Home Work 2

Due Date: 8am, 29th October 2019 (Attempt all questions. Total 50 marks)

You need to show detailed calculations / explainations for your answers.

Q. 1] Consider the function $f: \mathbb{R}^n \to \mathbb{R}$ given by

$$f(\mathbf{x}) = \frac{1}{2} \mathbf{x}^{\mathsf{T}} Q \mathbf{x} - \mathbf{b}^{\mathsf{T}} \mathbf{x},$$

where $\mathbf{b} \in \mathbb{R}^n$ and $Q \in \mathbb{R}^{n \times n}$ is symmetric & positive definite. Let \mathbf{x}^* be the unknown point at which f achieves minima. Assume that algorithms are initialized with an \mathbf{x}_0 such that $\mathbf{x}_0 - \mathbf{x}^*$ is parallel to an eigen vector of Q. How many iterations will be needed to find a minimizer of f if the following methods were used

- a. Gradient descent with exact line search
- b. Conjugate gradient
- c. Newton's method

[5+4+1 marks]

- **Q. 2**] Consider $f(x_1, x_2) = 100(x_2 x_1^2)^2 + (1 x_1)^2$. Report the function value at the end of each iteration of the following:
- (a) Apply two iterations of Newton's method initialized with (0,0). [4 marks
- (b) Apply two iterations of Gradient Descent with fixed stepsize 0.05 and initialized with (0,0).
- **Q.** 2] (c)

Consider $f(x_1, x_2) = \frac{5}{2}x_1^2 + \frac{1}{2}x_2^2 + 2x_1x_2 - 3x_1 - x_2$. Find the minimizer of f using the conjugate gradient algorithm starting from (0, 0). [4 marks

- **Q. 3**] Implement (in Matlab/Python) the conjugate gradient algorithm to solve the system of linear equations $\mathbf{A}\mathbf{x} = \mathbf{b}$, where (i, j)-th entry in $\mathbf{A} \in \mathbb{R}^{n \times n}$ is $\frac{1}{i+j-1}$ and all the coordinates in $\mathbf{b} \in \mathbb{R}^n$ are 1. Start the algorithm from zero vector. For n = 5, 8, 12, 20 report the number of iterations required to reduce the error $\|\mathbf{A}\mathbf{x} \mathbf{b}\|_2$ below 10^{-6} . For n = 20 plot the error (log scale) vs iteration. No need to submit code.
- Q. 4] Consider steepest descent algorithm (with exact line search) applied to the function

$$f(x_1, x_2) = 5x_1^2 + 5x_2^2 - x_1x_2 - 11x_1 + 11x_2 + 11.$$

- (a) What would be the rate of convergence of the algorithm? [5 marks
- (b) Starting with (0,0), how many iterations would it take (at most) to reduce the function value to 10^{-11} ? [5 marks

Q. 5] Consider the function $f: \mathbb{R}^2 \to \mathbb{R}$ given by

$$f(x_1, x_2) = \frac{3}{2}(x_1^2 + x_2^2) + (1+a)x_1x_2 - (x_1 + x_2) + b,$$

where $a, b \in \mathbb{R}$ are some unknown parameters.

- (a) Find the largest range of values of a and b such that the unique global minimizer of f exists, and write down the minimizer in terms of a, b. [4+2 marks
- (b) Consider gradient descent algorithm (with fixed stepsize 0.4) applied to f. Find the largest range of values of a and b for which the algorithm converges to a global minimizer of f from any arbitrary initial point. [4 marks