E0 298 | Linear Algebra and Its Applications

Assignment 5

Deadline: 11.30 AM, 16-11-23 — Full Marks: 25

1. Prove that the set of invertible matrices in $\mathbb{R}^{n\times n}$ forms an open set (with respect to the topology induced by some matrix norm $\|\cdot\|$). That is, prove that for any invertible $A \in \mathbb{R}^{n\times n}$, there exists $\delta > 0$ such that $\|X - A\| < \delta \Rightarrow X$ is invertible. (5)

2. Let $A \in \mathbb{R}^{n\times n}$ and $\mathbb{U}$ be a k-dimensional subspace of $\mathbb{R}^n$, where $1 \leqslant k \leqslant n$. For any orthonormal basis $\{v_1, \ldots, v_k\}$ of $\mathbb{U}$, define (5)

$$\varphi(v_1, \ldots, v_n) = \sum_{i=1}^{k} v_i^\top A v_i.$$

   Show that φ is the same for any choice of orthonormal basis $\{v_1, \ldots, v_k\}$ of $\mathbb{U}$.

3. Let $A, B \in \mathbb{S}^n$ and A be positive definite.
   (a) Prove that though $A^{-1}B$ is generally not symmetric, its eigenvalues are real. (3)
   (b) If B is positive definite, then prove that the eigenvalues of $A^{-1}B$ are positive. (2)

   (Hint: If A is positive semidefinite, we can construct a positive semidefinite matrix R such $R^2 = A$).

4. Let $A, B \in \mathbb{S}^n$ be positive semidefinite. Prove that if $A^2 = B^2$, then necessarily $A = B$. (4)

   (Hint: Use contradiction; e.g., assume that $A \neq B$ and v is an eigenvector of $A - B \in \mathbb{S}^n$ with nonzero eigenvalue, and proceed).

5. Consider numbers $r_1, \ldots, r_m > 0$, and define $G \in \mathbb{S}^m$ to be (3)

$$G_{ij} = \frac{1}{r_i + r_j + 1} \qquad (1 \leqslant i, j \leqslant m).$$

   Show that G is positive semidefinite.

6. Consider the Gaussian function $g(x) = \exp(-\lambda\|x\|^2)$ defined on $\mathbb{R}^n$. Given m points $x_1, \ldots, x_n \in \mathbb{R}^n$, define $K \in \mathbb{S}^n$ to be (3)

$$K_{ij} = g(x_i - x_j) \qquad (1 \leqslant i, j \leqslant n).$$

   Prove that K is positive semidefinite.

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