Computational Linear Algebra – Problem Set 6

(points: 35)

Problem 1 (points: 5)

Let $\mathbf{M}_1$ and $\mathbf{M}_2$ be two matrix representations of $\ell \in \mathcal{L}(\mathbb{V})$ w.r.t. two different bases of $\mathbb{V}$. Show that there exists an invertible matrix $\mathbf{P}$ such that $\mathbf{M}_2 = \mathbf{P}\mathbf{M}_1\mathbf{P}^{-1}$.

Problem 2 (points: 2+2 = 4)

Let $(\mathbb{V}, \langle\cdot,\cdot\rangle)$ be a real inner-product space. Prove from first-principles that, for any $\delta > 0$,

$$|\langle u, v\rangle| \leqslant \frac{1}{2}\left(\frac{\|u\|^2}{\delta} + \delta\|v\|^2\right).$$

Notice that the right-hand side depends on δ but not the left-hand side. Minimizing the right-hand side over $\delta > 0$, establish the Cauchy-Schwarz inequality: $|\langle u, v\rangle| \leqslant \|u\| \cdot \|v\|$.

Problem 3 (points: 3)

Let $(\mathbb{V}, \|\cdot\|)$ be a normed vector space and let $\ell \in \mathcal{L}(V)$ be such that $\|\ell(v)\| \leqslant \|v\|$ for all $v \in V$. Prove that $\ell - 2I$ is invertible where I is the identity map.

Problem 4 (points: 4)

Suppose $u, v \in \mathbb{R}^n$. Prove that $\langle u, v\rangle = 0$ if and only if $\|u\| \leqslant \|u + cv\|$ for all $c \in \mathbb{R}$.

Problem 5 (points: 4+2 = 6)

Let $(\mathbb{V}, \langle\cdot,\cdot\rangle)$ be an inner-product space and let $\ell \in \mathcal{L}(V)$ be injective. Prove that $\langle u, v\rangle_* = \langle \ell(u), \ell(v)\rangle$ is a valid inner-product on $\mathbb{V}$. Is $\langle u, v\rangle_{**} = \langle \ell(u), v\rangle$ also a valid inner-product on $\mathbb{V}$ for if $\ell \in \mathcal{L}(\mathbb{V})$ is injective?

Problem 6 (points: 4+4 = 8)

Let $\mathcal{P}_2$ be the space of real-valued polynomials on $[0, 1]$ whose degree is at most 2. Show that

$$\langle p, q\rangle = \int_0^1 p(t)q(t)\, dt \qquad (p, q \in \mathcal{P}_2)$$

is a valid inner-product on $\mathcal{P}_2$. Furthermore, apply the Gram-Schmidt process to the monomials $\{1, t, t^2\}$ to produce an orthonormal basis of $\mathcal{P}_2$.

Problem 7 (points: 5)

Let $\mathbb{U}$ and $\mathbb{W}$ be subspaces of a finite-dimensional space and let $P_{\mathbb{U}}$ and $P_{\mathbb{W}}$ be the orthogonal projectors onto $\mathbb{U}$ and $\mathbb{W}$. Prove that $P_{\mathbb{U}}P_{\mathbb{W}}$ is the zero operator if and only if $\langle u, w\rangle = 0$ for all $u \in \mathbb{U}$ and $w \in \mathbb{W}$.

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