E0 298 | Linear Algebra and Its Applications

Assignment 1

Deadline: 11:30 AM, 29 August, 2023 Max mark: 20

1. Let $(\mathbb{F}, +, *)$ be a field and let $a, b, c \in \mathbb{F}$. Show that (2)

   1. $a + b = a + c \Rightarrow b = c$
   2. $a * b = a * c,\ a \neq 0 \Rightarrow b = c$
   3. $(-a) * (-b) = ab$
   4. $a * 0 = 0$

2. Let $n \geqslant 2$ be an integer and $X_n = \{0, 1, \ldots, n-1\}$. Define the following operations on X_n:
$$\forall\, a, b \in X_n : \quad a \oplus b = a + b \pmod n, \quad a \otimes b = ab \pmod n.$$
   (a) Verify if $(X_n, \oplus, \otimes)$ is a field when n is prime. (4)

   (b) Let $(\mathbb{F}, +, *)$ be a field. If $a, b \in \mathbb{F}$ and $a * b = 0$, then show that either $a = 0$ or $b = 0$. (1)

   (c) Using (b), show that $(X_n, \oplus, \otimes)$ is not a field when n is not prime. (3)

3. Let $(\mathbb{V}, +, \cdot)$ be a vector space over a field $(\mathbb{F}, +, *)$. Let $0_{\mathbb{F}}$ and $1_{\mathbb{F}}$ be the identity elements of $\mathbb{F}$ and $0_{\mathbb{V}}$ be the identity element of $\mathbb{V}$. Verify that (3)

   1. $\forall\, v \in \mathbb{V} :\ 0_{\mathbb{F}} \cdot v = 0$
   2. $\forall\, a \in \mathbb{F} :\ a \cdot 0_{\mathbb{V}} = 0_{\mathbb{V}}$
   3. $\forall\, v \in \mathbb{V} :\ (-1_{\mathbb{F}}) \cdot v = -v$

4. Recall the definition of a vector space from the lecture notes, which consists of 8 axioms. Show that the commutativity axiom $(u + v = v + u)$ is not an independent axiom by deducing it from the other axioms. (3)

5. Let $\mathbb{V}$ be a vector space over $\mathbb{R}$. Let X be a nonempty subset of $\mathbb{V}$ with the property that (4)
$$\forall\, u, v \in X, \forall\, t \in \mathbb{R} : \quad tu + (1-t)v \in X.$$
   Prove that there exists $x_0 \in X$ and a subspace $\mathbb{U} \subseteq \mathbb{V}$ such that
$$X = \big\{x_0 + u :\ u \in \mathbb{U}\big\}.$$

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