Test 1: Computational Linear Algebra

(duration: 1hr 30min)

Problem 1 (points: 3+5+2 = 10)

We say that $\mathbb{A} \subset \mathbb{R}^n$ is an *affine* space if for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{A}$ and $t \in \mathbb{R}$, we must have $t\boldsymbol{x} + (1-t)\boldsymbol{y} \in \mathbb{A}$.

(i) Show that an affine space $\mathbb{A}$ is a subspace if and only if $\mathbf{0} \in \mathbb{A}$.

(ii) If $\mathbb{A}$ is nonempty and affine, show that there exists an *unique* subspace $\mathbb{U} \subset \mathbb{R}^n$ such that, for any $\boldsymbol{x}_0 \in \mathbb{A}$,

$$\mathbb{A} = \big\{\boldsymbol{x}_0 + \boldsymbol{x} : \boldsymbol{x} \in \mathbb{U}\big\}.$$

(iii) Show that the solutions of the equation $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ form an affine space, assuming the equations are solvable. Identify the unique subspace $\mathbb{U}$ in this case.

Problem 2 (points: 5)

Let $\mathbb{N}$ be the set of natural numbers and $\mathfrak{F}$ denote the set of functions $f : \mathbb{N} \to \mathbb{R}$. Show that $\mathfrak{F}$ is a vector space over $\mathbb{R}$ but is not finite dimensional.

Problem 3 (points: 5)

Let $\mathbb{V}$ be a vector space and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n \in \mathbb{V}$ be linearly independent. Show that for any $\boldsymbol{v} \in \mathbb{V}$, the dimension of the space spanned by $\boldsymbol{v}_1 + \boldsymbol{v}, \ldots, \boldsymbol{v}_n + \boldsymbol{v}$ is at least $n-1$.

Problem 4 (points: 2+1+2 = 5)

What is the dimension of $\mathbb{R}^n$ as a vector space over the field $\mathbb{Q}$ of rational numbers? What is the dimension of $\mathbb{C}^n$ as a vector space over the fields $\mathbb{C}$ and $\mathbb{R}$?

Problem 5 (points: 5)

Let $\mathbb{U}$ and $\mathbb{V}$ be 6-dimensional subspaces of a 10-dimensional vector space. Show that there exists vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ common to both $\mathbb{U}$ and $\mathbb{V}$ such that neither $\boldsymbol{x}$ nor $\boldsymbol{y}$ is a scalar multiple of the other.

Problem 6 (points: 4+1 = 5)

Let $\mathbb{U}_1, \ldots, \mathbb{U}_m$ be subspaces of a finite-dimensional vector space. Prove that

$$\dim(\mathbb{U}_1 + \cdots + \mathbb{U}_m) \leqslant \dim(\mathbb{U}_1) + \cdots + \dim(\mathbb{U}_m).$$

Give a condition on $\mathbb{U}_1, \ldots, \mathbb{U}_m$ for which equality holds.

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