E0 298 | Linear Algebra and Its Applications

Assignment 6

Deadline: 08-12-23 Full Marks: 25

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1. Suppose $A \in \mathbb{R}^{n\times n}$ has no real eigenvalues. Prove that $\det A > 0$. (2)

2. Using the fundamental theorem of algebra but without using determinants, prove that any $A \in \mathbb{C}^{n\times n}$ has an eigenvalue. (3)

3. For matrices, we generally have $AB - BA \neq 0$. However, show that there does not exist matrices $A, B \in \mathbb{C}^{n\times n}$ such that $AB - BA = I$. (2)

4. Let $A, B \in \mathbb{C}^{n\times n}$. Prove that (5)

   (i) $I - AB$ invertible $\iff$ $I - BA$ invertible.

   (ii) λ is an eigenvalue of AB $\iff$ λ is an eigenvalues of BA.

5. Let $A \in \mathbb{C}^{m\times m}, B \in \mathbb{C}^{n\times n}, C \in \mathbb{C}^{m\times n}$, and $0 \in \mathbb{C}^{n\times m}$. Prove that (5)

$$\det \left(\begin{array}{c|c} A & C \\ \hline 0 & B \end{array}\right) = \det(A)\ \det(B).$$

6. Let $A = (a_{ij}) \in \mathbb{C}^{n\times n}$ be such that (3)

   - $\forall\, 1 \leqslant i \leqslant n:\ a_{ii} = 1.$
   - $\forall\, 1 \leqslant i \leqslant n:\ \sum_{j=1, j\neq i}^{n} |a_{ij}| \leqslant 1/2.$

   Prove that A is invertible.

7. We say that $v \in \mathbb{C}^n \backslash \{0\}$ is a generalized eigenvector of $A \in \mathbb{C}^{n\times n}$ with eigenvalue $\lambda \in \mathbb{C}$ if there exists $k \geqslant 1$ such that $(A - \lambda I)^k v = 0$. Prove that generalized eigenvectors corresponding to distinct eigenvalues are linearly independent. (5)

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