E1 222 Stochastic Models and Applications
Problem Sheet 6

1. Let X, Y be continuous random variables with joint density

$$f_{XY}(x, y) = K(1 + xy),\ 0 \leq x, y \leq 1.$$

Find value of K, $E[Y|X]$ and $E[X^2 + Y^2]$.

2. Let X, Y be discrete random variables, taking non-negative integer values, with joint mass function

$$P[X = i, Y = j] = e^{-(a+bi)} \frac{(bi)^j\ a^i}{j!\ i!}$$

where a, b are positive real numbers. Find $\mathrm{Cov}(X, Y)$.

3. A total of m keys are to be put in n boxes with each key independently being put in box-i with probability p_i. (Note that $p_i \geq 0, \forall i$ and $\sum_{i=1}^{n} p_i = 1$). Each time a key is put in a non-empty box, we say a collison has occurred. Find the expected number of collisons.

4. Let X, Y be jointly normal with means zero, variances σ_1^2, σ_2^2 and correlation coefficient ρ. Let $W = X\cos\theta + Y\sin\theta$ and $Z = X\cos\theta - Y\sin\theta$. Find a value of θ so that Z, W are independent.

5. Let $X_1, \cdots, X_n$ be *iid* Poisson random variables with mean 1. Let $S_n = \sum_{k=1}^{n} X_k$. Find $\mathrm{Prob}[S_n \leq n]$. Show that

$$\lim_{n\to\infty} e^{-n} \sum_{k=0}^{n} \frac{n^k}{k!} \ = \ 0.5.$$

6. A fair die is rolled repeatedly till each of the six outcomes occur at least once. Let Z denote the number of rolls needed. Let Y denote the number of distinct outcomes in the first five rolls of the die. Find $E[Z|Y = 2]$.

7. Let $X_i,\ i = 1, 2, \cdots$ be a sequence of *iid* Gaussian random variables with $EX_i = 0$ and $\mathrm{Var}(X_i) = 1$. Let $S_n = \sum_{i=1}^{n} X_i$. Let F_n be the distribution function of S_n/n and let G_n be the distribution function of $S_n/\sqrt{n}$. Find F_n and G_n. Do these distribution functions converge as $n \to \infty$? If so, what are the limits?

8. Let X_i, $i = 1, 2, \cdots$, be *iid* random variables which are uniform over $[0, 1]$. Let $Z_n = \min(X_1, \cdots, X_n)$ and $W_n = \max(X_1, \cdots, X_n)$. Let $S_n = (Z_n + W_n)/2$. Does S_n converge in probability? Explain.

9. Let π be a stationary distribution of a Markov Chain. (i). Show that if $\pi(x) > 0$ and x leads to y then $\pi(y) > 0$. (ii). Suppose the chain has transition probabilities that satisfy the following: for some two states y and z, $P(x, y) = cP(x, z)$, $\forall x$, where c is a constant. Show that $\pi(y) = c\pi(z)$.

10. Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ and assume that it is independent of a non-negative random variable, T. Suppose the mean of T is μ and its variance is σ^2. Find (i). $E[N(T)]$, (ii). $\text{Var}(N(T))$, (iii). $\text{Cov}(T, N(T))$