E1 222 Stochastic Models and Applications
Test I

`Time: 75 minutes` `Max. Marks:40`
`Date: 20 Sept 2021`

Answer **ALL** questions. All questions carry equal marks

1. a. Two players A and B are playing a game that consists of a series of points. Each point is independently won by A with probability p and by B with probability $1-p$. When one player has won two points more than the other player, the game ends and the winner is the player with more points. Calculate the probability that the game would end after m points are played. Also, calculate the probability that player A wins the game.

   b. Take $\Omega = \{(x,\ y)\ :\ 0 \leq x, y \leq 1\}$ with the usual probability assignment where the probability of an event is proportional to its area. Every time this random experiment is performed, the outcome is a point in $\Omega \subset \Re^2$. Suppose this random experiment is repeated till the point obtained is such that its distance from the origin is less than 0.5. Let X denote the number of repetitions needed. Find the expected value of X.

2. a. Let X be a continuous random variable with pdf

$$f_X(x) = \frac{K}{x^4},\ x \geq 3$$

   Find value of K, $E[X]$, $\mathrm{Var}(X)$ and $P[X \leq 9]$.

   b. Let X be a non-negative integer valued random variable. Let $\Phi_X(t) = Et^X$ be its probability generating function and assume that $\Phi_X(t)$ is finite for all t. Show that for any positive integer, y,

$$P[X \leq y] \leq \frac{\Phi_X(t)}{t^y},\ 0 \leq t \leq 1$$

3. a. A total of m keys are to be put in n boxes with each key independently being put in box-i with probability p_i, $i = 1, \cdots, n$. Every time a key is put in a non empty box, we say a collison has occurred in that box. Find the expected number of collisons in box-1

b. Suppose X is a continuous random variable with pdf

$$f_X(x) = \frac{\lambda e^{-\lambda x}}{(1+e^{-\lambda x})^2}, \quad -\infty < x < \infty$$

where $\lambda > 0$. Define $Y = \frac{1}{1+e^{-\lambda x}}$. Find the pdf of Y.

4. a. Let X, Y be continous random variables with joint density, $f_{XY}(x,y) = 1$ if (x,y) is inside the triangle with vertices at $(-1,0)$, $(0,1)$ and $(1,0)$. (Otherwise f_{XY} is zero). Find the marginal densities f_X and f_Y and $P[Y > 0.6 \mid X = -0.1]$.

   b. Let X, Y be iid random variables having geometric distribution with parameter p. (That is, X, Y are independent and both have the mass function $f(i) = (1-p)^{i-1}p, \; i = 1, 2, \cdots$). Let $Z = X+Y$. Find the conditional mass function $f_{X|Z}$. (Hint: You need not explicitly calculate the mass function of Z).