Universal Enveloping Algebras and the PBW Theorem

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§0 Preface

The following documents the work conducted on Universal Enveloping Algebras and the PBW Theorem from October 2023 to June 2024. I would like to thank Prof. Haynes Miller wholeheartedly for his patience and encouragement while guiding me on this journey in abstract algebra. I would also like to thank the previous students that worked on this project: Nishant Dhankhar for this helpful insights and Ali Tahboub for providing some of his Gröbner basis code. The following assumes that the reader is familiar with the paper *Beck Modules and Alternative Algebras* (https://arxiv.org/abs/2309.07962). Several detailed calculations and additional materials are provided in this document. As of now, I am a rising sophomore at MIT majoring in mathematics. Having only taken introductory algebra courses, much of the deeper ideas in this project has, regrettably, alluded my attention. It is in my hopes that in the future, someone, possibly me, will continue this project and fill the blanks in our knowledge.

§1 Linearizing Relations

In section 6 of *Beck Modules and Alternative Algebras*, we have that I, the kernel of the mapping from the tensor algebra to the universal enveloping algebra is generated by the images of relations $l_x l_x$, $r_x r_x$, $l_x r_x - r_x l_x$, $l_x l_y + r_y r_x$, $l_y r_x + r_y l_x - r_x l_y$, for all x, y. We set l_i, r_i, $1 \leq i \leq n$ to be the images of the standard basis elements. Thus, any l_x, r_x is expressed as a linear combination of them due to linearity of l and r. Instead of studying the images of the relations for all $S \to K^n$, where S is the set of variables in the relations, we can study the images of the relations for all $S \to \{l_1, \cdots, l_n, r_1, \cdots, r_n\}$. However, the ideal generated is now smaller. For example, from $l_x l_x = 0$, substituting $l_x = l_1 + l_2$ we have $l_1 l_1 + l_1 l_2 + l_2 l_1 + l_2 l_2 = 0 \implies l_1 l_2 + l_2 l_1 = 0$. This is not in the ideal generated by the relations mapped to the basis. We introduce the notion of "linearizing" relations, which adds sufficient and necessary relations such that we only need to look at the relations mapped to the basis.

Reducing Relations: Suppose we have a relation R, say with variables $v_1, \cdots, v_r$, these variables have maximum degree of $(d_1, \cdots, d_r)$. To linearize this relation, we conduct a series of steps. Each step, we choose a variable v_i where $d_i \neq 1$. Regarding R as a polynomial of v_i, we obtain the new relation $R(v_i + v_{r+1}) - R(v_i) - R(v_{r+1})$. The $v_i^{d_i}$ and $v_{r+1}^{d_i}$ terms cancel out, so our new degrees are $(d_1, \cdots, d_i - 1, \cdots, d_r, d_i - 1)$.

Example 1.1

Reducing $R = a^2 + ab + b^2$: regarding R as a polynomial of a, we have

$$R' = R(a + c) - R(a) - R(c) = ac + ca - b^2$$

Our degree terms change from $(2, 2)$ to $(1, 2, 1)$. Then, regarding R' as a polynomial of b, we have

$$R'' = R'(b + d) - R'(b) - R'(d) = -ac - ca - bd - db$$

To show that these steps terminate, define $D(R) = \sum_{i=1}^{r} 3^{d_i}$, this decreases every step.

Generating Linear Relations: Now, the degree of each variable is 1, but the relation is not necessarily linear, as in the example, $R'' = -ac - ca - bd - db$. We show that the homogeneous components are linear in all variables. Here, we outline the proof. We induct on the number of variables the relation is not linear in. Suppose R is not linear in v_i, then $R(v_i) + R(v_{r+1}) - R(v_i + v_{r+1})$ gives us the component of R that does not include v_i. By induction hypothesis, this relation's homogeneous components are linear in all variables. So, we are left with the component of R where all monomials include v_i, but then this is linear in v_i and we can use induction hypothesis again.

Homogeneous Components are Relations: Note that we will make the important assumption throughout this document that the copy of $\mathbb{Z}\backslash\{0\}$ in our ring K consists of units. We outline the proof that homogeneous components of any relation R are still relations. We induct on the number of homogeneous components. Thus, it suffices to show that any one homogeneous component is a relation. Note that the reduced versions of homogeneous components are disjoint. Thus, it follows that there is a homogeneous component of R, such that after reducing a finite number of times, we get the sum of several components linear in all variables. By the last paragraph, these linear components are all relations.

In other words, there is a homogeneous component of R that reduces to a relation after a finite number of times. It suffices to show that if a homogeneous component reduces to a relation, then it is a relation. This is obvious via substitution, e.g,

Example 1.2

Reducing $R = a^3$ we get $aab + aba + baa + abb + bab + bba$. If the latter is a relation, substituting b for a we have $6a^3$ is a relation, so a^3 is one too[a].

[a] using the fact that $\mathbb{Z}\backslash\{0\}$ embeds into units

If $\mathbb{Z}\backslash\{0\}$ embeds into units in K, then we may replace our initial relations with our final linear relations. Using the images of all $S \to A$ (the algebra we are studying), we have the same ideal that cuts out $U_V(A)$. This is because all relations follow from these linear ones. Moreover, the ideal is the same if we map using $S \to Basis(A)$ since relations are linear in all variables. In the case in the paper, K does not necessarily have this property. Luckily, we have the "quick fix" by appending the relations $l_il_j + l_jl_i$ and $r_ir_j + r_jr_i$.

§2 An Analysis of the $U_V(A)$ Structure by Relations

This section studies how the non-commutative partial derivative mapping of relations and images of relations in the tensor algebra work with each other. We build up this theory of relations based on the previous section's notions of "linearizing relations" and end with the theorem that there are no "radicals" in the relations. The relations are thus, in some sense, well-behaved.

§2.1 The Relations that Cut out a Variety

We know that a set of relations $R_0 \subset KMag(S)$ cut out a variety of algebras V. For now, we are using $S = \mathbb{N}$ as a countable set of symbols for relations. We claim that the full set of relations $R \subset KMag(S)$ can be expressed as the ideal generated by

$$\{\varphi(\omega) \mid \omega \in R_0, \varphi : KMag(S) \to KMag(S)\}$$

where φ is an algebra homomorphism. Let us write R' as the ideal generated by this set. We claim that $R' \subset R$. It suffices to show that any element of the form $\varphi(\omega)$ where $\omega \in R_0$ is in R. $\forall \phi : KMag(S) \to A$, it maps $\varphi(\omega)$ to $\phi(\varphi(\omega)) = (\phi \circ \varphi)(\omega)$ which is mapped to 0 since ω is a relation satisfied by A.

Then we find that $KMag(S)/R'$ is an algebra in the variety V because it satisfies the relations in R_0. Moreover, since the only relations it satisfies are R', then we have $R \subset R'$.

So, we have that $R' = R$. This provides us with a concrete description of the full set of relations satisfied by V.

§2.2 The Non-commutative Partial Derivative of an Image

We will be working with mappings $\varphi : KMag(S) \to KMag(S)$ a lot. So, it will be useful to know how $\frac{\partial \omega}{\partial s}$ changes when ω is mapped via φ. The following theorem explains this.

Theorem 2.1

$\forall \omega \in KMag(S)$, $s \in S$, $\forall \varphi : KMag(S) \to KMag(S)$ is an algebra homomorphism, let $\{v_1, \cdots, v_n\}$ be the variables in ω, then

$$\frac{\partial \varphi(\omega)}{\partial s} = \sum_{j=1}^{n} \frac{\partial \varphi(v_j)}{\partial s} \Big(\frac{\partial \omega}{\partial v_j}(\varphi) \Big)$$

Proof. Since $\frac{\partial}{\partial s}$, φ, and $-(\varphi)$ are all compatible with scalar multiplication and addition, it suffices to prove this for $\omega \in Mag(S) \subset KMag(S)$ a monomial. We prove this by inducting on the degree of ω.

Base case: $deg(\omega) = 1 \implies \omega = v_1$. So,

$$RHS = \frac{\partial \varphi(v_1)}{\partial s}(1(\varphi)) = \frac{\partial \varphi(v_1)}{\partial s} = LHS$$

Now assume our equation holds true for $deg(\omega) < n$ where $n \geq 2$. For $deg(\omega) = n$, we have $\omega = \omega_1 \cdot \omega_2$ where $deg(\omega_1), deg(\omega_2) < n$, so

$$\frac{\partial \varphi(\omega)}{\partial s} = \frac{\partial \varphi(\omega_1 \cdot \omega_2)}{\partial s}$$

$$
\begin{aligned}
&= \frac{\partial(\varphi(\omega_1)\cdot\varphi(\omega_2))}{\partial s} \\
&= \frac{\partial\varphi(\omega_1)}{\partial s} r_{\varphi(\omega_2)} + \frac{\partial\varphi(\omega_2)}{\partial s} l_{\varphi(\omega_1)} \\
&= \frac{\partial\varphi(\omega_1)}{\partial s} r_{\omega_2}(\varphi) + \frac{\partial\varphi(\omega_2)}{\partial s} l_{\omega_1}(\varphi) \\
&= \sum_{j=1}^{n} \frac{\partial\varphi(v_j)}{\partial s}\left(\frac{\partial\omega_1}{\partial v_j}(\varphi)\right)\Big(r_{\omega_2}(\varphi)\Big) + \sum_{j=1}^{n} \frac{\partial\varphi(v_j)}{\partial s}\left(\frac{\partial\omega_2}{\partial v_j}(\varphi)\right)\Big(l_{\omega_1}(\varphi)\Big) \\
&= \sum_{j=1}^{n} \frac{\partial\varphi(v_j)}{\partial s}\left(\left(\frac{\partial\omega_1}{\partial v_j} r_{\omega_2} + \frac{\partial\omega_2}{\partial v_j} l_{\omega_1}\right)(\varphi)\right) \\
&= \sum_{j=1}^{n} \frac{\partial\varphi(v_j)}{\partial s}\left(\frac{\partial\omega}{\partial v_j}(\varphi)\right)
\end{aligned}
$$

Note that for the decomposition of $\frac{\partial\varphi(\omega_1)}{\partial s}$ into the sum we may iterate from 1 to n because if v_j is not in ω_1 as a variable, then $\frac{\partial\omega_1}{\partial v_j} = 0$ and that summand is 0. Same goes for $\frac{\partial\varphi(\omega_2)}{\partial s}$. Induction complete. □

§2.3 Linearizing Relations: Part II

We expand on the idea of linearizing relations. In the previous section, we showed how a monomial can be decomposed into a linearized form.

Example 2.2

Let us linearize $(a_1a_1)(b_1b_1)$. The process is to first replace a_1 with $(a_1 + a_2)$ and expand. We get

$$(a_1a_1)(b_1b_1) + (a_2a_1)(b_1b_1) + (a_1a_2)(b_1b_1) + (a_2a_2)(b_1b_1)$$

is a relation, and thus $(a_2a_1)(b_1b_1) + (a_1a_2)(b_1b_1)$ is a relation. We do the same, replacing b_1 and $(b_1 + b_2)$. Finally, the linearized form of this monomial is

$$(a_2a_1)(b_2b_1) + (a_2a_1)(b_1b_2) + (a_1a_2)(b_2b_1) + (a_1a_2)(b_1b_2)$$

Let us denote a monomial ω as having distinct variables $Var(\omega) = \{a_1, b_1, \cdots, z_1\}$ (these just denote variables not exactly 26 of them), let $n(v_1)$ denote the number of times v_1 appears in the ω. Then, the linearized form ω' has distinct variables

$$\{a_1, a_2, \cdots, a_{n(a_1)}, b_1, b_2, \cdots, b_{n(b_1)}, \cdots, z_1, z_2, \cdots, z_{n(z_1)}\}$$

Now we construct the mapping $\varphi : KMag(S) \to KMag(S)$ induced by $v_* \mapsto v_1$ and is the identity elsewhere. We observe that

$$\varphi(\omega') = \left(\prod_{v_1\in Var(\omega)} n(v_1)!\right)\omega$$

Example 2.3

Continuing the previous example, we have that

$$\varphi(\omega') = 4(a_1a_1)(b_1b_1) = (2!)(2!)\omega$$

Now, let us define the reduced linearized form of ω as

$$\psi = \omega' / \prod_{v_1 \in Var(\omega)} n(v_1)!$$

This can only be done since $\mathbb{Z}\backslash\{0\}$ embeds into units. It follows that $\omega = \varphi(\psi)$. Now, we present the following lemma regarding their partial derivatives.

Lemma 2.4

$\forall v_1 \in Var(\omega)$,

$$\frac{\partial \omega}{\partial v_1} = n(v_1)\frac{\partial \psi}{\partial v_1}(\varphi)$$

Proof. The statement is $\frac{\partial \varphi(\psi)}{\partial v_1} = n(v_1)\frac{\partial \psi}{\partial v_1}(\varphi)$. Since $\frac{\partial}{\partial v_1}$, $-(\varphi)$, and φ are all compatible with scalar multiplication, it suffices to show that $\frac{\partial \varphi(\omega')}{\partial v_1} = n(v_1)\frac{\partial \omega'}{\partial v_1}(\varphi)$. We now use Theorem 2.1. Note that $\frac{\partial \varphi(-)}{\partial v_1}$ is nonzero if it is $\frac{\partial \varphi(v_j)}{\partial v_1} = 1$ and is 0 otherwise. So,

$$\frac{\partial \varphi(\omega')}{\partial v_1} = \sum_{j=1}^{n(v_1)} \frac{\partial \omega'}{\partial v_j}(\varphi) = n(v)\frac{\partial \omega'}{\partial v_1}(\varphi)$$

because the $\frac{\partial \omega'}{\partial v_j}(\varphi)$ are symmetric with respect to the v_j. $\square$

In meetings, we have also shown that we can conduct the linearization procedure to a polynomial (element in $KMag(S)$) $\omega \in R$ as well. A result we got was that the homogeneous components of ω are also relations in R.

Example 2.5

Suppose we have $a_1a_1 + b_1a_1 + a_1b_1 \in R$. We replace a_1 with $(a_1 + a_2)$, expand, and reduce to get

$$a_1a_2 + a_2a_1 \in R$$

Since 2 is invertible in K, substituting a_2 for a_1 then dividing by 2 gives us $a_1a_1 \in R$. This component is in R. It follows that the other homogeneous component $b_1a_1 + a_1b_1$ is also in R since R is closed under addition and scalar multiplication.

So, $\forall \omega \in R$, we decompose into homogeneous components $\omega = \sum_{j=1}^{r} \omega_j$ where $\omega_j \in R$. Now, for each ω_j, we find a reduced linearized form ψ_j such that the additional elements for ψ_j are distinct for distinct j. These reduced linear forms are of course still in R. Hence, we can find a $\varphi : KMag(S) \to KMag(S)$ that acts on the additional elements for ψ_j as φ_j. This is thus well-defined. Then, we propose the Theorem for this section.

Theorem 2.6

$\forall \omega \in R$, $s \in Var(\omega)$, $s' \in S$, $\exists \psi \in R$ that has monomials which are linear in all their variables including s', $\varphi : KMag(S) \to KMag(S)$, such that

$$\frac{\partial \omega}{\partial s} = \frac{\partial \psi}{\partial s'}(\varphi)$$

Proof. When picking additional elements during the construction of ψ_j, let us be careful and not pick s'. Now, following our discussion preceding this theorem, we have that

$$\begin{aligned}\frac{\partial \omega}{\partial s} &= \sum_{j=1}^{r} \frac{\partial \omega_j}{\partial s} \\ &= \sum_{j=1}^{r} n_j(s) \frac{\partial \psi_j}{\partial s}(\varphi_j) \\ &= \sum_{j=1}^{r} n_j(s) \frac{\partial \psi_j}{\partial s}(\varphi) \\ &= \left(\frac{\partial \sum_{j=1}^{r} n_j(s)\psi_j}{\partial s} \right)(\varphi)\end{aligned}$$

We pick $\psi_s = \sum_{j=1}^{r} n_j(s)\psi_j$. Since each ψ_j has monomials linear in all their variables including s, then ψ_s satisfies this too. Moreover, since each $\psi_j \in R$, then $\psi' \in R$. We have

$$\frac{\partial \omega}{\partial s} = \frac{\partial \psi_s}{\partial s}(\varphi)$$

Now, let us construct the mapping φ_s that is the identity except for $s \mapsto s'$. Let us define $\psi = \varphi_s(\psi_s)$. Thus, using Theorem 2.1, we have

$$\frac{\partial \psi}{\partial s'} = \frac{\partial \varphi_s(s)}{\partial s'} \Big(\frac{\partial \psi_s}{\partial s}(\varphi_s) \Big) = \Big(\frac{\partial \psi_s}{\partial s}(\varphi_s) \Big) = \frac{\partial \psi_s}{\partial s}$$

this being true since the non-zero $\frac{\partial \varphi_s(v)}{\partial s'}$ is when $v = s$. We have essentially swapped out the s in ψ_s with s'. So, ψ is has monomials linear in all their variables including s'. We thus get

$$\frac{\partial \omega}{\partial s} = \frac{\partial \psi_s}{\partial s}(\varphi) = \frac{\partial \psi}{\partial s'}(\varphi)$$

□

§2.4 The Ideal in $Tens(A \oplus A)$

We are interested in the structure of $U_V(A)$, which is $Tens(A \oplus A)$ quotient an ideal generated by a set we denote I. So,

$$U_V(A) = Tens(A \oplus A)/(I)$$

Our paper covers the description of I as

$$I = \{\frac{\partial \omega}{\partial s}(\phi) \mid s \in S, \omega \in R, \phi : KMag(S) \to A\}$$

Now, Theorem 3.5 provides us with the following description of elements in I.

Corollary 2.7

$\forall i \in I$, $s' \in S$, $\exists \psi \in R$ that has monomials linear in all their variables including s', $\phi' : KMag(S) \to A$, such that

$$i = \frac{\partial \psi}{\partial s'}(\phi')$$

Proof. Note that $i = \frac{\partial \omega}{\partial s}(\phi)$. If $s \notin Var(\omega)$ then $i = 0$ and we can just pick $\psi = 0$. Otherwise, we use Theorem 3.5 and get $\psi \in R$ that has monomials linear in all their variables including s', $\varphi : KMag(S) \to KMag(S)$ such that

$$i = \frac{\partial \omega}{\partial s}(\phi) = \frac{\partial \psi}{\partial s'}(\varphi)(\phi)$$

Then, picking $\phi' = \phi \circ \varphi$ suffices. □

Now, with this in mind we may present the following theorem that characterizes the structure of I.

Theorem 2.8

I is an ideal, i.e., $(I) = I$.

Proof. First we show that I is closed under addition. Suppose that we have $i_1, i_2 \in I$. We pick a $s \in S$. Then, by Corollary 4.1, we can find ψ_1, ψ_2, with monomials linear in all their variables including s and ϕ_1, $\phi_2 : KMag(S) \to A$ such that

$$i = \frac{\partial \psi_1}{\partial s}(\phi_1), i_2 = \frac{\partial \psi_2}{\partial s}(\phi_2)$$

Now suppose that aside from s, ψ_2' has variables $\{v_1, \cdots, v_n\}$. We can map variables v_j to distinct variables v_j' that doesn't appear in ψ_1 through mapping φ that is identity elsewhere. Define $\psi_2' = \varphi(\psi_2)$. Now, let us define ϕ_2' such that $\phi_2'(v_j') = \phi(v_j)$. This is well-defined. Using Theorem 2.1, we have

$$\frac{\partial \psi_2'}{\partial s}(\phi_2') = \frac{\partial \varphi(\psi_2)}{\partial s}(\phi_2') = \frac{\partial \psi_2}{\partial s}(\varphi)(\phi_2') = \frac{\partial \psi_2}{\partial s}(\phi_2) = i_2$$

Suppose that ψ_1 has variables $\{u_1, \cdots, u_m\}$ other than s. As aforementioned,

$$\{u_1, \cdots, u_m\} \cap \{v_1', \cdots, v_n'\} = \emptyset$$

So, we construct $\phi : KMag(S) \to A$ induced by mapping $S \to A$ with $u_j \mapsto \phi_1(u_j)$, $v_j' \mapsto \phi_2'(v_j')$. So, we have

$$i_1 = \frac{\partial \psi_1}{\partial s}(\phi), i_2 = \frac{\partial \psi_2'}{\partial s}(\phi)$$

$$(i_1 + i_2) = \frac{\partial(\psi_1 + \psi_2')}{\partial s}(\phi)$$

Since $\psi_1, \psi_2' \in R$, then their sum is in R too.

For the following we work with any $i \in I$ and fine $i = \frac{\partial \psi}{\partial s}(\phi)$. Now, we claim that right multiplication of i by an element r_a is still in I. We construct $\psi' = \psi' \cdot v \in R$

where v is a variable not in ψ. We construct ϕ' that acts on the variables in ψ same as ϕ, but maps v to a. Then, we have that

$$\frac{\partial \psi'}{\partial s}(\phi') = (\frac{\partial \psi}{\partial s} r_v)(\phi') + (\frac{\partial v}{\partial s} l_\psi)(\phi') = (\frac{\partial \psi}{\partial s}(\phi)) r_a = i \cdot r_a$$

Right multiplication of i by l_a follows similarly from symmetry.

Now we claim that left multiplication of i by an element r_a is still in I. We construct $\psi' = \varphi(\psi)$ where φ is the identity except for $s \mapsto (sv)$ where v is a variable not in ψ. We use the same ϕ' as in the previous paragraph. Then, using Theorem 2.1., we have

$$\frac{\partial \psi'}{\partial s}(\phi') = \left(\frac{\partial \varphi(s)}{\partial s}(\frac{\partial \psi}{\partial s}(\varphi))\right)(\phi') = (r_v \cdot \frac{\partial \psi}{\partial s})(\phi') = r_a(\frac{\partial \psi}{\partial s}(\phi)) = r_a \cdot i$$

Left multiplication of i by l_a follows from symmetry. □

Remark: in our proof above we only use the fact that A is an algebra. So, this proof works for any algebra A, not necessarily in the variety V.

§2.5 $U_V(A)$ is Independent of R_0

This section gives an alternative proof that our definition of $U_V(A)$ is well-defined. We can first define I_0 as follows:

$$I_0 = \{\frac{\partial \omega_0}{\partial s}(\phi) \mid s \in S, \omega_0 \in R_0, \phi : KMag(S) \to A\}$$

Our definition of $U_V(A)$ is $Tens(A \oplus A)/(I_0)$. To show that this quotient is independent of the choice of R_0, it suffices to show that $(I_0) = I$.

Theorem 2.9

$I = (I_0)$.

Proof. Since $I_0 \subset I$ and I is an ideal, then we have $(I_0) \subset I$. It suffices to show that $I \subset (I_0)$. We note that I is

$$\left\{\frac{\partial \omega}{\partial s}(\phi) \mid \omega \in \big(\{\varphi(\omega_0)\}\big)\right\}$$

First we show that if $\omega = \varphi(\omega_0)$, then $\frac{\partial \omega}{\partial s}(\phi) \in (I_0)$. Let $\{v_1, \cdots, v_n\}$ be the variables in ω_0. Using Theorem 2.1, we have that

$$\frac{\partial \omega}{\partial s} = \frac{\partial \varphi(\omega_0)}{\partial s} = \sum_{j=1}^{n} \frac{\partial \varphi(v_j)}{\partial s}\left(\frac{\partial \omega_0}{\partial v_j}(\varphi)\right)$$

$$\implies \frac{\partial \omega}{\partial s}(\phi) = \sum_{j=1}^{n} \left(\frac{\partial \varphi(v_j)}{\partial s}(\phi)\right)\left(\frac{\partial \omega_0}{\partial v_j}(\phi \circ \varphi)\right)$$

since each $\frac{\partial \omega_0}{\partial v_j}(\phi \circ \varphi)$ is in I_0, then $\frac{\partial \omega}{\partial s}(\phi) \in (I_0)$. We say ω has property P if $\forall s \in S$, $\phi : KMag(S) \to A$, we have $\frac{\partial \omega}{\partial s}(\phi) \in (I_0)$. Since all $\varphi(\omega_0)$ have property P, then it suffices to show that property P is closed under addition and left multiplication (right multiplication follows from symmetry).

Addition: if ω_1 and ω_2 have property P, then we have

$$\frac{\partial(\omega_1+\omega_2)}{\partial s}(\phi) = \frac{\partial \omega_1}{\partial s}(\phi) + \frac{\partial \omega_2}{\partial s}(\phi) \in (I_0)$$

Left multiplication: if ω has property P, then $\forall \sigma \in KMag(S)$, we have

$$\frac{\partial(\sigma\omega)}{\partial s}(\phi) = (\frac{\partial \omega}{\partial s} l_\sigma)(\phi) + (\frac{\partial \sigma}{\partial s} r_\omega)(\phi) = \frac{\partial \omega}{\partial s}(\phi) l_{\phi(\sigma)} + \frac{\partial \sigma}{\partial s}(\phi) r_{\phi(\omega)} = \frac{\partial \omega}{\partial s}(\phi) l_{\phi(\sigma)} \in (I_0)$$

since $\phi(\omega) = 0$ as ω is a relation in R and thus satisfied by A. $\square$

§2.6 No Radicals in the Relations

We first investigate the specific case where $A = KMag(S)/R$. Note that the mapping sequence that ultimately maps to elements in I factor through $-(\pi)$, which is the mapping between the tensor algebras induced by the canonical mapping $\pi : KMag(S) \to KMag(S)/R = A$. So we have that the elements in I are precisely the image of this sequence of maps:

$$\begin{aligned} KMag(S) &\xrightarrow{\frac{\partial}{\partial s}} Tens(KMag(S) \oplus KMag(S)) \\ &\xrightarrow{-(\phi)} Tens(KMag(S) \oplus KMag(S)) \\ &\xrightarrow{-(\pi)} Tens(KMag(S)/R \oplus KMag(S)/R) \end{aligned}$$

$$I = \left\{ \frac{\partial \omega}{\partial s}(\phi)(\pi) \mid s \in S, \omega \in R, \phi : KMag(S) \to KMag(S) \right\}$$

Let us also write

$$J = \{ \frac{\partial \omega}{\partial s}(\phi) \mid s \in S, \omega \in R, \phi : KMag(S) \to KMag(S) \}$$

So, $I = J(\pi)$.

Lemma 2.10
$\forall j \in J$, $\exists s'' \in S$, $\psi \in R$ linear in s'', such that

$$j = \frac{\partial \psi}{\partial s''}$$

Proof. Note that $j = \frac{\partial \omega}{\partial s}(\phi)$. If $s \notin Var(\omega)$, then $j = 0$ and we pick $\psi = 0$. Otherwise, we can use Theorem 3.5 to find ψ' that has monomials linear in all their variables including s' and $\varphi : KMag(S) \to KMag(S)$ such that

$$j = \frac{\partial \omega}{\partial s}(\phi) = \frac{\partial \psi'}{\partial s'}(\varphi)(\phi) = \frac{\partial \psi'}{\partial s'}(\phi \circ \varphi)$$

Assume that ψ' has variables $\{s', v_1, \cdots, v_n\}$. Then, let us construct $\chi : KMag(S) \to KMag(S)$ defined by $v_k \mapsto (\phi \circ \varphi)(v_k)$ and $s' \mapsto s''$ that is not in any $(\phi \circ \varphi)(v_k)$. Define $\psi = \chi(\psi')$. Then, by Theorem 2.1, we have

$$\frac{\partial \psi}{\partial s''} = \frac{\partial \chi(s')}{\partial s''}(\frac{\partial \psi'}{\partial s'}(\chi)) = \frac{\partial \psi'}{\partial s'}(\chi) = \frac{\partial \psi'}{\partial s'}(\phi \circ \varphi) = j$$

Since ψ' is linear in s' and only s' is mapped to an expression containing s'', then $\psi = \chi(\psi')$ is linear in s''. $\square$

Now, we are interested in whether there are such "radicals" in the relation. The following theorem asserts that there are none.

Theorem 2.11

Suppose that we have ω such that

$$\{\frac{\partial \omega}{\partial s}(\phi) \mid s \in S, \phi : KMag(S) \to A\} \subset I$$

i.e., it always maps to 0 in $U_V(A)$ through the partial derivative and another map, then we have $\omega \in R$.

Proof. **Outline**: We denote ω has property Q if it satisfies the property stated in the theorem. We use induction to prove this. Base case: ω has 0 variables, then $\omega = 0 \in R$. Suppose that if a relation has $n-1$ variables or less and satisfies property Q then it is in R. Now, for a relation with n variables, suppose that one of them is s. So, $\omega = \omega_s + (\omega - \omega_s)$ where ω_s is the sum of all monomials including s. If we prove that $\omega_s \in R$, then we have that its partial derivatives always map into I. So, $(\omega - \omega_s)$ has property Q because I is an ideal. $(\omega - \omega_s)$ has $n-1$ variables or less. So, $\omega = \omega_s + (\omega - \omega_s) \in R$. It suffices to show that $\omega_s \in R$ then.

Step 0: We will only look at the $A = KMag(S)/R$ case (and that is all we need to look at). The mapping factors through $-(\pi)$, and we pick $\phi = id$. So, we have

$$\frac{\partial \omega}{\partial s}(\pi) \in I \implies \frac{\partial \omega}{\partial s} \in J + ker(-(\pi)) \implies \frac{\partial \omega_s}{\partial s} = \frac{\partial \omega}{\partial s} \in J + ker(-(\pi))$$

Step 1: We claim that $ker(-(\pi)) \subset J$.
Note that our remark after the proof of Theorem 4.2 implies that J is an ideal. I believe it is well-known that the kernel of mapping between tensor algebras is generated by the inclusion of the kernel of the base mapping. In this case,

$$ker(-(\pi)) = \Big(l(ker(\pi)) \cup r(ker(\pi))\Big)$$

$ker(\pi) = R$. So, it suffices to show that $l(R) \subset J$ as $r(R) \subset J$ follows from symmetry. Note that $\forall \theta \in R$, let us construct $\theta \cdot v$ where $v \notin Var(\theta)$. Then, we have

$$\frac{\partial(\theta \cdot v)}{\partial v}(id) = l_\theta$$

and we are done.

Step 2: We have $\frac{\partial \omega_s}{\partial s} = \frac{\partial \omega}{\partial s} = j \in J$. Lemma 6.1 gives us that there exists $s'' \in S$, $\psi \in R$ linear in s'' such that $j = \frac{\partial \psi}{\partial s''}$. However, s'' may be in $Var(\omega)$. We will fix this by mapping ψ via $\varphi : KMag(S) \to KMag(S)$ that is the identity on S except for $s'' \mapsto t \notin Var(\psi) \cup Var(\omega)$. Hence, define $\psi^t = \varphi(\psi)$. Of course, since $\psi \in R \implies \psi^t \in R$. Theorem 2.1 gives us that

$$\frac{\partial \psi^t}{\partial t} = \frac{\partial \varphi(s'')}{\partial t}(\frac{\partial \psi}{\partial s''}(\varphi)) = \frac{\partial \psi}{\partial s''}(id) = j$$

Since ψ is linear in s'', then ψ^t is linear in t.

Step 3: Finally we are motivated to convert ω_s to some form ω_s^t such that $\frac{\partial \omega_s}{\partial s} = \frac{\partial \omega_s^t}{\partial t}$. First, let us decompose

$$\omega_s = \sum_{k=1}^{r} \omega_{s,k}$$

where $\omega_{s,k}$ are homogeneous components. We demonstrate that for any monomial m, we can transform it into a polynomial m^t such that $\frac{\partial m}{\partial s} = \frac{\partial m^t}{\partial t}$. Let $n(s)$ be the number of times s shows up in m. We construct m^t by forming the $n(s)$ monomials, each one is the result of replacing one s in m with a t.

Example 2.12

If $m = (as)(bs)$, then we form $m^t = (at)(bs) + (as)(bt)$.

Let us construct $\xi : KMag(S) \to KMag(S)$ that is the identity on S except for $t \mapsto s$. We also have that have that $m = \frac{1}{n(s)}\xi(m^t)$. Now, we do the same for every homogeneous component of ω_s to get

$$\omega_s^t \overset{\text{def}}{=} \sum_{k=1}^{r} \omega_{s,k}^t \implies j = \frac{\partial \omega_s}{\partial s} = \frac{\partial \omega_s^t}{\partial t}$$

$$\implies \frac{\partial \psi^t}{\partial t} = \frac{\partial \omega_s^t}{\partial t}$$

Note that ψ^t and ω_s^t are both linear in t. Since the kernel of $\frac{\partial(-)}{\partial t} = KMag(S\backslash\{t\})$ consists of terms not linear in t, then we must have $\psi^t = \omega_s^t \implies \omega_s^t \in R$. So, the homogeneous components, $\omega_{s,k}^t$, of ω_s^t are in R too. Since $\omega_{s,k} = \frac{1}{n_k(s)}\xi(\omega_{s,k}^t)$, then $\omega_{s,k} \in R \implies \omega_s \in R$. We are finished here and the induction follows through. $\square$

§3 Examples of $U_V(A)$ for Inhomogeneous Alternative Algebras

Now, in this and the next section, we provide several detailed calculations for the structures of specific universal enveloping algebras. The relations among the maps l and r induced by the altenative algebra relations are

$$l_i^2 - l_{ii},\ r_i^2 - r_{ii}$$

$$l_{ij} - l_j l_i - r_j l_i + l_i r_j$$

$$r_i r_j - r_{ij} - r_j l_i + l_i r_j$$

The equations in the first row are not linear in i, so by replacing i with $i+j$ we get the new relations

$$l_i l_j + l_j l_i - l_{ij} - l_{ji}$$

$$r_i r_j + r_j r_i - r_{ij} - r_{ji}$$

Therefore, the initial ideal $I = R_0$ consists of [1]

$$l_i^2 - l_{ii},\ r_i^2 - r_{ii},\ -r_i l_i + l_i r_i \text{ for } 1 \le i \le n$$

$$l_i l_j + l_j l_i - l_{ij} - l_{ji},\ r_i r_j + r_j r_i - r_{ij} - r_{ji} \text{ for } 1 \le j < i \le n$$

$$l_{ij} - l_j l_i - r_j l_i + l_i r_j,\ r_i r_j - r_{ij} - r_j l_i + l_i r_j \text{ for } 1 \le j < i \le n$$

$$l_{ji} - l_i l_j - r_i l_j + l_j r_i,\ r_j r_i - r_{ji} - r_i l_j + l_j r_i \text{ for } 1 \le j < i \le n$$

I am ordering these terms first by degree and then lexicographically so for example $r_1^2 > l_1$ and $l_1 > r_1 > l_2$.

Example 3.1

The leading monomial of $l_1^2 + l_1 r_1 + l_3$ will be l_1^2.

§3.1 $\mathbb{C}$ over $\mathbb{R}$

Note: The terms l_1, r_1, l_2, r_2 in $\mathrm{Tens}(A \oplus A)$ correspond to $(1,0)$, $(0,1)$, $(i,0)$, $(0,i)$, respectively. Also, note that my code prints out all polynomials in reversed order, so the leading monomial is in fact the right-most term of a polynomial.

The minimal Grobner basis has 14 elements and is
$-l_1 + l_1^2$
$-r_1 + r_1^2$
$-r_1 l_1 + l_1 r_1$
$l_1 + l_2^2$
$r_1 + r_2^2$
$-r_2 l_2 + l_2 r_2$
$-r_2 - r_2 l_1 + r_1 r_2 + l_1 r_2$
$-r_2 + r_2 r_1 + l_2 r_1 - r_1 l_2$
$-2l_2 + l_2 l_1 + l_1 l_2$
$-2r_2 + r_2 r_1 + r_1 r_2$

[1] I'm not ordering these left-lexicographically for the sake of writing but they are indeed ordered in my code

$l_2 - l_2 l_1$
$r_2 - r_2 r_1$
$r_2 l_1 - r_1 l_2 - r_2 l_1^2 + l_2 r_1 l_1$
$- l_2 r_2 + l_2^2 + r_1 l_1 - l_1 r_1 + l_1^2 + r_2 l_2 r_1$

Thus, we may conclude results regarding the associated graded algebra. $gr(U(A))_1$ has basis $\{l_1, r_1, l_2, r_2\}$, $gr(U(A))_2$ has basis $\{r_1 l_1, l_2 r_1, r_2 l_1, r_2 l_2\}$, and $\forall k \geq 3$, $gr(U(A))_k$ has dimension 0.

§3.2 $\mathbb{H}$ over $\mathbb{R}$

Note: The terms a, c, e, g refer to l_1, l_i, l_j, l_k. The terms b, d, f, h refer to r_1, r_i, r_j, r_k.

The minimal Grobner basis has 73 elements and is
$[-a+a^2, -b+b^2, -ba+ab, a+c^2, b+d^2, -dc+cd, a+e^2, b+f^2, -fe+ef, a+g^2, b+h^2, -hg+gh, c-ca, -da+ad, cb-bc, -c+ac, -d+bd, e-ea, -fa+af, eb-be, -e+ae, -f+bf, g-ga, -ha+ah, gb-bg, -g+ag, -h+bh, h+g+fd-ec, -h-fd-fc+cf, h+fd+ed-de, h+g+fd+ce, fd+df, -f-e+hd-gc, f-hd-hc+ch, -f+hd+gd-dg, -f-e+hd+cg, hd+dh, d+c+hf-ge, -d-hf-he+eh, d+hf+gf-fg, d+c+hf+eg, hf+fh, d-db, f-fb, h-hb, cb-cba, dc-dcb, eb-eba, fe-feb, ed-eda, -fc+fcb, h-ha+fd-fda, gb-gba, hg-hgb, gd-gda, -hc+hcb, gf-gfa, -he+heb, -f+hd+fa-hda, d+hf-da-hfa, -f+hd+hc+fdc, -f+hd+gd-edc, -he-fed, -h-fd-fc+hdc, -h-fd-ed-gdc, d+hf+gf-hgd, -fe-gfc, -b+hg+hfd+gfd, fe-hed, ed+hfe, fc-gfe, f-hd-gd-hgf, b-dc-hfd-hfc]$

The normal monomials are
$[1, a, b, c, d, e, f, g, h, ba, cb, da, dc, eb, ed, fa, fc, fd, fe, gb, gd, gf, ha, hc, hd, he, hf, hg, hfd]$

Therefore we have that

$$\operatorname{rank}_{\mathbb{R}}(gr_i U(A)) = \begin{cases} 1 & i = 0 \\ 8 & i = 1 \\ 19 & i = 2 \\ 1 & i = 3 \\ 0 & i \geq 4 \end{cases}$$

§3.3 $\mathbb{Q}[x]/x^4$ over $\mathbb{Q}$

Note: I tried $\mathbb{Z}[x]/x^4$ but that didn't work since rationals popped up. The terms l_1, l_2, l_3, l_4 correspond to $(1, 0)$, $(x, 0)$, $(x^2, 0)$, $(x^3, 0)$, respectively. r_* is similarly defined.

The minimal Grobner basis has 68 elements and is
$-l_1 + l_1^2$, $-r_1 + r_1^2$, $-r_1 l_1 + l_1 r_1$, $-l_3 + l_2^2$, $-r_3 + r_2^2$, $-r_2 l_2 + l_2 r_2$, l_3^2, r_3^2, $-r_3 l_3 + l_3 r_3$, l_4^2, r_4^2, $-r_4 l_4 + l_4 r_4$, $-r_2 - r_2 l_1 + r_1 r_2 + l_1 r_2$, $-r_2 + r_2 r_1 + l_2 r_1 - r_1 l_2$, $-2l_2 + l_2 l_1 + l_1 l_2$, $-2r_2 + r_2 r_1 + r_1 r_2$, $-r_3 - r_3 l_1 + r_1 r_3 + l_1 r_3$, $-r_3 + r_3 r_1 + l_3 r_1 - r_1 l_3$, $-2l_3 + l_3 l_1 + l_1 l_3$, $-2r_3 + r_3 r_1 + r_1 r_3$, $-r_4 - r_4 l_1 + r_1 r_4 + l_1 r_4$, $-r_4 + r_4 r_1 + l_4 r_1 - r_1 l_4$, $-2l_4 + l_4 l_1 + l_1 l_4$, $-2r_4 + r_4 r_1 + r_1 r_4$, $-r_4 - r_3 l_2 + r_2 r_3 + l_2 r_3$, $-r_4 + r_3 r_2 + l_3 r_2 - r_2 l_3$, $-2l_4 + l_3 l_2 + l_2 l_3$, $-2r_4 + r_3 r_2 + r_2 r_3$, $-r_4 l_2 + r_2 r_4 + l_2 r_4$, $r_4 r_2 + l_4 r_2 - r_2 l_4$, $l_4 l_2 + l_2 l_4$, $r_4 r_2 + r_2 r_4$, $-r_4 l_3 + r_3 r_4 + l_3 r_4$, $r_4 r_3 + l_4 r_3 - r_3 l_4$, $l_4 l_3 + l_3 l_4$, $r_4 r_3 + r_3 r_4$, $r_4 - r_3 r_2$, $l_4 - l_3 l_2$, $r_2 l_1 - r_1 l_2 - r_2 l_1^2 + l_2 r_1 l_1$, $-r_3 + l_3 + r_3 r_1 - l_3 l_1$, $-r_2 + l_2 + r_2 r_1 - l_2 l_1$, $-l_3 r_1 +$

$l_2r_2 - l_2^2 + r_1l_3 + l_1l_3 - r_2l_2r_1,\ 2r_3 - 2r_3r_1,\ -r_2 + r_2r_1,\ r_3l_1 - r_1l_3 - r_3l_1^2 + l_3r_1l_1,\ l_3r_3 - l_3^2 - r_3l_3r_1,\ r_4l_1 - r_1l_4 - r_4l_1^2 + l_4r_1l_1,\ -r_4 + l_4 + r_4r_1 - l_4l_1,\ l_4r_4 - l_4^2 - r_4l_4r_1,\ -r_4+r_4r_1,\ -r_4r_2,\ 2r_4r_3,\ -2r_4r_1+2r_3r_2+2r_3l_2+l_3l_2+r_2r_3-l_2r_3-l_1l_4-r_3r_2r_1-r_3l_2r_1,\ r_4l_2+l_4l_2+l_3r_3-l_3^2-r_2l_4-l_2r_4-r_3l_2^2+l_3r_2l_2,\ -2r_4r_1-2r_4l_1-l_4l_1+l_3r_2+r_2r_3+r_2l_3+l_2l_3+l_1r_4+r_3r_2r_1+r_3r_2l_1-r_3l_2l_1+r_3l_1l_2-l_3r_2r_1-l_3r_2l_1,\ -l_4l_3+l_3r_4-r_3l_3r_2,\ -2r_4r_3-2l_4l_3,\ 2r_4r_2-l_4l_2,\ 2r_4r_2+2r_4l_2+l_4l_2+r_2r_4-l_2r_4-r_4r_2r_1-r_4l_2r_1,\ -l_4l_3+l_3r_4-r_4l_2^2+l_4r_2l_2,\ 2l_4r_2-l_4l_2+r_2r_4+r_2l_4+l_2l_4+r_4r_2r_1+r_4r_2l_1-r_4l_2l_1+r_4l_1l_2-2l_4r_2r_1-l_4r_2l_1,\ -r_4l_4r_2,\ 2r_4r_3+2r_4l_3+l_4l_3+r_3r_4-l_3r_4-r_4r_3r_1-r_4l_3r_1,\ r_4^2+r_4l_4-l_4r_4+l_4^2-r_4r_3r_2-r_4l_3r_2,\ -r_4l_3^2+l_4r_3l_3,\ 2l_4r_3-l_4l_3+r_3r_4+r_3l_4+l_3l_4+r_4r_3r_1+r_4r_3l_1-r_4l_3l_1+r_4l_1l_3-2l_4r_3r_1-l_4r_3l_1,\ -r_4^2+l_4r_4+r_4r_3r_2+r_4r_3l_2-r_4l_3l_2+r_4l_2l_3-l_4r_3r_2-l_4r_3l_2,\ -r_4l_4r_3$

The leading monomials are of the form

$$l_i^2,\ r_i^2,\ l_ir_i \text{ for } 1 \leq i \leq 4$$

$$l_jr_i,\ r_jl_i,\ l_jl_i,\ r_jr_i,\ r_ir_j,\ l_il_j,\ l_ir_jl_j,\ r_il_ir_j \text{ for } 1 \leq j < i \leq 4$$

$$l_ir_jl_k,\ r_il_jr_k \text{ for } 1 \leq k < j < i \leq 4$$

Thus, $gr(U(A))_1$ has basis $\{l_1, r_1, l_2, r_2, l_3, r_3, l_4, r_4\}$, $gr(U(A))_2$ has basis

$$\{r_1l_1, l_2r_1, r_2l_1, r_2l_2, l_3r_1, l_3r_2, r_3l_1, r_3l_2, r_3l_3, l_4r_1, l_4r_2, l_4r_3, r_4l_1, r_4l_2, r_4l_3, r_4l_4\}$$

And $\forall k \geq 3$, $gr(U(A))_k$ has dimension 0.

§3.4 $\mathbb{Q}[x]/(x^4+1)$ over $\mathbb{Q}$

Note: The terms l_1, l_2, l_3, l_4 correspond to $(1,0)$, $(x,0)$, $(x^2,0)$, $(x^3,0)$, respectively. r_* is similarly defined.

The minimal Grobner basis has 68 elements and is
$-l_1 + l_1^2,\ -r_1 + r_1^2,\ -r_1l_1 + l_1r_1,\ -l_3 + l_2^2,\ -r_3 + r_2^2,\ -r_2l_2 + l_2r_2,\ l_1 + l_3^2,\ r_1 + r_3^2,\ -r_3l_3+l_3r_3,\ l_3+l_4^2,\ r_3+r_4^2,\ -r_4l_4+l_4r_4,\ -r_2-r_2l_1+r_1r_2+l_1r_2,\ -r_2+r_2r_1+l_2r_1-r_1l_2,\ -2l_2+l_2l_1+l_1l_2,\ -2r_2+r_2r_1+r_1r_2,\ -r_3-r_3l_1+r_1r_3+l_1r_3,\ -r_3+r_3r_1+l_3r_1-r_1l_3,\ -2l_3+l_3l_1+l_1l_3,\ -2r_3+r_3r_1+r_1r_3,\ -r_4-r_4l_1+r_1r_4+l_1r_4,\ -r_4+r_4r_1+l_4r_1-r_1l_4,\ -2l_4+l_4l_1+l_1l_4,\ -2r_4+r_4r_1+r_1r_4,\ -r_4-r_3l_2+r_2r_3+l_2r_3,\ -r_4+r_3r_2+l_3r_2-r_2l_3,\ -2l_4+l_3l_2+l_2l_3,\ -2r_4+r_3r_2+r_2r_3,\ r_1-r_4l_2+r_2r_4+l_2r_4,\ r_1+r_4r_2+l_4r_2-r_2l_4,\ 2l_1+l_4l_2+l_2l_4,\ 2r_1+r_4r_2+r_2r_4,\ r_2-r_4l_3+r_3r_4+l_3r_4,\ r_2+r_4r_3+l_4r_3-r_3l_4,\ 2l_2+l_4l_3+l_3l_4,\ 2r_2+r_4r_3+r_3r_4,\ r_4-r_3r_2,\ l_4-l_3l_2,\ l_3-l_3l_1,\ r_3-r_3r_1,\ -l_2-l_4l_3,\ -r_2-r_4r_3,\ r_2l_1-r_1l_2-r_2l_1^2+l_2r_1l_1,\ -r_2+l_2+r_2r_1-l_2l_1,\ -l_3r_1+l_2r_2-l_2^2+r_1l_3+l_1l_3-r_2l_2r_1,\ -r_2+r_2r_1,\ r_3l_1-r_1l_3-r_3l_1^2+l_3r_1l_1,\ l_3r_3-l_3^2-r_1l_1+l_1r_1-l_1^2-r_3l_3r_1,\ r_4l_1-r_1l_4-r_4l_1^2+l_4r_1l_1,\ -r_4+l_4+r_4r_1-l_4l_1,\ l_4r_4-l_4^2+l_3r_1-r_1l_3-l_1l_3-r_4l_4r_1,\ -r_4+r_4r_1,\ -r_1-r_4r_2,\ -2r_4r_1+2r_3r_2+2r_3l_2+l_3l_2+r_2r_3-l_2r_3-l_1l_4-r_3r_2r_1-r_3l_2r_1,\ r_4l_2+l_4l_2+l_3r_3-l_3^2-r_2l_4-l_2r_4-r_3l_2^2+l_3r_2l_2,\ -2r_4r_1-2r_4l_1-l_4l_1+l_3r_2+r_2r_3+r_2l_3+l_2l_3+l_1r_4+r_3r_2r_1+r_3r_2l_1-r_3l_2l_1+r_3l_1l_2-l_3r_2r_1-l_3r_2l_1,\ -l_4l_3+l_3r_4-r_2l_1-l_2l_1+l_1r_2-r_3l_3r_2,\ 2r_1-l_1+2r_4r_2-l_4l_2,\ 2r_4r_2+2r_4l_2+l_4l_2+r_2r_4-l_2r_4+2r_1^2+l_1^2-r_4r_2r_1-r_4l_2r_1,\ -l_4l_3+l_3r_4+r_2l_1+l_2r_1-r_1l_2-l_1l_2-r_4l_2^2+l_4r_2l_2,\ 2l_4r_2-l_4l_2+r_2r_4+r_2l_4+l_2l_4+2r_1^2+2r_1l_1-l_1r_1+r_4r_2r_1+r_4r_2l_1-r_4l_2l_1+r_4l_1l_2-2l_4r_2r_1-l_4r_2l_1,\ -l_4r_1+l_3r_2-r_2l_3-l_2l_3+l_1l_4-r_4l_4r_2,\ 2r_4r_3+2r_4l_3+l_4l_3+r_3r_4-l_3r_4+2r_2r_1+l_1l_2-r_4r_3r_1-r_4l_3r_1,\ r_4^2+r_4l_4-l_4r_4+l_4^2-2r_3r_1+2r_2^2+l_2^2-r_1l_3-r_4r_3r_2-r_4l_3r_2,\ l_4l_1+r_3l_2+l_3r_2-r_2l_3-l_2l_3-l_1r_4-r_4l_3^2+l_4r_3l_3,\ l_4r_3-l_4l_3+r_3r_4+r_3l_4+l_3l_4+2r_2r_1+2r_2l_1-l_1r_2+r_4r_3r_1+r_4r_3l_1-r_4l_3l_1+r_4l_1l_3-l_4r_3r_1-l_4r_3l_1,\ -r_4^2+l_4r_4-2r_3r_1-r_3l_1-l_3l_1+2r_2^2+2r_2l_2-l_2r_2+l_1l_3+r_4r_3r_2+r_4r_3l_2-r_4l_3l_2+r_4l_2l_3-l_4r_3r_2-l_4r_3l_2,\ -l_4r_2-r_3l_3+l_3r_3-l_3^2+l_2l_4-r_4l_4r_3$

The leading monomials are the same as the last section, and thus the universal enveloping algebras are isomorphic.

§4 Examples of $U_V(A)$ for the Octonions over Fields

We use the variables $\{a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p\}$ to describe the 16 variables l_* and r_*. Specifically,

$$a = l_{(1,0)}, b = r_{(1,0)}, c = l_{(i,0)}, d = r_{(i,0)}, e = l_{(j,0)}, f = r_{(j,0)}, g = l_{(k,0)}, h = r_{(k,0)}$$

$$i = l_{(0,1)}, j = r_{(0,1)}, k = l_{(0,k)}, l = r_{(0,k)}, m = l_{(0,j)}, n = r_{(0,j)}, o = l_{(0,i)}, p = r_{(0,i)}$$

We first order them by degree. Then, for same degree we use alphabetical ordering $a > b > c > \cdots$. Note that my code always writes out the polynomials from smallest monomial to largest monomial, opposite to how one would traditionally write polynomials.

§4.1 $\mathbb{O}$ over $\mathbb{R}$

Consider the Octonions over $\mathbb{R}$, we use the same equations as described in the previous section and map them into the basis elements. Our Sage code gives us that the final simplified Grobner basis is
$[-b + b^2, b + c^2, b + d^2, -dc + cd, b + e^2, b + f^2, -fe + ef, b + g^2, b + h^2, -hg + gh,$
$b+i^2, b+j^2, -ji+ij, b+k^2, b+l^2, -lk+kl, b+m^2, b+n^2, -nm+mn, b+o^2, b+p^2,$
$-po + op, c - bc, -d + bd, e - be, -f + bf, g - bg, -h + bh, i - bi, -j + bj, k - bk,$
$-l + bl, m - bm, -n + bn, o - bo, -p + bp, -g - pn - lj - ec, 2g + pn + lj - fc + cf,$
$-2g-pn-pm+on-lj+fc-de, -g-pn-lj+ce, -h-2g-pn-lj+df, e-pl+nj-gc,$
$-2e+pl-nj-hc+ch, 2e-pl-pk+ol+nj+hc-dg, e-pl+nj+cg, f+2e-pl+nj+dh,$
$-o-nh+lf-ic, 2o+nh-lf-jc+cj, -2o-nh-ng+mh+lf+jc-di, -o-nh+lf+ci,$
$-p - 2o - nh + lf + dj, n + m + ld - kc, -n - ld - lc + cl, n + pg - oh + ld + lc - dk,$
$n+m+ld+ck, ld+dl, -l-k+nd-mc, l-nd-nc+cn, -l+pe-of+nd+nc-dm,$
$-l-k+nd+cm, nd+dn, -j-i+pd-oc, j-pd-pc+cp, -j+pd+od-do, -j-i+pd+co,$
$pd + dp, -c - pj - nl - ge, 2c + pj + nl - he + eh, -2c - pj - pi + oj - nl + he - fg,$
$-c - pj - nl + eg, -d - 2c - pj - nl + fh, -m + ph - ld - ie, 2m - ph + ld - je + ej,$
$-2m + ph + pg - oh - ld + je - fi, -m + ph - ld + ei, -n - 2m + ph - ld + fj,$
$-p-o+lf-ke, p-lf-le+el, -p+ng-mh+lf+le-fk, -p-o+lf+ek, lf+fl,$
$-j - i + nf - me, j - nf - ne + en, -j - pc + od + nf + ne - fm, -j - i + nf + em,$
$nf + fn, l + k + pf - oe, -l - pf - pe + ep, l + pf + of - fo, l + k + pf + eo, pf + fp,$
$-k - pf + nd - ig, 2k + pf - nd - jg + gj, -2k - pf - pe + of + nd + jg - hi,$
$-k - pf + nd + gi, -l - 2k - pf + nd + hj, i - pd - nf - kg, -2i + pd + nf - lg + gl,$
$2i - pd - pc + od - nf + lg - hk, i - pd - nf + gk, j + 2i - pd - nf + hl, p + o + nh - mg,$
$-p - nh - ng + gn, p + nh + mh - hm, p + o + nh + gm, nh + hn, -n - m + ph - og,$
$n - ph - pg + gp, -n + ph + oh - ho, -n - m + ph + go, ph + hp, h + g + lj - ki,$
$-h - lj - li + il, h - pm + on + lj + li - jk, h + g + lj + ik, lj + jl, f + e + nj - mi,$
$-f - nj - ni + in, f + pk - ol + nj + ni - jm, f + e + nj + im, nj + jn, d + c + pj - oi,$
$-d-pj-pi+ip, d+pj+oj-jo, d+c+pj+io, pj+jp, d+c+nl-mk, -d-nl-nk+kn,$
$d - pi + oj + nl + nk - lm, d + c + nl + km, nl + ln, -f - e + pl - ok, f - pl - pk + kp,$
$-f + pl + ol - lo, -f - e + pl + ko, pl + lp, h + g + pn - om, -h - pn - pm + mp,$
$h + pn + on - no, h + g + pn + mo, pn + np, d - db, f - fb, h - hb, j - jb, l - lb, n - nb,$
$p - pb, -1/2b + po - 1/2nm + lk - 1/2ji - 1/2hg + fe - 1/2dc - 3/2plf, -1/6b + 1/3po +$
$1/3nm - 1/6lk - 1/6ji + 1/3hg - 1/6fe - 1/6dc + 1/2pnh, -f + pl + ol - nj - ni - ljc,$
$2pk - 2ol - 2hc + 2gd, pn + pm - li + nle, 3/2h + 3/2g + 3/2pn + 3/2pm + 3/2lj + 3/2li -$
$3/2fc + 3/2pld, -4/3b + 2/3po - 4/3nm - 4/3lk + 2/3ji + 2/3hg + 2/3fe + 2/3dc + 2nlc,$
$2pc - 2od - 2ne + 2mf, 2pe - 2of - 2jg + 2ih, n - oh + lc + pli, -n + m + oh - lc - je + plj,$
$-2l - 4k - 2pf + 2nd - 2jh, 2n + 4m - 2ph + 2ld + 2jf, -l + nd + nc + ldc, 2pc - 2od - 2lg + 2kh,$
$ng + lf + le - nfc, p - o + mh - le + jc + nlj, 2f - 2ol + 2ni - 2pnc, 2j + 4i - 2pd - 2nf - 2lh,$

$-2p-4o-2nh+2lf-2jd$, $-2f-4e+2pl-2nj+2hd$, $-n-ld-lc+ndc$, $2ng-2mh-2jc+2id$, $3ng-3mh+3le-3kf$, $-i+pd-lg-mhc$, $-pm+on+li-kj$, $j+2i-pd-nf-ne+lg-nhc$, $-g-pn-pm+li+fc+nlf$, $2h+4g+2pn+2lj+2fd$, $2o+nh-lf-jc+pdc$, $2o+nh+ng-mh-lf-jc-odc$, $d+pj+oj-pod$, $m+ld-je-ofc$, $2m-ph-pg+oh-lc-je-olj$, $k-nd-jg-ohc$, $2k+pf+pe-of+nc-jg+onj$, $-ji-ojc$, $2/3b-1/3po-1/3nm+2/3lk-1/3ji-1/3hg+2/3fe-1/3dc+olf$, $g+pn+pm-on-li-olc$, $-lj-li+fc+old$, $e-pl-pk+ol-ni-onc$, $-nj-ni-hc+ond$, $-4/3b-4/3po+2/3nm-4/3lk+2/3ji+2/3hg+2/3fe+2/3dc-2ple$, $2pm-2on-2fc+2ed$, $nf+ne-lg+lfc$, $-8/3o+8/3ob$, $p+mh-le-nli$, $-l-pf-pe+lfe$, $-2l+2k-2of+2nc-2jg-2pnj$, $-2e+2eb$, $4pg-4oh+4lc-4kd$, $4pg-4oh+4je-4if$, $-2d-2oj-2nk-2pne$, $d+pj+oj+nl+nk-lje$, $2d+4c+2pj+2nl+2hf$, $2m-ph+ld-je+nfe$, $-1/2m+1/2mb$, $f+e-pl-pk+nj+ni+hc+pnd$, $k+pf-jg-mhe$, $-1/3b-1/3po+2/3nm+2/3lk-1/3ji-1/3hg-1/3fe+2/3dc+nld$, $-h-pn-on-lj-li+njc$, $-l-2k-pf+nd+nc+jg-nhe$, $2/3b-1/3po+2/3nm-1/3lk+2/3ji-1/3hg-1/3fe-1/3dc-nje$, $2g-2gb$, $-p+lf+le+pfe$, $-p+ng-mh+lf+le-ofe$, $f-pl-ol-pof$, $i-nf+lg-ohe$, $2i-pd-pc+od+ne+lg-onl$, $h+g+pn+lj+li-oje$, $2h+2on+2li-2plc$, $lk-ole$, $-2/3b+1/3po-2/3nm+1/3lk+1/3ji-2/3hg+1/3fe+1/3dc+onh$, $-c-pj-pi+oj+nk-one$, $nl+nk-he+onf$, $pm-lj-li-pje$, $4b+4po+4nm-2lk-2ji-2hg-2fe-2dc-6png$, $2c-2cb$, $-2j+2i+2od+2ne+2lg+2pnl$, $2/3k-2/3kb$, $2/3b-1/3po-1/3nm+2/3lk+2/3ji-1/3hg-1/3fe-1/3dc-ljg$, $-n+ph+pg+nhg$, $-n+ph+oh-mhg$, $-pm-nmh$, $-pj-pi+nk-njg$, $-pl-pk-ni-nlg$, $-p-nh-ng+phg$, $-p-nh-mh-ohg$, $h+pn+on-poh$, $-f-e+pl-nj-ni-ojg$, $-3pk+3ol-3ni+3mj$, $-d-c-pj-nl-nk-olg$, $3pi-3oj-3nk+3ml$, $-nm-ong$, $2/3b+2/3po-1/3nm-1/3lk+2/3ji-1/3hg-1/3fe-1/3dc-pjc$, $pk+nj+ni-pjg$, $-pi+nl+nk-plg$, $nm-pmh$, $-pe+of+jg+lji$, $pg-oh+je+nji$, $-j-2i+pd+pc+nf-lg-phe$, $-ng+mh+jc+pji$, $jc-oji$, $j-pd-od-poj$, $-m+ph+pg-oh+lc-oli$, $-n-2m+ph+pg-ld+je-pfc$, $k+pf+pe-of+nc-oni$, $l+2k+pf+pe-nd-jg+phc$, $-3pe+3of-3nc+3md$, $pg-oh+lc+nlk$, $2l+2of-2nc-2pni$, $-ng+mh-le+plk$, $-le-olk$, $l+pf+of-pol$, $-i+pd+pc-od-ne-onk$, $-2i+2ib$, $-j+od+ne-pnk$, $mh+pnm$, $ng-onm$, $n-ph-oh-pon$, $1/2pi-1/2oj-1/2he+1/2gf$, $b-a$, $d+c+pj+pi+nl+nk-he-pnf$]
271 elements.

The leading monomials (ordered) are
[a, p^2, pb, op, o^2, om, ok, oi, og, oe, oc, ob, np, no, n^2, nb, mp, mo, mn, m^2, ml, mk, mj, mi, mg, mf, me, md, mc, mb, lp, lo, ln, lm, l^2, lh, lb, kp, ko, kn, km, kl, k^2, kj, ki, kh, kg, kf, ke, kd, kc, kb, jp, jo, jn, jm, jl, jk, j^2, jh, jf, jd, jb, ip, io, in, im, il, ik, ij, i^2, ih, ig, if, ie, id, ic, ib, hp, ho, hn, hm, hl, hk, hj, hi, h^2, hf, hd, hb, gp, go, gn, gm, gl, gk, gj, gi, gh, g^2, gf, ge, gd, gc, gb, fp, fo, fn, fm, fl, fk, fj, fi, fh, fg, f^2, fd, fb, ep, eo, en, em, el, ek, ej, ei, eh, eg, ef, e^2, ed, ec, eb, dp, do, dn, dm, dl, dk, dj, di, dh, dg, df, de, d^2, db, cp, co, cn, cm, cl, ck, cj, ci, ch, cg, cf, ce, cd, c^2, cb, bp, bo, bn, bm, bl, bk, bj, bi, bh, bg, bf, be, bd, bc, b^2, pon, pol, poj, poh, pof, pod, pnm, pnl, pnk, pnj, pni, pnh, png, pnf, pne, pnd, pnc, pmh, plk, plj, pli, plg, plf, ple, pld, plc, pji, pjg, pje, pjc, phg, phe, phc, pfe, pfc, pdc, onm, onl, onk, onj, oni, onh, ong, onf, one, ond, onc, olk, olj, oli, olg, olf, ole, old, olc, oji, ojg, oje, ojc, ohg, ohe, ohc, ofe, ofc, odc, nmh, nlk, nlj, nli, nlg, nlf, nle, nld, nlc, nji, njg, nje, njc, nhg, nhe, nhc, nfe, nfc, ndc, mhg, mhe, mhc, lji, ljg, lje, ljc, lfe, lfc, ldc]
271 elements.

We find that the only normal monomials are of degree 0, 1, or 2, precisely those not contained in this list above. They are
$[1, p, o, n, m, l, k, j, i, h, g, f, e, d, c, b, po, pn, pm, pl, pk, pj, pi, ph, pg, pf, pe, pd, pc, on, ol, oj, oh, of, od, nm, nl, nk, nj, ni, nh, ng, nf, ne, nd, nc, mh, lk, lj, li, lg, lf, le, ld, lc, ji, jg, je, jc, hg, he, hc, fe, fc, dc]$
65 elements.

§4.2 The Associated Graded Algebra

Denote $Tens(\mathbb{O} \oplus \mathbb{O})$ and T and denote $U_V(\mathbb{O})$ as A. We have $\{F_iT\}$ the weight filtration of the tensor algebra T. To determine gr_*A we must first determine $F_iA = im(F_iT \to A)$. This mapping sends $x \in F_iT$ to $x + I = [x]$. First note that T/I is generated by the images of the normal monomials. Since these normal monomials are all degree 1 or 2, then we have that $\forall [x] \in F_iA$ where $i \geq 2$, $[x]$ is written as a linear combination of images of degree 1 or 2 monomials. So, $[x] \in F_2A$.

$$F_iA = F_2A, i \geq 2$$

Moreover, we have $F_0A = K$ and F_1A is generated by the basis

$$\{[1], [b], [c], [d], [e], [f], [g], [h], [i], [j], [k], [l], [m], [n], [o], [p]\}$$

since $[a] = [0]$. F_2A is generated by the basis that consists of the images of normal monomials. This is described in the last paragraph of the previous section. There are 65 of these. Therefore we have that

$$\text{rank}_K(gr_iU(A)) = \begin{cases} 1 & i = 0 \\ 15 & i = 1 \\ 49 & i = 2 \\ 0 & i \geq 3 \end{cases}$$

§4.3 $\mathbb{O}$ over $\mathbb{F}_2$

The final simplified Grobner basis is
$[a + a^2, b + b^2, ba + ab, a + c^2, b + d^2, dc + cd, a + e^2, b + f^2, fe + ef, a + g^2, b + h^2, hg + gh, a + i^2, b + j^2, ji + ij, a + k^2, b + l^2, lk + kl, a + m^2, b + n^2, nm + mn, a + o^2, b + p^2, po + op, c + ca, da + ad, cb + bc, c + ac, d + bd, e + ea, fa + af, eb + be, e + ae, f + bf, g + ga, ha + ah, gb + bg, g + ag, h + bh, i + ia, ja + aj, ib + bi, i + ai, j + bj, k + ka, la + al, kb + bk, k + ak, l + bl, m + ma, na + an, mb + bm, m + am, n + bn, o + oa, pa + ap, ob + bo, o + ao, p + bp, g + pn + lj + ec, pn + lj + fc + cf, pn + lj + ed + de, g + pn + lj + ce, h + pn + lj + df, e + pl + nj + gc, pl + nj + hc + ch, pl + nj + gd + dg, e + pl + nj + cg, f + pl + nj + dh, o + nh + lf + ic, nh + lf + jc + cj, nh + lf + id + di, o + nh + lf + ci, p + nh + lf + dj, n + m + ld + kc, n + ld + lc + cl, n + ld + kd + dk, n + m + ld + ck, ld + dl, l + k + nd + mc, l + nd + nc + cn, l + nd + md + dm, l + k + nd + cm, nd + dn, j + i + pd + oc, j + pd + pc + cp, j + pd + od + do, j + i + pd + co, pd + dp, c + pj + nl + ge, pj + nl + he + eh, pj + nl + gf + fg, c + pj + nl + eg, d + pj + nl + fh, m + ph + ld + ie, ph + ld + je + ej, ph + ld + if + fi, m + ph + ld + ei, n + ph + ld + fj, p + o + lf + ke, p + lf + le + el, p + lf + kf + fk, p + o + lf + ek, lf + fl, j + i + nf + me, j + nf + ne + en, j + nf + mf + fm, j + i + nf + em, nf + fn, l + k + pf + oe, l + pf + pe + ep, l + pf + of + fo, l + k + pf + eo, pf + fp, k + pf + nd + ig, pf + nd + jg + gj, pf + nd + ih + hi, k + pf + nd + gi, l + pf + nd + hj, i + pd + nf + kg, pd + nf + lg + gl, pd + nf + kh + hk, i + pd + nf + gk, j + pd + nf + hl, p + o + nh + mg, p + nh + ng + gn, p + nh + mh + hm, p + o + nh + gm, nh + hn,$

$n+m+ph+og,\ n+ph+pg+gp,\ n+ph+oh+ho,\ n+m+ph+go,\ ph+hp,$
$h+g+lj+ki,\ h+lj+li+il,\ h+lj+kj+jk,\ h+g+lj+ik,\ lj+jl,\ f+e+nj+mi,$
$f+nj+ni+in,\ f+nj+mj+jm,\ f+e+nj+im,\ nj+jn,\ d+c+pj+oi,$
$d+pj+pi+ip,\ d+pj+oj+jo,\ d+c+pj+io,\ pj+jp,\ d+c+nl+mk,\ d+nl+nk+kn,$
$d+nl+ml+lm,\ d+c+nl+km,\ nl+ln,\ f+e+pl+ok,\ f+pl+pk+kp,\ f+pl+ol+lo,$
$f+e+pl+ko,\ pl+lp,\ h+g+pn+om,\ h+pn+pm+mp,\ h+pn+on+no,$
$h+g+pn+mo,\ pn+np,\ d+db,\ f+fb,\ h+hb,\ j+jb,\ l+lb,\ n+nb,\ p+pb,\ cb+cba,$
$dc+dcb,\ eb+eba,\ fe+feb,\ ed+eda,\ fc+fcb,\ gb+gba,\ hg+hgb,\ gd+gda,\ hc+hcb,$
$gf+gfa,\ he+heb,\ ib+iba,\ ji+jib,\ id+ida,\ jc+jcb,\ if+ifa,\ je+jeb,\ ih+iha,$
$jg+jgb,\ kb+kba,\ lk+lkb,\ kd+kda,\ lc+lcb,\ kf+kfa,\ le+leb,\ kh+kha,\ lg+lgb,$
$kj+kja,\ li+lib,\ n+na+ld+lda,\ p+pa+lf+lfa,\ h+lj+ha+lja,\ mb+mba,$
$nm+nmb,\ md+mda,\ nc+ncb,\ mf+mfa,\ ne+neb,\ mh+mha,\ ng+ngb,\ mj+mja,$
$ni+nib,\ ml+mla,\ nk+nkb,\ l+nd+la+nda,\ j+nf+ja+nfa,\ p+pa+nh+nha,$
$f+nj+fa+nja,\ d+nl+da+nla,\ ob+oba,\ po+pob,\ od+oda,\ pc+pcb,\ of+ofa,$
$pe+peb,\ oh+oha,\ pg+pgb,\ oj+oja,\ pi+pib,\ ol+ola,\ pk+pkb,\ on+ona,\ pm+pmb,$
$j+pd+ja+pda,\ l+pf+la+pfa,\ n+ph+na+pha,\ d+pj+da+pja,\ f+pl+fa+pla,$
$h+pn+ha+pna,\ pl+nj+pnc+ljc,\ pl+nj+gd+edc,\ he+fed,\ pn+lj+plc+njc,$
$pn+lj+ed+gdc,\ pj+nl+gf+hgd,\ fe+gfc,\ dc+pnh+plf+pjc,\ fe+hed,$
$j+pc+nf+kh+pnk+lfc,\ j+pd+od+idc,\ pi+jid,\ of+nd+ih+pnj+pni+ifc,$
$pf+nd+nc+ih+pni+phc,\ oh+ld+if+plj+pli+ihc,\ ph+ld+lc+if+pli+pfc,$
$pe+nd+ih+pnj+pni+jed,\ pg+ld+if+plj+pli+jgd,\ l+nd+nc+ldc,$
$l+nd+md+kdc,\ nk+lkd,\ pd+mf+kh+pnl+pnk+kfc,\ pd+nf+ne+kh+pnk+phe,$
$p+nh+mh+lf+id+nlj+nli+khc,\ p+lf+jc+id+nli+nfc,\ f+mj+hc+pnd+pnc+kjc,$
$pd+ne+kh+pnl+pnk+led,\ p+nh+ng+lf+id+nlj+nli+lgd,\ f+ni+hc+pnd+pnc+lid,$
$n+ld+lc+ndc,\ n+ld+kd+mdc,\ d+nl+ml+nmd,\ p+nh+lf+kf+id+nlj+nli+mfc,$
$pd+pnl+pnk+mhc,\ h+kj+fc+pld+plc+mjc,\ h+pn+fc+ed+plc+nle,\ lk+mlc,$
$b+nm+nld+mld,\ p+nh+lf+le+id+nlj+nli+ned,\ pd+lg+kh+pnl+pnk+ngd,$
$h+li+fc+pld+plc+nid,\ lk+nkd,\ nh+lf+jc+pdc,\ nh+lf+id+odc,\ d+pj+oj+pod,$
$ld+plj+pli+ofc,\ nd+pnj+pni+ohc,\ ji+ojc,\ lj+li+fc+pld+plc+olc,$
$h+on+lj+fc+plc+old,\ nj+ni+hc+pnd+pnc+onc,\ f+ol+nj+hc+pnc+ond,$
$ld+je+if+plj+pli+ped,\ nd+jg+ih+pnj+pni+pgd,\ ji+pid,\ lj+kj+fc+pld+plc+pkd,$
$nj+mj+hc+pnd+pnc+pmd,\ fc+gfe,\ pl+nj+gd+hgf,\ j+nf+mf+ife,\ ni+jif,$
$l+pf+nd+jh,\ mh+lf+id+nlj+nli+ihe,\ ng+lf+id+nlj+nli+jgf,\ l+pf+pe+lfe,$
$l+pf+of+kfe,\ pk+lkf,\ j+pd+nf+lh,\ n+ph+oh+ld+if+plj+pli+khe,$
$d+oj+he+pnf+pne+kje,\ pj+nl+pne+lje,\ n+ph+pg+ld+if+plj+pli+lgf,$
$d+pi+he+pnf+pne+lif,\ ph+ld+je+nfe,\ ph+ld+if+mfe,\ f+nj+mj+nmf,$
$p+nh+lf+jd,\ pf+pnj+pni+mhe,\ ji+mje,\ fe+pnh+nld+nje,\ lj+li+$
$fc+pld+plc+mle,\ h+pn+pm+fc+plc+mlf,\ pf+jg+ih+pnj+pni+ngf,$
$ji+nif,\ lj+kj+fc+pld+plc+nkf,\ p+lf+le+pfe,\ p+lf+kf+ofe,$
$f+pl+ol+pof,\ n+ph+ld+jf,\ nf+pnl+pnk+ohe,\ h+kj+fc+pld+plc+oje,$
$h+li+fc+pld+plc+pif,\ lk+ole,\ b+po+plf+olf,\ nl+nk+he+pnf+pne+one,$
$d+oj+nl+he+pne+onf,\ nf+lg+kh+pnl+pnk+pgf,\ h+lj+fc+ed+plc+pje,$
$lk+pkf,\ nl+ml+he+pnf+pne+pmf,\ pd+pc+nf+kh+pnk+nhc,\ pd+nf+kh+ihg,$
$li+jih,\ pf+pe+nd+ih+pni+nhe,\ pf+nd+ih+khg,\ h+lj+kj+lkh,\ ji+kjg,$
$ji+lih,\ hg+plf+nld+ljg,\ n+ph+pg+nhg,\ n+ph+oh+mhg,\ pm+nmh,$
$d+oj+he+pnf+pne+mjg,\ d+pi+he+pnf+pne+nih,\ f+ol+hc+pnd+pnc+mlg,$
$f+pk+hc+pnd+pnc+nkh,\ d+pj+he+gf+pne+njg,\ f+pl+hc+gd+pnc+nlg,$
$p+nh+ng+phg,\ p+nh+mh+ohg,\ h+pn+on+poh,\ f+mj+hc+pnd+pnc+ojg,$
$f+ni+hc+pnd+pnc+pih,\ d+ml+he+pnf+pne+olg,\ d+nk+he+pnf+pne+pkh,$
$nm+ong,\ b+po+pnh+onh,\ f+nj+hc+gd+pnc+pjg,\ d+nl+he+gf+pne+plg,$
$nm+pmh,\ ih+lji,\ jg+kji,\ pd+nf+kh+lkj,\ f+pl+nj+hd,\ d+pj+nl+hf,$
$if+nji,\ je+mji,\ j+nf+mf+nmj,\ h+pn+lj+fd,\ nh+le+id+nlj+nli+mli,$

$p + nh + mh + id + nli + mlj$, $nh + kf + id + nlj + nli + nkj$, $id + pji$, $jc + oji$, $j + pd + od + poj$, $ph + lc + if + plj + pli + oli$, $n + ph + oh + if + pli + olj$, $pf + nc + ih + pnj + pni + oni$, $l + pf + of + ih + pni + onj$, $ph + kd + if + plj + pli + pkj$, $pf + md + ih + pnj + pni + pmj$, $kd + nlk$, $lc + mlk$, $l + nd + md + nml$, $kf + plk$, $le + olk$, $l + pf + of + pol$, $pd + ne + kh + pnl + pnk + onk$, $j + pd + od + kh + pnk + onl$, $pd + mf + kh + pnl + pnk + pml$, $mh + pnm$, $ng + onm$, $n + ph + oh + pon$, $h + lj + pld + nlf$, $b + dc + nld + nlc$, $nh + lf + nlj + pnld$, $b + fe + plf + ple$, $b + hg + pnh + png$, $pnh + plf + nld + pnlj$]
410 elements.

The normal monomials are
[1, a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, ba, cb, da, dc, eb, ed, fa, fc, fe, gb, gd, gf, ha, hc, he, hg, ib, id, if, ih, ja, jc, je, jg, ji, kb, kd, kf, kh, kj, la, lc, ld, le, lf, lg, li, lj, lk, mb, md, mf, mh, mj, ml, na, nc, nd, ne, nf, ng, nh, ni, nj, nk, nl, nm, ob, od, of, oh, oj, ol, on, pa, pc, pd, pe, pf, pg, ph, pi, pj, pk, pl, pm, pn, po, nld, nli, nlj, plc, pld, plf, pli, plj, pnc, pnd, pne, pnf, pnh, pni, pnj, pnk, pnl, $pnli$]
113 elements.

Therefore we have that

$$\operatorname{rank}_{\mathbb{F}_2}(gr_iU(A)) = \begin{cases} 1 & i=0 \\ 16 & i=1 \\ 78 & i=2 \\ 17 & i=3 \\ 1 & i=4 \\ 0 & i\geq 5 \end{cases}$$

§4.4 $\mathbb{O}$ over $\mathbb{F}_3$

The final simplified Grobner basis is
[$2b+b^2$, $b+c^2$, $b+d^2$, $b+po+nm+lk+ji+hg+fe+cd$, $b+e^2$, $b+f^2$, $2fe+ef$, $b+g^2$, $b+h^2$, $2hg+gh$, $b+i^2$, $b+j^2$, $2ji+ij$, $b+k^2$, $b+l^2$, $2lk+kl$, $b+m^2$, $b+n^2$, $2nm+mn$, $b+o^2$, $b+p^2$, $2po+op$, $c+2bc$, $2d+bd$, $e+2be$, $2f+bf$, $g+2bg$, $2h+bh$, $i+2bi$, $2j+bj$, $k+2bk$, $2l+bl$, $m+2bm$, $2n+bn$, $o+2bo$, $2p+bp$, $2g+2pn+2lj+2ec$, $2g+pn+lj+2fc+cf$, $g+2pn+2pm+on+2lj+fc+2de$, $2g+2pn+2lj+ce$, $2h+g+2pn+2lj+df$, $e+2pl+nj+2gc$, $e+pl+2nj+2hc+ch$, $2e+2pl+2pk+ol+nj+hc+2dg$, $e+2pl+nj+cg$, $f+2e+2pl+nj+dh$, $2o+2nh+lf+2ic$, $2o+nh+2lf+2jc+cj$, $o+2nh+2ng+mh+lf+jc+2di$, $2o+2nh+lf+ci$, $2p+o+2nh+lf+dj$, $n+m+ld+2kc$, $2n+2ld+2lc+cl$, $n+pg+2oh+ld+lc+2dk$, $n+m+ld+ck$, $ld+dl$, $2l+2k+nd+2mc$, $l+2nd+2nc+cn$, $2l+pe+2of+nd+nc+2dm$, $2l+2k+nd+cm$, $nd+dn$, $2j+2i+pd+2oc$, $j+2pd+2pc+cp$, $2j+pd+od+2do$, $2j+2i+pd+co$, $pd+dp$, $2c+2pj+2nl+2ge$, $2c+pj+nl+2he+eh$, $c+2pj+2pi+oj+2nl+he+2fg$, $2c+2pj+2nl+eg$, $2d+c+2pj+2nl+fh$, $2m+ph+2ld+2ie$, $2m+2ph+ld+2je+ej$, $m+ph+pg+2oh+2ld+je+2fi$, $2m+ph+2ld+ei$, $2n+m+ph+2ld+fj$, $2p+2o+lf+2ke$, $p+2lf+2le+el$, $2p+ng+2mh+lf+le+2fk$, $2p+2o+lf+ek$, $lf+fl$, $2j+2i+nf+2me$, $j+2nf+2ne+en$, $2j+2pc+od+nf+ne+2fm$, $2j+2i+nf+em$, $nf+fn$, $l+k+pf+2oe$, $2l+2pf+2pe+ep$, $l+pf+of+2fo$, $l+k+pf+eo$, $pf+fp$, $2k+2pf+nd+2ig$, $2k+pf+2nd+2jg+gj$, $k+2pf+2pe+of+nd+jg+2hi$, $2k+2pf+nd+gi$, $2l+k+2pf+nd+hj$, $i+2pd+2nf+2kg$, $i+pd+nf+2lg+gl$, $2i+2pd+2pc+od+2nf+lg+2hk$, $i+2pd+2nf+gk$, $j+2i+2pd+2nf+hl$, $p+o+nh+2mg$, $2p+2nh+2ng+gn$, $p+nh+mh+2hm$, $p+o+nh+gm$, $nh+hn$, $2n+2m+ph+2og$, $n+2ph+2pg+gp$, $2n+ph+oh+2ho$, $2n+2m+ph+go$, $ph+hp$,

$h+g+lj+2ki,\ 2h+2lj+2li+il,\ h+2pm+on+lj+li+2jk,\ h+g+lj+ik,\ lj+jl,$
$f+e+nj+2mi,\ 2f+2nj+2ni+in,\ f+pk+2ol+nj+ni+2jm,\ f+e+nj+im,$
$nj+jn,\ d+c+pj+2oi,\ 2d+2pj+2pi+ip,\ d+pj+oj+2jo,\ d+c+pj+io,$
$pj+jp,\ d+c+nl+2mk,\ 2d+2nl+2nk+kn,\ d+2pi+oj+nl+nk+2lm,$
$d+c+nl+km,\ nl+ln,\ 2f+2e+pl+2ok,\ f+2pl+2pk+kp,\ 2f+pl+ol+2lo,$
$2f+2e+pl+ko,\ pl+lp,\ h+g+pn+2om,\ 2h+2pn+2pm+mp,\ h+pn+on+2no,$
$h+g+pn+mo,\ pn+np,\ d+2db,\ f+2fb,\ h+2hb,\ j+2jb,\ l+2lb,\ n+2nb,$
$p+2pb,\ 2pk+ol+hc+2gd,\ pm+2on+2li+kj,\ nm+2lk+hg+2fe+pnh+plf,$
$k+2kb,\ pe+2of+nc+2md,\ 2f+pl+ol+2nj+2ni+2ljc,\ pk+2ol+ni+2mj,$
$2pi+oj+he+2gf,\ pn+pm+2li+nle,\ b+2po+lk+2hg+2pnh+2nlc,\ 2pc+od+ne+2mf,$
$2pe+of+jg+2ih,\ n+2oh+lc+pli,\ 2n+m+oh+2lc+2je+plj,\ l+2k+pf+2nd+jh,$
$2n+m+ph+2ld+2jf,\ 2pc+od+lg+2kh,\ ng+lf+le+2nfc,\ p+2o+mh+2le+jc+nlj,$
$2f+ol+2ni+pnc,\ 2j+i+pd+nf+lh,\ p+2o+nh+2lf+jd,\ f+2e+2pl+nj+2hd,$
$2ng+mh+jc+2id,\ 2o+ob,\ 2i+pd+2lg+2mhc,\ j+2i+2pd+2nf+2ne+lg+2nhc,$
$2pm+on+fc+2ed,\ 2h+g+2pn+2lj+2fd,\ d+pj+oj+2pod,\ m+ld+2je+2ofc,$
$2m+2ph+2pg+oh+2lc+2je+2olj,\ k+2nd+2jg+2ohc,\ 2k+pf+pe+2of+nc+2jg+onj,$
$2ji+2ojc,\ b+2po+2nm+lk+2hg+fe+2pnh+olf,\ g+pn+pm+2on+2li+2olc,$
$2lj+2li+fc+old,\ e+2pl+2pk+ol+2ni+2onc,\ 2nj+2ni+2hc+ond,\ pg+2oh+je+2if,$
$b+po+nm+lk+ji+hg+fe+dc,\ g+2gb,\ nf+ne+2lg+lfc,\ j+2i+2od+2ne+2lg+2pnl,$
$f+e+2pl+2pk+nj+ni+hc+pnd,\ ng+2mh+le+2kf,\ p+mh+2le+2nli,$
$2l+2pf+2pe+lfe,\ l+2k+of+2nc+jg+pnj,\ e+2eb,\ pg+2oh+lc+2kd,\ 2m+mb,$
$d+oj+nk+pne,\ d+pj+oj+nl+nk+2lje,\ 2d+c+2pj+2nl+2hf,\ 2m+2ph+ld+2je+nfe,$
$k+pf+2jg+2mhe,\ b+2nm+lk+2hg+2pnh+ple,\ 2h+2pn+2on+2lj+2li+njc,$
$2l+k+2pf+nd+nc+jg+2nhe,\ b+2po+ji+2hg+2pnh+2nje,\ g+pn+pm+2li+$
$2fc+2nlf,\ 2p+lf+le+pfe,\ 2p+ng+2mh+lf+le+2ofe,\ f+2pl+2ol+2pof,$
$i+2nf+lg+2ohe,\ 2i+2pd+2pc+od+ne+lg+2onl,\ h+g+pn+lj+li+2oje,$
$lk+2ole,\ 2b+po+pnh+onh,\ 2c+2pj+2pi+oj+nk+2one,\ nl+nk+2he+onf,$
$pm+2lj+2li+2pje,\ i+2ib,\ 2h+2on+2li+plc,\ b+2po+nm+ji+2hg+fe+pnh+2nld,$
$b+2po+2nm+lk+ji+2hg+2pnh+2ljg,\ 2n+ph+pg+nhg,\ 2n+ph+oh+2mhg,$
$2pm+2nmh,\ 2c+cb,\ 2pj+2pi+nk+2njg,\ 2pl+2pk+2ni+2nlg,\ 2p+2nh+2ng+phg,$
$2p+2nh+2mh+2ohg,\ h+pn+on+2poh,\ 2f+2e+pl+2nj+2ni+2ojg,$
$2d+2c+2pj+2nl+2nk+2olg,\ 2nm+2ong,\ b+2nm+ji+2hg+2pnh+2pjc,$
$pk+nj+ni+2pjg,\ 2pi+nl+nk+2plg,\ nm+2pmh,\ 2pe+of+jg+lji,\ pg+2oh+je+nji,$
$2j+i+pd+pc+nf+2lg+2phe,\ 2ng+mh+jc+pji,\ jc+2oji,\ j+2pd+2od+2poj,$
$2m+ph+pg+2oh+lc+2oli,\ 2n+m+ph+pg+2ld+je+2pfc,\ k+pf+pe+2of+nc+2oni,$
$l+2k+pf+pe+2nd+2jg+phc,\ pg+2oh+lc+nlk,\ 2l+2of+nc+pni,\ 2ng+mh+2le+plk,$
$2le+2olk,\ l+pf+of+2pol,\ 2i+pd+pc+2od+2ne+2onk,\ 2j+od+ne+2pnk,$
$mh+pnm,\ ng+2onm,\ n+2ph+2oh+2pon,\ 2b+a,\ d+c+pj+pi+nl+nk+2he+2pnf,$
$2pi+oj+nk+2ml,\ 2h+2g+2pn+2pm+2lj+2li+fc+2pld,\ 2b+hg+pnh+png]$
267 elements.

The normal monomials are
$[1,\ b,\ c,\ d,\ e,\ f,\ g,\ h,\ i,\ j,\ k,\ l,\ m,\ n,\ o,\ p,\ fc,\ fe,\ hc,\ he,\ hg,\ jc,\ je,\ jg,\ ji,\ lc,\ ld,\ le,\ lf,$
$lg,\ li,\ lj,\ lk,\ mh,\ nc,\ nd,\ ne,\ nf,\ ng,\ nh,\ ni,\ nj,\ nk,\ nl,\ nm,\ od,\ of,\ oh,\ oj,\ ol,\ on,\ pc,$
$pd,\ pe,\ pf,\ pg,\ ph,\ pi,\ pj,\ pk,\ pl,\ pm,\ pn,\ po,\ pnh]$
65 elements.

Therefore we have that

$$\mathrm{rank}_{\mathbb{F}_3}(gr_i U(A)) = \begin{cases} 1 & i = 0 \\ 15 & i = 1 \\ 48 & i = 2 \\ 1 & i = 3 \\ 0 & i \geq 4 \end{cases}$$

§4.5 Future work

G_2 is the group of automorphisms of $\mathbb{O}$, and thus $rank_K(gr_i U(A))$ inherits a representation of G_2. It remains to be determined how this representation can be broken down. For $i = 1$, the dimension is 15, so it is likely $1 + 7 + 7$. However, it is more difficult for $i = 2$. Moreover, one can also study this representation for $\mathbb{O}$ over $\mathbb{F}_2$ and $\mathbb{O}$ over $\mathbb{F}_3$.

§5 The PBW Theorem

We now turn our attention to the PBW Theorem. First, we prove that a Grobner basis always induces a PBW theorem. Then, we provide an example of a $U_V(A)$ that does not satisfy the PBW theorem.

§5.1 A Grobner Basis Induces a PBW Theorem

It suffices to show that the top weight components of the basis elements generates the ideal defining the associated graded algebra. Since we are discussing a Grobner basis, we work with T, the tensor algebra of a set with a partial order. We use the weight filtration $\{F_iT\}$. We use the shorthand $[x]_n \overset{\text{def}}{=} x + F_{n-1}T$ as the coset of x as an element of gr_nT for $n \geq 1$.

Theorem 5.1

A Grobner basis always induces a PBW theorem.

Proof. Let G be the Grobner basis and $I = (G)$. First note that the n-th component of the ideal defining the associated graded algebra I_n can be described as $I \cap F_nT + F_{n-1}T$.

Here the function d coincides with the degree of a (non-zero) polynomial. The top weight components of the basis elements is described by $\{[g]_{d(g)} \mid g \in G\}$. Moreover, note that $\forall x \in F_{d(x)} \subset F_{d(x)+1} \subset \cdots$, the inclusion of x into gr_*T $(* \geq d(x))$ is nonzero if and only if $* = d(x)$. So, J_n, the n-th component of the ideal generated by this set is described as

$$\left\{ \sum_{*=1}^{r} [x]_{d(x)}[g]_{g(x)}[y]_{d(y)} \mid g \in G, d(xgy) = n \right\}$$

Now let us consider any element $[f]_n \in I_n$. Assume for the sake of contradiction that $[f]_n \notin J_n \implies [0]_n \notin [f]_n + J_n$. Note that we have the mapping between sets $LM : F_nT \backslash F_{n-1}T \to B_n$. Here, the domain consists of degree n polynomials, and the image B_n consists of degree n monomials. We claim that this mapping descends onto the mapping (use the same notation)

$$LM : (F_nT \backslash F_{n-1}T)/F_{n-1}T = gr_nT \backslash \{[0]_n\} \to B_n, [x]_n \mapsto LM(x)$$

this is well-defined since the leading monomial is always degree n so adding polynomials of degree n_1 does not affect the leading monomial.

Because $[0]_n \notin [f]_n + J_n$, we may map all elements of $[f]_n + J_n$ into B_n. Since B is partially ordered, we select a $[h]_n \in [f]_n + J_n$ such that $LM([h]_n)$ is the smallest. Now, since G is a Grobner basis, then we can find element g such that $LM(g) \mid LM(h) \implies \exists x, y \in B, xLM(g)y = LM(h)$. Since $d(xgy) = d(LM(xgy)) = d(xLM(g)y) = d(LM(h)) = d(h) = n$, then $[xgy]_n = [x]_{d(x)}[g]_{g(x)}[y]_{d(y)} \in J_n$. Thus, $[h - xgy]_n = [h]_n - [xgy]_n \in [f]_n + J_n$. However, we have that $LM([h - xgy]_n) = LM(h - xgy) < LM(h) = LM([h]_n)$. Contradiction. ∎

§5.2 $U_V(A)$ with no PBW Theorem

Consider the algebra defined by the relations

$$x(x(xy)) + xx, \; ((xy)y)y + xy$$

$$\begin{aligned}\frac{\partial x(x(xy))+xx}{\partial x} &= \rho_{x(xy)} + \frac{\partial x(xy)}{\partial x}\lambda_x + \lambda_x + \rho_x \\ &= \rho_{x(xy)} + (\rho_{xy} + \rho_y\lambda_x)\lambda_x + \lambda_x + \rho_x \\ &= \rho_{x(xy)} + \rho_{xy}\lambda_x + \rho_y\lambda_x^2 + \lambda_x + \rho_x\end{aligned}$$

$$\begin{aligned}\frac{\partial x(x(xy))+x}{\partial y} &= \frac{\partial x(xy)}{\partial y}\lambda_x \\ &= \lambda_x^3\end{aligned}$$

$$\begin{aligned}\frac{\partial((xy)y)y+xy}{\partial x} &= \frac{\partial(xy)y}{\partial x}\rho_y + \rho_y \\ &= \rho_y^3 + \rho_y\end{aligned}$$

$$\begin{aligned}\frac{\partial((xy)y)y+xy}{\partial y} &= \lambda_{(xy)y} + \frac{\partial(xy)y}{\partial y}\rho_y + \lambda_x \\ &= \lambda_{(xy)y} + (\lambda_{xy} + \lambda_x\rho_y)\rho_y + \lambda_x \\ &= \lambda_{(xy)y} + \lambda_{xy}\rho_y + \lambda_x\rho_y^2 + \lambda_x\end{aligned}$$

Let A be a K-module regarded as such an algebra with trivial product. So, the ideal I in $\mathrm{Tens}(A \oplus A)$ defining $U_V(A)$ is generated by

$$r_y l_x^2 + l_x + r_x,\ l_x^3,\ r_y^3 + r_y,\ l_x r_y^2 + l_x,\ \text{for all } x, y \in A$$

Thus, we have $l_x r_y(r_y l_x^2 + l_x + r_x) - (l_x r_y^2 + l_x)l_x^2 + l_x^3 = l_x r_y l_x + l_x r_y r_x \in I$. Note that $l_x r_y l_x + l_x r_y r_x$ is in $F_3T \cap I$ but its residue class mod F_2T is not in the ideal generated by the classes of $r_y l_x^2$, l_x^3, r_y^3, and $l_x r_y^2$. It follows that $I_3 \neq J_3$ and PBW is not satisfied.

§6 References

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