Computational Methods of Optimization Final Exam-Part 1 (1st Dec, 2021)

Instructions:

- This is the first part of the Final test
- This is a closed book test. Please do not consult any additional material.
- Attempt all questions
- Total time is 70 mins.

Name:		
SRNO:	Degree·	Dent:

Question:	1	2	3	4	Total
Points:	10	10	10	10	40
Score:					

\mathcal{S}_d would denote the set of $d \times d$ Symmetric rea	d valued matrices. \mathcal{S}_d^+ denote the set of $d \times d$ Symmetric
real valued positive semidefinite matrices and \mathcal{S}_d^{++}	would denote the set of positive definite matrices.

4	a .	
1.	Consier	minimizing

$$min_{t \in \mathbb{R}} g(t) \left(= \mathbf{u}^{\top} L^2 \mathbf{u} + t^2 \mathbf{u}^{\top} L^{-2} \mathbf{u} - 2t(\mathbf{u}^{\top} \mathbf{u}) \right)$$

where $\mathbf{u} \in \mathbb{R}^d$, L is invertible and $L \in \mathcal{S}_d$.

- (a) (1 point) Answer True or False. There exists $t \in \mathbb{R}$ such that g(t) < 0. ____.
- (b) (2 points) Justify your answer.

(c) (2 points) Find the global minimum of g. Find optimal t and the optimal value

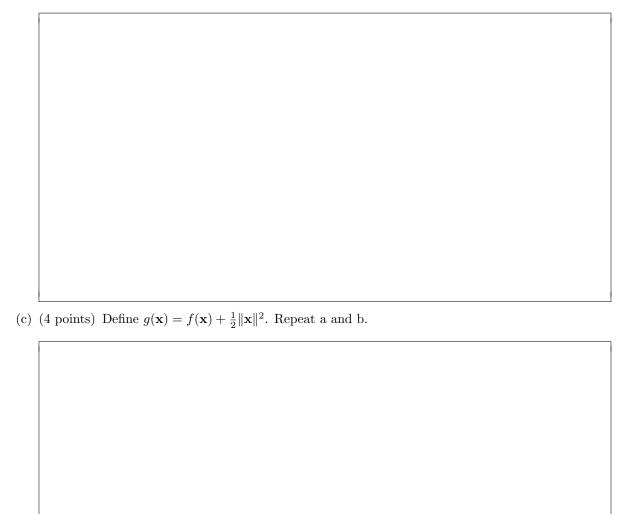
(d) Let $A \in \mathcal{S}_d^{++}$. Consider

$$\mathrm{max}_{\mathbf{u} \in \mathbb{R}^d} f(\mathbf{u}) \left(= \frac{\|\mathbf{u}\|^4}{(\mathbf{u}^\top A \mathbf{u}) \mathbf{u}^\top A^{-1} \mathbf{u}} \right)$$

i. (1 point) Find the global optimal value

ii. (2 points) Give reasons

iii.	(2 points) Find \mathbf{u} where the maximum is achieved?
2. Let v_1, v_2	$v_1^\top v_2 = 0$. Consider
	$\min_{\mathbf{x} \in \mathbb{R}^d} f(\mathbf{x}) \left(= \frac{1}{2} (\mathbf{x}^\top v_1)^2 + \alpha v_2^\top \mathbf{x} \right)$
	$\in \mathbb{R}$ is a constant.
(a) (2 p	points) Compute the Hessian? Is this function convex? Justify.
(b) (4 p	points) Compute the global minima of this function for all choices of α ? (Your answer should
clea	rly indicate the optimal value, f^* and the optimal point, \mathbf{x}^*)



- 3. Let $f: \mathbb{R}^d \times \mathbb{R} \in \mathcal{C}_L^1$ with L = 0.5 and f(0) = 0. It is given that $f^* = \min_{\mathbf{x} \in \mathbb{R}^d} f(\mathbf{x}) = -1$ Suppose we use the steepest descent procedure with exact line search for this algorithm with $\mathbf{x} = 0$. Let $\mathbf{x}^{(k)}, f(\mathbf{x}^{(k)})$ be the ouput of the algorithm after k iterations.
 - (a) (3 points) Derive a lower bound on the decrease of function value at each iteration, $\Delta_k = f(\mathbf{x}^{(k)}) f(\mathbf{x}^{(k+1)})$ in terms of Lipschitz constant and $\nabla f(\mathbf{x})$.
 - (b) (2 points) What is the smallest number of iterations required to guarantee that the algorithm outputs a point, $\hat{\mathbf{x}}$ such that

$$\|\nabla f(\hat{\mathbf{x}})\| \le 0.1$$

	(c)	(5 points) Redo a and b if constant stepsize was used instead of exact stepsize strategy
	()	
4.	(a)	(2 points) State one iteration of Newton method for minimizing a C^2 function.
	(b)	(8 points) Consider applying Newton Method to the following problem
		$\min_{\mathbf{x} \in \mathbb{R}^2} g(\mathbf{x}) = (1 - x_1)^3 + x_2^2$
		where $\mathbf{x} = [x_1, x_2]^{\top}$ starting from $\mathbf{x}^{(0)} = [\alpha, \beta]^{\top}$ It was found that after 5 iterations the value of $\mathbf{x}^{(3)} = [2, 0]^{\top}$. Can you find α, β .