Computational Linear Algebra – Problem Set 7

(points: 60)

Problem 1 (points: 5)
Let $\lambda_1, .., \lambda_n \in \mathbb{C}$ be the eigenvalues of $\mathbf{A} \in \mathbb{C}^{n\times n}$. Using the characteristic equation, prove that

$$\mathrm{trace}(\mathbf{A}) = \lambda_1 + \cdots + \lambda_n \qquad \text{and} \qquad \det(\mathbf{A}) = \lambda_1 \cdots \lambda_n.$$

Problem 2 (points: 10)
Let $f : \mathbb{M}_n \to \mathbb{R}$ be a functional defined on $\mathbb{M}_n$, the space of matrices of size $n \times n$. Suppose that

- $f(\mathbf{I}) = 1$, where $\mathbf{I} \in \mathbb{M}_n$ is the identity matrix,
- $f(\mathbf{A})$ is linear in each column of $\mathbf{A}$, if we keep the rest of the columns fixed, and
- f is alternating, i.e., $f(\hat{\mathbf{A}}) = -f(\mathbf{A})$ where $\hat{\mathbf{A}}$ is obtained by switching two columns of $\mathbf{A}$.

Show that $f(\mathbf{A}) = \det(\mathbf{A})$ for all $\mathbf{A} \in \mathbb{M}_n$.

Problem 3 (points: 2+3 = 5)
Let λ and μ be two eigenvalues of $\mathbf{A} \in \mathbb{C}^{n\times n}$. Suppose the corresponding eigenvectors are $\boldsymbol{x}_\lambda, \boldsymbol{x}_\mu \in \mathbb{C}^n$. Prove that if $\lambda \neq \mu$, then $\boldsymbol{x}_\lambda$ and $\boldsymbol{x}_\mu$ must be linearly independent.
Next, show that if the n eigenvalues of $\mathbf{A} \in \mathbb{C}^{n\times n}$ are distinct, then $\mathbf{A}$ can be diagonalized.

Problem 4 (points: 5+2+3 = 10)
The spectral norm of $\mathbf{A} \in \mathbb{R}^{n\times n}$ is defined as

$$\|\mathbf{A}\|_* = \max\big\{\|\mathbf{A}\boldsymbol{x}\| : \|\boldsymbol{x}\| = 1\big\},$$

where $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^n$.
(a) Show that

$$\|\mathbf{A}\|_* = \begin{cases} \max\big\{\|\mathbf{A}\boldsymbol{x}\| : \|\boldsymbol{x}\| \leqslant 1\big\}. \\ \max\big\{\boldsymbol{y}^\top\mathbf{A}\boldsymbol{x} : \|\boldsymbol{x}\| = 1 \text{ and } \|\boldsymbol{y}\| = 1\big\}. \\ \text{largest singular value of } \mathbf{A}. \end{cases}$$

(b) If $\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^m$ and $\mathbf{B} : \mathbb{R}^m \to \mathbb{R}^k$, then show that

$$\|\mathbf{B}\mathbf{A}\|_* \leqslant \|\mathbf{B}\|_* \|\mathbf{A}\|_* .$$

(c) If $\mathbf{A} = [\mathbf{A}_{ij}]$, then show that $|\mathbf{A}_{ij}| \leqslant \|\mathbf{A}\|_*$ for all i, j.

Problem 5 (points: 2+4+4 = 10)
Define the shift operator $S : \mathbb{C}^n \to \mathbb{C}^n$ to be

$$S(x_1, x_2, \ldots, x_n) = (x_n, x_1, \ldots, x_{n-1}).$$

1. Prove that S is unitary: $\|S\boldsymbol{x}\| = ||\boldsymbol{x}||$ for all $\boldsymbol{x} \in \mathbb{C}^n$.
2. Determine the eigenvalues and eigenvectors of S.
3. Verify that the eigenvectors are orthogonal.

Problem 6 (points: 5+5 = 10)
Let $\mathbf{A} \in \mathbb{C}^{n\times n}$ be Hermitian. Prove that the following are equivalent:

1. For all $\boldsymbol{x} \in \mathbb{C}^n$, $\boldsymbol{x}^{\mathrm{H}}\mathbf{A}\boldsymbol{x} \geqslant 0$.
2. The eigenvalues of $\mathbf{A}$ are non-negative.

Show that if the eigenvalues of $\mathbf{A} = [\mathbf{A}_{ij}]$ are non-negative, then $\mathbf{A}_{ii} \geqslant 0$ and $|\mathbf{A}_{ij}| \leqslant \max(\mathbf{A}_{ii}, \mathbf{A}_{jj})$.

Problem 7 (points: 5)
Let $\sigma_{\min}$ and $\sigma_{\max}$ be the smallest and largest singular values of $\mathbf{A} \in \mathbb{R}^{m\times n}$. Show that

$$\sigma_{\min} = \min_{\boldsymbol{x}\in S} \ \|\mathbf{A}\boldsymbol{x}\| \qquad \text{and} \qquad \sigma_{\max} = \max_{\boldsymbol{x}\in S} \ \|\mathbf{A}\boldsymbol{x}\|,$$

where $S = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| = 1\}$ is the unit sphere in $\mathbb{R}^n$.

Problem 8 (points: 5)
Let $\mathbf{A} = \sum_{k=1}^{n} \sigma_k \boldsymbol{u}_k \boldsymbol{v}_k^\top$ be the full SVD of $\mathbf{A} \in \mathbb{R}^{m\times n}$, where $r = \operatorname{rank}(\mathbf{A})$ and $\sigma_1 \geqslant \cdots \geqslant \sigma_n$. Show that

1. $\operatorname{range}(\mathbf{A}) = \operatorname{span}(\boldsymbol{u}_1, \ldots, \boldsymbol{u}_r)$.
2. $\operatorname{nullspace}(\mathbf{A}^\top) = \operatorname{span}(\boldsymbol{u}_{r+1}, \ldots, \boldsymbol{u}_m)$.
3. $\operatorname{range}(\mathbf{A}^\top) = \operatorname{span}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r)$.
4. $\operatorname{nullspace}(\mathbf{A}) = \operatorname{span}(\boldsymbol{v}_{r+1}, \ldots, \boldsymbol{v}_n)$.

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