Name:

S.R.Number:

E1 222 Stochastic Models and Applications
Test II

`Time: 90 minutes` `Max. Marks:40`
`Date: 26 Oct 2019`

Answer **ALL** questions. All questions carry equal marks
Answers should be written only in the space provided.

1. (a) Let X, Y have joint density given by

$$f_{XY}(x,y) = \lambda^2 e^{-\lambda y},\ 0 \leq x \leq y < \infty$$

Find marginal densities of X, Y, and $E[X|Y]$ and $\mathrm{Cov}(X,Y)$.

Answer : For the marginal of X we have

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)\ dy = \lambda \int_x^{\infty} \lambda e^{-\lambda y}\ dy = \lambda e^{-\lambda x},\ 0 \leq x < \infty$$

(Note that we need not actually do the integration. We know the integral because we know the distribution function of exponential random variable).
Thus, X is exponential with parameter λ.
Similarly, for the marginal of Y

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(x,y)\ dx = \int_0^{y} \lambda^2 e^{-\lambda y}\ dx = \lambda^2 y e^{-\lambda y}, 0 \leq y < \infty$$

This is easily seen to be gamma density with parameters 2 and λ.
The conditional density of X given Y is given by

$$f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)} = \frac{\lambda^2 e^{-\lambda y}}{\lambda^2 y e^{-\lambda y}} = \frac{1}{y},\ 0 \leq x \leq y < \infty$$

This is a uniform density over $[0, y]$. That is, conditioned on Y, X is uniform over $[0, Y]$. Hence $E[X|Y] = \frac{Y}{2}$.

We can get this conditional expectation by direct calculation too.

$$E[X|Y = y] = \int_{-\infty}^{\infty} x \; f_{X|Y}(x|y) \; dx = \int_0^y x \frac{1}{y} dx = \frac{y}{2}$$

Hence, $E[X|Y] = \frac{Y}{2}$

We know $\text{Cov}(X, Y) = EXY - EXEY$. From the marginal densities, we see that X is exponential while Y is gamma. We have $EX = \frac{1}{\lambda}$ and $EY = \frac{2}{\lambda}$.

$$EXY = \int_0^{\infty} \int_0^y xy \; \lambda^2 e^{-\lambda y} \; dx \; dy = \int_0^{\infty} y\lambda^2 e^{-\lambda y} \frac{y^2}{2} dy = \frac{1}{2} \int_0^{\infty} y^2 \; \lambda^2 y e^{-\lambda y} dy$$

We can see that the last integral above is the second moment of a Gamma random variable with parameters $\alpha = 2$ and λ. For any Z, $EZ^2 = \text{Var}(Z) + (EZ)^2$. hence, we get

$$EXY = \frac{1}{2} \left(\frac{2}{\lambda^2} + \frac{4}{\lambda^2} \right) = \frac{3}{\lambda^2}$$

(Evaluating EXY by direct integration is also not too difficult). Now,

$$\text{Cov}(X, Y) = \frac{3}{\lambda^2} - \frac{2}{\lambda} \frac{1}{\lambda} = \frac{1}{\lambda^2}$$

Comment: I hope at least some of you noticed that this is essentially an example problem that I solved in class. After defining conditional expectation, this problem was solved in class to illustrate how you calculate conditional expectation given the joint density. At that time also I remarked on how you can avoid doing integrations here by using known facts about distributions. In the class we used this joint density with $\lambda = 1$. Here you have a general λ. That is the only difference.

(b) Let X, Y be random variables with $EX = EY = 0$, $\text{Var}(X) = 1$, $\text{Var}(Y) = 2$. Let ρ be the correlation coefficiet of X, Y. Let $Z = X + Y$ and $W = X - Y$. Find $\text{Var}(Z)$, $\text{Var}(W)$ and $\text{Cov}(Z, W)$.

Answer: We have $\text{Cov}(X, Y) = \rho_{XY} \sqrt{\text{Var}(X)\text{Var}(Y)} = \rho\sqrt{1 * 2} = \rho\sqrt{2}$. Hence

$$\text{Var}(Z) = \text{Var}(X) + \text{Var}(Y) + 2\text{Cov}(X, Y) = 3 + \rho \; 2\sqrt{2}$$

$$\mathrm{Var}(W) = \mathrm{Var}(X) + \mathrm{Var}(Y) - 2\mathrm{Cov}(X,Y) = 3 - \rho\, 2\sqrt{2}$$

$\mathrm{Cov}(Z,W) = EZW - EZEW$. Since $EX = EY = 0$, we have $EZ = EW = 0$.

$EZW = E[(X+Y)(X-Y)] = E[X^2] - E[Y^2] = \mathrm{Var}(X) - \mathrm{Var}(Y) = -1$

Hence, $\mathrm{Cov}(Z,W) = -1$.

2. (a) Let $X_1, X_2, \cdots, X_n$ be *iid* random variables each of them being uniform over $[0,\ 1]$. Let $Y_1 = X_1$, $Y_2 = X_1X_2$, $Y_3 = X_1X_2X_3$, and so on with $Y_n = X_1X_2\cdots, X_n$. Find the joint density of $Y_1, \cdots, Y_n$ and the conditional density of Y_3 given Y_1, Y_2.

Answer: The given transformation is invertible. The inverse transformation is given by

$$\begin{aligned} X_1 &= Y_1 \\ X_2 &= Y_2/Y_1 \\ X_3 &= Y_3/Y_2 \\ &\cdot \\ &\cdot \\ &\cdot \\ X_n &= Y_n/Y_{n-1} \end{aligned}$$

The Jacobian of the transformation is given by

$$\left| \begin{array}{ccccc} 1 & 0 & 0 & \cdots & 0 \\ -y_2/(y_1^2) & 1/y_1 & 0 & \cdots & 0 \\ 0 & -y_3/(y_2^2) & 1/y_2 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & -y_n/(y_{n-1}^2) & 1/y_{n-1} \end{array} \right| = \frac{1}{y_1y_2\cdots y_{n-1}}$$

The determinant is easy to evaluate because it is a triangular matrix. Now we get

$$f_{Y_1Y_2\cdots Y_n}(y_1, y_2, \cdots, y_n) = \left|\frac{1}{y_1y_2\cdots y_{n-1}}\right| f_{X_1X_2\cdots X_n}(y_1, y_2/y_1, y_3/y_2, \cdots, y_n/y_{n-1})$$

Since X_i are iid uniform over $[0,1]$, the joint density of $X_1, \cdots, X_n$ is zero unless all arguments are between 0 and 1. When all arguments are between 0 and 1, the joint density is 1. Hence, the joint density of $Y_1, Y_2, \cdots, Y_n$ is zero unless $0 \leq y_1 \leq 1$, $0 \leq y_2/y_1 \leq 1$ and so on till $0 \leq y_n/y_{n-1} \leq 1$. This gives us the joint density of $Y_1, Y_2, \cdots, Y_n$ as

$$f_{Y_1 Y_2 \cdots Y_n}(y_1, y_2, \cdots, y_n) = \frac{1}{y_1 y_2 \cdots y_{n-1}},\ 0 \leq y_n \leq y_{n-1} \leq \cdots \leq y_2 \leq y_1 \leq 1$$

The above is true for all n. Hence we get

$$f_{Y_1 Y_2 Y_3}(y_1, y_2, y_3) = \frac{1}{y_1 y_2},\ 0 \leq y_3 \leq y_2 \leq y_1 \leq 1$$

$$f_{Y_1 Y_2}(y_1, y_2) = \frac{1}{y_1},\ 0 \leq y_2 \leq y_1 \leq 1$$

Hence the needed conditional density is

$$f_{Y3|Y_1 Y_2}(y_3|y_1, y_2) = \frac{f_{Y_1 Y_2 Y_3}(y_1, y_2, y_3)}{f_{Y_1 Y_2}(y_1, y_2)} = \frac{1}{y_2},\ 0 \leq y_3 \leq y_2 \leq y_1 \leq 1$$

(b) Let X_1, X_2 be iid geometric random variables. Let $Y = \min(X_1, X_2)$. Show that Y is also geometric and find its expectation.

Answer: Since X_i are geometric, we have $P[X_i > y] = (1-p)^y$ for all positive integers y. Hence for any positive integer y,

$$P[Y > y] = P[\min(X_1, X_2) > y] = P[X_1 > y, X_2 > y]$$

$$= P[X_1 > y]P[X_2 > y] = (1-p)^y(1-p)^y = (1-p)^{2y}$$

So, for any positive integer y,

$$P[Y > y] = (1-p)^{2y} = \left((1-p)^2\right)^y = \left(1 - (1 - (1-p)^2)\right)^y$$

This shows that Y is geometric with parameter $(1 - (1-p)^2)$. Now, since Y is geometric, its expectation is given by

$$EY = \frac{1}{(1 - (1-p)^2)} = \frac{1}{p(2-p)}$$

You can solve the problem by directly calculating the mass function of Y also.

$$P[Y = y] = P[X_1 = y, X_2 > y] + P[X_2 = y, X_1 > y] + P[X_1 = y, X_2 = y]$$

$= (1-p)^{y-1}p(1-p)^y + (1-p)^{y-1}p(1-p)^y + (1-p)^{y-1}p(1-p)^{y-1}p$

Hence,

$$f_Y(y) = 2p(1-p)^{2y-1} + p^2(1-p)^{2y-2} = p(1-p)^{2y-2}(2(1-p)+p) = p(1-p)^{2y-2}(2-p)$$

$$= \left((1-p)^2\right)^{y-1} p(2-p) = \left((1-p)^2\right)^{y-1} (1-(1-p)^2)$$

This shows that Y has geometric distribuion with parameter $(1-(1-p)^2)$.

3. (a) A rod of length 1 is broken at a random point. The piece containing the left end is once again broken at a random point. Let L be the length of the final piece containing the left end. Find $E[L]$.

Answer: Let us take the left end of rod as origin. Let X_1 denote the point where the rod is broken the first time. Then X_1 is uniform over $(0,1)$. Let X_2 denote the point where the rod is broken on the second time. Note that (since we took left end as origin) X_2 is also the length of the final piece containing the left end and hence $L = X_2$. Now, given X_1, we know X_2 is uniform over $(0, X_1)$. What we mean by this is

$$f_{X_2|X_1}(x_2|x_1) = \frac{1}{x_1}, \quad 0 < x_2 < x_1 < 1$$

What we need is $EL = EX_2$. From the above conditional density, it is easy to see that $E[X_2|X_1] = \frac{X_1}{2}$. Hence,

$$E[L] = E[X_2] = E\left[E[X_2|X_1]\right] = E\left[\frac{X_1}{2}\right] = \frac{1}{4}$$

because $EX_1 = \frac{1}{2}$.

(b) Let $X_1, X_2, \cdots$ be *iid* random variables having exponential density with parameter λ. Let N be a geometric random variable with parameter p. N is independent of all X_i. Let $S = X_1 + X_2 + \cdots + X_N$. Find ES and density of S

(You can use the following fact: If X has gamma density with parameters α_1 & λ, and Y has gamma density with parameters α_2 & λ, and X, Y are independent then $X + Y$ is gamma with parameters $\alpha_1 + \alpha_2$ & λ)

Answer: Using the formula for expectation of random number of random variables, we get

$$ES = EN\ EX_1 = \frac{1}{p}\frac{1}{\lambda} = \frac{1}{\lambda p}$$

because N is geometric and X_i are iid exponential.

We can find the density of S as follows. First let us derive an expression for the distribution function of S. For this, first consider

$$P\left[\sum_{i=1}^{N} X_i \le x | N = n\right] = \frac{P[\sum_{i=1}^{N} X_i \le x, N = n]}{P[N = n]}$$

$$= \frac{P[\sum_{i=1}^{n} X_i \le x, N = n]}{P[N = n]} = P\left[\sum_{i=1}^{n} X_i \le x\right]$$

because N and X_i are independent. We know that exponential is a special case of gamma with parameters 1 and λ. So, sum of n iid exponential random variables would be gamma with parameters n and λ. Let $F_{G(n,\lambda)}$ denote this distribution function and let $f_{G(n,\lambda)}$ denote the corresponding density. Then, from the above, we have

$$P\left[\sum_{i=1}^{N} X_i \le x | N = n\right] = P\left[\sum_{i=1}^{n} X_i \le x\right] = F_{G(n,\lambda)}(x)$$

Hence

$$P\left[\sum_{i=1}^{N} X_i \le x\right] = \sum_{n} P\left[\sum_{i=1}^{N} X_i \le x | N = n\right] P[N = n] = \sum_{n=1}^{\infty} f_N(n) F_{G(n,\lambda)}(x)$$

Differentiating this, we get the density of S as

$$f_S(x) = \sum_{n=1}^{\infty} f_N(n) f_{G(n,\lambda)}(x) = \sum_{n=1}^{\infty} (1-p)^{n-1} p \frac{1}{(n-1)!} \lambda^n x^{n-1} e^{-\lambda x}$$

where we have substituted for the gamma density and the mass function of a geometric random variable. We can simplify this as follows.

$$f_S(x) = \sum_{n=1}^{\infty} (1-p)^{n-1}\ p\ \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} = \lambda\ p\ e^{-\lambda x} \sum_{n=1}^{\infty} \frac{((1-p)\lambda x)^{n-1}}{(n-1)!}$$

$$= \lambda\, p\, e^{-\lambda x} \sum_{m=0}^{\infty} \frac{((1-p)\lambda x)^m}{m!} = \lambda\, p\, e^{-\lambda x}\, e^{(1-p)\lambda x} = \lambda\, p\, e^{-\lambda p x}$$

Thus, S is exponential with parameter λp. (Now we can easily see that its expectation is what we got earlier through the formula).

We can also solve this problem using the moment generating functions. The moment generating function of S is

$$M_S(t) = E\left[e^{t(X_1+X_2+\cdots+X_N)}\right]$$

Now we have

$$E\left[e^{t(X_1+X_2+\cdots+X_N)}|N=n\right] = E\left[e^{t(\sum_{i=1}^{n} X_i)}|N=n\right] = E\left[e^{t(\sum_{i=1}^{n} X_i)}\right] = \prod_{i=1}^{n} E\left[e^{tX_i}\right]$$

because X_i are independent of N and X_i are iid. Since X_i are exponential

$$M_{X_i}(t) = E\left[e^{tX_i}\right] = \frac{\lambda}{\lambda - t},\ t < \lambda$$

Hence we get

$$E\left[e^{t(X_1+X_2+\cdots+X_N)}|N=n\right] = \left(\frac{\lambda}{\lambda-t}\right)^n$$

Thus, we get the moment generating function of S as

$$M_S(t) = E\left[\ E\left[e^{t(X_1+X_2+\cdots+X_N)}|N\right]\right] = E\left[\left(\frac{\lambda}{\lambda-t}\right)^N\right] = \sum_{k=1}^{\infty}(1-p)^{k-1}\, p\, \left(\frac{\lambda}{\lambda-t}\right)^k$$

$$= p\left(\frac{\lambda}{\lambda-t}\right)\sum_{k=1}^{\infty}(1-p)^{k-1}\, \left(\frac{\lambda}{\lambda-t}\right)^{k-1}$$

What we have here is an infinite geometric series. That would converge if $\frac{\lambda(1-p)}{\lambda-t} < 1$ which is true if $t < \lambda - \lambda(1-p) = \lambda p$. That would be the range of t for which the moment generating function of S exists. Now, summing the infinite series, we get

$$M_S(t) = p\, \frac{\lambda}{\lambda-t}\, \frac{1}{1-\frac{\lambda(1-p)}{\lambda-t}} = \frac{p\lambda}{\lambda-t}\, \frac{\lambda-t}{(\lambda-t)-\lambda(1-p)} = \frac{\lambda p}{\lambda p - t}$$

This is the moment generating function of S and it exists for $t < \lambda p$. Hence S is exponential with parameter λp.

Comment: This is a standard result and it is worth remembering.

4. (a). Let X be Gaussian with mean zero and variance 1. Let Z be a discrete random variable that is independent of X and suppose $\text{Prob}[Z = 1] = \text{Prob}[Z = -1] = 0.5$. Let $Y = ZX$. Find density of Y. Are X, Y uncorrelated? Are X, Y jointly Gaussian?

Answer: We are given $Y = ZX$. The distribution function of Y is given by

$$P[Y \leq y] = P[ZX \leq y] = P[ZX \leq y|Z = 1]P[Z = 1] + P[ZX \leq y|Z = -1]P[Z = -1]$$

$$= \frac{1}{2}\left(P[X \leq y] + P[-X \leq y]\right) = \frac{1}{2}\left(P[X \leq y] + P[X \geq -y]\right)$$

Since X is standard Gaussian, its distribution function is Φ. So, we get

$$F_Y(y) = \frac{1}{2}\left(\Phi(y) + (1 - \Phi(-y))\right) = \Phi(y)$$

(Recall that $\Phi(-x) = 1 - \Phi(x)$). Hence Y is also Gaussian with mean zero and variance 1.

From above, we have $EX = EY = 0$. Now, since X and Z are independent, $EXY = EZX^2 = EZ\ EX^2 = 0$ because $EZ = 0$. Hence, X, Y are uncorrelated.

But X and Y are not independent. (Obvious because Y can be only X or $-X$. If you are not convinced, then: $P[Y > 2|X \in [-1, 1]] = 0$ but $P[Y > 2] \neq 0$!) If X, Y are jointly Gaussian then their uncorrelatedness should imply independence. Hence, X, Y are not jointly Gaussian.

(b). Suppose we have two decks of n cards each. Cards of each deck are numbered $1, 2, \ldots, n$. The two decks are separately shuffled and then the corresponding cards in each deck are compared one by one. We say a match has occurred at position i if the i^{th} card in each deck has the same number. Let S_n denote the total number of matches. Find $E[S_n]$ and $\text{Var}(S_n)$.

Answer: Let $X_1, \cdots X_n$ be indicator random variables with $X_i = 1$ if there is a match at position i. It is easy to see that $P[X_i = 1] = \frac{(n-1)!}{n!} = \frac{1}{n}$. The number of matches is given by $S_n = \sum_{i=1}^{n} X_i$. Hence $ES_n = n\,\frac{1}{n} = 1$.

To calculate the variance, note that X_i are not independent. But we have

$$P[X_i = 1, X_j = 1] = \frac{(n-2)!}{n!} = \frac{1}{n(n-1)}$$

Hence

$$\mathrm{Cov}(X_i, X_j) = E[X_i X_j] - EX_i\ EX_j = \frac{1}{n(n-1)} - \frac{1}{n^2} = \frac{1}{n^2(n-1)}$$

Also,

$$\mathrm{Var}(X_i) = \frac{1}{n}\left(1 - \frac{1}{n}\right)$$

Hence

$$\mathrm{Var}(S_n) = \sum_{i=1}^{n} \mathrm{Var}(X_i) - \sum_{i=1}^{n} \sum_{j \neq i} \mathrm{Cov}(X_i, X_j) = n\frac{1}{n}\left(1 - \frac{1}{n}\right) + n(n-1)\frac{1}{n^2(n-1)} = 1$$

Comment: I hope you can see the similarity with the problem of n men putting their hats together and making random selection. You can think of each man and his hat being given a number and then we randomly pair men with hats. Thus the matches in this problem correspond to man getting his own hat.