E1 222 Stochastic Models and Applications

Assignment 2

Submission Deadline: 24 October 9 PM
You need to submit solutions to three problems
The specific three problems would be communicated to you through Teams at 6 PM on 24 October.

1. Let X, Y be iid geometric random variables with parameter p. (That is, $f_X(k) = f_Y(k) = (1-p)^{k-1}p,\ k = 1, 2, \cdots$). Let $Z = X - Y$ and $W = \min(X, Y)$. Find the joint mass function of Z, W. Show that Z, W are independent.

2. Let X be a random variable having Gaussian density with mean zero and variance 1. Show that $Y = X^2$ has gamma density with parameters $\frac{1}{2}$ and $\frac{1}{2}$.
   Now, let $X_1, \cdots, X_n$ be iid random variables having Gaussian density with mean zero and variance σ^2. Show that $Y = \frac{X_1^2 + \cdots + X_n^2}{\sigma^2}$ has Gamma density with parameters $\frac{n}{2}$ and $\frac{1}{2}$. (This rv, Y, is said to have chi-squared distribution with n degrees of freedom).

3. Let X be uniform over $(0, 1)$ and let Y be a discrete random variable taking non-negative integer values. Suppose X, Y are independent. let $Z = X + Y$. Show that Z is a continuous random variable.

4. Let X, Y, Z be iid continuous random variables. Show that $P[X < Y] = 0.5$ irrespective of what is the common density function of these random variables. Now calculate $P[X < Y < Z]$ and show that its value is same irrespective of what is the common density function of these random variables. Based on all this, can you guess what is the value of $P[X < Y, Z < Y]$. Explain.

5. Let $X_1, X_2, \cdots, X_n$ be random variables with mean zero and variance unity. Suppose the correlation coefficient of any pair of random variables, X_i and X_j, $i \neq j$, is ρ. Show that $\rho \geq \frac{-1}{n-1}$. Will this result remain true if $EX_i = \mu_i$ and $\text{Var}(X_i) = \sigma_i^2$; but correlation coefficient between any pair of them is still ρ.

6. Let X and Y be two discrete random variables with

$$P[X = x_1] = p_1, \quad P[X = x_2] = 1 - p_1;$$

$$P[Y = y_1] = p_2, \quad P[Y = y_2] = 1 - p_2.$$

Show that X and Y are independent if and only if they are uncorrelated. (Hint: Consider the special case where $x_1 = y_1 = 0$ and $x_2 = y_2 = 1$).

7. Let X be a discrete random variable taking non-negative integer values with mass function, $p(i), \ i = 0, 1, \cdots$. Let $Y_1, Y_2, \cdots, Y_n$ be *iid* discrete random variables taking non-negative integer values and with mass function $q(i), i = 0, 1, \cdots$. Assume $p(i), q(i) > 0, \forall i$. Let $h : \Re \to \Re$ be some function. Define

$$S = \frac{1}{n} \sum_{k=1}^{n} \frac{p(Y_k) h(Y_k)}{q(Y_k)}.$$

Find ES.

8. Consider repeated independent tosses of a coin whose probability of heads is p, $0 < p < 1$. Let X denote the number of tosses needed to get at least one head and one tail. Let Y denote the number of tosses needed to get a head immediately followed by a tail. Find EX and EY.

9. An interval of length 1 is broken at a point uniformly distributed over $(0, 1)$. Let c be a fixed point in $(0, 1)$. Find the expected length of the subinterval that contains the point c. Show that this probability is maximized when $c = 0.5$.