Equity modelling with local stochastic volatility and stochastic discrete dividends

In this article, Pierre Henry-Labordère and Hamza Guennoun extend previous work on the calibration of local stochastic volatility models with discrete dividends by incorporating stochastic dividends. An exact calibration method is obtained using the particle algorithm

Modelling (discrete) dividends is a crucial issue. Equity products such as knockout dividend swaps for which the payoff is of the form:

$$1_{\min_{t\in[T_1,T_2]} S_t<B}\, F\left(\sum_{t_i\in[T_1,T_2]} D_{t_i}\right)$$

require us to handle separately the dynamics of the (future) dividends and the equity asset. The appearance of the digital payoff on $\min_{t\in[T_1,T2]} S_t$ calls for an equity model that is compatible with the smile, such as the local volatility model (Dupire 1994). The calibration of the local (stochastic) volatility model (LSVM) with stochastic dividends, despite its importance, has not received much attention thus far.

In Henry-Labordère (2009), an exact calibration method is found for a special (simple) stochastic dividend model in which the dividends are an affine function of the underlying equity. In this case, when the yield curve is non-zero, the dividend volatility is uniquely driven by the equity volatility, and the dividend forwards are perfectly correlated with the underlying equity. This model is not flexible enough for pricing derivatives on dividends.

In the present article, we show how to calibrate an LSVM exactly using a generic stochastic dividend model. The algorithm, based on the particle method introduced in Guyon & Henry-Labordère (2012), is detailed in the article. In the next section, we specify a dividend model, which is (1) easy to simulate and (2) consistent with the market dividend smile. In the last section, we illustrate our algorithm using various numerical experiments.

Dividend modelling

■ **Markov representation of dividend futures.** Let us consider a Markovian representation of the dividend forward curve:

$$\mathbb{E}_t[D_{t_i}] = f_i(t, X_t)$$

with X_t a low-dimensional Itô diffusion. For example, we take a one-dimensional Ornstein-Uhlenbeck process:

$$\mathrm{d}X_t = -kX_t\,\mathrm{d}t + \mathrm{d}B_t \quad \text{with } X_0 = 0$$

that can be simulated exactly. As $\mathbb{E}_t[D_{t_i}]$ is driftless by construction, we should have for each dividend:

$$\mathbb{E}_t[D_{t_i}] = \mathbb{E}_t\left[f_i\left(t_i, \mathrm{e}^{-k(t_i-t)}X_t + \int_t^{t_i} \mathrm{e}^{k(t_i-s)}\,\mathrm{d}B_s\right)\right]$$

depending uniquely on the boundary condition $f_i(t_i,\cdot)$ at t_i. Finally, $f_i(t_i,\cdot)$ is specified by assuming the cumulative distribution of the positive random variable D_{t_i} is that of a parametric cumulative distribution $F^{\vec{\lambda},i}_{\mathrm{param}}(\cdot)$ on $\mathbb{R}^+$ (depending on some parameters denoted generically as $\vec{\lambda}$), left unspecified for the moment:

$$\mathbb{E}[1_{D_{t_i}<K}] \equiv F^{\vec{\lambda},i}_{\mathrm{param}}(K)$$

By inversion, this gives

$$f_i^{-1}(t_i, K) = \left(\frac{\mathrm{e}^{2kt_i}-1}{2k}\right) N^{-1}(F^{\vec{\lambda},i}_{\mathrm{param}}(K)) \tag{1}$$

with $N(\cdot)$ the Gaussian cumulative distribution. Modulo the choice of $F^{\vec{\lambda},i}_{\mathrm{param}}$, this specifies completely our dividend model. Our construction is similar to Markov-functional models used in fixed-income modelling.

Note that in Buehler *et al* (2010) the dividend curve is also assumed to be driven by an Ornstein-Uhlenbeck process. Unfortunately, this model's main drawback is that its dividends can become negative, and it does not price in any skew dividends. Below, we will show how to choose $F^{\vec{\lambda},i}_{\mathrm{param}}$ such that (1) vanillas on dividends are matched and (2) dividends are positive. In particular, only three global parameters $\vec{\lambda}$ will be used for each dividend in (T_1, T_2).

■ **Calibration on dividend vanillas.** Vanillas on dividends are quoted on the market. They depend on realised dividends between two future dates, say, T_1 and T_2; in practice, T_1 corresponds to the first day of a year and $T_2 = T_1 + 1$ years:

$$\begin{aligned} C^{T_1,T_2}(K) &\equiv \mathbb{E}\left[\left(\sum_{T_1<t_i<T_2} D_{t_i} - K\right)^+\right] \\ &= \mathbb{E}\left[\left(\sum_{T_1<t_i<T_2} f_i(t_i, X_{t_i}) - K\right)^+\right] \end{aligned}$$

Below, we will assume the cumulative distribution $F^{\vec{\lambda},i}_{\mathrm{param}}$ comes from a stochastic volatility inspired (SVI) parameterisation of the implied volatility $\sigma^2_{\mathrm{SVI}}(K)$ (Gatheral & Jacquier 2014), with three parameters $\vec{\lambda} = \{\text{atmvol, leftslope, rightslope}\}$ that are identical for each $t_i \in (T_1, T_2)$:

$$F^{\vec{\lambda},i}_{\mathrm{param}}(K) \equiv 1 + \partial_K\,\mathrm{BS}(\sigma^2_{\mathrm{SVI}}(K)t_i, K, f_i(0,0)) \quad \forall t_i \in (T_1, T_2)$$

$\mathrm{BS}(v, K, S_0)$ denotes the Black-Scholes formula with variance v, strike K and spot S_0. As dividend vanilla options are mostly liquid around the forward:

$$\sum_{T_1<t_i<T_2} f_i(0,0)$$

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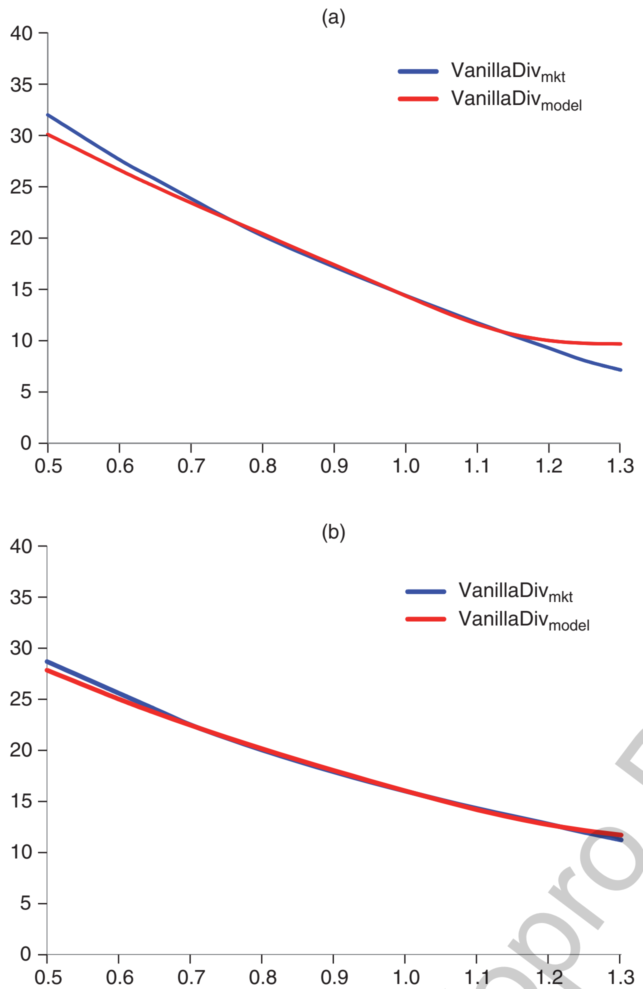
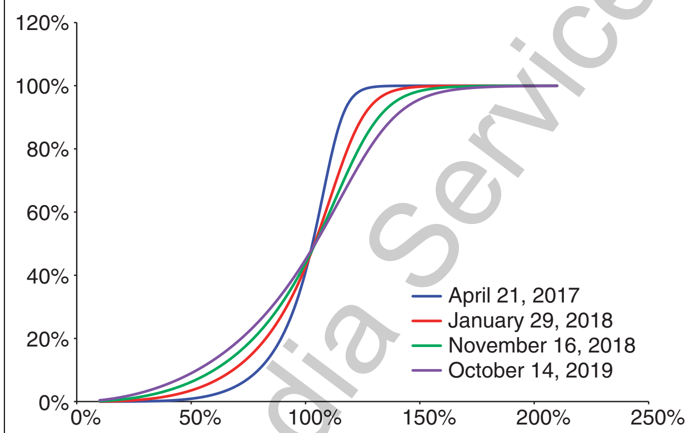
1 Market- versus model-implied volatilities of call options on Euro Stock dividends between January 1 and December 25 in (a) 2017 and (b) 2018, as a function of the relative strike $K / \sum_{T_1<t_i<T_2} f_i(0,0)$



We have taken $k = 0$. The implied volatility $\sigma(K)$ is the volatility σ such that $C^{T_1,T_2}(K) = \text{BS}(\sigma^2 T_2, K, \sum_{T_1<t_i<T_2} f_i(0,0))$

2 The cumulative distribution of dividends with different payment dates



We used the same SVI parameters that calibrated call options on Euro Stock dividends paid between January 1, 2018 and December 25, 2018

we have decided to use this simple low-dimensional parameterisation because it has the advantage of being arbitrage-free (and therefore the cumulative distribution $F_{\text{param}}^{\vec{\lambda},i}$ is well-defined) if and only if:

$$(\text{leftslope} + \text{rightslope}) \max(\text{leftslope}, \text{rightslope}) < 2$$
$$\text{atmvol}\sqrt{t_i} \max(\text{leftslope}, \text{rightslope}) < 2$$

Note that the parameters $\vec{\lambda}$ being the same does not mean the distributions of D_{t_i} are the same (see figure 2). It only means dividend options $\mathbb{E}[(D_{t_i} - K)_+]$ have the same smile for $t_i \in [T_j, T_{j+1}]$. Note that the dividends D_{t_i} are not perfectly correlated either since the process X_t is stochastic and $D_{t_i} = f_i(t_i, X_{t_i})$.

■ **Meta-algorithm 1.** To match (dividend) vanilla market prices, we use the following algorithm.

■ (1) Simulate (and store) N paths of X_{t_i}, $t_i \in (T_1, T_2)$. This can be done exactly without introducing any discretisation, as we are using an Ornstein-Uhlenbeck process. In practice, we use $N = 2^{14}$.

■ (2) Estimate the parameters {atmvol, leftslope, rightslope}. From (1), store the pairs $(f_i^{-1}(t_i, K), K)$ in a one-dimensional spline for K in a grid. Then, by Monte Carlo, compute $C^{T_1,T_2}(K_i)$ for the liquid strikes $(K_i)_{i=1,\dots,M}$ using the above precomputed paths.

■ (3) Optimise the parameters {atmvol, leftslope, rightslope} in order to match the market values:

$$\min_{\{\text{atmvol},\text{leftslope},\text{rightslope}\}} \sum_i (\sigma_{\text{Div}}(K_i) - \sigma_{\text{Div,Mkt}}(K_i))^2$$

where:

$$C^{T_1,T_2}(K_j) \equiv \text{BS}\left(\sigma_{\text{Div}}(K_j)^2 T_2, K_j, \sum_{T_1<t_i<T_2} f_i(0,0)\right)$$

This algorithm allows us to match the market values of vanillas on dividends and therefore define the functional $f_i(t_i, \cdot)$ for all $t_i \in (T_1, T_2)$. As the subsequent vanillas on dividends are written on some non-overlapping intervals $(T_a, T_{a+1})_{a=1,\dots,N}$, this specifies all the functionals $f_i(t_i, \cdot)$ for all t_i. In figure 1, we show some examples of calibration. As can be observed, the calibration is correct despite our use of a three-factor parameterisation. In practice, dividend vanillas are not very liquid, so we do not need a more evolved approach (such as a non-parametric one) to better fit dividend smiles.

LSVMs with stochastic dividends

We consider an underlying process S_t that decreases in line with the dividend amount D_{t_i} paid at time t_i and modelled as in the previous section. Its dynamics under a risk-neutral measure $\mathbb{P}$ is:

$$\left.\begin{aligned} \mathrm{d}S_t &= r(t)S_t\,\mathrm{d}t + \sigma_t\,\mathrm{d}W_t \\ S_{t_i^+} &= S_{t_i^-} - D_{t_i} \\ \mathrm{d}\langle W, B\rangle_t &= \rho_{S\text{Div}}\,\mathrm{d}t \end{aligned}\right\} \tag{2}$$

with $r(t)$ representing a deterministic rate. The extension of our work to stochastic interest rates would be straightforward and is therefore left to the

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reader. The stochastic volatility σ_t, left unspecified for the moment, will be chosen in order to match equity vanilla options with discrete maturities $T_1 < \cdots < T_{\mathcal{N}}$ (which is different, in practice, from a dividend date):

$$e^{-\int_0^{T_i} r(s)\,ds}\mathbb{E}[(S_{T_i} - K)^+] = C^{\text{mkt}}(T_i, K) \quad \forall K \in \mathbb{R}_+$$

The calibration of LSVMs with cash/yield dividends requires the interpolation of the implied volatility surface between the maturities T_i. In the case of discrete stochastic dividends, this is not obvious as call prices at ex-dividend dates should satisfy the matching condition:

$$\mathbb{E}[(S_{t_i^-} - D_{t_i} - K)^+] = C^{\text{mkt}}(t_i^+, K)$$

depending not only on the risk-neutral marginal of $S_{t_i^-}$ but also on the joint distribution of $(S_{t_i^-}, D_{t_i})$. We denote by $\mathbb{P}_i^{\text{mkt}}$ the marginal implied by T_i-(equity) vanillas:

$$\mathbb{P}_i^{\text{mkt}}(K) \equiv e^{\int_0^{T_i} r(s)\,ds}\partial_K^2 C^{\text{mkt}}(T_i, K)$$

For use below, we set:

$$S_t^{(n)} \equiv S_t e^{\int_t^{T_n} r(s)\,ds} - \sum_{t<t_i<T_n} e^{\int_{t_i}^{T_n} r(s)\,ds}\mathbb{E}_t^{\mathbb{P}}[D_{t_i}] \quad \forall t \in [0, T_n],\ n = 1, \ldots, \mathcal{N}$$

$S_t^{(n)}$ is a continuous martingale that satisfies $S_{T_n}^{(n)} = S_{T_n}$ and:

$$dS_t^{(n)} = e^{\int_t^{T_n} r(s)\,ds}\sigma_t\,dW_t - \sum_{t<t_i<T_n} e^{\int_{t_i}^{T_n} r(s)\,ds}\,d\mathbb{E}_t[D_{t_i}] \quad \forall t \in [0, T_n] \tag{3}$$

Financially, it is the T_n-forward price at time t of the underlying S.

Note that the process $S_t^{(n)}$ does not jump on ex-dividend dates. In fact, while the process $S_t \exp(\int_t^{T_n} r(s)\,ds)$ jumps with $\exp(\int_{t_j}^{T_n} r(s)\,ds)D_{t_j}$ at t_j, the process:

$$\sum_{t<t_i<T_n} e^{\int_{t_i}^{T_n} r(s)\,ds}\mathbb{E}_t^{\mathbb{P}}[D_{t_i}]$$

jumps by exactly the same amount, meaning:

$$S_t e^{\int_t^{T_n} r(s)\,ds} - \sum_{t<t_i<T_n} e^{\int_{t_i}^{T_n} r(s)\,ds}\mathbb{E}_t^{\mathbb{P}}[D_{t_i}]$$

does not jump.

Let σ_D^i denote the volatility of the t_i-dividend future:

$$d\mathbb{E}_t[D_{t_i}] = \sigma_D^i\,dB_t, \quad \sigma_D^i = \partial_x f_i(t, X_t) \tag{4}$$

Calibration on equity vanillas. The calibration is done in two steps. In step (1), we specify a continuous process $\bar{S}^{(n)}$ (with local volatility $\sigma_{\text{Dup}}^{(n)}(t, \bar{S}_t^{(n)})$) having the same marginals as $S_t^{(n)}$ at T_{n-1} and the market marginal $\mathbb{P}_n^{\text{mkt}}$ at T_n. In step (2), we specify σ_t as a function of $\sigma_{\text{Dup}}^{(n)}$.

Building a local volatility $\sigma_{\text{Dup}}^{(n)}$. Assume the dynamics of S_t are specified for $t \leqslant T_{n-1}$ and our model is calibrated to equity vanillas for maturities $T_1 < \cdots < T_{n-1}$. Then, we can compute (eg, by Monte Carlo; see details in the 'Meta-algorithm' section below) call options at T_{n-1} of $S^{(n)}$ for all strikes K (in a grid):

$$\begin{aligned} C_n^{\text{model}}&(T_{n-1}, K) \\ &\equiv e^{-\int_0^{T_{n-1}} r(s)\,ds}\mathbb{E}[(S_{T_{n-1}}^{(n)} - K)^+] \\ &= e^{-\int_0^{T_{n-1}} r(s)\,ds}\mathbb{E}\bigg[\bigg(S_{T_{n-1}} e^{\int_{T_{n-1}}^{T_n} r(s)\,ds} \\ &\qquad - \sum_{T_{n-1}<t_i<T_n} e^{\int_{t_i}^{T_n} r(s)\,ds}\mathbb{E}_{T_{n-1}}[D_{t_i}] - K\bigg)^+\bigg] \end{aligned}$$

At time T_n, as $S_{T_n}^{(n)} = S_{T_n}$, in order to be calibrated to T_n-(equity) vanillas, we need to impose:

$$C_n^{\text{model}}(T_n, K) \equiv e^{-\int_0^{T_n} r(s)\,ds}\mathbb{E}[(S_{T_n}^{(n)} - K)^+] = C^{\text{mkt}}(T_n, K)$$

From $C_n^{\text{model}}(T_{n-1}, K)$ and $C_n^{\text{model}}(T_n, K) = C^{\text{mkt}}(T_n, K)$, which are known respectively from a Monte Carlo computation and the market value of T_n-(equity) vanillas, we can derive the Black-Scholes-implied volatility at T_{n-1} (respectively, T_n), defined as the volatility $\sigma_n(T_{n-1}, K)$ (respectively, $\sigma_n(T_n, K)$) such that:

$$C_n^{\text{model}}(T_\alpha, K) = \text{BS}(\sigma(T_\alpha, K)^2 T_\alpha, K, S_0^{(n)}), \quad \alpha = \{n-1, n\}$$

Linearly interpolating the variances $\sigma(T_{n-1}, K)^2 T_{n-1}$ and $\sigma(T_n, K)^2 T_n$ specifies an implied volatility surface for all $t \in [T_{n-1}, T_n]$ by:

$$\sigma(t, K)^2 t \equiv (\sigma_n(T_n, K)^2 T_n - \sigma_n(T_{n-1}, K)^2 T_{n-1})\frac{(t - T_{n-1})}{T_n - T_{n-1}} + \sigma_n(T_{n-1}, K)^2 T_{n-1} \tag{5}$$

By assuming:

$$\sigma_n(T_n, K)^2 T_n \geqslant \sigma_n(T_{n-1}, K)^2 T_{n-1}$$

(this can be checked numerically), we can specify a (factious) process $\bar{S}_t^{(n)}$:

$$\begin{aligned} d\bar{S}_t^{(n)} &= \sigma_{\text{Dup}}^{(n)}(t, \bar{S}_t^{(n)})\,dW_t, \quad t \in [T_{n-1}, T_n] \\ \sigma_{\text{Dup}}^{(n)}(t, K)^2 &\equiv \frac{\partial_t C(t, K)}{\partial_K^2 C(t, K)} \end{aligned} \tag{6}$$

where $C(t, K) = \text{BS}(\sigma(t, K)^2 t, K, S_0)$. By construction, $\bar{S}_{T_n} \sim \mathbb{P}_n^{\text{mkt}}$. Note that, since the process $S^{(n)}$ does not jump as we go through an ex-dividend date, we can (1) linearly interpolate the variance (see (5)) and (2) use the Dupire formula (see (6)).

Building σ_t from $\sigma_{\text{Dup}}^{(n)}$. From Dupire (1996), the processes S_t^n and $\bar{S}_t^{(n)}$ have the same marginals for all $t \in [T_{n-1}, T_n]$ (in particular, $S_{T_n}^n = S_{T_n} \sim \mathbb{P}_n^{\text{mkt}}$) if and only if the following proposition holds.

Proposition 1 *$S_{T_n} \sim \mathbb{P}_n^{\text{mkt}}$ for all $n = 1, \ldots, \mathcal{N}$ if and only if σ_t satisfies:*

$$\mathbb{E}\bigg[\bigg(\frac{d\langle \ln S\rangle_t}{dt}\bigg) \bigg| S_t^{(n)} = K\bigg] = \sigma_{\text{Dup}}^{(n)}(t, K)^2 \quad \forall t \in [T_{n-1}, T_n]$$

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By taking $\sigma_t = a_t\sigma(t, S_t^{(n)})$ for all $t \in [T_{n-1}, T_n]$, with a_t a multi-factor stochastic volatility model (see, for example, Henry-Labordère 2009), we get a second-order algebraic equation on $\sigma(t, K)$ that can be solved explicitly:

$$\sigma(t, K) = I_1(t, K) + \sqrt{\Delta(t, K)} \tag{7}$$

with:

$$\Delta(t, K) \equiv I_1^2(t, K) + \mathrm{e}^{2\int_t^{T_n} r(s)\,\mathrm{d}s} I_0(t, K)(\sigma_{\mathrm{Dup}}^{(n)}(t, K)^2 - I_2(t, K))$$

and:

$$I_0(t, K) \equiv \mathbb{E}^{\mathbb{P}}[a_t^2 \mid S_t^{(n)} = K]$$

$$I_1(t, K) \equiv \rho_{S\mathrm{Div}} \mathrm{e}^{\int_t^{T_n} r(s)\,\mathrm{d}s} \sum_{t<t_i<T_n} \mathrm{e}^{\int_{t_i}^{T_n} r(s)\,\mathrm{d}s} \mathbb{E}^{\mathbb{P}}[a_t\sigma_D^i \mid S_t^{(n)} = K]$$

$$I_2(t, K) \equiv \sum_{t<t_i,t_j<T_n} \mathrm{e}^{2\int_{t_i}^{T_n} r(s)\,\mathrm{d}s} \mathbb{E}^{\mathbb{P}}[\sigma_D^i\sigma_D^j \mid S_t^{(n)} = K]$$

Here, we need to assume $\Delta(t, K) \geqslant 0$ (this can be checked numerically). Finally, the dynamics of S_t reads:

$$\mathrm{d}S_t = r(t)S_t\,\mathrm{d}t + \sigma(t, S_t^{(n)})a_t\,\mathrm{d}W_t \quad \forall t \in [T_{n-1}, T_n]$$

To compute the functions I_0, I_1 and I_2, we use the particle method, which was introduced in Guyon & Henry-Labordère (2012) and is detailed below. In principle, for a local volatility model with one-factor stochastic dividend, one can also rely on a numerical solution of the two-dimensional Fokker-Planck partial differential equation (PDE).

In figure 4, we have plotted the local volatility of stochastic and deterministic dividends, both calibrated on the same smile. We can see the local volatility of deterministic dividends is lower than that of stochastic dividends. This means if we calibrate a local volatility model with deterministic dividends on a market smile and then use a local volatility model with stochastic dividends, the model smile will be lower than the market smile. This is exactly what we see in figure 3.

Below, we explain why the local volatility in the case of stochastic dividends needs to be higher than that of deterministic dividends in order to calibrate the same smile. To calibrate vanillas with maturity T, say, the T-forwards in the two models need to have basically the same volatility. However, since the volatility of a T-forward is equal to the difference of the volatility of the underlying and the volatility of the dividend futures (in the case of 100% correlation), as the volatility of the dividends increases, so too does the volatility of the underlying in order to ensure it maintains the same volatility as the T-forward.

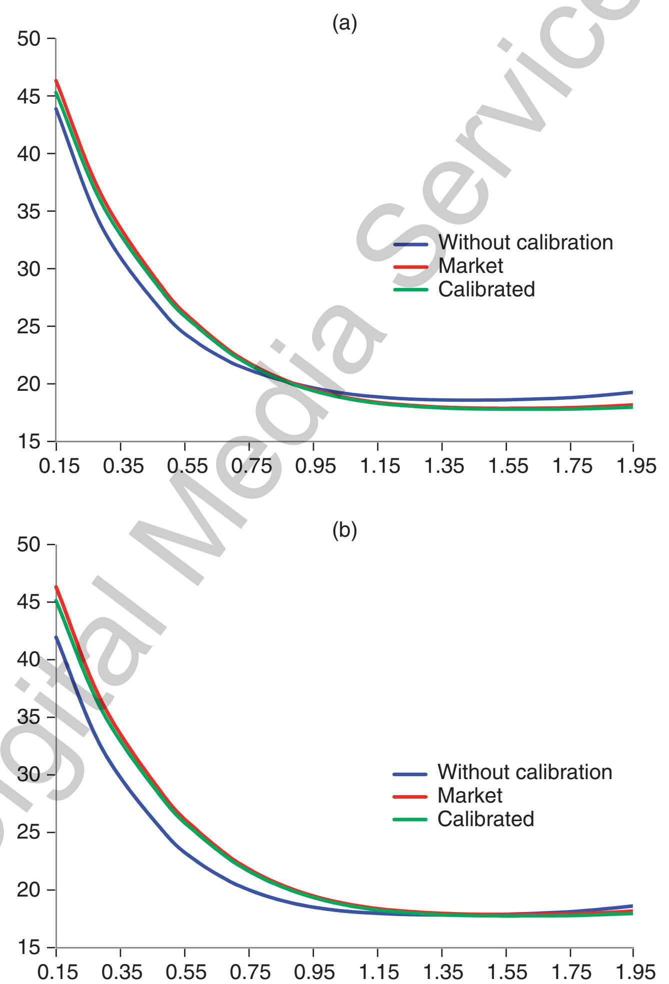
Particle system. We define a system composed of N processes $(S_t^{i,N}, X_t^{i,N}, a_t^{i,N})_{i=1,\dots,N}$ by:

$$\begin{aligned} \mathrm{d}S_t^{i,N} &= r(t)S_t^{i,N}\,\mathrm{d}t + \sigma_N(t, S_t^{i,N,(n)})a_t^{i,N}\,\mathrm{d}W_t^{i,N} \quad \forall t \in [T_{n-1}, T_n] \\ \mathrm{d}X_t^{i,N} &= -kX_t^{i,N}\,\mathrm{d}t + \mathrm{d}B_t^{i,N}, \quad \mathrm{d}\langle B^{i,N}, W^{i,N}\rangle_t = \rho_{S\mathrm{Div}}\,\mathrm{d}t \end{aligned} \tag{8}$$

where:

$$S_t^{i,N,(n)} = S_t^{i,N}\mathrm{e}^{\int_t^{T_n} r(s)\,\mathrm{d}s} - \sum_{t<t_j<T_n} \mathrm{e}^{\int_{t_j}^{T_n} r(s)\,\mathrm{d}s} f_j(t, X_t^{i,N})$$

3 Implied volatilities of the EuroStock as of September 22, 2016 for $T = 3.3$ years: (a) $\rho_{S\mathrm{Div}} = 0.5$; (b) $\rho_{S\mathrm{Div}} = 0.9$



The Brownian motions $(W_t^{i,N}, B_t^{i,N})$ and $(W_t^{j,N}, B_t^{j,N})$ are independent for all $i \neq j$ and $\sigma_N(t, K)$ is defined by:

$$\begin{aligned} \sigma_N(t, K) &= I_1^N(t, K) \\ &\quad + \sqrt{I_1^N(t, K)^2 + \mathrm{e}^{2\int_t^{T_n} r(s)\,\mathrm{d}s} I_0^N(t, K)(\sigma_{\mathrm{Dup}}^{(n)}(t, K)^2 - I_2^N(t, K))} \end{aligned} \tag{9}$$

$I_0^N(t, K)$, $I_1^N(t, K)$ and $I_2^N(t, K)$ are obtained by replacing in the definitions of $I_0(t, K)$, $I_1(t, K)$ and $I_2(t, K)$ the conditional expectations with respect to $\mathbb{P}$ with the empirical measure:

$$\mathbb{P}_t^N = \frac{1}{N}\sum_{i=1}^{N} \delta_{(S_t^{i,N}, a_t^{i,N}, X_t^{i,N})}$$

For example:

$$I_0^N(t, K) \equiv \frac{\sum_{i=1}^N (a_t^{i,N})^2 \delta_{t_i,N}(K - S_t^{i,N})}{\sum_{i=1}^N \delta_{t_i,N}(K - S_t^{i,N})}$$

where $\delta_{t_i,N}$ is a kernel. In practice, we take:

$$\delta_{t,N}(x) = \frac{1}{h_{t,N}} K\left(\frac{x}{h_{t,N}}\right)$$

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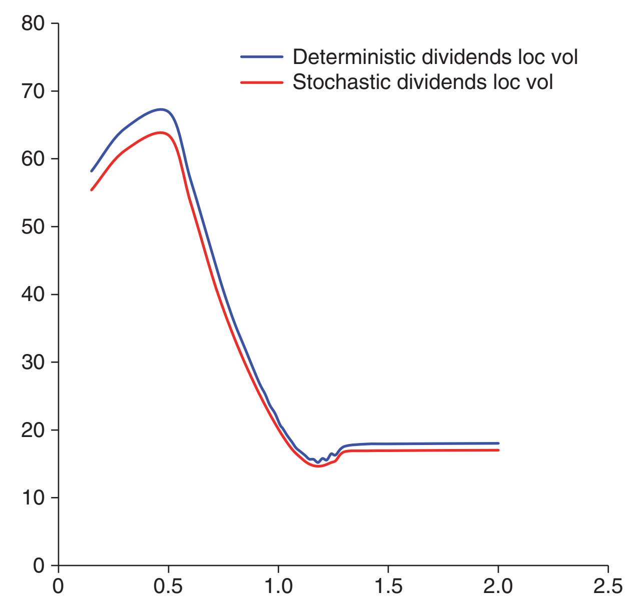
4 Calibrated local volatility at $t = 0.5$ of deterministic and stochastic dividends: $\rho_{S\mathrm{Div}} = 90\%$



where K is a fixed, symmetric kernel with a bandwidth $h_{t,N}$ that tends to zero as N grows to infinity. We use in our implementation the exponential kernel $K(x) = (1/\sqrt{2\pi})\mathrm{e}^{-x^2/2}$. We take:

$$h_{t,N} = \kappa S_0 \sigma_0 \sqrt{\max(t, t_{\min})}\, N^{-1/5}$$

The factor $N^{-1/5}$ comes from the Nadaraya-Watson estimator, which is the sum of two terms: bias and variance. The smaller the bandwidth, the smaller the bias, but the larger the variance. The critical bandwidth that minimises the sum of the bias and variance decreases with $N^{-1/5}$ for large N. The prefactor $\kappa S_0 \sigma_0 \sqrt{\max(t, t_{\min})}$ is of the order of the standard deviation of the regressor S_t. The fine-tuning of κ is crucial. In practice, we take $\kappa = 1.5$, $t_{\min} = \frac{1}{4}$ and σ_0 of the order of the mean volatility. Similar expressions are used for $I_1^N(t, K)$ and $I_2^N(t, K)$.

■ **Meta-algorithm.** To match equity vanilla market prices, we use the following algorithm.

■ (0) From T_1-vanillas, compute the local volatility $\sigma^{(1)}_{\mathrm{Dup}}(t, K)$ using Dupire's local volatility formula (6).

■ (1) Then compute the local volatility $\sigma(\cdot, \cdot)$ as given in (9) for all $t \in [0, T_1]$ using the particle method. This consists of:

Building $\sigma(t, \cdot)$**:**

■ (i) Set $k = 1$ and $\sigma(0, S_0) = \sigma^{(1)}_{\mathrm{Dup}}(0, S_0)$ for all $t \in [0, (k/M)T_1]$.

■ (ii) Simulate N processes $\{S_t^{i,N}, a_t^{i,N}, X_t^{i,N}\}_{1 \leqslant i \leqslant N}$ from t_{k-1} up to $t_k \equiv (k/M)T_1$ using a discretisation scheme for stochastic differential equation (8), say, an Euler scheme.

■ (iii) Compute the local volatility $\sigma_N(t, K)$ from (9).

■ (iv) Set $k := k + 1$. Iterate steps (ii) and (iii) up to maturity T_1.

■ (2) Compute T_1 call options on:

$$S^{(2)}_{T_1} = S_{T_1} \mathrm{e}^{\int_{T_1}^{T_2} r(s)\,\mathrm{d}s} - \sum_{T_1 < t_i < T_2} \mathrm{e}^{\int_{t_i}^{T_2} r(s)\,\mathrm{d}s} \mathbb{E}_{T_1}[D_{t_i}]$$

for all K (in a grid):

$$\mathbb{E}[(S^{(2)}_{T_1} - K)^+] \approx \frac{1}{N} \sum_{i=1}^{N} (S^{i,N,(2)}_{T_1} - K)^+$$

■ (3) From the implied volatility of $S^{(2)}_{T_1}$ at T_1 and the market-implied volatility at T_2, compute the local volatility $\sigma^{(2)}_{\mathrm{Dup}}(t, K)$ as outlined previously in the section on building a local volatility. Then, compute the local volatility $\sigma(t, K)$ (7) in the interval $[T_1, T_2]$ using the particle method (see (i)–(iv)). Compute the implied volatility of:

$$S^{(3)}_{T_2} = S_{T_1} \mathrm{e}^{\int_{T_2}^{T_3} r(s)\,\mathrm{d}s} - \sum_{T_2 < t_i < T_3} \mathrm{e}^{\int_{t_i}^{T_1} r(s)\,\mathrm{d}s} \mathbb{E}_{T_2}[D_{t_i}]$$

■ (4) Iterate up to $T_{\mathcal{N}}$.

Control variate. In step (2), we use a variate control in order to compute the call price $\mathbb{E}[(S^{(2)}_{T_1} - K)^+]$ with a low number of Monte Carlo paths (in practice, we take $N = 2^{12}$):

$$\begin{aligned}
&\mathbb{E}[(S^{(2)}_{T_1} - K)^+] \\
&\quad = \mathbb{E}\bigg[(S^{(2)}_{T_1} - K)^+ \\
&\qquad\qquad - \bigg(S_{T_1} \mathrm{e}^{\int_{T_1}^{T_2} r(s)\,\mathrm{d}s} - \sum_{T_1 < t_i < T_2} \mathrm{e}^{\int_{t_i}^{T_2} r(s)\,\mathrm{d}s} \mathbb{E}[D_{t_i}] - K\bigg)^+\bigg] \\
&\qquad + \mathrm{e}^{\int_0^{T_2} r(s)\,\mathrm{d}s} C^{\mathrm{mkt}}\bigg(T_1, K + \sum_{T_1 < t_i < T_2} \mathrm{e}^{\int_{t_i}^{T_1} r(s)\,\mathrm{d}s} \mathbb{E}[D_{t_i}]\bigg) \\
&\quad \approx \frac{1}{N} \sum_{i=1}^{N} (S^{i,N,(2)}_{T_1} - K)^+ \\
&\qquad - \bigg(S^{i,N}_{T_1} \mathrm{e}^{\int_{T_1}^{T_2} r(s)\,\mathrm{d}s} - \sum_{T_1 < t_i < T_2} \mathrm{e}^{\int_{t_i}^{T_2} r(s)\,\mathrm{d}s} f_i(t_i, 0) - K\bigg)^+ \\
&\qquad + \mathrm{e}^{\int_0^{T_2} r(s)\,\mathrm{d}s} C^{\mathrm{mkt}}\bigg(T_1, K + \sum_{T_1 < t_i < T_2} \mathrm{e}^{\int_{t_i}^{T_1} r(s)\,\mathrm{d}s} f_i(t_i, 0)\bigg)
\end{aligned}$$

This consists of replacing $\mathbb{E}_{T_1}[D_{t_i}]$ with $\mathbb{E}[D_{t_i}]$. In particular, in the case where the volatility $\sigma_D = 0$, our algorithm produces an exact calibration even if we use a single particle (ie, $N = 1$). As the volatility σ_D is low (around the equity at-the-money volatility), our control variate is very efficient, and we do not need to use a large number of particles.

Numerical experiments

We have checked the accuracy of our calibration procedure on the Euro Stock market-implied volatility (September 22, 2016). We have a dividend date every four days on average. We have considered only a local volatility model (for which $a_t = 1$), as this seems to be the most appropriate model for pricing dividend derivatives as outlined in the introduction. The stochastic dividend model has been calibrated on dividend Euro Stock vanillas as outlined in the section on dividend modelling (see also figure 1). We have chosen $\rho_{S\mathrm{Div}} = 50\%$ or $\rho_{S\mathrm{Div}} = 90\%$. The Euler time step has been set to $\Delta t = 1/250$, and we have used $N = 2^{12}$ particles. After calibrating the model using the procedure outlined above,

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we have computed equity vanilla-implied volatilities with $T = 3.3$ years using a (quasi) Monte Carlo pricer with $N = 2^{15}$ paths (see figure 3). We then compared this with our market inputs. We have a perfect match (errors around a few basis points). We have also plotted the prices of equity vanillas (see green curve) obtained by calibrating local volatility with the deterministic dividend curve $\mathbb{E}[D_{t_i}] = f_i(t_i, 0)$, using the method in Henry-Labordère (2009). We observe that we then have an error at-the-money of around 100 basis points, meaning the use of our algorithm is necessary.

The calibration on dividend vanillas takes around 1 second, and the calibration on equity vanillas takes approximately 1 minute. Note that the computational timing of the calibration on equity vanillas depends on the maturity and also on the number of intermediate vanilla smile maturities. In our numerical example, the market smile included 19 intermediate maturities before $T = 3.3$ years.

■ **Non-vanilla pricing example.** We price a three-year Athena autocall on an underlying S defined as follows.

- At each year i, before maturity, if $S_i > S_0$, pay $6\%\, i$ and the product ends.
- At maturity, if the product has not yet ended, pay $1 - 1_{(S_T/S_0)<70\%}(1 - (S_T/S_0))$.

Using the same model (calibrated on equity vanillas) as in our previous numerical results, we get the following prices:

$$\begin{aligned} \text{deterministic div}: &\quad 92.64\% \\ \text{stochastic div, } \rho_{S\mathrm{Div}} = 50\%: &\quad 92.82\% \\ \text{stochastic div, } \rho_{S\mathrm{Div}} = 90\%: &\quad 93.06\% \end{aligned}$$

To interpret these numerical results in an intuitive way, assume that, in the life of the product, at time t the underlying is very low. In this case, the mark-to-market is very close to $\mathbb{E}_t[S_T/S_0]$, which is proportional to the T-forward. Remember that in the absence of interest rates the forward is equal to the underlying value minus the dividend futures (for dividend dates between t and maturity T). Therefore, if the equity/dividend correlation is high, there is a high probability the dividend futures are small in the case of a decreasing underlying. This means the T-forward in this case will be higher, and consequently the autocall price will be higher. ■

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