E1 222 Stochastic Models and Applications
Problem Sheet 4

1. Given $P[X_n = 0] = 1 - 2^{-n+1}$, $P[X_n = 2^n] = 2^{-n}$, $P[X_n = -2^n] = 2^{-n}$. Examine the convergence of X_n.

2. Let $X_1, X_2, \cdots$ be iid Gaussian random variables with mean zero and variance unity. Let $\bar{X}_n = (X_1 + \cdots + X_n)/n$. Let F_n be the distribution function of $\bar{X}_n$. Find Lim F_n. Is this a distribution function?

3. Let $X_1, X_2, \cdots$ be a sequence of discrete random variables with X_n being uniform over the set $\{\frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n}\}$. Does the sequence $\{X_n\}$ converge in distribution?

4. Let $\{X_n\}$ be a sequence of random variables converging in distribution to a continuous random variable X. Let a_n be a sequence of positive numbers such that $a_n \to \infty$ as $n \to \infty$. Show that X_n/a_n converges to zero in probability.

5. A runner attempts to pace off 100 meters for an informal race. His paces are independently distributed with mean $\mu = 0.97$ meters and standard deviation $\sigma = 0.1$ meter. Find the probability that his hundred paces will differ from hundred meters by no more than 5 meters.

6. Find the characteristic function of X when X has (a). Poisson distribution, (b). Geometric distribution.

7. Let X be a continuous random variable having density $f_X(x) = \frac{1}{2}e^{-|x|}$, $-\infty < x < \infty$. Show that $\phi_X(t) = 1/(1+t^2)$. Use this to show that

$$e^{-|x|} = \int_{-\infty}^{\infty} e^{-itx} \frac{1}{\pi(1+t^2)} \, dt$$

(Recall that for a continuous random variable, the characteristic function is the Fourier transform of its density function).