5}(k) &:= \left((k^2 - k - 1) \cdot \binom{\mathbf{w}_{7.5}(k)}{k} \cdot k! + 1 \right) \cdot \ell_{6.10}(\mathbf{w}_{7.5}(k), \mathbf{w}_{7.5}(k)). & \ell_{7.5} \end{aligned}$$

Observe that $\mathbf{w}_{7.5}(k) \in O(k^2)$ and $\ell_{7.5}(k) \in 2^{1 \uparrow \uparrow \text{poly}^7(k)}$.

Theorem 7.5. Every path of 1-order-linked sets of width at least $w = \text{w}_{7.5}(k)$ and length at least $\ell_{7.5}(k)$ contains an acyclic (k, k) -grid.

Proof. Let $(\mathcal{S} := (S_0, S_1, \dots, S_\ell), \mathcal{P} := (\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{\ell-1}))$ be a path of 1-order-linked sets of width at least $w := \text{w}_{7.5}(k)$ and length $\ell \geq \ell_{7.5}(k)$. Let $D = D((\mathcal{S}, \mathcal{P}))$. By Observation 7.4, there is an $A(S_0)$ - $B(S_\ell)$ -linkage \mathcal{L} of order w in D . Note that every path in \mathcal{L} must intersect every $A(S_i)$ and every $B(S_i)$.

Let T be the routing temporal digraph of \mathcal{L} through \mathcal{S} . Since $A(S_i)$ is 1-order-linked to $B(S_i)$ for every $S_i \in \mathcal{S}$, we have that each layer of T is unilateral.

Let $k_1 = k(k-1)$, let $k_2 = (k_1 - 1) \cdot \binom{w}{k} \cdot k! + 1$. For each $1 \leq i \leq k_2$, let $t_i = (i-1) \cdot \ell_{6.10}(w, w)$ and let T_i be the temporal subgraph of T from time step t_i to $t_{i+1} - 1$. Note that $\ell(T_i) = \ell_{6.10}(w, w)$ and $|V(T_i)| = w$.

By Theorem 6.12, each T_i contains a \mathbf{P}_k -routing φ_i . Since $\ell(T) \geq \ell_{7.5}(k) = k_2 \cdot \ell_{6.10}(w, w)$, there are at least k_2 subgraphs T_i . By the pigeon-hole principle, there is some set \mathcal{T} of size k_1 of temporal subgraphs T_i of T such that $\varphi := \varphi_i = \varphi_j$ for all $T_i, T_j \in \mathcal{T}$.

Let $(T'_1, T'_2, \dots, T'_{k_1}) := \mathcal{T}$ be sorted according to the corresponding time steps, let \mathcal{Q} be the image of φ .

Let u_1, u_2, \dots, u_k be the vertices of the \mathbf{P}_k ordered according to their occurrence on the \mathbf{P}_k . We construct a sequence \mathcal{P} of k paths where, for each $1 \leq i \leq k$, the path P_i is constructed as follows.

For each $1 \leq j < k$, let $t_{i,j} = (i-1) \cdot (k-1) + j$ and let $R_{i,j}$ be a $\varphi(u_j)$ - $\varphi(u_{j+1})$ temporal path in $T'_{t_{i,j}}$ which does not contain any path in $\mathcal{Q} \setminus \{\varphi(u_j), \varphi(u_{j+1})\}$. Note that $t_{i,k-1} = t_{i+1,1} - 1$. Since φ is a \mathbf{P}_k -routing in $T'_{t_{i,j}}$, such a path $R_{i,j}$ exists. Finally, $R_{i,j}$ corresponds to a $V(\varphi(u_j))$ - $V(\varphi(u_{j+1}))$ path $P_{i,j,2}$ in D . Let $P_{i,j,1}$ be the $\text{end}(P_{i,j-1,2})$ - $\text{start}(P_{i,j,2})$ -path in $D(\varphi(u_j))$ (to simplify notation, we choose $\text{end}(P_{i,0,2})$ as $\text{start}(P_{i,1,2})$).

We now set $P_i = P_{i,1,1} \cdot P_{i,1,2} \cdot P_{i,2,1} \cdot P_{i,2,2} \cdot \dots \cdot P_{i,k-1,2}$. After constructing all P_i , set $\mathcal{P} = (P_1, P_2, \dots, P_k)$. Note that the paths in \mathcal{P} are pairwise disjoint. It is now immediate from the construction that $(\mathcal{P}, \mathcal{Q})$ is an acyclic (k, k) -grid. \square \square

The previous results show that we can convert an acyclic grid into a path of r -order linked sets and vice versa. We now turn to the problem of actually constructing a path of r -order-linked sets in a digraph. We show first how to construct a path of 1-order-linked sets from routing temporal digraphs containing \mathbf{P}_k -routings. Similar to a column in an acyclic grid such a \mathbf{P}_k -routing allows us to shift one path to its destination without intersecting the other paths in the linkage we construct.

Lemma 7.6. Let h, k be integers. Let T be the routing temporal digraph of some linkage \mathcal{L} through a sequence (H_1, H_2, \dots, H_h) of disjoint digraphs. Let $\mathcal{L}' \subseteq \mathcal{L}$ be a linkage of order at most k . If T contains a \mathbf{P}_k -routing on the paths $L_1, L_2, \dots, L_k \in \mathcal{L}'$, ordered according to their occurrence on the \mathbf{P}_k -routing, then A is 1-order-linked to B in $D(\mathcal{L} \cup \bigcup_{i=1}^h H_i)$, where $A = \{a_i \mid a_i \text{ is the first vertex of } L_i \text{ on } H_1\}$ and $B = \{b_i \mid b_i \text{ is the last vertex of } L_i \text{ on } H_h\}$.

Proof. Let $A' \subseteq A$ and $B' \subseteq B$ such that B' is a 1-shift of A' . Let $\pi : A' \rightarrow B'$ be the bijection witnessing that B' is a 1-shift of A' . If $\pi(a_x) = b_x$ for all $a_x \in A'$, then \mathcal{L}' contains an A' - B' -linkage \mathcal{R} such that for all $a_x \in A'$ there is an a_x - b_x path in \mathcal{R} , and so we are done.

Otherwise, let $x \in \{1, \dots, k\}$ be such that $a_x \in A'$ is the unique vertex such that $\pi(a_x) \neq b_x$ and let $b_y = \pi(a_x)$. As B' is a 1-shift of A' , we know that $x < y$ and $a_i \notin A'$ for all $i \in \{x+1, \dots, y-1\}$.

We construct an A' - B' -linkage \mathcal{R} satisfying $\pi(a_x) = \mathcal{R}(a_x)$ for all $a_x \in A'$ as follows. For each $a_j \in A'$, let W_j be the temporal walk in the \mathbf{P}_k -routing starting in L_j and ending in L_{j+1} and

let W be the concatenation of $W_x \cdot W_{x+1} \cdot \dots \cdot W_{y-1}$. The temporal walk W connects L_x to L_y in T and we can assume it starts on time step 1 and ends on time step h . If $\pi(a_j) = b_j$, we add the path L_j to \mathcal{R} . Construct L'_x as follows.

Let (v_i, t_i) and (v_j, t_j) be two consecutive elements in the sequence of W . We follow L_i from H_{t_i} to H_{t_j} , then take a path P in H_{t_j} connecting L_i to L_j . By construction of T and because $a_n \notin A'$ for all $n \in \{x+1, \dots, y-1\}$, the path P in H_{t_j} does not intersect any other path of \mathcal{L} . Further, by definition of \mathbf{P}_k -routing, L_i and L_j only intersect A' at a_x or a_y . Hence, L'_x is disjoint from all L_i we chose earlier. Thus, we obtain an A' - B' -linkage \mathcal{R} such that $\mathcal{R}(a_x) = \pi(a_x)$ for all $a_x \in A'$ as desired. \square \square

The previous lemma allows us to construct a path of 1-order-linked sets. We show next that we can increase the order-linkedness of the clusters at the expense of the length of the path of order-linked sets we obtain. The idea is to “merge” a set of consecutive clusters of a path of r -order-linked sets into a single cluster increasing the order-linkedness.

This idea is much easier to implement in uniform paths of r -order-linked sets than in the general case and also yields much better bounds. As the uniform case is sufficient for our application we only consider the uniform case here.

The next lemma essentially explains how to construct in a path of r -order-linked sets a single cluster of higher order-linkedness by merging the existing clusters into one.

Lemma 7.7. Let r, c, w be integers. Let $D = (\mathcal{S} = (S_0, S_1, \dots, S_\ell), \mathcal{P})$ be a uniform path of r -order-linked sets of width w and length at least $c - 1$. Then $A(S_0)$ is cr -order-linked to $B(S_\ell)$ in D .

Proof. Let $(\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{\ell-1}) := \mathcal{P}$. For each $0 \leq i \leq \ell$ let $\varphi_i : A(S_i) \rightarrow B(S_i)$ be the bijection witnessing that $A(S_i)$ is r -order-linked to $B(S_i)$.

We define for each $0 \leq i \leq \ell$ two bijections $\alpha_i : A(S_i) \rightarrow \{1, 2, \dots, w\}$ and $\beta_i : B(S_i) \rightarrow \{1, 2, \dots, w\}$ according to $\leq_{A(S_i)}$ and $\leq_{B(S_i)}$, that is, $\alpha_i(a_j) \leq \alpha_i(a_k)$ holds if and only if $a_j \leq_{A(S_i)} a_k$ holds (and analogously for β_i). In particular, we have $\varphi_i = \beta_i^{-1} \circ \alpha_i$. Since $(\mathcal{S}, \mathcal{P})$ is uniform, we also have that $\beta_i(b) = \alpha_{i+1}(\mathcal{P}_i(b))$ for all $0 \leq i \leq \ell - 1$ and all $b \in B(S_i)$.

Let $A' \subseteq A(S_0)$ and let $B' \subseteq B(S_\ell)$ be sets of size k such that B' is a cr -shift of A' as witnessed by the bijection $\varphi : A' \rightarrow B'$. We also define $\pi := \beta_\ell \circ \varphi \circ \alpha_0^{-1}$.

For each $0 \leq i \leq c - 1$ we construct an A' - $B(S_i)$ -linkage \mathcal{R}_i of order $|A'|$ satisfying the following,

$$(\mathbf{L1}) \quad |\{a \in A' \mid \beta_i(\mathcal{R}_i(a)) = \beta_\ell(\varphi(a))\}| \geq (i + 1)r.$$

To simplify notation we set $\text{end}(\mathcal{R}_{-1})$ as A' .

On step i , let $\mathcal{L}_i^1 \subseteq \mathcal{P}_{i-1}$ be such that $\text{start}(\mathcal{L}_i^1) = \text{end}(\mathcal{R}_{i-1})$ and let $\widehat{\mathcal{R}}_{i-1} = \mathcal{R}_{i-1} \cdot \mathcal{L}_i^1$.

Choose the largest possible $A'' \subseteq A'$ of size at most r by starting at the largest elements of A' with respect to $\leq_{A(S_0)}$ and proceeding in descending order such that $\alpha_i(\widehat{\mathcal{R}}_{i-1}(a)) \neq \pi(a)$ for all $a \in A''$. Let $\widehat{A} = \widehat{\mathcal{R}}_{i-1}(A'')$.

Let $B'' = \beta_i^{-1}(\pi(\alpha_i(\widehat{A})) \cup \beta_i^{-1}(\alpha_i(\text{end}(\widehat{\mathcal{R}}_{i-1}) \setminus \widehat{A}))$. Let $\varphi'_i : \text{end}(\widehat{\mathcal{R}}_{i-1}) \rightarrow B''$ be the bijection defined as follows

$$\varphi'_i(a) := \begin{cases} \beta_i^{-1}(\pi(\alpha_i(a))), & a \in \widehat{A} \\ \beta_i^{-1}(\alpha_i(a)), & a \in \text{end}(\widehat{\mathcal{R}}_{i-1}) \setminus \widehat{A}. \end{cases}$$

Because A'' was constructed by taking the largest elements of A' with respect to $\leq_{A(S_0)}$, we have that $\alpha_i(a) \notin \pi(\alpha_i(\widehat{A}))$ for all $a \in \text{end}(\widehat{\mathcal{R}}_{i-1}) \setminus \widehat{A}$. Hence, the set B'' is an r -shift of $\text{end}(\widehat{\mathcal{R}}_{i-1})$, witnessed by φ'_i .

Since $A(S_i)$ is r -order-linked to $B(S_i)$ in S_i , there is a linkage \mathcal{L}_i^2 in S_i such that $\varphi'_i(\text{end}(\widehat{\mathcal{R}}_{i-1})) = \mathcal{L}_i^2(\text{end}(\widehat{\mathcal{R}}_{i-1}))$. We now set $\mathcal{R}_i = \widehat{\mathcal{R}}_{i-1} \cdot \mathcal{L}_i^2$.

For every $a \in A''$ we now have $\alpha_{i+1}(\mathcal{R}_i(a)) = \varphi(a)$. Since (L1) holds for $i - 1$, we also have that (L1) holds for i .

After iterating c steps, we have that $\mathcal{R}_{c-1}(A') = \varphi(A')$ since (L1) holds for $i = c$ and B' is an cr -shift of A' . Thus, \mathcal{R}_{c-1} is an A' - B' -linkage. Hence, $A(S_0)$ is cr -order-linked to $B(S_\ell)$. $\square \square$

By applying the previous lemma repeatedly we obtain the following theorem.

Theorem 7.8. Every uniform path of r -order-linked sets $D = (\mathcal{S} = (S_0, S_1, \dots, S_\ell), \mathcal{P} = (\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{\ell-1}))$ of length at least $c\ell$ and width w contains a uniform path of cr -order-linked sets $(\mathcal{S}' = (S'_0, S'_1, \dots, S'_\ell), \mathcal{P}' = (\mathcal{P}'_0, \mathcal{P}'_1, \dots, \mathcal{P}'_{\ell-1}))$ of length ℓ and width w . Additionally, for every $0 \leq i \leq \ell$ we have $S'_i \subseteq D[c_i, c(i+1)-1]$, $A(S'_i) \subseteq A(S_{ci})$ and $B(S'_i) \subseteq B(S_{c(i+1)-1})$, and for $0 \leq i < \ell$ we have $\mathcal{P}'_i \subseteq \mathcal{P}_{(c-1)(i+1)}$.

Proof. For each $0 \leq i \leq \ell$ let $S'_i = D[c_i, c(i+1)-1]$ and set $A(S'_i) := A(S_{ci})$ and $B(S'_i) := B(S_{c(i+1)-1})$. Note that each S'_i is a path of r -order-linked sets of width w and length $c - 1$. From Lemma 7.7, we have that $A(S'_i)$ is cr -order-linked to $B(S'_i)$ in S'_i .

Let $\mathcal{P}' := (\mathcal{P}_{c-1}, \mathcal{P}_{2(c-1)}, \dots, \mathcal{P}_{(c-1)\ell})$. It is immediate that $(\mathcal{S}' := (S'_0, S'_1, \dots, S'_\ell), \mathcal{P}')$ is a uniform path of cr -order-linked sets of width w and length ℓ satisfying the requirements in the statement. $\square \square$

8 Paths of well-linked sets and fences

In the previous section we introduced *path of r -order-linked sets* as a suitable abstraction of acyclic grids. We now want to extend this idea to find a similar abstraction of fences as well.

The main difference between a fence and an acyclic grid $(\mathcal{P}, \mathcal{Q})$ is that if we are interested in routing from left to right, that is, from $\text{start}(\mathcal{Q})$ to $\text{end}(\mathcal{Q})$, then if $(\mathcal{P}, \mathcal{Q})$ is an acyclic grid the two sides are only order-linked whereas in a fence they are well-linked. Consequently we relax the requirement of the path of r -order-linked sets to obtain a suitable abstraction of fences.

Definition 8.1 (path of well-linked sets). A *path of well-linked sets* of width w and length ℓ is a tuple $(\mathcal{S}, \mathcal{P})$ such that

1. \mathcal{S} is a sequence of $\ell + 1$ pairwise disjoint subgraphs (S_0, \dots, S_ℓ) , called *clusters*,
2. for every $0 \leq i \leq \ell$ there are disjoint sets $A(S_i), B(S_i) \subseteq V(S_i)$ of size w such that $A(S_i)$ is well-linked to $B(S_i)$ in S_i , and
3. \mathcal{P} is a sequence of ℓ pairwise disjoint linkages $(\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{\ell-1})$ such that, for every $0 \leq i < \ell$, \mathcal{P}_i is a $B(S_i)$ - $A(S_{i+1})$ -linkage of order w which is internally disjoint from S_i and S_{i+1} and is disjoint from every $S \in \mathcal{S} \setminus \{S_i, S_{i+1}\}$.

We call $(\mathcal{S}, \mathcal{P})$ *strict* if within each cluster S_i every vertex $v \in V(S_i)$ occurs on a path from $A(S_i)$ to $B(S_i)$ in S_i .

As before we define $(\mathcal{S}, \mathcal{P})[i, j] := ((S_i, \dots, S_j), (\mathcal{P}_i, \dots, \mathcal{P}_{j-1}))$, where $0 \leq i \leq j \leq \ell$.

While acyclic grids and fences may look quite different, Reed et al. proved in [RRST96] that it is possible to construct a fence from any given acyclic grid.

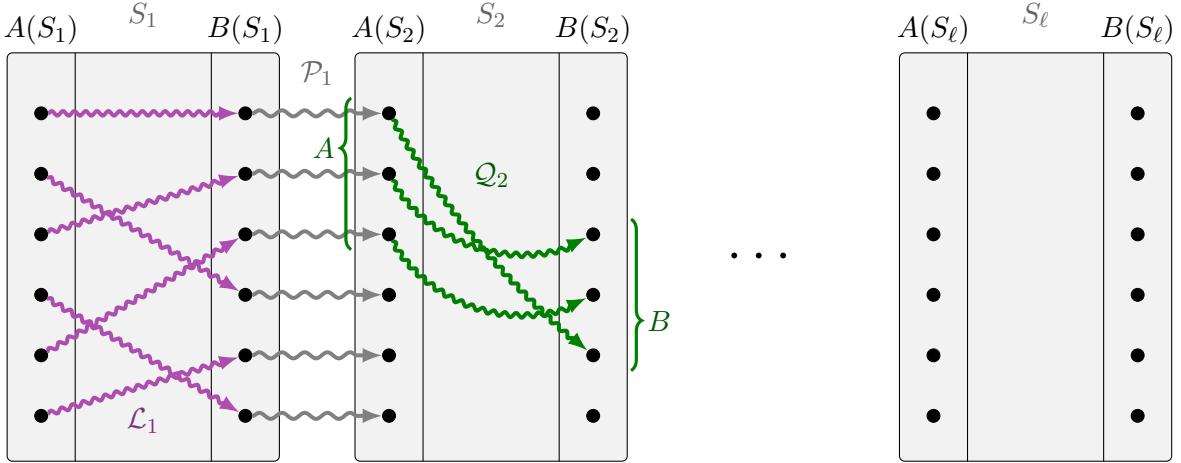


Figure 8: A path of well-linked sets of width w and length ℓ . The linkage \mathcal{L}_1 connects all of $A(S_1)$ to all of $B(S_1)$ while there are also linkages from every subset of $A(S_i)$ to every subset of $B(S_i)$ as \mathcal{Q}_2 in S_2 illustrates for example.

Lemma 8.2 ([RRST96, statement (4.7)]). Every acyclic $(pq+1, pq+1)$ -grid contains a (p, q) -fence.

We show next that a similar relation as proved in the previous lemma is also true for paths of order-linked sets and paths of well-linked sets.

Lemma 8.3. Let $w_{8.3}(w, \ell) := w(\ell+1)$. Every path of w -order-linked sets $(\mathcal{S} = (S_0, S_1, \dots, S_\ell), \mathcal{P} = (\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{\ell-1}))$ of width at least $w_{8.3}(w, \ell)$ and length at least ℓ contains a path of well-linked sets $(\mathcal{S}' = (S'_0, S'_1, \dots, S'_\ell), \mathcal{P}' = (\mathcal{P}'_0, \mathcal{P}'_1, \dots, \mathcal{P}'_{\ell-1}))$ of width w and length ℓ . Further, for every $0 \leq i \leq \ell$ we have $A(S'_i) \subseteq A(S_i)$, $B(S'_i) \subseteq B(S_i)$, $S'_i \subseteq S_i$ and for every $0 \leq i < \ell$ we have $\mathcal{P}'_i \subseteq \mathcal{P}_i$. w_{8.3}

Proof.

Recall that $\leq_{A(S_i)}$ is the order of the vertices of $A(S_i)$ witnessing that $A(S_i)$ is w -order-linked to $B(S_i)$ in S_i . Further, let π_i be the bijection witnessing this property.

In the following we construct the sets A'_i and B'_{i-1} for each $1 \leq i \leq \ell$ such that A'_i is a subset of the smallest $(i+1)w$ elements of $\leq_{A(S_i)}$ and \mathcal{P}_i contains a B'_{i-1} - A'_i -linkage of order $|A'_i|$, see Figure 9 for an illustration.

First let \widehat{A}_0 be the w smallest elements of $\leq_{A(S_0)}$. For $0 < i \leq \ell$, let \widehat{A}_i be the $(i+1)w$ smallest elements of $\leq_{A(S_i)}$. Since $|A(S_i)| \geq w(\ell+1)$ and $i \leq \ell$, such a set exists.

Now, let $\widehat{\mathcal{P}}_{i-1}$ be the paths of \mathcal{P}_{i-1} such that $\text{end}(\widehat{\mathcal{P}}_{i-1}) = \widehat{A}_i$. Since \widehat{A}_{i-1} contains the smallest iw elements of $\leq_{A(S_{i-1})}$, there is some $B'_{i-1} \subseteq \text{start}(\widehat{\mathcal{P}}_{i-1})$ of size w such that $\pi_{i-1}(a) \leq_{B(S_{i-1})} b$ for all $a \in \widehat{A}_{i-1}$ and all $b \in B'_{i-1}$.

Finally, choose $A'_i := \text{end}(\widehat{\mathcal{P}}_{i-1})$ for all $0 < i \leq \ell$ and $A'_0 := \widehat{A}_0$.

As $A'_i \subseteq \widehat{A}_i$ for all $0 \leq i \leq \ell$, we have $\pi_{i-1}(a) \leq_{B(S_{i-1})} b$ for all $a \in A'_{i-1}$ and all $b \in B'_{i-1}$. Hence, for every $A' \subseteq A'_{i-1}$ and every $B' \subseteq B'_{i-1}$ with $|A'| = |B'|$ we have that B' is an r -shift of A' . Thus, A'_{i-1} is well-linked to B'_{i-1} in S_{i-1} . Let S'_{i-1} be a minimal subgraph of S_{i-1} in which A'_{i-1} is well-linked to B'_{i-1} . We set $A(S'_{i-1}) := A'_{i-1}$ and $B(S'_{i-1}) := B'_{i-1}$. Choose $\mathcal{P}'_{i-1} \subseteq \mathcal{P}_{i-1}$ such that $\text{start}(\mathcal{P}'_{i-1}) = B'_{i-1}$ and set $A'_i := \text{end}(\mathcal{P}'_{i-1})$.

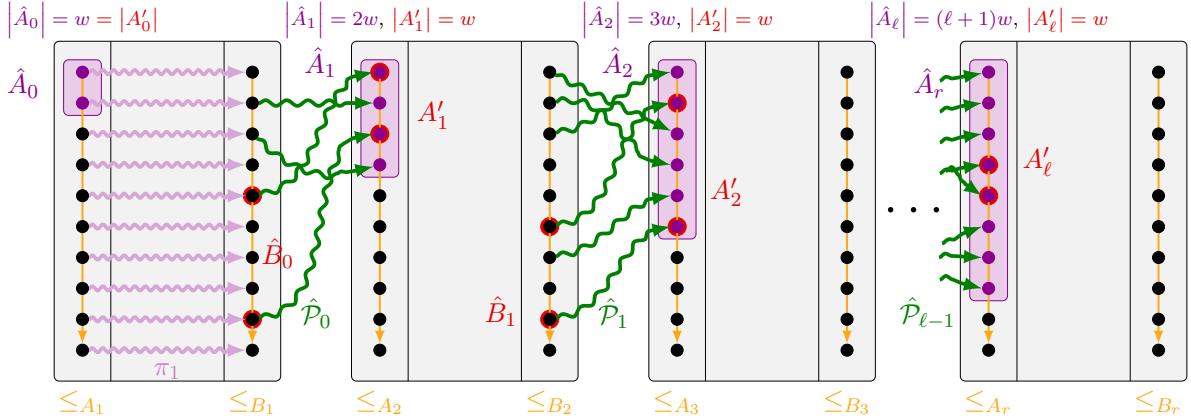


Figure 9: A path of well-linked sets of width w and length ℓ . The linkage \mathcal{L}_1 connects all of $A(S_1)$ to all of $B(S_1)$ while there are also linkages from every subset of $A(S_i)$ to every subset of $B(S_i)$ as \mathcal{Q}_2 in S_2 illustrates for example.

After constructing all sets above, we choose B'_ℓ as the w largest elements of $\leq_{B(S_\ell)}$. As argued above, the set A'_ℓ is well-linked to B'_ℓ in some $S'_i \subseteq S_i$, where we choose S'_i minimal.

We set $\mathcal{S}' = (S'_0, S'_1, \dots, S'_\ell)$ and $\mathcal{P}' = (\mathcal{P}'_0, \mathcal{P}'_1, \dots, \mathcal{P}'_{\ell-1})$. It is immediate from the construction above that $(\mathcal{S}', \mathcal{P}')$ is a path of well-linked sets of width w and length ℓ satisfying the conditions in the statement. \square \square

We show next that every path of well-linked sets contains a fence. Towards this aim we first show that the well-linkedness of $A(S_i)$ to $B(S_i)$ within an individual cluster S_i can be preserved when going from one cluster to the next, i.e. the set $A(S_i)$ is also well-linked to every $A(S_j)$ and $B(S_j)$ for clusters S_j with $j > i$ appearing later on the path of well-linked sets.

Lemma 8.4. Let $(\mathcal{S} := (S_0, \dots, S_\ell), \mathcal{P} := (\mathcal{P}_0, \dots, \mathcal{P}_{\ell-1}))$ be a path of well-linked sets of width w and length ℓ and let $0 \leq i < j \leq \ell$. Then for every $0 \leq i < j \leq \ell$, for each $A' \in \{A(S_i), B(S_i)\}$ and for each $B' \in \{B(S_j), A(S_j)\}$ we have that A' is well-linked to B' in $(\mathcal{S}, \mathcal{P})[i, j]$.

Proof. We show the case where $A' = A(S_i)$ and $B' = B(S_j)$. The other cases follow analogously.

Let $X \subseteq A'$ and $Y \subseteq B'$ be sets size k . We prove by induction on $j - i$ that there is an X - Y -linkage of order k in $(\mathcal{S}, \mathcal{P})$.

If $j - i = 1$, then let $B_i \subseteq B(S_i)$ be a set of size k and let $A_j \subseteq A(S_j)$ the set of size k such that $\mathcal{P}_i(B_i) = A_j$. Since $A(S_i)$ is well-linked to $B(S_i)$ in S_i , there is an A' - B_1 -linkage \mathcal{R}_i of order k in S_i . Similarly, there is an A_j - B' -linkage \mathcal{R}_j in S_j . Let $\mathcal{R}'_i \subseteq \mathcal{P}_i$ be the paths of \mathcal{P}_i such that $\text{start}(\mathcal{R}'_i) = \text{end}(\mathcal{R}_i)$. Clearly, $\mathcal{R}_i \cdot \mathcal{R}'_i \cdot \mathcal{R}_j$ is an A' - B' -linkage of order k .

Now consider the case where $j - i > 1$. Choose any subset $B_i \subseteq B(S_i)$ of order $|A'| = k$. As before there is an A' - B_i -linkage \mathcal{R}_1 of order k in S_i . Let $\mathcal{R}_2 \subseteq \mathcal{P}_i$ be the linkage with $\text{start}(\mathcal{R}_2) = \text{end}(\mathcal{R}_1)$. Note that $\text{end}(\mathcal{R}_2) \subseteq A(S_{i+1})$. By induction, there is an $\text{end}(\mathcal{R}_2)$ - B' -linkage \mathcal{R}_3 of order k , and so $\mathcal{R}_1 \cdot \mathcal{R}_2 \cdot \mathcal{R}_3$ is an A' - B' -linkage of order k , as desired. \square \square

We now apply our framework of \mathbf{P}_k -routings in temporal digraphs to construct a fence from a path of well-linked sets. The idea is to first construct an acyclic grid using \mathbf{P}_k -routings and then apply Lemma 8.3 to obtain a fence. Observe that $\mathbf{w}_{8.5}(p, q) \in O(p^5 q^5)$ and $\mathbf{\ell}_{8.5}(p, q) \in 2^{1 \uparrow \uparrow \text{poly}^5(p, q)}$.

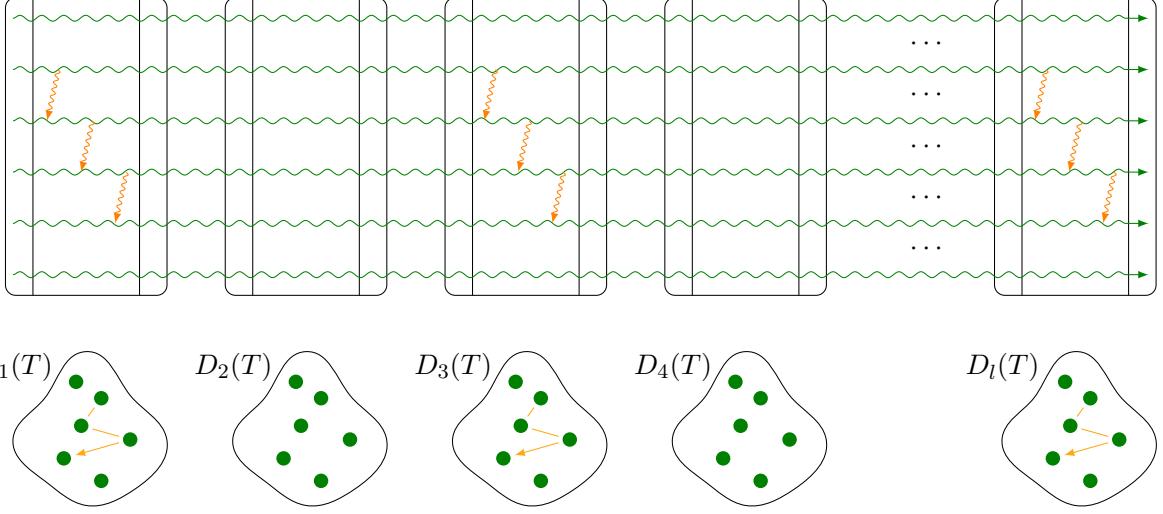


Figure 10: For each cluster S_i we obtain a digraph $D_i(T)$ from the temporal digraph T . Every D_i contains a path of length k_1 or a K_{k_1} -routing, which means it contains a P_{k_1} -routing in any case. As there are enough clusters we can find $k_4 - 1$ agreeing on the vertices and their order, shown in orange.

Theorem 8.5. Every path of a well-linked set $(\mathcal{S}, \mathcal{P})$ of width at least $\text{w}_{8.5}(p, q) := 2(pq + 1)^5$ and length $\ell \geq \ell_{8.5}(p, q) := ((pq + 1)(pq) - 1)\binom{2(pq+1)^5}{pq+1}(pq + 1)! + 1$ contains a (p, q) -fence. w_{8.5}
ℓ_{8.5}

Proof. Let $k_1 = pq + 1$ and $k_2 = 2(k_1)^5$. Let $D = D((\mathcal{S}, \mathcal{P}))$. Let $(S_0, S_1, \dots, S_\ell) = \mathcal{S}$ and let \mathcal{L} be an $A(S_0)$ - $B(S_\ell)$ -linkage of order k_2 in $(\mathcal{S}, \mathcal{P})$. By Lemma 8.4, such a linkage exists.

Let T be the routing temporal digraph of \mathcal{L} through \mathcal{S} . Note that $\ell(T) = \ell + 1$, see Figure 10 for an illustration. Since $A(S_i)$ is well-linked to $B(S_i)$ and every path in \mathcal{L} must intersect both $A(S_i)$ and $B(S_i)$ for every $S_i \in \mathcal{S}$, we have that every $D_i(T)$ is strongly connected.

By Theorem 6.8, every $D_i(T)$ contains a path of length k_1 or a \tilde{K}_{k_1} -routing. In both cases, $D_i(T)$ contains a \mathbf{P}_{k_1} -routing φ_i . Note that there are at most $k_3 := \binom{k_2}{k_1} \cdot (k_1)!$ distinct φ_i .

Let $k_4 = k_1(k_1 - 1)$. Because $\ell(T) \geq (k_4 - 1)k_3 + 1$, there is a subsequence \mathcal{S}' of \mathcal{S} of length k_4 such that $\varphi := \varphi_i = \varphi_j$ for every $S_i, S_j \in \mathcal{S}'$. Let $(S'_0, S'_1, \dots, S'_{k_4-1}) := \mathcal{S}'$, let \mathcal{Q} be the image of φ and let T' be the routing temporal digraph of \mathcal{Q} through \mathcal{S}' . Note that T' is a temporal subgraph of T and that φ is a P_{k_1} -routing in every $D_i(T')$. Further, $\ell(T') = k_4$ and $|\mathcal{Q}| = k_1$.

Let u_1, u_2, \dots, u_{k_1} be the vertices of the \mathbf{P}_{k_1} ordered according to their occurrence on the P_{k_1} . We construct a sequence \mathcal{P} of k_1 paths where, for each $1 \leq i \leq k_1$, the path P_i is constructed as follows.

For each $1 \leq j < k_1$, let $t_{i,j} = (i-1) \cdot (k_1-1) + j$ and let $R_{i,j}$ be a $\varphi(u_j)$ - $\varphi(u_{j+1})$ path in $D_{t_{i,j}}(T')$ which is disjoint from every path in $\mathcal{Q} \setminus \{\varphi(u_j), \varphi(u_{j+1})\}$. Note that $t_{i,k_1-1} = t_{i+1,1} - 1$. Since φ is a \mathbf{P}_{k_1} -routing in $D_{t_{i,j}}(T)$, such a path $R_{i,j}$ exists. Finally, $R_{i,j}$ corresponds to a $V(\varphi(u_j))$ - $V(\varphi(u_{j+1}))$ path $P_{i,j,2}$ in D . Let $P_{i,j,1}$ be the $\text{end}(P_{i,j-1,2})$ - $\text{start}(P_{i,j,2})$ -path in $D(\varphi(u_j))$ (to simplify notation, we choose $\text{end}(P_{i,0,2})$ as $\text{start}(P_{i,1,2})$).

We now set $P_i = P_{i,1,1} \cdot P_{i,1,2} \cdot P_{i,2,1} \cdot P_{i,2,2} \cdot \dots \cdot P_{i,k_1-1,2}$. After constructing all P_i , set $\mathcal{P} = (P_1, P_2, \dots, P_{k_1})$. Note that the paths in \mathcal{P} are pairwise disjoint.

It is immediate from the construction that $(\mathcal{P}, \mathcal{Q})$ is an acyclic (k_1, k_1) -grid. By Lemma 8.2, $D(\mathcal{P} \cup \mathcal{Q})$ contains a (p, q) -fence, as desired. \square

We close the section by exhibiting various routing properties of paths of well-linked sets that

are useful below.

We first observe the following simple property.

Observation 8.6. Let D be a digraph and let $A, B \subseteq V(D)$ be sets in D such that A is well-linked to B . Let $v \in V(D)$ be a vertex contained in some A - B path. Then there is an $(A \cup \{v\})$ - B -linkage \mathcal{L} of order $|A|$ such that $v \in \text{start}(\mathcal{L})$.

Proof. Let \mathcal{R} be some A - B -linkage of order $|A|$. Let P be some A - B path containing v and let P' be the v - B subpath of P . Let P'' be the largest subpath of P' with $\text{start}(P'') = \text{start}(P')$ which is internally disjoint from \mathcal{R} and let $R \in \mathcal{R}$ be the path of \mathcal{R} intersecting P'' . Finally, let R' be the $\text{end}(P'') - \text{end}(R)$ subpath of R . It is now immediate that $\mathcal{R}' := (\mathcal{R} \setminus \{R\}) \cup \{P'' \cdot R'\}$ is a linkage of order $|R|$ with $v \in \text{start}(\mathcal{R}')$. \square \square

When working with paths of well-linked sets below we are often in a situation where we are given two equal-sized sets X and Y of vertices in a path of well-linked sets $(\mathcal{S}, \mathcal{P})$ and we want to find a linkage connecting X to Y in $(\mathcal{S}, \mathcal{P})$. In the next lemma we identify several cases in which these linkages are guaranteed to exist. This lemma is frequently applied in the next steps of the proof.

Lemma 8.7. Let $(\mathcal{S} = (S_0, S_1, \dots, S_\ell), \mathcal{P} = (\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{\ell-1}))$ be a path of well-linked sets of width w and length ℓ . Let $X, Y \subseteq V((\mathcal{S}, \mathcal{P}))$ such that $|X| = |Y| = k$. Let $f : X \cup Y \rightarrow \mathbb{N}$ be a function such that $v \in S_{f(v)} \cup \mathcal{P}_{f(v)}$ for all $v \in X \cup Y$. There is an X - Y -linkage \mathcal{L} in $(\mathcal{S}, \mathcal{P})$ if $f(x) \leq f(y) - 2$ for all $x \in X$ and all $y \in Y$ and at least one of the following is true:

- (L1) there are $0 \leq i < j \leq \ell$ such that $X \subseteq B(S_i)$ and $Y \subseteq A(S_j)$,
- (L2) $|f(x_1) - f(x_2)| \geq 2$ for all $x_1, x_2 \in X$ with $x_1 \neq x_2$ and there is some $0 \leq i \leq \ell$ such that $Y \subseteq A(S_i)$,
- (L3) $|f(y_1) - f(y_2)| \geq 2$ for all $y_1, y_2 \in Y$ with $y_1 \neq y_2$ and there is some $0 \leq i \leq \ell$ such that $X \subseteq B(S_i)$, or
- (L4) $|f(x_1) - f(x_2)| \geq 2$ for all $x_1, x_2 \in X$ with $x_1 \neq x_2$ and $|f(y_1) - f(y_2)| \geq 2$ for all $y_1, y_2 \in Y$ with $y_1 \neq y_2$.

Furthermore, choose i minimal with S_i containing a vertex from X and j maximal with S_j containing a vertex from Y . Then, \mathcal{L} is contained inside $(\mathcal{S}, \mathcal{P})[i, j]$.

Proof. The case where (L1) holds follows directly from Lemma 8.4.

If (L2) holds, we construct an X - $A(S_{i-1})$ -linkage of order k as follows. We first rename the vertices of X such that $x_r \in V(S_r)$ for all $x_r \in X$. For each $x_r \in X$, let k_r be the number of vertices in X which appear before x_r along $(\mathcal{S}, \mathcal{P})$.

By Observation 8.6, there is an $(A(S_r) \cup \{x_r\})$ - $B(S_r)$ -linkage \mathcal{R}_r of order $k_r + 1$ in S_r such that $x_r \in \text{start}(\mathcal{R}_r)$. If $k_r > 0$, then by Lemma 8.4 there is an $\text{end}(\mathcal{R}_{r-1})$ - $\text{start}(\mathcal{R}_r)$ -linkage \mathcal{L}_{r-1} of order $k_r = |\mathcal{R}_r| - 1$.

Clearly, the concatenation of all \mathcal{R}_r and all \mathcal{L}_r above (in the only order possible) yields an X - $B(S_{r'})$ -linkage of order $|X|$, where r' is the smallest index such that all vertices of X appear before $S_{r'}$ along $(\mathcal{S}, \mathcal{P})$. Now by Lemma 8.4 we have an $\text{end}(\mathcal{R}_{r'-1})$ - Y -linkage of order k , as desired. The proof for the case where (L3) holds is analogous to the one of where (L2) holds and so we omit it.

If (L4) holds, let r_y be the smallest index such that S_{r_y} contains a vertex of Y and let r_x be the largest index such that S_{r_x} contains a vertex of X .

Construct linkages \mathcal{R}_r and \mathcal{L}_r as in the proof of the case when (L2) holds. Let \mathcal{X} be the linkage obtained by concatenating all \mathcal{R}_r and all \mathcal{L}_r (in the only possible order) belonging to vertices of X . Similarly, let \mathcal{Y} be the linkage obtained by concatenating all \mathcal{R}_r and all \mathcal{L}_r (in the only possible order) belonging to vertices of Y .

Note that $\text{end}(\mathcal{X}) \subseteq B(S_{r_x})$ and that $\text{start}(\mathcal{Y}) \subseteq A(S_{r_y})$. Hence, by Lemma 8.4 there is an $\text{end}(\mathcal{X})\text{-}\text{start}(\mathcal{Y})$ -linkage of order k , as desired. \square \square

The last statement we prove in this section helps us to deal with a situation where already have a path of well-linked sets $(\mathcal{S} := (S_0, \dots, S_\ell), \mathcal{P})$ but we would like to restrict the system so that it “starts” at a specific set $A \subseteq A(S_0)$ and ends at some fixed set $B \subseteq B(S_\ell)$.

Observation 8.8. Let $(\mathcal{S} = (S_0, S_1, \dots, S_\ell), \mathcal{P} = (\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{\ell-1}))$ be a path of well-linked sets of width at least w and length ℓ . Let $A_0 \subseteq A(S_0)$ and $B_\ell \subseteq B(S_\ell)$ with $|A(S_0)| = |B_\ell| = w$. Then, $(\mathcal{S}, \mathcal{P})$ contains a path of well-linked sets $(\mathcal{S}' = (S'_0, S'_1, \dots, S'_\ell), \mathcal{P}' = (\mathcal{P}'_0, \mathcal{P}'_1, \dots, \mathcal{P}'_{\ell-1}))$ of width w and length ℓ such that $B(S'_\ell) = B_\ell$, $A(S'_0) = A_0$, $S'_i \subseteq S_i$ for all $0 \leq i \leq \ell$ and $\mathcal{P}'_i \subseteq \mathcal{P}_i$ for all $0 \leq i < \ell$.

Proof. For each $0 \leq i < \ell$ choose some $B_i \subseteq B(S_i)$ of size w and let $\mathcal{P}'_i \subseteq \mathcal{P}_i$ be such that $\text{start}(\mathcal{P}'_i) = B_i$. For each $1 \leq i \leq \ell$ let $A_i = \mathcal{P}'_{i-1}(B_{i-1})$.

For each $0 \leq i \leq \ell$ let $S'_i \subseteq S_i$ be a maximal subgraph of S_i such that A_i is well-linked to B_i in S'_i and for each $v \in V(S'_i)$ there is some A_i - B_i path P in S'_i containing v . Clearly, if no such path P exists for some vertex v , then we can remove v from S'_i while preserving the property that A_i is well-linked to B_i . Hence, such a subgraph S'_i exists. We then set $A(S'_i) := A_i$ and $B(S'_i) := B_i$.

By construction, $((S'_0, \dots, S'_\ell), (\mathcal{P}'_0, \dots, \mathcal{P}'_{\ell-1}))$ is a path of well-linked sets of width w and length ℓ , as desired. \square \square

9 Cycles of well-linked sets and cylindrical grids

We have already seen how paths of r -order-linked sets can be seen as an abstraction of acyclic grids and paths of well-linked sets are an abstraction of fences. In this section we introduce the analogous abstraction of cylindrical grids. It is easily seen that a cylindrical grid is essentially the same as a fence together with a linkage which contains for each horizontal path Q_i of the fence a path connecting $\text{end}(Q_i)$ to $\text{start}(Q_i)$ but is otherwise disjoint from the fence.

Unsurprisingly, therefore, our abstractions of cylindrical grids, called cycle of well-linked sets, arise from a path of well-linked sets by adding a linkage from the last to the first cluster.

Definition 9.1 (cycle of well-linked sets). A *cycle of well-linked sets* of width w and length ℓ is a tuple $(\mathcal{S}, \mathcal{P})$ such that

1. \mathcal{S} is a sequence of ℓ pairwise disjoint subgraphs $(S_0, \dots, S_{\ell-1})$, which are called *clusters*,
2. for every $0 \leq i < \ell$ there are disjoint sets $A(S_i), B(S_i) \subseteq V(S_i)$ of size w such that $A(S_i)$ is well-linked to $B(S_i)$ in S_i ,
3. \mathcal{P} is a sequence of ℓ pairwise disjoint linkages $(\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{\ell-1})$ such that, for every $0 \leq i < \ell$, \mathcal{P}_i is a $B(S_i)$ - $A(S_{(i+1 \bmod \ell)})$ -linkage of order w which is internally disjoint from S_i and S_{i+1} and is disjoint from every $S \in \mathcal{S} \setminus \{S_i, S_{i+1}\}$.

As before we call $(\mathcal{S}, \mathcal{P})$ *strict* if in every cluster S_i every vertex $v \in V(S_i)$ is contained in an $A(S_i) - B(S_i)$ -path.

In the same way as a path of well-linked sets can be constructed from a fence, a cycle of well-linked sets can be constructed from a cylindrical grid. We now turn to the converse operation, i.e. how one can construct a cylindrical grid from a cycle of well-linked sets.

We first need the following lemma from [KK15].

Lemma 9.2 ([KK15, Lemma 6.3]). Let t be an integer, let $(\mathcal{P}, \mathcal{Q})$ be a (q, q) -fence where $q \geq \mathbf{q}_{9.2}(t) := (t-1)(2t-1) + 1$ and let \mathcal{R} be an $\text{end}(\mathcal{Q})$ - $\text{start}(\mathcal{Q})$ -linkage of order q which is internally disjoint from $(\mathcal{P}, \mathcal{Q})$. Then $(\mathcal{P}, \mathcal{Q})$ contains a cylindrical grid of order t as a butterfly minor.

We are now ready to show how a cylindrical grid can be obtained from a cycle of well-linked sets. We first define

$$\begin{aligned}\mathbf{w}_{9.3}(k) &:= \mathbf{w}_{8.5}(\mathbf{q}_{9.2}(k), \mathbf{q}_{9.2}(k)) && \text{W9.3} \\ \ell_{9.3}(k) &:= \ell_{8.5}(\mathbf{q}_{9.2}(k), \mathbf{q}_{9.2}(k)). && \text{L9.3}\end{aligned}$$

We note that $\mathbf{w}_{9.3}(k) \in \text{poly}^{20}(k)$ and $\ell_{9.3}(k) \in 2^{1 \uparrow \uparrow \text{poly}^9(k)}$.

Theorem 9.3. Every cycle of well-linked sets of width $w \geq \mathbf{w}_{9.3}(k)$ and length $\ell \geq \ell_{9.3}(k)$ contains a cylindrical grid of order k .

Proof. Let $k_1 = \mathbf{q}_{9.2}(k)$. Let $\ell_1 = \text{len}_{8.5}(k_1, k_1)$. Note that $w \geq \mathbf{w}_{8.5}(k_1, k_1)$ and $\ell \geq \ell_1 + 1$.

Let $(\mathcal{S} = (S_0, S_1, \dots, S_\ell), \mathcal{P} = (\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_\ell))$ be a cycle of well-linked sets of width w and length ℓ . Note that $D_1 := (\mathcal{S}' := (S_0, S_1, \dots, S_{\ell-1}), \mathcal{P}' := (\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{\ell-2}))$ is a path of well-linked sets of width w and length at least ℓ_1 .

By Theorem 8.5, D_1 contains a (k_1, k_1) -fence $(\mathcal{P}^1, \mathcal{Q}^1)$ such that $\text{start}(\mathcal{Q}^1) \subseteq A(S_0)$ and $\text{end}(\mathcal{Q}^1) \subseteq B(S_{\ell-1})$. Let $\mathcal{R}^1 \subseteq \mathcal{P}_\ell$ be the set of paths satisfying $\text{end}(\mathcal{R}^1) = \text{start}(\mathcal{Q}^1)$. Let \mathcal{R}^2 be an $\text{end}(\mathcal{Q}^1)$ - $\text{start}(\mathcal{R}^1)$ -linkage of order k_1 in $(\mathcal{S}, \mathcal{P})[\ell_1, \ell]$. By Lemma 8.7(L1), such a linkage \mathcal{R}^2 exists. Further, \mathcal{R}^2 is internally disjoint from $(\mathcal{P}^1, \mathcal{Q}^1)$. By Lemma 9.2, $(\mathcal{P}^1, \mathcal{Q}^1)$ and \mathcal{R}^2 together contain a cylindrical grid of order k as a butterfly minor. \square \square

With the results of this section we have now found suitable abstractions of acyclic grids, fences, and cylindrical grids. We have also seen how to obtain, e.g. a cylindrical grid from a cycle of well-linked sets. What remains to show is how one can find a cycle of well-linked sets in a given digraph. We address this problem in the remainder of the paper.

10 Constructing a path of well-linked sets

We show how to obtain a path of well-linked sets from splits and segmentations by using the results from Section 6, where we defined the *routing temporal digraph* of a linkage \mathcal{L} through a sequence of disjoint digraphs H_1, H_2, \dots, H_t .

In order to construct the routing temporal digraph, the linkage \mathcal{L} must intersect all H_i in an *ordered* fashion. This means that, if one of the linkages in a web $(\mathcal{H}, \mathcal{V})$ is *ordered* with respect to the other, then we can construct such a routing temporal digraph. This leads us to the following definition of *ordered web* (see Figure 11 for an example of an ordered web).

Definition 10.1. Let $(\mathcal{H}, \mathcal{V})$ be an (h, v) -web. We say that $(\mathcal{H}, \mathcal{V})$ is an *ordered web* if there is an ordering of $\mathcal{V} = (V_1, V_2, \dots, V_v)$ for which each path $H \in \mathcal{H}$ can be decomposed into $H = H_1 \cdot H_2 \cdots H_v$ such that H_i intersects V_j if and only if $i = j$.

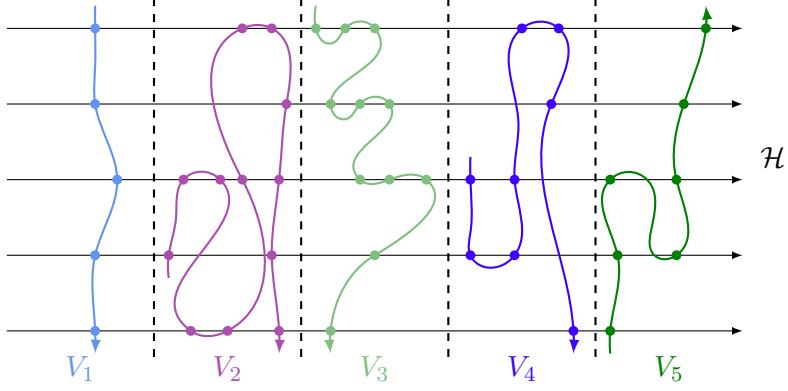


Figure 11: A $(5, 3)$ -ordered web $(\mathcal{H}, \{V_1, V_2, V_3, V_4, V_5\})$.

We show next how to construct a path of 1-order-linked sets from an ordered web using our framework of H -routings developed in [Section 6](#). We start by defining

$$\textcolor{blue}{h}_{10.2}(w) := w^2 - 1, \quad \textcolor{blue}{h}_{10.2}$$

$$\textcolor{blue}{v}_{10.2}(w, \ell) := (w\ell \cdot \binom{\textcolor{blue}{h}_{10.2}(w)}{w} \cdot w! + 1) \cdot \ell_{6.12}(w, \textcolor{blue}{h}_{10.2}(w)) - 1. \quad \textcolor{blue}{v}_{10.2}$$

Observe that $\textcolor{blue}{v}_{10.2}(w, \ell) \in 2^{1 \uparrow \uparrow \text{poly}^{13}(\ell, w)}$.

Lemma 10.2. Let $(\mathcal{H}, \mathcal{V})$ be an ordered (h, v) -web where $h = \textcolor{blue}{h}_{10.2}(w)$ and $v \geq \textcolor{blue}{v}_{10.2}(w, \ell)$. Then $(\mathcal{H}, \mathcal{V})$ contains a path of w -order-linked sets $(\mathcal{S} = (S_0, S_1, \dots, S_\ell), \mathcal{P})$ of width w and length ℓ with the following additional properties.

- There is a $\text{start}(\mathcal{H})$ - $\text{end}(\mathcal{H})$ -linkage $\mathcal{L} = \mathcal{L}_1 \cdot \mathcal{L}_2 \cdot \mathcal{L}_3$ of order w contained in \mathcal{H} such that \mathcal{L}_2 is an $A(S_0)$ - $B(S_\ell)$ -linkage and both \mathcal{L}_1 and \mathcal{L}_3 are internally disjoint from $(\mathcal{S}, \mathcal{P})$.
- There is a linkage $\mathcal{X} \subseteq \mathcal{V}$ of order $\ell + 1$ and a bijection $\pi : \mathcal{S} \rightarrow \mathcal{X}$ such that $A(S_i) \subseteq V(\pi(S_i))$ and $V(\pi(S_i)) \cap V((\mathcal{S}, \mathcal{P})) \subseteq V(S_i)$ for each $0 \leq i \leq \ell$.

Proof. Let $\ell_1 := w\ell + 1$ and let $\ell_2 := (\ell_1 - 1) \cdot \binom{h}{w} \cdot w! + 1$.

We define $f(i) := (i - 1) \cdot \ell_{6.12}(w, h) + 1$ and observe that $f(i) - f(i - 1) = \ell_{6.12}(w, h)$. Let $(V_1 \cdot V_2 \cdot \dots \cdot V_v) := \mathcal{V}$ be an ordering of \mathcal{V} witnessing that $(\mathcal{H}, \mathcal{V})$ is an ordered web. Observe that $f(\ell_2 + 1) - 1 = \textcolor{blue}{v}_{10.2}(w, \ell) \leq v$.

Decompose \mathcal{H} into $\mathcal{H} = \mathcal{H}^0 \cdot \mathcal{H}^1 \cdot \dots \cdot \mathcal{H}^{\ell_2}$, where $\text{start}(\mathcal{H}_0) = \text{start}(\mathcal{H})$, $\text{end}(\mathcal{H}_{\ell_2}) = \text{end}(\mathcal{H})$ and for each $1 \leq i \leq \ell_2 - 1$, the sublinkage \mathcal{H}_i starts at the first intersections of \mathcal{H} with $V_{f(i)}$ and ends at the first intersections of \mathcal{H} with $V_{f(i+1)}$. For each $1 \leq t \leq \ell_2$, let $\mathcal{V}^t := (V_{f(t)}, V_{f(t)+1}, \dots, V_{f(t)+\ell_{6.12}(w, h)-1})$ and let T_t be the routing temporal digraph of \mathcal{H} through \mathcal{V}^t .

Each layer of each T_i is unilateral since every path in \mathcal{H} intersects every path in \mathcal{V} . As $\ell(T_i) = \ell_{6.12}(h, w)$, by [Theorem 6.12](#) each T_i contains some \mathbf{P}_w -routing φ_i over some paths of \mathcal{H} .

There are at most $\binom{h}{w} \cdot w!$ distinct \mathbf{P}_w -routings φ_i . Hence, by the pigeon-hole principle, there is a subset $\mathcal{T} = \{T_{t_1}, T_{t_2}, \dots, T_{t_{\ell_1}}\}$ of the temporal digraphs above of size ℓ_1 such that $\varphi := \varphi_i = \varphi_j$ for all $T_i, T_j \in \mathcal{T}$.

Let (u_1, u_2, \dots, u_w) be the vertices of \mathbf{P}_w sorted according to their order along \mathbf{P}_w . For each

$i \in \{1, \dots, \ell_1\}$ let

$$S'_i = D(\mathcal{H}^{t_i} \cup \mathcal{V}^{t_i}),$$

$A(S'_i) = \{a_{i,j} \mid 1 \leq j \leq w \text{ and } a_{i,j} \text{ is the first vertex of } \varphi(u_j) \text{ on } V_{f'(t_i)}\}$ and

$B(S'_i) = \{b_{i,j} \mid 1 \leq j \leq w \text{ and } b_{i,j} \text{ is the last vertex of } \varphi(u_j) \text{ on } V_{f'(t_i+1)-1}\}.$

Let T'_i be the routing temporal digraph of \mathcal{H}^i through \mathcal{V}^i . Since T'_i is isomorphic to T_i , the bijection φ induces a \mathbf{P}_w -routing on T'_i as well. By Lemma 7.6, each $A(S'_i)$ is 1-order-linked to $B(S'_i)$ in S'_i . By choice of $b_{i,j}$ and $a_{i+1,j}$, the path $\varphi(u_j)$ contains a $b_{i,j}-a_{i+1,j}$ path. Hence, for each $1 \leq i \leq \ell_1$ there is a $B(S'_i)$ - $A(S'_{i+1})$ -linkage \mathcal{P}'_i such that $(\mathcal{S}' := (S'_1, S'_2, \dots, S'_{\ell_1}), \mathcal{P}' := (\mathcal{P}'_1, \mathcal{P}'_2, \dots, \mathcal{P}'_{\ell_1-1}))$ is a uniform path of 1-order-linked sets of width w and length $\ell_1 - 1 = \ell_w$.

By Theorem 7.8, $(\mathcal{S}', \mathcal{P}')$ contains as a subgraph a uniform path of w -order-linked sets $(\mathcal{S} = (S_0, S_1, \dots, S_\ell), \mathcal{P} = (\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{\ell-1}))$ of length ℓ and width w . Additionally, for every $0 \leq i \leq \ell$ we have $S_i \subseteq (\mathcal{S}', \mathcal{P}')[w_i + 1, w(i+1)]$, $A(S_i) \subseteq A(S'_{w_i+1})$ and $B(S_i) \subseteq B(S'_{w(i+1)})$, and for $0 \leq i < \ell$ we have $\mathcal{P}_i \subseteq \mathcal{P}'_{(w-1)(i+1)+1}$.

By construction of each S'_i , we have that $A(S_i) \subseteq V(V_{f(t_{w_i+1})})$. Let $\mathcal{X} = \{V_{f(t_{w_i+1})} \mid 0 \leq i \leq \ell\}$. Define the bijection $\pi : \mathcal{S} \rightarrow \mathcal{X}$ as $\pi(S_i) = V_{f(t_{w_i+1})}$. Hence, \mathcal{X} is a linkage of order $\ell + 1$ inside \mathcal{V} such that $A(S_i) \subseteq V(\pi(S_i))$ for all $0 \leq i \leq \ell$. Furthermore, by construction of each S_i it is immediate that $V(\pi(S_i)) \cap V((\mathcal{S}, \mathcal{P})) \subseteq V(S_i)$ for each $0 \leq i \leq \ell$.

We construct the linkage \mathcal{L} as follows. Let \mathcal{Q} be the image of φ and, for each $0 \leq i \leq \ell_2$, let $\mathcal{Q}^i \subseteq \mathcal{H}^i$ be the paths of \mathcal{H}^i which are subpaths of \mathcal{Q} .

Let $\mathcal{L}_1 := \mathcal{Q}^0$, $\mathcal{L}_2 := \mathcal{Q}^1 \cdot \mathcal{Q}^2 \cdot \dots \cdot \mathcal{Q}^{\ell_2}$ and let \mathcal{L}_3 be the $B(S_\ell)$ -end(\mathcal{Q})-linkage inside \mathcal{Q} . By construction, $\mathcal{L}_1 \cdot \mathcal{L}_2 \cdot \mathcal{L}_3$ is a start(\mathcal{H})-end(\mathcal{H})-linkage of order w , \mathcal{L}_2 is an $A(S_0)$ - $B(S_\ell)$ -linkage and both \mathcal{L}_1 and \mathcal{L}_3 are internally disjoint from $(\mathcal{S}, \mathcal{P})$, as desired. \square \square

Combining the previous lemma and Lemma 8.3 allows us to construct a path of well-linked sets from an ordered web. At a later part of our proof we need some additional information about how the linkage \mathcal{V} intersects the individual clusters of the path of well-linked sets. This is captured by the bijection π in the statement of the next result.

We define

$$\mathbf{h}_{10.3}(w, \ell) := \mathbf{h}_{10.2}(w(\ell + 1)),$$

$$\mathbf{v}_{10.3}(w, \ell) := \mathbf{v}_{10.2}(w(\ell + 1), \ell).$$

$\mathbf{h}_{10.3}$

$\mathbf{v}_{10.3}$

Note that $\mathbf{h}_{10.3}(w, \ell) \in O(w^2 \ell^2)$ and $\mathbf{v}_{10.3}(w, \ell) \in 2^{1 \uparrow \uparrow \text{poly}^{25}(w, \ell)}$.

Corollary 10.3. Let $(\mathcal{H}, \mathcal{V})$ be an ordered (h, v) -web such that $h \geq \mathbf{h}_{10.3}(w, \ell)$ and $v \geq \mathbf{v}_{10.3}(w, \ell)$. Then, there is a path of well-linked sets $(\mathcal{S} = (S_0, S_1, \dots, S_\ell), \mathcal{P})$ of width w and length ℓ in $D(\mathcal{H} \cup \mathcal{V})$ such that $B(S_\ell) \subseteq \text{end}(\mathcal{H})$. Finally, there is a linkage $\mathcal{X} \subseteq \mathcal{V}$ of order $\ell + 1$ and a bijection $\pi : \mathcal{S} \rightarrow \mathcal{X}$ such that $A(S_i) \subseteq V(\pi(S_i))$ and $V(\pi(S_i)) \cap V((\mathcal{S}, \mathcal{P})) \subseteq V(S_i)$ for each $0 \leq i \leq \ell$.

Proof. By Lemma 10.2, $(\mathcal{H}, \mathcal{V})$ contains a path of w -order-linked sets $(\mathcal{S}' = (S'_0, S'_1, \dots, S'_\ell), \mathcal{P}' = (\mathcal{P}'_0, \mathcal{P}'_1, \dots, \mathcal{P}'_{\ell-1}))$ of width $\ell(w+1)$ and length ℓ . Further, there is a linkage $\mathcal{X}' \subseteq \mathcal{V}$ of order $\ell + 1$ and a bijection $\pi' : \mathcal{S}' \rightarrow \mathcal{X}'$ such that $A(S'_i) \subseteq V(\pi'(S'_i))$ and $V(\pi'(S'_i)) \cap V((\mathcal{S}', \mathcal{P}')) \subseteq V(S'_i)$ for each $0 \leq i \leq \ell$.

By Lemma 8.3, $(\mathcal{S}', \mathcal{P}')$ contains a path of well-linked sets $(\mathcal{S} = (S_0, S_1, \dots, S_\ell), \mathcal{P} = (\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{\ell-1}))$ of length ℓ and width w . Additionally, for each $0 \leq i \leq \ell$ we have that $S_i \subseteq S'_i$ and that $A(S_i) \subseteq A(S'_i)$.

Finally, let $\mathcal{X} = \{\pi'(S'_i) \mid 0 \leq i \leq \ell\}$ and let $\pi : \mathcal{S} \rightarrow \mathcal{X}$ be the bijection given by $\pi(S_i) = \pi'(S'_i)$. It is immediate that \mathcal{X} and π satisfy the desired conditions in the statement. \square \square

We can manipulate the path of well-linked sets given by [Corollary 10.3](#) above in order to ensure that the extremities of the path of well-linked sets are contained in the extremities of \mathcal{H} . This will be useful later, when we need the end of the path of well-linked sets to be well-linked to its beginning.

We define

$$\mathbf{h}_{10.4}(w, \ell) = \mathbf{h}_{10.3}(w, \ell), \quad \mathbf{h}_{10.4}$$

$$\mathbf{v}_{10.4}(w, \ell) = \mathbf{v}_{10.3}(w, \ell + 4w). \quad \mathbf{v}_{10.4}$$

Note that $\mathbf{h}_{10.4}(w, \ell) \in O(w^2\ell^2)$ and $\mathbf{v}_{10.4}(w, \ell) \in 2^{1 \uparrow \text{poly}^{25}(w, \ell)}$.

Lemma 10.4. Let $(\mathcal{H}, \mathcal{V})$ be an ordered (h, v) -web. If $h \geq \mathbf{h}_{10.4}(w, \ell)$ and $v \geq \mathbf{v}_{10.4}(w, \ell)$, then $(\mathcal{H}, \mathcal{V})$ contains a path of well-linked sets $(\mathcal{S} = (S_0, S_1, \dots, S_\ell), \mathcal{P})$ of width w and length ℓ . Additionally, $A(S_0) \subseteq \text{start}(\mathcal{H})$ and $B(S_\ell) \subseteq \text{end}(\mathcal{H})$.

Proof. Let $\ell_1 = \ell + 4w$. By [Corollary 10.3](#), $D((\mathcal{H}, \mathcal{V}))$ contains a path of well-linked sets $(\mathcal{S}' = (S'_0, S'_1, \dots, S'_{\ell_1}), \mathcal{P}' = (\mathcal{P}'_0, \mathcal{P}'_1, \dots, \mathcal{P}'_{\ell_1-1}))$ of width w and length ℓ_1 . Additionally, there is a linkage $\mathcal{X} \subseteq \mathcal{V}$ of order $\ell_1 + 1$ and a bijection $\pi : \mathcal{S}' \rightarrow \mathcal{X}$ such that $A(S'_i) \subseteq V(\pi(S'_i))$ and $V(\pi(S'_i)) \cap V((\mathcal{S}', \mathcal{P}')) \subseteq V(S'_i)$ for each $0 \leq i \leq \ell_1$.

We construct a path of well-linked sets $(\mathcal{S}, \mathcal{P})$ of width w and length ℓ as follows. For each $0 \leq i \leq \ell_1$, let $X_i = \pi(S'_i)$, let a_i be the first intersection of X_i with $V(S'_i)$ and let b_i be the last intersection of X_i with $V(S'_i)$. Let $A' = \{a_0, a_2, a_{2(w-1)}\}$ and $B' = \{b_{\ell_1}, b_{\ell_1-2}, \dots, b_{\ell_1-2(w-1)}\}$, let \mathcal{X}_A be the $\text{start}(\mathcal{X})$ - A' linkage of order w inside \mathcal{X} and let \mathcal{X}_B be the B' - $\text{end}(\mathcal{X})$ linkage of order w inside \mathcal{X} .

By [Lemma 8.7\(L2\)](#), there is an A' - $A(S'_{2w})$ linkage \mathcal{L}_A of order w in $(\mathcal{S}', \mathcal{P}')[0, 2w]$. Analogously, by [Lemma 8.7\(L3\)](#) there is a $B(S'_{\ell_1-2w})$ - B' linkage \mathcal{L}_B of order w inside $(\mathcal{S}', \mathcal{P}')[\ell_1-2w, \ell_1]$.

Since $V(X_i) \cap V((\mathcal{S}', \mathcal{P}')) \subseteq V(S'_i)$ for all $0 \leq i \leq \ell_1$, we have that $\mathcal{Y}_A := \mathcal{X}_A \cdot \mathcal{L}_A$ is a $\text{start}(\mathcal{V})$ - $A(S'_{2w})$ linkage of order w and $\mathcal{Y}_B := \mathcal{L}_B \cdot \mathcal{X}_B$ is a $B(S'_{\ell_1-2w})$ - $\text{end}(\mathcal{V})$ linkage of order w .

Let $S_0 = D(S'_{2w} \cup \mathcal{Y}_A)$, $A(S_0) = \text{start}(\mathcal{Y}_A)$, $B(S_0) = B(S'_{2w})$, $S_\ell = D(S'_{2w+\ell} \cup \mathcal{Y}_B)$, $A(S_\ell) = A(S'_{2w+\ell})$ and $B(S_\ell) = \text{end}(\mathcal{Y}_B)$. Let $\mathcal{S} = (S_0, S'_{2w+1}, S'_{2w+2}, \dots, S'_{2w+\ell-1}, S_\ell)$ and $\mathcal{P} = (\mathcal{P}_{2w}, \mathcal{P}_{2w+1}, \dots, \mathcal{P}_{2w+\ell-1})$. Clearly, $(\mathcal{S}, \mathcal{P})$ is a path of well-linked sets of width w and length ℓ . Finally, we have $A(S_0) \subseteq \text{start}(\mathcal{L}_1) \subseteq \text{start}(\mathcal{V})$ and $B(S_\ell) \subseteq \text{end}(\mathcal{L}_3) \subseteq \text{end}(\mathcal{V})$. \square \square

Unfortunately, the construction from [Corollary 10.3](#) above does not guarantee that the paths in \mathcal{H} intersect many clusters of the resulting path of well-linked sets. The reason is that the layers of the routing temporal digraph constructed from an ordered web are only unilateral and not strongly connected. Therefore there is no guarantee that $\text{start}(\mathcal{H})$ and $\text{end}(\mathcal{H})$ are well-linked. In the next definition we exhibit a property of webs that allows us to overcome this problem (see [Figure 12](#) for an illustration of folded webs).

Definition 10.5. An (h, v) -web $(\mathcal{H}, \mathcal{V})$ is a *folded web* if every $V_i \in \mathcal{V}$ can be split as $V_i^a \cdot V_i^b := V_i$ such that both V_i^a and V_i^b intersect all paths of \mathcal{H} .

Folded ordered webs, correspond to splits from [Definition 5.1\(S1\)](#). The example shown in [Figure 13](#) illustrates the connection between splits and folded ordered webs, which we make precise in the following observation.

Observation 10.6. Let $(\mathcal{P}', \mathcal{Q}')$ be a $(2p, q)$ -split of $(\mathcal{P}, \mathcal{Q})$. Then there is some \mathcal{P}'' containing only subpaths of \mathcal{P} such that $(\mathcal{Q}', \mathcal{P}'')$ is a folded ordered (q, p) -web.

Proof. Let $(P_1, P_2, \dots, P_{2p}) := \mathcal{P}$ be an ordering of \mathcal{P}' witnessing that $(\mathcal{P}', \mathcal{Q}')$ is a $(2p, q)$ -split. For each $1 \leq i \leq p$ let $P'_i = P_{2i-1} \cdot e_{2i-1} \cdot P_{2i}$, where e_{2i-1} is the edge inside \mathcal{P}' such that

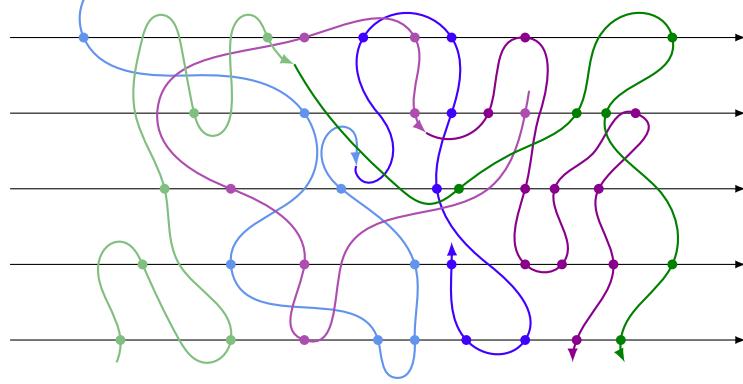


Figure 12: A folded $(5, 3)$ -web.

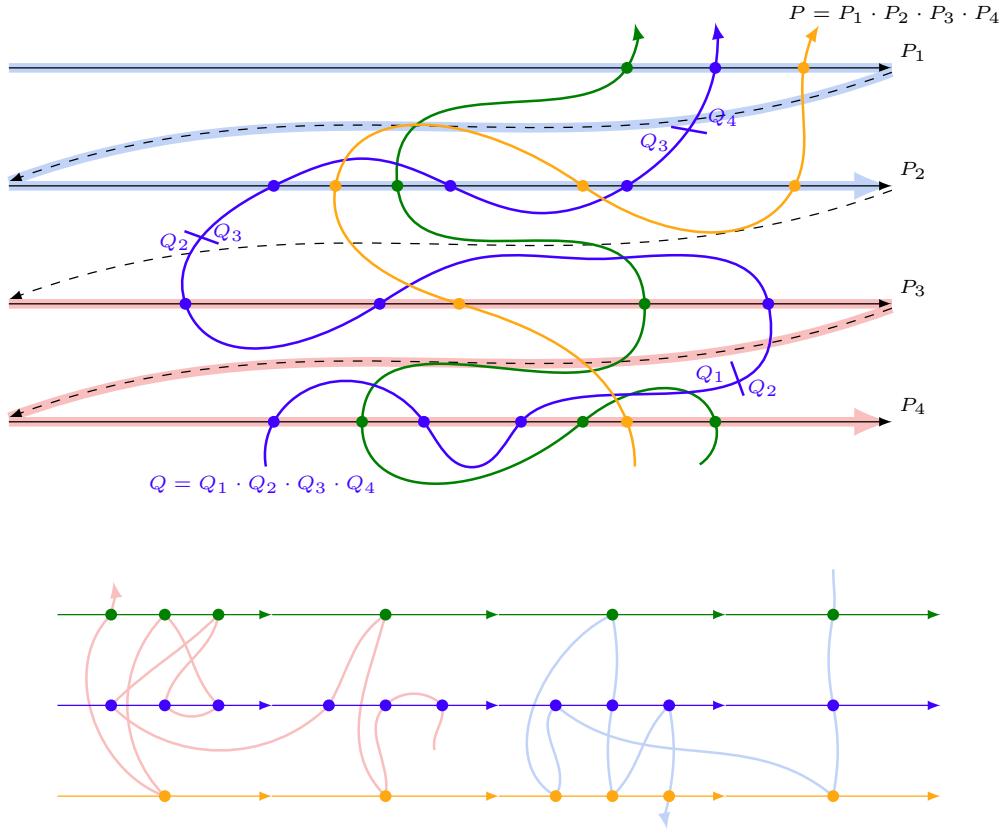


Figure 13: An example of how a $(3, 2)$ folded ordered web is obtained from a $(4, 3)$ -split.

$P_{2i-1} \cdot e_{2i-1} \cdot P_{2i}$ is a subpath of \mathcal{P}' . Let $\mathcal{P}'' = (P'_1, P'_2, \dots, P'_p)$. Now $(\mathcal{Q}', \mathcal{P}'')$ is a folded ordered (q, p) -web, which can be seen by partitioning P'_i into P_{2i-1} and P_{2i} . \square \square

We show next that if we construct a path of well-linked set starting from an ordered web that is also folded, then we can construct the path of well-linked sets in a way that the paths in \mathcal{H} a guaranteed to intersect the individual clusters of the resulting path of well-linked sets. The idea of the construction is similar to the proof of Lemma 10.2 but now we can use Theorem 6.16 in

the construction which yields the extra properties we need.

We define

$$\begin{aligned} \mathbf{h}_{10.7}(w) &:= \ell_{6.20}(w), & \mathbf{h}_{10.7} \\ \mathbf{v}_{10.7}(w, \ell) &:= \mathbf{h}_{6.20}(w) \left(\ell \binom{\mathbf{h}_{10.7}(w)}{w} + 1 \right). & \mathbf{v}_{10.7} \end{aligned}$$

We observe that $\mathbf{h}_{10.7}(w) \in O(w^{11})$ and $\mathbf{v}_{10.7}(w, \ell) \in 2^{1 \uparrow \text{poly}^2(w, \ell)}$.

Lemma 10.7. Let $(\mathcal{H}, \mathcal{V})$ be a folded ordered (h, v) -web. If $h \geq \mathbf{h}_{10.7}(w)$ and $v \geq \mathbf{v}_{10.7}(w, \ell)$, then $(\mathcal{H}, \mathcal{V})$ contains a path of well-linked sets $(\mathcal{S} = (S_0, S_1, \dots, S_\ell), \mathcal{P})$ of width w and length ℓ . Additionally, there is a $\text{start}(\mathcal{H})$ - $\text{end}(\mathcal{H})$ -linkage $\mathcal{L} = \mathcal{L}_1 \cdot \mathcal{L}_2 \cdot \mathcal{L}_3$ using only arcs of \mathcal{H} such that \mathcal{L}_2 is an $A(S_0)$ - $B(S_\ell)$ -linkage of order w inside $(\mathcal{S}, \mathcal{P})$ and \mathcal{L}_1 and \mathcal{L}_3 are internally disjoint from $(\mathcal{S}, \mathcal{P})$.

Proof. Assume, without loss of generality, that $h = \mathbf{h}_{10.7}(w)$, as any $\mathcal{H}' \subseteq \mathcal{H}$ of size $\mathbf{h}_{10.7}(w)$ also satisfies the assumptions of the statement above.

Let $\ell_1 = \mathbf{h}_{6.20}(w)$, and $\ell_2 = \ell \binom{h}{w} + 1$. Let $(V_0, V_1, \dots, V_{v-1})$ be an ordering of \mathcal{V} witnessing that $(\mathcal{H}, \mathcal{V})$ is a folded ordered web.

For each $1 \leq i \leq \ell_2$ let \mathcal{H}^i be the maximal linkage inside \mathcal{H} such that $\text{start}(\mathcal{H}^i) \subseteq V(V_{(i-1)\ell_1})$ and $\text{end}(\mathcal{H}^i) \subseteq V(V_{i\ell_1-1})$. Additionally, let T_i be the routing temporal digraph of \mathcal{H}^i through $\mathcal{V}^i := (V_{(i-1)\ell_1}, \dots, V_{i\ell_1-1})$. Because $(\mathcal{H}, \mathcal{V})$ is a folded web, for every $0 \leq j \leq \ell_1 - 1$ and every pair of paths $H_a^i, H_b^i \in \mathcal{H}^i$ there is a subpath of $V_{(i-1)\ell_1+j}$ from $V(H_a^i)$ to $V(H_b^i)$. Hence, each layer of T_i is strongly connected.

By construction, $\ell(T_i) = \ell_1$ and by assumption $|V(T_i)| = |\mathcal{H}| = \mathbf{h}_{10.7}(w)$. By [Proposition 6.20](#), for every $1 \leq i \leq \ell_2$ there is some $\mathcal{L}_i \subseteq \mathcal{H}^i$ of order w such that $\text{start}(\mathcal{L}_i)$ is well-linked to $\text{end}(\mathcal{L}_i)$ inside $\mathcal{D}(\mathcal{H}^i \cup \mathcal{V}^i)$.

By the pigeon-hole principle, there is some $\mathcal{H}' \subseteq \mathcal{H}$ of order w and some $\mathcal{I} \subseteq \{1, \dots, \ell_2\}$ of order $\ell + 1$ such that \mathcal{L}_i is a sublinkage of \mathcal{H}' of order w for all $i \in \mathcal{I}$. Let $(t_0, t_1, \dots, t_\ell) := \mathcal{I}$ be the ascending order of the elements of \mathcal{I} .

For each $0 \leq i \leq \ell$ let $S_i := \mathcal{D}(\mathcal{L}_{t_i} \cup \mathcal{V}_{t_i})$ and set $A(S_i) = \text{start}(\mathcal{L}_{t_i})$ and $B(S_i) = \text{end}(\mathcal{L}_{t_i})$. For each $1 \leq i \leq \ell - 1$ let \mathcal{P}_i be the $\text{end}(\mathcal{L}_{t_i})$ - $\text{start}(\mathcal{L}_{t_{i+1}})$ -linkage inside \mathcal{H} .

By construction, $A(S_i)$ is well-linked to $B(S_i)$ inside S_i for all i . This implies that $(\mathcal{S} = (S_0, S_1, \dots, S_\ell), \mathcal{P} = (\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{\ell-1}))$ is a path of well-linked sets of width w and length ℓ . Further, \mathcal{L} is an $A(S_0)$ - $B(S_\ell)$ -linkage of order w inside $(\mathcal{S}, \mathcal{P})$ using only arcs of \mathcal{H} . \square \square

In a way similar to [Lemma 10.4](#) above, we can manipulate the path of well-linked sets obtained from [Lemma 10.7](#) in order to ensure that the extremities of the path of well-linked sets are subsets of the extremities of \mathcal{H} .

Corollary 10.8. Let $(\mathcal{H}, \mathcal{V})$ be a folded ordered (h, v) -web. If $h \geq \mathbf{h}_{10.7}(w)$ and $v \geq \mathbf{v}_{10.7}(w, \ell)$, then $(\mathcal{H}, \mathcal{V})$ contains a path of well-linked sets $(\mathcal{S} = (S_0, S_1, \dots, S_\ell), \mathcal{P})$ of width w and length ℓ . Additionally, $A(S_0) \subseteq \text{start}(\mathcal{H})$ and $B(S_\ell) \subseteq \text{end}(\mathcal{H})$.

Proof. By [Lemma 10.7](#), $(\mathcal{V}', \mathcal{H}'')$ contains a path of well-linked sets $(\mathcal{S} = (S_0, S_1, \dots, S_\ell), \mathcal{P})$ of width w and length ℓ . Additionally, there is a $\text{start}(\mathcal{V}')$ - $\text{end}(\mathcal{V}')$ linkage $\mathcal{L} = \mathcal{L}_1 \cdot \mathcal{L}_2 \cdot \mathcal{L}_3$ of order w such that \mathcal{L}_2 is an $A(S_0)$ - $B(S_\ell)$ linkage of order w inside $(\mathcal{S}, \mathcal{P})$ and \mathcal{L}_1 and \mathcal{L}_3 are internally disjoint from $(\mathcal{S}, \mathcal{P})$.

Set $S'_0 = \mathcal{D}(S_0 \cup \mathcal{L}_1)$, $A(S'_0) = \text{start}(\mathcal{L}_1)$, $B(S'_0) = B(S_0)$, $S'_\ell = \mathcal{D}(S_\ell \cup \mathcal{L}_3)$, $A(S'_\ell) = A(S_\ell)$ and $B(S'_\ell) = \text{end}(\mathcal{L}_3)$. Because $\text{end}(\mathcal{L}_1) = A(S_0)$ and $\text{start}(\mathcal{L}_3) = B(S_\ell)$, we have that $(\mathcal{S}' := (S'_0, S_1, S_2, \dots, S'_\ell), \mathcal{P})$ is a path of well-linked sets of width w and length ℓ such that $A(S'_0) \subseteq \text{start}(\mathcal{V})$ and $B(S'_\ell) \subseteq \text{end}(\mathcal{V})$. \square \square

We conclude this section by showing that digraphs of high treewidth contain a large path of well-linked sets where the last cluster is well-linked to the first. As we will see later in [Section 11](#), we will use this well-linkedness property to construct the linkage required to *close* the cycle of well-linked sets.

Define

$$v'(w, \ell) = \mathbf{h}_{10.7}(w) + \mathbf{v}_{10.4}(w, \ell), \\ \mathbf{t}_{10.9}(w, \ell) = \mathbf{t}_{5.15}(2\mathbf{v}_{10.7}(w, \ell), \mathbf{p}_{5.3}(\mathbf{h}_{10.4}(w, \ell), v'(w, \ell)), v'(w, \ell), (\ell + 1)w). \quad \mathbf{t}_{10.9}$$

Note that $\mathbf{t}_{10.9}(w, \ell) \in 2^{7 \uparrow \text{poly}^{25}(w, \ell)}$.

Theorem 10.9. Every digraph D with $\text{dtw}(D) \geq \mathbf{t}_{10.9}(w, \ell)$ contains a path of well-linked sets $(\mathcal{S} = (S_0, S_1, \dots, S_\ell), \mathcal{P})$ of width w and length ℓ such that $B(S_\ell)$ is well-linked to $A(S_0)$ in D .

Proof. We define $\ell_1 = \ell + 1$, $h_3 = \mathbf{v}_{10.7}(w, \ell)$, $h_2 = 2h_3$, $v_2 = \mathbf{h}_{10.7}(w) + \mathbf{v}_{10.4}(w, \ell)$, $h_5 = \mathbf{h}_{10.4}(w, \ell)$. $h_4 = \mathbf{p}_{5.3}(h_5, v_2)$, Observe that $\mathbf{t}_{10.9}(w, \ell) = \mathbf{t}_{5.15}(h_2, h_4, v_2, w\ell_1)$.

By [Theorem 5.15](#), we obtain three cases.

If [Theorem 5.15\(D1\)](#) holds, then D contains a cylindrical grid of order $w\ell_1$, which in turn contains a cycle of well-linked sets $(\mathcal{S}^1 = (S_0^1, S_1^1, \dots, S_{\ell_1}^1), \mathcal{P}^1 = (\mathcal{P}_0^1, \mathcal{P}_1^1, \dots, \mathcal{P}_{\ell_1}^1))$ of width w and length ℓ_1 .

Let $S_0 = D(\mathcal{P}_{\ell_1}^1 \cup S_0^1)$ and $S_\ell = D(S_\ell^1 \cup \mathcal{P}_\ell^1)$. Set $A(S_0) = \text{start}(\mathcal{P}_{\ell_1}^1)$, $B(S_0) = B(S_0^1)$, $A(S_\ell) = A(S_\ell^1)$ and $B(S_\ell) = \text{end}(\mathcal{P}_\ell^1)$. For each $1 \leq i \leq \ell - 1$, set $A(S_i) = A(S_i^1)$ and $B(S_i) = B(S_i^1)$. It is immediate that $(\mathcal{S} := (S_0, S_1^1, \dots, S_{\ell-1}^1, S_\ell), \mathcal{P} := (\mathcal{P}_0^1, \mathcal{P}_1^1, \dots, \mathcal{P}_\ell^1))$ is a path of well-linked sets of width w and length ℓ . Further, as $B(S_\ell) \subseteq A(S_{\ell_1}^1)$ and $A(S_0) \subseteq B(S_{\ell_1}^1)$, we have that $B(S_\ell)$ is well-linked to $A(S_0)$, as desired.

If [Theorem 5.15\(D2\)](#) holds, then D contains a (h_2, v_2) -split $(\mathcal{H}_2, \mathcal{V}_2)$ where $\text{end}(\mathcal{V}_2)$ is well-linked to $\text{start}(\mathcal{V}_2)$. By [Observation 10.6](#), there is some $\mathcal{H}_3 \subseteq \mathcal{H}_2$ of order h_3 such that $(\mathcal{V}_2, \mathcal{H}_3)$ is a folded ordered (v_2, h_3) -web. Applying [Corollary 10.8](#) to $(\mathcal{V}_2, \mathcal{H}_3)$ yields a path of well-linked sets $(\mathcal{S} = (S_0, S_1, \dots, S_\ell), \mathcal{P})$ of width w and length ℓ such that $A(S_0) \subseteq \text{start}(\mathcal{V}_2)$ and $B(S_\ell) \subseteq \text{end}(\mathcal{V}_2)$. As $\text{end}(\mathcal{V}_2)$ is well-linked to $\text{start}(\mathcal{V}_2)$, we have that $A(S_0)$ is well-linked to $B(S_\ell)$, as desired. \square \square

Finally, if [Theorem 5.15\(D3\)](#) holds, then D contains an (h_4, v_2) -segmentation $(\mathcal{H}_4, \mathcal{V}_4)$ where $\text{end}(\mathcal{H}_4)$ is well-linked to $\text{start}(\mathcal{H}_4)$. By [Observation 5.3](#), there is some $\mathcal{H}_5 \subseteq \mathcal{H}_4$ of order h_5 such that $(\mathcal{H}_5, \mathcal{V}_4)$ is an ordered segmentation. By definition, $(\mathcal{H}_5, \mathcal{V}_4)$ is an ordered web. By [Lemma 10.4](#), $D((\mathcal{H}_5, \mathcal{V}_4))$ contains a path of well-linked sets $(\mathcal{S} = (S_0, S_1, \dots, S_\ell), \mathcal{P})$ of width w and length ℓ such that $A(S_0) \subseteq \text{start}(\mathcal{H}_5)$ and $B(S_\ell) \subseteq \text{end}(\mathcal{H}_5)$. As $\text{end}(\mathcal{H}_5)$ is well-linked to $\text{start}(\mathcal{H}_5)$, we have that $A(S_0)$ is well-linked to $B(S_\ell)$, as desired. \square \square

11 Constructing a cycle of well-linked sets

In this section we complete the proof of [Theorem 1.1](#). The results of the previous section allow us to construct in any given digraph of large enough directed treewidth a path of well-linked sets $(\mathcal{S}, \mathcal{P})$ where the last cluster S_ℓ is well-linked to the first cluster S_0 . Let w be the width of $(\mathcal{S}, \mathcal{P})$. The well-linkedness implies that there is a large $B(S_\ell)-A(S_0)$ -linkage \mathcal{R} . We refer to a $B(S_\ell)-A(S_0)$ -linkage \mathcal{R} as a *partial back-linkage*. \mathcal{R} is called a *(total) back-linkage* if it has order w .

We analyse how this back-linkage intersects the path of well-linked sets and identify different types of intersections that are possible. In each of these cases we are able to construct a cycle of well-linked sets but by different techniques in each case depending on the type of intersection.

11.1 Back-linkage intersecting cluster by cluster

The first case we consider is the case where the back-linkage is disjoint from a large part of the path of well-linked sets. But first we need the following simple observation which shows that if a path of well-linked sets has a back-linkage \mathcal{R} then we can use \mathcal{R} to construct a back-linkage for every subpath of well-linked sets.

Lemma 11.1. Let $w_{11.1}(w) := 2w$. Let $(\mathcal{S} = (S_0, S_1, \dots, S_\ell), \mathcal{P})$ be a path of well-linked sets of width at least $w_{11.1}(w)$ and length ℓ in a digraph D . Let \mathcal{R} be a $B(S_\ell)$ - $A(S_0)$ -linkage of order $w_{11.1}(w)$. Let $0 \leq i \leq j \leq \ell$. Then there is a $B(S_j)$ - $A(S_i)$ -linkage \mathcal{R}' of order w such that $D(\mathcal{R}') \cap (\mathcal{S}, \mathcal{P})[i, j] \subseteq D(\mathcal{R} \cup \text{start}(\mathcal{R}') \cup \text{end}(\mathcal{R}'))$.

Proof. If $j < \ell$, then by Lemma 8.4 there is a linkage \mathcal{L}_B from $B(S_j)$ to $B(S_\ell)$ in $(\mathcal{S}, \mathcal{P})[j, \ell]$ which is internally disjoint from S_j . If $j = \ell$, we set \mathcal{L}_B as the linkage containing only the vertices of $B(S_\ell)$ and no arcs.

Similarly, if $i = 0$ we set \mathcal{L}_A as the linkage containing only the vertices of $A(S_0)$ and no arcs. Otherwise, we set \mathcal{L}_A as an $A(S_0)$ - $A(S_i)$ -linkage of order $2w$ in $(\mathcal{S}, \mathcal{P})[0, i]$, which exists by Lemma 8.4.

Let $\mathcal{L} = \mathcal{L}_B \cdot \mathcal{R} \cdot \mathcal{L}_A$. As \mathcal{L}_B and \mathcal{L}_A are internally disjoint, we have that \mathcal{L} is a half-integral linkage from $B(S_j)$ to $A(S_i)$. By Lemma 3.3, $D(\mathcal{L})$ contains a $B(S_k)$ - $A(S_i)$ -linkage \mathcal{R}' of order w .

As both \mathcal{L}_A and \mathcal{L}_B are internally disjoint from $(\mathcal{S}, \mathcal{P})[i, j]$, we have that $D(\mathcal{R}') \cap (\mathcal{S}, \mathcal{P})[i, j] \subseteq D(\mathcal{R} \cup \text{start}(\mathcal{R}') \cup \text{end}(\mathcal{R}'))$. \square

As explained above, given a path of well-linked sets together with a back-linkage, we construct a cycle of well-linked sets by analysing how the back-linkage intersects the path of well-linked sets. The next lemma deals with the simplest possible case where the back-linkage is disjoint from the path of well-linked sets, or at least from a sufficiently large continuous subpath.

We define

$$\begin{aligned}\ell'_{11.2}(\ell) &:= \ell - 1, & \ell'_{11.2} \\ w'_{11.2}(w) &:= 2w, & r_{11.2} \\ r_{11.2}(w) &:= 2w. & w'_{11.2}\end{aligned}$$

Lemma 11.2. Let w, ℓ be integers, let $(\mathcal{S}, \mathcal{P})$ be a path of well-linked sets of length $\ell' \geq \ell'_{11.2}(\ell)$ and width $w' \geq w'_{11.2}(w)$ with a partial back-linkage \mathcal{R} of order $r \geq r_{11.2}(w)$ in a digraph D . If there is a $0 \leq i \leq \ell' - \ell + 1$ such that \mathcal{R} is internally disjoint from $(\mathcal{S}, \mathcal{P})[i, i + \ell - 1]$, then D contains a cycle of well-linked sets of length ℓ and width w as a subgraph.

Proof. Let $D' := (\mathcal{S}, \mathcal{P})[i, i + \ell - 1]$, $(S_0, S_1, \dots, S_{\ell'}) := \mathcal{S}$ and $(\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{\ell'-1}) := \mathcal{P}$. By Lemma 11.1, there is a $B(S_{i+\ell-1})$ - $A(S_i)$ -linkage \mathcal{R}' of order $r/2 = w$ such that $V(\mathcal{R}') \cap V(D') \subseteq V(\mathcal{R}) \cup \text{start}(\mathcal{R}') \cup \text{end}(\mathcal{R}')$. As \mathcal{R} is internally disjoint from D' , the linkage \mathcal{R}' is a partial back-linkage for D' of order w which is also internally disjoint from D' . By Observation 8.8, D' contains a path of well-linked sets $(\mathcal{S}' = (S'_0, S'_1, \dots, S'_{\ell-1}), \mathcal{P}' = (\mathcal{P}'_0, \mathcal{P}'_1, \dots, \mathcal{P}'_{\ell-2}))$ of width w and length $\ell - 1$ such that $S'_j \subseteq S_{i+j}$ for all $0 \leq j \leq \ell - 1$ and $\mathcal{P}'_j \subseteq \mathcal{P}_{i+j}$ for all $0 \leq j \leq \ell - 2$. Additionally, $A(S'_0) = \text{end}(\mathcal{R}')$ and $B(S'_{\ell-1}) = \text{start}(\mathcal{R}')$. Hence, by definition, $(\mathcal{S}', (\mathcal{P}'_0, \mathcal{P}'_1, \dots, \mathcal{P}'_{\ell-1}, \mathcal{R}'))$ is a cycle of well-linked sets of width w and length ℓ . \square \square

The previous lemma shows that if the back-linkage avoids a continuous part of the path of well-linked sets, then this allows us to construct a cycle of well-linked sets and we are done. So

we may now assume that this does not happen, i.e. that any large enough subpath of well-linked sets intersects the back-linkage.

Our next goal is to analyse this situation further and to draw some conclusions about the structure of the back-linkage if this happens. We show that in this case we obtain a path of well-linked sets and a back-linkage for it that essentially intersects the clusters one by one and in order from the last cluster to the first. To formalise this property we first introduce the concept of *jumps*.

Definition 11.3. Let $(\mathcal{S}, \mathcal{P})$ be a path of well-linked sets of length ℓ . A *jump* of length k over $(\mathcal{S}, \mathcal{P})$ is a path R with $\text{start}(R) \in V(S_i) \cup V(\mathcal{P}_i)$ and $\text{end}(R) \subseteq V(S_j) \cup V(\mathcal{P}_j)$ (if $j = \ell$, we require $\text{end}(R) \subseteq V(S_j)$ instead) such that $|j - i| = k$. If $i < j$, then R is a *forward jump*. If $i \geq j$ and R is internally disjoint from $(\mathcal{S}, \mathcal{P})$, then R is a *backward jump*.

Note that while a backward jump is required to be internally disjoint from the path of well-linked sets we do not require this from a forward jump. In fact, a forward jump could simply be a subpath of a path $R \in \mathcal{R}$ which R has in common with the path of well-linked sets $(\mathcal{S}, \mathcal{P})$.

Our next goal is to get rid of all jumps of length more than one in the back-linkage \mathcal{R} . We say that back-linkages without such jumps intersects $(\mathcal{S}, \mathcal{P})$ *cluster by cluster*.

Definition 11.4. Let $(\mathcal{S}, \mathcal{P})$ be a path of well-linked sets and let \mathcal{R} be a partial back-linkage for $(\mathcal{S}, \mathcal{P})$. We say that \mathcal{R} intersects $(\mathcal{S}, \mathcal{P})$ *cluster by cluster* if \mathcal{R} does not contain any forward or backward jump of length greater than one over $(\mathcal{S}, \mathcal{P})$.

Note that even if \mathcal{R} intersects $(\mathcal{S}, \mathcal{P})$ cluster by cluster this does not imply that the paths in \mathcal{R} visit the clusters strictly in reverse order S_l, S_{l-1}, \dots, S_0 . It is still possible that a path in \mathcal{R} intersects a cluster S_i then goes back to S_{i+1} and then intersects S_i again. So the paths in \mathcal{R} can go back and forth between two consecutive clusters numerous times. However, once R hits a vertex in S_{i-1} it can no longer go back to S_{i+1} .

Our next goal is to show that we can always construct a back-linkage that intersects the path of well-linked sets cluster by cluster. We do this in two steps. In the next lemma we eliminate forward jumps assuming that we have already eliminated all long backward jumps. In the second step, proved in [Lemma 11.6](#) below, we show how to get rid of backwards jumps.

The main technical tool we rely on in both steps is weak-minimality. Choosing the initial back-linkage \mathcal{R} to be weakly minimal gives us the tools we need to construct a path of well-linked sets and a back-linkage intersecting it cluster by cluster.

Lemma 11.5. Let $(\mathcal{S}, \mathcal{P})$ be a strict² path of well-linked sets of length $\ell' \geq \ell_{11.5}(j, \ell, m) := 3jm$ and width w in a digraph D and let \mathcal{R} be a partial back-linkage of order at least w for $(\mathcal{S}, \mathcal{P})$ which is weakly m -minimal with respect to $(\mathcal{S}, \mathcal{P})$ and does not induce any backwards jumps of length j or more. Then, there is a path of well-linked sets $(\mathcal{S}', \mathcal{P}')$ of length ℓ and width w within $D((\mathcal{S}, \mathcal{P}))$ with a back-linkage $\mathcal{R}' \subseteq \mathcal{R}$ such that \mathcal{R}' intersects $(\mathcal{S}', \mathcal{P}')$ cluster by cluster.

Proof. Let $(S_0, S_1, \dots, S_{\ell'}) := \mathcal{S}$ and $(\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{\ell'-1}) := \mathcal{P}$. First, we prove that \mathcal{R} does not contain any forward jumps of length more than $3mj$. Suppose there was a path $R \in \mathcal{R}$ containing a forward jump J of length more than $3mj$ with $\text{start}(J) \in V(S_s) \cup V(\mathcal{P}_s)$ and $\text{end}(J) \in V(S_t) \cup V(\mathcal{P}_t)$, for some s smaller than t . Let $R_a \cdot J \cdot R_b := R$ be a decomposition of

²In order to properly use the intersections of the back-linkage with the path of well-linked sets, we need the path of well-linked sets to be *strict* as defined in [Definition 8.1](#). Since we can always take a path of well-linked sets to be strict without losing width or height, we often implicitly assume that the path of well-linked sets we construct are strict.

R into subpaths. Let $e_J \in E(R)$ be the edge of R that has its tail in R_a and whose head is the first vertex of J . By going to a larger forward jump containing J if necessary we may assume w.l.o.g. that e_J is not contained in $(\mathcal{S}, \mathcal{P})$.

For each $1 \leq i \leq 3m$ let $\mathcal{H}_i = \{S_{s+(i-1)j+k} \cup D(\mathcal{P}_{s+(i-1)j+k}) \mid 0 \leq k \leq j-1\}$. As \mathcal{R} does not contain any backward jumps of length j or more, for every $1 \leq i \leq 3m$ there is a subgraph $H_i^a \in \mathcal{H}_i$ which intersects R_a and there is an $H_i^b \in \mathcal{H}_i$ which intersects R_b .

For each $1 \leq i \leq m$, let v_i^a be an arbitrary vertex of $V(\mathcal{H}_{3i-2}^a) \cap V(R_a)$, let v_i^b be an arbitrary vertex of $V(\mathcal{H}_{3i}^b) \cap V(R_b)$ and let L_i be a v_i^a - v_i^b path inside $(\mathcal{S}, \mathcal{P})[3i-2, 3i]$. By Lemma 8.7(L4), such a path L_i exists. Note that L_i is disjoint from L_j for all $1 \leq i, j \leq m$ where $i \neq j$. Thus, $\mathcal{L} = \{L_i \mid 1 \leq i \leq m\}$ is a $V(R_a)$ - $V(R_b)$ -linkage of order m in $(\mathcal{S}, \mathcal{P})$ which does not contain the edge e_J defined above, a contradiction to the assumption that \mathcal{R} is weakly m -minimal with respect to $(\mathcal{S}, \mathcal{P})$. Thus, \mathcal{R} does not contain any forward jumps of length greater than $3mj$.

Second, we construct the desired path of well-linked sets. For each $0 \leq k < \ell$ let $S'_k = S_{3kmj}$ and let \mathcal{P}'_k be a $B(S_{3kmj})$ - $A(S_{3(k+1)mj})$ -linkage of order m inside the path of well-linked sets $(\mathcal{S}, \mathcal{P})[3kmj, 3(k+1)mj]$. Further, let $S'_\ell = S_\ell$ and \mathcal{P}'_ℓ be a $B(S_{3(\ell-1)mj})$ - $A(S_\ell)$ -linkage of order w inside $(\mathcal{S}, \mathcal{P})[3(\ell-1)mj, \ell]$. By Lemma 8.7(L1), such linkages \mathcal{P}'_k exist.

Let $\mathcal{S}' = (S'_0, S'_1, \dots, S'_\ell)$ and let $\mathcal{P}' = (\mathcal{P}'_0, \mathcal{P}'_{3mj}, \dots, \mathcal{P}'_{(\ell-1)3mj})$. Note that $\text{start}(\mathcal{R}) \subseteq V(S'_\ell)$ and $\text{end}(\mathcal{R}) \subseteq V(S'_0)$. By construction, $(\mathcal{S}', \mathcal{P}')$ is a path of well-linked sets of width w and length ℓ . Furthermore, every jump over $(\mathcal{S}', \mathcal{P}')$ of length j' , for some j' , in \mathcal{R} is a jump over $(\mathcal{S}', \mathcal{P}')$ of length $3mj'$. Hence, \mathcal{R} does not contain any forward jumps or backwards jumps of length greater than one over $(\mathcal{S}', \mathcal{P}')$. Finally, the linkage \mathcal{R} is a back-linkage for $(\mathcal{S}', \mathcal{P}')$ and \mathcal{R} intersects $(\mathcal{S}', \mathcal{P}')$ cluster by cluster. \square \square

The previous lemma allows us to handle forward jumps assuming that there are no backward jumps. The next lemma we take care of backward jumps. The main idea is that if a back-linkage contains many long backward jumps that jump over the same part of the path of well-linked sets $(\mathcal{S}, \mathcal{P})$, then these backward jumps themselves essentially constitute a back-linkage for the part of $(\mathcal{S}, \mathcal{P})$ they jump over. As, by definition, backward jumps are internally disjoint from $(\mathcal{S}, \mathcal{P})$, we can apply Lemma 11.2 to obtain a cycle of well-linked sets in this case.

We define

$$\ell'_{11.6}(w_1, \ell_1, \ell_2, m) := 3\ell_2m((\ell_1 + 3)(3\ell_2m)^{w_1} + 6 \frac{(3\ell_2m)^{w_1} - 1}{w_1 - 1}), \quad \ell'_{11.6}$$

$$w'_{11.6}(w_1, w_2) := 2w_2 + w_1. \quad w'_{11.6}$$

Observe that $\ell'_{11.6}(w_1, \ell_1, \ell_2, m) \in 2^{1 \uparrow \text{poly}^2(w_1, \ell_1, \ell_2, m)}$.

Lemma 11.6. Let ℓ_1, w_1, ℓ_2, w_2 be integers, let $(\mathcal{S} = (S_0, S_1, \dots, S_{\ell'}), \mathcal{P} = (\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{\ell'-1}))$ be a strict path of well-linked sets of length $\ell' \geq \ell'_{11.6}(w_1, \ell_1, \ell_2, m)$ and width $w' \geq w'_{11.6}(w_1, w_2)$ with a partial back-linkage \mathcal{R} of order at least w_2 in a digraph D such that \mathcal{R} is weakly m -minimal with respect to $(\mathcal{S}, \mathcal{P})$. Then D contains at least one of the following:

- (C1) a cycle of well-linked sets of length ℓ_1 and width w_1 , or
- (C2) a path of well-linked sets of length ℓ_2 and width w_2 together with a partial back-linkage $\mathcal{R}' \subseteq \mathcal{R}$ of order w_2 intersecting it cluster by cluster.

Proof. We recursively define d_i by $d_1 = \ell_1 + 3$ and $d_i = 3\ell_2md_{i-1} + 6$. Solving the recurrence relation, we obtain that $d_{w_1} = ((\ell_1 + 3)(3\ell_2m)^{w_1} + 6 \frac{(3\ell_2m)^{w_1} - 1}{w_1 - 1})$ and thus $\ell' \geq 3\ell_2md_{w_1}$.

Let J_1, J_2 be two backward jumps over $(\mathcal{S}, \mathcal{P})$ and let x_1, x_2, y_1, y_2 be such that $\text{start}(J_1) \subseteq V(S_{y_1}) \cup V(\mathcal{P}_{y_1})$, $\text{end}(J_1) \subseteq V(S_{x_1}) \cup V(\mathcal{P}_{x_1})$, $\text{start}(J_2) \subseteq V(S_{y_2}) \cup V(\mathcal{P}_{y_2})$ and $\text{end}(J_2) \subseteq V(S_{x_2}) \cup V(\mathcal{P}_{x_2})$. We say that J_1 jumps over J_2 if $y_1 \geq y_2 + 2$ and $x_1 + 2 \leq x_2$.

If \mathcal{R} does not contain any jump of length at least d_{w_1} over $(\mathcal{S}, \mathcal{P})$, then by Lemma 11.5 there is a path of well-linked sets $(\mathcal{S}' = (S'_0, S'_1, \dots, S'_{\ell_2}), \mathcal{P}')$ of width w_2 and length ℓ_2 together with a $B(S'_{\ell_2})$ - $A(S'_0)$ -linkage \mathcal{R}' of order w_2 intersecting $(\mathcal{S}', \mathcal{P}')$ cluster-by-cluster, satisfying (C2).

Otherwise, let $r \in \{0, \dots, w_1 - 1\}$ be the highest number for which a set $\mathcal{J} = \{J_{w_1-r}, \dots, J_{w_1}\}$ of backward jumps over $(\mathcal{S}, \mathcal{P})$ exists such that for every $i \in \{w_1-r, \dots, w_1\}$ and every $i+1 \leq j \leq w_1$, J_i is a backward jump of length at least d_i and J_j jumps over J_i . We distinguish between two possible cases.

Case 1: $r < w_1 - 1$.

Then J_{w_1-r} is a backward jump of length at least d_{w_1-r} . Let i, j be such that $\text{start}(J_{w_1-r}) \subseteq V(S_j) \cup V(\mathcal{P}_j)$ and $\text{end}(J_{w_1-r}) \subseteq V(S_i) \cup V(\mathcal{P}_i)$. By Lemma 11.1, there is a $B(S_{j-2})$ - $A(S_{i+2})$ -linkage \mathcal{R}' of order w_2 such that $V(\mathcal{R}') \cap V((\mathcal{S}, \mathcal{P})[i+2, j-2]) \subseteq V(\mathcal{R}) \cup \text{start}(\mathcal{R}') \cup \text{end}(\mathcal{R}')$. Hence, any backward jump over $(\mathcal{S}, \mathcal{P})[i+3, j-3]$ contained in \mathcal{R}' is also contained in \mathcal{R} . Finally, \mathcal{R}' is also weakly m -minimal with respect to $(\mathcal{S}, \mathcal{P})[i+3, j-3]$.

By choice of i and j , if \mathcal{R}' contains a backward jump J' over $(\mathcal{S}, \mathcal{P})[i+3, j-3]$, then every jump in \mathcal{J} jumps over J' . Since r is maximal, there is no backward jump over $(\mathcal{S}, \mathcal{P})[i+3, j-3]$ of length at least d_{w_1-r-1} in \mathcal{R}' . And because $j-3-(i+3) \geq d_{w_1-r}-6 = 3\ell_2 m d_{w_1-r-1}$, by Lemma 11.5 there is a path of well-linked sets $(\mathcal{S}', \mathcal{P}')$ of width w_2 and length ℓ_2 together with a partial back-linkage $\mathcal{R}'' \subseteq \mathcal{R}'$ of order w_2 intersecting $(\mathcal{S}', \mathcal{P}')$ cluster by cluster, satisfying (C2).

Case 2: $r = w_1 - 1$, that is, $w_1 - r = 1$.

We construct a linkage \mathcal{R}' as follows. Let i, j be such that $\text{start}(J_{w_1-r}) \subseteq V(S_j) \cup V(\mathcal{P}_j)$ and $\text{end}(J_{w_1-r}) \subseteq V(S_i) \cup V(\mathcal{P}_i)$.

For every two distinct jumps $J_x, J_y \in \mathcal{J}$ we have that J_x jumps over J_y or J_y jumps over J_x . Further, \mathcal{J} is internally disjoint from $(\mathcal{S}, \mathcal{P})$. Hence, by Lemma 8.7(L3), there is a $B(S_{j-2})$ - $\text{start}(\mathcal{J})$ -linkage \mathcal{X}_1 of order w_1 in $(\mathcal{S}, \mathcal{P})[j-2, \ell']$ which is internally disjoint from S_{j-2} and from \mathcal{J} . Additionally, by Lemma 8.7(L2), there is an $\text{end}(\mathcal{J})$ - $A(S_{i+2})$ -linkage \mathcal{X}_2 of order w_1 in $(\mathcal{S}, \mathcal{P})[0, i+2]$ which is internally disjoint from S_{i+2} and from \mathcal{J} . Thus, $\mathcal{R}' := \mathcal{X}_1 \cdot \mathcal{J} \cdot \mathcal{X}_2$ is a linkage.

By construction, the linkage \mathcal{R}' above has order at least w_1 and is internally disjoint from $(\mathcal{S}, \mathcal{P})[i+2, j-2]$, which is a path of well-linked sets of length $j-2-(i+2) \geq \ell_1-1$ and width w_1 . Thus, by Observation 8.8, there is a path of well-linked sets $(\mathcal{S}' = (S'_0, S'_1, \dots, S'_{\ell_1-1}), \mathcal{P}' = (\mathcal{P}'_0, \mathcal{P}'_1, \dots, \mathcal{P}'_{\ell_1-2}))$ of length ℓ_1-1 and width w_1 inside $D((\mathcal{S}, \mathcal{P}))$ such that $B(S'_{\ell_1-1}) = \text{start}(\mathcal{R}')$ and $A(S'_0) = \text{end}(\mathcal{R}')$. By definition, $(\mathcal{S}', (\mathcal{P}'_0, \mathcal{P}'_1, \dots, \mathcal{P}'_{\ell_1-2}, \mathcal{R}'))$ is a cycle of well-linked sets of length ℓ_1 and width w_1 , satisfying (C1). \square \square

11.2 Obtaining a 2-horizontal web

By the results of the previous section we may now assume that we have a path of well-linked sets together with a back-linkage going through it cluster by cluster. We use this back-linkage to construct a new web that is in some way aligned with the cluster by cluster property of the back-linkage.

The next definition formalises the properties we require of this new web we aim to construct.

Definition 11.7. Let $(\mathcal{H}, \mathcal{V})$ be a web. We say that $(\mathcal{H}, \mathcal{V})$ is a q -horizontal web if every path $H_i \in \mathcal{H}$ can be decomposed into paths $H_i = H_i^1 \cdot H_i^2 \cdot \dots \cdot H_i^q$ and every path $V_j \in \mathcal{V}$ can be decomposed into paths $V_j = V_j^1 \cdot V_j^2 \cdot \dots \cdot V_j^q$ such that $V_j^x \cap H_i \subseteq H_i^{q-x+1} \cup H_i^{q-x}$ and $V_j^x \cap H_i^{q-x+1} \neq \emptyset$ for all $1 \leq x \leq q$, where for simplicity we define H_i^0 to be empty.

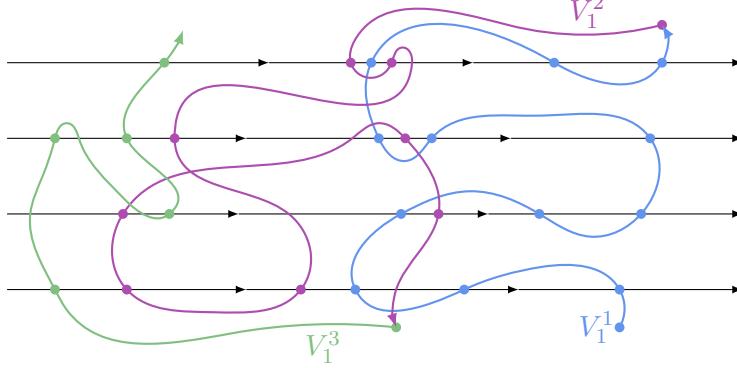


Figure 14: A $(4, 1)$ -web that is a 3-horizontal web. Four horizontal paths H_1, H_2, H_3 and H_4 partitioned into three subpaths each and one vertical path V_1 partitioned into three subpaths V_1^1 , V_1^2 and V_1^3 . V_1^1 only intersects the later two subpaths of the horizontal paths. V_1^2 only intersects the first two, note that it always intersects the second subpaths but not necessarily the first. Finally, V_1^3 only intersects the first subpath of the horizontal paths.

In the next lemma we construct an ordered web from a back-linkage and a path of well-linked sets.

Lemma 11.8. Let $(\mathcal{S}, \mathcal{P})$ be a strict path of well-linked sets of length ℓ and width at least 1 in a digraph D , and let \mathcal{R} be a partial back-linkage of order r intersecting $(\mathcal{S}, \mathcal{P})$ cluster by cluster.

If $\ell \geq \text{len}_{11.8}(r, v) := 2(r - 1) + 2r(v - 1)$, then there is a linkage $\mathcal{V} := (V_1, V_2, \dots, V_v)$ of order v inside $D((\mathcal{S}, \mathcal{P}))$ such that $(\mathcal{R}, \mathcal{V})$ is an ordered web and for all $1 \leq i \leq v$ there are $0 \leq s_i \leq t_i \leq \ell$ with $V_i \subseteq (\mathcal{S}, \mathcal{P})[s_i, t_i]$ such that $t_i < s_j$ for all $1 \leq i < j \leq v$.

Proof. Let $(S_0, S_1, \dots, S_\ell) := \mathcal{S}$ and $(\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{\ell-1}) := \mathcal{P}$. To simplify notation, we set $\mathcal{P}_\ell := \emptyset$. Since \mathcal{R} intersects $(\mathcal{S}, \mathcal{P})$ cluster by cluster, every $R \in \mathcal{R}$ intersects some vertex of $D(S_i \cup \mathcal{P}_i)$ for every $0 \leq i < \ell$. Let $\mathcal{R} = (R_1, R_2, \dots, R_r)$ be an arbitrary ordering of the paths in \mathcal{R} . For each $1 \leq i \leq v$, construct a path V^i intersecting every path in \mathcal{R} as follows.

For each $R_j \in \mathcal{R}$, let $k_j^i = 2(j - 1) + 2r(i - 1)$ and let u_j^i be some vertex in $V(R_j) \cap V(S_{k_j^i}) \cup V(\mathcal{P}_{k_j^i})$. Let Q_j^i be a path visiting u_j^i with $y_j^i := \text{start}(Q_j^i) \in A(S_{k_j^i})$ and $z_j^i := \text{end}(Q_j^i) \in A(S_{k_j^i+1})$. Since $k_j^i - k_{j-1}^i = 2$ for all $2 \leq j \leq r$, by Lemma 8.7(L4) there is a z_j^i - y_{j+1}^i path V_j^i inside $(\mathcal{S}, \mathcal{P})[k_j^i, k_{j+1}^i]$ for every $1 \leq j < r$. Since all V_j^i and all Q_j^i are pairwise internally disjoint and V_j^i intersects R_j at u_j^i , the path $V^i = Q_1^i \cdot V_1^i \cdot Q_2^i \cdot V_2^i \cdot \dots \cdot Q_r^i$ in $(\mathcal{S}, \mathcal{P})[k_1^i, k_r^i]$ intersects every path in \mathcal{R} .

Let $\mathcal{V} = \{V^i \mid 1 \leq i \leq v\}$. Since $k_1^i - k_r^{i-1} = 2$ for all $2 \leq i \leq v$, all paths in \mathcal{V} are pairwise disjoint. Further, such paths exist because $\ell \geq k_r^v = 2(r - 1) + 2r(v - 1)$. Finally, because \mathcal{R} intersects $(\mathcal{S}, \mathcal{P})$ cluster by cluster, $(\mathcal{R}, \mathcal{V})$ is an ordered web. \square \square

We now have a new web $(\mathcal{R}, \mathcal{V})$. Our next goal is to find a new ‘‘horizontal’’ linkage \mathcal{H} which is weakly $|\mathcal{H}|$ -minimal with respect to \mathcal{V} such that $(\mathcal{H}, \mathcal{V})$ is a 2-horizontal web $(\mathcal{H}, \mathcal{V})$. The idea is that \mathcal{H} goes *forwards* through the path of well-linked sets $(\mathcal{S}, \mathcal{P})$ from beginning to end, i.e. in the same direction as the path of well-linked sets itself. $(\mathcal{S}, \mathcal{P})$ contains a forward linkage from its beginning to its end and as a result of the way the linkage \mathcal{V} is constructed, we are

able to construct \mathcal{H} so that together with \mathcal{V} it forms a web. But we also want that \mathcal{H} is weakly $|\mathcal{H}|$ -minimal with respect to \mathcal{V} . If we simply choose \mathcal{H} as a \mathcal{V} -minimal forward linkage then we can no longer guarantee that this intersects \mathcal{V} as required for a 2-horizontal web.

To solve this problem we show that in this case we do get a cycle of well-linked sets immediately and are done.

Towards this aim, we prove in [Lemma 11.10](#) that any path of well-linked sets which contains a forward linkage disjoint from the back-linkage contains a cycle of well-linked sets.

The next lemma is the first step towards this goal. We define

$$\begin{aligned} w'(w, \ell, \ell^*) &:= \ell^* + \textcolor{blue}{w}_{8.3}(w, \ell), \\ \textcolor{blue}{q}_{11.9}(w, \ell, \ell^*) &:= \textcolor{blue}{s}_{6.16}(\ell^* + \textcolor{blue}{w}_{8.3}(w, \ell)), \\ \ell'_{11.9}(w, \ell, \ell^*) &:= (3w\ell \binom{\textcolor{blue}{q}_{11.9}(w, \ell, \ell^*)}{w'(w, \ell, \ell^*)})(w'(w, \ell, \ell^*))! + 3 \\ &\quad \cdot \textcolor{blue}{\ell}_{6.16}(\textcolor{blue}{q}_{11.9}(w, \ell, \ell^*), w'(w, \ell, \ell^*)) - 1. \end{aligned} \quad \begin{matrix} \textcolor{blue}{q}_{11.9} \\ \ell'_{11.9} \end{matrix}$$

We note that $\textcolor{blue}{q}_{11.9}(w, \ell, \ell^*) \in \text{poly}^{22}(w, \ell, \ell^*)$ and $\ell'_{11.9}(w, \ell, \ell^*) \in 2^{1 \uparrow \uparrow \text{poly}^{243}(w, \ell, \ell^*)}$.

Lemma 11.9. Let ℓ^*, w be integers, let $D = (\mathcal{S} = (S_0, S_1, \dots, S_{\ell'}), \mathcal{P} = (\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{\ell'-1}))$ be a strict path of well-linked sets of width $w' \geq 1$ and length $\ell' := \ell'_{11.9}(w, \ell^*)$. Let \mathcal{L} be an $A(S_0)$ - $B(S_{\ell'})$ -linkage of order at least $\textcolor{blue}{q}_{11.9}(w, \ell, \ell^*)$ such that every path in \mathcal{L} intersects every $S_i \in \mathcal{S}$. Then, there is an $\mathcal{L}^* \subseteq \mathcal{L}$ of order ℓ^* for which $D(\mathcal{S} \cup \mathcal{P})$ contains a path of well-linked sets $(\mathcal{S}' = (S'_0, S'_1, \dots, S'_{\ell}), \mathcal{P}' = (\mathcal{P}'_0, \mathcal{P}'_1, \dots, \mathcal{P}'_{\ell-1}))$ of width w and length ℓ which is disjoint from \mathcal{L}^* such that $A(S'_0) \subseteq A(S_0)$ and $B(S'_{\ell}) \subseteq B(S_{\ell'})$. Further, for every $0 \leq i < j \leq \ell$ there are $0 \leq i_0 < i_1 < j_0 < j_1 \leq \ell$ such that $S'_i \subseteq D[i_0, i_1]$ and $S'_j \subseteq D[j_0, j_1]$. Finally, $D(\mathcal{P}') \subseteq \mathcal{L} \setminus \mathcal{L}^*$.

Proof. Let $w_1 = \textcolor{blue}{w}_{8.3}(w, \ell)$, $w_2 = \ell^* + w_1$, $w_3 = \textcolor{blue}{s}_{6.16}(w_2)$. $\ell_1 = w\ell$, $\ell_2 = \ell_1 \binom{w_3}{w_2} w_2! + 1$ and $\ell_3 = \textcolor{blue}{\ell}_{6.16}(w_2)$. Note that $\ell' \geq 3\ell_2\ell_3 - 1$ and that $|\mathcal{L}| \geq w_3$.

For each $1 \leq i \leq \ell_2$ construct a temporal digraph T_i as follows. For each $1 \leq j \leq \ell_3$ let $s_{i,j} = 3(i-1)\ell_3 + 3(j-1)$ and note that $s_{i,1} = 1 + s_{i-1,\ell_3}$ and that $s_{\ell_2,\ell_3} + 2 \leq \ell'$. Let $H_j^i = D(S_{s_{i,j}} \cup S_{s_{i,j}+1} \cup S_{s_{i,j}+2})$.

Let $\mathcal{H}^i = (H_1^i, H_2^i, \dots, H_{\ell_2}^i)$. Let T_i be the routing temporal digraph of \mathcal{L} through \mathcal{H}^i as described in [Definition 6.3](#). Note that $\ell(T_i) = \ell_3$ for every i .

Let A_i be the set containing the first intersection of each path in \mathcal{L} with H_1^i and let B_i be the set containing the last intersection of each path in \mathcal{L} with $H_{\ell_2}^i$. Since every path in \mathcal{L} intersects every $S \in \mathcal{S}$, we have that $|A_i| = |B_i| = |\mathcal{L}|$.

We show that every layer of T_i is strongly connected. Let $D_j(T_i)$ be layer j of T_i . Let $L_a, L_b \in \mathcal{L}$ be two distinct paths. Since every path in \mathcal{L} intersects every cluster in \mathcal{S} , there is some $a_0 \in V(S_{s_{i,j}})$ and some $b_0 \in V(S_{s_{i,j}+2})$ such that L_a contains a_0 and L_b contains b_0 . Further, there is some $a_1 \in B(S_{s_{i,j}})$ and some $b_1 \in A(S_{s_{i,j}+2})$ such that a_0 can reach a_1 in $S_{s_{i,j}}$ and b_1 can reach b_0 in $S_{s_{i,j}+2}$. As $A(S_{s_{i,j}})$ is well-linked to $B(S_{s_{i,j}+1})$, there is some a_2 - b_2 path in $S_{s_{i,j}+1}$, where $a_2 = \mathcal{P}_{s_{i,j}}(a_1)$ and $b_2 = \mathcal{P}_{s_{i,j}+2}(b_1)$. Hence, a_0 can reach b_0 in H_j^i . Thus, there is a $V(L_a)$ - $V(L_b)$ path in H_j^i , which implies that $D_j(T_i)$ is strongly connected.

As $\ell(T_i) = \ell_3 = \textcolor{blue}{\ell}_{6.16}(w_3, w_2)$ and $|V(T_i)| = w_3 = \textcolor{blue}{s}_{6.16}(w_2)$, by [Theorem 6.16](#) T_i contains an R_i -routing φ_i for some $R_i \in \{\mathbf{C}_{w_2}, \tilde{\mathbf{P}}_{w_2}\}$. Since there are ℓ_2 temporal digraphs T_i , by the pigeon-hole principle there is a set $\mathcal{T} := \{T_{t_0}, T_{t_1}, \dots, T_{t_{\ell_1}}\}$ of size $\ell_1 + 1$ of temporal digraphs such that $R := R_i = R_j$ and $\varphi := \varphi_i = \varphi_j$ for all $T_i, T_j \in \mathcal{T}$.

Let R' be a path of length $w_2 - 1$ in R and let u_1, u_2, \dots, u_{w_2} be the vertices of R' sorted according to their order on R' . Let $\mathcal{L}' = \{\varphi(u_i) \mid 1 \leq i \leq w_1\}$, let $\mathcal{L}^* = \{\varphi(u_i) \mid w_1 + 1 \leq i \leq w_2\}$ and let $\varphi' = \varphi|_{\mathcal{L}'}$. Note that $|\mathcal{L}^*| = \ell^*$.

For each $0 \leq j \leq \ell_1$, we construct a subgraph S'_j of D and a linkage \mathcal{P}'_j as follows. Note that

φ' is a \mathbf{P}_{w_1} -routing in $T_{t_j} - V(\mathcal{L}^*)$. Let \mathcal{Q}^j be the set of digraphs obtained by deleting $V(\mathcal{L}^*)$ from each digraph in \mathcal{H}^{t_j} and let T'_j be the routing temporal digraph of \mathcal{L}' through \mathcal{Q}^j . Observe that φ' is also a \mathbf{P}_{w_1} -routing in T'_j .

Let $A'_j = A_j \cap V(\mathcal{L}')$ and let $B'_j = B_j \cap V(\mathcal{L}')$. By Lemma 7.6, we have that A'_j is 1-order-linked to B'_j in $D(\mathcal{Q}^j)$. We set $S'_j = D(\mathcal{Q}^j)$ and take \mathcal{P}'_j as the B'_j - A'_{j+1} -linkage of order w_1 inside \mathcal{L}' (to simplify notation, we set $A'_{\ell_1+1} = \text{end}(\mathcal{L}')$).

After finishing the construction, let $\mathcal{S}' = (S'_0, S'_1, \dots, S'_{\ell_1})$ and let $\mathcal{P}' = (\mathcal{P}'_0, \mathcal{P}'_1, \dots, \mathcal{P}'_{\ell_1-1})$. By construction, $(\mathcal{S}', \mathcal{P}')$ is a path of 1-order-linked sets of width w_1 and length ℓ_1 . Furthermore, \mathcal{L}^* is disjoint from $(\mathcal{S}', \mathcal{P}')$. Finally, we have that $D(\mathcal{P}'_j) \subseteq \mathcal{L} \setminus \mathcal{L}^*$ and that $S'_j \subseteq D[t_j, t_{j+1} - 1]$.

By Theorem 7.8, $(\mathcal{S}', \mathcal{P}')$ contains a path of w -order-linked sets $(\mathcal{S}^2, \mathcal{P}^2)$ of width w_1 and length ℓ . By Lemma 8.3, $(\mathcal{S}^2, \mathcal{P}^2)$ contains a path of well-linked sets $(\mathcal{S}^3, \mathcal{P}^3)$ of width w and length ℓ satisfying the requirements of the statement. \square \square

With the previous lemma at hand we can now proceed as follows. Given a path of well-linked sets and a large forward linkage \mathcal{L} , we can construct a new path of well-linked sets and a subset $\mathcal{L}^* \subseteq \mathcal{L}$ disjoint from it. Furthermore, \mathcal{L}^* is also disjoint from the back-linkage \mathcal{R} . We now apply Corollary 10.3 to obtain another path of well-linked sets which follows the direction of \mathcal{R} . With respect to this new path of well-linked sets the forward linkage \mathcal{L}^* now acts like a back-linkage which is disjoint from the path of well-linked sets. As we have already seen above, in this situation we can construct a cycle of well-linked sets.

We define

$$\mathbf{r}_{11.10}(w, \ell) := \mathbf{h}_{10.3}(2w, \ell - 1), \quad \mathbf{r}_{11.10}$$

$$\ell''(w, \ell) := \ell'_{11.6}(w, \ell, \ell_{11.8}(\mathbf{h}_{10.3}(2w, \ell - 1), 8w + \mathbf{v}_{10.3}(2w, \ell - 1) + 2), \mathbf{r}_{11.10}(w, \ell)),$$

$$\ell'_{11.10}(w, \ell) := \ell'_{11.9}(\mathbf{w}'_{11.6}(w, \mathbf{h}_{10.3}(2w, \ell - 1)), \ell''(w, \ell), 2w), \quad \ell'_{11.10}$$

$$\mathbf{q}_{11.10}(w, \ell) := \mathbf{q}_{11.9}(w, \ell''(w, \ell), \mathbf{h}_{10.3}(2w, \ell - 1)). \quad \mathbf{q}_{11.10}$$

Observe that $\mathbf{r}_{11.10}(w, \ell) \in O(w^2 \ell^2)$, $\ell'_{11.10}(w, \ell) \in 2^{3 \uparrow \uparrow \text{poly}^{25}(w, \ell)}$ and $\mathbf{q}_{11.10}(w, \ell) \in 2^{2 \uparrow \uparrow \text{poly}^{25}(w, \ell)}$.

Lemma 11.10. Let $(\mathcal{S} = (S_0, S_1, \dots, S_{\ell'}), \mathcal{P} = (\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{\ell'-1}))$ be a strict path of well-linked sets of width $w' \geq 1$ and length $\ell' \geq \ell'_{11.10}(w, \ell)$ with a partial back-linkage \mathcal{R} of order $r \geq \mathbf{r}_{11.10}(w, \ell)$ in a digraph D . Let \mathcal{L} be an $A(S_0)$ - $B(S_{\ell'})$ linkage of order $q \geq \mathbf{q}_{11.10}(w, \ell, m)$ which is internally disjoint from \mathcal{R} such that every $L \in \mathcal{L}$ intersects some vertex of $D(S_i \cup \mathcal{P}_i)$ for every $0 \leq i \leq \ell'$ and, for all $0 \leq i < j \leq \ell'$, \mathcal{L} does not intersect $D(S_i \cup \mathcal{P}_i)$ after intersecting $D(S_j \cup \mathcal{P}_j)$. Then, $D(\mathcal{S} \cup \mathcal{P} \cup \mathcal{R} \cup \mathcal{L})$ contains a cycle of well-linked sets of width w and length ℓ .

Proof. Let $\ell_4 = \ell - 1$, $\ell_3 = 4w + 1$, $v_1 = \mathbf{v}_{10.3}(2w, \ell_4) + 2\ell_3$, $w_2 = \mathbf{h}_{10.3}(2w, \ell_4)$, $w_1 = \mathbf{w}'_{11.6}(w, w_2)$, $\ell_2 = \ell_{11.8}(w_2, v_1)$ and $\ell_1 = \ell'_{11.6}(w, \ell, \ell_2, r)$. Note that $w_2 \geq 2w$, $\ell' \geq \ell'_{11.9}(w_1, \ell_1, 2w)$, $r \geq w_2$ and $q \geq \mathbf{q}_{11.9}(w, \ell_1, w_2)$.

By Lemma 11.9 there is some linkage $\mathcal{L}' \subseteq \mathcal{L}$ of order $2w$ and a path of well-linked sets $(\mathcal{S}^1 = (S_0^1, S_1^1, \dots, S_{\ell_1}^1), \mathcal{P}^1 = (\mathcal{P}_0^1, \mathcal{P}_1^1, \dots, \mathcal{P}_{\ell_1-1}^1))$ of width w_1 and length ℓ_1 inside $(\mathcal{S}, \mathcal{P})$ with $A(S_0^1) \subseteq A(S_0)$ and $B(S_{\ell_1}^1) \subseteq B(S_{\ell'})$ such that \mathcal{L}' is internally disjoint from $(\mathcal{S}^1, \mathcal{P}^1)$. Further, $D(\mathcal{P}^1) \subseteq D(\mathcal{L} \setminus \mathcal{L}')$ and $S_i^1 \subseteq S_i$ for all $0 \leq i \leq \ell_1$.

Let $\mathcal{R}^1 \subseteq D((\mathcal{S}^1, \mathcal{P}^1)) \cup D(\mathcal{R})$ be a $(\mathcal{S}^1, \mathcal{P}^1)$ -minimal linkage of order $|\mathcal{R}|$ such that $\text{start}(\mathcal{R}^1) = \text{start}(\mathcal{R})$ and $\text{end}(\mathcal{R}^1) = \text{end}(\mathcal{R})$. By Observation 3.6, \mathcal{R}^1 is weakly r -minimal with respect to $(\mathcal{S}^1, \mathcal{P}^1)$. Further, \mathcal{R}^1 is internally disjoint from \mathcal{L}' . Applying Lemma 11.6 to $(\mathcal{S}^1, \mathcal{P}^1)$ and \mathcal{R}^1 yields two cases.

If **(C1)** holds, then we have a cycle of well-linked sets of width w and length ℓ , as desired. Otherwise, **(C2)** holds. That is, $D((\mathcal{S}^1, \mathcal{P}^1) \cup \mathcal{R})$ contains a path of well-linked sets $(\mathcal{S}^2, \mathcal{P}^2)$ of length ℓ_2 and width w_2 and a linkage $\mathcal{R}^2 \subseteq \mathcal{R}^1$ of order w_2 such that \mathcal{R}^2 intersects $(\mathcal{S}^2, \mathcal{P}^2)$ cluster by cluster. Note that \mathcal{R}^2 is weakly r -minimal with respect to $(\mathcal{S}^2, \mathcal{P}^2)$.

By [Lemma 11.8](#), there is some linkage $\mathcal{V} = (V_1, V_2, \dots, V_{v_1})$ of order v_1 inside $(\mathcal{S}^2, \mathcal{P}^2)$ such that $(\mathcal{R}^2, \mathcal{V})$ is an ordered web. Additionally, for all $1 \leq i < j \leq v_1$ there are $0 \leq s_i \leq t_i < s_j \leq t_j \leq \ell_2$ such that $V_i \subseteq (\mathcal{S}^2, \mathcal{P}^2)[s_i, t_i]$ and $V_j \subseteq (\mathcal{S}^2, \mathcal{P}^2)[s_j, t_j]$.

Let $\mathcal{V}' = (V_{\ell_3+1}, V_{\ell_3+2}, \dots, V_{v_1-\ell_3})$ and observe that $|\mathcal{V}'| = \text{v}_{10.3}(2w, \ell_4)$. Decompose \mathcal{R}^2 as $\mathcal{R}_a^2 \cdot \mathcal{R}^3 \cdot \mathcal{R}_b^2 := \mathcal{R}^2$ such that \mathcal{R}^3 intersects all paths of \mathcal{V}' , $\text{end}(\mathcal{R}_a^2) \subseteq V(V_{v_1-\ell_3})$, $\text{start}(\mathcal{R}_b^2) \subseteq V(V_{\ell_3+1})$ and \mathcal{R}_a^2 and \mathcal{R}_b^2 are internally disjoint from \mathcal{V}' .

By [Corollary 10.3](#), $(\mathcal{R}^3, \mathcal{V}')$ contains a path of well-linked sets $(\mathcal{S}^3 = (S_0^3, S_1^3, \dots, S_{\ell_4}^3), \mathcal{P}^3)$ of width $2w$ and length ℓ_4 such that $A(S_0^3) \subseteq \text{start}(\mathcal{R}^3)$ and $B(S_{\ell_4}^3) \subseteq \text{end}(\mathcal{R}^3)$. Since \mathcal{L}' is internally disjoint from $(\mathcal{S}^2, \mathcal{P}^2)$ and from \mathcal{R} , it is also internally disjoint from $(\mathcal{S}^3, \mathcal{P}^3)$. We construct a partial back-linkage \mathcal{R}^4 for $(\mathcal{S}^3, \mathcal{P}^3)$ as follows.

Choose some $B'_0 \subseteq B(S_0)$ and some $A'_{\ell'} \subseteq A(S_{\ell'})$ of size $2w$. Let \mathcal{X}_1 be some $\text{end}(\mathcal{R}_b^2)$ - B'_0 -linkage of order $2w$ in S_0 and let \mathcal{X}_2 be some $A'_{\ell'}\text{-start}(\mathcal{R}_a^2)$ -linkage of order $2w$ in $S_{\ell'}$. Since $\text{end}(\mathcal{R}_b^2) \subseteq A(S_0)$ and $\text{start}(\mathcal{R}_a^2) \subseteq B(S_{\ell'})$, the linkages \mathcal{X}_1 and \mathcal{X}_2 exist.

Fix an arbitrary ordering of $\mathcal{L}' = \{L'_1, L'_2, \dots, L'_{2w}\}$. For each $L'_i \in \mathcal{L}'$ let $k_i = 2i + 1$ and choose some $v_i \in V(L_i) \cap V(S_{k_i} \cup \mathcal{P}_{k_i})$ and some $u_i \in V(L_i) \cap V(S_{\ell'-k_i} \cup \mathcal{P}_{\ell'-k_i})$. Let $Y_1 = \{v_i \mid 1 \leq i \leq 2w\}$ and $Y_2 = \{u_i \mid 1 \leq i \leq 2w\}$. Since $k_i - k_j \geq 2$ and $\ell' - k_i - (\ell' - k_j) \geq 2$ if $i < j$, by [Lemma 8.7\(L3\)](#) there is a B''_0 - Y_1 -linkage \mathcal{Z}_1 of order $2w$ inside $(\mathcal{S}, \mathcal{P})[0, k_{2w}]$ and by [Lemma 8.7\(L2\)](#) there is a $Y_2\text{-}A''_{\ell'}$ -linkage \mathcal{Z}_2 of order $2w$ inside $(\mathcal{S}, \mathcal{P})[\ell' - k_{2w}, \ell']$.

Let \mathcal{L}'' be the sublinkage of \mathcal{L}' from Y_1 to Y_2 . By construction, $\mathcal{X}_1, \mathcal{X}_2, \mathcal{Z}_1$ and \mathcal{Z}_2 are pairwise internally disjoint. Further, \mathcal{R}_a^2 and \mathcal{R}_b^2 are disjoint since they are both part of the linkage \mathcal{R} . Hence, $\mathcal{R}'' = \mathcal{R}_b^2 \cdot \mathcal{X}_1 \cdot \mathcal{Z}_1 \cdot \mathcal{L}'' \cdot \mathcal{Z}_2 \cdot \mathcal{X}_2 \cdot \mathcal{R}_a^2$ is a half-integral $B(S_{\ell_4}^3)\text{-}A(S_0^3)$ linkage of order $2w$ which is internally disjoint from $(\mathcal{S}^3, \mathcal{P}^3)$. By [Lemma 3.3](#), there is a $\text{end}(\mathcal{R}'')\text{-start}(\mathcal{R}'')$ linkage \mathcal{R}^4 of order w inside $D(\mathcal{R}'')$. Hence, \mathcal{R}^4 is a partial back-linkage of order w for $(\mathcal{S}^3, \mathcal{P}^3)$ which is internally disjoint from $(\mathcal{S}^3, \mathcal{P}^3)$.

By [Observation 8.8](#), $(\mathcal{S}^3, \mathcal{P}^3)$ contains as a subgraph a path of well-linked sets $(\mathcal{S}^4 = (S_0^4, S_1^4, \dots, S_{\ell_4}^4), \mathcal{P}^4)$ of width w and length ℓ_4 with $A(S_0^4) = \text{end}(\mathcal{R}^4)$ and $B(S_{\ell_4}^4) = \text{start}(\mathcal{R}^4)$. By definition, $(\mathcal{S}^4, (\mathcal{P}_0^4, \mathcal{P}_1^4, \dots, \mathcal{P}_{\ell_4-1}^4, \mathcal{R}^4))$ is a cycle of well-linked sets of width w and length ℓ . \square \square

Recall the outline of the next steps described in the paragraph before [Lemma 11.9](#). The previous lemma now allows us to choose the new forward linkage \mathcal{H} discussed there as we no longer need to worry about the new linkage not intersecting \mathcal{V} often enough. If that were the case, we could reduce to the previous lemma to obtain a cycle of well-linked sets. Before we prove this in detail in [Lemma 11.14](#) below we first define a variant of q -horizontal webs with less strict requirements that we call *semi-web*.

The reason for this is that when we choose the linkage \mathcal{H} so that it is weakly minimal with respect to \mathcal{V} then we may only get a semi-web. But at least we can preserve the horizontal property as proved below.

Definition 11.11. Let \mathcal{H}, \mathcal{V} be two linkages. We say that $(\mathcal{H}, \mathcal{V})$ is a *c-horizontal semi-web* if \mathcal{H} can be decomposed as $\mathcal{H} = \mathcal{H}^1 \cdot \mathcal{H}^2 \cdot \dots \cdot \mathcal{H}^c$ and \mathcal{V} can be decomposed as $\mathcal{V} = \mathcal{V}^1 \cdot \mathcal{V}^2 \cdot \dots \cdot \mathcal{V}^c$ such that $D(\mathcal{H}^i) \cap D(\mathcal{V}) \subseteq D(\mathcal{V}^{c-i+1} \cup \mathcal{V}^{c-i})$ (we set $\mathcal{V}^0 = \emptyset$ for simplicity).

We also need the following technical lemma.

Observation 11.12. Let $(\mathcal{H}, \mathcal{V})$ be a 3-horizontal semi-web in a digraph D . Then $D((\mathcal{H}, \mathcal{V}))$ contains a linkage $\mathcal{P} = \mathcal{P}^1 \cdot \mathcal{P}^2$ of order $|\mathcal{H}|$ such that $(\mathcal{P}, \mathcal{V})$ is a 2-horizontal semi-web where \mathcal{P} is weakly $|\mathcal{H}|$ -minimal with respect to \mathcal{V} . Further, $\text{start}(\mathcal{P}) = \text{start}(\mathcal{H})$, $\text{end}(\mathcal{P}) = \text{end}(\mathcal{H})$ and $\text{start}(\mathcal{P}^2) \subseteq V(\mathcal{H}^2)$.

Proof. Let \mathcal{P} be a $\text{start}(\mathcal{H})$ - $\text{end}(\mathcal{H})$ -linkage of order $|\mathcal{H}|$ which is \mathcal{V} -minimal. By Observation 3.6, \mathcal{P} is also weakly $|\mathcal{H}|$ -minimal with respect to \mathcal{V} .

Since $(\mathcal{H}, \mathcal{V})$ is a 3-horizontal semi-web, there is no path from \mathcal{H}^1 to \mathcal{H}^3 in \mathcal{V} . As $\text{start}(\mathcal{P}) = \text{start}(\mathcal{H}) = \text{start}(\mathcal{H}^1)$ and $\text{end}(\mathcal{P}) = \text{end}(\mathcal{H}) = \text{end}(\mathcal{H}^3)$, every path in \mathcal{P} must intersect some vertex of \mathcal{H}^2 . Let Y be the set containing, for each $P \in \mathcal{P}$, the last vertex of P which is also in \mathcal{H}^2 .

Decompose \mathcal{P} into $\mathcal{P}^1 \cdot \mathcal{P}^2 = \mathcal{P}$ such that $\text{start}(\mathcal{P}^2) = Y$. Let $\mathcal{V}^1 \cdot \mathcal{V}^2 \cdot \mathcal{V}^3 := \mathcal{V}$ be a decomposition of \mathcal{V} witnessing that $(\mathcal{H}, \mathcal{V})$ is a 3-horizontal semi-web.

By construction, we have that \mathcal{V}^2 and \mathcal{V}^3 do not intersect \mathcal{P}^2 . Hence $(\mathcal{P}, \mathcal{V})$ is a 2-horizontal semi-web where \mathcal{P} is weakly- $|\mathcal{H}|$ minimal with respect to \mathcal{V} . \square \square

Using the previous lemma we can construct a horizontal web $(\mathcal{H}', \mathcal{V}')$ from a semi-web $(\mathcal{H}, \mathcal{V})$ such that \mathcal{H}' is weakly $|\mathcal{H}|$ -minimal with respect to \mathcal{V} or find large $\mathcal{H}' \subseteq \mathcal{H}$ and $\mathcal{V}' \subseteq \mathcal{V}$ which are internally disjoint and satisfy some extra conditions specified in the next lemma. This is useful in the last result of this section where we finally construct the 2-horizontal web we are looking for unless we already find a cycle of well-linked sets while constructing the 2-horizontal web.

We define

$$\text{h}_{11.13}(h_1, h_2) := 2(h_2 - 1) + h_1, \quad \text{h}_{11.13}$$

$$\text{v}_{11.13}(h, h_1, v_1, h_2, v_2) := (v_2 - 1) \cdot 2 \binom{h}{h_2} + (v_1 - 1) \cdot \binom{h}{h_1} - 1. \quad \text{v}_{11.13}$$

Note that $\text{v}_{11.13}(h, h_1, v_1, h_2, v_2) \in 2^{1 \uparrow \uparrow \text{poly}^3(h, h_1, v_1, h_2, v_2)}$.

Lemma 11.13. Let $(\mathcal{H}, \mathcal{V})$ be a 3-horizontal semi-web such that $h := |\mathcal{H}| \geq \text{h}_{11.13}(h_1, h_2)$ and $v := |\mathcal{V}| \geq \text{v}_{11.13}(h, h_1, v_1, h_2, v_2)$. Then $(\mathcal{H}, \mathcal{V})$ contains one of the following:

(W1) a 2-horizontal web $(\mathcal{H}', \mathcal{V}')$ where $D(\mathcal{H}') \subseteq D(\mathcal{H} \cup \mathcal{V})$, $\mathcal{V}' \subseteq \mathcal{V}$, $|\mathcal{H}'| \geq h_1$, \mathcal{H}' is weakly h -minimal with respect to \mathcal{V} and $|\mathcal{V}'| \geq v_1$, or

(W2) some linkage $\mathcal{H}' \subseteq D(\mathcal{H} \cup \mathcal{V})$ of order h_2 and some linkage $\mathcal{V}' \subseteq \mathcal{V}$ of order v_2 such that \mathcal{H}' is internally disjoint from \mathcal{V}' . Additionally, $\text{start}(\mathcal{H}') \subseteq \text{start}(\mathcal{H})$ and $\text{end}(\mathcal{H}') \subseteq V(\mathcal{H}^2)$, or $\text{start}(\mathcal{H}') \subseteq V(\mathcal{H}^2)$ and $\text{end}(\mathcal{H}') \subseteq \text{end}(\mathcal{H})$.

Proof. By Observation 11.12, $(\mathcal{H}, \mathcal{V})$ contains a linkage \mathcal{P} of order $|\mathcal{H}|$ such that $(\mathcal{P}, \mathcal{V})$ is a 2-horizontal semi-web where \mathcal{P} is weakly $|\mathcal{H}|$ -minimal with respect to \mathcal{V} . In particular, we can decompose \mathcal{P} as $\mathcal{P} = \mathcal{P}^1 \cdot \mathcal{P}^2$ and we can decompose \mathcal{V} as $\mathcal{V} = \mathcal{V}^1 \cdot \mathcal{V}^2$ such that \mathcal{P}^2 is disjoint from \mathcal{V}^2 . Additionally, $\text{start}(\mathcal{P}) = \text{start}(\mathcal{H})$, $\text{end}(\mathcal{P}) = \text{end}(\mathcal{H})$ and $\text{start}(\mathcal{P}^2) \subseteq V(\mathcal{H}^2)$.

For each $V_j \in \mathcal{V}$ and each $1 \leq i \leq 2$ let $\mathcal{X}_j^i \subseteq \mathcal{P}^i$ be the paths of \mathcal{P}^i which intersect V_j and let $\mathcal{Y}_j^i \subseteq \mathcal{P}^i$ be the paths of \mathcal{P}^i which are disjoint from V_j . Let $\mathcal{M} \subseteq \mathcal{V}$ be the set of paths $V_j \in \mathcal{V}$ for which some i exists such that $|\mathcal{Y}_j^i| \geq h_2$. Let $\mathcal{N} = \mathcal{V} \setminus \mathcal{M}$.

Case 1: $|\mathcal{M}| \leq (v_2 - 1) \cdot 2 \binom{|\mathcal{H}|}{h_2}$.

Hence, $|\mathcal{N}| \geq |\mathcal{V}| - (v_2 - 1) \cdot 2 \binom{|\mathcal{H}|}{h_2} \geq (v_1 - 1) \cdot \binom{|\mathcal{H}|}{h_1} + 1$. For each $V_j \in \mathcal{N}$ let $\mathcal{X}_j = \{P^1 \cdot P^2 \in \mathcal{P} \mid P^1 \in \mathcal{X}_j^1 \text{ and } P^2 \in \mathcal{X}_j^2\}$. Since $\mathcal{X}_j^i \cup \mathcal{Y}_j^i = \mathcal{P}^i$ and $|\mathcal{Y}_j^i| < h_2$ for all $V_j \in \mathcal{N}$ and all $1 \leq i \leq 2$, we have that $|\mathcal{X}_j| \geq |\mathcal{H}| - 2 \cdot (h_2 - 1) \geq h_1$ for every $V_j \in \mathcal{N}$.

There are at most $\binom{|\mathcal{H}|}{h_1}$ distinct linkages $\mathcal{H}' \subseteq \mathcal{P}$ of order h_1 . By the pigeon-hole principle, there is some $\mathcal{V}' \subseteq \mathcal{V}$ of order v_1 for which some $\mathcal{H}' \subseteq \mathcal{P}$ of order h_1 exists such that $\mathcal{X}_j \supseteq \mathcal{H}'$ for all $V_j \in \mathcal{V}'$. Hence, $(\mathcal{H}', \mathcal{V}')$ is a 2-horizontal web with $|\mathcal{H}'| = h_1$, $|\mathcal{V}'| = v_1$ and \mathcal{H}' is weakly $|\mathcal{H}'$ -minimal with respect to \mathcal{V}' , satisfying (W1).

Case 2: $|\mathcal{M}| \geq (v_2 - 1) \cdot 2 \binom{|\mathcal{H}|}{h_2} + 1$.

By the pigeon-hole principle, there is some $i \in \{1, 2\}$ and some $\mathcal{H}' \subseteq \mathcal{P}^i$ of order h_2 for which there is a set $\mathcal{V}' \subseteq \mathcal{V}$ of order v_2 such that $\mathcal{Y}_j^i \supseteq \mathcal{H}'$ for all $V_j \in \mathcal{V}'$. Hence, \mathcal{H}' is a linkage of order h_2 which is internally disjoint from \mathcal{V}' .

If $i = 1$, then $\text{start}(\mathcal{H}') \subseteq \text{start}(\mathcal{P}) = \text{start}(\mathcal{H})$ and $\text{end}(\mathcal{H}') \subseteq \text{end}(\mathcal{P}^1) \subseteq V(\mathcal{H}^2)$.

If $i = 2$, then $\text{start}(\mathcal{H}') \subseteq \text{start}(\mathcal{P}^2) \subseteq V(\mathcal{H}^2)$ and $\text{end}(\mathcal{H}') \subseteq \text{end}(\mathcal{P}^2) = \text{end}(\mathcal{H})$.

Hence, (W2) holds. \square

\square

In the last lemma of this section we use the results established so far to construct a weakly minimal 2-horizontal web unless we already find a cycle of well-linked sets during the construction.

We define

$$\begin{aligned} w_{11.14}(h, w, \ell) &:= \textcolor{blue}{h}_{11.13}(h, \textcolor{red}{q}_{11.10}(w, \ell)) + 2\textcolor{red}{r}_{11.10}(w, \ell), & \text{w}_{11.14} \\ \ell_{11.14}(w, \ell) &:= 3(\ell'_{11.10}(w, \ell) + 1) - 1, & \text{ell}_{11.14} \\ r_{11.14}(h, w) &:= \textcolor{blue}{v}_{11.13}(\textcolor{blue}{h}_{11.13}(h, \textcolor{red}{q}_{11.10}(w, \ell)), h, v, \textcolor{red}{q}_{11.10}(w, \ell), 2\textcolor{red}{r}_{11.10}(w, \ell)), & \text{r}_{11.14} \\ m_{11.14}(h, w) &:= \textcolor{blue}{h}_{11.13}(h, \textcolor{red}{q}_{11.10}(w, \ell)). & \text{m}_{11.14} \end{aligned}$$

Observe that $w_{11.14}(h, w, \ell) \in 2^{2 \uparrow \uparrow \text{poly}^{25}(h, w, \ell)}$, $\ell_{11.14}(w, \ell) \in 2^{3 \uparrow \uparrow \text{poly}^{25}(w, \ell)}$, $r_{11.14}(h, w, \ell, v) \in 2^{3 \uparrow \uparrow \text{poly}^{26}(h, w, \ell, v)}$ and $m_{11.14}(h, w, \ell) \in 2^{2 \uparrow \uparrow \text{poly}^{25}(h, w, \ell)}$.

Lemma 11.14. Let w, ℓ, h, v be integers, let $(\mathcal{S}, \mathcal{P})$ be a strict path of well-linked sets of length $\ell' \geq \ell_{11.14}(w, \ell)$ and width $w' = w_{11.14}(h, w, \ell)$ with a back-linkage \mathcal{R} of order $r \geq r_{11.14}(h, w, \ell, v)$ intersecting $(\mathcal{S}, \mathcal{P})$ cluster by cluster. Then, $D(\mathcal{S} \cup \mathcal{P} \cup \mathcal{R})$ contains one of the following:

(H1) a cycle of well-linked sets of width w and length ℓ , or

(H2) an $m_{11.14}(h, w, \ell)$ -horizontally minimal 2-horizontal web $(\mathcal{H}, \mathcal{V})$ where $\mathcal{V} \subseteq \mathcal{R}$, $|\mathcal{H}| \geq h$ and $|\mathcal{V}| \geq v$.

Proof. Let $h_1 = \ell'_{11.10}(w, \ell)$, $h_2 = \textcolor{red}{q}_{11.10}(w, \ell)$, $h_3 = \textcolor{blue}{h}_{11.13}(h, h_2)$, $w_1 = \textcolor{red}{r}_{11.10}(w, \ell)$ and $v_1 = 2w_1$. Let $(S_0, S_1, \dots, S_{\ell'}) := \mathcal{S}$ and $(\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{\ell'-1}) := \mathcal{P}$. To simplify notation, set $\mathcal{P}_{\ell'} := \emptyset$.

Let \mathcal{H} be an $A(S_0)$ - $B(S_{\ell'})$ -linkage of order h_3 in $(\mathcal{S}, \mathcal{P})$. By Lemma 8.4, such a linkage exists.

Let $t_i = (i-1)(h_1+1)$ for each $i \in \{1, 2, 3, 4\}$. Decompose \mathcal{H} into $\mathcal{H} = \mathcal{H}^1 \cdot \mathcal{H}^2 \cdot \mathcal{H}^3$, where \mathcal{H}^i is the sublinkage of \mathcal{H} contained in $(\mathcal{S}, \mathcal{P})[t_i, t_{i+1}-1]$. Decompose \mathcal{R} iteratively as follows. Let $X_0 = \text{start}(\mathcal{R})$ and let $X_3 = \text{end}(\mathcal{R})$. For each $0 \leq i \leq 3$ let Y_i be the vertices of \mathcal{R} such that for each $R \in \mathcal{R}$ the last intersection of R with $D(S_{t_{i+1}-1} \cup \mathcal{P}_{t_{i+1}-1})$ lies in Y_i . Let X_i be the successors of the vertices of Y_i in \mathcal{R} . For each $1 \leq i \leq 3$, let \mathcal{R}^i be the X_{i-1} - X_i sublinkage of order $|\mathcal{R}|$ in \mathcal{R} . To simplify notation, set \mathcal{R}^0 as the linkage containing only the vertices of $\text{start}(\mathcal{R}^1)$.

Because \mathcal{R} intersects $(\mathcal{S}, \mathcal{P})$ cluster by cluster, we have $D(\mathcal{H}^i) \cap D(\mathcal{R}) \subseteq D(\mathcal{R}^{4-i} \cup \mathcal{R}^{3-i})$ for all $1 \leq i \leq 3$. Hence, $(\mathcal{H}, \mathcal{R})$ is a 3-horizontal semi-web. Further, $|\mathcal{H}| = h_3 = \textcolor{blue}{h}_{11.13}(h, h_2)$ and $|\mathcal{R}| \geq \textcolor{blue}{v}_{11.13}(h_3, h, v, h_2, v_1)$. By Lemma 11.13, we have two cases.

Case 1: (W1) holds. That is, $(\mathcal{H}, \mathcal{R})$ contains a 2-horizontal web $(\mathcal{H}', \mathcal{V}')$ where $D(\mathcal{H}') \subseteq D(\mathcal{H} \cup \mathcal{R})$, $\mathcal{V}' \subseteq \mathcal{R}$, $|\mathcal{H}'| \geq h$, $|\mathcal{V}'| \geq v$ and \mathcal{H}' is weakly $|\mathcal{H}'$ -minimal with respect to \mathcal{V}' . This satisfies (H2).

Case 2: (W2) holds. That is, there is some $\mathcal{H}' \subseteq D(\mathcal{H} \cup \mathcal{R})$ of order h_2 and some $\mathcal{V}' \subseteq \mathcal{R}$ of order v_1 such that \mathcal{H}' is internally disjoint from \mathcal{V}' . Additionally, $\text{start}(\mathcal{H}') \subseteq \text{start}(\mathcal{H})$ and $\text{end}(\mathcal{H}') \subseteq V(\mathcal{H}^2)$, or $\text{start}(\mathcal{H}') \subseteq V(\mathcal{H}^2)$ and $\text{end}(\mathcal{H}') \subseteq \text{end}(\mathcal{H})$. We assume, without loss of generality, that $\text{start}(\mathcal{H}') \subseteq \text{start}(\mathcal{H})$ holds. The other case follows analogously by considering \mathcal{H}^3 instead of \mathcal{H}^1 .

We show that every path in \mathcal{H}' intersects $D(S_i \cup \mathcal{P}_i)$ for every $0 \leq i \leq h_1$.

Assume towards a contradiction that this is not the case. As $\text{start}(\mathcal{H}') \subseteq A(S_0)$ and $\text{end}(\mathcal{H}') \subseteq V(S_{h_1+d})$ for some integer d , there is some $0 \leq j \leq h_1$ for which some $H \in \mathcal{H}'$ exists such that H is an $A(S_0)$ - $V(S_{h_1+d})$ path which does not intersect any vertex of $D(S_j \cup \mathcal{P}_j)$.

Let $j_0 < j$ be the largest index smaller than j such that H intersects $D(S_{j_0} \cup \mathcal{P}_{j_0})$. Similarly, let $j_1 > j$ be the smallest index larger than j such that H intersects $D(S_{j_1} \cup \mathcal{P}_{j_1})$. Since there is no $V(S_{j_0} \cup \mathcal{P}_{j_0})$ - $V(S_{j_1} \cup \mathcal{P}_{j_1})$ path which is disjoint from $D(S_j \cup \mathcal{P}_j)$ inside $(\mathcal{S}, \mathcal{P})$, H contains a $V(S_{j_0} \cup \mathcal{P}_{j_0})$ - $V(S_{j_1} \cup \mathcal{P}_{j_1})$ path H_x which is internally disjoint from $(\mathcal{S}, \mathcal{P})$ as a subpath. Hence, H_x is a subpath of $\mathcal{V}' \subseteq \mathcal{R}$. This, however, implies that H_x is a jump of length $j_1 - j_0 > 1$, a contradiction to the initial assumption that \mathcal{R} intersects $(\mathcal{S}, \mathcal{P})$ cluster by cluster.

Let $S'_{h_1} = S_{h_1} \cup D(\mathcal{P}_{h_1})$. Clearly, the digraph $(\mathcal{S}' := (S_0, S_1, \dots, S_{h_1-1}, S'_{h_1}), \mathcal{P}' := (\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{h_1-1}))$ is a path of well-linked sets of width $w' \geq 2w_1$ and length h_1 . By Lemma 11.1, there is a partial back-linkage \mathcal{R}' of order w_1 for $(\mathcal{S}, \mathcal{P})[0, h_1]$ (and, hence, for $(\mathcal{S}', \mathcal{P}')$ as well) such that $D(\mathcal{R}') \cap (\mathcal{S}', \mathcal{P}') \subseteq D(\mathcal{V}' \cup \text{start}(\mathcal{R}') \cup \text{end}(\mathcal{R}'))$. As \mathcal{V}' is weakly m -minimal with respect to $(\mathcal{S}', \mathcal{P}')$, the back-linkage \mathcal{R}' is also weakly m -minimal with respect to $(\mathcal{S}', \mathcal{P}')$.

As every path in \mathcal{H}' intersects $D(S_j \cup \mathcal{P}_j)$ for every $0 \leq j \leq h_1$, by Lemma 11.10 the digraph $D(\mathcal{S}' \cup \mathcal{P}' \cup \mathcal{R}' \cup \mathcal{H}')$ contains a cycle of well-linked sets of width w and length ℓ , implying (H1). \square \square

11.3 Finding a cycle of well-linked sets inside a 2-horizontal web

In this section we complete the proof of our main results. The remaining open case is (H2) of Lemma 11.14. That is, we already have constructed a 2-horizontal web $(\mathcal{H}, \mathcal{V})$ as specified in the lemma. The idea is to construct a new path of well-linked sets on the first half of \mathcal{H} and then use the other half of \mathcal{H} to complete the cycles.

We start with a few simple observations used in the sequel.

Observation 11.15. Let D be a digraph and let $(\mathcal{P}, \mathcal{Q})$ be a web where $|\mathcal{P}| = |\mathcal{Q}|$. Then $\text{start}(\mathcal{P})$ is well-linked to $\text{end}(\mathcal{Q})$ in $D(\mathcal{P} \cup \mathcal{Q})$.

Proof. Let $A \subseteq \text{start}(\mathcal{P})$ and $B \subseteq \text{end}(\mathcal{Q})$ such that $|A| = |B|$. Since $(\mathcal{P}, \mathcal{Q})$ is a web, there is no A - B separator of size less than $|A|$, as such a separator must hit $|A|$ paths of \mathcal{P} . Hence, by Theorem 3.2 there is an A - B -linkage of size $|A|$ in $D(\mathcal{P} \cup \mathcal{Q})$. Thus, $\text{start}(\mathcal{P})$ is well-linked to $\text{end}(\mathcal{Q})$. \square \square

Observation 11.16. Let $(\mathcal{H}, \mathcal{V})$ be a 2-horizontal web. Define $\mathcal{H}^2 := \{H_i^2 \mid H_i \in \mathcal{H}\}$ and $\mathcal{V}^1 := \{V_i^1 \mid V_i \in \mathcal{V}\}$. Then, $\text{start}(\mathcal{H}^2)$ is well-linked to $\text{end}(\mathcal{V}^1)$ inside $D(\mathcal{H}^2 \cup \mathcal{V}^1)$.

Proof. By definition $(\mathcal{H}^2, \mathcal{V}^1)$ is a web. Hence, by Observation 11.15 $\text{start}(\mathcal{H}^2)$ is well-linked to $\text{end}(\mathcal{V}^1)$ inside $D(\mathcal{H}^2 \cup \mathcal{V}^1)$. \square \square

We also need the following lemma from [KK15].

Lemma 11.17 ([KK15, Corollary 5.12]). Let H be a digraph and let \mathcal{Q}^* be a linkage in H and let $\mathcal{Q} \subseteq \mathcal{Q}^*$ be a linkage of order q . Let $P \subseteq H$ be a path intersecting every path in \mathcal{Q} . Let $c \geq 0$ be such that for every edge $e \in E(P) \setminus E(\mathcal{Q}^*)$ there are no c pairwise vertex disjoint paths in $H - e$ from P_1 to P_2 , where $P = P_1 \cdot e \cdot P_2$. For all $s, r \geq 0$, if $q \geq (r + c) \cdot s$, then

(R1) there is an s -segmentation $\mathcal{Q}' \subseteq \mathcal{Q}$ of P with respect to \mathcal{Q}^* or

(R2) a $(2, r)$ -split $((P_1, P_2), \mathcal{Q}'')$ of (P, \mathcal{Q}) with respect to \mathcal{Q}^* .

To construct a path of well-linked sets in the first half of \mathcal{H} we suitably adapt the construction in the proof of [KK15, Lemma 5.15], as we require somewhat different properties of the segmentation we obtain. In particular, we need the paths of the segmentation contain the end of the linkage \mathcal{H}^1 in our horizontal web $(\mathcal{H}^1 \cdot \mathcal{H}^2, \mathcal{V})$ as this allows us to continue from the last cluster of the path of well-linked sets we construct to \mathcal{H}^2 without intersecting the new path of well-linked sets.

We define

$$\mathbf{q}'_{11.18}(q, c, z) := (q(c+1))^{2^z} (2^{2^z-1})$$

and note that $\mathbf{q}'_{11.18}(q, c, z) \in 2^{2 \uparrow \uparrow \text{poly}^2(q, c, z)}$.

Lemma 11.18. Let $c, x, y, q, q' \geq 0$ and $p \geq x$ be integers. Let $\mathcal{W} = (\mathcal{P}, \mathcal{Q})$ be a (p, q') -web where \mathcal{P} is weakly c -minimal with respect to \mathcal{Q} . If $q' \geq \mathbf{q}'_{11.18}(q, c, xy)$, then there is some $\mathcal{Q}' \subseteq \mathcal{Q}$ such that \mathcal{W} contains one of the following

(S1) a (y, q) -split $(\mathcal{P}', \mathcal{Q}')$ of $(\mathcal{P}, \mathcal{Q})$ or

(S2) an (x, q) -segmentation $(\mathcal{P}', \mathcal{Q}')$ of $(\mathcal{P}, \mathcal{Q})$. Additionally, $\text{end}(\mathcal{P}') \subseteq \text{end}(\mathcal{P})$.

Proof. For all $0 \leq i \leq xy$ we define values q_i inductively as follows. We set $q_{xy} := q$ and $q_{i-1} := q_i \cdot (q_i + c)$. We first show that $q_0 \leq \mathbf{q}'_{11.18}(q, c, xy)$.

Claim 1. $q_i \leq (q(c+1))^{2^{xy-i}} (2^{2^{xy-i}-1})$ for all $0 \leq i \leq xy$

Proof. Clearly $q_{xy} = q \leq q(c+1)$. Assume the inequality holds from xy to $i > 0$. By definition of q_{i-1} we obtain

$$\begin{aligned} q_{i-1} &= q_i \cdot (q_i + c) \\ &\leq (q(c+1))^{2^{xy-i}} (2^{2^{xy-i}-1}) \cdot ((q(c+1))^{2^{xy-i}} (2^{2^{xy-i}-1}) + c) \\ &= ((q(c+1))^{2^{xy-i}} (2^{2^{xy-i}-1}))^2 + (q(c+1))^{2^{xy-i}} (2^{2^{xy-i}-1})c \\ &= (q(c+1))^{2^{xy-(i-1)}} (2^{2^{xy-(i-1)}-2}) + (q(c+1))^{2^{xy-i}} (2^{2^{xy-i}-1})c \\ &\leq (q(c+1))^{2^{xy-(i-1)}} (2^{2^{xy-(i-1)}-2} + 2^{2^{xy-i}-1}) \\ &= (q(c+1))^{2^{xy-(i-1)}} (2^{2^{xy-i}-1} \cdot 2^{2^{xy-i}-1} + 2^{2^{xy-i}-1}) \\ &= (q(c+1))^{2^{xy-(i-1)}} (2^{2^{xy-i}-1} \cdot (2^{2^{xy-i}-1} + 1)) \\ &\leq (q(c+1))^{2^{xy-(i-1)}} (2^{2^{xy-i}-1} \cdot 2^{2^{xy-i}}) \\ &= (q(c+1))^{2^{xy-(i-1)}} (2^{2^{xy-(i-1)}-1}). \end{aligned}$$

□

Hence, by **Claim 1**, we have $q_0 \leq q' \leq (q(c+1))^{2^{xy}} (2^{2^{xy}-1}) = \mathbf{q}'_{11.18}(q, c, xy)$.

For each $0 \leq i \leq xy$ we construct a tuple $\mathcal{M}_i := (\mathcal{P}^i, \mathcal{Q}^i, \mathcal{S}_{\text{seg}}^i, \mathcal{S}_{\text{split}}^i)$, satisfying all of the following

(M1) $\mathcal{Q}^i \subseteq \mathcal{Q}^*$ is a linkage of order q_i and and $\mathcal{P}^i \subseteq \mathcal{P}$ is a linkage such that, if there is no $P \in \mathcal{S}_{\text{split}}^i \setminus \mathcal{S}_{\text{seg}}^i$ with $\text{end}(P) \subseteq \text{end}(\mathcal{P})$, then $|\mathcal{P}^i \cup \mathcal{S}_{\text{seg}}^i| \geq x$,

(M2) $(\mathcal{S}_{\text{split}}^i, \mathcal{Q}^i)$ is a (y_i, q_i) -split of $(\mathcal{P}, \mathcal{Q})$ and

(M3) $(\mathcal{S}_{\text{seg}}^i, \mathcal{Q}^i)$ is an (x_i, q_i) -segmentation of $(\mathcal{P}, \mathcal{Q})$ where $\text{end}(\mathcal{S}_{\text{seg}}^i) \subseteq \text{end}(\mathcal{P})$.

Furthermore, $(\mathcal{P}^i, \mathcal{Q}^i)$ has linkedness c and $V(\mathcal{P}^i) \cap V(\mathcal{S}_{\text{seg}}^i) = \emptyset$, for all i . Recall that, in particular, this means that the paths in $\mathcal{S}_{\text{split}}^i$ are the subpaths of a single path in \mathcal{P} that is split by edges $e \in E(P) \setminus E(\mathcal{Q}^*)$.

We first set $\mathcal{P}^0 := \mathcal{P}$, $\mathcal{Q}^0 := \mathcal{Q}^*$, $\mathcal{S}_{\text{seg}}^0 := \emptyset$, $\mathcal{S}_{\text{split}}^0 := \emptyset$. Clearly, this satisfies the conditions **(M1)**, **(M2)** and **(M3)** defined above.

On step $i + 1 \geq 1$, we do the following. If $|\mathcal{S}_{\text{split}}^i| \geq y$ or if $|\mathcal{S}_{\text{seg}}^i| \geq x$, stop the construction. Otherwise, proceed as follows.

We first set $\mathcal{S}'_{\text{seg}} := \mathcal{S}_{\text{seg}}^i$. If there is no $P \in \mathcal{S}_{\text{split}}^i \setminus \mathcal{S}_{\text{seg}}^i$ such that $\text{end}(P) \subseteq \text{end}(\mathcal{P})$, we choose a path $P \in \mathcal{P}^i$ and set $\mathcal{S}'_{\text{split}} = \{P\}$ and $\mathcal{P}^{i+1} := \mathcal{P}^i \setminus \{P\}$. Clearly, $\text{end}(P) \subseteq \text{end}(\mathcal{P})$.

Otherwise, if there is some $P \in \mathcal{S}_{\text{split}}^i \setminus \mathcal{S}_{\text{seg}}^i$ with $\text{end}(P) \subseteq \text{end}(\mathcal{P})$, we set $\mathcal{S}'_{\text{split}} := \mathcal{S}_{\text{split}}^i$ and $\mathcal{P}^{i+1} := \mathcal{P}^i$.

Now, let $P \in \mathcal{S}'_{\text{split}} \setminus \mathcal{S}'_{\text{seg}}$ with $\text{end}(P) \subseteq \text{end}(\mathcal{P})$. We apply [Lemma 11.17](#) to P, \mathcal{Q}^i and \mathcal{Q}^* setting $r = s = q_{i+1}$ in the statement of the lemma. If **(R1)** holds and there is a q_{i+1} -segmentation $\mathcal{Q}_1 \subseteq \mathcal{Q}^i$ of P with respect to \mathcal{Q}^* , we set

$$\mathcal{Q}^{i+1} := \mathcal{Q}_1, \quad \mathcal{S}_{\text{seg}}^{i+1} := \mathcal{S}_{\text{seg}}^i \cup \{P\} \quad \text{and} \quad \mathcal{S}_{\text{split}}^{i+1} := \mathcal{S}_{\text{split}}^i.$$

Otherwise, **(R2)** holds and there is a $(2, q_{i+1})$ -split $((P_1, P_2), \mathcal{Q}_2)$ where $\mathcal{Q}_2 \subseteq \mathcal{Q}^i$. Then we set

$$\begin{aligned} \mathcal{Q}^{i+1} &:= \mathcal{Q}_2, \\ \mathcal{S}_{\text{seg}}^{i+1} &:= \mathcal{S}_{\text{seg}}^i \quad \text{and} \\ \mathcal{S}_{\text{split}}^{i+1} &:= (\mathcal{S}_{\text{split}}^i \setminus \{P\}) \cup \{P_1, P_2\}. \end{aligned}$$

If there is no $P \in |\mathcal{S}_{\text{split}}^{i+1}|$ with $\text{end}(P) \subseteq \text{end}(\mathcal{P})$, then we obtained a segmentation and so added a path to $\mathcal{S}_{\text{seg}}^{i+1}$. As **(M1)** holds for i , we have that $|\mathcal{P}^{i+1} \cup \mathcal{S}_{\text{seg}}^{i+1}| \geq x$ in this case.

It is easily verified that the conditions **(M1)**, **(M2)** and **(M3)** are maintained for \mathcal{P}^{i+1} , \mathcal{Q}^{i+1} , $\mathcal{S}_{\text{seg}}^{i+1}$ and $\mathcal{S}_{\text{split}}^{i+1}$. In particular, the linkedness c of $(\mathcal{P}^{i+1}, \mathcal{Q}^{i+1})$ is preserved as deleting or splitting paths cannot increase forward connectivity. This concludes the construction.

Note that in the construction, after every y steps, either **(R2)** holds after every application of [Lemma 11.17](#), and so we find a set $\mathcal{S}_{\text{split}}^i$ of size y or **(R1)** holds in at least one of the y steps, and so we add a path to $\mathcal{S}_{\text{seg}}^i$. Whenever this happens, we take a new path $P \in \mathcal{P}^i$ in the next iteration, which always exists because of **(M1)**.

Hence, in the construction above, in each step we either increase x_i and add the path P with $\text{end}(P) \subseteq \text{end}(\mathcal{P})$ to $\mathcal{S}_{\text{seg}}^i$ or we increase y_i . After at most $i \leq xy$ steps, either we have constructed a set $\mathcal{S}_{\text{seg}}^i$ of order x or a set $\mathcal{S}_{\text{split}}^i$ of order y .

Because **(M2)** holds, if we found a set $\mathcal{S}_{\text{split}}^i$ of order y , then we can choose any set $\mathcal{Q}' \subseteq \mathcal{Q}^i$ of order q and $(\mathcal{S}_{\text{split}}^i, \mathcal{Q}')$ satisfies **(S1)**.

If, instead, we get a set $\mathcal{S}_{\text{seg}} := \mathcal{S}_{\text{seg}}^i$ of order $x' \geq x$, then, by **(M3)**, $(\mathcal{S}_{\text{seg}}, \mathcal{Q}^i)$ is an (x, q) -segmentation of $(\mathcal{P}, \mathcal{Q})$ such that $\text{end}(\mathcal{S}_{\text{seg}}) \subseteq \text{end}(\mathcal{P})$, satisfying **(S2)**.

Finally, it is easily seen that if \mathcal{W} is well-linked then so is $(\mathcal{S}_{\text{split}}^i, \mathcal{Q}')$ (in case **(S1)** holds) or $(\mathcal{S}_{\text{seg}}, \mathcal{Q}^i)$ (in case **(S2)** holds). \square

In the next lemma we use the split or segmentation obtained from the previous lemma to construct a folded ordered web or an ordered web. This allows us to apply the results of

Section 10. We define

$$\begin{aligned} \mathbf{q}_{11.19}(q'', x) &:= (q'')^{2^{2x-1}} + 1, & \mathbf{q}_{11.19} \\ \mathbf{q}'_{11.19}(q, c, x, y) &:= \mathbf{q}'_{11.18}(q, c, 2(2x-1)(y-1)y), \text{ and} & \mathbf{q}'_{11.19} \\ \mathbf{p}_{11.19}(x) &:= 2x - 1. & \mathbf{p}_{11.19} \end{aligned}$$

Note that $\mathbf{q}_{11.19}(q'', x) \in 2^{2\uparrow\uparrow\text{poly}^2(q'', x)}$ and $\mathbf{q}'_{11.19}(q, c, x, y) \in 2^{2\uparrow\uparrow\text{poly}^5(q, c, x, y)}$.

Lemma 11.19. Let $c, x, y, q'', q' \geq 0$, $q \geq \mathbf{q}_{11.19}(q'', x)$ and $p \geq \mathbf{p}_{11.19}(x)$ be integers. Let $\mathcal{W} = (\mathcal{P}, \mathcal{Q})$ be a (p, q') -web where \mathcal{P} is weakly c -minimal with respect to \mathcal{Q} . If $q' \geq \mathbf{q}'_{11.19}(q, c, x, y)$, then there is some $\mathcal{Q}' \subseteq \mathcal{Q}$ and some \mathcal{P}' such that $D(\mathcal{P}') \subseteq D(\mathcal{P})$ and W contains one of the following

- (O1) a folded ordered (q, y) -web $(\mathcal{Q}', \mathcal{P}')$, or
- (O2) an ordered (x, q'') -web $(\mathcal{P}', \mathcal{Q}')$ such that $\text{end}(\mathcal{P}') \subseteq \text{end}(\mathcal{P})$.

Proof. Let $x_1 = 2x - 1$.

We apply Lemma 11.18 to \mathcal{W} . If (S1) holds, then by Observation 10.6 we obtain a folded ordered (q, y) -web and (O1) holds. Otherwise, (S2) holds and we obtain an (x_1, q) -segmentation $(\mathcal{P}^1, \mathcal{Q}')$ of $(\mathcal{P}, \mathcal{Q})$ such that $\text{end}(\mathcal{P}^1) \subseteq \text{end}(\mathcal{P})$.

Recursively define q_i by $q_{x_1} = q''$ and $q_i = \mathbf{len}_{3.1}(q_{i+1}, q_{i+1})$. We show that $q_i \leq (q'')^{2^{x_1-i}} + 1$ for all $1 \leq i \leq x_1$. Clearly, $q_{x_1} = q'' \leq (q'')^{2^0} + 1$. By definition, for an arbitrary $1 \leq i \leq x_1$ we have

$$\begin{aligned} q_i &= (q_{i+1} - 1)^2 + 1 \\ &\leq ((q'')^{2^{x_1-i-1}} + 1 - 1)^2 + 1 \\ &= (q'')^{2^{x_1-i}} + 1. \end{aligned}$$

Hence, $q \geq q_1$.

We construct a set $\mathcal{Q}'' \subseteq \mathcal{Q}'$ as follows. Let $\{P_1^1, P_2^1, \dots, P_{x_1}^1\} = \mathcal{P}^1$ be an arbitrary ordering of the paths in \mathcal{P}^1 . Set $\mathcal{Q}_1 = \mathcal{Q}'$ and then iterate from 2 to x_1 , constructing a set \mathcal{Q}_i of size q_i .

On step $i \leq x_1$, consider the ordering \preceq_{i-1} of the paths in \mathcal{Q}_{i-1} according to their occurrence along P_{i-1}^1 . By Theorem 3.1, there is a $\mathcal{Q}_i \subseteq \mathcal{Q}_{i-1}$ of order at least q_i such that P_i^1 intersects \mathcal{Q}_i in order or in reverse with respect to \preceq_{i-1} . Since $\mathcal{Q}_i \subseteq \mathcal{Q}_{i-1}$, we have that each $P_j^1 \in \mathcal{P}^1$ with $j \leq i$ also intersects \mathcal{Q}_i in order or in reverse with respect to \preceq_{i-1} .

After x_1 steps, we set $\mathcal{Q}'' := \mathcal{Q}_{x_1}$. By construction, there is an ordering \preceq of \mathcal{Q}'' such that each $P_i^1 \in \mathcal{P}^1$ intersects \mathcal{Q}'' in order or in reverse with respect to \preceq .

By the pigeon-hole principle, there is some $\mathcal{P}^2 \subseteq \mathcal{P}^1$ of order at least x such that every path in \mathcal{P}^2 intersects the paths of \mathcal{Q}'' in the same order. Hence, $(\mathcal{P}^2, \mathcal{Q}'')$ is an ordered (x, q'') -web where $\text{end}(\mathcal{P}^2) \subseteq \mathcal{P}$, satisfying (O2). \square \square

The previous lemma leaves us with two cases to consider when constructing a cycle of well-linked sets. If Lemma 11.19 returns a folded ordered web then we can use the tools established so far to construct a cycle of well-linked sets directly.

In the second case of Lemma 11.19 we obtain an ordered web $(\mathcal{H}', \mathcal{V}')$ which ends on $\text{end}(\mathcal{H}^1)$. We can use the final subpaths of the paths in \mathcal{H}' following the last path of \mathcal{V}' to construct a linkage to \mathcal{H}^2 which is disjoint from the path of order-linked sets constructed from the ordered web. To construct a back-linkage we use the paths in \mathcal{V}^1 . The paths in \mathcal{V}^1 , however, may intersect the path of order-linked sets we constructed which we need to avoid somehow. The key to solving this problem is the weak minimality of \mathcal{H} with respect to \mathcal{V} . We use the paths of \mathcal{V}^2 to construct a large linkage that contradicts the weak minimality of \mathcal{H} .

We first define

$$\begin{aligned} \mathbf{h}_{11.20}(w_2) &:= \mathbf{p}_{11.19}\left((w_2)^2 - 1\right), & \mathbf{h}_{11.20} \\ \mathbf{v}'(w_2, \ell_2) &:= \mathbf{q}_{11.19}\left(((w_2\ell_2 - 1)\binom{(w_2)^2 - 1}{w_2} w_2! + 1\right) \\ &\quad \cdot \mathbf{\ell}_{6.12}(w_2, (w_2)^2 - 1), (w_2)^2 - 1), \\ \mathbf{v}_{11.20}(w_1, \ell_1, w_2, \ell_2, c) &:= \mathbf{q}'_{11.19}\left(\mathbf{h}_{10.7}(w_1) + \mathbf{v}'(w_2, \ell_2), c, (w_2)^2 - 1, \mathbf{v}_{10.7}(w_1, \ell_1)\right). & \mathbf{v}_{11.20} \end{aligned}$$

Note that $\mathbf{h}_{11.20}(w_2) \in O((w_2)^2)$ and $\mathbf{v}_{11.20}(w_1, \ell_1, w_2, \ell_2, c) \in 2^{5\uparrow\uparrow\text{poly}^{15}(w_1, \ell_1, w_2, \ell_2, c)}$.

Lemma 11.20. Let $(\mathcal{H}, \mathcal{V})$ be an (h, v) -web where \mathcal{H} is weakly c -minimal with respect to \mathcal{V} . If $|\mathcal{H}| \geq \mathbf{h}_{11.20}(w_2)$ and $|\mathcal{V}| \geq \mathbf{v}_{11.20}(w_1, \ell_1, w_2, \ell_2, c)$, then one of the following is true:

(E1) $(\mathcal{H}, \mathcal{V})$ contains a path of well-linked sets $(\mathcal{S} = (S_0, S_1, \dots, S_{\ell_1}), \mathcal{P})$ of width w_1 and length ℓ_1 . Additionally, there is a $\text{start}(\mathcal{V})$ - $\text{end}(\mathcal{V})$ -linkage $\mathcal{L} = \mathcal{L}_1 \cdot \mathcal{L}_2 \cdot \mathcal{L}_3$ of order w_1 using only arcs of \mathcal{V} such that \mathcal{L}_2 is an $A(S_0)$ - $B(S_{\ell_1})$ -linkage and both \mathcal{L}_1 and \mathcal{L}_3 are internally disjoint from $(\mathcal{S}, \mathcal{P})$.

(E2) There is some $\mathcal{V}' \subseteq \mathcal{V}$ such that $(\mathcal{H}, \mathcal{V}')$ contains a uniform path of w_2 -order-linked sets $(\mathcal{S} = (S_0, S_1, \dots, S_{\ell_2}), \mathcal{P})$ of width w_2 and length ℓ_2 for which there are linkages $\mathcal{L}_1, \mathcal{L}_2$ such that

(L1) \mathcal{L}_1 is a $B(S_{\ell_2})$ - $\text{end}(\mathcal{H})$ -linkage of order w_2 inside \mathcal{H} which is internally disjoint from \mathcal{V}' and from $(\mathcal{S}, \mathcal{P})$, and

(L2) $\mathcal{L}_2 \subseteq \mathcal{V}'$ is a linkage of order $\ell_2 + 1$ where for each $L_{2,j} \in \mathcal{L}_2$ there is some $0 \leq i \leq \ell_2$ such that $A(S_i) \subseteq V(L_{2,j})$ and $V(L_{2,j}) \cap V(\mathcal{S}, \mathcal{P}) \subseteq V(S_i)$.

Proof. We define $h_2 = (w_2)^2 - 1$, $h_1 = \mathbf{v}_{10.7}(w_1, \ell_1)$, $\ell_4 = w_2\ell_2$, $\ell_3 = \mathbf{\ell}_{6.12}(w_2, h_2)$, $t_1 = (\ell_4 - 1)\binom{h_2}{w_2} w_2! + 1$, $v_2 = t_1\ell_3$, $v_1 = \mathbf{h}_{10.7}(w_1) + \mathbf{q}_{11.19}(v_2)$.

By Lemma 11.19, there is some $\mathcal{H}^1 \subseteq \mathcal{H}$ such that one of the following cases hold:

Case 1: (O1) holds.

That is, there is a sublinkage $\mathcal{V}^1 \subseteq \mathcal{V}$ for which $(\mathcal{V}^1, \mathcal{H}^1)$ is a folded ordered (v_1, h_1) -web. As $v_1 \geq \mathbf{h}_{10.7}(w_1)$, by Lemma 10.7 $(\mathcal{V}^1, \mathcal{H}^1)$ contains a path of well-linked sets $(\mathcal{S}, \mathcal{P})$ of width w_1 and length ℓ_1 . Additionally, there is a $\text{start}(\mathcal{V}^1)$ - $\text{end}(\mathcal{V}^1)$ -linkage $\mathcal{L} = \mathcal{L}_1 \cdot \mathcal{L}_2 \cdot \mathcal{L}_3$ of order w_1 using only arcs of \mathcal{V}^1 such that \mathcal{L}_2 is an $A(S_0)$ - $B(S_{\ell_1})$ -linkage and both \mathcal{L}_1 and \mathcal{L}_3 are internally disjoint from $(\mathcal{S}, \mathcal{P})$. This immediately satisfies (E1).

Case 2: (O2) holds.

That is, $(\mathcal{H}^1, \mathcal{V})$ contains an ordered (h_2, v_2) -web $(\mathcal{H}^2, \mathcal{V}^1)$ such that $\text{end}(\mathcal{H}^2) \subseteq \text{end}(\mathcal{H}^1)$.

Decompose \mathcal{H}^2 into $\mathcal{H}^2 = \mathcal{Q} \cdot \mathcal{L}'_1$ such that \mathcal{L}'_1 is internally disjoint from \mathcal{V}^1 and $\text{start}(\mathcal{L}'_1) \subseteq V(\mathcal{V}^1)$. Since \mathcal{L}'_1 is internally disjoint from \mathcal{V}^1 , we have that $(\mathcal{Q}, \mathcal{V}^1)$ is also an ordered (h_2, v_2) -web.

Let $(V_1^1, V_2^1, \dots, V_{v_2}^1) := \mathcal{V}^1$ be an ordering of \mathcal{V}^1 witnessing that $(\mathcal{Q}, \mathcal{V}^1)$ is an ordered web. For each $1 \leq i \leq t_1$ let T_i be the routing temporal digraph of \mathcal{Q} through $G_i := (V_{(i-1)\ell_3+1}^1, V_{(i-1)\ell_3+2}^1, \dots, V_{i\ell_3}^1)$. As every path in \mathcal{V}^1 intersects every path in \mathcal{Q} , we have that $D_j(T_i)$ is unilateral for all $1 \leq i \leq t_1$ and all $1 \leq j \leq \ell_3$. Since $\ell(T_i) = \ell_3$, by Theorem 6.12 we have that every T_i contains a \mathbf{P}_{w_2} -routing φ_i over some $\mathcal{Q}_i \subseteq \mathcal{Q}$.

As there are at most $\binom{h_2}{w_2} w_2!$ distinct φ_i , by the pigeon-hole principle there is some $\mathcal{I} \subseteq \{1, \dots, t_1\}$ of size ℓ_4 such that $\varphi := \varphi_i = \varphi_j$ and $\mathcal{Q}' := \mathcal{Q}_i = \mathcal{Q}_j$ hold for all $i, j \in \mathcal{I}$.

For each $i \in \mathcal{I}$ let \mathcal{R}_i be the maximal $V(V_{(i-1)\ell_3+1}^1)$ - $V(V_{i\ell_3}^1)$ -linkage of order w_2 inside \mathcal{Q}' and let T'_i be the routing temporal digraph of \mathcal{R}_i through G_i . Note that φ induces a \mathbf{P}_{w_2} -routing ψ over \mathcal{R}_i in T'_i .

By Lemma 7.6 we have that $\text{start}(\mathcal{R}_i)$ is 1-order-linked to $\text{end}(\mathcal{R}_i)$ inside $D(\mathcal{R}_i) \cup D(G_i)$ for every $i \in \mathcal{I}$. For every two consecutive $i < j \in \mathcal{I}$ (that is, there is no $k \in \mathcal{I}$ with $i < k < j$) let \mathcal{R}'_i be the $\text{end}(\mathcal{R}_i)$ - $\text{start}(\mathcal{R}_j)$ -linkage of order w_2 in \mathcal{Q}' . We define

$$\begin{aligned}\mathcal{P} &:= (\mathcal{R}'_i \mid i \in \mathcal{I} \setminus \max(\mathcal{I})) , \\ (\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{w_2\ell_2-1}) &:= \mathcal{P}, \\ \mathcal{S} &:= (D(\mathcal{R}_i) \cup D(G_i) \mid i \in \mathcal{I}) , \\ (S_0, S_1, \dots, S_{w_2\ell_2}) &:= \mathcal{S},\end{aligned}$$

whereas the order of the elements of \mathcal{S} and \mathcal{P} is given by the order of $i \in \mathcal{I}$. Finally, we set $A(S_j) = \text{start}(\mathcal{R}_i)$ and $B(S_j) = \text{end}(\mathcal{R}_i)$ for every $0 \leq j \leq \ell_2$, where \mathcal{R}_i is the sublinkage of \mathcal{Q}' which is inside S_j .

By choice of \mathcal{R}_i , $(\mathcal{S}, \mathcal{P})$ is a uniform path of 1-order-linked sets of width w_2 and length $w_2\ell_2$. By Theorem 7.8, there is a uniform path of w_2 -order-linked sets $(\mathcal{S}' = (S'_0, S'_1, \dots, S'_{\ell_2}), \mathcal{P}' = (\mathcal{P}'_0, \mathcal{P}'_1, \dots, \mathcal{P}'_{\ell_2-1}))$ of length ℓ_2 and width w_2 inside $(\mathcal{S}, \mathcal{P})$. Additionally, for every $0 \leq i \leq \ell_2$ we have $S'_i \subseteq (\mathcal{S}, \mathcal{P})[iw_2, (i+1)w_2 - 1]$, $A(S'_i) \subseteq A(S_{iw_2})$ and $B(S'_i) \subseteq B(S_{(i+1)w_2-1})$, and for $0 \leq i < \ell_2$ we have $\mathcal{P}'_i \subseteq \mathcal{P}_{(i+1)(w_2-1)}$.

Let $\mathcal{L}_1 \subseteq \mathcal{L}'_1$ be the paths of \mathcal{L}'_1 satisfying $\text{start}(\mathcal{L}_1) = B(S'_{\ell_2})$. Since \mathcal{L}'_1 is internally disjoint from \mathcal{V}^1 by construction and $\text{end}(\mathcal{L}_1) \subseteq \text{end}(\mathcal{H})$, we have that \mathcal{L}_1 is a $B(S'_{\ell_2})$ - $\text{end}(\mathcal{H})$ -linkage of order w_2 which is internally disjoint from \mathcal{V}^1 , satisfying (L1).

For each $0 \leq i \leq \ell_2$ let $L_i \in \mathcal{V}^1$ be the path of \mathcal{V}^1 which intersects $A(S_i)$. By construction of \mathcal{S} , each path in \mathcal{V}^1 intersects at most one $A(S_j)$. Since $S'_i \subseteq (\mathcal{S}, \mathcal{P})[iw_2, (i+1)w_2 - 1]$, we have that L_i intersects exactly one $A(S'_i)$ as well. Let $\mathcal{L}_2 = \{L_i \mid 0 \leq i \leq \ell_2\}$.

By construction we have $V(\mathcal{V}^1) \cap V(\mathcal{P}'_i) \subseteq \text{start}(\mathcal{P}'_i) \cup \text{end}(\mathcal{P}'_i)$ for every $\mathcal{P}'_i \in \mathcal{P}'$. Further, L_j intersects only the cluster S'_i . Hence, $\mathcal{L}_2 \subseteq \mathcal{V}^1$ is a linkage of order $\ell_2 + 1$ and for each $L_j \in \mathcal{L}_2$ there is some $0 \leq i \leq \ell_2$ such that $A(S'_i) \subseteq V(L_j)$ and $V(L_j) \cap V(\mathcal{S}', \mathcal{P}') \subseteq V(S'_i)$, satisfying (L2) and so (E2) as well. \square \square

We are now ready to prove the last intermediate step required to complete the proof of Theorem 11.22 from which Theorem 1.1 then follows.

We define

$$\begin{aligned}\ell'(w, c) &:= 4w + (\ell - 1)(c + 1) - 2, \\ w'(w, \ell) &:= \mathbf{q}_{11.10}(w, \ell) + 2\mathbf{r}_{11.10}(w, \ell), \\ v'(w) &:= \mathbf{h}_{11.20}(4w), \\ v''(w, \ell, c) &:= \mathbf{v}_{11.20}(w', \ell', \mathbf{r}_{11.10}(w, \ell), \mathbf{h}_{11.20}(4w), \ell'(w, c), c) - 1, \\ v'''(w, \ell, c) &:= v''(w, \ell, c)(v'(w) + 1), \\ \mathbf{h}_{11.21}(w, \ell) &:= 2\mathbf{r}_{11.10}(w, \ell) + v'(w) + 1, \\ \mathbf{v}_{11.21}(w, \ell, c) &:= v'''(w, \ell, c) \left(\frac{\mathbf{h}_{11.21}(w, \ell)}{2\mathbf{r}_{11.10}(w, \ell)} \right) + 1 + \mathbf{h}_{11.21}(w, \ell)c.\end{aligned}\tag{h_{11.21}}\tag{v_{11.21}}$$

Observe that $\mathbf{h}_{11.21}(w, \ell) \in O(w^2\ell^2)$ and $\mathbf{v}_{11.21}(w, \ell, c) \in 2^{8\uparrow\uparrow\text{poly}^{25}(w, \ell, c)}$.

Lemma 11.21. Let $(\mathcal{H}, \mathcal{V})$ be a 2-horizontal web where \mathcal{H} is weakly c -minimal with respect to \mathcal{V} . If $|\mathcal{H}| \geq \mathbf{h}_{11.21}(w, \ell)$ and $|\mathcal{V}| \geq \mathbf{v}_{11.21}(w, \ell, c)$, then $D((\mathcal{H}, \mathcal{V}))$ contains a cycle of well-linked sets of length ℓ and width w .

Proof. We define $z_1 = c + 1$, $q_1 = \mathbf{q}_{11.10}(w, \ell)$, $\ell_4 = \ell - 1$, $w_4 = 2w$, $w_3 = 2w_4$, $w_2 = \mathbf{r}_{11.10}(w, \ell)$, $w_1 = q_1 + 2w_2$, $\ell_3 = 2(w_4 - 1)$, $\ell_2 = \ell_3 + z_1\ell_4$, $\ell_1 = \ell'_{11.10}(w, \ell)$, $m_1 = 2w_2$, $m_2 = \mathbf{h}_{11.20}(w_3)$, $h_1 = m_1 + m_2 + 1$, $v_2 = \mathbf{v}_{11.20}(w_1, \ell_1, w_3, \ell_2, c)$, $v_1 = (v_2 - 1)(m_2 + 1)\binom{h_1}{m_1} + 1$. Observe that $\mathbf{h}_{11.21}(w, \ell) = m_1 + m_2 + 1$ and $\mathbf{v}_{11.21}(w, \ell, c) = v_1 + \mathbf{h}_{11.21}(w, \ell)c$.

Decompose \mathcal{H} into $\mathcal{H} = \mathcal{H}^1 \cdot \mathcal{H}^2$ and decompose \mathcal{V} into $\mathcal{V}^1 \cdot \mathcal{V}^2$ such that $\text{start}(\mathcal{V}^2) \subseteq V(\mathcal{H}^2)$ and \mathcal{V}^2 is internally disjoint from \mathcal{H}^2 . Hence, $V(\mathcal{V}^2) \cap V(\mathcal{H}) \subseteq V(\mathcal{H}^1) \cup \text{start}(\mathcal{V}^2)$. By [Definition 11.7](#), such a decomposition exist. For each $H_i \in \mathcal{H}$ we write H_i^1 for the subpath of H_i in \mathcal{H}^1 and H_i^2 for the subpath of H_i in \mathcal{H}^2 .

Let $\mathcal{V}' \subseteq \mathcal{V}$ be the paths $V_j \in \mathcal{V}$ for which there is some $H_i \in \mathcal{H}$ such that V_j contains a subpath V'_j with $\text{start}(V'_j) \in V(H_i^1)$ and $\text{end}(V'_j) \in V(H_i^2)$. Let $\mathcal{V}^* = \mathcal{V} \setminus \mathcal{V}'$. Since $(\mathcal{H}, \mathcal{V})$ is weakly c -minimal 2-horizontal web, for each $H_i \in \mathcal{H}$ there are at most c paths in \mathcal{V}' which contain a subpath as above. Hence, $|\mathcal{V}'| \leq |\mathcal{H}|c$ and thus $|\mathcal{V}^*| \geq v_1$. Further, \mathcal{V}^* satisfies the following by construction.

(V1) Let $Q_i^1 \cdot Q_i^2 \in \mathcal{V}^*$ be an arbitrary decomposition of a path in \mathcal{V}^* . If Q_i^2 intersects some $H_j^2 \in \mathcal{H}^2$, then Q_i^1 is disjoint from H_j^1 .

Let $\mathcal{V}^3 \subseteq \mathcal{V}^1$ and $\mathcal{V}^4 \subseteq \mathcal{V}^2$ be the subpaths of \mathcal{V}^1 and \mathcal{V}^2 such that $V(\mathcal{V}^3) \subseteq V(\mathcal{V}^*)$ and $V(\mathcal{V}^4) \subseteq V(\mathcal{V}^*)$. Note that $\mathcal{V}^* = \mathcal{V}^3 \cdot \mathcal{V}^4$.

For each subpath V_i of \mathcal{V}^* which contains some path of \mathcal{V}^4 as a subpath define a linear ordering \preceq_{V_i} of \mathcal{H}^1 such that $H_a \preceq_{V_i} H_b$ if V_i does not intersect H_b before the first intersection of V_i with H_a . As every $V_i^4 \in \mathcal{V}^4$ intersects every $H_a \in \mathcal{H}^1$, every \preceq_{V_i} is a linear ordering. Define $\mathcal{M}(V_i)$ as the set of $m_1 + m_2$ maximal elements of \preceq_{V_i} and $\mathcal{N}(V_i)$ as the set of m_1 maximal elements of \preceq_{V_i} .

For each $1 \leq i \leq |\mathcal{V}^*|$, decompose $V_i \in \mathcal{V}^3 \cdot \mathcal{V}^4$ and construct a set \mathcal{M}_i iteratively as follows. Start with $V_i = Q_i^1 \cdot Q_i^2$ such that $Q_i^1 \in \mathcal{V}^3$ and $Q_i^2 \in \mathcal{V}^4$. Set $\mathcal{H}' = \mathcal{H}^2$ and set $\mathcal{M}_i = \emptyset$. Repeat the following steps until stopping.

1. Let $H_j \in \mathcal{H}$ be such that $\text{start}(Q_i^2) \in V(H_j^2)$.
2. If $H_j^1 \notin \mathcal{M}(Q_i^2)$, stop the construction.
3. Otherwise, set $\mathcal{H}' := \mathcal{H}' \setminus \{H_j^2\}$ and let $Q_i^3 \cdot Q_i^4 := Q_i^1$ such that $\text{start}(Q_i^4) \subseteq V(\mathcal{H}')$ and Q_i^4 is internally disjoint from \mathcal{H}' .
4. Set $Q_i^1 := Q_i^3$, $Q_i^2 := Q_i^4 \cdot Q_i^2$ and $\mathcal{M}_i := \mathcal{M}_i \cup \{H_j^1\}$.

By [\(V1\)](#), whenever we add some H_j^1 to \mathcal{M}_i in the construction above, then $H_j^1 \in \mathcal{M}(Q_i^4 \cdot Q_i^2)$ as well. Hence, $\mathcal{M}_i \subseteq \mathcal{M}(Q_i^2)$. The construction above stops at step 2 for every $V_i \in \mathcal{V}^*$ after at most $|\mathcal{M}(V_i)|$ iterations because $|\mathcal{M}(V_i)| \leq |\mathcal{H}|$, every path in \mathcal{V}^3 intersects every path in \mathcal{H}^2 , and $|\mathcal{M}_i|$ increases after each iteration.

Since $|\mathcal{V}^*| \geq (v_2 - 1) \binom{h_1}{m_1} \binom{h_1 - m_1}{m_2} + 1$, by the pigeon-hole principle there is some $\mathcal{Q}^* \subseteq \mathcal{V}^*$ of order v_2 such that $\mathcal{M}(Q_i^*) = \mathcal{M}(Q_j^*)$ and $\mathcal{N}' := \mathcal{N}(Q_i^*) = \mathcal{N}(Q_j^*)$ for every $Q_i^*, Q_j^* \in \mathcal{Q}^*$. We set $\mathcal{M}' = \mathcal{M}(Q_i^*) \setminus \mathcal{N}(Q_i^*)$ for some $Q_i^* \in \mathcal{Q}^*$. Decompose \mathcal{Q}^* into $\mathcal{Q}^* = \mathcal{Q}^1 \cdot \mathcal{Q}^2 \cdot \mathcal{Q}^M \cdot \mathcal{Q}^N$ such that $\mathcal{Q}^1 = \{Q_i^1 \mid Q_i \in \mathcal{Q}^*\}$, $\text{end}(\mathcal{Q}^2) \subseteq V(\mathcal{M}')$, $\text{end}(\mathcal{Q}^M) \subseteq V(\mathcal{N}')$, $\mathcal{Q}^1 \cdot \mathcal{Q}^2$ is internally disjoint from $\mathcal{M}' \cup \mathcal{N}'$, and \mathcal{Q}^M is internally disjoint from \mathcal{N}' . By choice of \mathcal{M}' and \mathcal{N}' , such a decomposition exists.

$\mathcal{P}^1 = \{H_j^1 \in \mathcal{H}^1 \setminus (\mathcal{N}' \cup \mathcal{M}') \mid H_j^2 \in \mathcal{H}'\}$ and $\mathcal{P}^2 = \{H_j^2 \mid H_j^1 \in \mathcal{P}^1\}$. By construction of \mathcal{H}' , for each $Q_i^1 \in \mathcal{Q}^1$ we have that Q_i^2 intersects H_j^1 , where $\text{end}(Q_i^1) \subseteq V(H_j^2)$ and $Q_i^1 \cdot Q_i^2 \in \mathcal{Q}^1 \cdot \mathcal{Q}^2$. Finally, $(\mathcal{M}', \mathcal{Q}^M)$ is a weakly c -minimal 1-horizontal web where $|\mathcal{M}'| = m_2$ and $|\mathcal{Q}^M| = v_2$. From [Lemma 11.20](#), we obtain two cases.

Case 1: [\(E1\)](#) holds.

That is, $(\mathcal{M}', \mathcal{Q}^M)$ contains a path of well-linked sets $(\mathcal{S} = (S_0, S_1, \dots, S_{\ell_1}), \mathcal{P})$ of width w_1 and length ℓ_1 . Additionally, there is a $\text{start}(\mathcal{Q}^M)$ - $\text{end}(\mathcal{Q}^M)$ -linkage $\mathcal{L} = \mathcal{L}_1 \cdot \mathcal{L}_2 \cdot \mathcal{L}_3$ of order $w_1 \geq q_1$ using only arcs of \mathcal{Q}^M such that \mathcal{L}_2 is an $A(S_0)$ - $B(S_\ell)$ -linkage and both \mathcal{L}_1 and \mathcal{L}_3 are internally disjoint from $(\mathcal{S}, \mathcal{P})$.

We construct a $B(S_{\ell_1})$ - $A(S_0)$ -linkage \mathcal{R} of order w_2 which is internally disjoint from \mathcal{L}_2 as follows. Take an $\text{end}(\mathcal{L}_2)$ - $\text{end}(\mathcal{N}')$ -linkage \mathcal{X}_1 in $D(\mathcal{N}' \cup \mathcal{Q}^N \cup \mathcal{L}_3)$. Take an $\text{end}(\mathcal{X}_1)$ - $\text{start}(\mathcal{L}_1)$ -linkage \mathcal{X}_2 in $D(\mathcal{H}^2 \cup \mathcal{Q}^1)$. Since both $(\mathcal{M}', \mathcal{Q}^M)$ and $(\mathcal{H}^2, \mathcal{Q}^1)$ are webs and $\text{end}(\mathcal{L}_3) \subseteq \text{end}(\mathcal{Q}^M) = \text{start}(\mathcal{Q}^N)$, by Observation 11.15 the linkages \mathcal{X}_1 and \mathcal{X}_2 exist.

As (E1) holds, the linkages \mathcal{L}_1 , \mathcal{L}_2 and \mathcal{L}_3 are pairwise internally disjoint. Since \mathcal{L}_2 is contained in $(\mathcal{M}', \mathcal{Q}^M)$, we have that \mathcal{L}_2 is internally disjoint from \mathcal{M}' and, hence, from \mathcal{X}_1 . Further, as \mathcal{L}_2 only uses arcs of \mathcal{Q}^M , we have that \mathcal{L}_2 is internally disjoint from \mathcal{X}_2 . Hence, $\mathcal{R}' = \mathcal{X}_1 \cdot \mathcal{X}_2 \cdot \mathcal{L}'_1$ is internally disjoint from \mathcal{L}_2 , where $\mathcal{L}'_1 \subseteq \mathcal{L}_1$ are the paths with $\text{start}(\mathcal{L}'_1) = \text{end}(\mathcal{X}^2)$.

Because \mathcal{L}_1 , \mathcal{L}_2 and \mathcal{L}_3 only use arcs of \mathcal{Q}^M and \mathcal{Q}^M is internally disjoint from \mathcal{N}' and from \mathcal{H}^2 , we have that \mathcal{L}_1 and \mathcal{L}_2 are internally disjoint from \mathcal{X}_1 and from \mathcal{X}_2 . Hence, \mathcal{R}' is a half-integral $B(S_{\ell_1})$ - $A(S_0)$ -linkage, as $\text{end}(\mathcal{L}_1) = \text{start}(\mathcal{L}_2) \supseteq A(S_0)$ and $\text{start}(\mathcal{L}_2) = \text{end}(\mathcal{L}_1) \supseteq B(S_{\ell_1})$. By Lemma 3.3, there is a $B(S_{\ell_1})$ - $A(S_0)$ -linkage \mathcal{R}'' of order w_2 inside $D(\mathcal{R}')$. Hence, by Lemma 11.10, $D((\mathcal{S}, \mathcal{P}) \cup \mathcal{R}'')$ contains a cycle of well-linked sets of width w and length ℓ .

Case 2: (E2) holds.

That is, there is some $\mathcal{Q}'' \subseteq \mathcal{Q}^M$ such that $(\mathcal{M}', \mathcal{Q}'')$ contains a uniform path of w_3 -order-linked sets $(\mathcal{S} = (S_0, S_1, \dots, S_{\ell_2}), \mathcal{P} = (\mathcal{P}_0, \mathcal{P}_1, \dots, \mathcal{P}_{\ell_2-1}))$ of width w_3 and length ℓ_2 for which there are linkages \mathcal{L}_1 and \mathcal{L}_2 satisfying (L1) and (L2).

Let $\mathcal{L}'_2 \subseteq \mathcal{L}_2$ be the paths of \mathcal{L}_2 satisfying $V(\mathcal{L}'_2) \cap \bigcup_{i=0}^{\ell_3} V(S_{2i}) \neq \emptyset$, let \mathcal{L}'_3 be the paths of \mathcal{Q}^2 such that $\text{end}(\mathcal{L}'_3) = \text{start}(\mathcal{L}'_2)$. Finally, let $\mathcal{Q}^4 \subseteq \mathcal{Q}^2$ be the paths satisfying $\text{end}(\mathcal{Q}^4) = \text{start}(\mathcal{L}_2)$ and let $\mathcal{Q}^3 \subseteq \mathcal{Q}^1$ be the paths satisfying $\text{end}(\mathcal{Q}^3) = \text{start}(\mathcal{L}'_3)$.

Claim 1. There are i, j with $j - i > \ell_4$ and $i \geq \ell_3 + 1$ for which some $\mathcal{Q}^5 \subseteq \mathcal{Q}^3$ of order w_5 exists such that \mathcal{Q}^5 is internally disjoint from $(\mathcal{S}, \mathcal{P})[i, j]$.

Proof. Assume towards a contradiction that for every $\ell_3 + 1 \leq i \leq j \leq \ell_2$ with $j - i > \ell_4$ and every $\mathcal{Q}^5 \subseteq \mathcal{Q}^3$ of order at least w_5 there is a path $Q_x^5 \in \mathcal{Q}^5$ which intersects some vertex of $(\mathcal{S}, \mathcal{P})[i, j]$.

For each $1 \leq k \leq z_1$ we construct sets $\mathcal{S}_k^a, \mathcal{S}_k^b \subseteq \mathcal{S}$, $\mathcal{O}_k^1 \subseteq \mathcal{Q}^3$ and bijections $f_{a,k} : \mathcal{O}_k^1 \rightarrow \mathcal{S}_k^a$ and $f_{b,k} : \mathcal{O}_k^1 \rightarrow \mathcal{S}_k^b$ as follows.

Start with empty $\mathcal{S}_0^a, \mathcal{S}_0^b, \mathcal{O}_0^1, f_{a,0}$ and $f_{b,0}$. Iterate from 1 to z_1 . On step k , choose some $Q_j^1 \in \mathcal{Q}^3 \setminus \mathcal{O}_{k-1}^1$ such that Q_j^1 intersects some $S_i \in \mathcal{S}$ in $(\mathcal{S}, \mathcal{P})[\ell_3 + 1 + wk, \ell_3 + 1 + w(k+1) - 1]$, and then set $\mathcal{O}_k^1 = \mathcal{O}_{k-1}^1 \cup \{Q_j^1\}$, $\mathcal{S}_k^a = \mathcal{S}_{k-1}^a \cup \{S_{i-1}\}$ and $\mathcal{S}_k^b = \mathcal{S}_{k-1}^b \cup \{S_i\}$. Further, define $f_{a,k}$ and $f_{b,k}$ as the functions satisfying $f_{a,k}(Q_x^i) = f_{a,k-1}(Q_x^i)$ for all $Q_x^i \in \mathcal{S}_{k-1}^a$, $f_{b,k}(Q_x^i) = f_{b,k-1}(Q_x^i)$ for all $Q_x^i \in \mathcal{S}_{k-1}^b$, $f_{a,k}(Q_j^1) = S_{i-1}$, and $f_{b,k}(Q_j^1) = S_i$.

Because $|\mathcal{O}_{k-1}^1| = k - 1$, we have $|\mathcal{Q}^3 \setminus \mathcal{O}_{k-1}^1| \geq w_5$. Hence, in every step k , there is some $Q_j^1 \in \mathcal{Q}^3 \setminus \mathcal{O}_{k-1}^1$ which intersects $(\mathcal{S}, \mathcal{P})[\ell_3 + 1 + w(k-1), \ell_3 + 1 + wk - 1]$. Further, $(\mathcal{S}, \mathcal{P})$ has length $\ell_2 = \ell_3 + (c+1)\ell_4$. Thus, we can construct such sets $\mathcal{S}_k^a, \mathcal{S}_k^b$ and \mathcal{O}_k^1 . Let $\mathcal{S}^a = \mathcal{S}_{z_1}^a$, $\mathcal{S}^b = \mathcal{S}_{z_1}^b$, $\mathcal{O}^1 = \mathcal{O}_{z_1}^1$, $f_a = f_{a,z_1}$ and $f_b = f_{b,z_1}$.

Let $X = V(\mathcal{O}^1) \cap V((\mathcal{S}, \mathcal{P})[\ell_3 + 1, \ell_2])$. We construct an X - $\text{end}(\mathcal{Q}^3)$ -linkage \mathcal{Z} of order z_1 as follows. For each $O_j^1 \in \mathcal{O}^1$ choose some $x \in X \cap f_b(O_j^1)$ and add the x - $\text{end}(\mathcal{O}^1) \subseteq \text{end}(\mathcal{Q}^3)$ subpath of O_j^1 to \mathcal{Z} .

Note that $|X| = |\mathcal{Z}| \geq z_1$. By choice of P_e^2 , $\text{end}(\mathcal{Z}) \subseteq V(P_e^2)$. Let a be the last arc of P_e^2 .

Construct a $V(P_e^1)$ - $V(P_e^2)$ -linkage \mathcal{F} of order c avoiding a as follows. For each $O_i^1 \in \mathcal{O}^1$ let $S_j = f_b(O_i^1)$ and let $(a_{j,1}, a_{j,2}, \dots, a_{j,w_3}) := A(S_j)$ and $(a_{j-1,1}, a_{j-1,2}, \dots, a_{j-1,w_3}) := A(S_{j-1})$ be

ordered according to the orders witnessing that $A(S_j)$ is w_3 -order-linked to $B(S_j)$ and $A(S_{j-1})$ is w_3 -order-linked to $B(S_{j-1})$. Let $L_i^2 \in \mathcal{L}_2$ be the path with $V(L_i^2) \cap V((\mathcal{S}, \mathcal{P})) \subseteq f_a(O_i^1) = S_{j-1}$ and let F_i be a $V(P_e^1)$ - $a_{j-1,1}$ path in $\mathcal{Q}^4 \cdot \mathcal{L}_2$.

The path of w_3 -order-linked sets $(\mathcal{S}, \mathcal{P})$ is contained within $D(\mathcal{M}' \cup \mathcal{Q}^M)$. By (L2), $\mathcal{L}_2 \subseteq \mathcal{Q}^M$ holds. Further, \mathcal{Q}^4 is contained inside \mathcal{Q}^2 . By choice of \mathcal{Q}^4 , every path in \mathcal{Q}^4 intersects P_e^1 . For each $O_i^1 \in \mathcal{O}^1$ there is some $L_2^i \in \mathcal{L}_2$ such that $A(S_j) \subseteq V(L_2^i)$, where $S_j = f_a(O_i^1)$. Hence, there is some $Q_i^4 \in \mathcal{Q}^4$ such that $Q_i^4 \cdot L_2^i$ contains a $V(P_e^1)$ - $a_{j-1,1}$ path as desired. Thus, the linkage \mathcal{F}_1 above exists.

Construct an $\text{end}(\mathcal{F}_1)$ - $\text{start}(\mathcal{Z}')$ -linkage \mathcal{F}_2 as follows. For each $O_i^1 \in \mathcal{O}^1$, let $S_j = f_b(O_i^1)$ and let $F_{4,i}$ be an $A(S_j)$ - x_i path in S_j , where $x_i \in \text{start}(\mathcal{Z}) \cap V(O_i^1)$. Let $\{a_{j,k}\} = \text{start}(F_{4,i})$. Let $F_{3,i}$ be an $a_{j-1,1}$ - $b_{j-1,k}$ path in S_{j-1} . As $\{b_{j-1,k}\}$ is a 1-shift of $\{a_{j-1,1}\}$ and $A(S_{j-1})$ is ℓ_3 -order-linked to $B(S_{j-1})$ in S_{j-1} , such a path $F_{3,i}$ exists. Now set $F_{2,i} = F_{3,i} \cdot P_{j-1,k} \cdot F_{4,i}$, where $P_{j-1,k} \in \mathcal{P}_{j-1}$ is the $b_{j-1,k}$ - $a_{j,k}$ path in \mathcal{P}_{j-1} .

Since $(\mathcal{S}, \mathcal{P})$ is a uniform path of w_3 -order-linked sets, each $F_{2,i}$ is a path. Let $\mathcal{F}_3 = \{F_{3,i} \mid 0 \leq i \leq c\}$ and $\mathcal{F}_4 = \{F_{4,i} \mid 0 \leq i \leq c\}$. As each S_j contains at most one path of $\mathcal{F}_3 \cup \mathcal{F}_4$, we have that \mathcal{F}_3 and \mathcal{F}_4 are two disjoint linkages inside $(\mathcal{S}, \mathcal{P})$. Hence, $\mathcal{F}_2 = \{F_{2,i} \mid 0 \leq i \leq c\}$ is a $\text{end}(\mathcal{F}_1)$ - $\text{start}(\mathcal{Z})$ -linkage of order $c+1$ as desired. Finally, let \mathcal{F}_5 be the $\text{start}(\mathcal{F}_1)$ - $\text{end}(\mathcal{F}_2)$ -linkage contained inside $D(\mathcal{F}_1 \cup \mathcal{F}_2)$. Since each path in one linkage intersects exactly one path in the other, we have that $|\mathcal{F}_5| = |\mathcal{F}_1|$.

Construct an $\text{end}(\mathcal{F}_5)$ - $V(P_e^2)$ -linkage \mathcal{F}_3 of order z_1 by following the corresponding paths of \mathcal{Z} until the first intersection with P_e^2 . By choice of \mathcal{Z} , this is possible.

If there is some path in $\mathcal{F} := \mathcal{F}_5 \cdot \mathcal{F}_3$ using a , we delete this path from \mathcal{F} . Hence, we obtain a $V(P_e^1)$ - $V(P_e^2)$ -linkage \mathcal{F} of order at least c inside $D(\mathcal{H}^1 \cup \mathcal{Q}^2 \cup \mathcal{Q}^M) - a$, contradicting the initial assumption that $(\mathcal{H}, \mathcal{V})$ is a weakly c -minimal 2-horizontal web. \square

By [Claim 1](#), there is some $\mathcal{Q}^5 \subseteq \mathcal{Q}^3$ of order w_3 and some $\ell_3 + 1 \leq i < j \leq \ell_2$ such that \mathcal{Q}^5 is internally disjoint from $(\mathcal{S}, \mathcal{P})[i, j]$ and $j - i \geq \ell_4 - 1$.

By [Lemma 8.3](#), the path of w_3 -order-linked sets $(\mathcal{S}, \mathcal{P})[i, j]$ contains a path of well-linked sets $(\mathcal{S}' = (S'_0, S'_1, \dots, S'_{\ell_4}), \mathcal{P}' = (\mathcal{P}'_0, \mathcal{P}'_1, \dots, \mathcal{P}'_{\ell_4-1}))$ of width w_3 and length ℓ_4 such that $A(S'_0) \subseteq A(S_i)$ and $B(S'_{\ell_4}) \subseteq B(S_j)$.

Construct a $B(S'_{\ell_4})$ - $A(S'_0)$ -linkage \mathcal{R} of order w_3 as follows.

By [Observation 7.4](#), there is a $B(S'_{\ell_4})$ - $B(S_{\ell_2})$ -linkage \mathcal{Z}_5 of order w_3 inside $(\mathcal{S}, \mathcal{P})$.

Let $\mathcal{L}'_1 \subseteq \mathcal{L}_1$ be the linkage satisfying $\text{start}(\mathcal{L}'_1) = \text{end}(\mathcal{Z}_5)$ and let $\mathcal{L}''_3 \subseteq \mathcal{L}'_3$ be the linkage satisfying $\text{start}(\mathcal{L}''_3) = \text{end}(\mathcal{Q}^5)$. Take an $\text{end}(\mathcal{L}'_1)$ - $\text{start}(\mathcal{L}''_3)$ -linkage \mathcal{X}_1 of order w_3 in $D(\mathcal{H}^2 \cup \mathcal{Q}^5)$. Because $(\mathcal{H}^2, \mathcal{Q}^5)$ is a web, and because $\text{end}(\mathcal{L}'_1) \subseteq \text{end}(\mathcal{H}^2) = \text{start}(\mathcal{H}^2)$ and $\text{start}(\mathcal{L}''_3) = \text{end}(\mathcal{Q}^5)$ hold, by [Observation 11.15](#) such a linkage \mathcal{X}_1 exists.

For each $i \in \{0, \dots, w_3 - 1\}$ let $X_{2,i}$ be a path inside $D(\mathcal{L}''_3 \cup \mathcal{L}'_2)$ which starts on $\text{start}(\mathcal{X}_1)$ and ends on $a_{2i,i} \in A(S_{2i})$, where $(a_{2i,0}, a_{2i,1}, \dots, a_{2i,w_3-1}) := A(S_{2i})$ is sorted according to the order witnessing that $A(S_{2i})$ is w_3 -order-linked to $B(S_{2i})$ inside S_{2i} . Let $\mathcal{X}_2 = \{X_{2,0}, X_{2,1}, \dots, X_{2,w_5-1}\}$. By choice of \mathcal{L}'_2 and of \mathcal{L}''_3 and because (L2) holds, such a linkage \mathcal{X}_2 exists.

Construct an $\text{end}(\mathcal{X}_2)$ - $A(S_{2(w_3-1)})$ -linkage \mathcal{X}_3 inside $(\mathcal{S}, \mathcal{P})$ as follows. Towards this end, we construct, for each $0 \leq i \leq w_3 - 1$, an $A(S_{2(i-1)})$ - $A(S_{2i})$ -linkage \mathcal{X}_3^i of order $i+1$. Start with $\mathcal{X}_3^0 := \{a_{0,0}\} \subseteq A(S_0)$.

On step $i \geq 1$, let $(b_{2(i-1),0}, b_{2(i-1),1}, \dots, b_{2(i-1),w_3-1}) := B(S_{2(i-1)})$ be the ordering of the set $B(S_{2(i-1)})$ witnessing that $A(S_{2(i-1)})$ is w_3 -order linked to $B(S_{2(i-1)})$. Let \mathcal{Y}_i be an $\text{end}(\mathcal{X}_3^{i-1})$ - B_i -linkage of order i in $S_{2(i-1)}$, where $B_i = \{b_{2(i-1),j} \in B(S_{2(i-1)}) \mid 1 \leq j \leq i\}$. Since $A(S_{2(i-1)})$ is w_3 -order-linked to $B(S_{2(i-1)})$ in $S_{2(i-1)}$ and $\text{end}(\mathcal{X}_3^{i-1})$ contains the minimal i elements of the corresponding ordering, such a linkage \mathcal{Y}_i exists.

Let \mathcal{Z}_i be a B_i - A_i the linkage of order $i + 1$ in $\mathcal{P}_{2(i-1)}$ such that $\text{end}(\mathcal{Y}_i) \subseteq \text{start}(\mathcal{Z}_i)$, where $A_i = \{a_{2i,j} \mid 1 \leq j \leq i + 1\}$. Since $(\mathcal{S}, \mathcal{P})$ is a uniform path of w_3 -order-linked sets, such a linkage \mathcal{Z}_i exists. Set $\mathcal{X}_3^i = \mathcal{X}_3^{i-1} \cdot \mathcal{Y}_i \cdot \mathcal{Z}_i$. Since $|\mathcal{Y}_i \cdot \mathcal{Z}_i| = i + 1$, \mathcal{X}_3^i is a $\text{start}(\mathcal{X}_3^i)$ - A_i -linkage of order $i + 1$ (recall that, by definition of the concatenation operation \cdot , the additional path in \mathcal{Z}_i which does not have a corresponding endpoint in \mathcal{Y}_i is simply added to the result of the concatenation).

After iterating all the steps above, we obtain an $\text{end}(\mathcal{X}_2)$ - $A(S_{2(w_3-1)})$ -linkage $\mathcal{X}_3 := \mathcal{X}_3^{w_3-1}$ of order w_3 as desired. By Lemma 7.7, $A(S_{2(w_3-1)})$ is w_3 -order-linked to $A(S_i) \supseteq A(S'_0)$ in $(\mathcal{S}, \mathcal{P})[2(w_3-1), i]$. As $\text{start}(\mathcal{X}_3)$ contains the minimal w_3 elements of $A(S_{2(w_3-1)})$, the set $A(S'_0)$ is an w_3 -shift of $A(S_{2(w_3-1)})$. Hence, there is an $\text{end}(\mathcal{X}_3)$ - $A(S'_0)$ -linkage \mathcal{X}_4 of order w_3 in $(\mathcal{S}, \mathcal{P})[2(w_3-1), i]$.

The concatenation $\mathcal{X}_2 \cdot \mathcal{X}_3 \cdot \mathcal{X}_4$ produces a half-integral $\text{start}(\mathcal{L}'_3)$ - $A(S'_0)$ -linkage of order w_3 . By Lemma 3.3 there is a $\text{start}(\mathcal{X}_2)$ - $A(S'_0)$ -linkage \mathcal{X}_5 of order w_4 inside $D(\mathcal{X}_2 \cup \mathcal{X}_3)$.

Let $\mathcal{X}_6 \subseteq \mathcal{Z}_5 \cdot \mathcal{L}'_1 \cdot \mathcal{X}_1$ be the linkage of order w_4 with $\text{end}(\mathcal{X}_6) = \text{start}(\mathcal{X}_5)$. We claim that $\mathcal{X}_6 \cdot \mathcal{X}_5$ is a half-integral $B(S'_{\ell_4})$ - $A(S'_0)$ -linkage of order w_4 .

Assume towards a contradiction that there is some $v \in V(\mathcal{Z}_5 \cdot \mathcal{L}'_1) \cap V(\mathcal{X}_1) \cap V(\mathcal{X}_5)$. Since $\mathcal{Z}_5 \cdot \mathcal{L}'_1$ is contained in $D(\mathcal{M}' \cup \mathcal{Q}^M)$ and \mathcal{X}_1 is contained inside $D(\mathcal{Q}^5 \cup \mathcal{H}^2)$, we have that $v \in V(\mathcal{Q}^5) \cap V(\mathcal{M}')$. Furthermore, v is not in $(\mathcal{S}, \mathcal{P})[0, 2(w_5-1)]$ as $\mathcal{Z}_5 \cdot \mathcal{L}'_1$ is disjoint from $(\mathcal{S}, \mathcal{P})[0, 2(w_5-1)]$ by construction. As $v \in V(\mathcal{X}_5)$ and $\mathcal{X}_3 \cdot \mathcal{X}_4$ is contained inside the path of order-linked sets $(\mathcal{S}, \mathcal{P})[0, 2(w_5-1)]$, we have that $v \in V(\mathcal{X}_2) \subseteq V(\mathcal{Q}^2)$ as well. This however implies that $v \in V(\mathcal{Q}^2) \cap V(\mathcal{Q}^5) = \text{start}(\mathcal{Q}^2)$. However, $\text{start}(\mathcal{Q}^2) \cap V(\mathcal{M}') = \emptyset$ by choice of \mathcal{Q}^2 , a contradiction to the previous observation that $v \in V(\mathcal{Q}^5) \cap V(\mathcal{M}')$. Hence, by Lemma 3.3, $D(\mathcal{L}_1 \cdot \mathcal{X}_1 \cdot \mathcal{X}_6)$ contains a $B(S'_{\ell_4})$ - $A(S'_0)$ -linkage \mathcal{R} of order w .

We show that $V(\mathcal{R}) \cap V((\mathcal{S}, \mathcal{P})[i, j]) \subseteq B(S'_{\ell_4}) \cup A(S'_0)$.

Because (L1) holds, we have that \mathcal{L}'_1 is internally disjoint from $(\mathcal{S}, \mathcal{P})$. By construction we have that $V(\mathcal{X}_1) \cap V((\mathcal{M}', \mathcal{Q}^2)) \subseteq V(\mathcal{Q}^5)$. By choice of \mathcal{Q}^5 we have that $V(\mathcal{Q}^5) \cap V((\mathcal{S}, \mathcal{P})[i, j]) = \emptyset$. Hence, \mathcal{X}_1 is disjoint from $(\mathcal{S}, \mathcal{P})[i, j]$.

The linkage \mathcal{X}_2 is contained in $D(\mathcal{L}'_3 \cdot \mathcal{L}'_2)$ and is thus disjoint from $(\mathcal{S}, \mathcal{P})[i, j]$ because (L2) holds and $i > 2(w_3 - 1)$.

The linkage \mathcal{X}_5 is contained in $(\mathcal{S}, \mathcal{P})[0, i]$ and is thus internally disjoint from $(\mathcal{S}, \mathcal{P})[i, j]$. Hence, \mathcal{X}_5 is also internally disjoint from $(\mathcal{S}, \mathcal{P})[i, j]$. This implies that $V((\mathcal{S}, \mathcal{P})[i, j]) \cap V(\mathcal{R}) \subseteq B(S'_{\ell_4}) \cup A(S'_0)$, as desired. Hence, $(\mathcal{S}', (\mathcal{P}'_0, \mathcal{P}'_1, \dots, \mathcal{P}'_{\ell_4-1}, \mathcal{R}))$ is a cycle of well-linked sets of width w and length ℓ , as desired. \square \square

We are now ready to prove our main theorems. We state our main result both in terms of cylindrical grids and in terms of cycle of well-linked sets as each may be useful in a different context.

We define

$$\begin{aligned} m'(w, \ell) &:= \mathbf{m}_{11.14}(\mathbf{h}_{11.21}(w, \ell), w, \ell), \\ \mathbf{w}'_{11.22}(w, \ell) &:= \mathbf{w}'_{11.6}(w, \mathbf{w}_{11.14}(\mathbf{h}_{11.21}(w, \ell), w, \ell)), & \mathbf{w}'_{11.22} \\ \mathbf{r}'_{11.22}(w, \ell) &:= \mathbf{r}_{11.14}(\mathbf{h}_{11.21}(w, \ell), w, \ell, \mathbf{v}_{11.21}(w, \ell, \mathbf{m}_{11.14}(\mathbf{h}_{11.21}(w, \ell), w, \ell))), & \mathbf{r}'_{11.22} \\ \ell'_{11.22}(w, \ell) &:= \ell'_{11.6}(w, \ell, \ell_{11.14}(w, \ell), \mathbf{r}_{11.22}(w, \ell)). & \ell'_{11.22} \end{aligned}$$

Observe that

$$\begin{aligned} \mathbf{w}'_{11.22}(w, \ell) &\in 2^{2 \uparrow \uparrow \text{poly}^{97}(\ell, w)}, \\ \mathbf{r}_{11.22}(w, \ell) &\in 2^{13 \uparrow \uparrow \text{poly}^{97}(\ell, w)} \text{ and} \\ \ell'_{11.22}(w, \ell) &\in 2^{14 \uparrow \uparrow \text{poly}^{97}(\ell, w)}. \end{aligned}$$

Theorem 11.22. Let w, ℓ be integers, let $(\mathcal{S} = (S_0, S_1, \dots, S_{\ell'}), \mathcal{P})$ be a strict path of well-linked sets of width w' and length ℓ' and let \mathcal{R} be a $B(S_{\ell'})$ - $A(S_0)$ linkage of order r . If $w' \geq w'_{11.22}(w, \ell)$, $r \geq r_{11.22}(w, \ell)$ and $\ell' \geq \ell'_{11.22}(w, \ell)$, then $D((\mathcal{S}, \mathcal{P}) \cup \mathcal{R})$ contains a cycle of well-linked sets of width w and length ℓ .

Proof. Assume, without loss of generality, that \mathcal{R} is weakly r -minimal with respect to $(\mathcal{S}, \mathcal{P})$ and that $r = r_{11.22}(w, \ell)$. If this is not the case, we just choose a $\text{start}(\mathcal{R})$ - $\text{end}(\mathcal{R})$ linkage of order $r_{11.22}(w, \ell) \leq |\mathcal{R}|$ which is $(\mathcal{S}, \mathcal{P})$ -minimal. By [Observation 3.6](#), such a linkage is also weakly $r_{11.22}(w, \ell)$ -minimal with respect to $(\mathcal{S}, \mathcal{P})$.

We define $h = h_{11.21}(w, \ell)$, $w_1 = w_{11.14}(h, w, \ell)$, $m = m_{11.14}(h, w, \ell)$, $v = v_{11.21}(w, \ell, m)$ and $\ell_1 = \ell_{11.14}(w, r)$. Observe that $w' \geq w'_{11.6}(w, w_1)$, $\ell' \geq \ell'_{11.6}(w, \ell, \ell_1, r)$ and $r \geq r_{11.14}(h, w, v)$.

Applying [Lemma 11.6](#) to $(\mathcal{S}, \mathcal{P})$ and \mathcal{R} yields two cases. If [\(C1\)](#) holds, then we obtain a cycle of well-linked sets of width w and length ℓ as desired. Otherwise, [\(C2\)](#) holds, and $D((\mathcal{S}, \mathcal{P}) \cup \mathcal{R})$ contains a path of well-linked sets $(\mathcal{S}', \mathcal{P}')$ of width w_1 and length ℓ_1 with a back-linkage \mathcal{R}' of order w_1 intersecting $(\mathcal{S}', \mathcal{P}')$ cluster by cluster such that $\mathcal{R}' \subseteq \mathcal{R}$. Note that \mathcal{R}' is also weakly r -minimal with respect to $(\mathcal{S}', \mathcal{P}')$.

Applying [Lemma 11.14](#) to $(\mathcal{S}', \mathcal{P}')$ and \mathcal{R}' yields two further cases. If [\(H1\)](#) holds, then we obtain a cycle of well-linked sets of width w and length ℓ as desired. Otherwise, [\(H2\)](#) holds, and we obtain a 2-horizontal (h, v) -web $(\mathcal{H}, \mathcal{V})$ such that \mathcal{H} is weakly m -minimal with respect to \mathcal{V} .

By [Lemma 11.21](#), $(\mathcal{H}, \mathcal{V})$ contains a cycle of well-linked sets of width w and length ℓ , as desired. \square

We define $\text{dtw}_{1.1}(w, \ell) := t_{10.9}(w'_{11.22}(w, \ell) + r_{11.22}(w, \ell), \ell'_{11.22}(w, \ell))$ and note that $\text{dtw}_{1.1}(w, \ell) \in 2^{21\uparrow\uparrow\text{poly}^{97}(w, \ell)}$. The next theorem is our main result stated in terms of cycles of well-linked sets.

Theorem 1.1. Let w, ℓ be integers. Every digraph D with $\text{dtw}(D) \geq \text{dtw}_{1.1}(w, \ell)$ contains a cycle of well-linked sets $(\mathcal{S}, \mathcal{P})$ of width w and length ℓ .

Proof. Let $r_1 = r_{11.22}(w, \ell)$, $w_1 = w'_{11.22}(w, \ell, r) + r$ and $\ell_1 = \ell'_{11.22}(w, \ell)$.

By [Theorem 10.9](#), D contains a path of well-linked sets $(\mathcal{S} = (S_0, S_1, \dots, S_{\ell_1}), \mathcal{P})$ of width w_1 and length ℓ_1 where $B(S_{\ell_1})$ is well-linked to $A(S_0)$ in D . Hence, there is a $B(S_{\ell_1})$ - $A(S_0)$ linkage \mathcal{R} of order r_1 in D . By [Theorem 11.22](#), $D(\mathcal{S} \cup \mathcal{P} \cup \mathcal{R})$ contains a cycle of well-linked sets $(\mathcal{S}', \mathcal{P}')$ of width w and length ℓ . \square

We close this section by stating our main result in terms of cylindrical grids. Define $\text{dtw}_{1.2}(k) := \text{dtw}_{1.1}(w_{9.3}(k), \ell_{9.3}(k))$. Note that $\text{dtw}_{1.2}(k) \in 2^{22\uparrow\uparrow\text{poly}^9(k)}$.

Theorem 1.2. Every digraph D with $\text{dtw}(D) \geq \text{dtw}_{1.2}(k)$ contains a cylindrical grid of order k as a butterfly minor.

Proof. By [Theorem 1.1](#), D contains a cycle of well-linked sets $(\mathcal{S}, \mathcal{P})$ of width $w_{9.3}(k)$ and length $\ell_{9.3}(k)$. By [Theorem 9.3](#), $(\mathcal{S}, \mathcal{P})$ contains a cylindrical grid of order k . \square

12 Younger's Conjecture and the Erdős-Pósa property for directed graphs

In this section we obtain an elementary bound on the function required by the Erdős-Pósa property for directed graphs. We say that a graph H has the Erdős-Pósa property if there exists

a function l_H such that every graph G contains either k disjoint copies of H as a minor or there exists a set $S \subset V(G)$ such that $G - S$ contains no H -minor. In [AKKW16] the Erdős-Pósa property has been generalised to directed graphs. In particular the authors show the following theorem.

Theorem 12.1. [AKKW16, Theorem 4.1] Let H be a strongly connected digraph. H has the Erdős-Pósa property for butterfly (topological) minors if, and only if, there is a cylindrical grid (wall) of order c of which H is a butterfly (topological) minor. Furthermore, for every fixed strongly connected digraph H satisfying these conditions and every k there is a polynomial time algorithm which, given a digraph D as input, either computes k disjoint (butterfly or topological) models of H in D or a set S of $\leq l_H(k)$ vertices such that $D - S$ does not contain a model of H .

The same authors also prove the following lemma, which we restate to make the bounds explicit.

Lemma 12.2 ([AKKW16, Lemma 4.2]). Let G be a directed graph with $\text{dtw}(G) \leq w$. For each strongly connected directed graph H , the graph G has either k disjoint copies of H as a topological minor, or contains a set T of at most $k \cdot (w + 1)$ vertices such that H is not a topological minor of $G - T$.

We can now prove the main result of this section.

Theorem 12.3. Let H be a directed graph. Let H be a digraph with the Erdős-Pósa property for butterfly minors and let c be the order of a minimal cylindrical grid of which H is a butterfly minor. Then for any digraph D and any natural number k either D contains k disjoint H -butterfly minors or a set S of at most $k(\text{dtw}_{1.1}(\mathbf{w}_{9.3}(kc), \ell_{9.3}(kc)) + 1)$.

Proof. If $\text{dtw}(D) \geq \text{dtw}_{1.1}(\mathbf{w}_{9.3}(kc), \ell_{9.3}(kc))$ then by Theorem 1.2 D contains a cylindrical grid of order kc and hence k disjoint copies of H as a butterfly minor. Otherwise, we can apply Lemma 12.2. \square \square

We note that when $H = \vec{\mathbf{K}}_2$ the previous theorem is equivalent to Younger's Conjecture, which asks whether for every integer $k \geq 0$ there exists a function $l_{\vec{\mathbf{K}}_2}(k)$ such that for every digraph D , either D has k vertex-disjoint directed circuits, or D can be made acyclic by deleting at most $l_{\vec{\mathbf{K}}_2}(k)$ vertices. This conjecture was settled by Reed, Robertson, Seymour and Thomas in [RRST96], but the function they obtained was non-elementary. With our Theorem 12.3 we obtain an elementary bound for $l_{\vec{\mathbf{K}}_2}$.

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