Computational Methods of Optimization
Final Exam-Part 1 (1st Dec, 2021)

Instructions:

- This is the first part of the Final test
- This is a closed book test. Please do not consult any additional material.
- Attempt all questions
- Total time is 70 mins.

Name: ____________________

SRNO: Degree: Dept:

Question:	1	2	3	4	Total
Points:	10	10	10	10	40
Score:					

$\mathcal{S}_d$ would denote the set of $d \times d$ Symmetric real valued matrices. $\mathcal{S}_d^+$ denote the set of $d \times d$ Symmetric real valued positive semidefinite matrices and $\mathcal{S}_d^{++}$ would denote the set of positive definite matrices.

1. Consier minimizing

$$min_{t \in \mathbb{R}} g(t) \left(= \mathbf{u}^\top L^2 \mathbf{u} + t^2 \mathbf{u}^\top L^{-2} \mathbf{u} - 2t(\mathbf{u}^\top \mathbf{u})\right)$$

where $\mathbf{u} \in \mathbb{R}^d$, L is invertible and $L \in \mathcal{S}_d$.

(a) (1 point) Answer True or False. There exists $t \in \mathbb{R}$ such that $g(t) < 0$. ____.

(b) (2 points) Justify your answer.

(c) (2 points) Find the global minimum of g. Find optimal t and the optimal value

(d) Let $A \in \mathcal{S}_d^{++}$. Consider

$$\max_{\mathbf{u} \in \mathbb{R}^d} f(\mathbf{u}) \left(= \frac{\|\mathbf{u}\|^4}{(\mathbf{u}^\top A \mathbf{u}) \mathbf{u}^\top A^{-1} \mathbf{u}}\right)$$

i. (1 point) Find the global optimal value

ii. (2 points) Give reasons

iii. (2 points) Find $\mathbf{u}$ where the maximum is achieved?

2. Let $v_1, v_2 \in \mathbb{R}^d$ such that $v_1^\top v_2 = 0$. Consider

$$\min_{\mathbf{x}\in\mathbb{R}^d} f(\mathbf{x}) \left(= \frac{1}{2}(\mathbf{x}^\top v_1)^2 + \alpha v_2^\top \mathbf{x}\right)$$

where $\alpha \in \mathbb{R}$ is a constant.

(a) (2 points) Compute the Hessian? Is this function convex? Justify.

(b) (4 points) Compute the global minima of this function for all choices of α? (Your answer should clearly indicate the optimal value, f^* and the optimal point, $\mathbf{x}^*$)

(c) (4 points) Define $g(\mathbf{x}) = f(\mathbf{x}) + \frac{1}{2}\|\mathbf{x}\|^2$. Repeat a and b.

3. Let $f : \mathbb{R}^d \times \mathbb{R} \in \mathcal{C}_L^1$ with $L = 0.5$ and $f(0) = 0$. It is given that $f^* = min_{\mathbf{x}\in\mathbb{R}^d} f(\mathbf{x}) = -1$ Suppose we use the steepest descent procedure with exact line search for this algorithm with $\mathbf{x} = 0$. Let $\mathbf{x}^{(k)}, f(\mathbf{x}^{(k)})$ be the ouput of the algorithm after k iterations.
   (a) (3 points) Derive a lower bound on the decrease of function value at each iteration, $\Delta_k = f(\mathbf{x}^{(k)}) - f(\mathbf{x}^{(k+1)}$ in terms of Lipschitz constant and $\nabla f(\mathbf{x})$.
   (b) (2 points) What is the smallest number of iterations required to guarantee that the algorithm outputs a point, $\hat{\mathbf{x}}$ such that
   $$\|\nabla f(\hat{\mathbf{x}})\| \leq 0.1$$

(c) (5 points) Redo a and b if constant stepsize was used instead of *exact stepsize* strategy

4. (a) (2 points) State one iteration of Newton method for minimizing a $\mathcal{C}^2$ function.

   (b) (8 points) Consider applying Newton Method to the following problem

$$min_{\mathbf{x}\in\mathbb{R}^2} g(\mathbf{x}) = (1 - x_1)^3 + x_2^2$$

where $\mathbf{x} = [x_1, x_2]^\top$ starting from $\mathbf{x}^{(0)} = [\alpha, \beta]^\top$ It was found that after 5 iterations the value of $\mathbf{x}^{(3)} = [2, 0]^\top$. Can you find α, β.