E0 298 | Linear Algebra and Its Applications

Assignment 3

Deadline: 11.30 AM, 5 October, 2023 | Full Marks: 25

1. Let $\mathbb{V}$ be a nontrivial finite-dimensional vector space and $\mathbb{W}$ be an infinite-dimensional vector space. Prove that the space $\mathcal{L}(\mathbb{V}, \mathbb{W})$ is infinite-dimensional. (4)

2. Suppose $\mathbb{V}$ is finite-dimensional and $T_1, T_2 \in \mathcal{L}(\mathbb{V})$. Prove that $\mathcal{R}(T_1) \subseteq \mathcal{R}(T_2)$ if and only if there exists $T \in \mathcal{L}(\mathbb{V})$ such that $T_1 = T_2 \circ T$. (6)

3. Given an example of $A \in \mathbb{R}^{2\times 2}$ such that $\mathcal{N}(A) = \mathcal{R}(A)$. On the other hand, show that there cannot exist $A \in \mathbb{R}^{3\times 3}$ such that $\mathcal{N}(A) = \mathcal{R}(A)$. (4)

4. Suppose $\mathbb{V}$ is a vector space and $T \in \mathcal{L}(\mathbb{V})$. Prove that T is a scalar multiple of the identity map if and only if $R \circ T = T \circ R$ for all $R \in \mathcal{L}(\mathbb{V})$. (6)

5. Suppose $\mathbb{V}$ is finite-dimensional and $T_1, T_2 \in \mathcal{L}(\mathbb{V})$. Prove that (5)

$$\text{nullity}\,(T_1 \circ T_2) \leqslant \text{nullity}\,T_1 + \text{nullity}\,T_2.$$

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