Computational Methods of Optimization Final Exam-Part 1 (1st Dec, 2021)

Instructions:

- · This is the first part of the Final test
- This is a closed book test. Please do not consult any additional material.
- · Attempt all questions
- · Total time is 70 mins.

SRNO:

Name:

Dept:

Score:

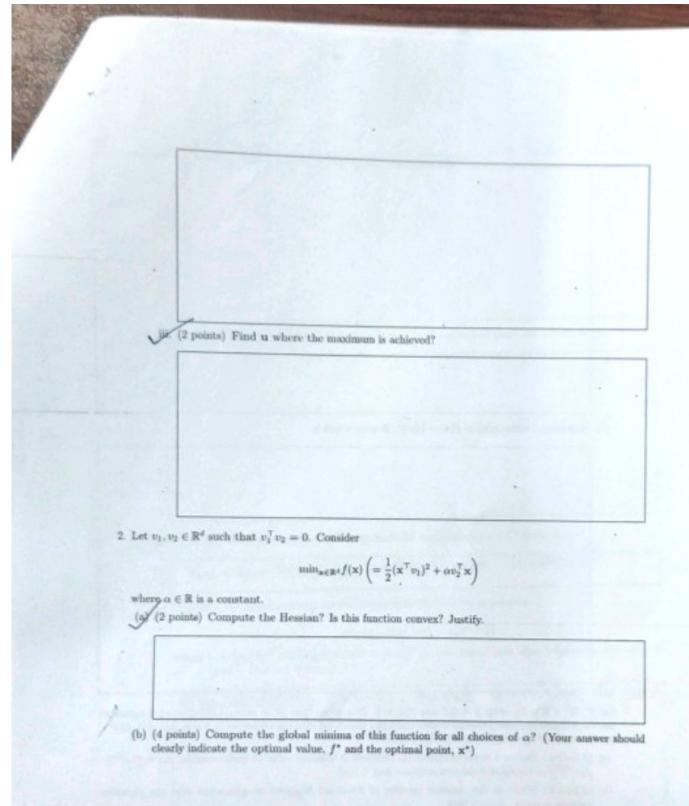
 Question:
 1
 2
 3
 4
 Total

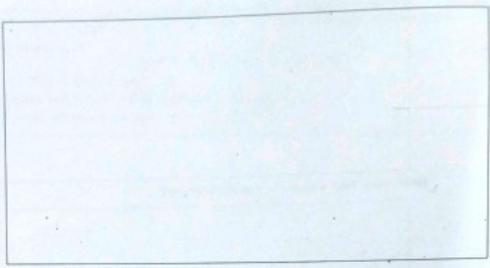
 Points:
 10
 10
 10
 10
 40

 S_d would denote the set of $d \times d$ Symmetric real valued matrices. S_d^+ denote the set of $d \times d$ Symmetric real valued positive semidefinite matrices and S_d^{++} would denote the set of positive definite matrices.

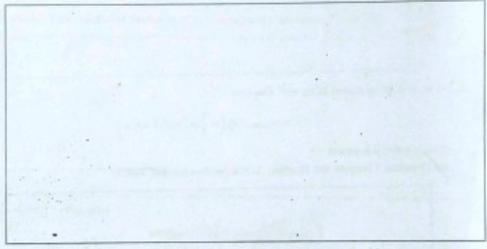
1 Consier minimizing $min_{t \in \mathbb{R}}g(t) \left(=\mathbf{u}^{\top}L^{2}\mathbf{u}+t^{2}\mathbf{u}^{\top}L^{-2}\mathbf{u}-2t(\mathbf{u}^{\top}\mathbf{u})\right)$ where $\mathbf{u} \in \mathbb{R}^d$, L is invertible and $L \in \mathcal{S}_d$. (1-point) Answer True or False. There exists $t \in \mathbb{R}$ such that g(t) < 0. (b) (2 points) Justify your answer. (c) (2 points) Find the global minimum of g. Find optimal t and the optimal value (d) Let $A \in \mathcal{S}_d^{++}$, Consider $\mathrm{max}_{\mathbf{u} \in \mathbb{R}^d} f(\mathbf{u}) \left(= \frac{\|\mathbf{u}\|^4}{(\mathbf{u}^\top A \mathbf{u}) \mathbf{u}^\top A^{-1} \mathbf{u}} \right)$ j (1 point) Find the global optimal value

ji: (2 points) Give reasons





(c) (4 points) Define $g(\mathbf{x}) = f(\mathbf{x}) + \frac{1}{2} ||\mathbf{x}||^2$. Repeat a and b.

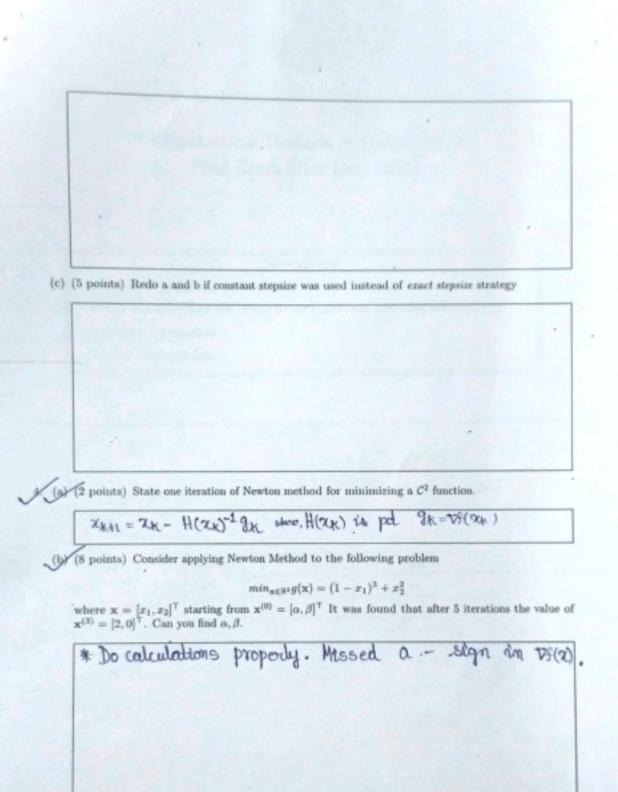


Let f: R^d × R ∈ C¹_L with L = 0.5 and f(0) = 0. It is given that f* = min_{x∈R^d}f(x) = −1 Suppose we use the steepest descent procedure with exact line search for this algorithm with x = 0. Let x^(k), f(x^(k)) be the output of the algorithm after k iterations.

(a) (3 points) Derive a lower bound on the decrease of function value at each iteration, Δ_k = f(x^(k)) - f(x^(k+1)) in terms of Lipschitz constant and ∇f(x).

(b) (2 points) What is the smallest number of iterations required to guarantee that the algorithm outputs a point, $\hat{\mathbf{x}}$ such that

 $\|\nabla f(\hat{\mathbf{x}})\| \le 0.1$



Computational Methods of Optimization Final Exam 2(1st Dec, 2021)

Instructions:

- · This is the second part of Final Test
- This is a closed book test. Please do not consult any additional material.
- Attempt all questions
- · Total time is 70 mins.

Name:

SRNO:

Degree:

Dept:

Question:	1	2	3	- 4 .	Total
Points:	10	10	10	10	40
Score:					10 10

 S_d would denote the set of $d \times d$ Symmetric real valued matrices. S_d^+ denote the set of $d \times d$ Symmetric real valued positive semidefinite matrices and S_d^{++} would denote the set of positive definite matrices.

 Let Q ∈ S⁺⁺_d, h ∈ R^d, a_i ∈ R^d, b_i ∈ R. We consider solving the following problem with the Active set strategy

$$min_{\mathbf{x} \in \mathbb{R}^d} f(\mathbf{x}) \left(= \frac{1}{2} \mathbf{x}^\top Q \mathbf{x} + h^\top \mathbf{x} \right) \ .$$

subject to $a_i^\top \mathbf{x} \ge b_i$, $i = \{1, ..., m\}$.

(a) (2 points) State the KKT conditions for this problem?

(b) (2 points) In a given iteration we find that the working set is W = {1,2,3}, and x is a feasible point. Moreover it is found that

$$u^* = 0, \mu^* = [1, 2, -3]^T$$
 is the KKT point for

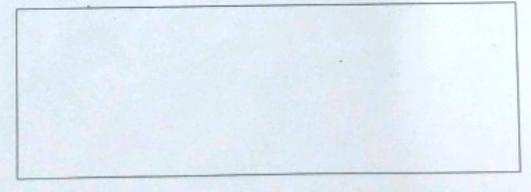
$$\mathbf{u}^* = \operatorname{argmin}_{\mathbf{u} \in \mathbb{R}^d} \frac{1}{2} \mathbf{u}^\top Q \mathbf{u} + \hat{g}^\top \mathbf{u}$$

subject to
$$a_i^T \mathbf{x} = b_i, i \in \hat{W}$$

The Lagrangian is

$$L(\mathbf{u}, \boldsymbol{\mu}) = \frac{1}{2}\mathbf{u}^{\top}Q\mathbf{u} + \hat{\boldsymbol{g}}^{\top}\mathbf{u} - \sum_{i \in \mathcal{W}} \mu_i(\boldsymbol{a}_i^{\top}\mathbf{x} - \boldsymbol{b}_i)$$

If the algorithm is stopped here what can be said about the optimality of \dot{x} .



(c) To continue the algorithm, as per active-set strategy, find a constraint l that needs to be dropped from \hat{W} to create a new working set $W=\hat{W}$. After identifying W compute

$$\hat{\mathbf{u}} = argmin_{\mathbf{u} \in \mathbb{R}^d} \frac{1}{2} \mathbf{u}^\top Q \mathbf{u} + \hat{g}^\top \mathbf{u}$$

subject to
$$a_i^\top \mathbf{u} = 0, i \in W$$

where $\hat{g} = \nabla f(\hat{\mathbf{x}})$.

- i. (2 points) What is the relationship between \hat{g} and μ^* ?
- (4 points) Suppose it is given that ûQû = 6 can you find the value of a_i^T û. Give reasons for your answer.
- 2 Let A = (r. a. l.c. Pd be a median and minimum and m
- Let A = {a₁,...,a_n} ⊂ R^d be a set of n vectors in R^d. We need to decide if z ∈ R^d lies in the convex hull of A, defined by

$$Conv(A) = \left\{ \sum_{i=1}^{n} \alpha_i a_i \| \alpha_i \ge 0, \sum_{i=1}^{n} \alpha_i = 1 \right\}$$

- (a) (5 points) Pose this problem as a convex optimization problem.
- (b) (5 points) Derive the KKT conditions of this problem
- 3. (10 points) Consider the set $C = \{\mathbf{x} \in \mathbb{R}^d | \|\mathbf{x} \mu\| \le \frac{1}{2} \|\mu\|^2 \}$. Find $h \in \mathbb{R}^d$ and $b \in \mathbb{R}$ such that

$$0 \in \{\mathbf{z} \in \mathbb{R}^d | h^{\mathsf{T}}\mathbf{z} \leq b\}, \text{ and } C \subset \{\mathbf{z} \in \mathbb{R}^d | h^{\mathsf{T}}\mathbf{z} > b\}$$

4. Consider the following problem

$$min_{\mathbf{x}}f(\mathbf{x})\left(=\frac{1}{2}\mathbf{x}^{\top}Q\mathbf{x}\right), \quad \text{ subject to } \|\mathbf{x}\|^{2} \leq 1$$

- Q is not necessas rily psd and can be indefinite. Let \mathbf{x}^* be the global optimal solution.
- (a) (3 points) Show that the dual is a function of one variable? You need to state the dual

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(2 points) Find the dom	nain of the dual function	
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(2 points) Find the dual	antimal colori - 2	
	openinal solution?	
	and the second s	
noints) Can 5.4 -	(0)	
points) can you mid x	(0) such that $f(\mathbf{x}^{(0)})$ is equal to the dual optimal?	
	- Liver and	
point) Find x* with jus	tification	

Computational Methods of Optimization Final Exam-Part 1(25th Jan, 2021)

Start Time: 9:15 AM End Time: 10:25 AM

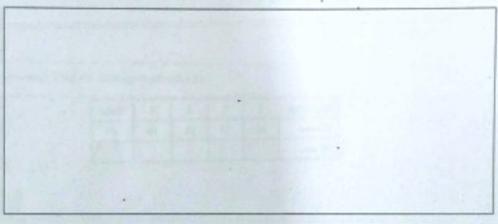
Instructions

- Answer all questions
- · See upload instructions in the form

Question:	1	2	3	4	Total
Points:	10	10	10	10	40
Score:			1		

In the following $\|\mathbf{x}\| = \sqrt{\mathbf{x}^T \mathbf{x}}$. For a real valued function, f, in one variable, f' denotes the first derivative, f'' denote the second derivative, and $f^{(3)}$ denotes the third derivative.

- (a) Let f: [-1,2] → R be a differentiable function.
 - i. (2 points) Suppose it was known that f(1) = -f(0) = 1. Which of the following is correct A. $|f'(x)| \le 1.5$ for all $x \in (0,1)$. B. $|f'(x)| \ge 1.5$ for all $x \in (0,1)$. C. |f'(x)| = 2 for some $x \in (0,1)$. D. None of the above
 - ii. (2 points) Suppose f(0.5) = f(0.8) and $f^*(x) > 0 \ \forall x \in (-1,2)$. A. There are no minima in [-1,2]. B. There is exactly one minimum in [-1,2]. C. There is at-least one minimum in [-1,2]. D. None of the above
 - iii. (2 points) Let f attain minimum at x = 2. Which of the following is true A. $f(x) \ge f(2)$ for all $x \in \mathbb{R}$ B. $f'(x) \le 0$ for all $x \in [-1, 2]$ C. $f'(x) \ge 0$ for all $x \in [-1, 2]$
 - (b) (4 points) Consider minimizing a convex quadratic function whose Hessian has largest and smallest eigenvalue 3 and 1 respectively. Suppose we implement the steepest descent procedure starting at a point x⁽⁰⁾ such that E(x⁽⁰⁾ = 1 where E(x) = ½(x-x*)H(x-x*) where x* is the global minimum. After how many iterations can you guarantee that ||x^(T) x*|| ≤ 10⁻².



- Let f: (a,b) → ℝ be thrice differentiable function such that f'(a)f'(b) < 0. Assume that for all x ∈ (a,b), |f^a(x)| ≥ β, |f⁽³⁾(x)| ≤ α where β, α > 0.
 - (a) i. (1 point) The Number of critical points in (a, b) isA. 1 B. 2 C. 3 D. 4
 - ii. (2 points) Justify your answer

(b) Consider Newton iterates

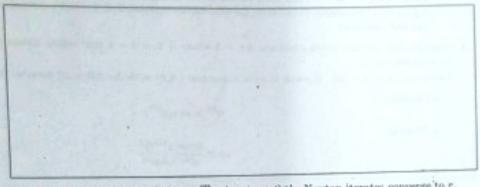
$$x^{(k+1)} = x^{(k)} - \frac{f'(x^{(k)})}{f^{\circ}(x^{(k)})}$$

It can be shown that there exists

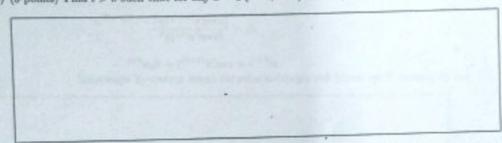
$$c_{k+1} \leq C c_k^2 \ e_k = |x^{(k)} - r|$$

Page 2

- i. (1 point) Choose the correct value of C from the following choices A. $\frac{\alpha}{\beta}$ B. $\frac{\beta}{\alpha}$ C. $\frac{\alpha}{2\beta}$ D. $\frac{\beta}{3\alpha}$
- ii. (3 points) Justify your answer



(c) (3 points) Find t > 0 such that for any $x^{(0)} \in (r - t, r + t)$ the Newton iterates converge to r.



 Let f: C ⊂ R^d → R be a differentiable function lowerbounded below and upperbounded by a function g as follows

 $f(\mathbf{y}) \le g(\mathbf{y}; \mathbf{x}) \left(\equiv f(\mathbf{x}) + \nabla f(\mathbf{x})^{\top} (\mathbf{y} - \mathbf{x}) + \frac{\beta}{2} ||\mathbf{y} - \mathbf{x}||^2 \right)$

holds for all $x, y \in C$

(a) (2 points) Under what condition on v

$$h(\alpha) = g(\mathbf{x}^{(k)} + \alpha \mathbf{v}; \mathbf{x}^{(k)}) - g(\mathbf{x}^{(k)}; \mathbf{x}^{(k)})$$

is strictly less than 0 for some $\alpha \ge 0$. For such a choice of v find

$$\alpha^* = \min_{\alpha \geq 0} h(\alpha)$$

(b) (3 points) Set up an iterative scheme

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \alpha_k \mathbf{v}^{(k)}$$

where $\mathbf{v}^{(k)}$ is chosen as in the previous question with $\mathbf{x} = \mathbf{x}^{(k)}$, and $\alpha_k = \alpha^*$. For such a choice find the smallest possible C_k such that

$$f(\mathbf{x}^{(k+1)}) - f(\mathbf{x}^{(k)}) \le C_k$$

Your answer should mention C_k with a very brief justification.

(c) (5 points) Starting from arbitrary x(0) and assuming that

$$\nabla f(\mathbf{x}^{(k)})^{\mathsf{T}} \mathbf{v}^{(k)} \ge \delta ||\nabla f(\mathbf{x}^{(k)})||^2$$

holds for all k = 0, 1, ..., how many iterations will be required to find an $\hat{\mathbf{x}}$ such that $|\nabla f(\hat{\mathbf{x}})| \le \epsilon$ for a given ϵ . (The answer should state the relationship between $T, \delta, \beta, f(\mathbf{x}^{(0)})$) and any other quantity you feel necessary.

Consider the Linear system equations Ax = b where A is a d × d real valued matrix and b is a d-dimensional vector.

Define res(x) = b - Ax. We wish to solve the system (A, b) with the following iterative procedure

· (Initialize)

$$u^{(0)} = res(x^{(0)})$$

· (Iterate)

$$\alpha_k = \frac{\|\text{res}(\mathbf{x}^{(k)})\|^2}{n^{(k)} \top An^{(k)}}$$

 $\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \alpha_k \mathbf{u}^{(k)}$

$$\beta_k = \frac{\lVert \operatorname{res}(\mathbf{x}^{(k+1)})\rVert^2}{\lVert \operatorname{res}(\mathbf{x}^{(k)})\rVert^2}$$

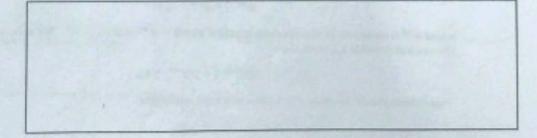
$$u^{(k+1)} = res(x^{(k+1)}) + \beta_k u^{(k)}$$

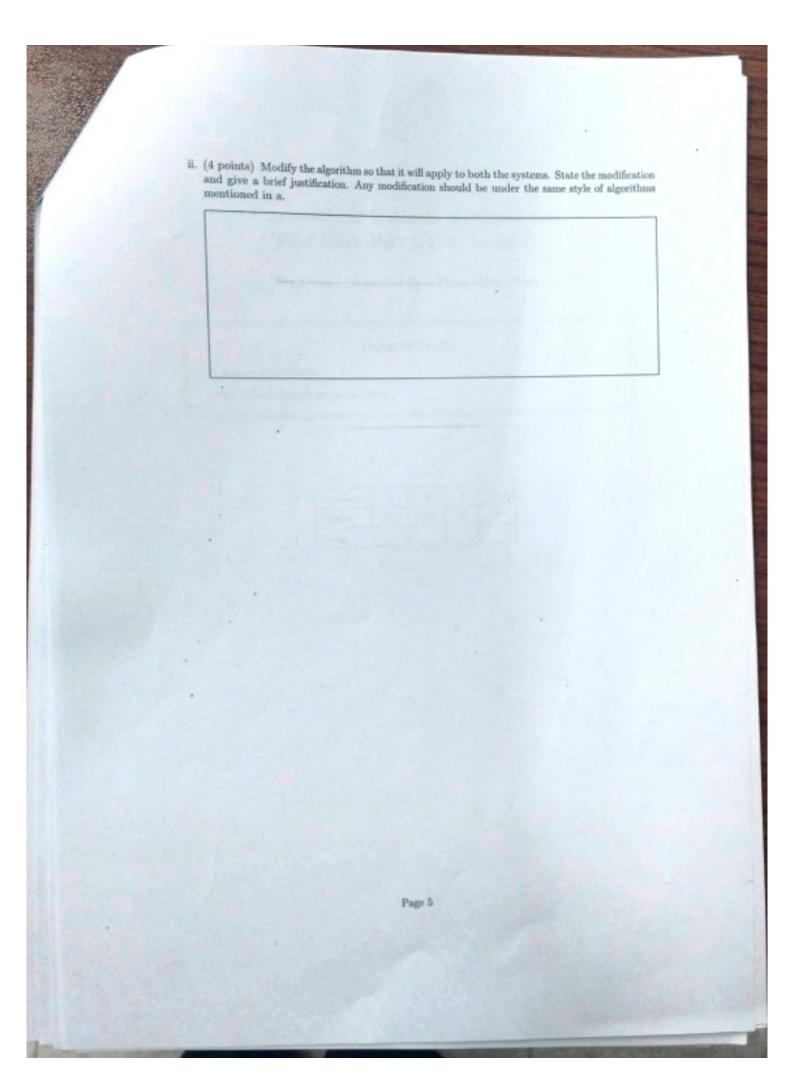
(a) (5 points) Why would this algorithm solve the linear system of equations?

(b) Consider two systems, (A1, b) and (A2, b) with same b but different matrices A1 and A2.

$$A_1 = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 2 \end{bmatrix}, \ A_2 = \begin{bmatrix} 4 & 1 & 1 \\ 1 & 3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

i. (1 point) The Algorithm applies to only one of the systems. Which one and why?





Computational Methods of Optimization Final Exam-Part 2(25th Jan, 2021)

Start Time: 10:30 AM End Time: 12:00 Noon

Instructions

- Answer all questions
- · See upload instructions in the form

Question:	1	2	3	Total
Points:	10	10	10	30
Score:				

Consider the gradient projection algorithm

$$\mathbf{x}^{(k+1)} = P_C \left(\mathbf{x}^{(k)} - \alpha \nabla f(\mathbf{x}^{(k)}) \right)$$

for

$$\min_{\mathbf{x} \in C} f(\mathbf{x})$$

where $P_C(\mathbf{z})$ is the projection of the point \mathbf{z} on the convex set C.

(a) (5 points) At a feasible point $\mathbf{x}^{(k)}$ suppose we use

$$\mathbf{x}^{(k+1}) = P_C \left(\mathbf{x}^{(k)} + \alpha \mathbf{u} \right)$$

where $\mathbf{u} \in \mathbb{R}^d$. For what values of \mathbf{u} is $\mathbf{x}^{(k+1)} - \mathbf{x}^{(k)}$ a feasible descent direction.

(b) (5 points) State an upper-bound on the stepsize if the derivative of f was continuous with Lipschitz constant L

2. Consider the following problem

$$p^* = min_{x,y \in \mathbb{R}} f(x,y) (\equiv x^2 - y^2 + 2(x+y))$$

subject to $x^2 + y^2 = 1$.

(a) (3 points) Define the dual function, $g(\mu)$ where μ is a dual variable? What is the domain of the function

1000	*				
(2 points	State the optimalia				
) State the optimalit	y criteria of t	he dual optimiz	ation problem	
Let d* be	the optimal value of sint) Is $p^* = d^*$? A.	the dual pro	blem.		
ii. (4 pc	ints) Give reasons	168 D. 140			
66					

 $min_{x_1,x_2}x_1+x_2$ subject $\mathrm{to}x_1+2x_2\leq 4, x_2\leq 1$

et (2 points)	Express the probl	lem in the standard form	17	
		$min_z e^{\top}z$, subject to	$Az = b, z \ge 0$	
Clearly S	ate A, b, c			
1) (0 - 1 - 1				
 (2 points) and x₂ < 	Find the Basis and 1 are active.	d BFS, \hat{z} , corresponding t	to the point where the constraints x_1+2	z ₂ ≤
c) (3 nointe)	Is the BFS optima	12 C:		
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which is le	Find a new BFS usvine?	sing the simplex method	? Identify the new basis vector and the	vect
			*	