E1 222 Stochastic Models and Applications
Problem Sheet 3–3

1. Let X, Y have joint density
$$f_{XY}(x, y) = e^{-y}, \;\; 0 < x < y < \infty$$
Find covariance of X, Y, $\text{Cov}(X, Y)$, and the correlation coefficient of X, Y, ρ_{XY}.

2. Let A, B be two events. Let I_A and I_B be the indicator random variables of these events. Find covariance of I_A, I_B and their correlation coefficient. Let $\rho(A, B)$ denote their correlation coefficient. Show that $\rho(A, B) > 0 \Rightarrow P(A|B) > P(A) \Rightarrow P(B|A) > P(B)$ and $\rho(A, B) < 0 \Rightarrow P(A|B) < P(A) \Rightarrow P(B|A) < P(B)$. What would be $\rho(A, B)$ if A and B are independent.

3. Let X_1, X_2, X_3 be independent random variables with finite variances $\sigma_1^2, \sigma_2^2, \sigma_3^2$ respectively. Find the correlation coefficient of $X_1 - X_2$ and $X_2 + X_3$.

4. Let X and Y be two discrete random variables with
$$P[X = x_1] = p_1, \quad P[X = x_2] = 1 - p_1;$$
$$P[Y = y_1] = p_2, \quad P[Y = y_2] = 1 - p_2.$$
Show that X and Y are independent if and only if they are uncorrelated. (Hint: First consider the special case where $x_1 = y_1 = 0$ and $x_2 = y_2 = 1$).

5. Let X, Y be continuous random variables with joint density
$$f_{XY}(x, y) = 2, \;\; 0 < x < y < 1$$
Find $E[X|Y]$ and $E[Y|X]$.

6. Find $E[X|Y]$ when X, Y have joint density given by
$$f_{XY}(x, y) = \frac{y}{2} e^{-xy}, \; x > 0, \; 1 < y < 3$$

7. Let X and Y be iid random variables having Poisson distribution with parameter λ. Let $Z = X + Y$. Find $E[X|Z]$ and $E[Z|Y]$.

8. Let $X_1, X_2, \cdots$ be *iid* discrete random variables with $P[X_i = +1] = P[X_i = -1] = 0.5$. Find EX_i. Let N be a positive integer-valued random variable (which is a function of all X_i) defined as $N = \min\{k : X_k = +1\}$. Find EX_N.