MA324 TOPICS IN COMPLEX ANALYSIS
ASSIGNMENT 3

Due date: March 10 (Th.) by 11:59 pm

Instructions. 1. Solutions may either be handwritten or typewritten. In either case, please upload a legible PDF file of your solutions to MS Teams.

2. As much as possible, try to attempt these problems on your own. If you end up taking any assistance (from any source — books, websites, friends, etc.) please give due credit.

Problem A. Suppose $f : X \to Y$ and $g : Y \to Z$ are holomorphic maps between compact Riemann surfaces. Show that $\deg(g \circ f) = \deg(f) \cdot \deg(g)$.

Problem B. Let $p(z)$ be a polynomial of degree $d \geq 2$ with distinct roots and let X be the affine curve $\{w^2 = p(z)\}$. Show that the projectivization of X in $\mathbb{P}^2$ (as defined in Lecture 10) is a smooth only when $d = 2, 3$.

Problem C. Let $\Pi : \mathbb{C}^2 \setminus \{0\} \to \mathbb{P}$ be the quotient map of the equivalence relation $w \sim w'$ if and only if $w' = \lambda w$ for some $\lambda \in \mathbb{C}$. Let $E := \{(z, w) \in \mathbb{P} \times \mathbb{C}^2 : w \in \Pi^{-1}(z)\}$ and $\pi : E \to \mathbb{P}$ be the projection onto the second coordinate.

(a) Show that E is a (smooth) manifold.
(b) Show that $\pi : E \to \mathbb{P}$ is a holomorphic line bundle over $\mathbb{P}$ (give a natural holomorphic atlas).
(c) Show that $\pi : E \to \mathbb{P}$ is isomorphic to the abstractly-defined bundle $\mathcal{O}(-1)$ (Lecture 14).

Problem D. Consider the conic

$$X_A := \{[x : y : z] \in \mathbb{P}^2 : F_A(x, y, z) = 0\},$$

where

$$F_A(x, y, z) = (x, y, z) \begin{pmatrix} a & b & c \\ b & e & d \\ c & d & f \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix},$$

and $A = \begin{pmatrix} a & b & c \\ b & e & d \\ c & d & f \end{pmatrix}$ is invertible.

(a) Show that X_A is a Riemann surface, i.e., nonsingular.
(b) Show that X_A is isomorphic to $\mathbb{P}$. (*Hint. First show that if A is the identity matrix, then X_A is isomorphic to $\mathbb{P}$*).