Computational Linear Algebra – Assignment 1

(points: 30)

Problem 1 (points: $4 + 3 + 3 = 10$)
Let X and Y be two sets. A function $f : X \to Y$ is said to be a bijection if:

- it is injective: if $f(x_1) = f(x_2)$ where $x_1, x_2 \in X$, then necessarily $x_1 = x_2$, and
- it is surjective: every $y \in Y$ can be written as $y = f(x)$ for some $x \in X$.

1. Let X and Y be of the form $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_m\}$. Prove that there exists a bijection $f : X \to Y$ if and only if $m = n$.
2. Note that an implication of the above result is that there cannot exist a bijection between a (finite) set X and a proper subset $Y \subset X$. Give a simple example to show that this is not true if X and Y are allowed to have infinitely many elements. (hint: consider the set of even integers).
3. If $f : X \to Y$ is a bijection, then its inverse $g : Y \to X$ is defined as $g(y) = x$ if $y = f(x)$. Check that g is also a bijection.

Problem 2 (points: $2 + 2 = 4$)
Let A be an $m \times n$ matrix. Its transpose $A^\top$ is the $n \times m$ matrix defined as

$$(A^\top)_{ij} = A_{ji} \qquad (1 \le i \le n, 1 \le j \le m),$$

where A_{ij} denotes the entry in the i-th row and j-th column of A.

1. Let the size of A and B be $m \times n$ and $n \times p$. Show that $(AB)^\top = B^\top A^\top$.
2. More generally, if matrices $A_1, \ldots, A_k$ are such that their product $A_1 \cdots A_k$ is defined, then show that
$$(A_1 \cdots A_k)^\top = {A_k}^\top \cdots {A_1}^\top.$$
(hint: use induction).

Problem 3 (points: $2 + 4 + 2 + 3 = 11$)
Let A be an $n \times n$ matrix. Recall that its inverse B is another $n \times n$ matrix such that $AB = BA = I$, where I is the $n \times n$ identity matrix. We say that a matrix is invertible if it has an inverse.

1. Using a simple argument, explain why the inverse of a matrix A (if it exists) is always unique, i.e., there cannot exist two distinct inverses of A. The unique inverse is denoted by A^{-1}.
2. Show that if A and B are invertible, then so is AB. In particular, show that $(AB)^{-1} = B^{-1}A^{-1}$.
3. More generally, show that if $A_1, \ldots, A_k$ are a set of invertible matrices, then the product $A_1 A_2 \cdots A_k$ is invertible and its inverse is the product $A_k^{-1} \cdots A_1^{-1}$.
4. If A is invertible, then show that $A^\top$ is invertible and its inverse is the transpose of A^{-1}.

Problem 4 (points: $2 + 2 + 1 = 5$)
Let A and B be two points on a plane whose coordinates are (a, b) and (c, d). Let θ be the angle (assumed to be acute) between line segments $\overline{OA}$ and $\overline{OB}$ where O is the origin. Then show that

$$ac + bd = \sqrt{a^2 + b^2} \cdot \sqrt{c^2 + d^2} \cdot \cos\theta. \tag{1}$$

From (1), deduce that

1. If $\overline{OA}$ and $\overline{OB}$ are perpendicular to each other, then $ac + bd = 0$.
2. For any choice of a, b, c, and d, $ac + bd \leq \sqrt{a^2 + b^2} \cdot \sqrt{c^2 + d^2}$.

Can (1) be extended to three dimensional space. What would be the corresponding formula?

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