Solutions for Test 2: Computational Linear Algebra

Problem 1 (points: 5)

Let $\mathbb{V}$ be a finite-dimensional vector space over $\mathbb{C}$. Prove that $\mathcal{L}(\mathbb{V},\mathbb{C})$ has finite dimension by explicitly constructing basis for $\mathcal{L}(\mathbb{V},\mathbb{C})$. Further prove that $\dim \mathcal{L}(\mathbb{V},\mathbb{C}) = \dim \mathbb{V}$.

SOLUTION:

Let $\dim \mathbb{V} = n$ and $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots \boldsymbol{v}_n$ be basis of vector space $\mathbb{V}$. Define $f_1, f_2, \ldots f_n \in \mathcal{L}(\mathbb{V},\mathbb{C})$ as,

$$f_i(\boldsymbol{v}_j) = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise} \end{cases}$$

and extend f_i to $\mathbb{V}$ by linearity.

To show, $\{f_1, f_2, \ldots f_n\}$ is a linear independent set, we need to prove, if $\sum_{i=1}^n c_i f_i = \mathtt{0}$ where $\mathtt{0} \in \mathcal{L}(\mathbb{V},\mathbb{C})$ is zero function, then $c_i = 0$ for all $i \in \{1, 2, \ldots n\}$.

In particular, for $j \in \{1, 2, \ldots n\}$,

$$\Big(\sum_{i=1}^n c_i f_i\Big)(\boldsymbol{v}_j) = 0.$$

This implies $0 = \sum_{i=1}^n c_i f_i(\boldsymbol{v}_j) = c_j$.

To show, for all $f \in \mathcal{L}(\mathbb{V},\mathbb{C})$, there exists $c_1, c_2 \ldots c_n \in \mathbb{C}$ such that $f = \sum_{i=1}^n c_i f_i$.

Given any $\boldsymbol{x} \in \mathbb{V}$, we can write $\boldsymbol{x} = \sum_{k=1}^n \alpha_k \boldsymbol{v}_k$, where $\alpha_k = f_k(\boldsymbol{x})$. By linearity,

$$f(\boldsymbol{x}) = \sum_{k=1}^n \alpha_k f(\boldsymbol{v}_k) = \sum_{k=1}^n c_k f_k(\boldsymbol{x}) = \Big(\sum_{k=1}^n c_k f_k\Big)(\boldsymbol{x}),$$

where $c_k = f(\boldsymbol{v}_k)$ for all $k \in \{1, 2, \ldots n\}$. Thus $\{f_1, f_2 \ldots f_n\}$ forms the basis for $\mathcal{L}(\mathbb{V},\mathbb{C})$. Hence, $\dim\,(\mathcal{L}(\mathbb{V},\mathbb{C})) = n = \dim\,(\mathbb{V})$.

Problem 2 (points: 5)

Let $\mathbb{V}$ be a vector space and $\ell_1, \ldots, \ell_n \in \mathcal{L}(\mathbb{V})$. Prove that the composition $\ell_1 \circ \ell_2 \circ \cdots \circ \ell_n$ is one-to-one if and only if $\ell_1, \ldots, \ell_n$ are one-to-one.

SOLUTION: We use following characterization of one-to-one linear function. Given $\ell \in \mathcal{L}(\mathbb{V})$, $\ell(\boldsymbol{x}) = 0 \Rightarrow \boldsymbol{x} = \boldsymbol{0}$.

Proof for $\ell_1, \ldots, \ell_n$ are one-to-one $\Rightarrow \ell_1 \circ \ell_2 \circ \cdots \circ \ell_n$ is one-to-one:

Given that $\ell_i(\boldsymbol{x}) = 0 \Rightarrow \boldsymbol{x} = \boldsymbol{0}$ for all $i \in \{1, 2, \ldots n\}$, we need to prove $\ell_1 \circ \ell_2 \circ \cdots \circ \ell_n(\boldsymbol{x}) = 0 \Rightarrow \boldsymbol{x} = \boldsymbol{0}$.

Suppose there exists $\boldsymbol{x} \in \mathbb{V}$ such that $\ell_1 \circ \cdots \circ \ell_n(\boldsymbol{x}) = 0$. Since ℓ_1 is one-to-one, we get $\ell_2 \circ \cdots \circ \ell_n(\boldsymbol{x}) = 0$. Further since ℓ_2 is one-to-one, $\ell_3 \circ \cdots \circ \ell_n(\boldsymbol{x}) = 0$. Continuing in the same way we get $\ell_n(\boldsymbol{x}) = 0$. Since ℓ_n is one-to-one, $\boldsymbol{x} = \boldsymbol{0}$.

Proof for $\ell_1 \circ \ell_2 \circ \cdots \circ \ell_n$ is one-to-one $\Rightarrow \ell_1, \ldots, \ell_n$ are one-to-one:

We will prove contrapositive statement. If some function in $\ell_1, \ldots, \ell_n \in \mathcal{L}(\mathbb{V})$ is not one-to-one, then $\ell_1 \circ \ell_2 \circ \cdots \circ \ell_n$ is not one-to-one.

Let p be the minimal element such that ℓ_p is not one-to-one. Thus $\ell_1 \circ \cdots \circ \ell_{p-1}$ is one-to-one by the statement, if $\ell_1, \ldots, \ell_{p-1}$ are one-to-one, then $\ell_1 \circ \cdots \circ \ell_{p-1}$ is one-to-one (proved above). Suppose ℓ_p is not one-to-one, there exists $\boldsymbol{y} \in \mathbb{V}$ such that $\boldsymbol{y} \neq \boldsymbol{0}$ and $\ell_p(\boldsymbol{y}) = 0$. Since $\ell_1 \circ \cdots \circ \ell_{p-1}$ is surjective, there exists $\boldsymbol{x} \in \mathbb{V}$ such that $\ell_1 \circ \cdots \circ \ell_{p-1}(\boldsymbol{x}) = \boldsymbol{y}$ and $\boldsymbol{x} \neq \boldsymbol{0}$. Thus there exists $\boldsymbol{x} \in \mathbb{V}$ such that $\boldsymbol{x} \neq \boldsymbol{0}$ but $\ell_1 \circ \ell_2 \circ \cdots \circ \ell_n(\boldsymbol{x}) = 0$. Thus $\ell_1 \circ \ell_2 \circ \cdots \circ \ell_n$ is not one-to-one.

Problem 3 (points: 5)

Let $\mathbf{A} \in \mathbb{R}^{n\times n}$ be a non-zero matrix and suppose that there exist $k \geqslant 1$ such that $\mathbf{A}^k$ is the zero matrix. If k_0 is the smallest such index, then show that k_0 cannot be larger than n.
SOLUTION:
Suppose k_0 is the smallest such index such that $\mathbf{A}^{k_0} = \mathbf{0}$, there exists $\boldsymbol{x} \in \mathbb{R}^n$ such that $\mathbf{A}^{k_0-1}\boldsymbol{x} \neq \mathbf{0}$. We will first show that set of vectors $S = \{\boldsymbol{x}, \mathbf{A}\boldsymbol{x}, \ldots \mathbf{A}^{k_0-1}\boldsymbol{x}\}$ is linearly independent. Given that

$$c_0\boldsymbol{x} + \sum_{i=1}^{k_0-1} c_i \mathbf{A}^i \boldsymbol{x} = \mathbf{0}, \tag{1}$$

we need to prove that $c_i = 0$ for all $i \in \{0, 1, \ldots k_0 - 1\}$. Multiplying (1) by $\mathbf{A}^{k_0-1}$, we get $c_0\mathbf{A}^{k_0}\boldsymbol{x} = \mathbf{0}$ which implies $c_0 = 0$. Further, multiplying (1) by $\mathbf{A}^{k_0-2}$ and using fact that $c_0 = 0$, we get $c_1\mathbf{A}^{k_0}\boldsymbol{x} = \mathbf{0}$ which implies $c_1 = 0$. Similarly we can prove $c_i = 0$ for all $i \in \{0, 1, \ldots k_0 - 1\}$. Thus set S is linearly independent whose cardinality is k_0. But there can be only n linearly independent vectors in $\mathbb{R}^n$. Hence, $k_0 \leqslant n$.

Problem 4 (points: 5+2 = 7)

Let $\ell : \mathbb{R}^n \to \mathbb{R}$ be such that $\ell(\mathbf{0}) = \mathbf{0}$, and for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $0 \leqslant c \leqslant 1$,

$$\ell(c\boldsymbol{x} + (1-c)\boldsymbol{y}) = c\ell(\boldsymbol{x}) + (1-c)\ell(\boldsymbol{y}). \tag{2}$$

a) Show that $\ell \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$.
b) Now, suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is such that (2) holds. Show that f must be of the form $f(\boldsymbol{x}) = \ell(\boldsymbol{x}) + \beta$, where $\beta \in \mathbb{R}$ and $\ell \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$.
SOLUTION:
a) Given that $\ell(\mathbf{0}) = 0$. Hence using (2),

$$\ell(c\boldsymbol{x}) = \ell(c\boldsymbol{x} + (1-c)\mathbf{0}) = c\ell(\boldsymbol{x}) + (1-c)\ell(\mathbf{0}) = c\ell(\boldsymbol{x}). \tag{3}$$

Given $\lambda > 1$, $1/\lambda \in (0, 1)$.

$$\ell(\boldsymbol{x}) = \ell(\frac{1}{\lambda}\lambda\boldsymbol{x}) = \frac{1}{\lambda}\ell(\lambda\boldsymbol{x}) \text{ (from (3))}. \tag{4}$$

Thus from (3) and (4), for $\lambda > 0$,

$$\ell(\lambda\boldsymbol{x}) = \lambda\ell(\boldsymbol{x}). \tag{5}$$

$$\begin{aligned} \ell(\boldsymbol{x}+\boldsymbol{y}) = \ell(\frac{1}{2}2\boldsymbol{x} + \frac{1}{2}2\boldsymbol{y}) == \frac{1}{2}\ell(2\boldsymbol{x}) + \frac{1}{2}\ell(2\boldsymbol{y}) \text{ (from (2))} \\ = \ell(\boldsymbol{x}) + \ell(\boldsymbol{y}) \text{ since } \ell(2\boldsymbol{x}) = 2\ell(\boldsymbol{x}) \text{ (from (5))}. \end{aligned}$$

Thus,

$$\ell(\boldsymbol{x}+\boldsymbol{y}) = \ell(\boldsymbol{x}) + \ell(\boldsymbol{y}). \tag{6}$$

$$0 = \ell(\mathbf{0}) = \ell(\boldsymbol{x} + (-\boldsymbol{x})) = \ell(\boldsymbol{x}) + \ell(-\boldsymbol{x}) \text{ (from (6))}.$$

Hence $\ell(-\boldsymbol{x}) = -\ell(\boldsymbol{x})$. Hence by (5), for all $\lambda \in \mathbb{R}$,

$$\ell(\lambda\boldsymbol{x}) = \lambda\ell(\boldsymbol{x}). \tag{7}$$

Hence by (5) and (7), $\ell \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$.
b) It suffices to show that there exists $\beta \in \mathbb{R}$, where $\ell : \mathbb{R}^n \to \mathbb{R}$ is defined as $\ell(\boldsymbol{x}) = f(\boldsymbol{x}) - \beta$ for all $\boldsymbol{x} \in \mathbb{R}^n$ with ℓ satisfying (2) and $\ell(\mathbf{0}) = 0$. Necessary condition for $\ell(\mathbf{0}) = 0$ is $\beta = f(\mathbf{0})$. Thus

the only statement required to prove is $\ell : \mathbb{R}^n \to \mathbb{R}$, defined as $\ell(\boldsymbol{x}) = f(\boldsymbol{x}) - f(\mathbf{0})$, is such that $\ell \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$. For all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$,

$$\begin{aligned}
\ell(c\boldsymbol{x} + (1-c)\boldsymbol{y}) &= f(c\boldsymbol{x} + (1-c)\boldsymbol{y}) - f(\mathbf{0}) \\
&= cf(\boldsymbol{x}) + (1-c)f(\boldsymbol{y}) - f(\mathbf{0}) \\
&= c(f(\boldsymbol{x}) - f(\mathbf{0})) + (1-c)(f(\boldsymbol{y}) - f(\mathbf{0})) \\
&= c\ell(\boldsymbol{x}) + (1-c)\ell(\boldsymbol{y}).
\end{aligned}$$

Problem 5 (points: 5)

Let $\mathbb{U}$ be a subspace of an inner-product space $(\mathbb{V}, \langle \cdot, \cdot \rangle)$ and P denote the orthogonal projection onto $\mathbb{U}$. For any $\boldsymbol{x} \in \mathbb{V}$ and $\boldsymbol{z} \in \mathbb{U}$, show that $\boldsymbol{z} = P\boldsymbol{x}$ if and only if $\langle \boldsymbol{x} - \boldsymbol{z}, \boldsymbol{u} \rangle = 0$ for all $\boldsymbol{u} \in \mathbb{U}$.
SOLUTION: $\boldsymbol{z} = P\boldsymbol{x} \Rightarrow \langle \boldsymbol{x} - \boldsymbol{z}, \boldsymbol{u} \rangle = 0$ for all $\boldsymbol{u} \in \mathbb{U}$:
Since P denote the orthogonal projection onto $\mathbb{U}$, $\boldsymbol{x} - \boldsymbol{z} = \boldsymbol{x} - P\boldsymbol{x} \in \mathbb{U}^\perp$, where $\mathbb{U}^\perp$ is orthogonal complement of $\mathbb{U}$. Hence if $\boldsymbol{u} \in \mathbb{U}$, then $\langle \boldsymbol{x} - \boldsymbol{z}, \boldsymbol{u} \rangle = 0$.
$\langle \boldsymbol{x} - \boldsymbol{z}, \boldsymbol{u} \rangle = 0$ for all $\boldsymbol{u} \in \mathbb{U} \Rightarrow \boldsymbol{z} = P\boldsymbol{x}$:
Since P denote the orthogonal projection onto $\mathbb{U}$, for all $\boldsymbol{u} \in \mathbb{U}$, $\langle \boldsymbol{x} - P\boldsymbol{x}, \boldsymbol{u} \rangle = 0$. Given that $\langle \boldsymbol{x} - \boldsymbol{z}, \boldsymbol{u} \rangle = 0$ for all $\boldsymbol{u} \in \mathbb{U}$. This shows that $\langle P\boldsymbol{x} - \boldsymbol{z}, \boldsymbol{u} \rangle = 0$ for all $\boldsymbol{u} \in \mathbb{U}$. Also $P\boldsymbol{x} - \boldsymbol{z} \in \mathbb{U}$ because of the fact that $\mathbb{U}$ is a subspace. Hence $\|P\boldsymbol{x} - \boldsymbol{z}\|^2 = 0$. This shows that $P\boldsymbol{x} = \boldsymbol{z}$.

Problem 6 (points: 3)

Let $(\mathbb{V}, \langle \cdot, \cdot \rangle)$ be an inner-product space and $\|\cdot\|$ be the norm induced by the inner product on $\mathbb{V}$. For any $\ell \in \mathcal{L}(\mathbb{V})$, show that the following are equivalent:

1. $\|\ell(\boldsymbol{x})\| = \|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \mathbb{V}$.
2. $\langle \ell(\boldsymbol{x}), \ell(\boldsymbol{y}) \rangle = \langle \boldsymbol{x}, \boldsymbol{y} \rangle$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{V}$.

SOLUTION:
$2 \Rightarrow 1$:
From 2, we know that $\langle \ell(\boldsymbol{x}), \ell(\boldsymbol{x}) \rangle = \langle \boldsymbol{x}, \boldsymbol{x} \rangle$ for all $\boldsymbol{x} \in \mathbb{V}$. Also as $\|\cdot\|$ be the norm induced by the inner product on $\mathbb{V}$,

$$\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x} \rangle} = \sqrt{\langle \ell(\boldsymbol{x}), \ell(\boldsymbol{x}) \rangle} = \|\ell(\boldsymbol{x})\|.$$

$1 \Rightarrow 2$:

$$\begin{aligned}
\langle \boldsymbol{x}, \boldsymbol{y} \rangle &= 1/2\big(\|\boldsymbol{x} + \boldsymbol{y}\|^2 - \|\boldsymbol{x}\|^2 - \|\boldsymbol{y}\|^2\big) \\
&= 1/2\big(\|\ell(\boldsymbol{x} + \boldsymbol{y})\|^2 - \|\ell(\boldsymbol{x})\|^2 - \|\ell(\boldsymbol{y})\|^2\big) \text{ (using given statement)} \\
&= 1/2\big(\|\ell(\boldsymbol{x}) + \ell(\boldsymbol{y})\|^2 - \|\ell(\boldsymbol{x})\|^2 - \|\ell(\boldsymbol{y})\|^2\big) \text{ (as } \ell \in \mathcal{L}(\mathbb{V})) \\
&= \langle \ell(\boldsymbol{x}), \ell(\boldsymbol{y}) \rangle.
\end{aligned}$$

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