E1 222 Stochastic Models and Applications
Problem Sheet 3–5

1. Let $\mathbf{X} = (X_1, X_2, X_3)$. Let $\mathbf{Y}$ be a function of $\mathbf{X}$ given by $\mathbf{Y} = (X_2, X_1, X_3)$. That is, Y_1, Y_2, Y_3 is a specific permutation of X_1, X_2, X_3 (where Y_1, Y_2, Y_3 are components of $\mathbf{Y}$). Show that we can write $\mathbf{Y} = A\mathbf{X}$ where A is a 3×3 matrix. Show that $f_{Y_1Y_2Y_3}(y_1, y_2, y_3) = f_{X_1X_2X_3}(y_2, y_1, y_3)$.

2. Recall that random variables $X_1, X_2, \cdots X_n$ are said to be *exchangeable* if any permutation of them has the same joint density. Show that exchangeable random variables are identically distributed. Show that random variables X, Y, Z with joint density given by

$$f_{XYZ}(x, y, z) = \frac{2}{3}(x + y + z), \; 0 \le x, y, z \le 1$$

are exchangeable but are not independent.

3. Suppose X_1, X_2, X_3 are exchangeable random variables. Show that

$$E\left[\frac{X_1 + X_2}{X_1 + X_2 + X_3}\right] = \frac{2}{3}$$

Can you generalize this result?
(Hint: Is there a relation between $E\left[\frac{X_1+X_2}{X_1+X_2+X_3}\right]$ and $E\left[\frac{X_2+X_3}{X_1+X_2+X_3}\right]$?)

4. Consider a probability space with $\Omega = [0,\ 1]$ and with the usual probability assignment where probability of an interval is the length of the interval. Consider two random variables defined on this probability space by:

$$\begin{aligned} X &= I_{[0,\ a]} + I_{[0.5,\ 0.5+a]} \\ Y &= I_{[b,\ 0.5]} + I_{[0.5+b,\ 1]} \end{aligned}$$

where I_A denotes indicator random variable of event A and a, b are numbers in the interval $(0,\ 0.5)$. Find f_X, f_Y, the mass functions of X, Y. Show that X, Y are not independent for any choice of $a, b \in (0,\ 0.5)$.

5. Let $X_1, \cdots, X_n$ be independent random variables with X_i being exponential with parameter λ_i, $i = 1, \cdots, n$. (i). Show that Prob[$X_1 <$

$X_2] = \frac{\lambda_1}{\lambda_1+\lambda_2}$. (ii). Let $Z = \min(X_1, \cdots, X_n)$. Find $E[Z]$. (iii). Let J be a random variable defined by: $J = k$ if X_k happens to be the minimum among X_1 to X_n. (That is, $J = \arg\min_i\{X_i\}$). Find distribution of J.

6. Let $X_1, X_2, \cdots, X_N$ be *iid* continuous random variables. We say a record has occurred at m $(1 \leq m \leq N)$ if $X_m > \max(X_{m-1}, \cdots, X_1)$. Show that (i). Probability that a record has occurred at m is equal to $\frac{1}{m}$. (ii). The expected number of records till k is $\sum_{m=1}^{k} \frac{1}{m}$. (iii). The variance of the number of records till k is $\sum_{m=1}^{k} \frac{m-1}{m^2}$.

7. Let X be a binomial random variable with parameters n and p. Let $Y = \max(0,\ X-1)$. Show that $EY = np - 1 + (1-p)^n$.

8. Suppose a box contains m white balls and n black balls. Balls are drawn one at a time from the box without replacement. Let X_k be the indicator random variable that takes value 1 if the k^{th} ball drawn is white. Argue that $P[X_k = 1] = \frac{m}{m+n}$. Let S_r denote the number of white balls drawn by the time we drew r balls from the box. Find ES_r. Argue that $P[X_k = 1, X_j = 1] = \frac{m(m-1)}{(m+n)(m+n-1)}, j \neq k$. Find $\mathrm{Var}(S_r)$.