E1 222 Stochastic Models and Applications
Problem Sheet 5.1

1. Let $\{X_n,\ n = 0, 1, \cdots\}$ be a Markov chain. Show that

$$\text{Prob}[X_{n+1} = z, X_{n-1} = y | X_n = x] = \text{Prob}[X_{n+1} = z | X_n = x]\ \text{Prob}[X_{n-1} = y | X_n = x]$$

   (That is, conditioned on the 'present' the 'past' is conditinally independent of the 'future')

2. Suppose we have two boxes and $2d$ balls, of which d are black and d are red. Initially d of the balls are placed in box-1 and the remaining in box-2. At each instant n, $n = 1, 2, \cdots$, a ball is chosen at random from each box and the two balls are placed in the opposite boxes. Let X_o denote number of black balls initially in box-1 and let X_n denote number of black balls in box-1 after the exchange at n, $n = 1, 2, \cdots$. Argue that $\{X_n\}$ is a Markov chain. Find the transition probabilities of the Markov chain $\{X_n\}$. State which are transient states and which are recurrent states. Is the chain irreducible?

3. Consider a Markov chain with the following transition probability matrix:

$$P = \begin{bmatrix} 0.7 & 0.3 & 0 & 0 & 0 & 0 \\ 0.8 & 0.2 & 0 & 0 & 0 & 0 \\ 0.2 & 0.2 & 0.2 & 0.2 & 0.2 & 0 \\ 0 & 0.3 & 0.2 & 0.2 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0 & 0.5 & 0.5 \end{bmatrix}$$

   Specify which are the transient and recurrent states and find all the closed irreducible subsets of recurrent states. Find the absorption probabilities from each of the transient states to each of the closed irreducible subsets of recurrent states.