E0 298 | Linear Algebra and Its Applications

Assignment 4

Deadline: 11.30 AM, 17-10-23 Full Marks: 25

1. Let $(\mathbb{V}, \langle \cdot, \cdot \rangle)$ be a complex inner-product space. Prove that if $\delta > 0$, then (5)

$$\forall\, u, v \in \mathbb{V}: \quad |\langle u, v \rangle| \leqslant \frac{1}{2}\left(\frac{\|u\|^2}{\delta} + \delta \|v\|^2\right),$$

where $\|\cdot\|$ is the norm induced by $\langle \cdot, \cdot \rangle$.

Notice that the right-hand side depends on δ but not the left-hand side. Minimizing the right-hand side over all possible $\delta > 0$, establish the Cauchy-Schwarz inequality:

$$\forall\, u, v \in \mathbb{V}: \quad |\langle u, v \rangle| \leqslant \|u\|\, \|v\|.$$

2. (a) Let $(\mathbb{V}, \langle \cdot, \cdot \rangle)$ be an inner-product space. Let $v_1, \ldots, v_m \in \mathbb{V}$ be linearly independent. We define $q_1, \ldots, q_m \in \mathbb{V}$ recursively as follows: (3)

$$r_1 = v_1, \quad q_1 = \frac{r_1}{\|r_1\|},$$

and

$$\forall\, j = 2, \ldots, m: \quad r_j = v_j - \sum_{i=1}^{j-1} \langle v_j, q_i \rangle\, q_i, \quad q_j = \frac{r_j}{\|r_j\|},$$

Show that $q_1, \ldots, q_m$ are orthonormal vectors and $\mathrm{span}(e_1, \ldots, e_m) = \mathrm{span}(v_1, \ldots, v_m)$.

(b) Use part (a) to show that every finite-dimensional inner-product space has an orthonormal basis. (2)

3. Let $(\mathbb{V}, \langle \cdot, \cdot \rangle)$ be an inner-product space and let $T \in \mathcal{L}(\mathbb{V})$ be injective. (3)

Define $\langle u, v \rangle_* : \mathbb{V} \times \mathbb{V} \to \mathbb{R}$ to be

$$\forall\, u, v \in \mathbb{V}: \quad \langle u, v \rangle_* = \langle T(u), T(v) \rangle.$$

Show that $\langle \cdot, \cdot \rangle_*$ is a valid inner-product on $\mathbb{V}$.

4. Let $(\mathbb{V}, \langle \cdot, \cdot \rangle)$ be a real inner-product space. Let $F : \mathbb{V} \to \mathbb{V}$ be a function such that (6)

$$\forall\, x, y \in \mathbb{V}: \quad \langle F(x), F(y) \rangle = \langle x, y \rangle.$$

Prove that F must be linear.

On the other hand, give a simple counterexample to show that if

$$\forall\, x \in \mathbb{V}: \quad \|F(x)\| = \|x\|,$$

where $\|\cdot\|$ is the norm induced by the inner-product on $\mathbb{V}$, then F need not be linear.

5. Let $A \in \mathbb{C}^{n \times n}$. Suppose that for all $x \in \mathbb{C}^n$, $\langle x, Ax \rangle_{\mathbb{C}^n} = 0$, where $\langle \cdot, \cdot \rangle_{\mathbb{C}^n}$ is the standard dot product on $\mathbb{C}^n$. Show that $A = 0$. (6)

On the other hand, give a counterexample to show that if $A \in \mathbb{R}^{2 \times 2}$, then the condition

$$\forall\, x \in \mathbb{R}^2: \quad \langle x, Ax \rangle_{\mathbb{R}^2} = 0,$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}^2}$ is the standard dot product on $\mathbb{R}^2$, does not imply that $A = 0$.

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