MA324 TOPICS IN COMPLEX ANALYSIS
ASSIGNMENT 2

Due date: February 13 (Sun.) by 11:59 pm

Instructions. 1. Solutions may either be handwritten or typewritten. In either case, please upload a legible PDF file of your solutions to MS Teams.
2. As much as possible, try to attempt these problems on your own. If you end up taking any assistance (from any source — books, websites, friends, etc.) please give due credit.

Problem A. Given a compact subset $X \subset \mathbb{C}$, let

$$\operatorname{harm}(X) = \{f \in C(X;\mathbb{R}) : f \text{ extends to a harmonic function on some open neighborhood of } X\}.$$

Show that for any $\alpha \in M_{\mathbb{R}}(X)$, $\mathscr{L}[\alpha] \equiv 0$ on $\mathbb{C} \setminus X$ if and only if α is orthogonal to $\operatorname{harm}(X)$.

Problem B. Let $X \subset \mathbb{C}$ be a compact set such that $[\operatorname{Re} R(X)]\,|_{bX} = \mathcal{C}(bX;\mathbb{R})$. Suppose $\mu \in M(X)$ is such that $\mathscr{C}[\mu] = 0$ on $\mathbb{C} \setminus X$. Show that $\mathscr{C}[\mu](z_0) = 0$ for any $z_0 \in bX$ satisfying

$$\int_X \frac{d|\mu|(z)}{|z - z_0|} < \infty.$$

Hint. First approximate $|z - z_0|$ by real parts of rational functions with no poles on X. Then, modify those rational approximants to produce a family of functions $\{f_n\} \subset \mathcal{O}(X)$ such that $f_n \to (z - z_0)^{-1}$ almost everywhere on X. Keep in mind that you wish to apply dominated convergence theorem at this stage.

Problem C. Suppose $\Omega \subset \mathbb{C}$ is a domain whose boundary is a simple closed curve, and $0 \in \Omega$. Show that for any $f \in \mathcal{C}(b\Omega)$ and $\varepsilon > 0$, there exists a function g of the form $g(z) = \sum_{j=-n}^{n} a_j z^j$ such that $|f(z) - g(z)| < \varepsilon$ for all $z \in b\Omega$.