E1 222 Stochastic Models and Applications

Assignment

Submission Deadline: 6 November 11:30 AM. (No extension possible)
(You can submit it in the class on 6^{th} or give it to Mr. Chandan in my lab)
You need to submit solutions for only two problems
The specific two problems would be communicated to you through e-mail and Teams at 6 PM on 5^{th} November.

1. Let X, Y be iid geometric random variables with parameter p. (The geometric pmf is: $f_X(k) = f_Y(k) = (1-p)^{k-1}p, \ k = 1, 2, \cdots$). Let $Z = X - Y$ and $W = \min(X, Y)$. Find the joint mass function of Z, W. Show that Z, W are independent.

2. Let $X_1, \cdots, X_n$ be iid random variables having Gaussian density with mean zero and variance σ^2. Show that $Y = \frac{X_1^2 + \cdots + X_n^2}{\sigma^2}$ has Gamma density with parameters $\frac{n}{2}$ and $\frac{1}{2}$. (This rv, Y, is said to have chi-squared distribution with n degrees of freedom).

3. Let X be uniform over $(0, 2)$ and let Y be a discrete random variable taking non-negative integer values. Suppose X, Y are independent. let $Z = X + Y$. Show that Z is a continuous random variable.

4. Let X and Y be two discrete random variables with
$$P[X = x_1] = p_1, \quad P[X = x_2] = 1 - p_1;$$
$$P[Y = y_1] = p_2, \quad P[Y = y_2] = 1 - p_2.$$
Show that X and Y are independent if and only if they are uncorrelated. (Hint: First consider the special case where $x_1 = y_1 = 0$ and $x_2 = y_2 = 1$).

5. Let X be a discrete random variable taking non-negative integer values with mass function, $p(i), \ i = 0, 1, \cdots$. Let $Y_1, Y_2, \cdots, Y_n$ be *iid* discrete random variables taking non-negative integer values and with mass function $q(i), i = 0, 1, \cdots$. (Assume $p(i), q(i) > 0, \forall i$). Let $h : \Re \to \Re$ be some function. Define
$$S = \frac{1}{n} \sum_{k=1}^{n} \frac{p(Y_k) h(Y_k)}{q(Y_k)}.$$
Find ES.

6. Let X, Y be continuous random variables with joint density

$$f_{XY}(x, y) = e^{-x},\ 0 < y < x < \infty$$

Let $Z = X - Y$. Show that Z has exponential density and that $Z \& Y$ are independent.

7. Consider repeated independent tosses of a coin whose probability of heads is p, $0 < p < 1$. Let X denote the number of tosses needed to get at least one head and one tail. Let Y denote the number of tosses needed to get a head immediately followed by a tail. Find EX and EY.

8. An interval of length 1 is broken at a point uniformly distributed over $(0, 1)$. Let c be a fixed point in $(0, 1)$. Find the expected length of the subinterval that contains the point c. Show that this is maximized when $c = 0.5$.

9. Let $X_1, X_2, \cdots$ be *iid* continuous random variables. We say a record has occurred at m, $m \geq 2$, if $X_m > \max(X_{m-1}, \cdots, X_1)$. Show that the probability that a record occurred at m is $\frac{1}{m}$.
Let $N = \min\{n : n > 1 \text{ and a record occurs at time } n\}$.
Show that $EN = \infty$.

10. Let $X_1, \cdots, X_n$ be independent Gaussian random variables with $EX_i = 0$ and $\mathrm{Var}(X_i) = \sigma_i^2$. Let $a_1, \cdots a_n$ and $b_1, \cdots, b_n$ be fixed (non-zero) real numbers which satisfy $\sum_{i=1}^n a_i b_i \sigma_i^2 = 0$. Let $Y_1 = \sum_{i=1}^n a_i X_i$ and $Y_2 = \sum_{i=1}^n b_i X_i$. Show that Y_1 and Y_2 are independent.