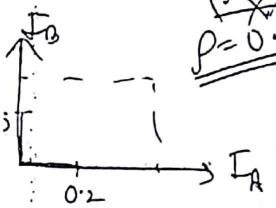
E1 222 Stochastic Models and Applications
Final Examination

Date: *10 December 2014* Time: *2-5 PM*

Answer any FIVE questions

1. a. Let A and B be events with $P(A) = 0.2$, $P(B) = 0.5$ and $P(A \cup B) = 0.6$. Let I_A and I_B be the indicator random variables of events A and B. Find the correlation coefficient of I_A and I_B.

$\rho = 0.1$



b. There are three chests each having two drawers. Chest 1 has a gold coin in each of the drawers while chest 2 has a silver coin in each of the drawers. Chest 3 has a gold coin in one drawer and a silver coin in the other drawer. A chest is selected at random and one of the drawers opened. It is found to contain a gold coin. Find the propbability that the other drawer has (i). a silver coin, $\frac{2}{3}$ (ii). a gold coin. $\frac{1}{3}$

2. a. Let X, Y be nonnegative integer-valued random variables with the joint mass function given by

$$P[X = i, Y = j] = e^{-(a+bi)} \frac{b^j\, a^i\, i^j}{i!\, j!}, \quad i \geq 0, j \geq 0$$

Find the conditional distribution of Y given X and covariance of X and Y.

b. Let X and Y be independent random variables with means μ_x and μ_y and variances σ_x^2 and σ_y^2. Show that

$$\text{Var}(XY) = \sigma_x^2\sigma_y^2 + \mu_y^2\sigma_x^2 + \mu_x^2\sigma_y^2$$

3. a. Let X be a continuous random variable with density

$$f_X(x) = Kx^2(1-x), \; 0 \leq x \leq 1.$$

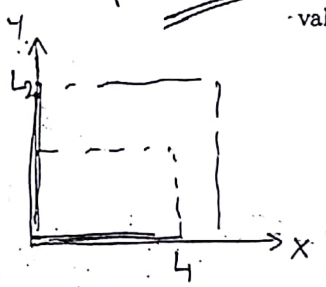
Find value of K, EX, and $P[X > 0.5]$.

$K = 12$

$EX = 3/5$

$P = 11/16$

b. Two numbers are drawn independently with the first one from a uniform distribution over $[0, L_1]$ and the second one with uniform distribution over $[0, L_2]$. Assume $L_1 < L_2$. Find the expected value of the smaller of the two numbers.



$\frac{L_1^2}{6L_2}(2L_2 - L_1)$. ??

4. a. Let X, Y be *iid* Gaussian random variables with mean zero and variance unity. Show that

$$P[\,|X| + |Y| \leq c\,] = (\,P[\,|X+Y| \leq c\,]\,)^2, \ \forall c.$$

(Hint: Drawing a diagram would help).

b. A coin, whose probability of heads is p, is tossed repeatedly till we get r heads. Find the expected ?? number of tosses needed.

D mean & variance ??

5. a. Let X, Y be jointly gaussian with means zero and variances σ_x^2 and σ_y^2. Let ρ be the correlation coefficient of X and Y. Calculate $E[X|Y]$ and show that it is a linear function of Y.

$f_{XY}(x,y) = ?$

b. Define convergence of a sequence of random variables (i) in r^{th} mean and (ii) with probability one. Give example of a sequence of random variables that converges almost surely but does not converge in r^{th} mean.

6. a. Consider a Markov Chain with $S = \{0, 1, 2\}$ and the transition probability matrix given by

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1-p & 0 & p \\ 0 & 1 & 0 \end{bmatrix}$$

where $0 < p < 1$. Is the chain periodic? Suppose g is a real valued function defined on S with $g(0) = -1$, $g(1) = 0$ and $g(2) = +1$. Suppose the Markov chain has initial probability distribution given by $\pi_0(0) = (1-p)/2$, $\pi_0(1) = 1/2$, and $\pi_0(2) = p/2$. Find expected value of $g(X_n)$ as $n \to \infty$, if it exists. Will this be the same if the initial probability distribution is changed?

b. What is a null recurrent state in a Markov chain? What would be the probability of a null recurrent state in a stationary distribution of a Markov chain?

7. a. Let $\{N(t),\ t \geq 0\}$ be a Poisson process with rate λ. Let X_0 be a random variable that is independent of this process and let $P[X_0 = 1] = P[X_0 = -1] = 0.5$. Define a process $X(t)$ by $X(t) = (-1)^{N(t)} X_0$. Find the mean and autocorrelation functions of this process.

NOT WSS

b. Consider a random process defined by $V(t) = (Y+1)\cos t + X \sin t$ where X and Y are independent random variables with $EX = EY = 0$ and $EX^2 = EY^2 = 1$. Find the mean and autocorrelation of $V(t)$. Is $V(t)$ wide-sense stationary?

Some standard Distributions

- Binomial: $P[X = x] = {}^nC_x\, p^x\,(1-p)^{n-x}$, $0 \le p \le 1$, $x = 0, 1, \cdots, n$.
  Parameters: n, p. Mean $= np$ and Variance $= np(1-p)$.
  Moment generating function, $M_X(s) = (pe^s + 1 - p)^n$.

- Poisson: $P[X = x] = e^{-\lambda}\,\frac{\lambda^x}{x!}$, $\lambda > 0$, $x = 0, 1, \cdots$
  Parameter: λ. Mean $= \lambda$ and Variance $= \lambda$.
  Moment generating function, $M_X(s) = \exp[\lambda(e^s - 1)]$.

- Geometric: $P[X = x] = p(1-p)^{x-1}$, $0 < p < 1$, $x = 1, 2, \cdots$.
  Parameter p. Mean $= \frac{1}{p}$ and Variance $= \frac{1-p}{p^2}$.
  Moment generating function, $M_X(s) = \frac{pe^s}{1-(1-p)e^s}$.

- Uniform: $f_X(x) = \frac{1}{b-a}$, $a \le x \le b$.
  Parameters: a, b. Mean $= \frac{a+b}{2}$ and Variance $= \frac{(b-a)^2}{12}$.
  Moment Generating function, $M_X(s) = \frac{e^{sb} - e^{sa}}{s(b-a)}$.

- exponential: $f_X(x) = \lambda\, e^{-\lambda x}$, $x \ge 0$, $\lambda > 0$.
  Parameter: λ. Mean $= \frac{1}{\lambda}$ and Variance $= \frac{1}{\lambda^2}$.
  Moment generating function, $M_X(s) = \frac{\lambda}{\lambda - s}$.

- Normal: $f_X(x) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{-(x-\mu)^2/2\sigma^2}$, $\sigma > 0$, $-\infty < x < \infty$
  Parameters: μ and σ. Mean $= \mu$ and Variance $= \sigma^2$.
  Moment generating function, $M_X(s) = \exp[\mu s + \frac{1}{2}\sigma^2 s^2]$.

- gamma: $f_X(x) = \lambda e^{-\lambda x}(\lambda x)^{n-1}/\Gamma(n)$, $x \ge 0$
  Parameters: n, $\lambda > 0$. Mean $= \frac{n}{\lambda}$ and Variance $= \frac{n}{\lambda^2}$.
  Moment generating function, $M_X(s) = (\frac{\lambda}{\lambda - s})^n$.

$Cov(U,V)$

$Cov(X+Y, X^2+Y^2)$

$, Cov(X, X^2)$ $+ Cov(\ldots)$ $+ Cov(Y \ldots)$

E1 222 Stochastic Models and Applications
Final Examination

Date: *6 December 2013* Time: *2-5 PM*

Answer any FIVE questions

1. a. Let A and B be independent events with $P(A) = 0.2$ and $P(B) = 0.5$. Find $P(A \cap B)$, $P(A \cup B)$, $P(A|A \cup B)$ and $P(A|A \cap B)$.

   b. A fair dice is rolled repeatedly. Find the probability that (in the sequence of outcomes) a 2 would appear before a 5 or a 6. $\frac{1}{3}$ $\frac{P(E)}{P(E)+P(F)}$

2. a. Consider a continuous random variable with density function

$$f_X(x) = K(x+1)\, e^{-\frac{x}{\theta}}, \quad x > 0$$

where $\theta > 0$ is a parameter. Find the value of K and EX. $\theta(1+\theta)$ $\theta > 0$

   b. Let X, Y, Z be binary random variables (taking values 0 or 1). Their joint mass function is given by

$$f_{XYZ}(1,0,0) = f_{XYZ}(0,1,0) = f_{XYZ}(0,0,1) = f_{XYZ}(1,1,1) = 0.25.$$

Find the marginal mass functions of X, Y, Z. Are X, Y, Z independent? Are they pairwise independent? yes No

$V = X^2 + Y$ $[Z|Y=3]$ $E[Z \ldots$

$P((A \cup B) \cap \theta)$

3. a. Let X, Y be *iid* Gaussian variables with mean zero and variance unity. Let $U = X + Y$ and $V = X^2 + Y^2$. Find the moment generating functions of U and V. Find the correlation coefficient of U and V. $\boxed{\text{Are } U \text{ and } V \text{ independent?}}$ NO → check $= 0$

$V = U^2 - 2XY$

SOLVE

   b. A fair dice is rolled repeatedly till each of the six outcomes occurs at least once. Let Z denote the number of rolls needed. Let Y denote the number of distinct outcomes in the first six rolls of the dice. Find $E[Z|Y=3]$. 17 17

$X = -Y$ $V = X^2 + Y^2$

4. a. Let X, Y be *iid* Gaussian random variables with mean zero and variance unity. Show that

$$P[\,|X| + |Y| \le c\,] = (\,P[\,|X+Y| \le c\,]\,)^2, \ \forall c$$

(Hint: Drawing a diagram would help).

$f_{UV}(u,v)$

$f_U(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{4}}$

$f_V(v) = \frac{1}{2} e^{-v/2}$

$X = U - Y$

$U^2 + Y^2 - 2UY + Y^2 = V^2$

$(U - Y)^2 + Y^2 = V^2$ $(X+Y)^2 - 2XY$

$\frac{X^2+Y^2}{2} \ge \frac{X+Y}{2}$ $f_{UV} =$

$V > U$

$U = 0$
$X = -Y$

$\frac{1}{r/p}$

V. Imp.

~~6.~~ A coin, whose probability of heads is p, is tossed repeatedly till we get r heads. Find the expected number of tosses needed.

5. a. Let $X_1, \cdots, X_n$ be *iid* geometric random variables with parameter p. Let $N_n = \min(X_1, \cdots, X_n)$. Find the mass function of N_n and hence show that it is also a geometric random variable. What happens to the distribution of N_n as $n \to \infty$?

   b. Consider a random process $V(t) = Y \cos t + X \sin t$, where X and Y are independent random variables with $EX = EY = 0$ and $EX^2 = EY^2 = 1$. Find the mean and autocorrelation of $V(t)$.

6. a. Let $X_1, X_2, \cdots$ be a sequence of discrete random variables with X_n being geometric with parameter λ/n where we have $0 < \lambda < 1$. Let $Z_n = X_n/n$. Does Z_n converge in distribution?

   b. Give example of a sequence of random variables that converges almost surely but does not converge in r^{th} mean.

V.V.V.V. IMP

7. a. Consider a Markov Chain with state space $S = \{0, 1, 2, 3\}$ and the transition probability matrix given by

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Is the chain periodic? Suppose g is a real valued function defined on S with $g(0) = -2$, $g(1) = -1$, $g(2) = 1$ and $g(3) = 2$. Suppose the Markov chain has initial probability distribution given by $\pi_0(0) = 1/6$, $\pi_0(1) = 1/3$, $\pi_0(2) = 1/3$ and $\pi_0(3) = 1/6$. Find expected value of $g(X_n)$ as $n \to \infty$, if it exists. Will this be the same if the initial probability distribution is changed?

$\lim_{n\to\infty} \frac{\sum P^n(x,y)}{n}$

$\sum \pi_0(x) P^n$

   b. What is a null recurrent state in a Markov chain? Show that a finite Markov chain can not have any null recurrent states.

8. a. Let $\{N(t),\ t \geq 0\}$ be a Poisson process with rate λ. Let X_0 be a random variable that is independent of this process and let $P[X_0 = 1] = P[X_0 = -1] = 0.5$. Define a process $X(t)$ by $X(t) = (-1)^{N(t)} X_0$. Find the mean and autocorrelation functions of this process.

$\frac{G_n(x,y)}{n} = \frac{\sum P^n(x,y)}{n}$

Confirm

Lemma:- A positive recurrent Markov chain converges to $\frac{\pi}{\dots}$ via π_j if and only if the chain is aperiodic.

$\frac{1}{n} \sum_{y=1}^{n} \sum_{m=1}^{n} P^m(x,y)$

2

3 → a, b

4 → a, b

5 (a) ~~Confirm~~

6 ?

$E[Z \mid Y=y)$

Suppose the arrival of customers at a bank is a Poisson process with rate 10 per hour. Independent of everything else, an arriving customer would be male with probability 0.6 and female with probability 0.4. (i). Given that there were 2 male customers in the first hour, what is the expected number of female customers during the same time interval? (ii). Given that there were 6 customers during the first half-hour, what is the probability that all of them came during the first twenty minutes?

$2/6$

-4

$\cancel{P\,E\,(\,)}\; P(N(t+20)) - P(t+60))$

$= 0$

$\cancel{P[N(t)>40]=0}$

$P[N((t+20)) - N(t+60)) = 0]$

$P[N[\frac{40}{60}] = 0]$

$= \qquad = e^{-\frac{400}{60}} \cdot \frac{(10\lambda t)^0}{0!}$

$= e^{-2/3}$

E1 222 Stochastic Models and Applications
Final Examination

Date: 12 December 2011
Time: 2-5 PM

Answer any FIVE questions

1. a. Let A and B be two events that are independent and let $P(A) = 0.3$ and $P(B) = 0.6$. Find $P(A \cap B), P(A \cup B), P(A|A \cup B)$ and $P(A|A \cap B)$.

   b. Three shops A, B, C have, respectively, 70, 100 and 150 employees. Of these, 30, 40 and 50 percent are women, respectively. An employee is selected at random and is found to be a women. What is the probability that she is from shop C.

2. a. Let random variables X, Y, Z have joint density function given by

$$f_{XYZ}(x, y, z) = K, \quad 0 < x < y < z < 1.$$

   Find the value of K and $E[Y|X]$. Are X, Y, Z independent?

   b. Let $\Omega = [0, 1]$ with the usual probability assignment (where probability of an interval is length of the interval). Let $X = I_{[0, a]}$ and $Y = I_{[0, b]}$ with $0 < a < b < 1$. (Here, I_A is the indicator function of set or event A). Let $W = X + Y$ and $Z = XY$. Find the correlation coefficient of Z and W.

3. a. Let X_1, X_2, X_3 be *iid* continuous random variables having exponential distribution with $\lambda = 1$. Let $Y_1 = X_1 + X_2 + X_3$, $Y_2 = \frac{X_1+X_2}{X_1+X_2+X_3}$ and $Y_3 = \frac{X_1}{X_1+X_2}$. Find the joint density of Y_1, Y_2, Y_3. Are Y_1, Y_2, Y_3 independent?

   b. A coin, whose probability of heads is p, is tossed repeatedly till we get r heads. Let Y denote the number of tosses needed. Find EY.

4. a. Let Z_i, $i = 1, 2, \cdots, n$ be *iid* Gaussian random variables with mean zero and variance one. Let $Y = \sum_{i=1}^{n} Z_i^2$. Show that the moment generating function of Y is $M_Y(t) = (1 - 2t)^{-n/2}$.

b. A fair dice is rolled repeatedly till each of the six outcomes occur at least once. Let Z denote the number of rolls needed. Let Y denote the number of distinct outcomés in the first six rolls of the dice. Find $E[Z|Y=3]$.

5. a. Let $X_1, X_2, \cdots$ be a sequence of *iid* random variables uniform over $[0, \theta]$. Let $Y_n = \max(X_1, X_2, \cdots, X_n)$. Show that the sequence Y_n converges in probability to θ.

b. Give an example each to show that $X_n \to X$ *almost surely* does not imply that $X_n \to X$ in the *rth mean* and vice versa.

6. a. Consider a Markov Chain with state space $S = \{0, 1, 2, 3\}$ and the transition probability matrix given by

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Is the chain periodic? Suppose g is a real valued function defined on S with $g(0) = -2$, $g(1) = -1$, $g(2) = 1$ and $g(3) = 2$. Suppose the Markov chain has initial probability distribution given by $\pi_0(0) = 1/6$, $\pi_0(1) = 1/3$, $\pi_0(2) = 1/3$ and $\pi_0(3) = 1/6$. Find expected value of $g(X_n)$ as $n \to \infty$, if it exists. Will this be the same if the initial probability distribution is changed?

b. Let π be a stationary distribution of a Markov Chain. (i). Show that if $\pi(x) > 0$ and x leads to y then $\pi(y) > 0$. (ii). Suppose the chain has transition probabilities that satisfy the following: for some two states y and z, $P(x, y) = cP(x, z)$; $\forall x$, where c is a constant. Show that $\pi(y) = c\pi(z)$.

7. a. Let $\{N(t),\ t \geq 0\}$ be a Poisson process with rate λ. Let X_0 be a random variable that is independent of this process and let $P[X_0 = 1] = P[X_0 = -1] = 0.5$. Define a process $X(t)$ by $X(t) = (-1)^{N(t)} X_0$. Find the mean and autocorrelation functions of this process.

b. Let $\{N(t),\ t \geq 0\}$ be a Poisson process with rate λ. Let S_n denote the time of the n^{th} event. Find (i). $E[S_4|N(1) = 2]$, and (ii). $E[N(4) - N(2)|N(1) = 3]$.

$1 + \frac{2}{\lambda}$

$E[N(2)] = 2\lambda$

2

$a_n = \frac{x}{n^2}$

$c_n = e^n$

$X(t) = (-1)^{N(t)} X_0$

$E[(-1)^{N(t)} X_0]$

E1 222 Stochastic Models and Applications
Final Examination

Date: 11 December 2009
Time: 2-5 PM

Answer any FIVE questions

1. a. Let A and B be two events that are independent and let $P(A) = \frac{1}{3}$ and $P(B) = \frac{3}{4}$. Find $P(A \cap B), P(A \cup B), P(A|A \cup B)$ and $P(A|A \cap B)$. =1

   b. Three shops A, B, C have, respectively, 70, 100 and 150 employees. Of these, 30, 40 and 50 percent are women, respectively. An employee is selected at random and is found to be a women. What is the probability that she is from shop C. 5/12....

2. a. Let X be a continuous random variable with density function given by

$$f_X(x) = C(1 - x^3), \quad 0 < x < 1.$$

   Find value of C, the distribution function F_X and EX. Let $Y = 2X^2 + 3X + 4$. Find EY.

   b. A random experiment results in one of r possible outcomes. The probability that the i^{th} outcome occurs is p_i, $i = 1, 2, \cdots, r$. (Hence, $p_1 + p_2 + \cdots + p_r = 1$). Consider N independent repetitions of this experiment. Let X_i denote the number of times the i^{th} outcome occurs in the N repetitions of the experiment; $i = 1, 2, \cdots, r$. (Hence $X_1 + X_2 + \cdots + X_r = N$). Let k be an integer such that $k < r$. Let $Y = X_1 + X_2 + \cdots + X_k$. Find the probability mass function of Y. Binomial with $(n, p_1+p_2+\cdots+p_k)$

3. a. Let X, Y be iid geometric random variables. Let $Z = \min(X, Y)$ and $W = X - Y$. Show that Z and W are independent.

   b. ~~A point is selected at random from inside the circle with center~~ at origin and radius 1 (in $\Re^2$). Let Y denote the distance of this point from the origin. Find $P[Y \leq a]$ where a is a real number with $0 < a < 1$.

$\frac{\pi a^2}{\pi 1^2} = a^2$

$w = 0 \quad x > y$

< 0

$w = x - y$

$z = y$

$|J| = 1$

$f_{W,Z}(w, z) = |J|^{-1} f_{X,Y}(x, y) = (1-p)^{w-z-1} p (1-p)^{z-1} p$

4. a. Let $X_1, X_2, \cdots, X_n$ be *iid* random variables each of them being uniform over $[0,\ 1]$. Let $Y_1 = X_1$, $Y_2 = X_1X_2$, $Y_3 = X_1X_2X_3$, and so on with $Y_n = X_1X_2\cdots, X_n$. Show that Y_1 is uniform over $[0,\ 1]$, Y_2 is uniform over $[0,\ Y_1]$, Y_3 is uniform over $[0,\ Y_2]$ and so on with Y_n uniform over $[0,\ Y_{n-1}]$.

b. A coin, whose probability of heads is p, is tossed repeatedly till we get r heads. Let Y denote the number of tosses needed. Find EY.

$r, r+1, \cdots, n$

5. a. Suppose we have two decks of n cards each. Cards of each deck are numbered $1, 2, \ldots, n$. The two decks are separately shuffled and then the corresponding cards in each deck are compared one by one. We say a match has occurred at position i if the i^{th} card in each deck has the same number. Let S_n denote the total number of matches. Find the mean of S_n.

b. Suppose X, Y are random variables with $E[Y|X] = 1$. Show that $EXY = EX$ and $\mathrm{Var}(XY) \geq \mathrm{Var}(X)$.

c. Let X be Gaussian with mean zero and variance σ^2. Find EX^4. $3\sigma^4$

6. a. Let X_n be geometric with parameter $\frac{\lambda}{n}$, $n = 1, 2, \cdots$. (We assume $\lambda < 1$). Let $Z_n = \frac{1}{n}X_n$, $n = 1, 2, \cdots$. Show that the sequence Z_n converges in distribution to the exponential distribution with parameter λ.

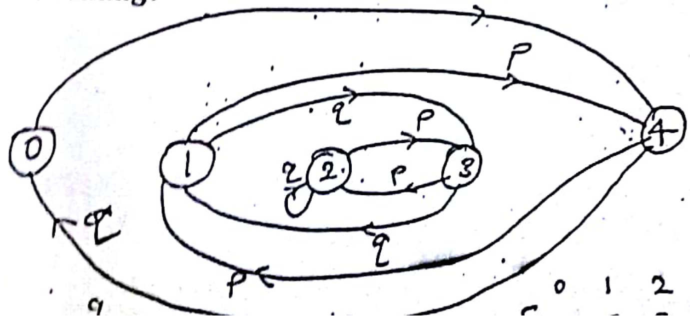
b. Let $X_1, X_2, \cdots$ be a sequence of *iid* random variables uniform over $[0,\ \theta]$. Let $Y_n = \max\ (X_1, X_2, \cdots, X_n)$. Show that the sequence Y_n converges in probability to θ.

7. a. A man has 4 umbrellas. Everyday in the morning, if it is raining and he has an umbrella at home then he takes it to his office. If it is not raining he goes to office without an umbrella. Similarly, in the evening if it is raining and he has an umbrella in his office then he takes it home; If it is not raining then he goes home without an umbrella. Assume that the probability of rain in the morning or in the evening is the same and it is equal to p where $0 < p < 1$. How often (i.e, what percentage of the time) is he without an umbrella when it is raining?

$E(XY) = E[E(XY|X)]$

$= E[X\, E[Y|X]]$

$= E[X]$



0 1 2 3

b. The arrivals of customers at a bank is a Poisson process with rate λ. Suppose two customers arrived during the first hour. What is the probability that (i). both arrived during the first 20 minutes, and (ii). at least one arrived during the first 20 minutes.

8. a. Let $\{N(t),\ t \geq 0\}$ be a Poisson process with rate λ. Let X_0 be a random variable that is independent of this process and let $P[X_0 = 1] = P[X_0 = -1] = 0.5$. Define a process $X(t)$ by $X(t) = (-1)^{N(t)} X_0$. Find the mean and autocorrelation functions of this process. Is this process weak-sense stationary?

b. Define the Brownian motion process. Show that all n^{th} order distributions of the process are Gaussian.

$E(N(t)) = \lambda t$

$\frac{e^{-\lambda t}(\lambda t)^n}{n!}$

$n = 2$

$T = 20$

$P[N(t)] = \frac{(\lambda t)^n e^{-(\lambda t)}}{n!}$

$X(t) = (-1)^{N(t)} X_0$

$E[X(t)]$ $\quad E[X(t) \mid X_0]$

$= E[(-1)^{N(t)} X_0 \mid X_0]$

$\sum_{n=0}^{\infty} (-1)^n e^{-\lambda t} \frac{(\lambda t)^n}{n!} X_0$

$e^{-2\lambda t} X_0$

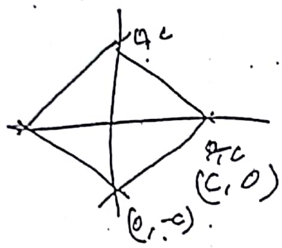
$P[N(t) \geq 1]$

$= 1 - P[N(t) = 0]$

$P[N(t) = 2]$

$= \frac{(\lambda t)^2}{2!} e^{-\lambda t}$

E1 222 Stochastic Models and Applications
Final Examination



Date: 12 December 2008
Time: 2-5 PM

Answer any FIVE questions

1. a. Let A, B be events with $P(A) = \frac{2}{5}$, $P(A|B) = \frac{2}{3}$ and $P(B|A) = \frac{1}{2}$. Find $P(B)$. Let I_A and I_B be random variables which are indicator functions of A and B respectively. Find the correlation coefficient of I_A and I_B.

   b. Let $\Omega = [0, 1]$ with the usual probability assignment so that probability of an interval is the length of the interval. Let A_n, $n = 1, 2, \cdots$, be events defined by $A_n = (2^{-n}, 2^{-n+1}]$. Find $P(\cup_{n=1}^{\infty} A_n)$. $= 1$

2. a. Let X be a continuous random variable with density function

$$f_X(x) = K(2x - x^2), \; 0 < x < 2 \qquad K = \frac{3}{4},$$

   Find value of K. Let $Y = 2X^2 + X$. Find EY. $\frac{47}{15}$

   b. Let X, Y be discrete random variables with joint probability mass function given by

$$P[X = 2, Y = 1] = P[X = 3, Y = 1] = P[X = c, Y = c] = \frac{1}{3}$$

   where c is some real number. Find all possible values for c so that (i) X, Y are uncorrelated, (ii) X, Y are independent. $c > 1$

3. a. Let X, Y be iid Gaussian random variables with mean zero and variance unity. Show that

$$P[|X| + |Y| \le c] = (P[|X + Y| \le c])^2, \; \forall c.$$

   (Hint: Drawing a diagram would help).

   b. Let X be an exponential random variable with parameter λ. Find EX^4.

4. a. Two pulses arrive at the input of an electronic counter and the arrival times can be assumed to be independent random variables which are uniform over $[0, 1]$. The counter can count them as two distinct pulses only if the time between their arrivals is more than 0.1. Find the probability that the counter would count them as two pulses.

b. Let X be a Poisson random variable with parameter λ. Show that

$$P[X > 3\lambda] \leq \frac{1}{4\lambda}.$$

5. a. Let X, Y be randomvariables with joint density function

~~$f_{XY}(x,y) = 4y^2 e^{-2xy}, \; 0 < y < 1, \; x > 0.$~~

Find $E[X|Y]$. $= \frac{1}{2y}$

b. A coin having probability p of coming up heads is tossed repeatedly until atleast 2 of the most recent 3 tosses are heads. Let N denote the number of tosses needed. If the first two tosses are heads then we take $N = 2$. Find expected value of N.

6. a. Let $\Omega = \{1, 2, \ldots\}$. Let P be a probability assignment with $P(\{i\}) = p_i, \; i = 1, 2, \ldots$ (Note that $p_i \geq 0, \; \forall i$ and $\sum p_i = 1$). Define a sequence of random variables, $X_k, \; k = 1, 2, \ldots$, by

$$\begin{aligned} X_k(\omega) &= 1 \quad \text{if } \omega \leq k \\ &= 0.5 \quad \text{if } k < \omega \leq k^2 \\ &= 0 \quad \text{Otherwise.} \end{aligned}$$

Does the sequence X_k converge (i). almost surely, (ii) in probability?

b. Briefly explain: (i) Stationary stochastic process, and (ii) Hastings-Metropolis algorithm for sampling from a distribution using a Markov chain.

7. a. Let $\{N(t), \; t \geq 0\}$ be a Poisson process with rate λ. Let X_0 be a random variable that is independent of this process and let $P[X_0 = 1] = P[X_0 = -1] = 0.5$. Define a process $X(t)$ by $X(t) = (-1)^{N(t)} X_0$. Find the mean and autocorrelation functions of this process. Is this process stationary?

$X_0^2 = 1$ $\quad 1 \quad \frac{|Yes|}{2}$

$E[X(t)] = E[(-1)^{N(t)} X_0]$
$= E[X_0] \, E[(-1)^{N}$
$\quad 0$
$= 0$

$E[(-1)^{2N(t)} \cdot (-1)^{N(t+s)-N(t)}]$ $\quad 2y \quad 2y e^{-2xy}$

$E[(-1)^{N(t+s)-N(t)}]$ $\quad \lambda e^{-\lambda \tau}$

$2 \quad E[(-1)^{N(s)}] = \sum (-1)^i \; (\lambda s)^i$ $\quad d = 2y \quad \lambda s$

$Cov[X(t), X(t+s)] = E[X(t) \, X($
$= E[(-1)^{N(t)} X_0 (-1)^{N(t+s)}$

8. a. Consider a Markov Chain with state space $S = \{0, 1, 2, 3\}$ and the transition probability matrix given by.

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Is the chain periodic? Suppose g is a real valued function defined on S with $g(0) = -1$, $g(1) = 0$, $g(2) = 1$ and $g(3) = 2$. Suppose the Markov chain has initial probability distribution given by $\pi_0(0) = 1/6$, $\pi_0(1) = 1/3$, $\pi_0(2) = 1/3$ and $\pi_0(3) = 1/6$. Find expected value of $g(X_n)$ as $n \to \infty$, if it exists. Will this be the same if the initial probability distribution is changed?

b. Let π be a stationary distribution of a Markov Chain. (i). Show that if $\pi(x) > 0$ and x leads to y then $\pi(y) > 0$. (ii). Suppose the chain has transition probabilities that satisfy the following: for some two states y and z, $P(x, y) = cP(x, z)$, $\forall x$, where c is a constant. Show that $\pi(y) = c\pi(z)$.

$\mathrm{Max}(a,b) + \mathrm{Min}(a,b) = a+b$

$\mathrm{Max}(a,b) - \mathrm{Min}(a,b) = |a-b|$

$E(e^{itXY})$

$= E(E(e^{itXY}/Y))$

$= E(\phi(tY))$

$e^{-\frac{t^2}{2}}$

E1 222 Stochastic Models and Applications
Final Examination

Date: 12 December 2006
Time: 2-5 PM

Answer any FIVE questions

1. a. Consider the probability space having $\Omega = [0, 1]$ with the usual σ- algebra and probability measure such that probability of an interval is the length of the interval. Define random variables X, Y as

$$X = I([0, a] \cup [0.5, 0.5 + a])$$
$$Y = I([b, 0.5] \cup [0.5 + b, 1])$$

where $I(A)$ denotes indicator of event A, and a, b are numbers such that $0 < a, b < \frac{1}{2}$. Will X, Y be independent for some choice of a, b? If yes, find the values for a, b that make X, Y independent; otherwise show that X, Y are not independent for any choice of a, b (with $0 < a, b < \frac{1}{2}$).

b. ~~A college is composed of 70% men and 30% women students. Suppose 5% of women and 10% of men are left-handed.~~ Find the probability that a student who is left-handed is a man.

2. a. Let X be a continuous random variable having density function

$$f_X(x) = C(1 - x^2), \quad -1 < x < 1.$$

Find the value of C, and the mean and variance of X.

b. Let X, Y be independent continuous random variables with X being uniform over $[-1, 1]$ and Y being uniform over $[-1, 3]$. Find $P[|X| > |Y|]$.

3. a. Suppose the joint density of two continuous random variables is given by

$$f_{XY}(x, y) = Ce^{-(9x^2 - 6xy + 4y^2)/2}, \quad -\infty < x, y < \infty.$$

What should be the value of C? Find the marginal densities of X and Y, the correlation coefficient and $E[X|Y]$.

$C = \frac{3\sqrt{3}}{2\pi}$

$C = \frac{1}{2\pi|\Sigma|^{1/2}}$ $|\Sigma| = \frac{1}{27}$

$\Sigma^{-1} = \begin{bmatrix} 9 & -3 \\ -3 & 4 \end{bmatrix}$ $\Sigma = \begin{bmatrix} 4/27 & 3/27 \\ 3/27 & 9/27 \end{bmatrix}$

$\sigma_x^2 = 4/27$ $\sigma_y^2 = \frac{1}{3}$

$P(X = x) = 1/2$

1. Let X, Y be iid Gaussian random variables. Show that $X+Y$ and $X-Y$ are independent .

2. Let X be a continuous random variable with density, $f_X(x) = e^{-x}$, $x \geq 0$. Find a function g such that the random variable Y, defined by $Y = g(X)$, would be uniform over $[0, 1]$. Is such a function g unique? If so prove that it is unique; otherwise give another function h such that $Y = h(X)$ also has the same uniform density.

e^{-2X}

$1-e^{-2X}$

$P[X \leq \alpha]$

3. Let $X_1, \cdots, X_n, Y_1, \cdots, Y_m, Z_1, \cdots, Z_r$ be random variables such that variance of each of them is 1. All the random variables are uncorrelated. (That is, every pair of them are uncorrelated). Let $U = X_1 + \cdots + X_n + Y_1 + \cdots + Y_m$ and $V = X_1 + \cdots + X_n + Z_1 + \cdots + Z_r$. Show that the correlation coefficient of U and V is $\frac{n}{\sqrt{(n+m)(n+r)}}$.

5. a. Let X be a discrete random variable that takes values $+1$ and -1 with equal probability. Let Y be a Gaussian random variable with mean zero and variance σ^2. Let X, Y be independent. Let $Z = X + Y$. Find $f_Z, f_{X|Z}$.

$P[X=1] = 1/2$

$P[X=-1] = 1/2$

b. The conditional variance of X given Y is defined by

$$\mathrm{Var}(X|Y) = E[(X - E[X|Y])^2|Y].$$

Show that

$$E[(X - EX)^2] = E[\mathrm{Var}(X|Y)] + \mathrm{Var}(E[X|Y]).$$

6. a. Let X be a Gaussian random variable with mean zero and variance unity. Let $Y_n = XI[n \leq X \leq n+1]$, $n \geq 0$, where $I[A]$ is the indicator function of event A. Show that the sequence Y_n converges to zero almost surely and in mean.

b. Consider a continuous random variable X with density function given by.

$$\begin{aligned} f(x) &= \frac{x^\lambda}{\lambda!} e^{-x}, \quad \text{if } x > 0 \\ &= 0 \qquad \text{otherwise} \end{aligned}$$

$EUV - EU\,EV$

$\mathrm{cov}[\Sigma, \Sigma]$

$\Sigma\Sigma \mathrm{Cov}(\ )$

$\mathrm{Cov}(U, V) = \mathrm{Cov}\left[\sum_{i=1}^{n} X_i + \sum_{j=1}^{m} Y_j, \sum_{i_1=1}^{n} X_{i_1} + \sum_{j_1=1}^{r} Z_{j_1}\right]$

$= \sum \sum \mathrm{Cov}(X_i, X_{i_1})$

discrete

where $\lambda \geq 0$ is an integer. (Note that $\int_0^\infty f(x)\, dx = 1$ for all $\lambda \geq 0$, λ an integer). Show that

$$P[0 < X < 2(\lambda+1)] > \frac{\lambda}{\lambda+1}.$$

use Chebyshev inequality.

7. a. Explain briefly (i). Markov chain Monte Carlo technique (of using a Markov chain to sample from a distribution), and (ii). Simulated annealing for optimization.

b. Consider a random process defined by $V(t) = (Y+1)\cos t + X \sin t$ where X and Y are independent random variables with $EX = EY = 0$ and $EX^2 = EY^2 = 1$. Find the mean, autocorrelation and autocovariance of $V(t)$. Is $V(t)$ wide-sense stationary?

$E\, V(t_1) V(t_2)$

$= E\, (Y+1) \cos t_1$ ...

$E\, X_{t_1} X_{t_2} - (E X_{t_1})(E X_{t_2})$

$E\, V(t)$

8. a. Consider a Markov Chain with state space $S = \{0, 1, 2, 3\}$ and the transition probability matrix given by

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Is the chain periodic? Suppose g is a real valued function defined on S with $g(0) = -1$, $g(1) = 0$, $g(2) = 1$ and $g(3) = 2$. Suppose the Markov chain has initial probability distribution given by $\pi_0(0) = 1/6$, $\pi_0(1) = 1/3$, $\pi_0(2) = 1/3$ and $\pi_0(3) = 1/6$. Find expected value of $g(X_n)$ as $n \to \infty$, if it exists. Will this be the same if the initial probability distribution is changed?

b. Consider a Markov Chain with state space S which is a subset of $\{0, 1, 2, \ldots\}$ and whose transition function P satisfies

$$\sum_y y P(x, y) = ax + b \; \forall x \in S$$

where a, b are constants. Show that (i). $EX_{n+1} = aEX_n + b$, (ii). if $a \neq 1$ then

$$EX_n = \frac{b}{1-a} + a^n \left(EX_0 - \frac{b}{1-a}\right).$$

7. (b) $V(t) = (Y+1)\cos t + X \sin t$

$E\,V(t) = E((Y+1)\cos t + X \sin t) = E(Y \cos t) + E \cos t + E(X \sin t)$

$= 0 + \cos t + 0 = \cos t$

HP-15AY Q79PX

E1 222 Stochastic Models and Applications
Test #2

Time: 90 mts

28 Oct. 2014

18 mys

Answer ALL questions. All questions carry equal marks.

1. a. Let X, Y, Z be continuous random variables with joint density

$$f_{XYZ}(x, y, z) = \frac{1}{xy}, \quad 0 \le z \le y \le x \le 1.$$

Find $f_{Y|X}$, $P[Z > 0.5Y]$ and EZ.

b. Let $X' = aX + b$ and $Y' = cY + d$. Find the relationship between the correlation coefficient of X and Y and that of X' and Y'.

2. a. Let X, Y be *iid* geometric random variables. Let $Z = \min(X, Y)$ and $W = X - Y$. Find the (marginal) probability mass functions of Z and W. Are Z and W independent?

b. Let X, Y be jointly normal with $EX = EY = 0$, $\text{Var}(X) = \sigma_1^2$ and $\text{Var}(Y) = \sigma_2^2$. Are $X + Y$ and $X - Y$ independent?

3. a. Consider repeated tossing of a coin. Let T_n denote the number of heads in the first n tosses. Let X denote the toss number where the first head appears. Show that $P[X = i \mid T_n = 1] = \frac{1}{n}$, $1 \le i \le n$.

b. Let X, Y be *iid* geometric random variables. Guess the value of $P[X = i \mid X + Y = n]$. Justify your guess. $1/n$

c. Let X be uniform over $[0, 1]$. Find $E[X|X > 0.5]$. ?

4. a. The conditional variance of X given Y is defined as $\text{Var}(X|Y) = E[\,(X - E[X|Y])^2 \mid Y]$. Show that

$$\text{Var}(X) = E[\text{Var}(X|Y)] + \text{Var}(E[X|Y])$$

b. Let X and Y be independent random variables each having normal distribution with mean zero and variance 1. Let $Z = \rho X + \sqrt{1 - \rho^2}\, Y$ where $-1 < \rho < 1$. (i). Find density of Z. (ii). Find the joint density of X, Z. (iii). Let $U = \mu_1 + \sigma_1 X$ and $W = \mu_2 + \sigma_2 Z$, where $\mu_1, \mu_2, \sigma_1, \sigma_2 \in \Re$ and $\sigma_1, \sigma_2 > 0$. Find the correlation coefficient of U and W.

Handwritten marks: (3) beside 1b; ✗ (2) beside 2a; (3) beside 2b; (2) beside 3a; English; (2) beside 3b; (3) beside 4a; (3) beside 4b.

Right margin: 2 + 3 + 1 + 3 + 1 + 0 + 2 + 3 + 3; Mon, ; 3/2.

$E[T=?]$

$F_X = x - 0 = x$

$F_X(x) = x$

$f_{X|T}\, dx$

$\frac{d}{dx} F_{X|T}(x|T=1)$

$$\frac{\int_{0.5}^{1} x f_X\, dx}{P(X > 0.5)} = \frac{\int_{0.5}^{x} x\, dx \;\frac{1}{n}}{1 - F(0.5)}$$

$x^2/2 \,\big|$

E1 222 Stochastic Models and Applications
Test #2

Time: 90 mts

25 Oct. 2013

Answer ALL questions. All questions carry equal marks.

1. a. Let X, Y be continuous random variables with joint density

$$f_{XY}(x, y) = \frac{1}{1-x}, \quad 0 \leq x \leq y \leq 1.$$

Find marginal densities of X & Y, $\boxed{E[Y|X]}$, and $E[Y - X]$.

b. Show that $\mathrm{Cov}(X, Y) = \mathrm{Cov}(X, E[Y|X])$.

2. a. Let X, Y be *iid* continuous random variables having exponential distribution with $\lambda = 1$. Let $Z = X + Y$ and $W = \frac{X}{X+Y}$. Show that Z and W are independent.

b. At a party N people make a heap of all their hats and then everyone picks a hat at random. Let X denote the number of people who get their own hat. Show that variance of X is 1.

3. a. Let X, Y be *iid* geometric random variables with parameter p. Let $Z = \min(X, Y)$. Find the probability mass function of Z.

b. Let X be a continuous random variable and let Y be a discrete random variable taking non-negative integer values. Suppose X and Y are independent. Let $Z = X+Y$. Is Z a continuous random variable? Provide a convincing explanation of your answer.

4. a. Let X_1, X_2, X_3, X_4 be *iid* Normal random variables with mean zero and variance 1. Let $Y = X_1X_2 + X_3X_4$. Show that the density of Y is given by

$$f_Y(y) = \frac{1}{2} e^{-|y|}, \quad -\infty < y < \infty.$$

b. Let $I_1, \cdots, I_N$ be independent binary random variables with $P[I_j = 0] = P[I_j = 1] = 0.5$, $\forall j$. Let $g_n(k) = P[\sum_{j=1}^{n} jI_j \leq k]$, $1 \leq n \leq N$. Show that $g_n(k) = 0.5g_{n-1}(k) + 0.5g_{n-1}(k - n)$.

$E[Y] = \int_0^1 y\, f_Y(y)$ $X = ZW$

$Y = Z - ZW$

E1 222 Stochastic Models and Applications

Test #2

Time: 90 mts 25 Oct. 2012

Answer ALL questions. All questions carry equal marks.

1. a. Let X, Y be continuous random variables with joint density

$$f_{XY}(x,y) = \frac{1}{1-x}, \quad 0 \le x \le y \le 1.$$

Find marginal densities of X & Y, $E[Y|X]$, and $E[Y-X]$.

b. Show that $\mathrm{Cov}(X,Y) = \mathrm{Cov}(X, E[Y|X])$.

2. a. Let X, Y be *iid* continuous random variables having exponential distribution with $\lambda = 1$. Let $Z = X+Y$ and $W = \frac{X}{X+Y}$. Show that Z and W are independent.

b. At a party N people make a heap of all their hats and then everyone picks a hat at random. Let X denote the number of people who get their own hat. Show that variance of X is 1.

3. a. Let X, Y be *iid* geometric random variables with parameter p. Let $Z = \min(X,Y)$. Find the probability mass function of Z.

b. Let X be a continuous random variable and let Y be a discrete random variable taking non-negative integer values. Suppose X and Y are independent. Let $Z = X+Y$. Is Z a continuous random variable? Provide a convincing explanation of your answer.

4. a. Let X_1, X_2, X_3, X_4 be *iid* Normal random variables with mean zero and variance 1. Let $Y = X_1X_2 + X_3X_4$. Show that the density of Y is given by

$$f_Y(y) = \frac{1}{2}e^{-|y|}, \quad -\infty < y < \infty.$$

b. Let $I_1, \cdots, I_N$ be independent binary random variables with $P[I_j = 0] = P[I_j = 1] = 0.5$, $\forall j$. Let $g_n(k) = P[\sum_{j=1}^{n} jI_j \le k]$, $1 \le n \le N$. Show that $g_n(k) = 0.5g_{n-1}(k) + 0.5g_{n-1}(k-n)$.

$E[Y|X] = y \cdot \frac{f_{Y|X}(y|x)}{f_X(x)}$

EI 222 Stochastic Models and Applications
Test #2

Time: 90 mts

31 Oct. 2012

Answer ALL questions. All questions carry equal marks.

1. a. Let X, Y be continuous random variables with joint density

$$f_{XY}(x,y) = K, \quad -1 \leq x \leq +1, \ |x| \leq y \leq 1.$$

Find K, marginal densities of X & Y, $E[X|Y]$ and $E[Y|X]$.

b. Let X, Y have a joint distribution that is uniform over the quadrilateral with vertices at $(-1, 0)$, $(1, 0)$, $(0, -1)$ and $(0, 1)$. Find $P[X^2 + Y^2 \leq 0.25]$.

2. a. Calculate $E[X|Y]$ where X, Y are discrete random variables with the joint probability mass function given by

$$f_{XY}(x,y) = \frac{1}{Ny}, \quad x, y \in \{1, 2, \cdots, N\}, \ x \leq y.$$

b. Let Z_i, $i = 1, 2, \cdots, n$, be iid Gaussian random variables with mean zero and variance one. Let $Y = \sum_{i=1}^{n} Z_i^2$. Show that the moment generating function of Y is $M_Y(t) = (1-2t)^{-n/2}$.

3. Let X be a geometric random variable with parameter p and let Y be uniform over (0, 1). Also, X and Y are independent. Let $Z = X + Y$. Find the distribution function of Z. Is Z a continuous random variable? If so, find its density function.

4. a. Let $X_1, X_2, \cdots, X_N$ be iid continuous random variables. We say that a record has occurred at k if $X_k > \max(X_1, X_2, \cdots, X_{k-1})$, $1 \leq k \leq N$. (By convention, we always have a record at $k = 1$). Let Z_n denote the number of records till n, $1 \leq n \leq N$. Show that (i). Prob[a record occurs at k] $= \frac{1}{k}$ and (ii). $E[Z_n] = \sum_{i=1}^{n} \frac{1}{i}$.

b. Let (X_1, X_2) be jointly normal with means μ_1 & μ_2, variances σ_1^2 & σ_2^2, and correlation coefficient ρ. Find a necessary and sufficient condition for $X_1 + X_2$ and $X_1 - X_2$ to be independent. ? understand

$\sigma_1^2 = \sigma_2^2$

$E[(X_1+X_2)(X_1-X_2)] - E[X_1+X_2]\,E[X_1-X_2]$

$\sigma_1^2 - \sigma_2^2 = 0$

$\sigma_1^2 = \sigma_2^2$

E1 222 Stochastic Models and Applications

Test #2

Time: 90 mts

29 Oct. 2011

Answer ALL questions. All questions carry equal marks.

1. a. Let X, Y be continuous random variables with joint density

$$f_{XY}(x, y) = K(1 + xy(x - y)) \quad 0 \le x, y \le 1$$

Find the value of K, the marginal densities of X and Y and covariance of X and Y.

b. Let X, Y be *iid* continuous random variables each having a uniform density over $[-1,\ 1]$. Find $P[X^2 + Y^2 \le 1]$. $\pi/4$

2. a. Let X, Y be continuous random variables with joint density

$$f_{XY}(x, y) = K \exp\left(-\frac{1}{10}(9x^2 + y^2 - 4xy)\right), \quad -\infty < x, y < \infty.$$

Find the value of K and the correlation coeeficient of X and Y.

b. Let X_1, X_2 be *iid* discrete random variables with $P[X_i = 1] = P[X_i = -1] = 0.5,\ i = 1, 2$. Let $X_3 = X_1 X_2$. Show that X_1, X_2, X_3 are pairwise independent but not independent.

3. a. Show that $\mathrm{Cov}(X, Y) = \mathrm{Cov}(X, E[Y|X])$. Using this, show that if X, Y are two random variables with $E[Y|X] = a + bX$ for some constants a and b, then $b = \frac{\mathrm{Cov}(X,Y)}{\mathrm{Var}(X)}$.

b. Let X, Y be independent random variables with means μ_x, μ_y and variances σ_x^2, σ_y^2. Show that $\mathrm{Var}(XY) = \sigma_x^2\sigma_y^2 + \mu_x^2\sigma_y^2 + \mu_y^2\sigma_x^2$.

4. Let $X_1, X_2, \cdots, X_n$ be iid continuous random variables each having uniform distribution over $(0,\ 1)$. Let $Y_1 = X_1$, $Y_2 = X_1X_2$, $Y_3 = X_1X_2X_3, \cdots, Y_n = X_1X_2\cdots X_n$. Find joint density of $Y_1, Y_2, \cdots Y_n$; and conditional density of Y_k conditioned on $Y_1, Y_2, \cdots Y_{k-1}$. Let t be a fixed number in the interval $[0,\ 1]$. Let Z denote the number of Y_i that are in the interval $[t,\ 1]$. Find $P[Z = 1]$.

E1 222 Stochastic Models and Applications
Test #2

Time: 90 mts

1 Nov. 2010

Answer ALL questions. All questions carry equal marks.

1. Let X, Y be continuous random variables with joint density given by

$$f_{XY}(x,y) = K \quad \text{if } 0 \le x, y \le 1, \text{ and } x^2 + y^2 \le 1$$
$$= 0 \quad \text{Otherwise.}$$

$\frac{4}{\pi}$

Find the value of K. Also calculate the marginal densities of X and Y and $E[Y|X]$.

2. Let X and Y be *iid* random variables both having uniform distribution over $\{1, 2, \cdots, N\}$. Let $Z = \min(X, Y)$ and $W = X - Y$. Find the distributions of Z and W and also the joint distribution of Z, W. Are Z and W independent? *

$\frac{1}{2}t$

3. Let X be a geometric random variable with parameter p and let Y be uniform over $(0, 1)$. Also, X and Y are independent. Let $Z = X + Y$. Find the distribution function of Z. Is Z a continuous random variable? *

4. a. Let Z_i, $i = 1, 2, \cdots, n$, be *iid* Gaussian random variables with mean zero and variance one. Let $Y = \sum_{i=1}^{n} Z_i^2$. Show that the moment generating function of Y is $M_Y(t) = (1 - 2t)^{-n/2}$.

   b. Let X and Y be independent random variables each having normal distribution with mean zero and variance 1. Let $Z = \rho X + \sqrt{1-\rho^2}\, Y$ where $-1 < \rho < 1$. (i). Find density of Z. (ii). Find the joint density of X, Z. (iii). Let $U = \mu_1 + \sigma_1 X$ and $W = \mu_2 + \sigma_2 Z$, where $\mu_1, \mu_2, \sigma_1, \sigma_2 \in \Re$ and $\sigma_1, \sigma_2 > 0$. Find the correlation coefficient of U and W.

5. a. Suppose X is a Poisson random variable with parameter λ where λ is itself a random variable having exponential distribution with mean 1. Show that $P[X = n] = \left(\frac{1}{2}\right)^{n+1}$.

   b. Let $X_1, X_2, \cdots, X_n$ be *iid* random variables with mean μ and Variance σ^2. Let $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Show that $\text{Cov}(\bar{X}, X_i - \bar{X}) = 0$ for $i = 1, 2, \cdots, n$.

$\alpha - 1 = \lambda$
$\alpha = \lambda$

$Y = e^{-y}$
$X: \frac{y^x e^{-y}}{x!}$ $\quad f_{X|Y} = \frac{y^x e^{-y}}{x!}$ $\quad f_Y(y) = e^{-y}$ $\quad \int y^x e^{-2y}$

$f_{XY} = \frac{y^x e^{-2y}}{x!}$ $\quad \lambda = e^{-y}$ $\quad P[F(\lambda) = n] | \lambda = e^{-x} y]$

$(2x)^{\alpha-1} \lambda e^{-}$
$(2y)^{\alpha-1} 2 \cdot e$
$(2y)^x 2 e^{-2}$
$\overline{\qquad 4(x+1)}$
$= 2^{x+1} \boxed{\frac{y^x e^{-}}{|x!}}$

$n = 10$
9

$f_x = \int_0^\infty f_{xy}\, dy = \int_0^\infty \frac{y^x e^{-2y}}{x!} dy = \frac{1}{n!} \int_0^\infty y^x e^{-2y}$

10

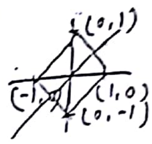
E1 222 Stochastic Models and Applications
Test #2

Time: 90 mts
Max Marks: 50

24 Oct 2009

Answer ALL questions.
All questions carry equal marks.

1. Let X, Y be *iid* geometric random variables (with parameter p). Let $Z = \max(X, Y)$ and $W = X - Y$. Find distribution of Z, W and their joint distribution. Are Z and W independent?



2. a. Let X, Y have a joint distribution that is uniform over the quadrilateral with vertices at $(-1,\ 0)$, $(1,\ 0)$, $(0,\ -1)$ and $(0,\ 1)$. Find $P[X > Y]$. Are X and Y independent? NO

   b. Let X, Y be *iid* exponential random variables. Show that $Z = \frac{X}{X+Y}$ is uniform over $[0,\ 1]$.

3. a. Let $X_1, X_2, \cdots, X_n$ be random variables with mean zero and variance unity. Suppose the correlation coefficient of any pair of random variables, X_i and X_j, $i \neq j$, is ρ. Show that $\rho \geq \frac{-1}{n-1}$. Will this result remain true if $EX_i = \mu_i$ and $\mathrm{Var}(X_i) = \sigma_i^2$; but correlation coefficient between any pair of them is still ρ.

   ~~b. Let (X_1, X_2) be jointly normal with means μ_1 & μ_2, variances σ_1^2~~ & σ_2^2, and correlation coefficient ρ. Find a necessary and sufficient condition for $X_1 + X_2$ and $X_1 - X_2$ to be independent.

4. a. Let X, Y have joint density given by

$$f_{XY}(x, y) = \frac{y}{\sqrt{\pi}} e^{-\frac{1}{4}(x^2+y^2-2xy)}, \quad \underline{0 < y < 1} \; x \in \Re$$

   Find $E[X|Y]$.

   $E[X|Y] = y$

   b. An interval of length 1 is broken at a point uniformly distributed over $(0,\ 1)$. Let c be a fixed point in $(0,\ 1)$. Find the expected length of the subinterval that contains the point c. Show that this ~~probability~~ is maximized when $c = 0.5$.

5. Answer the following very briefly.

a. Let X, Y be *iid* continuous random variables. Does the value of $P[X > Y]$ change when we change the joint density of X, Y? No

b. If X_i is normal for $i = 1, 2, \cdots, n$, then, is $\sum_{i=1}^{n} X_i$ normal? NO

Independent is must

c. Let $Y = X + \xi$ where ξ is a random variable independent of X and with mean μ and variance σ^2. Let $g(X)$ be the best mean square estimate of Y as a function of X. What is $g(X)$ and $E[Y - g(X)]^2$?

$g(X) = X + \mu$

$E[\xi^2] = \sigma^2$

d. Let $X' = aX + b$ and $Y' = cY + d$. Is the correlation coefficient of X' and Y' same as that of X and Y? Yes

$P(X=2) \cdot P(Y=1) = P(X=3) \cdot P(Y=1)$.

$P(X=2) = P(X=3) = 1/3$.

~~$P(X=c) = 1/3$.~~

X, Y: uniform iid $U[0,1]$

$Z = \frac{X}{X+Y}$

$P(X=c) \cdot P(Y=c) = 1/3$

$W = X+Y$. $\quad w > 0$

$P(Y=c) = 1$.

$w > zw \quad 1 > z$

$X = ZW$ $\quad 0 < z < 1$

$Y = W - ZW$

$J_{XY} = \begin{matrix} w & z \\ -w & 1-z \end{matrix}$

$w(1-z) + wz$

$\underline{w}$

$f_{ZW} = J \cdot f_{XY}(x, y)$ density

$= w \cdot f_X(zw) \cdot f_Y(w - zw)$

$= w\lambda^2 e^{-\lambda zw} e^{-\lambda(w - zw)}$

$f_Z(z) = \int_0^\infty w\lambda^2 e^{-\lambda w}\, dw \quad \boxed{0 < z < 1}$

EI 222 Stochastic Models and Applications
Test #2

Time: 90 mts
Max Marks: 50

30 Oct 2008

Answer ALL questions.
All questions carry equal marks.

1. Let random variables X, Y, Z have joint density function given by

$$f_{XYZ}(x, y, z) = K, \quad 0 < x < y < z < 1.$$

Find the value of K and $E[Y|X]$. Are X, Y, Z independent?

2. Let X and Y be discrete random variables such that the distribution of X is uniform over $\{1, 2, \cdots, 20\}$ and the conditional distribution of Y given $X = x$ is uniform over $\{1, 2, \cdots, x\}$. Find $E[Y]$. = 23/4

3. a. Let X, Y be random variables with $EX = EY = 0$ and $\text{Var}(X) = \text{Var}(Y) = 1$. Let ρ be the correlation coefficient of X and Y. Show that

$$E[\max\{X^2, Y^2\}] \leq 1 + \sqrt{1 - \rho^2}.$$

(Hint: $\max\{a, b\} = (a + b + |a - b|)/2$).

b. Let X, Y be independent random variables with means μ_x, μ_y and variances σ_x^2, σ_y^2. Show that $\text{Var}(XY) = \sigma_x^2\sigma_y^2 + \mu_y^2\sigma_x^2 + \mu_x^2\sigma_y^2$.

4. Let X_1, X_2, X_3, X_4 be iid continuous random variables each with distribution function F and density function f. Let $q = \text{Prob}[X_1 < X_2, X_4 > X_3]$. Write an expression for q as an integral of the joint density function. By evaluating this integral find the value of q and conclude that its value is same for all density functions f. Show that the value of q you got would be same if you had evaluated this probability by assuming that all 4! possible orderings of X_1, X_2, X_3, X_4 are equally likely.

5. Let X and Y be independent random variables with X having Poisson distribution with parameter λ and Y being uniform over $[0, 1]$. Let $Z = X + Y$. Find the distribution function of Z. Is Z a continuous random variable?

Verify

E1 222 Stochastic Models and Applications
Test #2

Time: 90 mts
Max Marks:50

29 Oct 2007

Answer ALL questions.
All questions carry equal marks.

1. a. Let A, B be two events such that

$$P(A) = \frac{1}{2}; \quad P(A \mid B) = \frac{5}{6}; \quad P(B \mid A) = \frac{1}{3}$$

Let X, Y be the indicator variables: $X = I_A$ and $Y = I_B$. (That is, $X(\omega) = 1$ if $\omega \in A$ and $X(\omega) = 0$ otherwise; and similarly for Y). Calculate expectations and variances of X and Y and the correlation coefficient of X, Y.

b. Let X_i, $i = 1, 2, \cdots n$, be independent random variables with $EX_i = 0$ and $\mathrm{Var}(X_i) = \sigma^2, \forall i$. Calculate $E[(X_1 + X_1X_2 + \cdots + X_1X_2\cdots X_n)^2]$. $\left(\frac{\sigma^{2n}-1}{\sigma^2-1}\right)\sigma^2$

2. Let X, Y be continuous random variables with joint density

$$f_{XY}(x, y) = 1 \quad \text{for } |y| \le x, \ 0 < x < 1.$$

Calculate the marginal densities f_X, f_Y, the conditional densities $f_{X|Y}$, $f_{Y|X}$, and the conditional expectation $E[X|Y]$. Verify that $E[\,E[X|Y]\,] = E[X]$.

3. a. Let X, Y be continuous random variables with $EX = 3$, $EY = 0$ and joint density

$$f_{XY}(x, y) = \frac{1}{2\pi\sqrt{3}} \exp\left[-\frac{1}{6}(4x^2 + y^2 - 2xy - 24x + 6y + 36)\right], \quad -\infty < x, y < \infty.$$

Find the marginal densities and the correlation coefficient of X, Y.

b. Let X, Y be iid Gaussian random variables. Let $U = X + Y$ and $W = X - Y$. Are U and W independent?

$E(X+Y)(X-Y) = E(X^2 - Y^2) = 0$
$E(X+Y)\,E(X-Y) = (2\mu)(0) = 0$ $\Rightarrow$ uncorrelated

$f_Y(y) = \int$

$\int_0^1 \int_{-x}^{x} 1 \, dy \, dx$

$\int_0^1 2x \, dx$

$\frac{(2x-6)^2}{6} + \frac{(y+3)^2}{6} - \frac{9}{6}$

$4(x-3)^2 + y^2 - 2(x-3)y$

7\. a. Let Z_i, $i = 1, 2, \cdots, n$ be *iid* Gaussian random variables with mean zero and variance one. Let $Y = \sum_{i=1}^{n} Z_i^2$. Show that the moment generating function of Y is $M_Y(t) = (1 - 2t)^{-n/2}$.

b. Let X_i, $i = 1, 2, \cdots, n$ be *iid* Gaussian random variables with mean μ and variance σ^2. Let

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i, \quad S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2.$$

(Recall that $\bar{X}$ and S^2 are independent). Show that

$$\sum_{i=1}^{n}(X_i - \bar{X})^2 = \sum_{i=1}^{n}(X_i - \mu)^2 - n(\bar{X} - \mu)^2.$$

Use this to find the moment generating function of $\frac{(n-1)S^2}{\sigma^2}$?

8\. a. Let X, Y be *iid* geometric random variables with parameter p. Let $Z = X + Y$. Calculate the mass function of Z and $E[Z \mid Y]$.

b. Let $I_1, I_2, \ldots, I_n$ be independent random variables each of which take values 0 or 1 with equal probability. Let

$$f_n(k) = P\left[\sum_{j=1}^{n} jI_j \le k\right].$$

Show that

$$f_n(k) = \frac{1}{2}f_{n-1}(k) + \frac{1}{2}f_{n-1}(k-n).$$

$E[X+Y/Y]$

$= E[X/Y] + E(Y$

$= \frac{1}{p} + Y$

$Y + \frac{1}{p}$

$$\sum_{k=1}^{z-1} P\{X=k\}\, P\{Y=z-k\}$$

$E[Y/Z] = E[Z - X/Z]$

$$\sum_{i=1}^{\infty}(1-p)^{i-1}$$

EI 222 Stochastic Models and Applications

Test #2

Time: 90 mts

Max Marks: 50

27 Oct 2006

Answer ALL questions.

All questions carry equal marks.

1. Let X, Y be continuous random variables with joint density

$$f_{XY}(x, y) = \frac{4x}{y}, \quad 0 < x < y < 1$$

Find the marginal densities, f_X & f_Y, the conditional densities, $f_{X|Y}$ & $f_{Y|X}$, $E[X|Y]$, and $E[X]$.

$f_X(x) = -4x \ln x$

$\frac{2x}{y^2}$

2. Let X, Y be iid random variables having geometric distribution with parameter p. (That is, $f_X(k) = p(1-p)^{k-1}$, $k = 1, 2, \ldots$). Let $M = \max(X, Y)$. Find the distribution function of M, the joint mass function of M, X, and the conditional mass function of M given X.

3. a. Let $X_1, X_2, \ldots, X_n$ be *iid* random variables with mean zero and variance σ^2. Calculate $E[(X_1+X_1X_2+X_1X_2X_3+\cdots+X_1X_2\ldots X_n)^2]$.

   b. Let X, Y be independent random variables having exponential distribution with parameters λ_1 and λ_2 respectively. Find $E[(X+Y)^2|X]$.

4. Let X and Y be independent random variables each having normal distribution with mean zero and variance 1. Let $Z = \rho X + \sqrt{1-\rho^2}Y$ where $-1 < \rho < 1$. (i). Find density of Z. (ii). Find the joint density of X, Z. (iii). Let $U = \mu_1 + \sigma_1 X$ and $W = \mu_2 + \sigma_2 Z$, where $\mu_1, \mu_2, \sigma_1, \sigma_2 \in \Re$ and $\sigma_1, \sigma_2 > 0$. Find the correlation coefficient of U and W.

5. A coin having probability p of coming up heads is tossed repeatedly until atleast 2 of the most recent 3 tosses are heads. Let N denote the number of tosses needed. (Note that by the definition of N, if, for example, the first two tosses are heads then $N = 2$). Find expected value of N.

$x^2 + \frac{x}{\lambda_2^2} + \frac{x}{\lambda_2}$

$f_{X|}$:

E1 222 Stochastic Models and Applications
Test #3

Max. Marks: 40

25 Nov 2014

Answer ALL questions. All questions carry equal marks.

1. Consider a Markov Chain on $S = \{0, 1, 2, 3, 4\}$ with the following transition probabilities:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} 0.2 & 0.2 & 0 & 0.6 & 0 \\ 0 & 0.6 & 0.4 & 0 & 0 \\ 0 & 0.4 & 0.6 & 0 & 0 \\ 0 & 0 & 0 & 0.5 & 0.5 \\ 0 & 0 & 0 & 0.8 & 0.2 \end{array} \right] \end{array}$$

$\pi = [0\ 0\ 0\ \pi(3)\ \pi(4)]$

Determine which states are transient and which states are recurrent and also find all the different closed irreducible sets of recurrent states. Find the absorption probabilities from different transient states to different closed irredicible sets. Find any one stationary distribution of the chain. Does this chain have a unique stationary distribution?

2. a. Let $X_1, X_2, \ldots$ be a sequence of *independent* random variables with probability distribution given by: $P[X_n = n] = \frac{1}{2n}$, $P[X_n = -n] = \frac{1}{3n}$, $P[X_n = 0] = 1 - \frac{5}{6n}$. Does the sequence $\{X_n\}$ converge (i) in probability, (ii). almost surely, (iii). in r^{th} mean ?

n	$\frac{1}{2n}$
$-n$	$\frac{1}{3n}$
0	$1-\frac{5}{6n}$

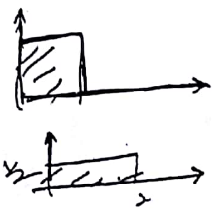
1. b. Let X be a random variable with mean μ and variance σ^2. We know σ^2 but do not know μ. Suppose we estimate μ by taking the sample mean of n *iid* realiztions of X. Discuss how we can decide on the number of observations, n. What would be a good indicator of the error in your final estimate.

$P\{|X_n - X| > \epsilon\} \to$

3. a. Let $\Omega = \Re^2$. Consider a sequence of events defined by

$$A_n = \{\ (x, y)\ :\ 0 \leq x < n,\ 0 \leq y < (1/n)\ \}.$$

Is this sequence monotone? Find lim sup A_n and lim inf A_n. Does the sequence A_n converge?



error $> \epsilon$:

error

3. Let X, Y be *iid* random variables having uniform distribution over $[-1, 1]$. Let $Z = X - Y$. Find the characteristic function of Z.

4. a. Let $\{X_n,\ n \geq 0\}$ be a Markov Chain. Show that X_{n-1} and X_{n+1} are conditionally independent given X_n. That is, show that for any n and any x, y, z,

$$\Pr[X_{n-1} = x,\ X_{n+1} = z | X_n = y] = \Pr[X_{n-1} = x | X_n = y] \Pr[X_{n+1} = z | X_n = y]$$

b. Explain why a finite Markov chain cannot have a null recurrent state.

E1 222 Stochastic Models and Applications
Test #3

Max. Marks: 40

28 Nov 2013

Answer ALL questions. All questions carry equal marks.

1. Consider a Markov Chain on $S = \{0, 1, 2, 3, 4\}$ with the following transition probabilities:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0.25 & 0.25 & 0 & 0.5 & 0 \\ 0. & 0.5 & 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 & 0 & 0 \\ 0 & 0 & 0 & 0.6 & 0.4 \\ 0 & 0 & 0 & 0.75 & 0.25 \end{bmatrix} \end{array}$$

Determine which states are transient and which states are recurrent and also find all the different closed irreducible sets of recurrent states. Find the absorption probabilities from different transient states to different closed irredicible sets. Find any one stationary distribution of the chain. Does this chain have a unique stationary distribution?

2. a. Let $X_1, X_2, \ldots,$ be a sequence of *iid* random variables with density function $f(x) = 2x,\ 0 \le x \le 1$. Define $Z_n = \frac{1}{n}\sum_{i=1}^{n} X_i^2$. Does Z_n converge almost surely? (Answer Yes/No and give a short justification). If your answer is 'Yes' (that is, Z_n converges almost surely), what is the limit? If your answer is 'No' (that is, it does not converge almost surely), does it converge in any other sense?

CLT

b. Let $X_1, \ldots, X_n$ be independent random variables each having a uniform distribution over $[-1, 1]$. Let $S_n = X_1 + \ldots + X_n$. Find $P[S_{10} \le 0.1]$ using central limit theorem approximation.

3. a. Let $X_1, X_2, \ldots$ be a sequence of *independent* random variables with probability distribution given by: $P[X_n = n] = P[X_n = 0] = \frac{1}{2}$. Let $Y_n = \Pi_{k=1}^{n} X_k,\ n = 1, 2, \ldots$. Does the sequence Y_n converge to zero (i) in probability, (ii) almost surely, (iii) in r^{th} mean ?

b. Let X, Y be *iid* random variables having uniform distribution over $[-1, 1]$. Let $Z = X - Y$. Find the characteristic function of Z.

$E[x^2] = \frac{1}{2}$

$Var(X) = \frac{1}{12}$

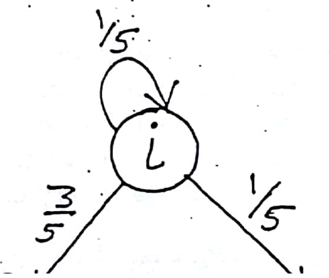
$EX^2 = \frac{1}{12}$ $2t\sin t\cos t - 2t$

$x^2\ \frac{x^3}{3}\Big|_0^1$ $\int e^{-itx}$

4. a. Let $\{X_n;\ n \geq 0\}$ be a Markov Chain. Show that X_{n-1} and X_{n+1} are conditionally independent given X_n. That is, show that for any n and any x, y, z,

$$\Pr[X_{n-1} = x,\ X_{n+1} = z | X_n = y] = \Pr[X_{n-1} = x | X_n = y]\ \Pr[X_{n+1} = z | X_n = y]$$

b. Given below is part of the state transition graph of a Markov chain, $\{X_n\}$. The chain is started in state i. Let (the random variable) N denote the first time instant when the chain leaves state i. Find the distribution of (i). N, and (ii). X_N.



$N = 1, 2, 3, \ldots$ X_N:

E1 222 Stochastic Models and Applications
Test #3

Max. Marks: 40

23 Nov 2012

Answer ALL questions. All questions carry equal marks.

1. Consider a Markov Chain on $S = \{0, 1, 2, 3, 4, 5\}$ with the following transition probabilities:

$$P = \begin{array}{c|cccccc} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & 0.2 & 0.3 & 0 & 0.3 & 0.2 & 0 \\ 1 & 0.1 & 0.2 & 0.3 & 0 & 0 & 0.4 \\ 2 & 0 & 0 & 0.3 & 0 & 0.7 & 0 \\ 3 & 0 & 0 & 0 & 0.2 & 0 & 0.8 \\ 4 & 0 & 0 & 0.6 & 0 & 0.4 & 0 \\ 5 & 0 & 0 & 0 & 0.8 & 0 & 0.2 \end{array}$$

$\beta_{\{2,4\}}(1) = 26/61$ $35/61$

$\beta_{\{2,4\}}(0) = \frac{25}{61}$ $\frac{36}{61}$

Determine which states are transient and which states are recurrent and also find all the different closed irreducible sets of recurrent states. Find the absorption probabilities from different transient states to different closed irreducible sets. Does this chain have a unique stationary distribution? Find any one stationary distribution of the chain.

$E[X_i^2] =$

$\int_0^1 x^2 \cdot 6x(1-x)$

$\int 6x^3 - 6x^4$

2. a. Let $X_1, X_2, \ldots,$ be a sequence of *iid* random variables with density function $f(x) = 6x(1-x)$; $0 \le x \le 1$. Define $Z_n = \frac{1}{n}\sum_{i=1}^{n} X_i^2$. Does Z_n converge almost surely? (Answer Yes/No and give a short justification). If your answer is 'Yes' (that is, Z_n converges almost surely), what is the limit? If your answer is 'No' (that is, it does not converge almost surely), does it converge in any other sense?

$3/10$

$Z_n \xrightarrow{a.s.} E[X_i^2]$

$6\left[\frac{x^4}{4} - \frac{x^5}{5}\right]$

$\frac{6}{20} = \frac{3}{10}$

b. Let $X_1, \ldots, X_n$ be independent random variables each having a uniform distribution over $[-1, 1]$. Let $S_n = X_1 + \ldots + X_n$. Find the characteristic function of S_n. Find $P[S_{10} \le 0.1]$ using central limit theorem approximation. $\phi(\frac{\sqrt{3}}{10})$

$\frac{\sin^n \theta}{\theta^n}$

$Z_n \to 3/10$

3. a. Let $X_1, X_2, \ldots$ be a sequence of *independent* random variables with probability distribution given by: $P[X_n = n] = P[X_n = 0] = \frac{1}{2}$. Let $Y_n = \prod_{k=1}^{n} X_k$, $n = 1, 2, \ldots$. Does the sequence Y_n converge to zero (i) in probability, (ii). almost surely, (iii). in r^{th} mean ?

Yes   Yes   No

b. Let X_n be a sequence of geometric random variables with parameter λ/n (where $\lambda < 1$). Let $Z_n = X_n/n$. Show that Z_n converge in distribution to an exponential random variable.

4. a. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ. Show that the time between successive events is exponentially distributed with parameter λ.

b. Consider a Markov Chain with state space $S = \{0, 1, 2\}$ and the transition probability matrix given by

$$\begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ P = \begin{bmatrix} 0 & 1 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Find the stationary distribution of this chain. Is this chain periodic? Suppose the initial probability distribution is: $\pi_0(0) = a$, $\pi_0(1) = 0.5$, $\pi_0(2) = (0.5 - a)$ where $0 \leq a \leq 0.5$. Find the distribution π_1 (that is, distribution of X_1). For this chain, will π_n (that is, distribution of X_n) have a limit as $n \to \infty$.

$\pi_n = \pi_1$

...coming
...n values
...are purely
...ginary.

$$\boxed{P^n_{(i,i)} \to 0}$$

$\pi_{\gamma}(0) = a$

.5

E1 222 Stochastic Models and Applications
Test #3

Max. Marks: 40

21 Nov 2011

Answer ALL questions. All questions carry equal marks.

1. Consider a Markov Chain on $S = \{0,1,2,3,4,5,6,7\}$ with the following transition probabilities:

$$
\begin{array}{c|cccccccc}
 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
\hline
0 & 0.2 & 0.3 & 0 & 0 & 0.3 & 0.2 & 0 & 0 \\
1 & 0 & 0.3 & 0.4 & 0.3 & 0 & 0 & 0 & 0 \\
2 & 0 & 0.5 & 0 & 0.5 & 0 & 0 & 0 & 0 \\
3 & 0 & 0.2 & 0.6 & 0.2 & 0 & 0 & 0 & 0 \\
4 & 0.1 & 0 & 0 & 0.3 & 0.3 & 0.3 & 0 & 0 \\
5 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5 \\
6 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0.4 & 0.3 \\
7 & 0 & 0 & 0 & 0 & 0 & 0 & 0.5 & 0.5
\end{array}
$$

Rec Class $\{1,2,3\}$ $\{5,6,7\}$ one rec class

Determine which states are transient and which states are recurrent and also find all the different closed irreducible sets of recurrent states. Find the absorption probabilities from different transient states to different closed irredicible sets. Find a stationary distribution of the chain.

2. a. Let $X_1, X_2, \ldots$, be a sequence of *iid* random variables with density function $f(x) = 6x(1-x)$, $0 \le x \le 1$. Define $Z_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Does Z_n converge in probability? (Answer Yes/No along with a justification not exceeding two lines). If Z_n converges, what is the limit? $1/2$

b. Let $X_1, X_2, \ldots$ be a sequence of *independent* random variables with probability distribution given by: $P[X_n = n] = \frac{1}{2n^2}$, $P[X_n = -n] = \frac{1}{3n^2}$, $P[X_n = 0] = 1 - \frac{5}{6n^2}$. Does the sequence $\{X_n\}$ converge (i) in probability, (ii). almost surely, (iii). in r^{th} mean ?

3. a. Let $X_1, \ldots, X_n$ be independent random variables each having a geometric distribution with parameter 1/3. Let $S_n = X_1 + \ldots + X_n$. Find the characteristic function of S_n. Find $P[S_{10} \le 20]$ using central limit theorem approximation.

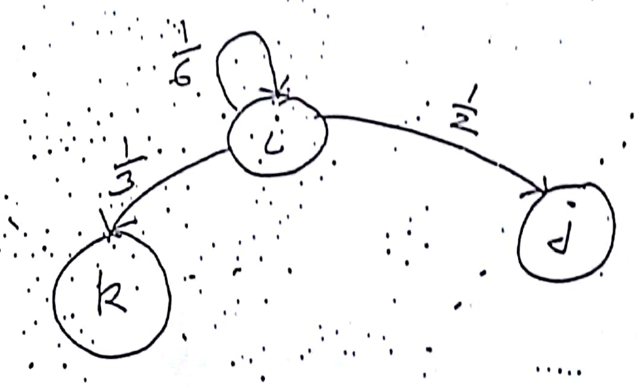
$(1-p)^n p$ $\quad (1-p)^{n-1} p$

b. Let X_n be a sequence of geometric random variables with parameter λ/n (where $\lambda < 1$). Let $Z_n = X_n/n$. Show that Z_n converge in distribution to an exponential random variable.

4. a. Explain why a finite Markov chain can not have a null recurrent state.

b. Given below is part of the state transition graph of a Markov chain, $\{X_n\}$. The chain is started in state i. Let (the random variable) N denote the first time instant when the chain leaves state i. Find the distribution of (i) N, and (ii) X_N.

Geometric R.V.



E1 222 Stochastic Models and Applications
Test #3

Max. Marks: 50

23 Nov 2010

Answer ALL questions. All questions carry equal marks.

1. Consider a Markov Chain on $S = \{0, 1, 2, 3, 4, 5, 6\}$ with the following transition probabilities:

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0.2 & 0.2 & 0.1 & 0 & 0.1 & 0.1 & 0.3 \\ 0 & 0.2 & 0 & 0.4 & 0 & 0.4 & 0 \\ 0 & 0 & 0.5 & 0 & 0.5 & 0 & 0 \\ 0 & 0.3 & 0 & 0.4 & 0 & 0.3 & 0 \\ 0 & 0 & 0.25 & 0 & 0.75 & 0 & 0 \\ 0 & 0.5 & 0 & 0.2 & 0 & 0.3 & 0 \\ 0.3 & 0.2 & 0.2 & 0 & 0 & 0 & 0.3 \end{bmatrix} \end{array}$$

C_1 $\{2, 4\}$ C_2 $\{1, 3, 5\}$

1, 2, 3, 4, 5

0, 6

Determine which states are transient and which states are recurrent and also find all the different closed irreducible sets of recurrent states. Find the absorption probabilities from different transient states to different closed irredicible sets.

2. In a system there are two switches each of which can be either OFF or ON. On day n, each switch would independently be ON with probability $(1 + a)/4$ where a is the number of switches that are ON during day $n - 1$. (Thus, for eaxmple, if both switches are OFF during day $n - 1$, then, on day n, each switch would independently be ON with probability 0.25). Taking the state to be the number of switches that are ON, represent this as a Markov chain and specify the state transition probabilities. For what fraction of days are both switches ON? Find S.D. $\Rightarrow \pi(2)$

22

3. a: Let $X_1, X_2, \ldots,$ be a sequence of *iid* random variables with density function $f(x) = 2e^{-2x}$, $x > 0$. Define $Z_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Does Z_n converge in probability? (Answer Yes/No along with a justification not exceeding two sentences) ~~If Z_n converges, what is the~~ limit? (Only say what the limit will be and provide a one sentence justification for the answer). Chibysh

S1
S2

b. Let $X_1, X_2, \ldots$ be a sequence of *independent* random variables with probability distribution given by: $P[X_n = n] = \frac{1}{2n}$, $P[X_n = -n] = \frac{1}{3n}$, $P[X_n = 0] = 1 - \frac{5}{6n}$. Does the sequence $\{X_n\}$ converge (i) in probability, (ii). almost surely, (iii). in r^{th} mean ?

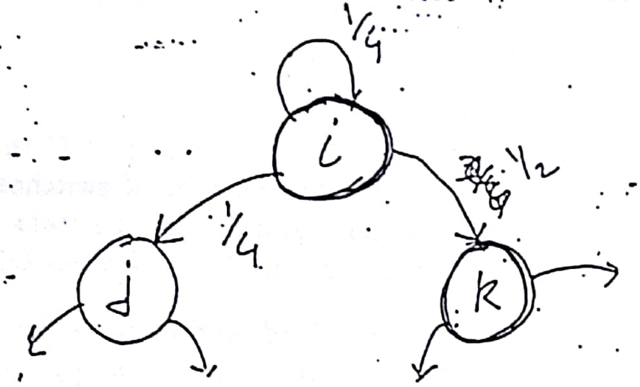
C.F 4. (a.) Let $X_1, \ldots, X_n$ be independent random variables each having a poisson distribution with parameter λ. Let $S_n = X_1 + \ldots + X_n$. Find the characteristic function of S_n. What is the approximation for $P[S_n \leq x]$ in terms of standard Normal distribution?

b. Let X be a random variable with mean μ and variance σ^2. We know σ^2 but do not know μ. Suppose we estimate μ by taking the sample mean of n *iid* realiztions of X. Discuss how we can decide on the number of observations, n, that we should use for our estimate. What would be a good indicator of the error in your final estimate.

5. a. Explain why a finite Markov chain can not have a null recurrent state.

(b.) Given below is part of the state transition graph of a Markov chain, $\{X_n\}$. The chain is started in state i. Let (the random variable) N denote the first time instant when the chain leaves state i. Find the distribution of (i). N, and (ii). X_N.

Geometr



EE 222 Stochastic Models and Applications

Test #3

Max. Marks: 50

25 Nov 2008

Answer ALL questions. All questions carry equal marks.

1. Consider a Markov Chain on $S = \{0, 1, 2, 3, 4, 5, 6\}$ with the following transition probabilities:

$$
\begin{array}{cc}
 & \begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
P = \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix}
0.125 & 0.25 & 0.25 & 0.125 & 0 & 0 & 0.25 \\
0 & 0.25 & 0 & 0.5 & 0 & 0.25 & 0 \\
0 & 0 & 0.25 & 0 & 0.75 & 0 & 0 \\
0 & 0.5 & 0 & 0.25 & 0 & 0.25 & 0 \\
0 & 0 & 0.75 & 0 & 0.25 & 0 & 0 \\
0 & 0.25 & 0 & 0.25 & 0 & 0.5 & 0 \\
0.2 & 0 & 0 & 0.2 & 0.4 & 0 & 0.2
\end{bmatrix}
\end{array}
$$

$\{1,3,5\}$
$\{2,4\}$

$0, 6.$ $\{1,3,5\}$, $\{2,4\}$

Determine which states are transient and which states are recurrent and also find all the different closed irreducible sets of recurrent states. Find the absorption probabilities from different transient states to different closed irredicible sets.

2. a. Let $X_1, X_2, \ldots,$ be a sequence of *iid* random variables with density function $f(x) = 2e^{-2(x-1)}$, $x \geq 1$. Define $Z_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Does Z_n converge in probability? (Answer Yes/No along with a justification not exceeding two sentences) If Z_n converges, what is the limit? Chebyshev

does it converge a.s.?

b. Let X be a random variable with mean μ and variance σ^2. We know σ^2 but do not know μ. Suppose we estimate μ by taking the sample mean of n *iid* realiztions of X. Discuss how we can decide on the number of observations, n, that we should use for our estimate.

3. a. Let $X_1, \ldots, X_n$ be independent random variables each having a geometric distribution with parameter p. Let $S_n = X_1 + \ldots + X_n$. ~~Find the characteristic function of S_n. What is the approximation~~ for $P[S_n \leq x]$ in terms of standard Normal distribution?

C·F

CLT

CLT

CLT

(b) Twenty numbers are rounded off to the nearest integer and then added. Assume that the individual round-off errors are independent and uniformly distributed over $(-0.5, 0.5)$. Find the (approximate) probability that this sum will differ from the sum of the original twenty numbers by more than 3. (You can give the final answer in terms of the distribution function of the standard normal distribution).

4: Let $X_1, X_2, \ldots$ be a sequence of *independent* random variables with probability distribution given by: $P[X_n = n] = P[X_n = 0] = \frac{1}{2}$. Let $Y_n = \prod_{k=1}^{n} X_k$, $n = 1, 2, \ldots$. Does the sequence Y_n converge to zero (i) in probability, (ii). almost surely, (iii). in r^{th} mean ? Suppose we change the distribution of X_n to $P[X_n = 1] = P[X_n = 0] = \frac{1}{2}$. Will the convergence properties of Y_n change?

5. (a) Let π be a stationary distribution of a Markov Chain. (i). Show that if $\pi(x) > 0$ and x leads to y then $\pi(y) > 0$. (ii). Suppose the chain has transition probabilities that satisfy the following: for some two states y and z, $P(x, y) = cP(x, z)$, $\forall x$, where c is a constant. Show that $\pi(y) = c\pi(z)$.

b. What are null recurrent states in a Markov Chain? Explain why a finite Markov chain can not have any null recurrent states.

$P(x,y) = c \cdot P(x,z)$ $\forall x$

$P(w,y) = cP(w,z)$ $\forall w$

$\pi(y) = \sum_{x} \pi(x) P(x,y)$

$\cancel{\pi(z) = \sum_{v} \pi(v) P(v,z)}$

$\pi(y) = \sum_{x} \pi(x) \cdot c \cdot P(x,z)$

$= c \left(\sum_{x} \pi(x) P(x,z) \right)$

if $\pi(x) > 0$

$\exists$ $x \to y$

$\rho_{xy} > 0$

$\pi(y) = \sum \pi(z$

EI 222 Stochastic Models and Applications
Test #3

Max. Marks: 50

22 Nov 2007

Answer ALL questions. All questions carry equal marks.

1. Consider a Markov Chain on $S = \{0, 1, 2, 3, 4, 5\}$ with the following transition probabilities:

$$P = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{bmatrix} 0.25 & 0 & 0 & 0.25 & 0 & 0.5 \\ 0 & 0.25 & 0.25 & 0 & 0.25 & 0.25 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0.25 & 0 & 0.25 & 0.5 & 0 & 0 \\ 0.5 & 0 & 0 & 0.25 & 0 & 0.25 \end{bmatrix} \end{array}$$

S_R $\{2\}$; $\{0, 3, 5\}$
S_T : 1, 4.

Determine which states are transient and which states are recurrent and also find all the different closed irreducible sets of recurrent states. Find the absorption probabilities from different transient states to different closed irredicible sets.

① ④
$\frac{5}{12}$ $\frac{1}{4}$
$\frac{7}{12}$ $\frac{3}{4}$

2. Consider a Markov Chain with $S = \{0, 1, 2\}$ and the transition probability matrix given by

$$P = \begin{array}{c|c} & \begin{matrix} 0 & 1 & 2 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \end{matrix} & \begin{bmatrix} 0 & 1 & 0 \\ 1-p & 0 & p \\ 0 & 1 & 0 \end{bmatrix} \end{array}$$

Yes

where $0 < p < 1$. Is the chain periodic? Suppose g is a real valued function defined on S with $g(0) = -1$, $g(1) = 0$ and $g(2) = +1$. Suppose the Markov chain has initial probability distribution given by $\pi_0(0) = (1-p)/2$, $\pi_0(1) = 1/2$, and $\pi_0(2) = p/2$. Find expected value of $g(X_n)$ as $n \to \infty$, if it exists. Will this be the same if the initial probability distribution is changed? Explain.

3. a. Let X, Y be *iid* random variables and let $Z = X - Y$. Show that $\phi_Z(t) = |\phi_X(t)|^2$ where ϕ_Z and ϕ_X are the characteristic functions of Z and X respectively.

e^{it}

$e^{itZ} = e^{it(X-Y)} = e^{itX} \cdot e^{-itY} = \phi_x(t) \cdot \phi$

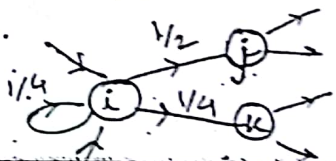
b. Let X be a random variable with mean μ and variance σ^2. We know σ^2 but do not know μ. Suppose we estimate μ by taking the sample mean of n *iid* realiztions of X. How can we quantify our confidence in this estimate.

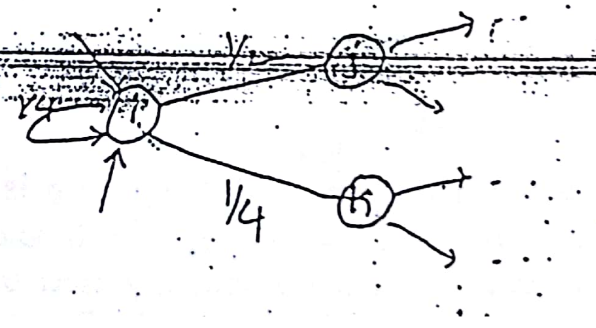
4. a. Let $X_1, X_2, \ldots$ be a sequence of *independent* random variables with probability distribution given by: $P[X_n = n] = P[X_n = 0] = \frac{1}{2}$. Let $Y_n = \prod_{k=1}^{n} X_k$, $n = 1, 2, \ldots$. Does the sequence Y_n converge (i). in probability, (ii). almost surely, (ii). in r^{th} mean ?

   b. Briefly exaplain (i). Strong law of large numbers, (ii). wide sense stationary stochastic process.

5. a. A transition probability matrix is said to be doubly stochastic if $\sum_i p_{ij} = 1$; $\forall j$. Let $\{X_n; n \geq 0\}$ be an irreducible Markov chain over a finite state space whose transition probability matrix is doubly stochastic. What can you say about the stationary probabilities of this chain.

   b. Given below is part of the state transition graph of a Markov chain X_n. Suppose at some time m the chain is in state i. Define the random variable T by $T = \min\{k > m : X_k \neq i\}$. Find the distribution of the random variables T and X_T.





E1 222 Stochastic Models and Applications

Test #3

Max. Marks: 50

18 Nov 2006

Answer ALL questions. All questions carry equal marks.

1. Consider a random process $\{Y(t),\ t \geq 0\}$ defined by $Y(t) = A\cos(t+B)$ where A, B are independent continuous random variables. The mean and variance of A are μ and σ^2 respectively. The density function of B is given by $f_B(b) = \frac{1}{2\pi},\ 0 \leq b \leq 2\pi$. Find $EY(t)$ and the autocorrelation function of $Y(t)$. Is this process mean ergodic?.

2. Let $\{N(t),\ t \geq 0\}$ be a Poisson process with rate λ. Let X_0 be a random variable independent of all $N(t)$ such that $P[X_0 = 1] = P[X_0 = -1] = 0.5$. Define a random process $\{X(t),\ t \geq 0\}$ by $X(t) = X_0(-1)^{N(t)}$. Find the mean and autocorrelation function of $X(t)$. Is this process wide-sense stationary?

3. Briefly explain (i). Strict-sense stationary process, (ii). wide-sense stationary process, (iii). Power Spectrai density, (iv). ergodicity.

4. a. Let $X_1, X_2, \ldots,$ be a sequence of *iid* random variables with density function $f(x) = e^{-x+\theta},\ x \geq \theta$. Define $Z_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Does Z_n converge in probability? If so, find the limit.

   b. Let $X_1, X_2, \ldots$ be a sequence of *independent* random variables with probability distribution as given by: $P[X_n = -n] = P[X_n = n] = \frac{1}{n^2}$ and $P[X_n = 0] = 1 - \frac{2}{n^2}$. Does the sequence converge (i). almost surely, (ii). in r^{th} mean ?

5. a. Let $X_1, \ldots, X_n$ be independent random variables each having a geometric distribution with parameter p. Let $S_n = X_1 + \ldots + X_n$. Find the characteristic function of S_n. What is the approximation for $P[S_n \leq x]$ in terms of standard Normal distribution?

   b. Candidates A and B are running for election and 65% of the electorate favour B. What is the probability that in a sample of size 200, at least one half will favour A?

LT

Q : $P(A \cap B \cap C) \geq$

2, 8.

E1 222 Stochastic Models and Applications

Test #1

Time: 90 mts
Max Marks:40

27 Sept 2011

Answer ALL questions. All questions carry equal marks.

1. a. Ninety eight percent of all babies survive delivery. Fifteen percent of all births involve a Cesarean section. When a Cesarean section is performed, the baby survives 96 percent of the time. If a random pregnant woman does not have a Cesarean section, what is the probability that her baby survives? .9835

   b. A rod of length 1 is broken at two random points. Find the probability that the three pieces so formed would make a triangle.

$P(S)=$
$P(C)\, S|C$
$P(\bar{C})\, P(S|\bar{C})$
All given, $P(S|\bar{C})$

2. a. Define the probability density function of a continuous random variable. Let a continuous random variable have density function given by

$$f_X(x) = Cx(2-x), \quad 0 < x < 2.$$

   Find value of C, Prob$[|X - 0.5| > 0.5]$, EX and variance of X. 1/5

   3/4 1/2 1

   b. Let X be a positive random variable having density function $f(x)$. If $f(x) \leq c$, $\forall x$, show that for any $a > 0$, $P[X > a] \geq 1 - ac$.

3. a. Let p be a number such that $0 < p < 1$ and let U be a random variable distributed uniformly over $[0,1]$. Let X be a random variable defined by

$$X = \text{Int}\left(\frac{\log(1-U)}{\log(1-p)}\right) + 1$$

   where $\text{Int}(x)$ is the largest integer smaller than or equal to x. Find the distribution of X.

   b. Let X, Y have joint density given by

$$f_{XY}(x,y) = ye^{-xy}, \quad x > 0,\ 0 < y < 1. \qquad ye^{-xy}$$

   Find the marginal density f_Y, the conditional density $f_{X|Y}$ and EY.

Doubt

$\frac{f_{XY}}{f_X}$ 1 $d_Y = 1$ same ye^{-xy} $1/2$

= $1/2e$

x, y
$(x+y)$
$(x+y) < 1-(x+y)$
$2(x+y) < 1$
$(x+y) < 1/2$
$1-(x+y)$

$= \frac{1}{2(b-a)\sqrt{y}}$

$\frac{(b-a)^2}{3}$

4\. a. Let X be uniformly distributed over $[a,\ b]$ and let $Y = (X - c)^2$ where $c \in [a,\ \frac{a+b}{2})$. Find density of Y and EY.

b. Let X_1 be uniform over $[0,\ 1]$ and X_2 be uniform over $[0,\ X_1]$. Find joint density of X, Y and marginal density of X. X_2

$\int_0^1 \int_0^x \frac{1}{x}\, dy\, dx = \int_0^1 \left(\frac{y}{x}\right) dx = 1$

E1 222 Stochastic Models and Applications
Test #1

Time: 90 mts
Max Marks:50

16 Sept 2010

Answer ALL questions. All questions carry equal marks.

1. a. It is known that 5% of women and 15% of men are colour-blind. What is the probability that a randomly selected person who is colour-blind is man? $3/4$

b. Two numbers are chosen at random from the interval [0, 1]. What is the probability that their difference is greater than 0.25? $\left(\frac{3}{4}\right)$

c. If A and B are independent events then show that A^c and B^c are independent.

2. a. Find the probability of (i). getting at least one head in two tosses of a fair coin, (ii). getting at least two heads in four tosses of a fair coin, (iii). getting at least three heads in six tosses of a fair coin. What is your guess about the probability of getting at least n heads in $2n$ tosses of a fair coin, for large n?

$\frac{2^{2n} + {}^{2n}C_n}{2^{2n+1}}$

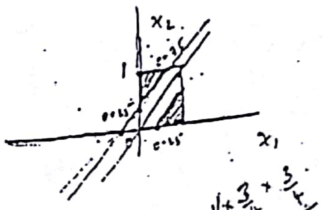
b. In a school, students went on a picnic in four buses. Different buses carried different number of students. Consider X, Y defined as follows. We select a student at random and value of X is the total number of students that travelled in the same bus as the selected student. We select a bus driver at random and the value of Y is the total number of students that travelled in the bus driven by the selected driver. Which of the following is true: $EX > EY$, or $EY > EX$ or $EX = EY$ or we can not compare EX and EY because the information provided is not sufficient. (Provide a very brief explanation of your answer in at most two sentences).

3. Let X be a continuous random variable with density function given by

$$f_X(x) = Cx(1-x), \quad 0 \le x \le 1.$$

$1/2$

Find the value of C. (6) Calculate Prob$[\,|X - 0.5| > 0.25\,]$. (10/32) Find EX and variance of X. (1/20) Let $Y = 2X^2 + 3X + 4$. Find EY. ($\frac{61}{10}$) Let $Z = X^2$. Find density function of Z. $3 - 3z^{1/2}$



$\frac{1}{2} + \frac{3}{4} \times \frac{3}{4}$

$= \frac{1}{16}$

4. Let X be a continuous random variable with density function

$$f_X(x) = \frac{e^{-x}x^n}{n!}, \quad x > 0$$

where n is a fixed integer. Show that

$$P[\, 0 < X < 2(n+1)\,] \geq \frac{n}{n+1}.$$

5. A match between two players A and B consists of a series of games. Each game is independently won by A with probability p and by B with probability $(1-p)$. When one of the players wins N games the match ends. Let X denote the number of games played before the match ends. (i). For $N = 4$, find the probability mass function of X, and (ii). for $N = 2$, find EX and show that this is maximized when $p = 0.5$.

n = 4.

n = 5

n = 6

n = 7.

$13 = \frac{6!}{2!\,4!} - \frac{3!}{2!} + 1$ Add 8 occur

$n+1 = E[X]$

$2E[X]$

E1 222 Stochastic Models and Applications
Test #1

Time: 90 mts
Max Marks: 50.

17 Sept 2009

Answer ALL questions. All questions carry equal marks.

1. a. Let $\Omega = [0,\ 1]$ with the usual probability assignment where probability of an interval is the length of the interval. Let $A_i = [2^{-i-1},\ 2^{-i})$, $i = 1, 2, \cdots$. Let $B = \cup_{i=1}^{\infty} A_i$. Find $P[B]$. 1/2

   b. A box contains four coins with probability of head for the i^{th} coin being $\frac{2i}{10}$, $i = 1, 2, 3, 4$. A coin is selected at random and tossed twice. Both times, it showed heads. What is the probability it is the third coin? 3/10.

2. Let X be a continuous random variable with density given by

$$f_X(x) = Cx(1-x),\ 0 < x < 1$$

Find the value of the constant C. Calculate $P[X > 0.5]$. Let $Y = (X-1)^2$. Find the density of Y. Calculate EY and $\mathrm{Var}(Y)$. $3(1-\sqrt{y})$

3. Let U be a continuous random variable with uniform distribution over $[0,\ 1]$. Define X by

$$X = \mathrm{Int}\left(\frac{\ln(1-U)}{\ln(1-p)}\right) + 1$$

where $\mathrm{Int}(x)$ is the integer part of x and p is a number such that $0 < p < 1$. Find the probability mass function of X.

4. Let X be a continuous random variable with density

$$f_X(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\ x^{a-1}(1-x)^{b-1},\ 0 < x < 1$$

where $a, b \in \Re$ with $a, b \geq 0$, are parameters. Here Γ is the gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1}e^{-x}dx.$$

$(a+b-1)!$

$1-x = t$

Recall that this function satisfies $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$. Show that the Expectation and Variance of X are given by

$$EX = \frac{a}{a+b} \quad \text{and} \quad \mathrm{Var}(X) = \frac{ab}{(a+b+1)(a+b)^2}$$

(Hint: Since f_X is given to be a density function we can use the fact that $\int_0^1 f_X(x)dx = 1$ for all a, b).

5. Answer the following very briefly. (You need to just write down the answer.. No explanations or justifications are necessary).

   a. X is a continuous random variable which is uniform over (0, 1). Let $Y = aX + b$. It is known that Y is uniform over (3, 5). Write down values for a, b.

   b. X is a random variable such that $\mathrm{Var}(X) = 0$. Sketch the distribution function of X. density

   c. In a general probability space, what is the maximum possible number of mutually exclusive events such that each of the events has probability greater than or equal to 0.2. 5

   d. In a school, students went on a picnic in four buses. Different buses carried different number of students. Consider X, Y defined as follows. We select a student at random and value of X is the total number of students that travelled in the same bus as the selected student. We select a bus driver at random and the value of Y is the total number of students that travelled in the bus driven by the selected driver. Which of the following is true: $EX > EY$, or $EY > EX$ or $EX = EY$ or we can not compare EX and EY because the information provided is not sufficient: EX

$f_x(x) = c$ No. of students $x = \frac{y-b}{a}$

$\frac{3-b}{a} = 0$

E1 222 Stochastic Models and Applications.
Test #1

Time: 90 mts
Max Marks:50

22 Sept 2008

Answer ALL questions. All questions carry equal marks.

5

1. a. Two players A and B alternately roll a dice starting with A. The first player to get a six wins. Find the probability that A is the winner. $6/11$

   b. A college is composed of 70% men and 30% women. It is known that 5% of the women and 20% the men are left-handed. What is the probability that a student who is left-handed is a man? $\frac{14}{15.5}$

10

2. Let X be a continuous random variable with density function

$$f_X(x) = C(x-3)^2, \quad 0 \leq x \leq 3. \qquad c = 1/9$$

Find the value of C and calculate EX and variance of X. $3/4$ $27/80$

Let $Y = (X-2)(X-1)$. Find Prob$[Y \leq 0]$. $7/27$ $1 \leq X \leq 2$

$y < -1/8$

$\frac{1}{8} < y \leq 1$

3. Let X be a random variable uniformly distributed over $[0, 2]$. Let $Y = 2X^2 - 5X + 3$. Find the distribution function of Y and EY. $2/3$

$y \leq 3$

$y > 3$

4. Consider the random experiment of five independent tosses of a fair coin. In any outcome (of this random experiment) we say a change-over has occurred at the i^{th} toss if the result of the i^{th} toss differs from that of the $(i-1)^{th}$ toss. Let X be a random variable whose value is the number of change-overs. For example, if the outcome of the random experiment is $HTTHH$ then the value of X would be 2. Note that the minimum value of X is 0 (e.g., when the outcome is $HHHHH$) and the maximum value of X is 4 (e.g., when the outcome is $HTHTH$). Find the probability mass function of X.

5. Consider the following algorithm for generating (samples of) a random variable X. In the following λ is a positive real number.

   1. Generate a random number U which is uniform over $(0, 1)$.

$2(x - \frac{5}{4})^2$

3

$F_X(y) =$

$\left[\frac{\sqrt{8y+1}}{4}, \frac{3}{4}\right]$

2. Set $i = 0$, $p = e^{-\lambda}$ and $F = p$.
3. If $U < F$ then set $X = i$, output X and stop. Otherwise go to step 4.
4. Set $p = \lambda p/(i+1)$, $F = F + p$ and $i = i + 1$.
5. Go to step 3.

What is the mass function of X as generated by the above algorithm? Let N denote the number of times we execute step 3 in the above algorithm. Note that N is a random variable because it depends on U which is random. Calculate EN.

$N = X+1$

$EN = EX + 1$

EI 222 Stochastic Models and Applications

Test #1

Time: 90 mts

Max Marks: 50

20 Sept 2007

Answer ALL questions. All questions carry equal marks.

Check

1. a. Two points are chosen at random from the interval [0, 1]. Find the probability that the smaller of the two is less than 0.5. 3/8

   b. A man has in his pocket a fair coin and a two-headed coin (i.e., a coin which always lands heads). He selects one of the coins at random. (a). When the selected coin is tossed it showed heads. What is the probability that it is the fair coin? (b). The same coin is tossed again and it showed heads again. Now what is the probability that it is the fair coin? (c). The same coin is tossed a third time and it showed tails. Now, what is the probability that it is the fair coin?

   1/3

   1/5

   1

2. Consider the random experiment of two independent rolls of a dice. Let X denote the maximum of the two numbers obtained. Find the probability mass function of X and EX.

3. Let X be a continuous random variable with density function

$$f_X(x) = Cx(2-x), \quad 0 \le x \le 2.$$

Find the value of C and calculate EX and variance of X. Let $Y = 2X^2 + 3X - 1$. Find EY.

4. Let U be a continuous random variable with uniform distribution over $[0, 1]$. Define X by

$$X = \mathrm{Int}\left(\frac{\ln(1-U)}{\ln(1-p)}\right) + 1$$

where $\mathrm{Int}(x)$ is the integer part of x and p is a number such that $0 < p < 1$. Find the probability mass function of X.

5. Let X be a random variable uniformly distributed over $[a, b]$. Let $Y = (X-c)^2$ where c is a constant such that $c \in (\frac{a+b}{2}, b)$. Find the density of Y and EY.

$$\frac{\ln(1-u)}{\ln(1-p)}$$

$$0 \le \frac{\ln(1-u)}{\ln(1-p)} < 1$$

$$0 \ge \ln(1-u)$$

$$x \ln(1-p)$$

E1 222 Stochastic Models and Applications
Test #1

Time: 90 mts
Max Marks: 50

22 Sept 2006

Answer ALL questions. All questions carry equal marks.

1. a. Two players A and B alternately roll a dice starting with A. The first player to get a six wins. Find the probability that A is the winner.

   b. A coin, whose probability of coming up heads is p, is tossed n times. Let X be the absolute value of difference of heads and tails. Write down the probability mass function of X.

2. Smoking increases risk of cancer. Let us suppose there is 1% chance that a random person in the population would have cancer while the chance becomes 20% if it is given that this random person is a smoker. Suppose there is a test such that it is positive for 95% of the people having cancer and it is also positive for 10% of the people who do not have cancer. Smoking has no effect on the test result. Suppose the test result for a smoker is positive. What is the probability that he has cancer. Suppose the test done on a random person is positive. What is the probability that he has cancer.

3. a. Let X be a continuous random variable with density function

$$f_X(x) = Cx^2, \quad 0 \le x \le 1.$$

   Find the value of C and calculate EX and variance of X.

   b. Children from a school went for a picnic in four buses. The number of students in different buses are different. Define two random variables, X, Y, as follows. A student is selected at random. The number of students who travelled in the same bus as this student is the value of X. One of the four bus drivers is selected at random. The number of students who travelled in the bus driven by that driver is the value of Y. Can you say which of EX and EY would be larger? Explain.

$n \quad n-1 \quad n-2 \quad n-3$
$0 \quad 1 \quad 2 \quad 3$

$n =$ odd $x = 1, 3 \ldots$

$1, \ldots\ldots n-5\ n-3\ n-1$

$n = e$

4. Suppose an urn has r red balls and b black balls. A ball is drawn at random. Then the drawn ball along with c balls of the same colour as the drawn ball are added to the urn where $c > 0$ is a constant. This procedure is repeated indefinitely. Let R_n denote the event that at the n^{th} draw the ball drawn is red and similarly let B_n denote the event that the ball drawn is black at the n^{th} draw. Calculate $P(R_2)$ and $P(B_2)$. Guess the value of $P(R_n)$ for any n.

5. Let X be a random variable uniformly distributed over $[a, b]$. Let $Y = (X - c)^2$ where c is a constant such that $c \in (a, \frac{a+b}{2})$. Find the density of Y.

$X = U[0,1]$

$y = ax + b$

$3x + 5 . 5$

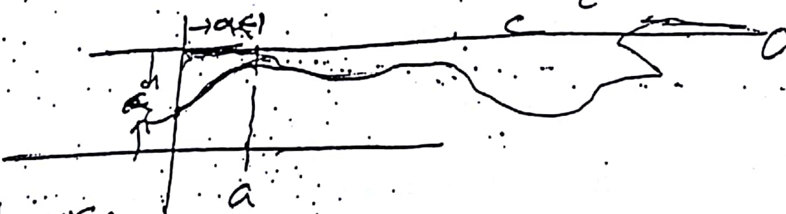
$F_X\left(\frac{y-b}{a}\right) \le y$

$\frac{1}{2}\int_3$

$f_X(x)$ $\frac{1}{2}$

$P(ax+b \le y)$

$ax+b \le y$



$X \le \frac{y-b}{a}$

$3 \le y \le 5$

$f_X(x \le a)$

$0 \le \frac{y-b}{a}$

$0 \le \frac{y-b}{a} \le 1$

$P[X \ge a]$

$= 1 - P[X \le a]$

$b \le y \le c$

$P[X^2 \le a] \le ac$

EI 222 Stochastic Models and Applications
Test #1

Time: 90 mts
Max Marks: 50

21 Sept 2005

Answer ALL questions. All questions carry equal marks.

1. a. Ninety eight percent of all babies survive delivery. Fifteen percent of all births involve a Cesarean section. When a Cesarean section is performed, the baby survives 96 percent of the time. If a random pregnant woman does not have a Cesarean section, what is the probability that her baby survives?

b. A man has in his pocket a fair coin and a two-headed coin (i.e., a coin which always lands heads). He selects one of the coins at random. (i). When the selected coin is tossed twice, it showed heads on both tosses. What is the probability that it is the fair coin? (ii). The same coin is tossed a third time and it showed tails. Now, what is the probability that it is the fair coin?

2. Suppose an urn has r red balls and b black balls. A ball is drawn at random. Then the drawn ball along with c balls of the same colour as the drawn ball are added to the urn where $c > 0$ is a constant. This procedure is repeated indefinitely. Let R_n denote the event that at the n^{th} draw the ball drawn is red and similarly let B_n denote the event that the ball drawn is black at the n^{th} draw. Calculate $P(R_2)$ and $P(B_2)$. Guess the value of $P(R_n)$ for any n.

3. Let X be a continuous random variable with density given by

$$f_X(x) = C(8x - 4x^2), \quad 0 < x < 2.$$

3/16 32/5 6/5

Find the value of C and Prob$[X \geq 1]$. Let $Y = X^2$. Find EY.

4. Let p be a number such that $0 < p < 1$ and let U be a random variable distributed uniformly over $[0, 1]$. Let X be a random variable defined by

$$X = \text{Int}\left(\frac{\log(1-U)}{\log(1-p)}\right) + 1$$

Integration

1. ↳ proper

. Improp

where Int(x) is the largest integer smaller than or equal to x. Find the distribution of X.

5. a. Let X be a positive random variable having density function $f(x)$. If $f(x) \leq c$, $\forall x$, show that for any $a > 0$,

$$P[X > a] \geq 1 - ac$$

b. Let X be a continuous random variable having uniform distribution over (0, 1). Find a function g such that $Y = g(X)$ is uniformly distributed over (3, 5). Find expectation and variance of Y.

$$f(y) = \frac{f(g^{-1}(y))}{\left|\frac{\partial y}{\partial x}\right|}$$

$$\frac{1}{2} \left|\frac{\partial y}{\partial x}\right| = f(g^{-1}(y))$$

$3 \leq Y \leq 5$

$$g'(y) = (f(g^{-1}(y)))$$

$$g'(y) = 2$$

$3 \leq y \leq 5$

$$Y = 2X + b$$

$$y = 2x + b$$

$P[\cancel{f(x) \leq c}]$ $\quad P(X > a) = 1 - P(X \leq a)$

$= 1 -$

$E(X^3) = \quad X(X-1)(X-2) \quad k! \quad \Sigma \; e^{(k t - \lambda)} \; \frac{\lambda^k}{k!}$

$X^2(X-1)$

$1 \; \Sigma \; k^2(k-1)$

$E\boxed{X^3 - X^2}$

$e^{kt} \cdot \frac{\lambda^k}{k!}$

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

E1 222 Stochastic Models and Applications
Test I

Time: 90 minutes

Date: 16 Sept 2016

Max. Marks:40

Answer ALL questions. All questions carry equal marks

1. a. Let A and B be two events such that $P(A) = 0.3$, $P(B) = 0.2$ and $P(A|B) = 0.4$. Find $P(A \cap B), P(A \cup B), P(A|A \cup B)$ and $P(A|A \cap B)$.

   b. A fair dice is rolled repeatedly. What is the probability that a 2 would appear before a 5 or 6?

   c. Two numbers are chosen at random from the interval $[0, 1]$. What is the probability that their sum is less than 0.5.

$x + y = 0.5$

$(0, 0.5)$

$(0.5 \;\; 0)$

$\frac{1}{2}\, 0.5 \times 0.5$

2. a. Let X be a continuous random variable which is uniform over $[a, b]$. Let c be such that $a < c < (a+b)/2$. Let $Y = (X - c)^2$. Find density of Y.

   b. A man has two packets of chewing gum, one in his left pocket and another in his right pocket. Everytime he wants a gum, he randomly chooses one of the packets and takes out one. To start with each packet has N pieces. Calculate the probability that when he finds that the chosen packet is empty, the other one contains x pieces.

$\frac{1}{2}$

3. a. Consider the following procedure generating a random variable Y. (Here, $\lambda > 0$ is a constant).

   1. Generate X uniformly over $[0, 1]$.
   2. Set $k = 0$; $S = e^{-\lambda}$.
   3. If $(X \le S)$ then {set $Y = k$; EXIT}

   Else {set $k = k + 1$; $S = S + \frac{\lambda^k}{k!}e^{-\lambda}$; go back to step 3}

   Note that after the first two steps, the else part of the 'If statement' is repeatedly executed till the condition $(X \le S)$ is satisfied and the process exits with a value for Y. Let N denote the number of times the condition in the 'IF statement' is executed. Find EN. What is the distribution of Y?

$F(2c - a)$

$c - a + c$

$c + c - a$

$2c - a$

$(a-c)^2$

$(2c - a - c)^2$

$\frac{2c - a - a}{b - a}$

$\frac{-a - b - 2c + 2a}{b - a}$

$c + \sqrt{y} - a \; - \; c - \sqrt{y} + a \; (c-a)^2$

$c - \sqrt{y}$

1

$y = 0 \; 1$

$\frac{1}{a}$

$c - x + c$

$-x^2 \quad c + x - c$

$y^{\frac{1}{2}}$

$x - c$

$2 \cdot \frac{1}{2} y^{-\frac{1}{2}}$

$-2\sqrt{y}$

b. Let X be Poisson radom variable with parameter λ. Find EX^3.

4. a. Let X, Y, Z be continuous random variables with joint density given by

$$f_{XYZ}(x, y, z) = \frac{1}{xy}, \ 0 \le z \le y \le x \le 1.$$

Calculate f_{YZ}, the joint density of Y, Z.

b. Let X, Y be discrete random variables with joint mass function

$$f_{XY}(x, y) = \frac{1}{Nx}, \ x, y \in \{1, \cdots, N\}, \ y \le x$$

where N is a positive integer. Calculate EY.

$EY = \sum y f_Y(y)$

$f_Y(y) = \sum_{1}^{N} \frac{1}{Nx}$ $x >$

$P(2c-a \le X \le b) = F_X(b) - F_X(2c-a)$

$= \frac{b-a}{b-a} - \frac{2c-a-a}{b-a}$

$= 1 - \frac{2c+a+a}{b-a}$

$2c-a \le x \le b$ $\quad y \le x < 1$

$Y = (X-c)^2$

$(c-a)^2 \le Y \le (b-a)^2$

$(c-a) \le \sqrt{Y} \le (b-a)$

$2c-a \le \sqrt{Y}+c \le b-a+c$

$Y = (X-c)^2$

$\sqrt{Y}+c$

$P(Y \le y)$

$P((X-c)^2 \le y)$

$P($

$\frac{b-a+c-2a}{b-a} + \frac{2c-a-a}{b-a}$

$\frac{b-2a+c+2c-2a}{b-a}$

$\int_{x}^{y} \int_{0}^{y} -\frac{\ln x}{y} \, dz \, dy$

$-\frac{\ln x}{y}$

$F_X(b) - F_X(2c-a)$

$\frac{b-a}{b-a} - \frac{2c-a-a}{b-a}$

$1 - 2c-a-a$

$a-2c+b$

$\frac{y-a}{b-a} - \frac{2c-2a}{b-a}$

$\frac{y-a-2c+2a}{1}$

$\int -\ln y$

1+

-1

E1 222 Stochastic Models and Applications
Test II

Time: 90 minutes Max. Marks:40
Date: 24 Oct 2016

Answer ALL questions. All questions carry equal marks

1. Let X, Y have joint density given by

$$f_{XY}(x, y) = \lambda^2 e^{-\lambda y},\ 0 \le x \le y < \infty$$

Find the marginal densities of X, Y, $E[X|Y]$ and $\mathrm{Cov}(X, Y)$.

$Z = 2X + 3Y$
$W = 2X - 3Y$
$Z + W = 2X$

2. a. Let A, B be two events and I_A, I_B be the indicator random variables of these two events. Let $X = \max(I_A, I_B)$ and $Y = \min(I_A, I_B)$. Find the joint probability mass function of X, Y and $\mathrm{Cov}(X, Y)$.

   b. Let X, Y be *iid* random variables with uniform density over $(0, 1)$. Let $Z = 2X + 3Y$. Find the density of Z.

3. a. Let X be a binomial random variable with parameters n and p. Let $Y = \max(0,\ X - 1)$. Show that $EY = np - 1 + (1 - p)^n$.

   b. A total of m keys are to be put in n boxes with each key independently being put in box-i with probability p_i. (Note that $p_i \ge 0, \forall i$ and $\sum_{i=1}^{n} p_i = 1$). Each time a key is put in a non-empty box, we say a collison has occurred. Find the expected number of collisons.

$-p)^m = EY$

4. a. Let $X_1, X_2, \cdots$ be *iid* random variable each being uniform over $(0,\ 1)$. Let

$$N = \min\{n\ :\ n \ge 2,\ X_n > X_{n-1}\}$$

Show that $\mathrm{Prob}[N > n] = \frac{1}{n!}$. Find EN.

   b. Consider the game discussed in the class where N men put all their mobiles in a heap and then everyone chooses a mobile at random. Let X denote the number of persons that got their own mobiles. Show that $\mathrm{Var}(X) = 1$.

$2X = Z + W$

$6Y = Z - W$

$Z = W + 6Y$

E1 222 Stochastic Models and Applications
Test III

Time: 90 minutes
Date: 17 Nov 2016

Max. Marks:40

Answer **ALL** questions. All questions carry equal marks

1. a. Consider a 3-state Markov chain with state space $S = \{0, 1, 2\}$. Let the transition probability matrix be

$$P = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

Is the chain irredicible? Is the chain periodic and, if so, what is the period? Find the stationary distribution of the chain.

b. Define transient and recurrent states in a Markov chain. When is a Markov chain said to be irreducible.

2. a. Let $\{X_n,\ n \geq 0\}$ be a Markov Chain. Let s_0, s_1, s_2 be some specific three states. Suppose the probabilities of transition out of s_0 are given by: $P(s_0, s_0) = 0.3; P(s_0, s_1) = 0.2; P(s_0, s_2) = 0.5$. Suppose the chain is started in s_0. Let T denote the first time instant when the chain left state s_0. (That is, $T = \min\{n : n \geq 1,\ X_n \neq s_0\}$). Find the distribution of T and X_T.

b. Define positive recurrent and null recurrent states in a Markov chain. Explain why a finite Markov chain cannot have a null recurrent state.

3. a. Let $X_1, X_2, \cdots, X_n, \cdots$ be *iid* random variables with density function $f(x) = e^{-x+\theta}, x > \theta > 0$. let $Y_n = \frac{1}{n}\sum_{i=1}^{n} X_i$. Show that Y_n converges in probability to $(1 + \theta)$.

b. Let $X_1, X_2, \cdots$ be a sequence of independent random variables with

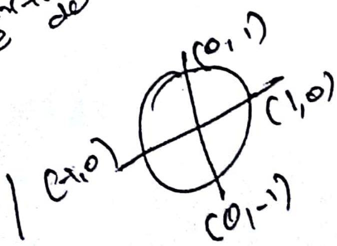
$$\Pr[X_n = \sin(n\pi/2)] = \frac{1}{3n}; \Pr[X_n = \cos(n\pi/2)] = \frac{1}{6n}; \ \Pr[X_n = 0] = \frac{2n-1}{2n}$$

Does this sequence converge (i) in probability, (ii) with probability one, (iii) in the r^{th} mean?

$\int x^2 e^{-x+\theta} dx$

$\int e$

$\int_\theta^\infty x^2 e^{-x+\theta} dx$



$\sin(0)$

$(1/3)^{m-1} \cdot \frac{1}{5} + (\frac{1}{3})(\frac{1}{2})$

$(\frac{1}{3})^{m-1}$

$n = 1$

$\frac{2}{10} \cdot \frac{1}{4}$

$\frac{2}{5} + \frac{1}{4} + \frac{9}{20}$

$\frac{8 + 5 + 9}{20}$

$\int x e^{-x+a} dx \quad e^{a}\int_0^\infty x e^{-x} dx = e^{a}\left[x \frac{e^{-x}}{-1}\Big|_0^\infty + \int_0^\infty e^{-x} dx\right]$

$e^{a}\left[0 + \frac{e^{-x}}{-1}\Big|_0^\infty\right]$

$e^{a}[1] =$

4. a. Suppose X is normal with mean μ and variance 1. We do not know μ. We are estimating it by taking sample average of n *iid* samples. Explain how we can use central limit theorem to decide on the value of n.

   b. Let $X_1, X_2, \cdots$ be a sequence of *iid* exponential random variables with parameter λ. Let $S_n = \frac{1}{n}\sum_{i=1}^{n} X_i^3$. Does the sequence S_n converge almost surely? If so, what is the limit?

$-a \le \frac{S_n - n\mu}{\sqrt{n}} \le a$

$-a \le \frac{S_n}{\sqrt{n}} - \frac{n\mu}{\sqrt{n}} \le a$

$\frac{a}{\sqrt{n}} \le \frac{S_n}{n\sqrt{n}} - \frac{\mu}{\sqrt{n}} \le \frac{a}{\sqrt{n}}$

$-\frac{a}{\sqrt{n}} \le$

- Uniform: Parameters: $a, b,\ b > a$.

$$f_X(x) = \frac{1}{b-a},\ a \le x \le b.$$

$EX = \frac{a+b}{2}$, $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$, $M_X(s) = \frac{e^{sb}-e^{sa}}{s(b-a)}$.

- exponential: Parameter: $\lambda > 0$.

$$f_X(x) = \lambda e^{-\lambda x},\ x \ge 0.$$

$EX = \frac{1}{\lambda}$, $\mathrm{Var}(X) = \frac{1}{\lambda^2}$, $M_X(s) = \frac{\lambda}{\lambda - s}$, $s < \lambda$.

- Normal: Parameters: $\sigma > 0, \mu$.

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}},\ -\infty < x < \infty$$

$EX = \mu$, $\mathrm{Var}(X) = \sigma^2$, $M_X(s) = exp[\mu s + \frac{1}{2}\sigma^2 s^2]$.

- gamma: Parameters: $\alpha, \lambda > 0$.

$$f_X(x) = \lambda e^{-\lambda x}(\lambda x)^{\alpha-1}/\Gamma(\alpha), x \ge 0$$

$EX = \frac{\alpha}{\lambda}$, $\mathrm{Var}(X) = \frac{\alpha}{\lambda^2}$, $M_X(s) = \left(\frac{\lambda}{\lambda-s}\right)^{\alpha}$.

$\frac{d}{ds}\frac{\lambda}{\lambda - s}$

$\lambda \cdot -\frac{1}{(\lambda-s)^2}(-1)$

$\frac{\lambda}{(\lambda-s)^2}$

$\frac{\lambda(-2)}{(\lambda-s)^3}(-1)$

$\frac{2\lambda}{(\lambda-s)^3}$

$\frac{2\lambda(-3)}{(\lambda-s)^4}(-1)$

$\frac{6\lambda}{(\lambda-s)^4}$

$\frac{S_n - n\mu}{\sigma\sqrt{n}}$

$\frac{S_n - n\mu}{\sqrt{n}}$

$EX^2 - (EX)^2$

$\frac{2}{\lambda^2} - \frac{1}{\lambda^2}$

$P(|\bar{X}_n - \mu| > \epsilon) < \delta$

$\frac{2\lambda}{\lambda^3}$

$\frac{2}{\lambda^2} + \frac{1}{\lambda^2}$

$P(|\bar{X}_n$

$P(|\frac{S_n}{n} - \mu| > \epsilon) < var$

$\phi\left(\frac{a}{\sqrt{n}}\right) + \phi\left(\frac{a}{\sqrt{n}}\right)$

$2\phi($