Computational Linear Algebra – Assignment 2

(points: 50)

Problem 1 (points: $4+2+3+3+4=16$)
Recall the definition of a "field" covered in the lecture. In this regard, verify the following:

1. The additive and multiplicative inverses of a field are unique.
2. The set of integers $\mathbb{Z}$ is not a field.
3. The set of complex numbers $\mathbb{C}$ form a field.
4. The set $\mathbb{Z}_2 = \{0, 1\}$, with addition $\oplus$ and multiplication $\otimes$ defined as $a \oplus b = \mathrm{mod}(a+b, 2)$ and $a \otimes b = \mathrm{mod}(ab, 2)$, is a field. ($\mathrm{mod}(c, 2)$ is the remainder on dividing c by 2).
5. The set $\mathbb{Z}_4 = \{0, 1, 2, 3\}$ with addition and multiplication modulo 4 is a not field.

Problem 2 (points: $2+3+2+3+4+4=18$)
Check if the following statements are true or false:

1. In a vector space, the additive inverse (of an element) is unique.
2. Complex numbers $\mathbb{C}$ form a vector space over the field of real numbers $\mathbb{R}$.
3. $\mathbb{R}$ is a vector space over $\mathbb{C}$.
4. The set of real symmetric matrices of size $n \times n$ is a vector space over $\mathbb{R}$.
5. The set of sequences $(a_i)_{i\in\mathbb{N}}, a_i \in \mathbb{C}$, such that $\sum_{i=1}^{\infty} |a_i| < \infty$ is a vector space over $\mathbb{C}$.
6. The set of continuous functions $f : [0, 1] \to \mathbb{R}$ such that
$$\int_0^1 f(t)^2 dt < \infty$$
is a vector space.

Problem 3 (points: $4+2=6$)
Let a, b and c be elements of a field $\mathbb{F}$. Show that

1. $a * (-b) = -(a * b)$ and $(-a) * b = -(a * b)$.
2. $(-a) * (-b) = a * b$.

Problem 4 (points: $2+2+2=6$)
Let $\mathbb{V}$ be a vector space over $\mathbb{F}$. Show that

1. $0 \cdot \boldsymbol{v} = \mathbf{0}$ for all $\boldsymbol{v} \in \mathbb{V}$,
2. $a \cdot \mathbf{0} = \mathbf{0}$ for all $a \in \mathbb{F}$,
3. $(-1) \cdot \boldsymbol{v} = -\boldsymbol{v}$ for all $\boldsymbol{v} \in \mathbb{V}$,

where 0 and 1 are the identity elements of $\mathbb{F}$ and $\mathbf{0}$ is the identity element of $\mathbb{V}$.

Problem 5 (points: 4)
Let $\mathbb{U}_1, \dots, \mathbb{U}_m$ be a subspaces of a vector space $\mathbb{V}$. Show that

$$\mathbb{U}_1 + \cdots + \mathbb{U}_m = \bigcap \Big\{ \mathbb{W} : \mathbb{W} \text{ is a subspace of } \mathbb{V} \text{ containing } \mathbb{U}_1, \dots, \mathbb{U}_m \Big\}.$$

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