E1 222 Stochastic Models and Applications
Problem Sheet 5.2

1. Consider a Markov chain on nonnegative integers having transition probabilities, $P(x, x+1) = p$ and $P(x, 0) = 1 - p$ where $0 < p < 1$. Show that the chain has unique stationary distribution and find that distribution.

2. A transition probability matrix is called doubly stochastic if both the rows as well as columns sum to one. Consider a finite irreducible Markov chain whose transition probability matrix is doubly stochastic. Show that the chain has a unique stationary distribution given by $\pi(y) = \frac{1}{n}$, $\forall y$, where n is the number of states.

3. On a road, three out of every four trucks are followed by a car while only one out of every five cars is followed by a truck. Find the ratio of trucks to cars on the road.

4. A professor keeps giving a sequence of exams to the class. The exams are of three types. Let q_i denote the probability that the class does well on exam of type i. It is known that $q_1 = 0.4$, $q_2 = 0.6$, $q_3 = 0.8$. If the class does well in the current exam, the next exam is equally likely to be any of the three types. If the class does badly on the current exam, then the next exam would always be of type 3. What proportion of exams are of type i, $i = 1, 2, 3$?

5. Each of two switches is either ON or OFF during a day. On day n each switch would independently be ON with proability $(1 + m_{n-1})/4$ where m_{n-1} is the number of switches that are ON on day $n-1$. What is the fraction of days on which both switches are (i). ON, (ii). OFF.

6. Suppose that whether or not it rains today depends on the whether or not it rained for the previous three days. Explain how we can set up a Markov chain model for this. Suppose that if it rained on each of the previous three days then it will rain today with probability 0.6; if it did not rain on any of the previous three days then it will rain today with probability 0.2; in all other cases the weather today would be same as that of yesterday with probability 0.5. Now find the transition probability matrix for the chain.

7. Let Y_n be the sum of numbers obtained on n independent rolls of a fair die, $n = 1, 2, \cdots$. Find

$$\lim_{n\to\infty} P[Y_n \text{ is divisible by } 3]$$

(Hint: Think of a 3-state Markov chain where the state at n could be the reminder obtained when Y_n is divided by 3).