Computational Methods of Optimization
Final Exam-Part 1 (1st Dec, 2021)

Instructions:

- This is the first part of the Final test
- This is a closed book test. Please do not consult any additional material.
- Attempt all questions
- Total time is 70 mins.

Name: ____________________

SRNO: Degree: Dept:

Question:	1	2	3	4	Total
Points:	10	10	10	10	40
Score:					

$\mathcal{S}_d$ would denote the set of $d \times d$ Symmetric real valued matrices. $\mathcal{S}_d^+$ denote the set of $d \times d$ Symmetric real valued positive semidefinite matrices and $\mathcal{S}_d^{++}$ would denote the set of positive definite matrices.

1. Consier minimizing

$$min_{t \in \mathbb{R}} g(t) \left(= \mathbf{u}^\top L^2 \mathbf{u} + t^2 \mathbf{u}^\top L^{-2} \mathbf{u} - 2t(\mathbf{u}^\top \mathbf{u})\right)$$

where $\mathbf{u} \in \mathbb{R}^d$, L is invertible and $L \in \mathcal{S}_d$.

(a) (1 point) Answer True or False. There exists $t \in \mathbb{R}$ such that $g(t) < 0$. ____.

(b) (2 points) Justify your answer.

(c) (2 points) Find the global minimum of g. Find optimal t and the optimal value

(d) Let $A \in \mathcal{S}_d^{++}$. Consider

$$\max_{\mathbf{u} \in \mathbb{R}^d} f(\mathbf{u}) \left(= \frac{\|\mathbf{u}\|^4}{(\mathbf{u}^\top A\mathbf{u})\mathbf{u}^\top A^{-1}\mathbf{u}}\right)$$

i. (1 point) Find the global optimal value

ii. (2 points) Give reasons

iii. (2 points) Find u where the maximum is achieved?

2. Let $v_1, v_2 \in \mathbb{R}^d$ such that $v_1^\top v_2 = 0$. Consider

$$\min_{\mathbf{x}\in\mathbb{R}^d} f(\mathbf{x}) \left(= \frac{1}{2}(\mathbf{x}^\top v_1)^2 + \alpha v_2^T \mathbf{x}\right)$$

where $\alpha \in \mathbb{R}$ is a constant.

(a) (2 points) Compute the Hessian? Is this function convex? Justify.

(b) (4 points) Compute the global minima of this function for all choices of α? (Your answer should clearly indicate the optimal value, f^* and the optimal point, $\mathbf{x}^*$)

(c) (4 points) Define $g(\mathbf{x}) = f(\mathbf{x}) + \frac{1}{2}\|\mathbf{x}\|^2$. Repeat a and b.

3. Let $f : \mathbb{R}^d \times \mathbb{R} \in C_L^1$ with $L = 0.5$ and $f(0) = 0$. It is given that $f^* = min_{\mathbf{x} \in \mathbb{R}^d} f(\mathbf{x}) = -1$ Suppose we use the steepest descent procedure with exact line search for this algorithm with $\mathbf{x} = 0$. Let $\mathbf{x}^{(k)}, f(\mathbf{x}^{(k)})$ be the ouput of the algorithm after k iterations.

   (a) (3 points) Derive a lower bound on the decrease of function value at each iteration, $\Delta_k = f(\mathbf{x}^{(k)}) - f(\mathbf{x}^{(k+1)})$ in terms of Lipschitz constant and $\nabla f(\mathbf{x})$.

   (b) (2 points) What is the smallest number of iterations required to guarantee that the algorithm outputs a point, $\hat{\mathbf{x}}$ such that

$$\|\nabla f(\hat{\mathbf{x}})\| \leq 0.1$$

(c) (5 points) Redo a and b if constant stepsize was used instead of *exact stepsize* strategy

(a) (2 points) State one iteration of Newton method for minimizing a C^2 function.

$x_{k+1} = x_k - H(x_k)^{-1} g_k$ where, $H(x_k)$ is pd $g_k = \nabla f(x_k)$

(b) (8 points) Consider applying Newton Method to the following problem

$$min_{\mathbf{x}\in\mathbb{R}^2} g(\mathbf{x}) = (1 - x_1)^3 + x_2^2$$

where $\mathbf{x} = [x_1, x_2]^\top$ starting from $\mathbf{x}^{(0)} = [\alpha, \beta]^\top$ It was found that after 5 iterations the value of $\mathbf{x}^{(3)} = [2, 0]^\top$. Can you find α, β.

* Do calculations properly. Missed a - sign in $\nabla f(x)$.

Computational Methods of Optimization
Final Exam 2(1st Dec, 2021)

Instructions:

- This is the second part of Final Test
- This is a closed book test. Please do not consult any additional material.
- Attempt all questions
- Total time is 70 mins.

Name: ______________________________

SRNO: Degree: Dept:

Question:	1	2	3	4	Total
Points:	10	10	10	10	40
Score:					

S_d would denote the set of $d \times d$ Symmetric real valued matrices. S_d^+ denote the set of $d \times d$ Symmetric real valued positive semidefinite matrices and S_d^{++} would denote the set of positive definite matrices.

1. Let $Q \in S_d^{++}, h \in \mathbb{R}^d, a_i \in \mathbb{R}^d, b_i \in \mathbb{R}$. We consider solving the following problem with the Active set strategy

$$min_{x \in \mathbb{R}^d} f(\mathbf{x}) \left(= \frac{1}{2}\mathbf{x}^\top Q\mathbf{x} + h^\top \mathbf{x} \right)$$

subject to $a_i^\top \mathbf{x} \geq b_i, \quad i = \{1, \dots, m\}$.

(a) (2 points) State the KKT conditions for this problem?

(b) (2 points) In a given iteration we find that the working set is $\hat{W} = \{1, 2, 3\}$, and $\hat{\mathbf{x}}$ is a feasible point. Moreover it is found that

$$\mathbf{u}^* = 0, \mu^* = [1, 2, -3]^\top \text{ is the KKT point for}$$

$$\mathbf{u}^* = argmin_{\mathbf{u} \in \mathbb{R}^d} \frac{1}{2}\mathbf{u}^\top Q\mathbf{u} + \hat{g}^\top \mathbf{u}$$

$$\text{subject to } a_i^\top \mathbf{x} = b_i, i \in \hat{W}$$

The Lagrangian is

$$L(\mathbf{u}, \mu) = \frac{1}{2}\mathbf{u}^\top Q\mathbf{u} + \hat{g}^\top \mathbf{u} - \sum_{i \in \hat{W}} \mu_i (a_i^\top \mathbf{x} - b_i)$$

If the algorithm is stopped here what can be said about the optimality of $\hat{\mathbf{x}}$.

(c) To continue the algorithm, as per active-set strategy, find a constraint l that needs to be dropped from $\hat{W}$ to create a new working set $W = \hat{W}$. After identifying W compute

$$\hat{\mathbf{u}} = argmin_{\mathbf{u}\in\mathbb{R}^d}\frac{1}{2}\mathbf{u}^\top Q\mathbf{u} + \hat{g}^\top\mathbf{u}$$

$$\text{subject to } a_i^\top\mathbf{u} = 0, i \in W$$

where $\hat{g} = \nabla f(\hat{\mathbf{x}})$.

i. (2 points) What is the relationship between $\hat{g}$ and μ^*?

ii. (4 points) Suppose it is given that $\hat{\mathbf{u}}Q\hat{\mathbf{u}} = 6$ can you find the value of $a_l^\top\hat{\mathbf{u}}$. Give reasons for your answer.

2. Let $A = \{a_1, \ldots, a_n\} \subset \mathbb{R}^d$ be a set of n vectors in $\mathbb{R}^d$. We need to decide if $\mathbf{z} \in \mathbb{R}^d$ lies in the convex hull of A, defined by

$$Conv(A) = \left\{\sum_{i=1}^{n}\alpha_i a_i | \alpha_i \geq 0, \sum_{i=1}^{n}\alpha_i = 1\right\}$$

(a) (5 points) Pose this problem as a convex optimization problem.

(b) (5 points) Derive the KKT conditions of this problem

3. (10 points) Consider the set $C = \{\mathbf{x} \in \mathbb{R}^d | \|\mathbf{x} - \mu\| \leq \frac{1}{2}\|\mu\|^2\}$. Find $h \in \mathbb{R}^d$ and $b \in \mathbb{R}$ such that

$$0 \in \{\mathbf{z} \in \mathbb{R}^d | h^\top\mathbf{z} \leq b\}, \text{ and } C \subset \{\mathbf{z} \in \mathbb{R}^d | h^\top\mathbf{z} > b\}$$

4. Consider the following problem

$$min_{\mathbf{x}} f(\mathbf{x}) \left(= \frac{1}{2}\mathbf{x}^\top Q\mathbf{x}\right), \quad \text{subject to } \|\mathbf{x}\|^2 \leq 1$$

Q is not necessarily psd and can be indefinite. Let $\mathbf{x}^*$ be the global optimal solution.

(a) (3 points) Show that the dual is a function of one variable? You need to state the dual

(b) (2 points) Find the domain of the dual function

(c) (2 points) Find the dual optimal solution?

(d) (2 points) Can you find $\mathbf{x}^{(0)}$ such that $f(\mathbf{x}^{(0)})$ is equal to the dual optimal?

(e) (1 point) Find $\mathbf{x}^*$ with justification

Computational Methods of Optimization
Final Exam-Part 1(25th Jan,2021)

Start Time: 9:15 AM End Time: 10:25 AM

Instructions

- Answer all questions
- See upload instructions in the form

Question:	1	2	3	4	Total
Points:	10	10	10	10	40
Score:					

1

In the following $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T\mathbf{x}}$. For a real valued function, f, in one variable, f' denotes the first derivative, f'' denote the second derivative, and $f^{(3)}$ denotes the third derivative.

1. (a) Let $f : [-1, 2] \to \mathbb{R}$ be a differentiable function.
   i. (2 points) Suppose it was known that $f(1) = -f(0) = 1$. Which of the following is correct A. $|f'(x)| \leq 1.5$ for all $x \in (0, 1)$. B. $|f'(x)| \geq 1.5$ for all $x \in (0, 1)$. C. $|f'(x)| = 2$ for some $x \in (0, 1)$. D. None of the above
   ii. (2 points) Suppose $f(0.5) = f(0.8)$ and $f''(x) > 0\ \forall x \in (-1, 2)$. A. There are no minima in $[-1, 2]$. B. There is exactly one minimum in $[-1, 2]$ C. There is at-least one minimum in $[-1, 2]$ D. None of the above
   iii. (2 points) Let f attain minimum at $x = 2$. Which of the following is true A. $f(x) \geq f(2)$ for all $x \in \mathbb{R}$ B. $f'(x) \leq 0$ for all $x \in [-1, 2]$ C. $f'(x) \geq 0$ for all $x \in [-1, 2]$

   (b) (4 points) Consider minimizing a convex quadratic function whose Hessian has largest and smallest eigenvalue 3 and 1 respectively. Suppose we implement the steepest descent procedure starting at a point $\mathbf{x}^{(0)}$ such that $E(\mathbf{x}^{(0)} = 1$ where $E(\mathbf{x}) = \frac{1}{2}(\mathbf{x} - \mathbf{x}^*)H(\mathbf{x} - \mathbf{x}^*)$ where $\mathbf{x}^*$ is the global minimum. After how many iterations can you guarantee that $\|\mathbf{x}^{(T)} - \mathbf{x}^*\| \leq 10^{-2}$.

2. Let $f : (a, b) \to \mathbb{R}$ be thrice differentiable function such that $f'(a)f'(b) < 0$. Assume that for all $x \in (a, b), |f''(x)| \geq \beta, |f^{(3)}(x)| \leq \alpha$ where $\beta, \alpha > 0$.

   (a) i. (1 point) The Number of critical points in (a, b) is
   A. 1 B. 2 C. 3 D. 4
   ii. (2 points) Justify your answer

   (b) Consider Newton iterates

$$x^{(k+1)} = x^{(k)} - \frac{f'(x^{(k)})}{f''(x^{(k)})}$$

   It can be shown that there exists

$$e_{k+1} \leq Ce_k^2 \quad e_k = |x^{(k)} - r|$$

i. (1 point) Choose the correct value of C from the following choices A. $\frac{\alpha}{\beta}$ B. $\frac{\beta}{\alpha}$ C. $\frac{\alpha}{2\beta}$ D. $\frac{\beta}{2\alpha}$

ii. (3 points) Justify your answer

(c) (3 points) Find $t > 0$ such that for any $x^{(0)} \in (r - t, r + t)$ the Newton iterates converge to r.

3. Let $f : C \subset \mathbb{R}^d \to \mathbb{R}$ be a differentiable function lowerbounded below and upperbounded by a function g as follows

$$f(\mathbf{y}) \leq g(\mathbf{y}; \mathbf{x}) \left(= f(\mathbf{x}) + \nabla f(\mathbf{x})^\top (\mathbf{y} - \mathbf{x}) + \frac{\beta}{2} \|\mathbf{y} - \mathbf{x}\|^2 \right)$$

holds for all $\mathbf{x}, \mathbf{y} \in C$

(a) (2 points) Under what condition on $\mathbf{v}$

$$h(\alpha) = g(\mathbf{x}^{(k)} + \alpha \mathbf{v}; \mathbf{x}^{(k)}) - g(\mathbf{x}^{(k)}; \mathbf{x}^{(k)})$$

is strictly less than 0 for some $\alpha \geq 0$. For such a choice of $\mathbf{v}$ find

$$\alpha^* = min_{\alpha \geq 0} h(\alpha)$$

(b) (3 points) Set up an iterative scheme

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \alpha_k \mathbf{v}^{(k)}$$

where $\mathbf{v}^{(k)}$ is chosen as in the previous question with $\mathbf{x} = \mathbf{x}^{(k)}$, and $\alpha_k = \alpha^*$. For such a choice find the smallest possible C_k such that

$$f(\mathbf{x}^{(k+1)}) - f(\mathbf{x}^{(k)}) \leq C_k$$

Your answer should mention C_k with a very brief justification.

(c) (5 points) Starting from arbitrary $\mathbf{x}^{(0)}$ and assuming that

$$\nabla f(\mathbf{x}^{(k)})^\top \mathbf{v}^{(k)} \geq \delta \|\nabla f(\mathbf{x}^{(k)})\|^2$$

holds for all $k = 0, 1, \ldots$, how many iterations will be required to find an $\hat{\mathbf{x}}$ such that $|\nabla f(\hat{\mathbf{x}})| \leq \epsilon$ for a given ϵ. (The answer should state the relationship between $T, \delta, \beta, f(\mathbf{x}^{(0)})$ and any other quantity you feel necessary.

4. Consider the Linear system equations $A\mathbf{x} = b$ where A is a $d \times d$ real valued matrix and b is a d-dimensional vector.

Define $\mathtt{res}(\mathbf{x}) = b - A\mathbf{x}$. We wish to solve the system (A, b) with the following iterative procedure

- (Initialize)

$$\mathbf{u}^{(0)} = \mathtt{res}(\mathbf{x}^{(0)})$$

- (Iterate)

$$\alpha_k = \frac{\|\mathtt{res}(\mathbf{x}^{(k)})\|^2}{\mathbf{u}^{(k)\top} A \mathbf{u}^{(k)}}$$

-

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \alpha_k \mathbf{u}^{(k)}$$

-

$$\beta_k = \frac{\|\mathtt{res}(\mathbf{x}^{(k+1)})\|^2}{\|\mathtt{res}(\mathbf{x}^{(k)})\|^2}$$

-

$$\mathbf{u}^{(k+1)} = \mathtt{res}(\mathbf{x}^{(k+1)}) + \beta_k \mathbf{u}^{(k)}$$

(a) (5 points) Why would this algorithm solve the linear system of equations?

(b) Consider two systems,(A_1, b) and (A_2, b) with same b but different matrices A_1 and A_2.

$$A_1 = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

i. (1 point) The Algorithm applies to only one of the systems. Which one and why?

ii. (4 points) Modify the algorithm so that it will apply to both the systems. State the modification and give a brief justification. Any modification should be under the same style of algorithms mentioned in a.

Computational Methods of Optimization
Final Exam-Part 2(25th Jan,2021)

Start Time: 10:30 AM End Time : 12:00 Noon

Instructions

- Answer all questions
- See upload instructions in the form

Question:	1	2	3	Total
Points:	10	10	10	30
Score:				

1. Consider the gradient projection algorithm

$$\mathbf{x}^{(k+1)} = P_C\left(\mathbf{x}^{(k)} - \alpha\nabla f(\mathbf{x}^{(k)})\right)$$

for

$$\min_{\mathbf{x}\in C} f(\mathbf{x})$$

where $P_C(\mathbf{z})$ is the projection of the point $\mathbf{z}$ on the convex set C.

(a) (5 points) At a feasible point $\mathbf{x}^{(k)}$ suppose we use

$$\mathbf{x}^{(k+1)} = P_C\left(\mathbf{x}^{(k)} + \alpha\mathbf{u}\right)$$

where $\mathbf{u} \in \mathbb{R}^d$. For what values of $\mathbf{u}$ is $\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}$ a *feasible descent direction*.

(b) (5 points) State an upper-bound on the stepsize if the derivative of f was continuous with Lipschitz constant L

2. Consider the following problem

$$p^* = \min_{x,y\in\mathbb{R}} f(x,y)\ (\equiv x^2 - y^2 + 2(x+y))$$

subject to $x^2 + y^2 = 1$.

(a) (3 points) Define the dual function, $g(\mu)$ where μ is a dual variable? What is the domain of the function

(b) (2 points) State the optimality criteria of the dual optimization problem

(c) Let d^* be the optimal value of the dual problem.

i. (1 point) Is $p^* = d^*$? A. Yes B. No

ii. (4 points) Give reasons

3. Consider the following Linear Program

$$min_{x_1,x_2} x_1 + x_2 \text{ subject to} x_1 + 2x_2 \leq 4, x_2 \leq 1$$

$$x_1, x_2 \geq 0$$

(a) (2 points) Express the problem in the standard form?

$$min_x c^T x, \text{subject to} Ax = b, x \geq 0$$

Clearly State A, b, c

(b) (2 points) Find the Basis and BFS, $\bar{x}$, corresponding to the point where the constraints $x_1 + 2x_2 \leq 4$, and $x_2 \leq 1$ are active.

(c) (3 points) Is the BFS optimal? Give reasons

(d) (3 points) Find a new BFS using the simplex method? Identify the new basis vector and the vector which is leaving?