Computational Linear Algebra – Assignment 5

(points: 40)

Problem 1 (points: 4)

Suppose $\mathbb{V}$ is finite-dimensional with $\dim(\mathbb{V}) \geqslant 1$ and $\mathbb{W}$ is infinite-dimensional. Prove that $\mathcal{L}(\mathbb{V}, \mathbb{W})$ is infinite-dimensional.

Problem 2 (points: 2+2 = 4)

Given an example of $\mathbf{A} \in \mathbb{R}^{2\times 2}$ such that $N(\mathbf{A}) = R(\mathbf{A})$. On the other hand, show that there does not exist an $\mathbf{A} \in \mathbb{R}^{3\times 3}$ such that $N(\mathbf{A}) = R(\mathbf{A})$.

Problem 3 (points: 2)

Let $\ell_1, \ell_2 \in \mathcal{L}(\mathbb{V})$ be such that $R(\ell_2) \subseteq N(\ell_1)$. Prove that $(\ell_2 \circ \ell_1)^2 = 0$ (zero transform).

Problem 4 (points: 4+3+2 = 9)

Suppose $\mathbb{V}$ is finite-dimensional and $\ell \in \mathcal{L}(\mathbb{V})$. Prove that ℓ is one-to-one (injective) if and only if there exists $\ell' \in \mathcal{L}(\mathbb{V})$ such that $\ell' \circ \ell$ is the identity map on $\mathbb{V}$. Moreover, prove that $\ell' \circ \ell$ is the identity transform if and only if $\ell \circ \ell'$ is the identity transform. State the above results in terms of matrices.

Problem 5 (points: 3)

Do the set of invertible $n \times n$ matrices form a subspace of the set of $n \times n$ matrices? What about the set of invertible $n \times n$ matrices with $n \geqslant 2$?

Problem 6 (points: 7)

Suppose $\mathbb{V}$ is finite-dimensional and $\ell \in \mathcal{L}(\mathbb{V})$. Prove that ℓ is a scalar multiple of the identity map if and only if $\ell' \circ \ell = \ell \circ \ell'$ for every $\ell' \in \mathcal{L}(\mathbb{V})$.

Problem 7 (points: 5)

Suppose $\mathbb{V}$ is finite-dimensional and $\ell', \ell \in \mathcal{L}(\mathbb{V})$. Prove that

$$\dim(N(\ell' \circ \ell)) \leqslant \dim(N(\ell)) + \dim(N(\ell')).$$

Problem 8 (points: 6)

Suppose $\mathbb{V}$ is finite-dimensional and $\ell_1, \ell_2 \in \mathcal{L}(\mathbb{V})$. Prove that $R(\ell_1) \subset R(\ell_2)$ if and only if there exists $\ell' \in \mathcal{L}(\mathbb{V})$ such that $\ell_1 = \ell_2 \circ \ell'$.

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