# Computational Linear Algebra – Assignment 4

(points: 40)

**Problem 1** (points: 2 + 5 + 3 = 10)
Let $\mathbb{V}$ and $\mathbb{W}$ be finite-dimensional vector spaces and suppose $\ell \in \mathcal{L}(\mathbb{V}, \mathbb{W})$.

1. Suppose $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathbb{V}$ are such that $\ell(\boldsymbol{v}_1), \ldots, \ell(\boldsymbol{v}_n) \in \mathbb{W}$ are linear independent. Show that $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ must necessarily be linear independent.
2. Is the converse of (1) true? If no, can you suggest a condition on $\ell$ for which the converse is true?
3. Using (1), prove that if there exists a bijection $\ell \in \mathcal{L}(\mathbb{V}, \mathbb{W})$, then $\dim(\mathbb{V}) = \dim(\mathbb{W})$.

**Problem 2** (points: 6)
Let $\mathbb{V}$ and $\mathbb{W}$ be finite-dimensional vector spaces. Suppose $\ell \in \mathcal{L}(\mathbb{V}, \mathbb{W})$ and that $\dim(\mathbb{V}) = \dim(\mathbb{W})$. Prove that the following are equivalent[1]:

1. $\ell$ is one-to-one (injective).
2. $\ell$ is onto (surjective).
3. $\ell$ is bijective (both injective and surjective).

**Problem 3** (points: 6)
Let $\mathbb{V}$ be an abstract vector space over a field $\mathbb{F}$, and $\dim(\mathbb{V}) = n$. Prove that $\mathbb{V}$ is isomorphic to $\mathbb{F}^n$, i.e., one can construct a linear bijection from $\mathbb{V}$ to $\mathbb{F}^n$.

**Problem 4** (points: 1+3+4+3 = 11)
Let $\mathbb{U}, \mathbb{V}$ and $\mathbb{W}$ be finite-dimensional vector spaces. Suppose $f \in \mathcal{L}(\mathbb{U}, \mathbb{V})$ and $g \in \mathcal{L}(\mathbb{V}, \mathbb{W})$, and $\mathbf{A}_f$ and $\mathbf{A}_g$ be their matrix representations (for fixed choice of bases of $\mathbb{U}, \mathbb{V}$ and $\mathbb{W}$). Recall that the composition $g \circ f : \mathbb{U} \to \mathbb{W}$ is defined as $g \circ f(\boldsymbol{x}) = g(f(\boldsymbol{x}))$ for all $\boldsymbol{x} \in \mathbb{U}$. Prove the following:

1. $g \circ f \in \mathcal{L}(\mathbb{U}, \mathbb{W})$.
2. The matrix representation of $g \circ f$ is the matrix product $\mathbf{A}_g \mathbf{A}_f$.
3. If $f$ is a bijection, then the matrix representation of its inverse is $\mathbf{A}_f^{-1}$.
4. If $f$ and $g$ are both bijections, then the matrix representation of $(g \circ f)^{-1}$ is $\mathbf{A}_f^{-1} \mathbf{A}_g^{-1}$.

[1] work from first principles only and do not use matrices or their inverses.

**Problem 5** (points: 4+3 = 7)

Suppose $\mathbb{V}$ is a finite-dimensional vector space and $\mathbb{U}$ a proper subspace of $\mathbb{V}$ (i.e., $\mathbb{U} \neq \mathbb{V}$). Let $\ell \in \mathcal{L}(\mathbb{U}, \mathbb{W})$ for some vector space $\mathbb{W}$. Show that we can extended the domain of $\ell$ from $\mathbb{U}$ to $\mathbb{V}$ without changing its action on $\mathbb{U}$, i.e., there exists $\ell' \in \mathcal{L}(V, W)$ such that $\ell'(\boldsymbol{x}) = \ell(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{U}$. We call $\ell'$ the extension of $\ell$.

Suppose $\ell$ is not the zero map, i.e. $\ell(\boldsymbol{x})$ is non-zero for some $\boldsymbol{x} \in \mathbb{U}$. Let $\ell' : \mathbb{V} \to \mathbb{W}$ be defined as $\ell'(\boldsymbol{x}) = \ell(\boldsymbol{x})$ if $\boldsymbol{x} \in \mathbb{U}$; and $\ell'(\boldsymbol{x}) = 0$ if $\boldsymbol{x} \in \mathbb{V} \backslash \mathbb{U}$. Show that $\ell'$ is not a linear extension of $\ell$.

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