E1 222 Stochastic Models and Applications
Test II

`Time: 75 minutes` `Max. Marks:40`
`Date: 1 Nov 2021`

Answer **Any FOUR** questions. All questions carry equal marks

1. a. Let X_1, X_2 be *iid* random variables with (common) density function

$$f(x) = e^{-x},\ x > 0$$

Define random variables Y_1, Y_2 by

$$Y_1 = X_1 + X_2, \quad Y_2 = \frac{X_1}{X_1 + X_2}$$

Find joint density of Y_1, Y_2. Are Y_1, Y_2 independent?

b. Let X be uniform over $[1,\ 3]$ and let Y be uniform over $[2,\ 4]$. Suppose X, Y are independent. Find $P[X > Y]$.

2. a. Let U, V be independent Gaussian random variables with mean 0 and variance 1. Let $Z = \rho U + \sqrt{(1-\rho^2)}V$, where $-1 < \rho < 1$. Find density of Z and $\mathrm{Cov}(Z, U)$. Define random variables X, Y by $X = \sigma_1 U$ and $Y = \sigma_2 Z$ where $\sigma_1, \sigma_2 > 0$. Find the joint density of X, Y.

b. Let A, B be events such that $P(A) = 0.3$, $P(B) = 0.5$ and $P(A \cup B) = 0.6$. Let I_A, I_B be indicator random variables of these two events. Find $E[I_A I_B]$, $\mathrm{Cov}(I_A, I_B)$ and the correlation coefficient of I_A and I_B.

3. a. Let $X_1, X_2, \cdots$ be *iid* random variables having exponential density with parameter λ. Let N be a geometric random variable with parameter p. N is independent of all X_i. Find density of S where

$$S = X_1 + X_2 + \cdots + X_N$$

(Hint: For any specific positive integer n, if $X_1, \cdots, X_n$ are iid exponential random variables with parameter λ, then $X_1 + X_2 + \cdots + X_n$ has gamma density with parameters n and λ. Note that $\Gamma(n) = (n-1)!$ when n is a positive integer).

b. For any two random variables, X, Y, show that $\mathrm{Cov}(X,Y) = \mathrm{Cov}(X, E[Y|X])$

4. a. Let $X_i,\ i = 1, 2, \cdots$ be a sequence of *iid* Gaussian random variables with $EX_i = 0$ and $\mathrm{Var}(X_i) = 1, \forall i$. Let $S_n = \sum_{i=1}^{n} X_i$. Let F_n be the distribution function of $\frac{S_n}{n}$ and let G_n be the distribution function of $\frac{S_n}{\sqrt{n}}$. Find F_n and G_n. Do these distribution functions converge as $n \to \infty$? If so, what are the limits?

   b. Suppose X is a random variable with mean μ and variance 1. We do not know μ. We are estimating it by taking sample average of n *iid* samples. Explain how we can use central limit theorem to decide on the value of n.

5. a. Define transient and recurrent states of a Markov chain.

   b. Consider a Markov chain on the state space $S = \{0, 1, 2, 3, 4\}$ with the following transition probability matrix:

$$P = \begin{bmatrix} 0.15 & 0.22 & 0.1 & 0.28 & 0.25 \\ 0 & 0.25 & 0.75 & 0 & 0 \\ 0 & 0.75 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0.65 & 0.35 \\ 0 & 0 & 0 & 0.55 & 0.45 \end{bmatrix}$$

   Specify which are the transient and recurrent states and find all the closed irreducible subsets of recurrent states. Find absorption probabilities from any one of the transient states to any one of the closed irreducible sets of recurrent states. Find a stationary distribution of the chain.