MA324 TOPICS IN COMPLEX ANALYSIS
ASSIGNMENT 1

Due date: January 20 (Thur.) by 11:59 pm

Instructions. 1. Solutions may either be handwritten or typewritten. In either case, please upload a legible PDF file of your solutions to MS Teams.

2. As much as possible, try to attempt these problems on your own. If you end up taking any assistance (from any source — books, websites, friends, etc.) please give due credit.

Problem A. Let A be a (commutative, complex, unital) Banach algebra. Given $f \in A$, the *spectrum* of f is the set

$$\mathrm{spec}(f) = \{z \in \mathbb{C} : z - f \text{ is not invertible in } A\}.$$

The spectrum of f is a nonempty compact subset of $\mathbb{C}$ satisfying

$$\sup\{|z| : z \in \mathrm{spec} f\} = \lim_{n\to\infty} ||f^n||^{1/n}.$$

Now, prove the following:

(a) An element $g \in A$ is noninvertible if and only if there is a character $\varphi \in M_A$ such that $\varphi(g) = 0$. (*Note. You may freely use the fact stated in class that every maximal ideal of A is the kernel of some character of A.*)

(b) Given any $f \in A$,
$$\hat{f}(M_A) = \mathrm{spec} f,$$
where M_A is the space of characters of A (as defined in Lecture 1).

(c) The Gelfand transform is an isometry from A into $\mathcal{C}(M_A)$ if and only if
$$||f^2|| = ||f||^2 \quad \forall f \in A.$$

Problem B. Let S^1 and $\mathbb{D}$ denote the unit circle and open unit disk, respectively, in $\mathbb{C}$. Recall that $A(\overline{\mathbb{D}})$ is the algebra of continuous functions on $\overline{\mathbb{D}}$ whose restrictions to $\mathbb{D}$ are holomorphic. Consider

$$A := \left\{ f \in \mathcal{C}(S^1) : \text{there is an } F \in A(\overline{\mathbb{D}}) \text{ such that } F|_{S^1} = f \right\}.$$

(a) Show that A is a uniform algebra on S^1.

(b) Show that for $f \in \mathcal{C}(S^1)$, $f \in A$ if and only if the negative Fourier coefficients of f vanish. (*Note. You may use Fejér's theorem without proof.*)

(*c*) (Wermer's maximality theorem) Complete the following steps to show that if $B \subset \mathcal{C}(S^1)$ is a closed subalgebra such that $A \subset B$, then either $B = A$ or $B = \mathcal{C}(S^1)$.

(*i*) Assume $A \subsetneq B$, and let $f \in B \setminus A$. Argue that there exist holomorphic polynomials p, q, r such that

$$\left\| z(pf) - 1 - zq - \overline{zr} \right\|_{\mathcal{C}(S^1)} < \frac{1}{2}.$$

(*Note. Again, use Fejér's theorem.*)

(*ii*) Let $F = pf - q - r$. Argue that $zF \in B$, and for sufficiently small $t > 0$, tzF admits an inverse in B. (*Hint. Look at* $||(1+t) - tzF||$.)

(*iii*) Conclude from Step (*ii*) that $\overline{z} \in B$. Why does this complete the proof?

Problem C. Given a measurable function g on $\mathbb{C}$, define

$$\mathscr{P}[g](z) = \frac{1}{\pi} \int_K g(\zeta) \left(\frac{1}{z - \zeta} + \frac{1}{\zeta} \right) d\lambda(\zeta), \qquad z \in \mathbb{C},$$

where λ is the Lebesgue measure on $\mathbb{C}$.

(*a*) Let $g \in L^p(\mathbb{C})$ for some $2 < p < \infty$. Argue that $P[g](z) < \infty$ for all $z \in \mathbb{C}$.

(*b*) Under the same assumption on g as in (*a*), show that there is a constant $K_p > 0$ such that

$$|\mathscr{P}[g](z)| \leq K_p ||g||_{L^p} |z|^{1-\frac{2}{p}}, \qquad z \in \mathbb{C}.$$

(*c*) Under the same assumption on g as in (*a*), produce a $\widetilde{g} \in L^p(\mathbb{C})$ so that

$$P[g](z) - P[g](w) = P[\widetilde{g}](z - w), \qquad z, w \in \mathbb{C}.$$

Now, conclude that $P[g]$ is $\left(1 - \frac{2}{p}\right)$-Hölder on $\mathbb{C}$.

(*d*) What goes wrong with the above arguments if $g \in L^\infty(\mathbb{C})$?

(*e*) Now prove the following statement made in class. Let $K \subset \mathbb{C}$ be a compact set, and $g \in L^p(K)$, $2 < p \leq \infty$. Then,

$$\mathscr{C}[g](z) = \frac{1}{\pi} \int_K \frac{g(\zeta)}{z - \zeta} \, d\lambda(\zeta),$$

is a continuous function on $\mathbb{C}$.

(*f*) **(Bonus)** The above arguments show that for $g \in L^\infty(K)$, $\mathscr{C}[g]$ is α-Hölder for any $\alpha \in (0, 1)$. Produce an example of a bounded function g on some compact set K so that $\mathscr{C}[g]$ is not Lipschitz, i.e., 1-Hölder. (*Hint. The characteristic function of an appropriately chosen compact set does the job.*)