# E1 222 Stochastic Models and Applications

**Assignment 3**

Submission Deadline: 29 November 9 PM

You need to submit solutions to three problems

The specific three problems would be communicated to you through Teams at 6 PM on 29 November.

1. Let $X_1, X_2, \cdots$ be a sequence of discrete random variables with $X_n$ being geometric with parameter $\lambda/n$ where we have $0 < \lambda < 1$. Let $Z_n = X_n/n$. Does $Z_n$ converge in distribution?

2. Let $X_n, \; n = 1, 2, \cdots$ be discrete random variables taking values in $\{0, 1, 2, \cdots, K\}, \; K < \infty$. Suppose $X_n \xrightarrow{P} 0$. Then show that the sequence converges in $r^{th}$ mean to zero.

3. Let $X_1, X_2, \cdots$ be *iid* random variables which are all uniform over $(0, \; 1)$. Let $Z_n = (\prod_{i=1}^n X_i)^{\frac{1}{n}}$. Show that $Z_n \xrightarrow{P} c$ and find the constant $c$. (Hint: Use (i) the weak law of large numbers and (ii). the fact that if $X_n \xrightarrow{P} X$ then $g(X_n) \xrightarrow{P} g(X)$ for any continuous $g$).

4. Each of two switches is either ON or OFF during a day. On day $n$, each switch would independently be ON with proability $(1 + m_{n-1})/4$ where $m_{n-1}$ is the number of switches that are ON on day $n-1$. What is the fraction of days on which both switches are (i). ON, (ii). OFF.

5. Let $\{X_n, \; n \geq 0\}$ be a Markov Chain. Let $s_0, s_1, s_2$ be some specific three states. Suppose the probabilities of transition out of $s_0$ are given by: $P(s_0, s_0) = 0.3; P(s_0, s_1) = 0.2; P(s_0, s_2) = 0.5$. Suppose the chain is started in $s_0$. Let $T$ denote the first time instant when the chain left state $s_0$. (That is, $T = \min\{n : n \geq 1, \; X_n \neq s_0\}$). Find the distribution of $T$ and $X_T$.

6. Consider the following situation. There is a box with a number of particles. At each time instant, $n$, we introduce $\xi_n$ new particles into the box. We assume that $\xi_n, n = 1, 2, \cdots$ are iid having Poisson distribution with parameter $\lambda$. Each particle in the box at time instant $n$ will, independently of all other particles and independently of all $\xi_n$, leave the box by time instant $n+1$ with probability $p, 0 < p < 1$. With probability $(1-p)$, the particle will stay in the box. Let $X_n$ denote

the number of particles in the box at time $n$ with $X_0$ being the initial number of particles in the box. Show that $\{X_n, n \geq 0\}$ is a Markov chain and calculate its transition probabilities.

(This model is useful in some applications. For example, the box could be a cell in a cellular network and the particles could be calls in progress. We count time by discrete intervals and in each time interval some random number of new calls are added and some random number of current calls would end).

7. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate $\lambda$ and assume that it is independent of a non-negative random variable, $T$. Suppose the mean of $T$ is $\mu$ and its variance is $\sigma^2$. Find (i). $E[N(T)]$, (ii). $\mathrm{Var}(N(T))$

8. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate $\lambda$. Let $X_0$ be a discrete random variable that is independent of $N(t)$ and with mass function $P[X_0 = +1] = P[X_0 = -1] = 0.5$. Define a stochastic process: $X(t) = X_0 \ (-1)^{N(t)}$. Find the mean and autocorrelation function of $X(t)$. Is this process wide-sense stationary?