E1 222 Stochastic Models and Applications
Test III

`Time: 90 minutes` `Max. Marks:40`
`Date: 17 Nov 2019`

Answer **ALL** questions. All questions carry equal marks
Answers should be written only in the space provided.

1. a. Consider a Markov chain with the following transition probability matrix:

$$P = \begin{bmatrix} 0.25 & 0.2 & 0 & 0.3 & 0.25 \\ 0 & 0.25 & 0.75 & 0 & 0 \\ 0 & 0.75 & 0.25 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0 & 0.7 & 0.3 \end{bmatrix}$$

Specify which are the transient and recurrent states and find all the closed irreducible subsets of recurrent states. Find the absorption probabilities from each of the transient states to each of the closed irreducible subsets of recurrent states.

Answer: Let us take the state space as $\{0, 1, 2, 3, 4\}$. It is easy to see that $\{1, 2\}$ is a closed irreducible set. (The two states communicate with each other and from 1 or 2 we cannot go to any other states). Similarly $\{3, 4\}$ is a closed irreducible set. Since from state 0 we can go to state 1 or state 3, it is a transient state. Hence, the decomposition of state space is as follows: $S_T = \{0\}$, $S_R = \{1, 2\} + \{3, 4\}$.
Let $C = \{1, 2\}$. Since there is only one transient state we need to compute $\rho_C(0)$. We have

$$\rho_C(0) = P(0, 1) + P(0, 2) + P(0, 0)\rho_C(0) = 0.2 + 0.25\rho_C(0)$$

This gives us $\rho_C(0) = 0.2/0.75 = 4/15$.
Let $C' = \{3, 4\}$. Then $\rho_{C'}(0) = 1 - \rho_C(0) = 11/15$.

b. Define positive recurrent and null recurrent states in a Markov chain. If π is a stationary distribution of the chain and i is a null recurrent state then explain why we must have $\pi(i) = 0$. Explain why a finite Markov chain cannot have a null recurrent state.

Answer: Let y be a recurrent state. Let T_y be the hitting time (or first passage time). That is, $T_y = \min\{n : n > 0, X_n = y\}$. Let $m_y = E_y[T_y]$ where E_y is expectation conditioned on $X_0 = y$.

The recurrent state y is called positive recurrent if $m_y < \infty$; otherwise (that is, if $m_y = \infty$) it is called null recurrent.

Let $N_n(y)$ denote the number of visitis to y till time n. Let $G_n(x, y) = E_x[N_n(y)]$. Then we have the result that $\frac{G_n(x,y)}{n} \to \frac{\rho_{xy}}{m_y}$ as $n \to \infty$. Hence, if y is null recurrent, then, $\frac{G_n(x,y)}{n} \to 0$.

If we let $I_y(X_n)$ be the indicator for X_n being y, then, $N_n(y) = \sum_{m=1}^{n} I_y(X_m)$. Since $E_x[I_y(X_n)] = P^n(x, y)$, we have $G_n(x, y) = \sum_{m=1}^{n} P^m(x, y)$. Thus

$$\frac{G_n(x, y)}{n} = \frac{1}{n} \sum_{m=1}^{n} P^m(x, y)$$

Hence, if y is null recurrent then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} P^m(x, y) = 0$$

If π is a stationary distribution, then we have $\pi(y) = \sum_x \pi(x) P^m(x, y)$ for all m. Summing this over $m = 1, \cdots, n$ and dividing by n, we get

$$\pi(y) = \frac{1}{n} \sum_{m=1}^{n} \sum_{x} \pi(x) P^m(x, y) = \sum_{x} \pi(x) \frac{1}{n} \sum_{m=1}^{n} P^m(x, y), \forall n$$

Since this holds for all n, it also holds in the limit. The limit of the RHS above, as $n \to \infty$, is zero if y is null recurrent, as explained earlier. (Note that we can take the limit inside the summation over x because of bounded convergence theorem). Thus, if y is null recurrent and π is a stationary distribution, then $\pi(y) = 0$.

Suppose, A is a finite closed set. Since it is closed, $\sum_{j\in A} P^m(i, j) = 1,\ \forall i \in A,\ \forall m$. Hence, we have

$$1 = \frac{1}{n} \sum_{m=1}^{n} \sum_{j\in A} P^m(i, j) = \sum_{j\in A} \frac{1}{n} \sum_{m=1}^{n} P^m(i, j), \forall n$$

Since the above holds for all n, it also holds as $n \to \infty$. Suppose all states in A are null recurrent. Then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} P^m(i,j) = 0, \forall j \in A$$

Thus, if all states in the finite closed set A are null recurrent, then,

$$1 = \lim_{n\to\infty} \sum_{j\in A} \frac{1}{n} \sum_{m=1}^{n} P^m(i,j) = \sum_{j\in A} \lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} P^m(i,j) = 0$$

which is a contradiction. In the above, we could take the limit inside the outer summation because A is a finite set.

The above shows that in a finite closed set, not all states can be null recurrent.

We have a result that if x leads to y and x is positive recurrent then y is positive recurrent. Hence, in a closed irreducible set of recurrent states, either all states positive recurrent or all states are null recurrent.

Suppose a finite chain has a null recurrent state. That state must be in one of the closed irreducible sets of recurrent states. In a finite chain, all closed irreducible sets of recurrent states have to be finite and hence none of them can be wholly null recurrent. Thus, a finite chain cannot have a null recurrent state.

2. a. Let $\{X_n,\ n \geq 0\}$ be an irreducible, positive recurrent aperiodic Markov chain whose stationary probabilities are given by π. Define another process $\{Y_n,\ n \geq 1\}$ by $Y_n = (X_{n-1}, X_n)$. (That is, Y_n keeps track of the last two states of the original chain). Is Y_n a Markov Chain? If so, find its transition probabilities and $\lim_{n\to\infty} \text{Prob}[Y_n = (i,j)]$.

Answer: Given $\{X_n\}$ is a Markov chain. Hence, conditioned on X_{n-2}, X_n, X_{n-1} are conditinally independent of $X_{n-3}, X_{n-4}, \cdots$. Hence, it is easy to see that $\{Y_n\}$ is a Markov chain.

We can calculate the transition probabilities as follows:

$$\begin{aligned}
\text{Prob}[Y_n = (i,j) | Y_{n-1} = (k,l)] &= \text{Prob}[X_{n-1} = i, X_n = j | X_{n-2} = k, X_{n-1} = l] \\
&= \begin{cases} 0 & \text{if } i \neq l \\ \text{Prob}[X_n = j | X_{n-1} = i] & \text{otherwise} \end{cases}
\end{aligned}$$

We can calculate the limiting probabilities as follows. Since $\{X_n\}$ is an irreducible positive recurrent aperiodic chain withstationary probabilities π, we have $\lim_{n\to\infty} \text{Prob}[X_n = i] = \pi(i)$

$$\begin{aligned}
\lim_{n\to\infty} \text{Prob}[Y_n = (i,j)] &= \lim_{n\to\infty} \text{Prob}[X_{n-1} = i, X_n = j] \\
&= \lim_{n\to\infty} \text{Prob}[X_n = j | X_{n-1} = i]\text{Prob}[X_{n-1} = i] \\
&= P(i,j)\pi(i)
\end{aligned}$$

where $P(i,j)$ are the transition probabilities of X_n.

b. Define transient and recurrent states in a Markov chain. When is a Markov chain said to be irreducible. Can an irreducible markov chain with countably infinite state space have some transient states and some recurrent states?

Answer: For any state y the hitting time is $T_y = \min\{n : n > 0, X_n = y\}$. Define $\rho_{xy} = \text{Prob}[T_y < \infty | X_0 = x]$. ρ_{xy} is the probability that starting in x, sometime or the other we hit y.

A state y is defined to be transient if $\rho_{yy} < 1$ and is defined to be recurrent if $\rho_{yy} = 1$.

A Markov chain is said to be irreducible if x leads to y for all x, y. That is, $\rho xy > 0$ for all states x, y.

We have a result that if x is recurrent and x leads to y then y is recurrent. In an irreducible chain, every state leads to every other state. Hence, in an irreducible chain, if any one state is recurrent then all states are recurrent. Thus, in an irreducible chain either all states are transient or all states are recurrent; we cannot have some transient and some recurrent states.

3. a. Let $X_1, X_2, \cdots$ be a sequence of discrete random variables with X_n being geometric with parameter λ/n where we have $0 < \lambda < 1$. Let $Z_n = X_n/n$. Does Z_n converges in distribution?

Answer: We have

$$F_{Z_n}(x) = P[Z_n \le x] = P[X_n \le nx] = 1 - \left(1 - \frac{\lambda}{n}\right)^{[nx]}$$

where $[nx]$ is the greatest integer smaller than or equal to nx. We need to find the limit of the RHS above as $n \to \infty$. For that we have

$$\left(1 - \frac{\lambda}{n}\right)^{[nx]} = \left(1 - \frac{\lambda}{n}\right)^{nx} \left(1 - \frac{\lambda}{n}\right)^{[nx]-nx}$$

For all n, $[nx] - nx$ is between -1 and zero and hence, as $n \to \infty$, the second factor on the RHS above goes to 1. The first term goes to $e^{-\lambda x}$. Thus,

$$\lim_{n\to\infty} F_{Z_n}(x) = 1 - e^{-\lambda x}$$

which shows that Z_n converges in distribution to the exponential distribution.

b. Let $X_1, X_2, \cdots$ be a sequence of independent random variables with

$$\text{Prob}[X_n = n] = \text{Prob}[X_n = -n] = \frac{1}{2n^2}; \quad \text{Prob}[X_n = 0] = \frac{n^2 - 1}{n^2}$$

Does this sequence converge (i) in probability, (ii) with probability one, (iii) in the r^{th} mean?

Answer: The probability of X_n being zero is increasing with n and hence a good candidate for limit is zero. To test convergence in probability, we have

$$\lim_{n\to\infty} P[|X_n - 0| > \epsilon] = \lim_{n\to\infty} P[X_n = n \text{ or } X_n = -n] = \lim_{n\to\infty} \frac{2}{2n^2} = 0$$

Hence $X_n \xrightarrow{P} 0$.

Now, if X_n converges in any other mode, it should converge to zero. To test for convergence almost surely, we can use Borel-Cantelli lemma. Let $A_n^\epsilon = [|X_n| > \epsilon]$. Then, as calculated above, $P[A_n^\epsilon] = \frac{1}{n^2}$, $\forall \epsilon > 0$. Thus, for all $\epsilon > 0$, $\sum_n P[A_n^\epsilon] < \infty$. Hence, by Borel-Cantelli lemma, X_n converges to zero almost surely.

To test for convergence in r^{th} mean we have

$$E[|X_n|^r] = (|n|^r + |-n|^r)\,\frac{1}{2n^2} = n^{r-2}$$

Hence, X_n converges in r^{th} mean for $r < 2$ and it does not converge for $r \geq 2$.

4. a. Twenty real numbers are rounded off to the nearest integer and added. Assume that the individual rounding-off errors are independent and uniformly distributed over $(-0.5,\ 0.5)$. Find the (approximate) probability that the sum obtained like this differes from the sum of the original twenty numbers by more than 3.

Answer: Let $X_1, \cdots, X_{20}$ represent the roundoff errors in the twenty numbers. It is given that X_i are iid uniform over $(-0.5,\ 0.5)$. Hence, $EX_i = 0$ and $\text{Var}(X_i) = 1/12$. Let $S = \sum_{i=1}^{20} X_i$. Then, $ES = 0$ and $\text{Var}(S) = 20/12 = 5/3$. The difference between the original sum and the sum of rounded-off numbers is $|S|$. Hence, the probability we want is

$$\begin{aligned} P[|S| > 3] &= P[S < -3] + P[S > 3] \\ &= P\left[\frac{S}{\sqrt{5/3}} < \frac{-3}{\sqrt{5/3}}\right] + P\left[\frac{S}{\sqrt{5/3}} > \frac{3}{\sqrt{5/3}}\right] \\ &\approx \Phi\left(\frac{-3}{\sqrt{5/3}}\right) + 1 - \Phi\left(\frac{3}{\sqrt{5/3}}\right) \end{aligned}$$

b. Suppose $X_1, X_2, \cdots$ be *iid* continuous random variables with density $f(x) = 2x,\ 0 \le x \le 1$. Let $S_n = \sum_{i=1}^{n} X_i^2$. Does S_n/n converge almost surely? (Answer Yes/No with a short justification). If your answer is yes, find x_0 such that the sequence S_n/n converges to x_0 with probability one. Also, explain how we can approximately find the probability of $[-a \le \frac{S_n}{n} - x_0 \le b]$ for some $a, b > 0$

Answer: Since X_i are iid, X_i^2 would also be iid. Hence, by strong law of large numbers, S_n/n converges almost surely to EX_1^2.

From the given density of X_i, we get

$$EX_1^2 = \int_0^1 x^2\ 2x\ dx = 0.5$$

Hence $\frac{S_n}{n} \overset{a.s.}{\to} 0.5$.

Let σ^2 be the variance of X_i^2. We can calculate it using the density for X_i. (Note that $\sigma^2 = EX_i^4 - (EX_i^2)^2$).

We have: $ES_n = 0.5n$ and $\text{Var}(S_n) = n\sigma^2$.

Now we have

$$
\begin{aligned}
P\left[-a \le \frac{S_n}{n} - 0.5 \le b\right] &= P[-na \le S_n - 0.5n < nb] \\
&= P\left[\frac{-na}{\sigma\sqrt{n}} \le \frac{S_n - 0.5n}{\sigma\sqrt{n}} \le \frac{nb}{\sigma\sqrt{n}}\right] \\
&\approx \Phi\left(\frac{nb}{\sigma\sqrt{n}}\right) - \Phi\left(\frac{-na}{\sigma\sqrt{n}}\right)
\end{aligned}
$$

This is how we can approximately find the required probability.