E0 298 | Linear Algebra and Its Applications

Assignment 2

Deadline: 11:30 AM, 12 September, 2023 Max mark: 20

1. Suppose a vector space $\mathbb{V}$ has the property that (2)

$$\forall\, k \geqslant 1, \exists\, v_1, \ldots, v_k \in \mathbb{V} \text{ which are linearly independent.}$$

   Argue why $\mathbb{V}$ cannot be finite-dimensional.

   Consider the vector space $\mathfrak{F}$ of functions $f : \mathbb{R} \to \mathbb{R}$ (you don't have to show that $\mathfrak{F}$ is a vector space). Using the above argument, show that $\mathfrak{F}$ cannot be finite-dimensional.

2. Let $\mathbb{V}$ be a vector space and $v_1, \ldots, v_n \in \mathbb{V}$ be linearly independent. Show that (2)

$$\forall\, v \in \mathbb{V}, \quad \dim\, \big(\mathrm{span}\{v_1 + v, \ldots, v_n + v\}\big) \geqslant n - 1.$$

3. Let $\mathbb{U}_1$ and $\mathbb{U}_2$ be two subspaces of a vector space $\mathbb{V}$. If $\dim\, \mathbb{U}_1 + \dim\, \mathbb{U}_2 > \dim\, \mathbb{V}$, then prove that there must exist a nonzero vector in $\mathbb{U}_1 \cap \mathbb{U}_2$. (2)

4. Let $\mathcal{C}(\mathbb{R})$ be the vector space of real-valued continuous function on $\mathbb{R}$. (3)

   (i) Show that $\cos x, \sin x \in \mathcal{C}(\mathbb{R})$ are linearly independent.

   (ii) Are $\cos^2 x$ and $\sin^2 x$ linearly independent?

   (iii) Are $1, \cos^2 x$, and $\sin^2 x$ linearly independent?

5. Let $\mathbb{U}$ and $\mathbb{W}$ be two subspaces of a vector space $\mathbb{V}$. Let $u_1, \ldots, u_m$ be a basis of $\mathbb{U}$ and $w_1, \ldots, w_n$ be a basis of $\mathbb{W}$. Prove that (3)

$$\dim\, \mathrm{span}\left(\Big\{u_i + w_j :\; i = 1, \ldots, m,\; j = 1, \ldots, n\Big\}\right) \leqslant \dim\, \mathbb{U} + \dim\, \mathbb{W}.$$

6. Let $\mathbb{U}_1, \mathbb{U}_2, \mathbb{U}_3$ be subspaces of an n-dimensional vector space. Prove that (3)

$$\dim\, (\mathbb{U}_1 \cap \mathbb{U}_2 \cap \mathbb{U}_3) \geqslant \dim\, \mathbb{U}_1 + \dim\, \mathbb{U}_2 + \dim\, \mathbb{U}_3 - 2n.$$

7. Let $(\mathbb{V}, +, \cdot)$ be a vector space over a field $\mathbb{F}$. Let $\mathbb{U}$ be some fixed subspace of $\mathbb{V}$. Define (5)

$$\forall\, v \in \mathbb{V} : \quad [v] = \big\{x \in \mathbb{V} :\; x - v \in \mathbb{U}\big\},$$

   and

$$\mathbb{W} = \Big\{[v] : v \in \mathbb{V}\Big\}.$$

   Define the operations $\circ : \mathbb{F} \times \mathbb{W} \to \mathbb{W}$ and $+ : \mathbb{W} \times \mathbb{W} \to \mathbb{W}$ as follows:

$$\forall\, \lambda \in \mathbb{F},\, \forall\, u, v \in \mathbb{W} : \quad \lambda \circ [u] := [\lambda \cdot x] \quad \text{and} \quad [u] + [v] := [x + y],$$

   where $x \in [u]$ and $y \in [v]$.

   (i) Argue why $[v]$ is nonempty for all $v \in \mathbb{V}$.

(ii) Show that the operations $+$ and $\circ$ are well-defined, i.e., they do not depend on the choice of x and y as long as $x \in [u]$ and $y \in [v]$.

(iii) It can be shown that $\mathbb{W} = (\mathbb{W}, +, \circ)$ is a vector space over $\mathbb{F}$. Verify just the following two properties:

$$\forall\, [u], [v] \in \mathbb{W} : \quad [u] + [v] = [v] + [u],$$

and

$$\forall\, [u], [v] \in \mathbb{W},\, \forall\, a \in \mathbb{F} : \quad a \circ \big([u] + [v]\big) = a \circ [u] + a \circ [v].$$

(iv) Verify that the identity element of $\mathbb{W}$ is $[0]$, where 0 is the identity element of $\mathbb{V}$.

(v) Prove that $\dim\, \mathbb{W} = \dim\, \mathbb{V} - \dim\, \mathbb{U}$.

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