Computational Methods of Optimization
Third Midterm(27th Nov, 2023)

Instructions:

- This is a closed book test. Please do not consult any additional material.
- Attempt all questions
- Total time is 70 mins.
- Answer the questions in the spaces provided. Answers outside the spaces provided will not be graded.
- Rough work can be done in the spaces provided at the end of the booklet

Name: ____________________

SRNO: Degree: Dept:

Question:	1	2	Total
Points:	15	15	30
Score:			

1. Consider the following two dimensional problem where $\mathbf{x} = [x_1, x_2]^\top$,

$$min_{\mathbf{x}\in\mathbb{R}^2} g(\mathbf{x}) \left(= -\frac{1}{2}(x_1-1)^2 + \frac{1}{2}(x_2-2)^2 + 3\right) \text{ subject to } x_1^2 + x_2^2 \leq 1 \quad \textcircled{P}$$

(a) (2 points) Is g a convex function of $\mathbf{x}$? Give reasons

Solution: No. The Hessian is indefinite.

(b) (2 points) Let λ be the dual variable. For what values of λ, if any, the Lagrangian is convex in $\mathbf{x}$? Justify your answer?

Solution: The Lagrangian is

$$L(\mathbf{x}, \lambda) = -\frac{1}{2}(x_1-1)^2 + \frac{1}{2}(x_2-2)^2 - \frac{3}{2} + \lambda(x_1^2 + x_2^2 - 1)$$

The Hessian w.r.t $\mathbf{x}$ is diagonal with eigenvalues of $-1+2\lambda$ and $1+2\lambda$. For the function to be convex all eigenvalues need to be positive which is only possible if $\lambda \geq \frac{1}{2}$.

(c) (3 points) Compute the dual function, $h(\lambda)$ of $\textcircled{P}$ in terms of λ. State the domain of the function.

Solution: In the dual feasible region $L(\mathbf{x}, \lambda)$ is convex in $\mathbf{x}$. Minima of such function w.r.t. $\mathbf{x}$ is attained whenever $\nabla_{\mathbf{x}} L(\mathbf{x}, \lambda) = 0$. The equation is satisfied when $x_1 = -\frac{1}{-1+2\lambda}, x_2 = \frac{2}{1+2\lambda}$.

$$h(\lambda) = min_{\mathbf{x}\in\mathbb{R}^2} L(\mathbf{x}, \lambda) = \begin{cases} -\frac{1}{2}\left(\frac{1}{-1+2\lambda} + \frac{4}{2\lambda+1}\right) - \lambda & \lambda > \frac{1}{2} \\ -\infty & \text{otherwise} \end{cases}$$

(d) (3 points) State the Dual problem. Is the Dual problem convex or concave? Give reasons

Solution: The Dual problem is

$$max_{\lambda > \frac{1}{2}} - \frac{1}{2}(\frac{4}{1+2\lambda} + \frac{1}{2\lambda - 1}) - \lambda$$

This is a problem in one variable and the second derivative is negative everywhere and hence the problem is concave.

(e) (5 points) Let λ^* be the dual optimal point. Can you find a primal feasible point, $\mathbf{x}^*$, whose primal objective value matches the dual objective value evaluated at λ^*. $\mathbf{x}^*$ can be expressed in terms of λ^*. If such a point does not exist give reasons?

Solution: Since $\lambda^* > \frac{1}{2}$, it is an interior point and thus $h'(\lambda^*) = 0$ is sufficient for optimlity due to concavity of h. This leads to $1 = \frac{1}{(-1+2\lambda^*)^2} + \frac{4}{(2\lambda^*-1)^2}$. Thus if we choose $x_1^* = \frac{-1}{-1+2\lambda^*}, x_2^* = \frac{2}{2\lambda^*+1}$, the point $\mathbf{x}^*$ is feasible and by previous question it solves $min_{\mathbf{x}\in\mathbb{R}^2} L(\mathbf{x}, \lambda^*)$. By direct substitution

$$f(\mathbf{x}^*) = -\frac{1}{2}x_1^{*2} + x_1^* + \frac{1}{2}x_2^{*2} - 2x_2^* = (\lambda^* - \frac{1}{2})x_1^{*2} + x_1^* + (\lambda^* + \frac{1}{2})x_2^{*2} - 2x_2^* - \lambda^*(x_1^{*2} + x_2^{*2}) = L(\mathbf{x}^*, \lambda^*)$$

$$= -\frac{1}{2}(\frac{4}{1+2\lambda^*} + \frac{1}{2\lambda^* - 1}) - \lambda^* = h(\lambda^*)$$

`Note: This is an example of a non-convex problem where strong duality holds`

2. Consider the following problem

$$\min_{\mathbf{x}\in\mathbb{R}^d} f(\mathbf{x}) \left(= \frac{1}{2}\mathbf{x}^\top Q\mathbf{x} - h^\top \mathbf{x} \right)$$

$$\text{subject to } \mathbf{a}^{(i)}\mathbf{x} \geq b_i, i \in \{1, \ldots, n\}$$

Let $h \in \mathbb{R}^d, Q \in \mathbb{R}^{d\times d}$ be symmetric and positive definite. Consider solving this through Active set method. After k iterations we have $\mathbf{x}^{(k)}$ a feasible point and $W^{(k)} = \{1, 2, 3\}$ be the corresponding working set. Following information is available.

- Let $g_k = \nabla f(\mathbf{x}^{(k)})$ and $g_k = \sum_{i=1}^{3} \mu_i \mathbf{a}^{(i)}$ for scalars $\mu_i, i = 1, 2, 3$.
- Following the Active set algorithm, it was observed that $\{3\}$ needs to be removed from $W^{(k)}$ for making progress. $\mathbf{u} = r\mathbf{e}^1$ is the new direction obtained after removing the constraint where r is a scalar. The vector $\mathbf{e}^1$ is the the first column of $d \times d$ identity matrix.
- Some numerical values are given $Q_{11} = 2, |r| = 2, \mathbf{a}_1^{(3)} = -\frac{1}{2}$.

Based on the above can you answer the following

(a) Answer True or False

i. (1 point) $\{1, 2\}$ is the Active set at optimality. **F**

ii. (1 point) $\mathbf{u}$ is a feasible direction **T**

iii. (1 point) $\mathbf{x}^{(k)} + \mathbf{u}$ maynot be feasible to constraint 3. **F**

iv. (1 point) $\mathbf{x}^{(k)} + \mathbf{u}$ is a feasible point. **F**

v. (1 point) $\mathbf{u}$ is a descent direction **T**

(b) (5 points) Compute $\mathbf{u}^\top \mathbf{a}^{(i)}$ for $i = 1, 2, 3$? Justify.

Solution: Note that $\mathbf{u}$ solves the following problem

$$min_{\mathbf{y}\in\mathbb{R}^d} g_k^\top \mathbf{y} + \frac{1}{2}\mathbf{y}^\top Q\mathbf{y}. \quad \text{subject to } \mathbf{y}^\top \mathbf{a}^{(1)} = \mathbf{y}^\top \mathbf{a}^{(2)} = 0$$

By feasibility $\mathbf{u}^\top \mathbf{a}^{(1)} = \mathbf{u}^\top \mathbf{a}^{(2)} = 0$ At optimality $g_k + Q\mathbf{u} = \tilde{\mu}_1 \mathbf{a}^{(1)} + \tilde{\mu}_2 \mathbf{a}^{(2)}$

$$Q\mathbf{u} = (\tilde{\mu}_1 - \mu_1)\mathbf{a}^{(1)} + (\tilde{\mu}_2 - \mu_2)\mathbf{a}^{(2)} - \mu_3 \mathbf{a}^{(3)}$$

$$\mathbf{u}Q\mathbf{u} = (\tilde{\mu}_1 - \mu_1){\mathbf{a}^{(1)}}^\top \mathbf{u} + (\tilde{\mu}_2 - \mu_2)\mathbf{u}^\top \mathbf{a}^{(2)} - \mu_3 \mathbf{u}^\top \mathbf{a}^{(3)}$$

$$\mathbf{u}Q\mathbf{u} = -\mu_3 \mathbf{u}^\top \mathbf{a}^{(3)} = -\mu_3 r \mathbf{a}_1^{(3)} > 0$$

As per active set strategy, Constraint 3 was discarded and hence $\mu_3 < 0$. Since LHS is positive, $r\mathbf{a}_1^{(3)} > 0$, which requires that $r = -2$. Hence $\mathbf{u}^\top \mathbf{a}^{(3)} = r\mathbf{a}_1^{(3)} = 1$

(c) (5 points) Find $g_k^\top \mathbf{u}$ and $f(\mathbf{x}^{(k)} + \mathbf{u}) - f(\mathbf{x}^{(k)})$?

Solution: From the above exercise we find that

$$g_k^\top \mathbf{u} + \mathbf{u}^\top Q\mathbf{u} = 0$$

Hence $g_k^\top \mathbf{u} = -\mathbf{u}^\top Q\mathbf{u} = -r^2 Q_{11} = -8$.

$$f(\mathbf{x}^{(k)} + \mathbf{u}) - f(\mathbf{x}^{(k)}) = g_k^\top \mathbf{u} + \frac{1}{2}\mathbf{u}^\top Q\mathbf{u} = -4$$