

E1 222 Stochastic Models and Applications

Problem Sheet 3-1

1. Let

$$\begin{aligned} F(x, y) &= 0, \text{ if } x < 0, \text{ or } y < 0, \text{ or } x + y < 1 \\ &= 1, \text{ otherwise} \end{aligned}$$

Show that F satisfies the following: $F(-\infty, -\infty) = 0$; $F(\infty, \infty) = 1$; F is non-decreasing in each variable. Is $F(x, y)$ a distribution function? (Hint: If it were the joint distribution of two random variables, X, Y , what would be $P[1/3 < X \leq 1, 1/3 < Y \leq 1]$).

2. Let F_1 and F_2 be two one dimensional continuous distribution functions with f_1 and f_2 being the corresponding densities. Define a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ by

$$f(x, y) = f_1(x)f_2(y) [1 + \alpha(2F_1(x) - 1)(2F_2(y) - 1)]$$

where α is a real number. Show that $f(x, y)$ is a two dimensional density function for all $\alpha \in (-1, 1)$. Show that the two marginals of $f(x, y)$ are f_1 and f_2 . What does this imply about determining the joint density from the marginals? (Note that $\int_{-\infty}^{\infty} F_1(x)f_1(x) dx = \frac{1}{2}$).

3. Let F_{XY} be a joint distribution function with F_X and F_Y being the corresponding marginal distribution functions. Show that

$$1 - (1 - F_X(x) + 1 - F_Y(y)) \leq F_{XY}(x, y) \leq \min(F_X(x), F_Y(y)), \forall x, y$$

4. Consider n Bernoulli trials. Let X denote the number of successes and let Y denote the trial number of first success. Find $f_{Y|X}(y|1)$.

5. Let (X, Y) have joint density

$$f_{XY}(x, y) = \frac{1}{4}[1 + xy(x^2 - y^2)], \quad |x| \leq 1, |y| \leq 1.$$

Find the marginal and conditional densities.

6. Let $f(x, y) = e^{-x-y}$, $x > 0, y > 0$. Is this a density function? If yes, find the marginals.

7. Let X, Y be independent discrete random variables each being uniform over $\{0, 1, \dots, N\}$. Find $P[X > Y]$ and $P[X < Y]$.
8. Let X, Y be iid random variables which are uniform over $(0, 1)$. Find $P[X > Y]$.
9. Let X be uniform from 0 to 1, let Y be uniform from 0 to X and let Z be uniform from 0 to Y . What is the joint density of X, Y, Z ? Find the marginal densities and the joint density of Y, Z .