Computational Methods of Optimization
Final Exam 2(1st Dec, 2021)

Instructions:

- This is the second part of Final Test
- This is a closed book test. Please do not consult any additional material.
- Attempt all questions
- Total time is 70 mins.

Name: ______________________________

SRNO: Degree: Dept:

Question:	1	2	3	4	Total
Points:	10	10	10	10	40
Score:					

$\mathcal{S}_d$ would denote the set of $d \times d$ Symmetric real valued matrices. $\mathcal{S}_d^+$ denote the set of $d \times d$ Symmetric real valued positive semidefinite matrices and $\mathcal{S}_d^{++}$ would denote the set of positive definite matrices.

1. Let $Q \in S_d^{++}, h \in \mathbb{R}^d, a_i \in \mathbb{R}^d, b_i \in \mathbb{R}$. We consider solving the following problem with the Active set strategy

$$min_{\mathbf{x}\in\mathbb{R}^d} f(\mathbf{x}) \left(= \frac{1}{2}\mathbf{x}^\top Q\mathbf{x} + h^\top \mathbf{x} \right)$$

subject to $a_i^\top \mathbf{x} \geq b_i, \quad i = \{1, \ldots, m\}$.

(a) (2 points) State the KKT conditions for this problem?

(b) (2 points) In a given iteration we find that the working set is $\hat{W} = \{1, 2, 3\}$, and $\hat{\mathbf{x}}$ is a feasible point. Moreover it is found that

$$\mathbf{u}^* = 0, \mu^* = [1, 2, -3]^\top \text{is the KKT point for}$$

$$\mathbf{u}^* = argmin_{\mathbf{u}\in\mathbb{R}^d} \frac{1}{2}\mathbf{u}^\top Q\mathbf{u} + \hat{g}^\top \mathbf{u}$$

$$\text{subject to } \; a_i^\top \mathbf{x} = b_i, i \in \hat{W}$$

The Lagrangian is

$$L(\mathbf{u}, \mu) = \frac{1}{2}\mathbf{u}^\top Q\mathbf{u} + \hat{g}^\top \mathbf{u} - \sum_{i\in\hat{W}} \mu_i(a_i^\top \mathbf{x} - b_i)$$

If the algorithm is stopped here what can be said about the optimality of $\hat{\mathbf{x}}$.

(c) To continue the algorithm, as per active-set strategy, find a constraint l that needs to be dropped from $\hat{W}$ to create a new working set $W = \hat{W}$. After identifying W compute

$$\hat{\mathbf{u}} = argmin_{\mathbf{u}\in\mathbb{R}^d} \frac{1}{2}\mathbf{u}^\top Q\mathbf{u} + \hat{g}^\top \mathbf{u}$$

$$\text{subject to } \; a_i^\top \mathbf{u} = 0, i \in W$$

where $\hat{g} = \nabla f(\hat{\mathbf{x}})$.

i. (2 points) What is the relationship between $\hat{g}$ and μ^*?

ii. (4 points) Suppose it is given that $\hat{\mathbf{u}}Q\hat{\mathbf{u}} = 6$ can you find the value of $a_l^\top \hat{\mathbf{u}}$. Give reasons for your answer.

2. Let $A = \{a_1, \ldots, a_n\} \subset \mathbb{R}^d$ be a set of n vectors in $\mathbb{R}^d$. We need to decide if $\mathbf{z} \in \mathbb{R}^d$ lies in the convex hull of A, defined by

$$Conv(A) = \left\{ \sum_{i=1}^{n} \alpha_i a_i \| \alpha_i \geq 0, \sum_{i=1}^{n} \alpha_i = 1 \right\}$$

(a) (5 points) Pose this problem as a convex optimization problem.

(b) (5 points) Derive the KKT conditions of this problem

3. (10 points) Consider the set $C = \{\mathbf{x} \in \mathbb{R}^d | \|\mathbf{x} - \mu\| \leq \frac{1}{2}\|\mu\|^2\}$. Find $h \in \mathbb{R}^d$ and $b \in \mathbb{R}$ such that

$$0 \in \{\mathbf{z} \in \mathbb{R}^d | h^\top \mathbf{z} \leq b\}, \text{ and } C \subset \{\mathbf{z} \in \mathbb{R}^d | h^\top \mathbf{z} > b\}$$

4. Consider the following problem

$$min_{\mathbf{x}} f(\mathbf{x}) \left(= \frac{1}{2}\mathbf{x}^\top Q\mathbf{x} \right), \quad \text{subject to } \|\mathbf{x}\|^2 \leq 1$$

Q is not necesasarily psd and can be indefinite. Let $\mathbf{x}^*$ be the global optimal solution.

(a) (3 points) Show that the dual is a function of one variable? You need to state the dual

(b) (2 points) Find the domain of the dual function

(c) (2 points) Find the dual optimal solution?

(d) (2 points) Can you find $\mathbf{x}^{(0)}$ such that $f(\mathbf{x}^{(0)})$ is equal to the dual optimal?

(e) (1 point) Find $\mathbf{x}^*$ with justification