**E1 222 Stochastic Models and Applications**
**Final Examination**

`Time: 3 hours (9 AM to 12 noon)` `Max. Marks:50`
`Date: 2 December 2021`

Answer any **FIVE** questions. All questions carry equal marks

1. a. Consider a gambling game like this. The player rolls three fair dice. If all show the same number the player wins. Otherwise if exactly two dice have same number, the player rolls the third one again; if all three dice have different numbers in the first roll, the player rolls all of them again. If the player gets same number on all three dice after the second round, then also the player wins. Otherwise the player looses. Calculate the probability of winning.

   b. Let $X$ be a continuous random variable with density function given by
$$f_X(x) = Kx(1 - x^2), \;\; 0 \le x \le 1$$
   Find the value of $K$, $Pr[X > 0.5]$ and $E[X]$. Let $Y = 3X + 5$. Find the pdf of $Y$ and $EY$.

2. a. Let $X, Y$ be continuous random variables with joint density
$$f_{XY}(x, y) = \frac{1}{8}\;(y^2 - x^2)e^{-y}, \;\; -y \le x \le y, \; 0 < y < \infty$$
   Calculate the marginal density of $Y$, $Pr[X > 0 \mid Y = y]$ and $E[X \mid Y]$. Are $X, Y$ independent?

   b. Let $X, Y$ be random variables that satisfy
$$E[Y \mid X] = a + bX$$
   for some constants, $a, b$. Show that $b = \text{Cov}(X, Y)/\text{Var}(X)$.

3. a. Let $X_1, X_2, \cdots, X_n$ be independent exponential random variables with $EX_i = 1/\lambda_i$, $i = 1, 2, \cdots, n$. Let $J$ be random index defined by $X_J = \min(X_1, X_2, \cdots, X_n)$. Find distribution of $J$.

b. A doctor's clinic opens at 10:00 AM. The doctor has given two appointments, one at 10:00 AM and another at 10:30 AM. The consultation time of any patient is exponentially distributed with mean of 20 minutes. Consultation times of different patients are independent. Assuming that both patients came at their appointed time, calculate the expected total time spent at the clinic by the second patient (that is, the patient with appointment at 10:30 AM).

4. a. Let $X, Y$ be iid Gaussian random variables with mean zero and variance 1. Show that

$$Pr[|X| + |Y| \leq x] = (Pr[|X + Y| \leq x])^2, \quad \forall x \in \Re$$

(Hint: You may want to sketch the regions: $\{(x, y) \ : \ |x|+|y| \leq a\}$ and $\{(x, y) \ : \ |x + y| \leq a\}$).

b. Let $X, Y$ be discrete random variables with joint pmf given by

$$Pr[X = 2, Y = 1] = Pr[X = 3.Y = 1] = Pr[X = A, Y = A] = \frac{1}{3}$$

where $A$ is some real number. Find all values of $A$ for which (i). $X, Y$ would be uncorrelated, (ii). $X, Y$ would be independent

5. a. Let $X_1, X_2, \cdots, X_{n_1}, X_{n_1+1}, \cdots, X_{n_1+n_2}$ be *iid* continuous random variables with density $f$ and distribution function $F$. Define

$$U = \min(X_1, \cdots, X_{n_1}); \qquad V = \max(X_{n_1+1}, \cdots, X_{n_1+n_2})$$

Calculate $Pr[U > V]$ and hence show that this probability does not depend on $f$, that it depends only on $n_1, n_2$ and that it can be expressed using the Gamma function.

b. Let $X$ be a continuous random variable having Cauchy density given by

$$f(x) = \frac{1}{\pi} \frac{1}{1 + x^2}, \quad -\infty < x < \infty$$

Let $Y = \frac{K}{X}$ where $K$ is a constant. Find density of $Y$. Does $Y$ also have Cauchy density?

6. a. Let $X_1, X_2, \cdots$ be a sequence of *iid* random variables with mean $\mu$ and variance $\sigma^2$. Define

$$Y_n = \frac{2}{n(n+1)} \sum_{i=1}^{n} iX_i, \ n = 1, 2, \cdots$$

Show that $Y_n \xrightarrow{P} \mu$.

b. Suppose that the time in hours between successive trains at a station is uniformly distributed over $(0,\ 1)$. Passengers arrive as a Poisson process with rate of 10 per hour. When a train comes, all passengers currently waiting would get on that train. Assume that a train has left just now. Let $X$ denote the number of passengers that would get on the next train. Find $E[X]$ and $\mathrm{Var}(X)$.

7. a. Let $\{X_n,\ n = 0, 1, \cdots\}$ be an irreducible, positive recurrent aperiodic Markov chain whose stationary probabilities are given by $\pi$. Define another process $\{Y_n,\ n \geq 1\}$ by $Y_n = (X_{n-1}, X_n)$. (That is, $Y_n$ keeps track of the last two states of the original chain). Is $Y_n$ a Markov Chain? If so, find its transition probabilities and $\lim_{n\to\infty} Pr[Y_n = (i, j)]$.

b. Suppose $\{X_n,\ n = 0, 1, \cdots\}$ is an irreducible positive recurrent birth-death Markov chain with stationary distribution $\pi$. Let $P$ denote its transition probability function. Suppose the chain is started in its stationary distribution. (That is, the distribution of $X_0$ is $\pi$). Show that

$$Pr[X_0 = y|X_1 = x] = P(x, y)$$

8. a. Let $\{B(t),\ t \geq 0\}$ be a standard Brownian motion process. Consider a process defined by

$$V(t) = e^{-\alpha t/2}\, B\left(e^{\alpha t}\right)$$

where $\alpha > 0$ is a parameter. Show that the process $\{V(t),\ t \geq 0\}$ is a stationary process.

b. Let $\{X(t),\ t \geq 0\}$ be a Brownian motion process with drift $\mu$ and variance parameter $\sigma$. (That is, $E[X(t)] = \mu,\ \ \mathrm{Var}(X(t)) = \sigma^2 t,\ \forall t$). Find the distribution of $X(t_1) + X(t_2) + X(t_3)$, with $t_1 < t_2 < t_3$.