Alan L. Lewis

Option Valuation under Stochastic Volatility

With Mathematica Code

Option Valuation under Stochastic Volatility
With Mathematica Code

Published by: Finance Press, Newport Beach, California, USA
Cover design by: Brian Burton Design

For the latest available updates and corrections to this book and publisher contact information, visit the Internet sites:
http://www.financepress.com or http://members.home.net/financepress

International Standard Book Number: 0-9676372-0-1
Library of Congress Card Number: 99-91935

Printed in the United States of America
Printed on acid-free paper

Contents

1 Introduction and Summary of Results

Suppose we use the standard deviation ... of possible future returns on a stock ... as a measure of its volatility. Is it reasonable to take that volatility as constant over time? I think not.

Fischer Black

The problem of how to value an option is a fascinating one, with a relatively long history for a financial topic. By 1900, Bachelier suggested a fair game approach using a normal distribution for the underlying security price—almost the modern model. But Bachelier's approach became suspect once the ideas of utility theory and risk-aversion became central in economic theory. If most investors were averse to risk-taking, shouldn't that influence an option's price? Shouldn't an option be worth less than its fair game value, just like a stock?

This puzzle was resolved by Black and Scholes (B-S 1973) for a world where security prices followed a geometric Brownian motion process with constant volatility. Their elegant solution for option pricing is, surprisingly, completely independent of investors' risk attitudes and expectations.

However, the assumption of constant volatility was suspect from the beginning, as the 1976 quotation above suggests. The facing figure shows the monthly volatility realized by the S&P 500 Index returns from 1928-1999. Statistical tests strongly reject the idea that a constant volatility process could have generated these data. It also became clear, although this was less immediate, that the B-S model was in conflict with evolving patterns in observed option pricing data. In particular, after the 1987 market crash, a persistent pattern emerged, called the "smile" that shouldn't exist under the B-S theory.

In this book we study option valuation when security prices evolve with stochastic (random) volatility. Stochastic volatility models lead to generalizations of the B-S option pricing formula. The generalized models are both mathematically interesting and useful because they can explain the real-world patterns that are missing from the B-S theory. Not only do stochastic volatility models explain the basic shapes of smile patterns, but they also allow for more realistic theories of the "term structure" of implied volatility. Consequently, the material should be of interest to both financial academics and traders.

Option valuation is still an extremely active area in finance, with many references cited in the notes. We don't create a comprehensive survey—instead, our goal is to create a uniform and fairly comprehensive theoretical treatment of the case where the stock price and volatility are described by a two-dimensional diffusion process. We work exclusively in continuous-time with continuous sample paths.

Once you allow random volatility into the theory, utility functions and risk attitudes come back into the theory also. In a B-S world, where volatility is constant, option prices are determined purely by arbitrage arguments and risk attitudes play only an indirect role. (That is, to set the price of the primary securities). But, with stochastic volatility, the absence of arbitrage alone does not fix option prices. We discuss this in detail in Chapter 7 and summarize our results below.

We consider a number of particular stochastic processes for volatility. However, we have a particular interest in the case where volatility is described as the diffusion limit of a GARCH-type process—a GARCH diffusion, for short. Since the seminal work of Engle (1982), discrete-time ARCH models have become a proven approach to modeling security price volatility. For a review of the substantial literature, see Bollerslev, Chou, and Kroner (1992). Since Nelson (1990), it has been understood that GARCH-type models have well-defined continuous-time limits.

From a practical point of view, the advantage of the GARCH diffusion model is that you can estimate its parameters by using off-the-shelf software for GARCH processes.(See Appendix 1 to this chapter). Although no closed-form solution is available for options priced under the particular process we call a GARCH diffusion, we are able, nevertheless, to develop a fairly complete picture.

1 Summary of Results

Our security model for most of this book is an (equity) price process P of the general form

$$(1.1) \qquad P: \begin{cases} dS_t = (\alpha_t S_t - D_t)dt + \sigma_t S_t dB_t \\ dV_t = b(V_t)dt + \xi\,\eta(V_t)\,dZ_t \end{cases},$$

where dB_t and dZ_t are Brownian motion processes with correlation $\rho(V_t)$. In (1.1) α_t is the instantaneous expected total return of the stock, which pays the owner dividends at the dollar rate D_t. The volatility $V_t = \sigma_t^2 \geq 0$, where σ_t is the instantaneous standard deviation of the price returns. Of course, we usually call σ_t the volatility also. In addition to using P as a simple label, as in (1.1), we will sometimes use P to mean a probability (measure): namely, the one under which dB_t and dZ_t are Brownian motions. That alternative interpretation should cause no confusion.

The "volatility of volatility" parameter ξ sets the scale for the random nature of volatility and plays a very important role in the theory. The functions $b(V)$ and $a(V) = \xi\,\eta(V)$ are the drift and diffusion coefficients of the (actual) volatility process. We consider only actual volatility processes that are time-homogeneous. Sometimes the risk-adjusted volatility process will pick up a time dependence from the risk-adjustment.

Examples of stochastic volatility models

I. The GARCH Diffusion Model. The GARCH diffusion model is important because it's the continuous-time limit of many GARCH-type processes.[1]. In this book, we use 'GARCH diffusion' to mean an actual volatility process of the form:

$$(1.2) \qquad dV_t = (\omega - \theta V_t)\,dt + \xi V_t\,dZ_t\,.$$

---

[1] In fact, the term GARCH is a loose term that accommodates many types of discrete-time financial models and various continuous-time limits are possible For example, Heston and Nandi (1997) have recently shown that a degenerate case of Heston's (1993) square root model (see below) can be obtained as a limit of a particular GARCH-type process.

The volatility drift parameters, ω and θ, are assumed to be constants and capture the mean-reverting nature of the volatility process. Since θ has the dimensions of inverse time, $1/\theta$ represents a "half-life" for volatility shocks.

Empirically, estimates for $1/\theta$ are quite variable against stock indices and can range from a few weeks to more than a year. For U.S. broad-based stock indices, ξ is typically in the range of one to two on an annualized basis, which represent volatility uncertainty of 100 to 200% over a year. The correlation, ρ, captures the association between security price changes and volatility changes. Typically, negative price shocks are associated with higher volatility than positive shocks of the same magnitude. For the same indices, we find $\rho \cong -0.5$ to -0.8.

See Appendix 1 of this chapter for a simple discrete-time GARCH model that admits (1.2) as a continuous-time limit. In that Appendix, we provide more examples of parameter estimates and formulas for estimators.

II. The Square Root Model. The square root model is described by the volatility process

$$dV_t = (\omega - \theta\, V_t)\, dt + \xi\, \sqrt{V_t}\, dZ_t\,. \tag{1.3}$$

This model is very important because of two facts. First, as shown by Heston (1993), it has essentially a closed-form solution for option prices which is easy to implement. Second, that solution is *typical*: it displays the same qualitative properties that we expect in general time-homogenous cases. Consequently, it tells us, for example, how the (unknown) solution to the GARCH diffusion behaves in many respects.

For the square root model, clearly ω and θ play the same role as in the GARCH diffusion. Also ξ again sets the scale for the volatility of volatility, but its numerical magnitude will be quite different when estimated against the same pricing data because it now multiplies a factor of $\sqrt{V_t}$ instead of the factor of V_t in the GARCH case. From dimensional considerations, $\xi_{1/2} \approx \xi_1\, \bar{\sigma}_t$, where $\xi_{1/2}$ is the square root model parameter, ξ_1 is the GARCH diffusion parameter, and $\bar{\sigma}_t$ is the average volatility of the data series being considered. For example, for the S&P 500 Index, if ξ_1 is in the range (1,2) annualized, and $\bar{\sigma}_t \approx 10\% = 0.1$ annualized, then $\xi_{1/2}$ is in the approximate range (0.1, 0.2) annualized.

III. The 3/2 model. The 3/2 model is described by the volatility process

$$dV_t = (\omega V_t - \theta V_t^2)\,dt + \xi V_t^{3/2} dZ_t\,. \tag{1.4}$$

This model is important because, not only does it have a closed-form solution almost as simple as the square root model, but it displays a feature of many stochastic volatility models that you don't see in the square root model. That is, even after a change of measure to the risk-adjusted process, option prices (relative to the bond price) under the 3/2 model are sometimes not martingales, but merely local martingales. When option prices are not martingales, this means that they are not given by the standard expected value formula—for example, $e^{-rT}\mathbb{E}_t[(S_T - K)^+]$ for a call option. Here $\mathbb{E}_t$ denotes an expectation under the pricing process (see the subsection below titled "the risk-adjusted process").

This failure of the usual martingale pricing relation is actually quite common in (unbounded) stochastic volatility models under very typical risk-adjustments, such as log-utility. The failure can also occur in the GARCH diffusion, for example, but only a specialized (and very complicated) closed-form solution is currently available for that one. So the 3/2 model is one of the simplest illustrations of this important phenomenon for financial theory, and our results for that model illustrating this behavior, mostly discussed in Chapter 9, are new.[2]

Again, from dimensionality, one expects $\xi_{3/2} \cong \xi_1 \, (\bar{\sigma}_t)^{-1}$. In this case, for the S&P 500 Index, if ξ_1 is in the range $(1,2)$, and $\bar{\sigma}_t \cong 0.1$, then $\xi_{3/2}$ is in the approximate range (10, 20), again with all values annualized.

IV. The Ornstein-Uhlenbeck process. With $y_t = \ln V_t$, this model is described by the process

$$dy_t = (\tilde{\omega} - \theta\, y_t) + \xi\, dZ_t\,, \quad (-\infty < y_t < \infty)$$

---

[2] Other aspects of the 3/2 model have been independently developed by Heston (1997). The failure of the martingale formula for the stock price in the GARCH diffusion model was first shown by Sin (1998). These failures are specific examples of the notion that the absence of arbitrage implies that financial claim prices are, *in general*, only strictly local martingales—not martingales. This refinement of earlier notions about financial claim prices as martingales is due to Delbaen and Schachermayer (1994).

By using the Ito formula (explained in Sec. 2), and letting $\omega = \tilde{\omega} + \frac{1}{2}\xi^2$, the process for V_t is

$$dV_t = (\omega V_t - \theta V_t \ln V_t)\,dt + \xi V_t\, dZ_t.$$

We don't treat this model specifically, but it's a very popular one among researchers. Many of our general results apply to it.

The risk-adjusted process. To value an option, you don't use (1.1), but a closely related process $\tilde{P}$ which is often called the risk-adjusted process. Throughout this book, it has a form similar to (1.1):

$$\tilde{P}: \begin{cases} dS_t = (rS_t - D_t)dt + \sigma_t S_t\, d\tilde{B}_t \\ dV_t = \tilde{b}(V_t,t)dt + \xi\,\eta(V_t)\, d\tilde{Z}_t \end{cases}, \tag{1.5}$$

To get from (1.1) to (1.5) we have made two changes: (i) we replaced the equity expected return by an interest rate r and (ii) we replaced the volatility drift by another function $\tilde{b}(V_t,t)$, the risk-adjusted volatility drift. This procedure is carried out explicitly for a class of equilibrium models in Chapter 7: representative agent models with power utility functions. In addition, we will often assume that the representative agents have very distant planning horizons.[3] This allows one to further simplify (1.5) to a volatility process with a time-homogenous drift $\tilde{b}(V_t)$.

Equation (1.5) has an associated partial differential equation (PDE) that determines option prices (see below). In the special case where the dividend yield is constant, then (1.5) is proportional in the stock price and the PDE may be solved with a transform technique.

The transform idea is explained in Chapter 2, where we introduce the *fundamental transform* $\hat{H}(k,V,\tau)$. Besides the volatility V, the transform depends upon k, the (generalized) Fourier transform variable and τ, the time to the option's expiration. The fundamental transform is determined by the volatility process and not by the particulars of any option contract. Once you have it, the *same* function $\hat{H}(k,V,\tau)$ is used to determine the value of *every*

[3] The planning horizon is the "date of death", not to be confused with the option expiration date, which may be very near.

European-style financial claim. This last step only requires an integration in the complex k-plane. So you can see why this function deserves its title.

While the idea of a transform-based approach is not new, previous applications have tended to be model-specific. Not only are our results more general, but they encompass the situation when option prices, relative to a numeraire, are not martingales, but only strictly local martingales. Specifically, we develop two formulas for option prices, which we call Solution I and Solution II. Solution I is the usual martingale-style or expected value formula. Solution II is more general and sometimes includes an additional term that relates to volatility "explosions". When a call option price is not a martingale, the Solution II formula yields the desired arbitrage-free fair value.

The fundamental transform has a lot of special properties, many of which follow from the fact that not only is it a characteristic function, but it's an *analytic characteristic function*. In particular, the function is regular in strips in the complex k-plane and also has the *ridge property*. The ridge property, which is more or less what it sounds like, plays a very important role in the theory of the term structure of implied volatility. This property is explained in Chapter 2 and applied in Chapter 6.

In Chapter 3, we show how to solve the valuation PDE associated with (1.5) by a power series in ξ, the volatility of volatility. The first two terms of that series, which can be rapidly evaluated on any desk-top computer, may suffice for most trading applications.

In Chapter 4, we discuss "mixing theorems", which show that solutions to the valuation PDE are weighted sums of the B-S solutions. This important concept has a number of applications, including a new Monte Carlo method.

In Chapter 5, we take up the theory of the smile. The volatility of volatility series expansion generates an explicit formula for the smile through order ξ^2 that explains the qualitative patterns seen in the marketplace. Also we explore the accuracy of "quadratic approximations" to the smile by developing correction terms.

The term structure of implied volatility is the subject of Chapter 6. One new result is that we show that the asymptotic theory is very simple: the asymptotic implied volatility is given by $V_\infty^{imp} = 8\lambda(k_0)$, where λ is the first eigenvalue of a differential operator and k_0 is a complex number. Regardless of the current volatility and regardless of the degree of "moneyness", the smile flattens as a function of the time to expiration to this same value V_∞^{imp}.

In Chapter 7, we present a utility-based equilibrium theory. This theory, with a distant planning horizon, provides several new and explicit examples of transformations of the volatility drift function $b(V)$ into a risk-adjusted drift $\tilde{b}(V)$. The model is the representative agent theory. The representative has a power utility function with a constant proportional risk-aversion (CPRA) parameter γ, where $\gamma = 1$ corresponds to risk-neutrality. Under this CPRA equilibrium, we find a simple pattern for the effect of risk attitudes on option prices: with $\rho \le 0$, call or put option prices are raised in an interval $\gamma_1(\rho) < \gamma < 1$ and lowered in an interval $\gamma < \gamma_1(\rho)$ relative to their risk-neutral values.

Another key, and novel result from Chapter 7 is the following. When the representative agent is a pure investor (no consumption until a distant planning horizon), then the risk-adjustment problem has a simple solution. The volatility risk premium (defined below in Sec. 5) is given by

$$\lambda^v(V) = (1-\gamma)\rho(V)\sqrt{V} - a(V)\psi(V), \quad \text{where} \quad \psi(V) = u'(V)/u(V).$$

And $u(V)$ is the first eigenfunction of the *same* differential operator that determines λ above. The eigenfunction depends upon γ. For a pure investor with any planning horizon, and $\gamma < 1$, we prove that when $\psi(V,t)$ exists, it has the same sign as γ. The leading eigenfunction is easily computed in Mathematica in just a few seconds.

In Chapter 8, we discuss the change-of-numeraire transformation or duality for short. This transformation generalizes a known symmetry (put-call symmetry) under constant volatility. Under duality, two representative agents who are risk-neutral and have log-utility, respectively, switch places. The general effect of the change-of-numeraire transformation on stochastic volatility models has been previously noted by Schroder (1999) and is implicit in Sin (1998); what is new here is placing the transformation in the context of our CPRA equilibrium.

In Chapter 9, we give a detailed account of the effect of volatility explosions and the failure of the martingale pricing formula. In economically reasonable models, the actual volatility is recurrent and, in many models, stationary. But, after risk-adjustment either the risk-adjusted volatility, or a closely related process that we call the auxiliary volatility process, can explode. If the risk-adjusted volatility process is written $dV = \tilde{b}\,dt + a\,d\tilde{Z}$, then the auxiliary volatility process is $dV = (\tilde{b} + \rho a \sigma)dt + a\,d\tilde{Z}$. An explosion means that the volatility can reach plus infinity in finite expected time. Our main result is that, when the auxiliary volatility process can explode prior to τ, with probability $\hat{P}_e(V,\tau)$, then the call option price is not a martingale. Instead, the price is given by

$$C(S,V,\tau) = e^{-r\tau}\mathbb{E}_t[(S_T - K)^+] + Se^{-\delta\tau}\hat{P}_e(V,\tau)\,.$$

We also show that this formula is identical to the Solution II formula of Chapter 2. These results, which are new, are also extended to any "call-like" claim.

In Chapter 10, we study both the fundamental transform and option prices at large volatility. This behavior is important in a number of contexts in previous chapters. We find that, in contrast to the B-S model, it is not always true that the call option price tends to the stock price as the volatility increases to infinity. A specific counter-example is found.

Finally, in Chapter 11, we develop the closed-form solutions for the fundamental transform in our three running examples: the square root model, 3/2 model, and a special case of the GARCH diffusion (geometric Brownian motion).

Mathematica code. For readers who are Mathematica users, many code fragments are included, usually in Appendices. All of the Mathematica routines can be evaluated in short periods, ranging from a few seconds to a few minutes on a desk-top machine. But the book can also be read without these sections. No attempt is made to explain the Mathematica system or its built-in objects, but I do attempt to explain how the code fragments work in a general sense.

Notations. Frequently used notations are given on a page at the end of the book. I refer to equation numbers within a chapter by their label within that chapter: say (2.3) for the third equation in Sec. 2 of Chapter 7. But, in another chapter, I refer to that same equation as (7.2.3). The more important equations are placed in boxes for emphasis.

Subjects not covered

Discrete-time theory. Virtually all of the development in this book can be translated into a discrete-time setting. But space and time limitations, for one thing, have forced me to omit this. Also, because the results do readily translate, the discrete-time setting for the option valuation equations don't really tell you anything new.

Having said that, there is an important caveat. The continuous-time setting used here, with its continuous sample paths may not be "variable enough" for the real world. The discrete-time world, especially GARCH models, accommodates "wider than normal distributions" in a straightforward way. So this may lead one back to a discrete-time theory.

One-dimensional models. There is a large literature that considers one-dimensional diffusions of the form $dS_t = rS_t dt + \kappa(S_t) dB_t$ as an explanation for smile and term structure patterns. These models are difficult to interpret over long time periods in a world with exponentially growing stock prices—so, for the most part, we don't discuss them. The empirical performance of this type of model is reviewed by Dumas, Fleming, and Whaley (1998). One exception is that, in Chapter 9, we study $dS = S^{\varphi} dB_t$, $\varphi > 1$ as a simple example of the failure of the martingale pricing formula.

Empirical comparisons. This important area deserves a book in itself, and by those more qualified. One very nice study, which is important to the models here, is a careful examination of put-call parity by Kamara and Miller (1995). All of the theoretical models in this book satisfy the put-call parity relation, which relates European-style put and call values by an arbitrage argument. In practice, there are many small violations, but in investigating all violations in intraday transaction data, Kamara and Miller found that almost half of the "arbitrages" result in a loss when execution delays are accounted for. Moreover, the mean ex post profit in trying to exploit the violations was negative.

American-style options. The effect of stochastic volatility on the early exercise premium for American-style options is an important subject. With one exception, we only treat European-style options. The exception is Chapter 8, which presents the duality relation mentioned above. This relation, as it does in the constant volatility world, connects the early exercise boundaries of the call option and its dual put partner.

The remaining sections in this chapter review some hedging arguments under both constant and stochastic volatility and the martingale pricing argument. This material is mostly standard. Advanced readers might want to look at some comments about risk premiums in Sec. 4, comments about local martingales in Sec. 5, and then skip directly to Chapter 2.

2 The Hedging Argument of Black and Scholes

This section reviews the Black and Scholes' arbitrage argument for option valuation under constant volatility. This allows us to introduce some frequently used notation and provides a basis for the generalization to stochastic volatility.

The security market model. We begin with a stochastic differential equation (SDE) that describes the price evolution of a traded security. We call this the stock price S_t, although the underlying security may really be a bond, a currency, or a future. We assume that the instantaneous change in the stock price is described by the *actual* price process

(2.1) $$P: \ dS_t = (\alpha_t S_t - D_t)dt + \sigma S_t dB_t \, ,$$

where α_t is the expected *total* rate of return on the stock, $D_t = D(S_t, t)$ is the dollar dividend rate, σ is a *constant* stock price volatility, and dB_t is a *Brownian motion process*. Stock prices are limited liability securities: hence $S_t \geq 0$; to preserve this, an admissible dividend process must drop to zero at $S = 0$, although it need not do so in a continuous way. When we generalize the theory to stochastic volatility, the dividend may also become a function of volatility.

If you own one share of the stock over the short period dt, your total return consists of two pieces. The first is the capital gains or price change dS_t given by (2.1). In addition, you receive dividends $D_t dt$. Your expected total return is $\alpha_t dt$.

The problem is to determine a rational value F_t for an option or generalized claim on S_t. In this book a generalized *European-style* financial claim is contract that pays $g(S_T)$ to the buyer and $-g(S_T)$ to the seller at time $T \geq t$, the *expiration date*. Other than the money exchanges when a contract is opened or closed, we assume that the claim provides no other cash flows. The function $g(S)$ is called the *payoff function* and is determined by the details of the contract. For call options, the contract gives the buyer the right, but not the obligation, to acquire the stock from the seller at the fixed strike (exercise) price K on expiration. Since the rational buyer will only exercise this right if $S_T > K$, then $g(S) = \max[S - K, 0] = (S - K)^+$. Similarly, put option buyers have the right to sell at K and so $g(S) = (K - S)^+$.

Besides the stock, we need a money market security. Throughout this book, we adopt the following two assumptions:

ASSUMPTION 1 *There is a money market security that pays at the continuously compounded annual rate r, a constant*. This means that if you hold the money market security for τ years, every dollar will grow to $e^{r\tau}$.

ASSUMPTION 2 *Security markets are perfect*. This means that you can trade continuously with no transaction costs, and there are no arbitrage opportunities. Arbitrage opportunities are, loosely, a way of earning a rate of return greater than the riskless rate r with no risk.

The replication argument. The insight of Black and Scholes was that, of the three securities: the stock, the option, and the money market security, any two could be used to exactly replicate the third by a trading strategy. In their original *Journal of Political Economy* paper, they replicate the money market security by creating a portfolio consisting of the stock and the financial claim. Modern treatments tend to stress the redundant nature of the option by replicating it from the stock and money market account, but we shall follow the original idea.

The replicating portfolio consists of $n_t^{(1)}$ shares of the claim and $n_t^{(2)}$ shares of the stock; these share amounts are random variables. The replicating portfolio must be *self-financing*, which means you neither consume from it nor add money to it beyond an initial deposit W_0.

The dynamics of the replicating portfolio are easier to understand by setting it up in discrete-time $t, t + \Delta t, \cdots$, and then taking a continuous-time limit. At discrete time-t, we assume a marketplace in which events occur in the following sequence: (i) just prior to a possible dividend, the stock price has value S_t and the replicating portfolio has value W_t; (ii) the stock pays a dividend $D_t \Delta t$ per share and the stock price drops *ex-dividend* to a value $S_t^+ = S_t - D_t \Delta t$; (iii) you invest W_t in $n_t^{(1)}$ shares of the claim and $n_t^{(2)}$ shares of the (ex-dividend) stock. That is,

$$W_t = n_t^{(1)} F_t + n_t^{(2)} S_t^+$$

and entering the next period:

$$W_{t+\Delta t} = n_t^{(1)} F_{t+\Delta t} + n_t^{(2)} S_{t+\Delta t}.$$

Subtracting the last two equations yields

$$W_{t+\Delta t} - W_t = n_t^{(1)}(F_{t+\Delta t} - F_t) + n_t^{(2)}(S_{t+\Delta t} - S_t) + n_t^{(2)} D_t \Delta t.$$

In the continuous-time limit $\Delta t \to dt$ and the instantaneous change in the value of the portfolio becomes:

$$dW_t = n_t^{(1)} dF_t + n_t^{(2)} dS_t + n_t^{(2)} D_t dt. \quad \textit{(Self-financing condition)} \tag{2.2}$$

We know dS_t from (2.1), but what is dF_t? Accept for the moment that $F_t = F(S_t, t)$, some function of the stock price and time. Then, dF_t has a type of stochastic Taylor expansion, which we shall use repeatedly. It's based on the following formula.

Ito's (change-of-variable) formula. Suppose $f(X,t)$ is a twice differentiable function of the process $X(t)$. And suppose $X(t)$ evolves according to the formula $dX = b(X,t)\,dt + a(X,t)\,dB(t)$. Then $df = b_f dt + a_f dB(t)$, where

$$\begin{cases} b_f(X,t) = \dfrac{\partial f}{\partial t} + b(X,t)\dfrac{\partial f}{\partial X} + \tfrac{1}{2} a^2(X,t)\dfrac{\partial^2 f}{\partial X^2} \\ a_f(X,t) = a(X,t)\dfrac{\partial f}{\partial X} \end{cases} \tag{2.3}$$

It's more compact to use subscripts for partial derivatives, writing $a_f = a\, f_X$, etc. But to avoid confusion with the subscript t meaning just a time-dependence, we shall continue to write out time-derivatives. Applying Ito's formula to dF_t and inserting it into the self-financing condition yields

$$dW = \left[n^{(1)} \left(\frac{\partial F}{\partial t} + (\alpha S - D) F_S + \tfrac{1}{2}\sigma^2 S^2 F_{SS} \right) + n^{(2)} \alpha S \right] dt \\ + \sigma S \left(n^{(1)} F_S + n^{(2)} \right) dB. \tag{2.4}$$

The central idea of the B-S argument is that all the dependence in (2.4) on the source of randomness dB_t (the second line) can be eliminated by choosing the share allocation so that $n_t^{(2)} = -F_S n_t^{(1)}$. After this choice, what remains in (2.4) has no risk. Hence, the absence of arbitrage in the marketplace implies that the portfolio should behave like a bank account in which you earn a riskless rate r. This argument gives us a second expression for dW_t:

$$dW = rWdt = r(n^{(1)}F + n^{(2)}S)dt = rn^{(1)}(F - SF_S)dt\,.$$

Equating the two expressions for dW_t yields a PDE that the claim price must satisfy:

$$rF - rSF_S = \frac{\partial F}{\partial t} + (\alpha S - D_t)F_S + \tfrac{1}{2}\sigma^2 S^2 F_{SS} - \alpha S\, F_S\,. \tag{2.5}$$

In (2.5), a remarkable cancellation of the expected stock return α_t occurs, and one is left with the

The Black-Scholes PDE for generalized European-style financial claims:

$$-\frac{\partial F}{\partial t} = -rF + \{rS - D(S,t)\}\frac{\partial F}{\partial S} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} \tag{2.6}$$

Option values are found by solving (2.6) subject to the terminal condition $F(S,T) = g(S)$.

3 The Drift Cancellation and Option Sensitivities.

One of the attractions of the B-S theory is that the option price function is independent of the expected return on the stock. For one thing, it's notoriously hard to estimate the expected return on a stock. But, at the same time, this independence can appear puzzling because it seems natural that a stock with a very high expected return (say instantaneously following a favorable news release) would have a more valuable call option than an equally risky stock without the same prospects.

Of course, one answer is simply that if option prices didn't follow the B-S formula, then there would be an arbitrage opportunity. This is especially transparent in the discrete-time versions of the theory, nicely discussed in Pliska (1997) for example.

A second answer is that the call option price *is* very sensitive to the expected return on the stock—it's just that it's indirect, through movements in the stock price. In other words, just because the parameter α (the stock expected return) doesn't appear in the option price formula, that doesn't mean that an option price can't change dramatically when α changes. There is news; α changes;

the stock price moves up; the option price moves up. In fact, one can be much more specific. If $\alpha_c - r$ is the return premium on the call option, and $\alpha_e - r$ is the return premium on the stock, then from Ito's formula and the B-S PDE, one can show that

$$\alpha_c - r = \Omega(S,t)(\alpha_e - r)\ , \quad \text{where} \quad \Omega(S,t) = \frac{SC_S}{C} > 1$$

In other words, the return premium on the call is always greater than the return premium on the stock[4] . This last relation also shows that if you have two, say at-the-money calls on equally risky stocks, then indeed the stock with the more favorable outlook will have a call with a more favorable outlook. Even though both calls will have the same price, and the same elasticity Ω, the return premium will be higher on the call whose stock has the higher expected return.

Related considerations apply to all of the option sensitivities or so-called "Greeks". Even when a parameter appears explicitly in the B-S formula, there are both direct and indirect sensitivities. For example, the B-S call option value is positively related to increases in interest rates; $\partial c / \partial r > 0$. But, if interest rates increase, this does not mean that your call options will advance in value, since stock prices themselves will usually decline. And the decline in the stock price will often be more influential on the option price than the direct effect. The effect of an increase in the volatility parameter can be similar: a positive direct effect and (sometimes) negative indirect effect through the stock price.

4 The Hedging Argument under Stochastic Volatility

In this section we generalize, to the extent possible, the B-S hedging argument of Sec. 2.[5] The result will be a PDE that determines option values, and by implication, a generalized form of the risk-adjustment. But, as we shall see, the argument does not completely determine that equation or that risk adjustment. For that, we will need some assumptions about risk preferences.

---

[4] Thanks to Sheen Kassouf for noting this relation. For a proof, see Bergman, Grundy, and Wiener (1996), p. 1582.

[5] This generalization is due to Wiggins (1987), who specialized earlier work.

The stochastic volatility model. We now assume that the stock price and the volatility follow the actual bivariate diffusion process P of the form (1.1), which we shall write as:

$$(4.1) \qquad P: \begin{cases} dS_t = (\alpha_t^e S_t - D_t)dt + \sigma_t S_t \, dB_t \\ dV_t = b(V_t)dt + a(V_t)\, dZ_t \end{cases}$$

The B-S constant volatility theory is recovered when $a = b = 0$, which implies that $\sigma_t = \sigma_0$, a constant. We call the case where $a = 0$ but $b \neq 0$, *deterministic volatility*.

Upon what does the option price depend? The natural assumption at this point is that the financial claim has the functional form $F = F(S_t, V_t, t)$. This assumption turns out to be a very strong one because, with stochastic volatility, financial claim prices are not determined by an arbitrage argument alone. They are ultimately dependent upon investor time preferences and risk attitudes.

As it turns out, in equilibrium models of the financial markets, financial claims can easily become dependent upon lengthy lists of so-called "state variables." For example, by allowing all securities to have stochastic volatilities and then trying to "clear the market" in equilibrium, one can find that the state variables include the volatilities of *all* of the traded securities. And the pricing formula for each option depends upon the entire list! Needless to say, this can become hopelessly complex. So, we are making the following simplifying assumption:

ASSUMPTION 3 *A complete set of state variables for the financial claim problem under stochastic volatility consists of time t, the price of the underlying security S_t, and the price volatility of the underlying security V_t.*

There are important applications where this assumption can be fully justified. One example is the case of a financial claim on the price of an index which represents the aggregate market. Then, in that example and under an equilibrium with a representative investor, Assumption 3 will hold. This is the case we develop in Chapter 7.

In addition, one can also develop models of individual security option valuation under stochastic volatility where this assumption holds. For example, in the well-known single-index model for security returns, which is closely related to the CAPM model, individual security volatilities naturally decompose into the sum of two pieces: a security-specific volatility and a systematic market-related volatility. If you take the security-specific volatility to be a (security-specific) constant (or even deterministic) and the market volatility to be stochastic, then Assumption 3 will hold for that model also. That model, which we do not develop further, would be a stochastic volatility generalization of the CAPM model.

The hedging portfolio. The results are compact if we define the differential generator $\mathcal{A}$ with

$$\mathcal{A}F = (\alpha^e S - D)\, F_S + bF_V - \tfrac{1}{2}\sigma^2 S^2 F_{SS} + \tfrac{1}{2}a^2 F_{VV} + \rho\,\sigma a\,S\,F_{SV}$$

As before, create a self-financing portfolio $W_t = n_t^{(1)} F_t + n_t^{(2)} S_t$. Applying both the self-financing condition (2.2) and the Ito formula, the change in the portfolio over a small time increment is now

$$dW = \left[n^{(1)}\left(\frac{\partial F}{\partial t} + \mathcal{A}F\right) + n^{(2)}\alpha^e S\right]dt + \sigma\left(n^{(1)} S F_S + n^{(2)}\right)dB + \left(n^{(1)} a F_V\right)dZ$$

The portfolio can be made independent of one of the two risks, or some linear combination of the two[6]. Since we have already indicated an application to market index options, consider this case. If dS_t is a market return, a close relative of a riskless portfolio is one uncorrelated with the market return. The uncorrelated portfolio, while not riskless, is also a *zero beta* portfolio. That's because beta is the name for a standardized measure of a portfolio's covariance with the market. If the correlation is zero, so is the covariance. We should emphasize that, since we are not going to be able to price the option by an arbitrage argument, this choice of a reference portfolio is really quite arbitrary. Any other portfolio could be used as well. The nice thing about the zero beta portfolio is that its risks are orthogonal to the stock price and this simplifies some formulas later.

To construct the zero beta portfolio, solve for the share amounts in

$$dW dS = 0 = (n^{(1)} dF + n^{(2)} dS + n^{(2)} D dt) dS$$

---

[6] Equivalently, one cannot replicate the option by dynamic trading in the stock and money market security: the market is *incomplete*, and is completed by the introduction of an option. Of course, any second option on the same security can then be replicated.

$$= \left[n^{(1)}(\sigma S F_S + \rho\, a F_V) + n^{(2)} \sigma S \right] \sigma S dt \,.$$

That is

$$n^{(2)} = -\left(F_S + \frac{\rho a}{\sigma S} F_V \right) n^{(1)} . \tag{4.2}$$

Equation (4.2) generalizes the B-S hedge ratio formula $n_t^{(2)} = -F_S n_t^{(1)}$ from Sec. 2. Even if dS_t represents an individual security return, we can still construct a self-financing portfolio which is continuously re-balanced by formula (4.2). These portfolios are not riskless because they still have exposure to fluctuations in volatility: $dW_t dV_t \neq 0$. Since they don't replicate the money market account, we shouldn't call them replicating portfolios. But they do hedge against fluctuations in the equity price, so its appropriate to call them *hedging* portfolios. With (4.2), the evolution of the hedging portfolio is described by

$$dW = n^{(1)} \left\{ \left[\frac{\partial F}{\partial t} + \mathcal{A} F - \alpha^e S F_S - \rho \alpha^e \sigma^{-1} a F_V \right] dt + \left[a F_V (dZ - \rho dB) \right] \right\}$$

The hedging portfolio's expected *total* return α^h and volatility σ^h are determined as follows:

$$\mathbb{E}_t[dW] = \alpha^h W dt = n^{(1)} \left[\frac{\partial F}{\partial t} + \mathcal{A} F - \alpha^e S F_S - \rho \alpha^e \sigma^{-1} a F_V \right] dt \,, \tag{4.3}$$

$$\mathbb{E}_t[(dW)^2] = (\sigma^h W)^2 dt = (n^{(1)})^2 (1-\rho^2) a^2 F_V^2 dt \,,$$

where $\mathbb{E}_t$ is the time-t expectation. This last equation yields $\sigma^h W = \pm n^{(1)} \sqrt{1-\rho^2}\, a F_V$ or

$$\sigma^h = \pm \sqrt{1-\rho^2} \left(F - S F_S - \rho \sigma^{-1} a F_V \right)^{-1} a F_V \,. \tag{4.4}$$

The sign above is determined by the requirement that $\sigma^h > 0$. Consider the deterministic volatility limit $a \to 0$ in the last equation. For a call option, from the B-S solution, $F_V \geq 0$ and $F - SF_S \leq 0$. On the other hand, for a put option, $F_V \geq 0$ but $F - SF_S \geq 0$. So the sign is ambiguous because it's payoff dependent.

The risk premiums. The final results are best expressed in terms of standardized ratios. These ratios are called risk premiums or market prices of risk, and usually are denoted by λ. Specifically, a market price of risk is an expected total return in excess of the riskless rate, per unit volatility. For the hedging portfolio and the equity respectively:

(4.5) $$\lambda_t^h = \frac{\alpha_t^h - r}{\sigma_t^h} \quad \text{and} \quad \lambda_t^e = \frac{\alpha_t^e - r}{\sigma_t}$$

Under our setup, the risk premiums depend only upon our state variables because of our Assumption 3. We have a lot more to say about these functions in Chapter 7. For now, here are some important points:

(i) In general, the risk premiums are *not* constants, but state-dependent. For the particular representative agent models that we adopt in Chapter 7, the risk premiums depend upon the volatility and time.

(ii) In our stochastic volatility models $dV = b\,dt + a\,dZ$, the coefficients are independent of the stock price and time-homogeneous. This can lead, with some further assumptions, to risk premiums with the same properties.

(iii) While λ^e is typically positive for risk-averse investors, the sign of λ^h depends upon aggregate risk-aversion relative to an investor with *logarithmic* utility.

(iv) Having said all that, sometimes it's a sensible approximation to ignore these functions altogether. For example, the implied volatility smile for short-dated options is primarily a variance-covariance effect and the two drifts can be neglected.

The option valuation PDE. Using the expressions above, we obtain

$$\alpha^h - r = \lambda^h \sigma^h \text{ or } \alpha^h W - r(n^{(1)} F + n^{(2)} S) = \lambda^h \sigma^h W .$$

This yields $$\alpha^h W - r\, n^{(1)} \left[F - \left(F_S + \frac{\rho\, a}{\sigma S} F_V \right) S \right] = \lambda^h \sigma^h W .$$

Substituting (4.3)-(4.4) into the last equation yields

$$\left[\frac{\partial F}{\partial t} + \boldsymbol{A} F - \alpha^e S F_S - \alpha^e \left(\frac{\rho}{\sigma} \right) a F_V \right] + \left[-rF + rSF_S + r\left(\frac{\rho}{\sigma} \right) a F_V \right]$$

$$= \pm \lambda^h \sqrt{1 - \rho^2}\, a F_V$$

That is, the price of any financial claim must be a solution to

The PDE for generalized European-style claims under stochastic volatility:

(4.10)
$$-\frac{\partial F}{\partial t} = -rF + \tilde{\mathcal{A}}\, F$$

where
$$\tilde{\mathcal{A}} F = (r-\delta)S\,\frac{\partial F}{\partial S} + \tfrac{1}{2} V S^2 \frac{\partial^2 F}{\partial S^2} + \tilde{b}(V,t)\,\frac{\partial F}{\partial V} + \tfrac{1}{2} a^2(V)\frac{\partial^2 F}{\partial V^2} + \rho(V) a(V) V^{1/2} S \frac{\partial^2 F}{\partial S \partial V}\,.$$

using
$$\tilde{b}(V,t) = b - \left[\rho \lambda^e \pm \sqrt{1-\rho^2}\lambda^h\right] a\,.$$

Equation (4.10) is to be solved subject to the terminal condition $F(S,V,T) = g(S)$.[7] The operator $\tilde{\mathcal{A}}$ is the generator for the risk-adjusted process $\tilde{P}$ defined by (1.5). In writing (4.10), we have used the fact that, under the equilibrium model we shall adopt, the risk premiums are functions of the volatility and time.

Summary of results from the hedging argument. We have shown:

- The risk-adjusted process has the same correlation ρ and diffusion coefficients as the actual process, but the drifts are modified.
- The expected total rate of return for the stock is modified by $\alpha_t^e \to r$.
- The volatility drift is modified by

$$\tilde{b} = b - \left[\rho \lambda^e \pm \sqrt{1-\rho^2}\lambda^h\right] a$$

  where $a(V)$ is the volatility diffusion coefficient, $\lambda^e(V,t)$ is the equity risk premium, and $\lambda^h(V,t)$ is a hedging risk premium. The hedging risk premium represents an orthogonal risk because the hedging portfolio is uncorrelated with the equity price changes.

Of course, the above development is incomplete because the risk premiums have been left unspecified. Fundamentally, they are not determined by the hedging argument.

---

[7] In principle, in a continuous-time world, payoff functions can also be volatility dependent. An interesting example, which our general approach supports, is a volatility future.

5 The Martingale Approach

The results in Sec. 4 are well understood to be a specific example of a general phenomenon for security pricing in continuous-time finance. Namely, the general setup starts with a set of prices of traded securities. And the evolution of this system is described by some (multivariate) diffusion process P, which may include SDEs for various non-traded factors or state variables in addition to the prices.

Then, under very weak conditions (the absence of arbitrage opportunities, appropriately defined), Delbaen and Schachermayer (1994, Corollary 1.2) have shown that traded security prices, relative to some numeraire, are *local martingales* under some probability measure $\tilde{P}$ (the *local martingale measure*).

For our purposes, the phrase 'X_t is a local martingale under $\tilde{P}$' for some process X_t means that (i) the Ito expansion for dX_t has no drift term under $\tilde{P}$ and (ii) the coefficients of the Brownian motion terms do not anticipate the future. In other words, under $\tilde{P}$, we have $dX_t = \vec{a}_t \cdot d\vec{B}_t$, a purely linear combination of Brownian motions. This follows the usage in Baxter and Rennie (1996, p. 79), for example. There is a more technical definition of local martingale, which involves stopping the process, [see Durrett (1996) or Revuz and Yor (1999)]. An example of stopping the process is discussed in Chapter 9. Sometimes, but not always, a local martingale is also a martingale, which means that $X_t = \mathrm{E}_t[X_T]$. That is, when a process is a martingale, its expectation is time-invariant. When a local martingale is not a martingale, it's called a *strictly local martingale.*

So, when an option price is a local martingale, its price evolution, relative to the bond price, has an Ito expansion under $\tilde{P}$ with no drift term. We will show that this holds for our model below.

In addition, when the option price is a martingale, then its price is the discounted expectation under $\tilde{P}$. As we have mentioned, the martingale property does not always hold under our models. If the volatility were bounded, then the property would hold, but it can and does fail for *some* claims under *some* unbounded volatility processes. A test to determine whether or not the call option price in particular is a martingale involves testing for explosions in a certain auxiliary volatility process. This is explained in Chapter 9.

The two diffusion processes, the real-world process under P and the risk-adjusted process under $\tilde{P}$, are closely related. They have the same random innovation terms and covariance structure, but the drifts are different. In particular, the instantaneous expected *total* return for every traded security under $\tilde{P}$ (in our case, the stock or generalized option) becomes the interest rate. If there are diffusion equations for non-traded factors or state variables, such as volatility, their drift modification is more complex, and is ultimately determined by investor risk attitudes.

The two expected returns of the option. In our application, $\tilde{P}$ is given by (1.5), which shows directly that the expected *price* return for the stock, under $\tilde{P}$, is given by $r - D_t / S_t$. So the expected total return for the stock is given by α_t^e under P and r under $\tilde{P}$. What about the expected rate of return for the option? From Ito's formula again

(5.1) $$\text{Under } P: \; dF = \left[\frac{\partial F}{\partial t} + \mathcal{A}\, F\right] dt + \sigma S F_S \, dB + a F_V dZ$$

So the coefficient of dt in (5.1) gives $\alpha^F F$ under P. What about under $\tilde{P}$? This is easily accomplished by a simple formal substitution Here are the two processes again:

$$P: \begin{cases} dS = (\alpha^e S - D)dt + \sigma S \, dB \\ dV = b dt + a \, dZ \end{cases} \qquad \tilde{P}: \begin{cases} dS = (rS - D)dt + \sigma S \, d\tilde{B} \\ dV = (b - \lambda^v a)dt + a \, d\tilde{Z} \end{cases}$$

Now we have defined a volatility risk premium

$$\lambda^v = \left[\rho \lambda^e + (1-\rho^2)^{1/2} \lambda^h\right].$$

We went through a fairly lengthy development to discover how to transform $P \to \tilde{P}$. But when they are placed side-by-side as we have done, it's clear that there is a very quick shorthand trick: just make the following substitutions in P:

(5.2) $$dB = d\tilde{B} - \lambda^e dt \qquad \text{and} \qquad dZ = d\tilde{Z} - \lambda^v dt.$$

This substitution, called the Girsanov transformation, is discussed further in the references cited in the notes and in Chapter 9. Applying the same substitution to (5.1) yields, under $\tilde{P}$,

(5.3) $$dF = \left[\frac{\partial F}{\partial t} + \mathcal{A}\, F - \lambda^e \sigma S F_S - \lambda^v a F_V\right] dt + \sigma S F_S d\tilde{B} + a F_V d\tilde{Z}$$

But we have already shown at (4.10) that

(5.4) $$\mathcal{A} F - \lambda^e \sigma S F_S - \lambda^v a F_V = \tilde{\mathcal{A}} F .$$

So, from the option valuation equation (4.10), the expression in brackets in (5.3) is $\partial F / \partial t + \tilde{\mathcal{A}} F = rF$, and (5.3) can be rewritten,

(5.5) $$\text{Under } \tilde{P}: \quad dF = rFdt + \sigma S F_S d\tilde{B} + a F_V \, d\tilde{Z} .$$

As we asserted, (5.5) shows that the instantaneous expected total return of the option, under the risk-adjusted process, is also equal to the interest rate. Put another way, $G = e^{-rt} F$ evolves under $\tilde{P}$ as $dG = \sigma S G_S d\tilde{B}_t + a G_V d\tilde{Z}_t$, which means that $G = e^{-rt} F$ is indeed a local martingale under $\tilde{P}$.

When the claim price is also a true martingale, then $G_t = \tilde{\mathbb{E}}_t[G_T]$, which gives us the martingale pricing solution $F_t = e^{-r(T-t)} \tilde{\mathbb{E}}_t[F_T]$. Even when the claim price is only a strictly local martingale, the martingale formula is still a solution to the valuation PDE. (However, it's not the *only* solution, nor the arbitrage-free fair value for the claim). This martingale PDE solution, which is sometimes called a Feynman-Kac formula, is developed more generally in Appendix 2.

The option PDE in 3 steps. The power of the martingale approach is that, once you accept the basic theory, you can develop the option PDE in only 3 steps by just repeating the last arguments. The lengthy hedging argument of Sec. 4 is unnecessary. Let's review just how quickly that really works. The absence of arbitrage means that there exists a transformation of measures $P \to \tilde{P}$ under which the claim price is a local martingale. Price evolutions under the two measures have the same variance-covariance structure but transformed drifts. So here are the three steps.

(1) $P \to \tilde{P}$ is achieved by a Girsanov transformation of the type (5.2) for *some* function λ^v.

Then an aside: The stock price equation must transform by $dB = d\tilde{B} - \lambda^e dt$, where $\lambda^e = (\alpha^e - r)/\sigma$, in order to have its expected total return under $\tilde{P}$ equal to r. Under Assumption 3 about the state variables of the problem, both λ^e and λ^v must be some functions of (S_t, V_t, t).

(2) Ito's formula applied to the option price yields (5.3).

Another aside: Because the option is also a traded security, and pays no dividends, the coefficient of the dt term under $\tilde{P}$ must be equal to rF.

(3) This yields the PDE $\partial F / \partial t + \tilde{\mathcal{A}} F = rF$, where $\tilde{\mathcal{A}}$ is defined by (5.4). ■

The state-pricing density. Earlier in this section, we constructed a hedging portfolio that was uncorrelated with the equity returns. Another way to present this is to express dB_t and dZ_t in terms of two *independent* Brownian motions dB_t^1 and dB_t^2. (The superscripts are just labels, not powers). The new algebra is $(dB_t^1)^2 = (dB_t^2)^2 = dt$ and $dB_t^1 dB_t^2 = 0$. There are two possible representations that preserve the old algebra $(dB_t)^2 = (dZ_t)^2 = dt$ and $dB_t dZ_t = \rho_t dt$, namely

(5.6) $$dB_t = dB_t^1 \quad \text{and} \quad dZ_t = \rho_t \, dB_t^1 \pm \sqrt{1-\rho_t^2} dB_t^2.$$

Combined with (5.2), then (5.6) implies the transformations

(5.7) $$dB_t^1 = d\tilde{B}_t^1 - \lambda_t^e \, dt \quad \text{and} \quad dB_t^2 = d\tilde{B}_t^2 - \lambda_t^h \, dt$$

which again emphasizes the orthogonal nature of the two risks associated with λ^e and λ^h. Now create 2-vectors

$$\vec{\lambda}_t = (\lambda_t^e, \lambda_t^h) \quad \text{and} \quad \vec{B}_t = (B_t^1, B_t^2).$$

and consider the scalar process X_t defined by the SDE

$$dX_t = -\tfrac{1}{2} | \vec{\lambda}_t |^2 \, dt - \vec{\lambda}_t \cdot d\vec{B}_t, \quad X_0 = 0.$$

We are using $| \vec{\lambda}_t |^2 = \vec{\lambda}_t \cdot \vec{\lambda}_t$, where the dot is the usual vector dot product. In other words,

$$X_t = -\int_0^t \vec{\lambda}_s \cdot d\vec{B}_s - \tfrac{1}{2} \int_0^t | \vec{\lambda}_s |^2 \, ds.$$

From the Ito formula,

$$M_t = e^{X_t} \text{ satisfies } dM_t = -M_t \vec{\lambda}_t \cdot d\vec{B}_t.$$

Hence M_t is a local martingale under the actual price process P with $M_0 = 1$. Now we want to show that $\exp(-rt) M_t F_t$ is a also a local martingale under P. Consider $d(M_t F_t) = M_t dF_t + dM_t F_t + dM_t dF_t$. From (5.1) and (5.6) we have, under P,

$$dF = \left[\frac{\partial F}{\partial t} + \mathcal{A} F \right] dt + \sigma S F_S dB^1 + a F_V \left(\rho \, dB^1 \pm \sqrt{1-\rho^2} dB^2 \right).$$

We also have $dM = \; M(\lambda^e dB^1 + \lambda^h dB^2)$, so the net result is

$$d(MF) = M\left[\frac{\partial F}{\partial t} + \mathcal{A}\,F - \lambda^e \sigma S F_S - \left(\rho\,\lambda^e \pm \sqrt{1-\rho^2}\lambda^h\right) a F_V\right] dt$$

$$- (\cdots) dB^1 + (\cdots) dB^2 .$$

But again from (5.4) and the option valuation PDE, the expression in the bracket just above is rF, and so $d(MF) = rMFdt + (\cdots)dB^1 + (\cdots)dB^2$. This shows that $\exp(-rt)M_t F_t$ is indeed a local martingale under P.

When $\exp(-rt)M_t F_t$ is, in fact, also a martingale, then we have a second equivalent pricing formula

(5.8) $$F(t) = e^{-r(T-t)} \frac{1}{M(t)} \mathbb{E}_t[M(T)F(T)],$$

where $M(t)$ is the (undiscounted) state-pricing density process

(5.9) $$M(t) = \exp\left(-\int_0^t \vec{\lambda}_s \cdot d\vec{B}_s - \tfrac{1}{2}\int_0^t |\vec{\lambda}_s|^2 \, ds\right) \text{ and } \quad \vec{\lambda}_t = (\lambda_t^e, \lambda_t^h).$$

Admittedly, these expressions are very formal; we give them more content as explicit integrals in Chapter 2.

Summary. The martingale approach is completely equivalent to the hedging approach, but the results have a slightly different flavor. Option values depend upon certain prices of risk. The price of volatility risk can be decomposed into an equity risk premium λ^e and a hedging portfolio risk premium λ^h. These risks are orthogonal because they transform, through the Girsanov transformation, two independent Brownian motions into two independent Brownian motions. The option valuation PDE can be quickly derived and arbitrage-free financial claim prices are solutions to that PDE.

When claim prices are true martingales, then they are given by the two equivalent solutions to the valuation PDE:

$$F(t) = e^{-r(T-t)} \frac{1}{M(t)} \mathbb{E}_t[M(T)F(T)] = e^{-r(T-t)} \tilde{\mathbb{E}}_t[F(T)],$$

where the state-pricing density $M(t)$ is given above. When the claim prices are only strictly local martingales, then you must look for other solutions to the valuation PDE.

Additional Notes

Fundamental references are Black and Scholes (1973) and Merton (1990). A good overview of modern financial theory is Ingersoll (1987). A helpful introduction to stochastic calculus for finance is Neftci (1996). Girsanov's Theorem and the martingale approach to financial modeling is explained in Duffie (1992), Baxter and Rennie (1996) and Musiela and Rutkowski (1997). An option FAQ (frequently asked questions) is offered by Carr (1999).

Some mathematical methods we use (transforms, eigenvalue equations, variational methods) are presented very clearly in Mathews and Walker (1965), which is still in print as of this writing.

The connection between solutions to parabolic PDEs and summing over Brownian motion paths is treated in Friedman (1975) and Durrett (1984); the idea was invented by Feynman (1948) and developed by Kac (1949). The formulas for the cases we use are developed in Appendix 2 to this chapter.

Since the work of Kendall (1953), Mandelbrot (1963) , and others, it has been known that (log) stock price return distributions typically have more extreme events than stationary normal distributions. Constant volatility is always strongly rejected when tested directly, and constant or deterministic volatility is indirectly rejected by biases in the pricing of options relative to the B-S price, as reported by MacBeth and Merville (1980), Rubinstein (1985) and others.

ARCH-related literature that is relevant to the diffusion limit include the GARCH model of Bollerslev (1986), and the Threshold ARCH model of Glosten, Jagannathan, Runkle (1993) and Zakoian (1990). Option valuation under discrete-time GARCH-type processes, with varying assumptions on preferences, has been considered by Duan (1995), Amin and Ng (1993), and Engle and Rosenberg (1995), using Monte Carlo estimates.

To improve the B-S theory, Merton (1976) added stock price jumps. Alternatively, as we do here, many researchers keep a continuous sample path model (a diffusion), but postulate stochastic volatility. General equilibrium theories with non-traded state variables, such as volatility, were developed by Merton (1973), Garman (1976), and Cox, Ingersoll, and Ross (1985). Specific models for options under stochastic volatility and numerical examples were provided by Wiggins (1987), Hull and White (1987), Johnson and Shanno (1987), and Scott (1987). Closed-form examples were created by Stein and Stein (1991), Heston (1993), and Ball and Roma (1994). A review has been offered by Taylor (1994).

Appendix 1.1 Parameter Estimators for the GARCH Diffusion Model

Parameter estimators may be obtained from interpreting (1.2) as the continuous-time limit of a discrete-time GJR-GARCH process[8].

(A1.1)
$$\Delta \ln S_t = \mu_t + \sigma_t Z_t ,$$
$$V_{t+1} = \varpi + \alpha V_t Z_t^2 + \beta V_t + \gamma V_t (Z_t^-)^2 .$$

In (A1.1), Z_t is an independent unit normal variate, and $Z_t^- = \min(Z_t, 0)$. Nelson (1990) created a moment matching procedure showing that (A1.1) can have the GARCH diffusion model (1.2) as a continuous-time limit. Suppose the estimates of the parameters in (A1.1) are denoted by the same letters, but with a hat. Then, Nelson's procedure also yields estimators for the parameters of the GARCH diffusion model:

(A1.2)
$$\hat{\omega} = \frac{\hat{\varpi}}{\delta t^2} , \qquad \hat{\theta} = \frac{1}{\delta t}(1 - \hat{\alpha} - \hat{\beta} - \tfrac{1}{2}\hat{\gamma}),$$
$$\hat{\xi} = \sqrt{\frac{1}{\delta t}\left(2\hat{\alpha}^2 + 2\hat{\alpha}\hat{\gamma} + \tfrac{5}{4}\hat{\gamma}^2\right)}, \text{ and } \hat{\rho} = -\frac{\hat{\gamma}}{\sqrt{\pi\left(\hat{\alpha}^2 + \hat{\alpha}\hat{\gamma} + \tfrac{5}{8}\hat{\gamma}^2\right)}} .$$

Typically, the GJR-GARCH parameters $\hat{\alpha}$, $\hat{\beta}$, $\hat{\gamma}$, etc. are measured on a daily basis. Often, the GARCH diffusion parameters $\hat{\theta}$, $\hat{\xi}$, etc. are quoted on an annualized basis. In that case, take $\delta t = 1/252$, and we will write $\hat{\theta}_a$, $\hat{\xi}_a$, etc. The correlation estimate is independent of the time scale.

An advantage of the GARCH diffusion, relative to the other models, is the existence of these simple estimators. These estimators, which are based upon maximum likelihood functions of discrete data, are implemented in a number of widely used econometric software packages.

Examples of parameter estimates for the S&P 100 Index, the S&P 500 Index, and the component stocks of the S&P100 Index are shown in Table A1.1. The various estimates are quite variable and sensitive to the inclusion of the Oct. 87

---

[8] This is Model 2 of Glosten, Jagannathan, and Runkle (1993).

crash. For the data shown, $\xi_a \simeq 0.5$ to 2.5, ρ is significantly negative, and θ_a ranges from 3 to 18 (annualized), which corresponds to a half-life of 4 months (1/3 year) to 3 weeks (1/18 year).

There doesn't seem to be any particular economic rationale for these parameter values, with one exception. The negative sign for ρ is consistent with the so-called "leverage effect". The idea is that, as a stock price drops, debt to equity ratios will rise, making the firm more levered financially and hence, riskier. Hence, this should be associated with a greater price variability. In practice, the actual changes in debt to equity ratios are too small for this to be the whole story, but at least the sign makes sense. And, it leads one to expect that ρ will be close to zero for financial time series where such leverage does not exist, as in the case of foreign exchange rates or commodity prices. For equity indices, the importance of having a model that allows for non-zero correlation has been stressed by Nandi (1998).

If the continuous-time limit exists in the way described by Nelson's theory, then the empirical parameters should scale with δt in the way that (A1.2) describes. This question has been investigated for symmetric GARCH ($\gamma = \rho = 0$) models by Chou (1988). Roughly speaking, the agreement is pretty good. See Drost and Nijman (1993) for the closely related theory of the behavior of the GARCH parameters under temporal aggregation.

Table A1.1 Examples of GARCH and GARCH Diffusion Parameter Estimates

Panel I: S&P 100 Index 1983-1992

GJR-GARCH parameter (daily basis)	S&P 100 Index With crash	S&P 100 Index Without crash	Stocks in Index (Median) With crash	Stocks in Index (Median) Without crash
$\hat{\alpha}$	0.0357	0.0157	0.0651	0.0367
$\hat{\beta}$	0.8391	0.9511	0.8396	0.9284
$\hat{\gamma}$	0.1081	0.0174	0.0656	0.0191
GARCH diffusion parameters (annualized)				
$\hat{\theta}_a$	17.9	6.2	15.8	6.39
$\hat{\xi}_a$	2.50	0.60	2.38	1.07
$\hat{\rho}$	-0.55	-0.37	-0.35	-0.23

Panel II: S&P 500 Index: Jan 2, 1969-Apr 9, 1996 (With crash)

GJR-GARCH parameters (daily basis)	Coefficient	Standard Error	t-Statistic
$\hat{\varpi}$	$1.12\ 10^{-6}$	$1.1\ 10^{-7}$	10.1
$\hat{\alpha}$	0.0323	0.0044	7.4
$\hat{\beta}$	0.9254	0.0032	286
$\hat{\gamma}$	0.0602	0.0047	12.8

GARCH diffusion parameters (annualized): $\hat{\theta}_a = 3.1$, $\hat{\xi}_a = 1.63$, $\hat{\rho} = -0.47$

Notes. Panel I GJR-GARCH parameters are adapted from "Asymmetric and Crash Effects in Stock Volatility for the S&P 100 Index and its Constituents" by B. Blair, S-H. Poon, and S.J. Taylor (1998, Table 4, by permission). Estimates "without crash" are based upon the GJR-GARCH model with an additional dummy variable for Oct. 19, 1987. Also shown in Panel I are the GARCH diffusion parameter estimates based upon (A1.2). Panel II shows the same parameters for the S&P 500 Index from 1969-1996 (author's data).

Appendix 1.2 Solutions to PDEs

The following three PDE problems occur repeatedly throughout the book:

P1: Consider the region $t \le T$. With $f(x,T) = \varphi(x)$, find a solution $f(x,t)$ to the following PDE:

(A2.1) $$\frac{\partial f}{\partial t} + \tfrac{1}{2}a^2(x)\frac{\partial^2 f}{\partial x^2} + b(x)\frac{\partial f}{\partial x} - c(x)f + h(x,t) = 0,$$

P2: Consider the region $\tau \ge 0$. With $g(x,0) = \varphi(x)$, find a solution $g(x,\tau)$ to the following PDE:

(A2.2) $$-\frac{\partial g}{\partial \tau} + \tfrac{1}{2}a^2(x)\frac{\partial^2 g}{\partial x^2} + b(x)\frac{\partial g}{\partial x} - c(x)g + k(x,\tau) = 0,$$

P3: Consider the region $\tau \ge 0$. With $g(x,0) = \varphi(x)$, find the solution $g(x,\tau)$ to the following PDE

(A2.3) $$-\frac{\partial g}{\partial \tau} + b(x)\frac{\partial g}{\partial x} - c(x)g + k(x,\tau) = 0,$$

In our applications of **P1** and **P2,** the x-interval is typically $(0,\infty)$. The solutions below may not be unique. When $c(x) < 0$, solutions may not exist or may exist only for $T - t < \tau^*$ or $\tau < \tau^*$. For more discussion of technical conditions, see Durrett (1996, Sec. 4.3).

When solutions do exist, we will solve **P1** using Ito's formula and then manipulate the solution to obtain the solution to **P2** and **P3**. To solve **P1**, first define X_t to be a diffusion process with the SDE

(A2.4) $$dX_t = b(X_t)dt + a(X_t)dB_t.$$

If $f(x,t)$ is the solution to **P1**, then define a function $M(X_t,t)$ of the diffusion process (A2.4) by

(A2.5)

$$M(X_t,t) = f(X_t,t)\exp\left(-\int_0^t c(X_\lambda)d\lambda\right) + \int_0^t h(X_s,s)\exp\left(-\int_0^s c(X_\lambda)d\lambda\right)ds$$

From Ito's formula,

$$\text{(A2.6)}\quad dM = \left[\frac{\partial f}{\partial t} + \tfrac{1}{2} a^2 \frac{\partial^2 f}{\partial x^2} + b \frac{\partial f}{\partial x} - c f + h \right] (X_t, t) \exp\left(-\int_0^t c(X_\lambda) d\lambda \right) dt + a(X_t) \frac{\partial M}{\partial X} dB_t .$$

So the expression in brackets in (A2.6) vanishes because of (A2.1), and $M(X_t, t)$ is a local martingale. Let's assume that, in fact, $M(X_t, t)$ is actually a martingale; then we have $M(X_t, t) = \mathbb{E}_t [M(X_T, T)]$ or

$$\text{(A2.7)}\quad f(X_t, t) \exp\left(-\int_0^t c(X_\lambda) d\lambda \right) + \int_0^t h(X_s, s) \exp\left(-\int_0^s c(X_\lambda) d\lambda \right) ds$$
$$= \mathbb{E}_t \left[\varphi(X_T) \exp\left(-\int_0^T c(X_\lambda) d\lambda \right) + \int_0^T h(X_s, s) \exp\left(-\int_0^s c(X_\lambda) d\lambda \right) ds \right].$$

Any of the terms on the left-hand-side are all known at time t and can be freely moved through the time t expectation symbol $\mathbb{E}_t [\cdots]$ on the right-hand-side. Initialize the diffusion process so that $X_t = x$. Then, after some rearrangement, (A2.7) becomes the

(A2.8) Feynman-Kac style solution to P1:

$$f(x,t) = \mathbb{E}_t \left[\varphi(X_T) \exp\left(-\int_t^T c(X_\lambda) d\lambda \right) + \int_t^T h(X_s, s) \exp\left(-\int_t^s c(X_\lambda) d\lambda \right) ds \right]$$

Now in (A2.8) let $f(x,t) = g(x, T-t)$ and $h(x,t) = k(x, T-t)$.
Clearly $g(x,\tau)$ satisfies **P2** and we have

$$g(x, T-t) =$$
$$\mathbb{E}_t \left[\varphi(X_T) \exp\left(-\int_t^T c(X_\lambda) d\lambda \right) + \int_t^T k(X_s, T-s) \exp\left(-\int_t^s c(X_\lambda) d\lambda \right) ds \right].$$

Setting $t = 0$ and relabeling $T \to \tau$ gives us the

(A2.9) Feynman-Kac style solution to P2:

$$g(x,\tau) = \mathbb{E}_0\left[\varphi(X_\tau)\exp\left(-\int_0^\tau c(X_\lambda)d\lambda\right) + \int_0^\tau k(X_s,\tau-s)\exp\left(-\int_0^s c(X_\lambda)d\lambda\right)ds\right].$$

where now the diffusion is initialized so that $X_0 = x$.

If there are additional solutions to P1 and P2, then the difference between the additional solutions and the ones shown must have vanishing initial conditions. In other words, when the solution you are looking for is, in fact, only a strictly local martingale, then you can look for additional solutions, with vanishing initial condition, to add to the solutions already shown.

Finally, the solution to **P3** is given by taking $a(x) = 0$. In that case, X_s is the deterministic solution to the *characteristic* equation $dX_s = b(X_s)ds$ on $0 \le s \le \tau$, where $X_0 = x$. Then (A2.9) still holds, but there is no need for the expectation symbol. The result is the

(A2.10) Method of characteristics solution to P3:

$$g(x,\tau) = \varphi(X_\tau)\exp\left(-\int_0^\tau c(X_\lambda)d\lambda\right) + \int_0^\tau k(X_s,\tau-s)\exp\left(-\int_0^s c(X_\lambda)d\lambda\right)ds.$$

This last solution should be unique.

2 The Fundamental Transform

In this chapter we introduce a transform-based approach to solving the option valuation PDE that we developed in Chapter 1. The method is based on a generalized Fourier transform. A particular function, which we call the fundamental transform, plays an important role throughout the book. While the idea of a transform-based approach is not new, previous applications have tended to be model-specific. Not only are our results more general, but they encompass the situation when option prices, relative to a numeraire, are not martingales, but only strictly local martingales.

1 Assumptions

In Chapter 1, we developed a PDE for valuing options under stochastic volatility at (1.4.10). Now we specialize to time-homogeneous volatility processes of the form $dV_t = b(V_t)dt + a(V_t)d\tilde{W}_t$. In other words, the volatility changes in time only through the Brownian noise and level-dependent coefficients; but there is no explicit time dependence.

Indeed, most models of the *actual* volatility process that are proposed by researchers are time-homogeneous. In particular, both GARCH-style models and their continuous-time limits are time-homogeneous. And, as we show later in Chapter 7, the time-homogeneity property can be preserved after risk adjustment. Briefly, this can be achieved with a power utility function using an infinite consumption horizon or a pure investor model with a distant planning horizon.

We take as constant both the dividend yield on the underlying security and the short-term interest rate. This too can be made consistent with a risk adjustment model. Finally, we make a smoothness assumption that we use in later chapters. In summary, we employ in this chapter and throughout much of the book the basic model given by:

Assumption 1. The martingale pricing process $\tilde{P}$ has the general form

$$\tilde{P}: \begin{cases} dS_t = (r-\delta)S_t\,dt + \sigma_t S_t\,d\tilde{B}_t \\ dV_t = \bar{b}(V_t)\,dt + a(V_t)\,d\tilde{W}_t \end{cases}, \tag{1.1}$$

where $d\tilde{B}_t$ and $d\tilde{W}_t$ are correlated Brownian motions under $\tilde{P}$, with correlation $\rho(V_t)$. The interest rate r and the dividend yield δ are constants. The coefficient functions $\tilde{b}(V)$ and $a(V)$ may be differentiated any number of times on $0 < V < \infty$.

Under Assumption 1, we can rewrite the PDE (1.4.10) for generalized European-style claims with price $F(S_t, V_t, t)$ and expiration T. That equation, defined in the region $0 < (S,V) < \infty$, $t < T$, becomes

$$\boxed{\begin{aligned} & -\frac{\partial F}{\partial t} = -rF + \tilde{\mathcal{A}}\,F\,, \\ \text{where}\qquad & \tilde{\mathcal{A}}F = (r-\delta)S\,\frac{\partial F}{\partial S} + \tfrac{1}{2}V S^2\,\frac{\partial^2 F}{\partial S^2} \\ & +\bar{b}(V)\,\frac{\partial F}{\partial V} + \tfrac{1}{2}a^2(V)\frac{\partial^2 F}{\partial V^2} + \rho(V)a(V)V^{1/2}S\,\frac{\partial^2 F}{\partial S\,\partial V}. \end{aligned}} \tag{1.2}$$

We almost always assume the payoff function is independent of volatility[1]. Then, European-style option prices are solutions to (1.2) with terminal condition $F(S,V,t=T) = g(S)$. As we will see below, sometimes there are multiple solutions to (1.2) with the same payoff function; briefly, this occurs because of volatility explosions. When that happens, we have to determine which solution is the "fair-value". Note that the first line defining the operator

---

[1] Our approach also accommodates very naturally a pure volatility-dependent payoff, such as a volatility future. The demands of traders for hedging and replication strategies under stochastic volatility would make such securities quite useful, although there are many real-world design issues

$\tilde{\mathcal{A}}$ is the linear operator of the B-S theory and the second line contains the stochastic volatility corrections.

2 The Transform-based Solution

In this section, we reduce (1.2) from two "space" variables to one. There are fundamental solutions to the reduced equation that provides a representation for the price of every (volatility independent) payoff function. As we will show, those fundamental solutions have a number of special properties.

This reduction to 1D is not the proverbial free lunch because the one variable PDE is then dependent upon a continuous transform parameter. Nevertheless, the reduction is extremely useful and it provides the basis for much of our subsequent development.

The first step is simply a change of variable from S to $x = \ln S$ in (1.2), letting $F(S,V,t) = f(x,V,t)$. Then f must solve, using subscripts for derivatives

$$(2.1) \qquad -f_t = -rf + (r - \delta - \tfrac{1}{2}V) f_x + \tfrac{1}{2} V f_{xx} + \tilde{b} f_V + \tfrac{1}{2} a^2 f_{VV} + \rho a V^{1/2} f_{xV} .$$

Now consider the Fourier transform of $f(x,V,t)$ with respect to x:

$$(2.2) \qquad \hat{f}(k,V,t) = \int_{-\infty}^{\infty} e^{ikx} f(x,V,t)\, dx ,$$

where $i = \sqrt{-1}$ and k is the transform variable. The first issue is to determine under what conditions (2.2) exists for typical option solutions. The simplest case is $t \to T$ (expiration), where we know the functional form $f(x,V,T)$

For example, call option solutions are given at expiration by $C(S,V,T) = \mathrm{Max}[S - K, 0] = (S - K)^{+}$, where K is the strike price. Hence, $f(x,V,T) = (e^x - K)^+$ and by a simple integration in (2.2),

$$(2.3) \qquad \hat{f}(k,V,T) = \left(\frac{\exp[(ik+1)x]}{ik+1} - K \frac{\exp(ikx)}{ik} \right) \Bigg|_{x = \ln K}^{x=\infty}$$

The upper limit $x = \infty$ in (2.3) does not exist unless $\mathrm{Im}\, k > 1$, where Im means Imaginary part. Assuming this restriction holds, then (2.3) is well-defined, giving the payoff transform

$$\hat{f}(k,V,T) = -\frac{K^{1-ik}}{k^2 - ik}. \tag{2.4}$$

So the key to the existence of (2.2) is that the Fourier transform variable k has to have an imaginary part making $k = k_r + ik_i$ a complex number[2]. Because k has been generalized to complex values, (2.2) is called a *generalized* Fourier transform[3]. In general, (2.2) exists for typical option payoffs only when $\operatorname{Im} k$ is restricted to a strip $\alpha < \operatorname{Im} k < \beta$. The reason that strips occur as a general feature of the theory is explained in Sec. 4. Given the transform $\hat{f}(k,V,t)$, the inversion formula is

$$f(x,V,t) = \frac{1}{2\pi} \int_{ik_i-\infty}^{ik_i+\infty} e^{-ikx} \hat{f}(k,V,t)\, dk. \tag{2.5}$$

This is an integral along a straight line in the complex k-plane parallel to the real axis. In the case of the call option at expiration, this line can lie anywhere in the region $\operatorname{Im} k > 1$: say along $k_i = 3/2$ for example. Actually selecting a contour for computations is discussed further below. We can go through the same exercise for various standard payoff functions and see what restrictions are necessary for their Fourier transforms to exist. The results are summarized in Table 2.1 below.

Table 2.1 Generalized Fourier Transforms for Various Financial Claims

Financial Claim	**Payoff Function**	**Payoff Transform**	***k*-plane Restrictions**
Call option	$\max[S_T - K, 0]$	$-\dfrac{K^{ik-1}}{k^2 - ik}$	$\operatorname{Im} k > 1$
Put option	$\max[K - S_T, 0]$	$-\dfrac{K^{ik+1}}{k^2 - ik}$	$\operatorname{Im} k < 0$
Covered call or cash-secured put	$\min[S_T, K]$	$\dfrac{K^{ik+1}}{k^2 - ik}$	$0 < \operatorname{Im} k < 1$
Delta function	$\delta(\ln S_T - \ln K)$	K^{ik}	none
Money market	1	$2\pi\delta(k)$	$\operatorname{Im} k = 0$

---

[2] If $z = x + iy = \operatorname{Re} z + i \operatorname{Im} z$ is any complex number, we write $|z|$ for the modulus or absolute value of z, and $z^* = x - iy$ for the complex conjugate.

[3] Sometimes the term *complex* Fourier transform is used. A comprehensive reference is Titchmarsh (1975).

The delta function. Two of the entries in the table use the Dirac delta function $\delta(x-y)$, which can be thought of as the limit of a function of x that is sharply peaked at $x=y$. In the limit, the function is zero everywhere else, while maintaining unit area under its "graph". More rigorously, the delta function is really a linear "functional" because it transforms well-behaved functions into numbers via $\int_{-\infty}^{\infty}\delta(x-y)f(x)dx=f(y)$. This function occurs naturally in the theory; for example, to prove the inversion formula, you insert (2.2) into (2.5) and rely upon this last equation and $\int_{ik_i-\infty}^{ik_i+\infty}\exp[-ik(x-y)]dk=2\pi\delta(x-y)$.

Continuing with the development, we next translate (2.1) into a PDE for $\hat{f}(k,V,t)$. That's done by taking the time derivative of both sides of (2.2), and inside the integral replacing f_t by the (negative of the) right-hand-side of (2.1). Then, after parts integrations, the net effect is that x-derivatives of f in (2.1) become multiplications of $\hat{f}$ by $(-ik)$.

An important point is that we assumed that the boundary terms associated with the parts integrations can be neglected. This is similar to the issue that we discovered at (2.3) and led to our introduction of the generalized transform. Typically, there exists a strip $\alpha<\operatorname{Im}k<\beta$ such that the boundary terms vanish. This is proved in the subsection "Neglected boundary terms" below. It's also typical that α and β depend upon the parameters of the problem as well, such as the time to expiration. We also show examples of $\alpha(\tau)$ and $\beta(\tau)$ below. With $\operatorname{Im}k$ appropriately restricted, the PDE satisfied by $\hat{f}(k,V,t)$ is

$$-\hat{f}_t=[-r-ik(r-\delta)]\hat{f}-\tfrac{1}{2}V(k^2-ik)\hat{f}+(\tilde{b}-ik\rho aV^{1/2})\hat{f}_V+\tfrac{1}{2}a^2\hat{f}_{VV}$$

We remove the dependence on r and δ, using $\tau=T-t$, and letting

$$\hat{f}(k,V,t)=\exp\{[-r-ik(r-\delta)]\tau\}\hat{h}(k,V,\tau)\ . \tag{2.6}$$

Also, introducing $c(k)=(k^2-ik)/2$, we see that $\hat{h}(k,V,\tau)$ satisfies the initial-value problem

$$\frac{\partial\hat{h}}{\partial\tau}=\tfrac{1}{2}a^2(V)\frac{\partial^2\hat{h}}{\partial V^2}+\left[\tilde{b}(V)-ik\rho(V)a(V)V^{1/2}\right]\frac{\partial\hat{h}}{\partial V}-c(k)V\hat{h} \tag{2.7}$$

The initial condition is that $\hat{h}(k,V,\tau=0)$ is given by the Fourier transform of the payoff function—the entries in Table 2.1.

The fundamental transform. Notice that the entries in Table 2.1 do not depend upon V. They don't because we have restricted our theory to volatility independent payoffs. Because of this assumption, it suffices to take the special case $\hat{h}(k,V,\tau=0)=1$. To obtain the solution to (2.7) for any other payoff of this type, multiply the solution for the special case by the "Payoff Transform" entry in Table 2.1 . This deserves a formal definition and some distinguishing notation:

Definition. A solution $\hat{H}(k,V,\tau)$ to (2.7) at a (complex-valued) point k, which satisfies the initial condition $\hat{H}(k,V,\tau=0)=1$, is called a *fundamental transform.*

Given the fundamental transform, to obtain a (not necessarily unique) solution $F(S,V,t)$ for a particular payoff, here are the steps:

- multiply the fundamental transform by the expiration payoff transform;
- further multiply by the factor that we removed in (2.6);
- invert the result with the k-plane integration (2.5), keeping $\operatorname{Im} k$ in an appropriate strip; this gives a solution $f(x,V,t)$ to (2.1);
- in terms of S, the solution is $F(S,V,t)=f(\ln S,V,t)$

For this procedure to work, we need a strip for which a fundamental solution to (2.7) exists; then we can carry out the inversion along any line contained within. Let's define a class of problems where this procedure is especially well-defined:

Definition. We call the initial-value problem (2.7) *regular*[4] if there exists a fundamental solution to (2.7) which is regular as a function of k within a strip $\alpha<\operatorname{Im} k<\beta$, where α and β are real numbers. We call this strip the *fundamental strip of regularity.* In typical examples, $\alpha<0$ and $\beta>1$.

Given the fundamental transform, the steps above are quite straightforward. For an example using Mathematica, see Appendix 2 to this chapter. For closed-form examples of the fundamental transform, see Sec. 3 below.

Call option Solution I. The call option payoff transform is given in Table 2.1 and it exists for $\operatorname{Im} k>1$. The call option solution in this subsection exists only under the following assumption: the initial-value problem (2.7) is regular in a

---

[4] A function $f(k)$ is *analytic* at a complex-valued point k if it has a derivative there. If it's both analytic and single-valued in a region, it's called *regular*

strip $\alpha < \text{Im}\, k < \beta$ *and* $\beta > 1$. In other words, we are assuming that the strip associated with the payoff transform and the fundamental strip intersect. If they don't, then this particular solution formula does not exist (but see below—there will always be an alternative formula that does exist). See Example II below for an example where there is such an intersection and further examples in Sec. 3. Carrying out the prescription above yields the solution representation

$$C_I(S,V,\tau) = -\frac{e^{-r\tau}}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty} e^{-ik\ln S} e^{-ik(r-\delta)\tau}\,\frac{K^{ik+1}}{k^2-ik}\,\hat{H}(k,V,\tau)\,dk\,,\ 1<\text{Im}\,k<\beta\,.$$

We continue to employ $\tau = T - t$. This equation can be simplified by introducing the dimensionless variable

$$X = \ln\left[\frac{Se^{-\delta\tau}}{Ke^{-r\tau}}\right].$$

Then, in terms of X, we have Solution I:

$$\text{(2.8)}\qquad C_I(S,V,\tau) = -Ke^{-r\tau}\,\frac{1}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty} e^{-ikX}\,\frac{\hat{H}(k,V,\tau)}{k^2-ik}\,dk\,,\qquad 1<\text{Im}\,k<\beta\,.$$

Frequently, $\hat{H}(k,V,\tau)$ is the Fourier transform of a norm-preserving transition density for the risk-adjusted process. This is discussed further below. For now, we simply note that when $\hat{H}$ is norm-preserving, then one can show, by Fourier inversion, that

$$C_I(S,V,\tau) = e^{-r\tau}\mathbb{E}_t\left[(S_T-K)^+\right],$$

which is the martingale-style solution. As we will see, sometimes there are other solutions and sometimes the martingale-style solution is not the arbitrage-free fair value.

Homogeneity. One immediate property of (2.8) is that the call option price is homogeneous of degree 1 in the stock price and the strike. That is, $C(S,V,\tau) = K\,c(S/K)$. If we multiply both the stock price and the strike by the same constant: $K \to \lambda K$ and $S \to \lambda S$, then $C \to \lambda C$. This is a well-known consequence of starting, as we did at Assumption (1.1), with a proportional stock price process. That is, the (risk-adjusted) stock price *return* distribution, although dependent upon the initial volatility, is independent of the level of S.[5]

---

[5] See Theorem 8.9 of Merton (1973).

Call option Solution II. In practice, we often do the k-plane integrations in $0<\text{Im}\,k<1$: usually along $k_i = 1/2$. In this strip, $\hat{H}$ is often free of singularities—see Example II below and the discussion in Sec. 4. The reason that this strip is the "regular" one is that solutions to (2.7) are usually quite well-behaved as long as $\text{Re}\,c(k) \geq 0$, which is true when $0 \leq \text{Im}\,k \leq 1$. This strip is especially important both in the asymptotic $\tau \to \infty$ behavior of the theory, which is explained in Chapter 6, and when the martingale-style solution is not the fair value, which is explained in Chapter 9.

We can obtain a formula for the call option with this restriction by using the put/call parity relation

(2.9) $$C(S,V,\tau) = S\exp(-\delta\tau) - [K\exp(-r\tau) - P(S,V,\tau)],$$

where $P(S,V,\tau)$ is the put option value. The expression in brackets in (2.9) is the cash-secured put entry in Table 2.1. As you can see from the table, the payoff function for the cash-secured put has (i) the same Fourier transform as the call option, except for a minus sign, and (ii) the different restriction $0<\text{Im}\,k<1$. Now we assume that $\hat{H}$ is regular in a fundamental strip which intersects $0<\text{Im}\,k<1$. With that assumption, we have solution II:

(2.10) $$\boxed{\begin{gathered} C_{II}(S,V,\tau) = Se^{-\delta\tau} - Ke^{-r\tau}\frac{1}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty} e^{-ikX}\,\frac{\hat{H}(k,V,\tau)}{k^2-ik}\,dk\,, \\ \max[0,\alpha] < \text{Im}\,k < \min[1,\beta] \end{gathered}}$$

In the same way, we define P_I to be the put option solution in its natural domain of definition, using Table 2.1:

$$P_I(S,V,\tau) = -Ke^{-r\tau}\frac{1}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty} e^{-ikX}\,\frac{\hat{H}(k,V,\tau)}{k^2-ik}\,dk\,, \qquad \alpha < \text{Im}\,k < 0\,.$$

Again, when $\hat{H}(k,V,\tau)$ is the Fourier transform of a norm-preserving transition density, then

$$P_I(S,V,\tau) = e^{-r\tau}\mathbb{E}_t\left[(K-S_T)^+\right],$$

And, using (2.9) and (2.10), we also have the second put option solution in the same strip as C_{II}

$$P_{II}(S,V,\tau) = Ke^{-r\tau}\left[1 - \frac{1}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty} e^{-ikX}\,\frac{\hat{H}(k,V,\tau)}{k^2 - ik}\,dk\right]$$

$$\max[0,\alpha] < \operatorname{Im} k < \min[1,\beta]$$

Relationships between the solutions. There is a very simple relationship between the Solution I and Solution II formulas under the assumption that the fundamental strip of regularity for $\hat{H}$ extends at least slightly above $\operatorname{Im} k = 1$ and at least slightly below $\operatorname{Im} k = 0$. In that case, one can apply the Residue Theorem (see Appendix 2.1) to show that

$$C_{II} = C_I + Se^{-\delta\tau}\left[1 - \hat{H}(k = i,V,\tau)\right]$$
$$P_{II} = P_I + Ke^{-r\tau}\left[1 - \hat{H}(k = 0,V,\tau)\right]$$

The meaning of these relationships is discussed further below and extensively in Chapter 9. For now, we simply note that in many situations, the fundamental transform is the transform of a norm-preserving transition density that is also *martingale-preserving.* These properties are defined below; when they hold, then

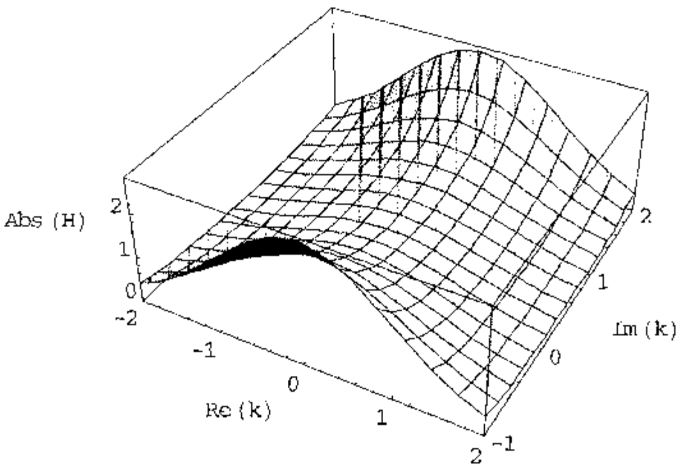
$$\hat{H}(k=0,V,\tau) = \hat{H}(k=i,V,\tau) = 1 \quad \text{and so} \quad C_{II} = C_I \ \text{and} \ P_{II} = P_I\,.$$

Example 1. *Constant or deterministic volatility.* In the case of constant volatility, the volatility process is $dV_t = 0$ and the fundamental transform satisfies $\hat{H}_\tau = -c(k)V\hat{H}$. Applying the initial condition, it's elementary to find $\hat{H}(k,V,\tau) = \exp[-c(k)V\tau]$. This is an entire function of k; i.e., analytic in the entire k-plane. So the only singularities of the integrands in both (2.8) and (2.10) are simple poles at $k = 0$ and $k = i$. In this case, (2.8) holds for the entire strip $1 < \operatorname{Im} k < \infty$ and (2.10) holds for the strip $0 < \operatorname{Im} k < 1$ and $C_{II} = C_I$. Of course, we should recover the B-S formula from both (2.8) or (2.10). This is shown in the Appendix 2.1 to this chapter.

In the case of deterministic volatility, the volatility process is $dV_t = b(V_t)dt$. The fundamental transform satisfies $\hat{H}_\tau = b(V)\hat{H}_V - c(k)V\hat{H}$. The solution to this equation is obtained by first finding $Y(u,V)$, which is defined as the solution to $dY/du = b(Y)$, $Y(0) = V$. Then, the fundamental transform is given by $H(k,V,\tau) = \exp[-c(k)U(V,\tau)]$, where $U(V,\tau) = \int_0^\tau Y(u,V)\,du$. So the k-plane behavior is identical to the case of constant volatility. Again the B-S formula is recovered, but the volatility V that appears in the formula is replaced by $v(V,\tau) = U(V,\tau)/\tau$. Again, see Appendix 2.1

Fig. 2.1 shows a plot of the modulus $|\hat{H}(k,1,1)|$, for the constant volatility case. Notice the saddle shape. Also the modulus is symmetrical about the $\mathrm{Im}(k)$ axis; we show below that this *reflection symmetry* is a general feature of the fundamental transform:

Fig. 2.1 $|\hat{H}|$ for the Constant Volatility Case



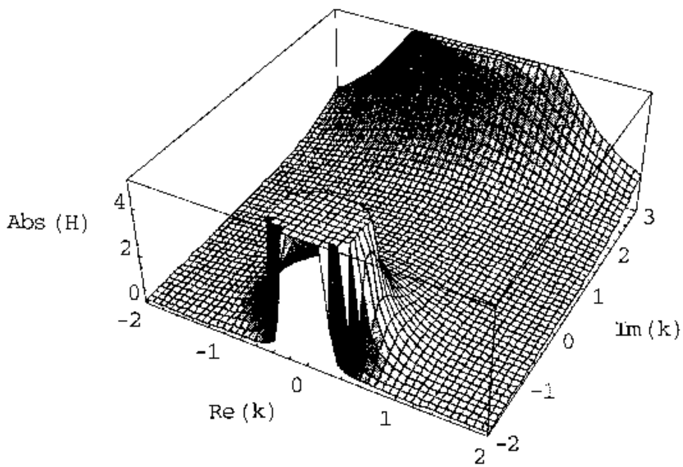
Along the pure imaginary axis, let $k = iy$ so that $c(k) = (y - y^2)/2$. This last expression becomes negative for $y < 0$ or $y > 1$, which means that the argument of the hyperbolic tangent, $(c/2)^{1/2}\xi\tau$, will be purely imaginary. So write $(c/2)^{1/2}\xi\tau = i\varphi$, where φ is a real number. But $\tanh(i\varphi) = i\tan\varphi$ which will of course diverge whenever $\varphi = (2n+1)\pi/2$, for $n = 0, \pm 1, \pm 2, \cdots$. Let k_n be the locations of the k-plane singularities of $\hat{H}$. The singularities in the figure correspond to the case $n = 0$. Setting $\varphi = \pi/2$, we find

$$k_0 = i y_{\pm}, \text{ where } y_{\pm}(\tau) = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \frac{\pi^2}{\xi^2\tau^2}} \cong \begin{cases} 3.68113 \\ -2.68113 \end{cases} \quad (\xi^2 = \tau = 1)$$

In the limits where $\xi^2 \to 0$ or $\tau \to 0$, we recover our previous results (an entire function) because the singularities move off to infinity. In the opposite limit where $\xi^2 \to \infty$, the singularities move to $y_{\pm} = 0,1$. So as long as ξ^2 is finite, we see that for this model, the integrand $\hat{H}(k,V,\tau)/(k^2 - ik)$ is free of singularities for the strips (i) $\alpha < \operatorname{Im} k < 0$ (ii) $0 < \operatorname{Im} k < 1$, and (iii) $1 < \operatorname{Im} k < \beta$, where $\alpha = y_-(\tau)$ and $\beta = y_+(\tau)$. This is typical.

In Fig 2.2, the line $\operatorname{Im} k = 1/2$ is symmetrically located between the two singularities. This occurs whenever $\rho = 0$. The square root model can also be solved when $\rho \neq 0$ (see Sec. 3 for formulas). Fig 2.3 shows the same model with the same parameters except that now $\rho = -1/2$; the reflection symmetry about $\operatorname{Re} k = 0$ is still present but now the symmetry about $\operatorname{Im} k = 1/2$ is lost.

Fig. 2.3 $|\hat{H}|$ for the Square Root Model ($\rho = -1/2$)



A Green function. Consider the entry in Table 2.1 for the delta function claim $\delta(\ln S_T - \ln K)$, but with $K = 1$. From the table, the transform of the payoff function is 1. So the fundamental transform is a solution to the problem with a delta function payoff and it's not too surprising that general claims can be developed in terms of this special one.

A closely related payoff function is $\delta(S_T - K)$, which has a fair value which is sometimes called a Green function or Arrow-Debreu security price. To get from one delta function to the other, apply the formula

$$\delta(f(x)) = \frac{\delta(x - x_0)}{|f'(x_0)|}, \quad \text{where } f(x_0) = 0\,.$$

Applying this in our case tells us that

$$\delta(S_T - K) = \frac{1}{K}\delta(\ln S_T - \ln K)\,.$$

That is, $\delta(S_T - K)$ has the payoff transform K^{ik-1} where k is any complex number. But, for times prior to expiration, we may still have a finite strip where the transform exists. So, a solution to the PDE (1.2) for this payoff, which we denote by $G(S,V,K,\tau)$ for Green function, is given by

$$G(S,V,K,\tau) = \frac{e^{-r\tau}}{2\pi} \int_{ik_i-\infty}^{ik_i+\infty} e^{-ik\ln S} e^{-ik(r-\delta)\tau} K^{ik-1} \hat{H}(k,V,\tau)\,dk$$

$$= \frac{e^{-r\tau}}{2\pi K} \int_{ik_i-\infty}^{ik_i+\infty} e^{-ikX} \hat{H}(k,V,\tau)\,dk\,, \quad \alpha < \operatorname{Im} k < \beta$$

Interpretation of the fundamental transform. The last equation can be interpreted as follows. Associated with the martingale pricing process (1.1) is a risk-adjusted transition density $\tilde{p}(S,V,S_T,\tau)$. Specifically $\tilde{p}\,dS_T$ is the probability that the stock price S with instantaneous variance V will, after the elapse of time-τ, reach the interval $(S_T, S_T + dS_T)$ with *any* variance. Since the stock price must end up *somewhere,* $\tilde{p}(S,V,S_T,\tau)$ is norm-preserving with respect to S_T. That is, $\int_0^\infty \tilde{p}(S,V,S_T,\tau)\,dS_T = 1$. Also, we have the initial value $\tilde{p}(S,V,S_T,0) = \delta(S - S_T)$. From the above, we know that both $G(S,V,S_T,\tau)$ and $e^{-r\tau}\tilde{p}(S,V,S_T,\tau)$ satisfy the same PDE, (1.1), with the same initial condition. Are these two functions equal? The answer is yes, *if* $G(S,V,S_T,\tau)$ is norm-preserving. As we now show, there is a very simple test to determine when $G(S,V,S_T,\tau)$ is norm-preserving.

We can simply relabel $K \to S_T$ and re-write the last equation, using $\tilde{X} = \ln\left[\frac{S}{S_T}\right] + (r-\delta)\tau$, as

$$G(S,V,S_T,\tau) = \frac{e^{-r\tau}}{2\pi S_T} \int_{ik_i-\infty}^{ik_i+\infty} e^{-ik\tilde{X}} \hat{H}(k,V_t,\tau)\, dk\,, \tag{2.12}$$

Inversion. Multiply both sides of (2.12) by $\exp(ik'\tilde{X})$ and integrate with respect to S_T from $S_T = 0$ to $S_T = \infty$. On the right-hand-side this is accomplished by changing variables to $y = \ln S_T$ and using the delta function formula given above. The result is

$$\hat{H}(k,V,\tau) = e^{r\tau} \int_0^\infty e^{ik\tilde{X}} G(S,V,S_T,\tau)\, dS_T\,.$$

This last formula shows that $\hat{H}(k=0,V,\tau) = e^{r\tau}\int_0^\infty G(S,V,S_T,\tau)\,dS_T$; hence $\tilde{p}(S,V,S_T,\tau)$ is norm-preserving if, and only if, $\hat{H}(k=0,V,\tau)=1$. That is, we can identify the fundamental transform as the Fourier transform of the norm-preserving transition density in S_T if and only if $\hat{H}(k=0,V,\tau)=1$. In addition, the last formula shows that the martingale property for the stock price:

$$Se^{-\delta\tau} = e^{-r\tau} \int_0^\infty S_T\, \tilde{p}(S,V,S_T,\tau)\, dS_T\,,$$

is preserved by G, if and only if $\hat{H}(k=i,V,\tau)=1$. These results prompt the following definitions:

Definitions. A fundamental transform $\hat{H}(k,V,\tau)$ is called *norm-preserving* if it has the property $\hat{H}(k=0,V,\tau)=1$. If a fundamental transform is not norm-preserving, it's called *norm-defective*. A fundamental transform is called *martingale-preserving* if it has the property $\hat{H}(k=i,V,\tau)=1$; otherwise it's called *martingale-defective*.

Examples. The fundamental transform solution for the square root model is both norm-preserving and martingale-preserving. The fundamental transform solutions for the 3/2 model and the GARCH diffusion solution (see Sec. 3 below and Ch. 11) are sometimes norm-defective or martingale-defective.

With these definitions, we can assert that, when a fundamental transform is norm-preserving, then it's the Fourier transform of the risk-adjusted transition density $\tilde{p}(S,V,S_T,\tau)$; i.e.,

(2.13)
$$\boxed{\begin{gathered}\hat{H}(k,V,\tau)=\int_0^\infty e^{ik\tilde{X}}\,\tilde{p}(S,V,S_T,\tau)\,dS_T\,,\\ \text{where } \tilde{X}=\ln\left[\frac{S}{S_T}\right]+(r-\delta)\tau\,,\\ \text{if and only if } \hat{H}(k=0,V,\tau)=1\end{gathered}}$$

Failure of the martingale pricing formula. We shall find that it's possible for a fundamental transform, in very typical models, to be norm-preserving, but martingale-defective. Since it's norm-preserving, it's the Fourier transform of the risk-adjusted transition density $\tilde{p}(S,V,S_T,\tau)$. In that case, as we noted earlier, we can interpret call option Solution I as an expectation

$$C_I(S,V,\tau)=e^{-r\tau}\mathbb{E}_t\left[(S_T-K)^+\right].$$

The expectation is taken with respect to the norm-preserving density of the risk-adjusted process: $\tilde{p}(S,V,S_T,\tau)$. But, as we showed earlier, because the fundamental transform is martingale-defective, we have a *second* PDE solution $C_{II}\neq C_I$. Moreover, we show in Chapter 9 that the arbitrage-free fair value is given by C_{II}. In other words, the usual martingale pricing formula $e^{-r\tau}\mathbb{E}_t\left[(S_T-K)^+\right]$, while always a solution to the valuation PDE, does *not* always give the fair value of an option. Sometimes, option prices are not martingales, but only strictly local martingales.

Relationship to volatility explosions. When a fundamental transform is norm-preserving but martingale-defective, we also show in Chapter 9 that $1-\hat{H}(k=i,V,\tau)=\hat{P}_{\exp}(V,\tau)$, where the right-hand-side is an *explosion* probability. Specifically, $\hat{P}_{\exp}(V,\tau)$ is the probability that a particular volatility process, the *auxiliary* volatility process, reaches $V=+\infty$ prior to time τ. Very briefly, to get a sense of what it going on in these cases, take $k=i$ in (2.7) and consider solutions to (2.7) $\hat{P}_{\exp}(V,\tau)$ with vanishing initial condition and with $\hat{P}_{\exp}(V=\infty,\tau)=1$. If you can find such solutions, the auxiliary process can explode. Similarly, if the risk-adjusted volatility process can explode, then there exists a norm-defective fundamental transform such that $1-\hat{H}(k=0,V,\tau)=P_{\exp}(V,\tau)$, where the right-hand-side is the explosion probability for the risk-adjusted process. In this case, take $k=0$ in (2.7) Again, see Chapter 9 for a detailed discussion.

Reflection symmetry. Note that we always have the property, because the fundamental transform is the transform of a real-valued function, $\hat{H}^*(k,V,\tau) = \hat{H}(-k^*,V,\tau)$. This always holds, whether or not the transform is defective.

With the exception of the results for the 3/2 model given in Sec. 3 below, we generally assume without further comment that for the remaining development in this chapter, the fundamental transform is both norm- and martingale-preserving.

Power law behavior and scaling. Since $\tilde{X}$ is a function of the ratio S/S_T, then (2.13) shows that the transition density satisfies the scaling behavior

(2.14) $$\tilde{p}(S,V,S_T,\tau) = \frac{1}{S_T}\varphi(u), \quad \text{where } u = S/S_T,$$

and $\varphi(u)$ is some *scaling* function. So if we know the behavior of $\tilde{p}(S,V,S_T,\tau)$ for $S_T \to \infty$, $(S_T \to 0)$ then we also know the behavior as $S \to 0$, $(S \to \infty)$ respectively. In fact, if the problem is regular, then we can deduce a lot about that behavior. For example, (2.13) exists for $k = iy$, where $y = \beta - \varepsilon$ for every $\varepsilon > 0$. That is, for any $S > 0$

$$\int_0^\infty S_T^{\beta-\varepsilon}\,\tilde{p}(S,V,S_T,\tau)\,dS_T < \infty.$$

This implies that $\tilde{p}(S,V,S_T,\tau) = O(S_T^{-1-\beta-\varepsilon})$ as $S_T \to \infty$, for every $\varepsilon > 0$. Similarly, taking $y = \alpha + \varepsilon$ implies that $\tilde{p}(S,V,S_T,\tau) = O(S_T^{-1+\alpha-\varepsilon})$ as $S_T \to 0$ [6]. In turn, these relations may be restated in terms of the scaling function:

(2.15) $$\varphi(u) = \begin{cases} O(u^{\beta+\varepsilon}) & \text{as } u \to 0 \\ O(u^{-\alpha+\varepsilon}) & \text{as } u \to \infty \end{cases} \quad \text{for every } \varepsilon > 0.$$

Neglected boundary terms. One application of (2.14) is to show that the neglected boundary terms associated with the call option solution (2.8) can indeed be neglected. Writing $f(x) = C(e^x,V,t)$, where $x = \ln S$, the two neglected boundary terms from parts integrations were

---

[6] The notation, as $x \to x_0$, $f(x) = O(g(x))$, means that $f(x)/g(x)$ is bounded as $x \to x_0$. For a more rigorous discussion of the power law order behavior for $\tilde{p}$, see Fourier's theorem for analytic functions (Titchmarsh 1975, Theorem 26, p.44)

$$e^{ikx} f(x)\Big|_{x=-\infty}^{x=+\infty} \quad \text{and} \quad e^{ikx} \frac{\partial f(x)}{\partial x}\Big|_{x=-\infty}^{x=+\infty} \quad \text{where } 1 < \operatorname{Im} k < \beta$$

In Appendix 2.3 to this chapter, we show the general arbitrage bounds $C \le S = e^x$, and $C_S \le 1$, which implies that both $f \le e^x$ and $\partial f / \partial x \le e^x$ for large enough x. So, since $\operatorname{Im} k > 1$, both boundary terms vanish at the upper limit $x = +\infty$.

When option prices are martingales, they are given by the pricing formula

$$C(S,V,\tau) = e^{-r\tau} \int_0^\infty \tilde{p}(S,V,S_T,\tau) \max[S_T - K, 0]\, dS_T$$
$$= e^{-r\tau} S \int_0^{S/K} \varphi(u) \Big[1 - \frac{K}{S} u\Big] \frac{du}{u^2},$$

where we substituted from (2.14). Letting $S \to 0$, we have from (2.15) that $\varphi(u) = O(u^{\beta+\varepsilon})$ as $u \to 0$. Substituting this expression into the above integral implies that $C(S) = O(S^{\beta+\varepsilon})$ as $S \to 0$ for every $\varepsilon > 0$. Or, in other words both $f = O(e^{x(\beta+\varepsilon)})$ and $\partial f / \partial x = O(e^{x(\beta+\varepsilon)})$ as $x \to -\infty$. Since $\operatorname{Im} k < \beta$, both boundary terms also vanish at the lower limit $x = -\infty$. ■

The fundamental transform as a characteristic function. By a characteristic function, we mean any function that has the form

$$\hat{H}(k) = \int_{-\infty}^{\infty} e^{ikx} dG(x) = \int_{-\infty}^{\infty} e^{ikx} g(x) dx, \tag{2.16}$$

where $G(x)$ is a *cumulative distribution function* and $g(x) = dG/dx$ is its probability density. For our purposes in this chapter, a cumulative distribution function is function of a real variable x that is (i) non-decreasing, and (ii) satisfies $G(-\infty) = 0$, $G(+\infty) = 1$. Of course, for this to occur, then $g(x)$ must be non-negative and integrable.

To show that $\hat{H}$ is a characteristic function, change integration variables in (2.13) from S_T to $\tilde{X} = \ln(S/S_T) + (r-\delta)\tau$ and define a new function $g(\tilde{X}; S,V,\tau)$ by

$$\tilde{p}(S,V,S_T,\tau) S\, e^{(r-\delta)\tau - \tilde{X}} = g(\tilde{X}; S,V,\tau).$$

Or, suppressing arguments again,

$$dG(\tilde{X}) = g(\tilde{X})\, d\tilde{X} = \tilde{p}\Big(S,V,Se^{(r-\delta)\tau - \tilde{X}}, \tau\Big) S\, e^{(r-\delta)\tau - \tilde{X}} d\tilde{X}.$$

This shows that $H(\tilde{X})$ is non-negative and now (2.13) reads

$$\hat{H}(k) = \int_{-\infty}^{\infty} e^{ik\tilde{X}} dG(\tilde{X}),$$

where $$G(\tilde{X}) = \int_{-\infty}^{\tilde{X}} \tilde{p}\left(S, V, Se^{(r-\delta)\tau-x}, \tau\right) S\, e^{(r-\delta)\tau-x} dx$$

$$= \int_{S\exp[(r-\delta)\tau-\tilde{X}]}^{\infty} \tilde{p}(S, V, S_T, \tau) dS_T.$$

This last equation shows that $G(\tilde{X})$ is indeed non-decreasing and satisfies $G(-\infty) = 0$, $G(+\infty) = 1$. And, since $\hat{H}(k)$ is of the form (2.16), with $x = \tilde{X}$, this shows that $\hat{H}(k)$ is a characteristic function. In fact, the examples show that $\hat{H}(k)$ can typically be further characterized as an *analytic characteristic function*. This important topic is discussed in Sec. 4.

The martingale pricing density. We can also consider the probability density $p(S_t, V_t, S_T, \tau)$ that the *actual* volatility process, starting from (S_t, V_t) reaches S_T with any variance. The ratio of the two probabilities

$$M_t = M(S_t, V_t, S_T, \tau) = \frac{\tilde{p}(S_t, V_t, S_T, \tau)}{p(S_t, V_t, S_T, \tau)}$$

also values arbitrary payoffs. That is, we have two general pricing formulas that work for any volatility-independent claim price, when it's a martingale:

$$\text{(2.17)} \qquad F(S_t, V_t, \tau) = e^{-r\tau} \int_0^{\infty} \tilde{p}(S_t, V_t, S_T, \tau) g(S_T) dS_T$$

$$= e^{-r\tau} \int_0^{\infty} M(S_t, V_t, S_T, \tau) p(S_t, V_t, S_T, \tau) g(S_T) dS_T.$$

These are explicit integral kernel versions of the martingale pricing formulas presented in Chapter 1. As a general rule, (2.17) is the long way around, however, from the Solution I and II formulas based upon a direct k-plane integration, since it forces you to do an extra integration. So we don't recommend (2.17) for most computations—but we have seen already that it was useful in considering the $S \to 0$ and $S \to \infty$ limits of the theory.

Forward contracts and options on forwards. The formulas are easily modified to handle forwards. For example, the forward stock price F_t is defined to be the fair value at time t for delivery of one share of the stock at time T. As usual, this price is determined by arbitrage to be $F_t = e^{(r-\delta)\tau} S_t$, where $\tau = T - t$. Hence by Ito's formula, the martingale pricing process $\tilde{P}$ of (1.1) becomes $dF_t = \sigma_t F_t d\tilde{B}_t$, with the same volatility evolution. Under $\tilde{P}$, the forward price behaves like a stock with a dividend yield of r. Using this idea, a call option on the forward, say solution II at $\mathrm{Im}\, k = 1/2$, becomes

$$C_{II}(F,V,\tau) = e^{-r\tau}\left[F - K\frac{1}{2\pi}\int_{i/2-\infty}^{i/2+\infty} e^{-ikX}\,\frac{\hat{H}(k,V,\tau)}{k^2 - ik}\,dk\right],$$

where $X = \ln(F/K)$.

Summary. If the initial-value problem in the box below is regular in a strip $\alpha < \mathrm{Im}\, k < \beta$ in the complex k-plane, then the solution can be used to determine option prices by a k-plane integration:

$$(2.19)\qquad \frac{\partial \hat{H}}{\partial \tau} = \tfrac{1}{2}a^2(V)\frac{\partial^2 \hat{H}}{\partial V^2} + \left[\tilde{b}(V) - ik\rho(V)a(V)V^{1/2}\right]\frac{\partial \hat{H}}{\partial V} - c(k)V\hat{H}$$

where $c(k) = (k^2 - ik)/2$. In addition to $\hat{H}(k,V,\tau = 0) = 1$, the fundamental solution has the following properties:

$$(2.20)\qquad \begin{aligned} &\text{(i)} && \hat{H}^*(k,V,\tau) = \hat{H}(-k^*,V,\tau) \\ &\text{(ii)} && \hat{H}(k=0,V,\tau) = 1 - P_{\exp}(V,\tau) \\ &\text{(iii)} && \hat{H}(k=i,V,\tau) = 1 - \hat{P}_{\exp}(V,\tau), \end{aligned}$$

where $\hat{P}_{\exp}$ and $P_{\exp}$ are the probabilities that the auxiliary volatility process and risk-adjusted volatility process can explode to $+\infty$

3 Some Models with Closed-form Solutions

In general, even with the assumption of a simple process for the *actual* volatility, the simplest risk-adjustments (via utility theory) can produce complex results for the martingale pricing process. Risk-adjustment is discussed in detail in Chapter 7. To obtain a model that can be solved in closed-form generally requires two assumptions: (i) a relatively simple process for the actual volatility, and (ii) a relatively simple preference model, such as the representative agent model with power utility.

Making both of these assumptions, here is a short list of models that can be solved in closed-form. Each volatility process has constant correlation ρ with the stock price process. All other parameters are also constants. The agent is assumed to be a pure investor (no consumption until a final date) with a distant planning horizon. The parameter γ is the representative's risk-aversion parameter. It's restricted to $\gamma < 1$ plus some additional restrictions that are shown. The risk-aversion adjustments are derived in Chapter 7.

Some solvable models and their volatility processes

Square root model

$$P:\ dV = (\omega - \theta V)\,dt + \xi\sqrt{V}\,dW$$

$$\tilde{P}:\ dV = \left\{\omega - \tilde{\theta} V\right\} dt + \xi\sqrt{V}\,dW\,,$$

where $\tilde{\theta} = (1-\gamma)\rho\xi + \sqrt{\theta^2 - \gamma(1-\gamma)\xi^2}$

Conditions: $\gamma(1-\gamma)\xi^2 \le \theta^2$

3/2 model

$$P:\ dV = (\omega V - \theta V^2)\,dt + \xi V^{3/2}\,dW$$

$$\tilde{P}:\ dV = \left\{\omega V - \tilde{\theta} V^2\right\} dt + \xi V^{3/2}\,dW\,,$$

where $\tilde{\theta} = -\tfrac{1}{2}\xi^2 + (1-\gamma)\rho\xi + \sqrt{(\theta + \tfrac{1}{2}\xi^2)^2 - \gamma(1-\gamma)\xi^2}$

Conditions: $\gamma(1-\gamma)\xi^2 \le (\theta + \tfrac{1}{2}\xi^2)^2$

Geometric Brownian motion

$$P:\ dV = -\theta V\,dt + \xi V\,dW$$

$$\tilde{P}:\ dV = \left\{-(1-\gamma)\rho\xi V^{3/2} + \tfrac{1}{2}\xi^2 V\left[1 + y\frac{K_\mu'(y)}{K_\mu(y)}\right]\right\} dt + \xi V\,dW\,,$$

where $y = \dfrac{2}{\xi}\sqrt{-\gamma(1-\gamma)V}$ and $\mu = 1 + \dfrac{2\theta}{\xi^2}$.

Conditions: ($-2\theta < \xi^2$ and $\gamma \le 0$) or $\gamma = 1$

The solution for the fundamental transform under geometric Brownian motion is quite complex and difficult to work with when the correlation is non-zero. In contrast, both the square root model and the 3/2 model have short solutions that we now show. Both models use the reduced variables

$$(3.1)\qquad t = \tfrac{1}{2}\xi^2\tau\,, \qquad \tilde{\omega} = \frac{2}{\xi^2}\omega\,, \qquad \tilde{c} = \frac{2}{\xi^2}c(k)\,.$$

In terms of these variables, the fundamental transforms are given below. The results for all three models are derived in Chapter 11.

The square root model[7] $[\,\gamma \le 1$ and $\gamma(1-\gamma)\xi^2 \le \theta^2\,]$

$$(3.2)\qquad \hat{H}(k,V,\tau) = \exp[\,f_1(t) + f_2(t)V\,]\,, \quad \text{using}$$

$$f_1(t) = \tilde{\omega}\,\left[t\,g - \ln\left(\frac{1-h\exp(d\,t)}{1-h}\right)\right], \qquad f_2(t) = g\,\left(\frac{1-\exp(d\,t)}{1-h\exp(d\,t)}\right)$$

$$d = [\hat{\theta}^2 + 4\tilde{c}]^{1/2}\,, \qquad g = \tfrac{1}{2}(\hat{\theta} + d)\,, \qquad h = \frac{\hat{\theta}+d}{\hat{\theta}-d}\,,$$

where $$\hat{\theta}(k) = \frac{2}{\xi^2}\left[(1-\gamma+ik)\rho\xi + \sqrt{\theta^2 - \gamma(1-\gamma)\xi^2}\,\right]$$

The 3/2 model[8] $[\,\gamma \le 1$ and $\gamma(1-\gamma)\xi^2 \le (\theta + \tfrac{1}{2}\xi^2)^2\,]$

$$(3.3)\qquad \hat{H}(k,V,\tau) = \frac{\Gamma(\beta-\alpha)}{\Gamma(\beta)}\left[X\left(\frac{\tilde{\omega}}{V},\omega\,\tau\right)\right]^{\alpha} M\left[\alpha,\beta,-X\left(\frac{\tilde{\omega}}{V},\omega\,\tau\right)\right],$$

using $$X(x,t) = \frac{x}{e^t-1}\,, \quad \mu = \tfrac{1}{2}(1+\hat{\theta})\,, \quad \delta = \left[\mu^2 + \tilde{c}\,\right]^{1/2},$$

$$\alpha = -\mu + \delta\,, \quad \beta = 1 + 2\delta\,,$$

where $$\hat{\theta}(k) = -1 + \frac{2}{\xi^2}\left[\sqrt{(\theta + \tfrac{1}{2}\xi^2)^2 - \gamma(1-\gamma)\xi^2} + (1-\gamma+ik)\rho\xi\,\right].$$

---

[7] Heston's (1993) call option solution is also achieved with a transform-based approach: an ordinary Fourier transform with respect to the log-strike price. In Heston's approach, there are two transforms instead of the one here.

[8] Caution: this fundamental transform is sometimes either norm-defective or martingale-defective. Using risk-neutral preferences only, the 3/2 model has been independently developed by Heston (1997), using an approach similar to his 1993 paper.

In (3.3), $\Gamma(z)$ is the Gamma function and $M(\alpha,\beta,z)$ is a confluent hypergeometric function[9]. Also, note that the second argument for $X(\cdot,\cdot)$ in (3.3) uses $\omega\,\tau = \tilde{\omega}\,t$.

Determining the fundamental strip. Once you have $\hat{H}$ for a model, then you can analyze it to determine the fundamental strip of regularity: $\alpha < \operatorname{Im} k < \beta$ and whether it's norm- and/or martingale-preserving. Once you know that, you know the regions of validity for all of the option formulas presented previously. As an example, consider the square root model above. Rather than a complete analysis, let's just establish that the strip $0 < \operatorname{Im} k < 1$ is free from singularities—this places the boundaries of the fundamental strip outside this region.

The singularities occur where $1 = he^{dt}$, which causes divergences in both $f_1(t)$ and $f_2(t)$. We know the singularities occur along the imaginary axis, so consider $k = iy$, where y is real. We see from (3.1) that $\tilde{\theta}$ is real along that axis. Moreover, for $0 < \operatorname{Im} k < 1$, then $\tilde{c} > 0$ (and real). Hence d is real and satisfies $d > |\tilde{\theta}|$, which implies that $\tilde{\theta} + d > 0$ and $\tilde{\theta} - d < 0$. In other words, $h < 0$. Since d is real and $h < 0$, there can be no solutions to $h = e^{-dt}$ inside the strip $0 < \operatorname{Im} k < 1$. Hence $0 < \operatorname{Im} k < 1$ is free from singularities. ■

Integrating. Once you know where you can legally integrate, then you're a k-plane integration away from the call option price. For these remaining steps, see Appendix 2.2 to this chapter. When you obtain those prices, you'll find that both models exhibit the typical qualitative behavior that we discuss in subsequent chapters: implied volatility smile patterns (see Chapter 5) and an implied volatility term structure that flattens to a constant as $\tau \to \infty$ (see Chapter 6). For the derivation of the formulas (3.2) and (3.3) see Chapter 11.

4 Analytic Characteristic Functions

We have seen from examples that $\hat{H}(k,V,\tau)$ is often an analytic function of k in some neighborhood. In general, a characteristic function $\hat{f}(k)$ is *any* function which has the representation

---

[9] See Abramowitz and Stegun (1970) for properties of these and other special functions.

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{ikx} p(x)\,dx \,,$$

where $p(x)$ is a probability density for some cumulative distribution function. Lukacs (1970, Chapter 7) proves two theorems that are relevant to our application. To achieve a more symmetrical notation, we write $\hat{H}(k)$ for our fundamental transform and $\hat{f}(k)$ for a generic characteristic function

If $\hat{f}(k)$ is regular in a neighborhood of $k = iy$, where y is real, then we call $\hat{f}(k)$ an analytic characteristic function (Lukacs takes $y = 0$). We have shown in a number of examples that the regions of regularity for $\hat{H}(k)$ are typically strips in the complex k-plane. And, we have suggested strips as regions of regularity in general. The rationale for the general case lies in the following theorem, quoted without proof:

THEOREM 2.1 (Lukacs Theorem 7.1.1): *If a characteristic function $\hat{f}(k)$ is regular in the neighborhood of $k = 0$, then it is also regular in a horizontal strip and can be represented in this strip by a Fourier integral. This strip is either the whole plane, or it has one or two horizontal boundary lines. The purely imaginary points on the boundary of the strip of regularity (if this strip is not the whole plane) are singular points of $\hat{f}(k)$.*

Discussion. In our application, we have often found that the fundamental transform $\hat{H}(k)$ is regular in the horizontal strip $\alpha < \operatorname{Im} k < \beta$, where $\alpha < 0$ and $\beta > 1$. We have already pointed out that the PDE (2.19) is especially well-behaved when $\operatorname{Re} c(k) > 0$, which occurs when $0 < \operatorname{Im} k < 1$. In this subsection, we try to understand a little better why the strip $0 < \operatorname{Im} k < 1$ is often free of singularities of $\hat{H}(k)$. We know from (2.13) that $\hat{H}(k)$ has the representation

$$\hat{H}(k,V,\tau) = \int_0^{\infty} e^{ik\tilde{X}(S_T)}\, \tilde{p}(S,V,S_T,\tau)\,dS_T \,,$$

where $\tilde{X}(S_T) = \ln\left[\dfrac{S}{S_T}\right] + (r-\delta)\tau$. Therefore

$$\hat{H}^{(m)}(k) = \frac{d^m}{dk^m}\hat{H}(k) = i^m \int_0^{\infty} \left[\tilde{X}(S_T)\right]^m e^{ik\tilde{X}}\, \tilde{p}(S,V,S_T,\tau)\,dS_T \,.$$

Let $k = k_r + iy$, where k_r and y are real. Then,

$$\hat{H}^{(m)}(k_r + iy) = i^m e^{-y(r-\delta)\tau} \int_0^{\infty} \left[\tilde{X}(S_T)\right]^m e^{ik_r\tilde{X}} \left(\frac{S_T}{S}\right)^{y} \tilde{p}(S,V,S_T,\tau)\,dS_T$$

Along the purely imaginary axis, we have

$$(4.1) \qquad \hat{H}^{(m)}(iy) = i^m e^{-y(r-\delta)\tau} \int_0^\infty [\tilde{X}(S_T)]^m \left(\frac{S_T}{S}\right)^y \tilde{p}(S,V,S_T,\tau)\, dS_T \,.$$

And in particular for the fundamental transform itself, we have

$$(4.2) \qquad \hat{H}(iy) = e^{-y(r-\delta)\tau} \int_0^\infty \left(\frac{S_T}{S}\right)^y \tilde{p}(S,V,S_T,\tau)\, dS_T \,.$$

Now it's known from complex variable theory that if a function is analytic in a region R, then it has derivatives of all orders and a Taylor series in R. Consequently, if $\hat{H}(k)$ is regular near the point $k = iy$, then the series

$$\hat{H}(k) = \sum_{m=0}^{\infty} \frac{\hat{H}^{(m)}(iy)}{m!}(k - iy)^m$$

is convergent. This means that $\hat{H}(k)$ is an analytic characteristic function near $k = iy$ if and only if the following two conditions are satisfied:

$$(4.3) \qquad \text{(i)} \quad \hat{H}^{(m)}(iy) \text{ exists for all } m = 0,1,2,\ldots$$

$$(4.4) \qquad \text{(ii)} \quad \lim_{m\to\infty} \left[\frac{|\hat{H}^{(m)}(iy)|}{m!}\right]^{1/m} = \frac{1}{\Delta} \text{ is finite.}$$

Then if these conditions hold, $\hat{H}(k)$ is regular in the strip $(y - \Delta) < \operatorname{Im} k < (y + \Delta)$.

Now recall the normalization and martingale identity:

$$\text{(a)} \quad \int_0^\infty \tilde{p}(S,V,S_T,\tau)\, dS_T = 1 \quad \text{and} \quad \text{(b)} \quad S e^{-\delta\tau} = e^{-r\tau} \int_0^\infty S_T\, \tilde{p}(S,V,S_T,\tau)\, dS_T \,.$$

These two relations strongly restrict the possible behavior of $\tilde{p}(S_T)$ near $S_T = 0$ and $S_T = \infty$, where we suppress the other arguments in $\tilde{p}(S,V,S_T,\tau)$. Because of (a), it must be true that $\tilde{p}(S_T) = O(S_T^{-1-\varepsilon})$ for every $\varepsilon > 0$ as $S_T \to 0$. In other words $\tilde{p}(S_T)$, if it diverges at all as $S_T \to 0$, diverges no faster than $S_T^{-1+\varepsilon}$. Similarly, because of (b), it must be true that $\tilde{p}(S_T) = O(S_T^{-2-\varepsilon})$ for every $\varepsilon > 0$, as $S_T \to \infty$. Because of these two end-point behaviors, if you keep y in (4.2) in the range $0 < y < 1$, then you will have a convergent integral. Similarly, with the same restriction, (4.1) should exist for any m because, (I) as $x \to \infty$, $x^y \,|\ln(1/x)|^m = O(x)$ for any $y < 1$ and (II) as $x \to 0$, $x^y \,|\ln(1/x)|^m = O(1)$ for any $y > 0$

Unfortunately, this argument establishes (4.3) but not (4.4). Nevertheless, it provides some additional insight into why $0 < \operatorname{Im} k < 1$ is the "natural" strip for the financial claim problem.

Stationary points. In Chapter 6, "The Term Structure of Implied Volatility", we examine the asymptotic $\tau \to \infty$ behavior of the theory. It turns out that the asymptotic implied volatility is determined by an eigenvalue of a differential operator. This eigenvalue is also a stationary or saddle point of $\hat{H}(k)$ in the k-plane (recall the saddle shapes from the figures). We discover, in particular models, that these stationary points always lie along the purely imaginary axis. The general reason for this behavior lies in the following theorem:

THEOREM 2.3 (Lukacs Theorem 7.1.2): *Let $\hat{f}(k)$ be an analytic characteristic function. Then $|\hat{f}(k)|$ attains its maximum along any horizontal line contained in the interior of its strip of regularity on the imaginary axis. The derivatives $d^{2j}\hat{f}/dk^{2j}$ of even order of $\hat{f}$ have the same property.*

PROOF: We know that $\hat{f}(k)$ has the representation

$$\hat{f}(k) = \int_{-\infty}^{\infty} e^{ikx} p(x)\,dx\,, \qquad \alpha < \operatorname{Im} k < \beta\,.$$

Therefore $\hat{f}^{(m)}(k) = \dfrac{d^m}{dk^m}\hat{f}(k) = i^m \displaystyle\int_{-\infty}^{\infty} x^m e^{ikx} p(x)\,dx\,.$

Let $k = k_r + iy$, where k_r and y are real and where $\alpha < y < \beta$. Then,

$$|\hat{f}^{(m)}(k_r + iy)| \;\le\; \int_{-\infty}^{\infty} |x|^m e^{-yx} p(x)\,dx\,.$$

If $m = 2j$ $(j = 0,1,2,\ldots)$ is an even integer, then this becomes

$$|\hat{f}^{(2j)}(k_r + iy)| \;\le\; \int_{-\infty}^{\infty} x^{2j} e^{-yx} p(x)\,dx \;=\; |\hat{f}^{(2j)}(iy)|,$$

so that
$$\max_{-\infty < k_r < \infty} |\hat{f}^{(2j)}(k_r + iy)| \;=\; |\hat{f}^{(2j)}(iy)|.$$

The ridge property. The relation

$$|\hat{f}(k_r + iy)| \;\le\; |\hat{f}(iy)|$$

is very important in the theory of analytic characteristic functions, and is called the "ridge property". It plays an important role in our application in the asymptotic $\tau \to \infty$ theory. So we have learned that if the fundamental transform $\hat{H}(k)$ is an analytic characteristic function, then it is also a "ridge function".

5 A Bond Price Analogy and Option Price Bound

In this section, we specialize to the case of zero correlation between the stock price and volatility process. We relate the problem of determining the fundamental transform to the theory of bond prices. Then, we show that there is a useful upper bound for at-the-money option prices involving the expected volatility and Black-Scholes prices, both of which are usually easy to compute.

With $\rho = 0$, the initial value problem for the fundamental transform (2.19) becomes

$$(5.1) \quad \frac{\partial \hat{H}}{\partial \tau} = \tfrac{1}{2} a^2(V) \frac{\partial^2 \hat{H}}{\partial V^2} + \tilde{b}(V) \frac{\partial \hat{H}}{\partial V} - c(k) V \hat{H}, \quad \text{where } \hat{H}(k, V, \tau = 0) = 1.$$

The Feynman-Kac style solution to (5.1), when it exists, is given by[10]:

$$(5.2) \qquad \hat{H}(k, V, \tau) = \mathbb{E}_0\left[\exp\left(-c(k)\int_0^\tau Y_t^V \, dt\right)\right].$$

The expectation is taken with respect to the volatility process from (1.1), namely $dY_t = \tilde{b}(Y_t)dt + a(Y_t)dW_t$, and Y_t^V is a solution to that SDE that begins at $Y_0 = V$.

We integrate along the contour $\operatorname{Im} k = 1/2$. Along that contour, $k^2 = (1/4) + k_r^2$, where k_r is real. Along the integration contour in (5.2) $c(k) = (k^2 - ik)/2 = (k_r^2 + 1/4)/2 > 0$. Since Y_t is also positive, the exponent in (5.2) is real and negative for every realization of a volatility path. That tells us that $0 \le \hat{H}(k, V, \tau) \le 1$ along the integration path, which suggests a bond price analogy.

That is, $\hat{H}$ behaves very much like a discount bond price in a (single factor) theory of the term structure of interest rates. You can make the analogy precise by defining the analogous interest rate $\tilde{r} = cV$, which makes sense for all k along the contour since $c > 0$. The tilde distinguishes this "pseudo" interest rate

---

[10] See Chapter 1, Appendix 2. In Chapter 4, we generalize this style of solution to the $\rho \neq 0$ case.

from the real interest rate r in (1.1). That is, a way to think about $\hat{H}$ is to make the connection $\hat{H}(V,\tau) \leftrightarrow B(\tilde{r},\tau)$, where $B(\tilde{r},\tau)$ is a riskless bond price (a T-bill). Then (5.1) becomes

$$\frac{\partial B}{\partial \tau} = \tfrac{1}{2}\sigma^2(\tilde{r})\frac{\partial^2 B}{\partial \tilde{r}^2} + \mu(\tilde{r})\frac{\partial B}{\partial \tilde{r}} - \tilde{r}B\,, \qquad B(\tilde{r},\tau=0)=1\,,$$

which is the bond problem[11]. Now from experience with bond prices, you expect that as interest rates rise, then bond prices fall, so you would have $B_{\tilde{r}}(\tilde{r},\tau) \le 0$, or in our case $\hat{H}_V(V,\tau) \le 0$. This can be proved by a very useful method discussed in Appendix 2.3 to this chapter: namely taking the V-derivative of the PDE and invoking the Feynman-Kac formulas. Instead, let's use an alternative, although closely related method.

The alternative method was used by Bergman, Grundy, and Wiener (1996) to prove some general properties of option prices. (General properties of option prices are the subject of Appendix 2.3). They noted a very powerful property of one-dimensional diffusions, which they called the "no crossing" property. The idea is to consider a fixed realization of a Brownian motion path in $dY_t = \tilde{b}(Y_t)dt + a(Y_t)dW_t$. This realization is $\{W_t \mid 0 \le t \le \tau\}$. It's a single (very jagged!) continuous curve that starts at $W_0 = 0$ and ends up somewhere at $t = \tau$. (See Fig. 1 in their paper).

Now, holding that path fixed, consider two paths for the process Y_t, which only differ by where they start. Path 1 starts at V_1 and Path 2 starts at V_2. Then, the no-crossing idea is simply that these two Y-paths can never cross. If they do happen to intersect at some point in time $t = s$, then they must merge from that point onward, because the evolution is Markovian. That is,

$$\text{if } Y_s^{V_1} = Y_s^{V_2} \text{ then } dY_s = \tilde{b}(Y_s)dt + a(Y_s)dW_s$$

will be the same for the two paths and will also be the same for all $t \ge s$. In other words, with a fixed Brownian motion path, and $V_1 \ge V_2$, then we have $Y_t^{V_1} \ge Y_t^{V_2}$ for $0 \le t \le \tau$ and so $\int_0^\tau Y_t^{V_1}\,dt \ge \int_0^\tau Y_t^{V_2}\,dt$.

---

[11] Of course, everything depends on the parameter k, but think of k as fixed along $\operatorname{Im} k = 1/2$. Analogies between bond and stochastic volatility models for options have been frequently noted in specific models.

When you sum over *all* the paths in (5.2), you obtain $\hat{H}(k,V_1,\tau) \le \hat{H}(k,V_2,\tau)$, if $V_1 \ge V_2$ and $c \ge 0$. This implies that $\hat{H}_V \le 0$. More generally, the expectation in (5.6) sometimes exists when $c < 0$ also and, in that case, the argument implies $\hat{H}_V \ge 0$. We can summarize the two cases by the following theorem:

THEOREM 2.4 *Let c be a real number, independent of τ and V, and let $H(c,V,\tau)$ be the solution (5.2), when it exists,, to the PDE problem*

$$(5.3) \qquad \frac{\partial H}{\partial \tau} = \tfrac{1}{2} a^2(V) \frac{\partial^2 H}{\partial V^2} + b(V) \frac{\partial H}{\partial V} - cVH \qquad \text{on } 0 < V < \infty,$$

where $H(c,V,\tau=0) = 1$. Then $H_V(c,V,\tau)$ has the opposite sign of c.

Remark. Theorem 2.4 proves to be very important when we take up the equilibrium theory in Chapter 7. That theory is used to translate the actual volatility process into a pricing process. It turns out that for the equilibrium theory, (5.3) describes the evolution of a function that determines the volatility risk premium in a representative agent model. In that case, the parameter $c = -\gamma(1-\gamma)/2$, where γ is the risk-aversion parameter and Theorem 2.4 fixes the *sign* of the risk premium.

A useful upper bound. Standard arbitrage arguments imply that call option prices, under the process (1.1), must lie in the range

$$\max\left[0, Se^{-\delta\tau} - Ke^{-r\tau}\right] \le C(S,V,\tau) \le Se^{-\delta\tau}.$$

See Appendix 2.3. These bounds are very important theoretically, but useless for any trading applications because they are too wide. In contrast, here's a much stronger bound that can actually be used for "quick and dirty" pricing in the symmetric ($\rho = 0$) case. Define the random variable

$$U = U(V,\tau) = \int_0^\tau Y_t^V \, dt,$$

which is called the integrated volatility. Then, another way of expressing (5.2) is

$$\hat{H}(k,V,\tau) = \mathbb{E}_0\left[e^{-c(k)U}\right] = \int_0^\infty e^{-c(k)U} P(U) dU,$$

where $P(U) = P(U,V,\tau)$ is the probability density of U. Then, our useful bound is given by the following theorem:

THEOREM 2.5 *Suppose the martingale pricing process follows* (1.1) *with* $\rho = 0$. *Let* $C(S,V,\tau)$ *be the at-the-money* ($Ke^{-r\tau} = Se^{-\delta\tau}$) *call option price and let*

$$\mu(\tau,V) = \frac{1}{\tau}\mathbb{E}_0(U) = \frac{1}{\tau}\int_0^\tau \mathbb{E}_0(V_s)\,ds\,, \qquad (V_0 \equiv V)$$

be the time average of the expected volatility. Then,

$$C(S,V,\tau) \le c(S,\mu(\tau,V),\tau)\,, \tag{5.4}$$

where $c(S,V,\tau)$ *is the Black-Scholes formula.*

PROOF: Since $e^{-c(k)U}$ is convex, we can apply Jensen's inequality[12] to find:

$$\hat{H}(k,V,\tau) = \mathbb{E}_0\left[e^{-c(k)U}\right] \ge \exp[-c(k)\mathbb{E}_0(U)] = \exp[-c(k)\mu(\tau,V)\tau]\,. \tag{5.5}$$

From (2.10) with $X = \ln[S/K] + (r-\delta)\tau = 0$, and $\operatorname{Im} k = 1/2$, we have

$$C(S,V,\tau)\big|_{X=0} = Se^{-\delta\tau}\left[1 - \frac{1}{2\pi}\int_{i/2-\infty}^{i/2+\infty} \frac{\hat{H}(k,V,\tau)}{k_r^2 + \frac{1}{4}}\,dk\right]. \tag{5.6}$$

From (5.2), the integrand is real and positive. The lower bound (5.5) for $\hat{H}$ yields an upper bound for $C(S,V,\tau)$. Substituting the bound (5.5) into (5.6) yields the formula for the *deterministic* volatility problem. The result is just the B-S formula with the volatility $\mu(\tau,V)$. (See Appendix 2.1 for the proof that the B-S formula is recovered). Hence $C(S,V,\tau) \le c(S,\mu(\tau,V),\tau)$, assuming $Ke^{-r\tau} = Se^{-\delta\tau}$. ■

Example. Consider the class of models defined by a linear drift, namely

$$dV = (\omega - \theta V)\,dt + a(V)\,dW(t)\,.$$

Then, it's easy to show (see Chapter 5, Appendix 5.1) that

$$\mu(\tau,V) = \frac{\omega}{\theta} + \left(V - \frac{\omega}{\theta}\right)\left(\frac{1-e^{-\theta\tau}}{\theta\tau}\right)$$

regardless of the diffusion coefficient $a(V)$ [subject to some growth restraints.]

---

[12] See, for example Ingersoll (1987), p16.

Remark. The implied volatility $V^{imp}(X,V,\tau)$, is defined to be the solution to the relation $C(S,V,\tau) = c(S,V^{imp},\tau)$. [Again $X = \ln[S/K]+(r-\delta)\tau$]. Hence (5.4) is equivalent to the bound:

$$V^{imp}(X=0,V,\tau) \leq \mu(\tau,V) \quad (\rho = 0) \tag{5.7}$$

It's important to realize that the bounds (5.4) and (5.7) are *not* valid when $\rho \neq 0$. Computational examples showing $V^{imp} > \mu$, when $\rho \neq 0$, may be found in Chapter 6.

Appendix 2.1
Recovery of the Black and Scholes Solution

In this Appendix, we show that (2.10) and (2.8) each generate the B-S solution when the fundamental transform has the form $\hat{H}(k,V,\tau) = \exp[-c(k)U(V,\tau)]$, where $c(k) = (k^2 - ik)/2$. In the case of constant volatility, $U(V,\tau) = V\tau$. In the case of deterministic volatility $U(V,\tau) = \int_0^\tau Y(u,V)\,du$, where $Y(u,V)$ is the deterministic volatility evolution function.

(I) Equation (2.10) We want to show that the B-S solution is recovered and that the k-plane integral can be anywhere within the strip $0 < \operatorname{Im} k < 1$. First define the integral

(A1.1) $$f(X,Z) = \frac{1}{2\pi} \int_{ik_i-\infty}^{ik_i+\infty} \exp[-ikX - c(k)Z] \frac{dk}{k^2 - ik}, \quad 0 < \operatorname{Im} k < 1,$$

where $X = \ln(S/K) + (r-\delta)\tau$. In terms of this integral, (2.10) reads

(A1.2) $$C(S,V,\tau) = Se^{-\delta\tau} - Ke^{-r\tau} f(X, U(V,\tau)).$$

Now $f(X,Z) = I(X,Z;a=0) - I(X,Z;a=i)$, where

$$I(X,Z;a) = \frac{i}{2\pi} \int_{ik_i-\infty}^{ik_i+\infty} \exp[-ikX - c(k)Z] \frac{dk}{k-a}$$

In $I(X,Z;a=0)$, consider the substitution

$$\frac{1}{k} = -\int_{-i\infty}^{0} e^{-sk}\,ds = \frac{1}{k} - \lim_{b\to+\infty} \left.\frac{e^{-sk}}{k}\right|_{s=-ib}.$$

The exponential factor in the last term is $\exp(ibk)$, which will vanish as $b \to +\infty$ because $\operatorname{Im} k > 0$ everywhere along the k-plane contour. Similarly,

$$\frac{1}{k-i} = \int_{0}^{i\infty} e^{-s(k-i)}\,ds = \frac{1}{k-i} - \lim_{b\to+\infty} \frac{e^{-ib(k-i)}}{k-i}.$$

Now the exponential factor vanishes because $\operatorname{Im} k < 1$ everywhere along the k-plane contour. If you make these substitutions, then the k-plane integrals become Gaussians and these are easily integrated by completing the square. This leaves s-plane integrals, which are easily recognized as cumulative normal functions $\Phi(\bullet)$ under the substitution $s = iy$, where y is real. The result is

$$I(X,Z;a=0)=\Phi(d_2) \text{ and } I(X,Z;a=i)=-e^X\Phi(-d_1),$$

where $\quad d_1(X,Z)=\frac{X}{\sqrt{Z}}+\frac{1}{2}\sqrt{Z} \quad$ and $\quad d_2(X,Z)=\frac{X}{\sqrt{Z}}-\frac{1}{2}\sqrt{Z}$.

After substitution into (A1.2) and using $\Phi(-d_1)=1-\Phi(d_1)$ and $\exp(X)=(Se^{-\delta\tau})/(Ke^{-r\tau})$, we obtain the Black and Scholes formulas:

$$C(S,V,\tau)=\begin{cases} Se^{-\delta\tau}\Phi(d_1(X,V\tau))-Ke^{-r\tau}\Phi(d_2(X,V\tau)) & \text{(volatility is constant)} \\ Se^{-\delta\tau}\Phi(d_1[X,U(V,\tau)])-Ke^{-r\tau}\Phi(d_2[X,U(V,\tau)]) & \text{(deterministic)} \end{cases}$$

(II) Equation (2.8) We can show that (2.10) implies (2.8) by a simple application of the *Residue Theorem.* This theorem concerns any integral in the complex k-plane, in a counter-clockwise fashion, around a closed contour $\mathcal{C}$. It states that, if $f(k)$ is analytic and single-valued (hence, regular) everywhere inside a contour $\mathcal{C}$, except at a number of isolated singularities, then

RESIDUE THEOREM:

(A1.3) $$\oint_{\mathcal{C}} f(k)\,dk = 2\pi i \sum \text{Residues inside } \mathcal{C}$$

If the integrand $f(k)$ behaves as $r_n/(k-a_n)$ near the points a_n, then the numbers a_n are called the (simple) poles and the numbers r_n are called the residues. (See, for example, any introductory text on complex variables).

In our application, take the integral to be (A1.1) but extend the integration to a closed contour $\mathcal{C}$ which is a rectangle with sides as infinity. The bottom side of the rectangle is the path for the original integral (A1.1). The top side of the rectangle has $\operatorname{Im} k>1$. There is no contribution from the sides at infinity since the integrand vanishes rapidly as $\operatorname{Re} k \to \pm\infty$. The contour encloses a simple pole at $k=i$ and the residue at that pole, using $c(k=i)=0$, is $(-i/2\pi)\exp(X)$. So $2\pi i$ times the residue is $\exp(X)=(Se^{-\delta\tau})/(Ke^{-r\tau})$, which knocks out the first term in (2.10) and leaves (2.8). ■

An important special case of the residue theorem is where $f(k)$ is regular entirely within the contour $\mathcal{C}$. In that case, the residue theorem becomes

CAUCHY'S THEOREM (1825)

(A1.4) $$\oint_{\mathcal{C}} f(k)\,dk = 0.$$

This result is equivalent to the statement that the integral between any two points is independent of the path linking those points, provided that all paths to be considered do not leave the region of regularity of $f(k)$. We need Cauchy's theorem in Chapter 6 in developing the asymptotic term structure theory. The Residue Theorem also plays an important role when option prices are not martingales.

Appendix 2.2 Mathematica Code for Chapter 2

This Appendix shows how to evaluate the option solutions, such as (2.10) in Mathematica. For example, for the square root model, the fundamental transform is given by (3.2), which is directly coded:

```
(* The fundamental transform H for the sqrt model *)
(* Expects gam (1-gam) ksi^2 <= theta^2 *)

H[k_,V0_,tau_,omega_,theta_,ksi_,rho_,gam_]:=
  Module[{Hval,t,a,b,c,d,f,h, thetaadj,A,B},
      If[ksi == 0,Return[E^(-(k^2-I k) V0 tau/2)],Null];
      t = ksi^2 tau/2;
      a = 2 omega/ksi^2;
      If[gam == 1, thetaadj = theta,
           thetaadj = (1-gam)rho ksi +
                    Sqrt[theta^2 - gam(1-gam)ksi^2]];
      b = 2 (thetaadj + I k rho ksi)/ksi^2;
      c = (k^2 - I k)/ksi^2;
      d = Sqrt[b^2 + 4 c];
      f = (b+d)/2;
      h = (b+d)/(b-d);
      A = f a t - a  Log[(1-h E^(d t))/(1-h)];
      B = f (1-E^(d t))/(1 - h E^(d t));
      Hval = E^(A + B V0);
      Return[Hval]]
```

Then, call option values are a pretty direct translation of (2.10). Some arguments are explained below:

```
(* Call option values-Sqrt model *)
(* Needs 0 < ki < 1 *)
(* Utility: Pure Investment-Distant Horizon *)

Cvalue[S_,K_,r_,y_,V0_,tau_,
   ki_,omega_,theta_,ksi_,rho_,gam_,pflag_,rflag_]:=

Module[{X,kc,bsval,val,impsigma},
 Clear[kc]; kc[kr_]:= ki I + kr;
 If[pflag == 1,Print["S=",S," K=",K," r=",r," y=",y,"
    V0=",V0," tau=",tau];
 Print["ki=",ki," omega=",omega," theta=",theta,"
  ksi=",ksi," rho=",rho," gam=",gam],Null];

If[ki <= 0 || ki >= 1,
  Print["Illegal ki value"];Abort[],Null];
If[gam (1 - gam) ksi^2 > theta^2,
  Print["Illegal Risk parameter"];Abort[],Null];

kmax = Max[1000,10/Sqrt[V0 tau]];
X = Log[S /K] + (r-y)tau;
bsval = N[cbs[S,K,r,y,Sqrt[V0],tau]];

val = N[ S E^(-y tau) - K E^(-r tau)/Pi *
 NIntegrate[ Re[E^(-X I kc[kr])/(kc[kr]^2-I kc[kr])*
  H[kc[kr],V0,tau,omega,theta,ksi,rho,gam]],
    {kr,0,10,kmax},MaxRecursion->20]];

impsigma = impsig[S,K,r,y,val,tau] * 100;
If[pflag == 1,
   Print["kmax=",kmax," Call price=",val,
   " Black-Scholes price=",bsval,
   " Implied sigma=",impsigma],Null];

If[rflag == 1,Return[val],Return[impsigma]]]
```

Comments. Most of the arguments in **`Cvalue`** have an obvious meaning from (2.10) or (3.2). The dividend yield δ in (2.10) has become y. The argument **`rflag`** determines what is returned: if **`rflag = 1,`** then the call price is returned; otherwise the B-S implied sigma (in percent) is returned. The utility routines **`cbs`** (Black-Scholes call price) and **`impsigma`** (B-S implied sigma) are straightforward and not given. The routine will fail if you take $V_0 = 0$ or $\tau = 0$, but $\xi = 0$ (ksi = 0) is fine.

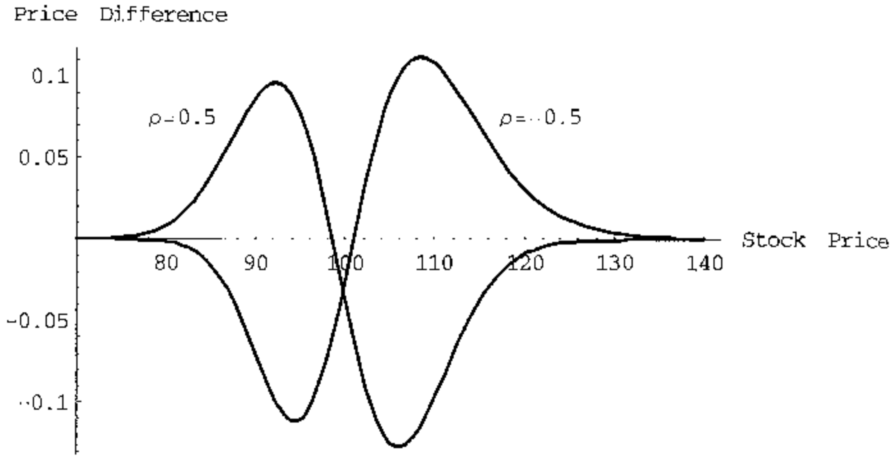
Note that the **`NIntegrate`** function is called with a purely real argument and real integration range. Although Mathematica can integrate in the complex plane, a short test with the integration range set to **`{k,I/2,I/2+kmax}`** ran 10 times slower. The integration range is $0 \leq \mathrm{Re}\, k < k_{\max}$. The value for **`kmax`** is usually large enough, but a sensitivity check should be performed.

Run times are usually about a second:

```
In[85]:= Timing[Cvalue[90, 100, 0, 0, .01, .5,
                        1/2, .02, 2, .1, -.5, 0, 1, 1]]

        S=90 K=100 r=0 y=0 V0=0.01 tau=0.5

        ki=1/2 omega=0.02 theta=2 ksi=0.1 rho=-0.5 gam=0

        kmax=1000 Call price=0.130698 Black-Scholes price=0.20102
          Implied sigma=9.11299

Out[85]= {1.21 Second, 0.130698}
```

Here's a **`Plot`** of the difference between the call option price and the B-S price for the parameters shown just above, but $\rho = \pm 0.5$ It's similar to Figure 2 in Heston (1993); the difference is that Heston takes $\gamma = 1$ (risk neutrality) and we take $\gamma = 0$ (logarithmic utility) for the risk-adjustment. Each of the two plots took about a minute to run:

Fig A2.1 Price Differences between the Square Root Stochastic Volatility Model and the Black-Scholes Model (K=100)



The figure shows a typical pattern under stochastic volatility. With negative stock-volatility correlation, in-the-money calls have prices higher than B-S prices and out-of-the money calls have lower prices. With positive correlation, the pattern is reversed.

Appendix 2.3
General Properties of Option Prices

A few authors have explored general properties of option prices under stochastic volatility models. In this Appendix, we discuss bounds on option prices and signs of derivatives of option prices that must hold under our general form stochastic volatility model at (1.1).

The results that we report are due to Bergman, Grundy, and Wiener (BGW 1996) and Romano and Touzi (RT 1997). BGW explored the general properties of option prices with one and two dimensional martingale pricing processes. RT consider 2D processes and proved similar results and some additional results using the mixing theorem idea, which is explained in Chapter 4. Both groups make additional assumptions on the coefficients.[1] In the theorem, we combine results from BGW (their Theorems 1-2) and RT (their Theorem 3.1).

THEOREM 2.6 *Let* $F(S,V,t)$ *be a claim price for the payoff function* $g(S)$, *in a world where the pricing process follows* (1.1). *We assume that the claim price, relative to the bond price, is a martingale, and the Feynman-Kac style PDE solutions used below exist and are unique. Then,*

(i) $$e^{-\delta\tau} \min_S g'(S) \le F_S \le e^{-\delta\tau} \max_S g'(S);$$

(ii) *If* $g(S)$ *is convex (concave), then* $F(S,V,t)$ *is convex (concave) as a function of S;*

(iii) *If* $F(S,V,t)$ *is convex (concave) as a function of* S, *then* $F_V(S,V,t) > 0$ ($F_V(S,V,t) \le 0$). *In other words,* F_V *and* F_{SS} *have the same sign.*

[1] RT require that the volatility remains in a range bounded away from zero and infinity and the volatility process coefficients are bounded. Technically, this rules out many of our example models, but it enforces the martingale property. BGW assume that a Feynman-Kac solution is the fair value; we know this assumption also fails when claim prices are not martingales.

PROOF: Each property will be established by differentiating the valuation PDE (1.2) and then invoking a Feynman-Kac representation for the solution of the new PDEs that result (see Appendix 2, Chapter 1).

For (i), differentiate (1.2) with respect to S. Letting $G = F_S$, the resulting PDE is

$$-\frac{\partial G}{\partial t} = -\delta G + \mathcal{A}_1 G\,,$$

where
$$\mathcal{A}_1 G = (r - \delta + V)S\,\frac{\partial G}{\partial S} + \tfrac{1}{2} V S^2 \frac{\partial^2 G}{\partial S^2}$$
$$+\left[\bar{b}(V) + \rho(V)a(V)V^{1/2}\right]\frac{\partial G}{\partial V} + \tfrac{1}{2} a^2(V)\frac{\partial^2 G}{\partial V^2} + \rho(V)a(V)V^{1/2}S\,\frac{\partial^2 G}{\partial S \partial V}\,.$$

By assumption, this equation has the unique solution representation

$$G(S_t, V_t, t) = e^{-\delta(T-t)}\mathbb{E}_t^{(1)}[G_T] = e^{-\delta(T-t)}\mathbb{E}_t^{(1)}[g'(S_T)]\,,$$

where the expectation is taken with respect to the stochastic process whose generator is $\mathcal{A}_1$. Since $\min_S g'(S) \le \mathbb{E}_t[g'(S_T)] \le \max_S g'(S)$, then property (i) follows.

For (ii), differentiate (1.2) twice with respect to S. Letting $H = F_{SS}$, the resulting PDE is

$$-\frac{\partial H}{\partial t} = (r - 2\delta + V)H + \mathcal{A}_2 H\,,$$

where
$$\mathcal{A}_2 H = (r - \delta + 2V)\,S\,\frac{\partial H}{\partial S} + \tfrac{1}{2} V S^2 \frac{\partial^2 H}{\partial S^2}$$
$$+\left[\bar{b}(V) + 2\rho(V)\,a(V)V^{1/2}\right]\frac{\partial H}{\partial V} + \tfrac{1}{2} a^2(V)\frac{\partial^2 H}{\partial V^2} + \rho(V)a(V)V^{1/2}S\,\frac{\partial^2 H}{\partial S \partial V}\,.$$

From Chapter 1, Appendix 2, this equation has the solution

$$H(S_t, V_t, t) = e^{-(2\delta - r)(T-t)}\mathbb{E}_t^{(2)}\left[\exp\left(\int_t^T Y_u\,du\right)H_T\right]$$

$$= e^{-(2\delta - r)(T-t)}\mathbb{E}_t^{(2)}\left[\exp\left(\int_t^T Y_u\,du\right)g''(S_T)\right].$$

In the last equation, $\mathbb{E}_t^{(2)}$ is the expectation with respect to the 2D process generated by $\mathcal{A}_2$, whose volatility component we have labeled Y_u. The volatility component follows, for $t \le u \le T$, the SDE

$$dY_u = \left[\bar{b}(Y_u) + 2\rho(Y_u)\,a(Y_u)Y_u^{1/2}\right]du + a(Y_u)dW_u \quad \text{and} \quad Y_t = V_t\,.$$

So we see that if $g''(S)$ exists and has a fixed sign, then F_{SS} inherits the same sign. This establishes property (ii).

Finally, for (iii), differentiate (1.2) with respect to V. Letting $U = F_V$, the resulting PDE is

$$\frac{\partial U}{\partial t} - \left[r + \tilde{b}'(V) \right] U + \mathcal{A}_3 U = -\tfrac{1}{2} S^2 \frac{\partial^2 F}{\partial^2 S},$$

where

$$\mathcal{A}_3 U = \left[(r-\delta) + \frac{d}{dV}\left(\rho(V) a(V) V^{1/2} \right) \right] S \frac{\partial U}{\partial S} + \tfrac{1}{2} V S^2 \frac{\partial^2 U}{\partial S^2}$$
$$+ \left[\tilde{b}(V) + a(V) a'(V) \right] \frac{\partial U}{\partial V} + \tfrac{1}{2} a^2(V) \frac{\partial^2 U}{\partial V^2} + \rho(V) a(V) V^{1/2} S \frac{\partial^2 U}{\partial S \partial V}.$$

Since we are only considering volatility-independent payoffs, then $U_T = 0$, and the above equation has the Feynman-Kac solution from (1.A2.8). That solution is

$$U(S_t, V_t, t) = e^{-r(T-t)} \mathbb{E}_t^{(3)} \left[\int_t^T h(X_u, Y_u, u) \exp\left(-\int_t^u c(Y_\lambda) d\lambda \right) du \right],$$

where

$$h(S,V,t) = \tfrac{1}{2} S^2 \frac{\partial^2}{\partial S^2} F(S,V,t) \quad \text{and} \quad c(V) = \tilde{b}'(V).$$

The 2D process generated by $\mathcal{A}_3$ has a stock price component labeled X_u and a volatility component labeled Y_u on $t \le u \le T$, where $X_t = S_t$ and $Y_t = V_t$. This solution establishes that F_V and F_{SS} have the same sign. ■

Discussion. These results must be applied carefully to our models when there are volatility explosions, because this causes some claim prices to fail to be martingales. In Chapter 9, we show that if the auxiliary volatility process can explode with probability $\hat{P}_{\exp}(V,\tau)$, then the call option price is not a martingale. We also propose the following generalized fair value formula:

$$C(S,V,\tau) = e^{-r\tau} \mathbb{E}_t\left[(S_T - K)^+ \right] + S e^{-\delta\tau} \hat{P}_{\exp}(V,\tau).$$

However, put option prices are generally martingales, even in the face of volatility explosions, because put option prices are bounded local martingales, hence martingales. In addition, even when call option prices are not martingales, they satisfy put-call parity.

So, the safe way to apply Theorem 2.6 is to apply it to a put option and then invoke put-call parity. For a put option, with the payoff $g(S) = (K - S)^+$, the theorem implies (i) $-e^{-\delta\tau} \le P_S \le 0$, (ii) $P_{SS} \ge 0$, and (iii) $P_V \ge 0$.

Then, combined with put-call parity $C = P + Se^{-\delta\tau} - Ke^{-r\tau}$, the theorem implies that (i) $0 \le C_S \le e^{-\delta\tau}$, (ii) $C_{SS} \ge 0$, and (iii) $C_V \ge 0$, even when the call price is not a martingale.

3 The Volatility of Volatility Series Expansion

This chapter introduces a powerful solution technique for the option problem: the volatility of volatility series expansion, or the $\xi -$ *expansion* for brevity. In this chapter, we discuss the general steps for the expansion, show some results, and give the expansion formulas through $O(\xi^2)$ for a particular model. There are really two expansions, one for the price and one for the implied volatility, which we call Series I and II respectively. Detailed calculations are found in the Appendix to this chapter.

1 Assumptions

As usual, we assume that the volatility process is time-homogeneous. The volatility of volatility is the constant parameter ξ when the actual volatility process is parameterized as follows:

$$P: \quad dV_t = b(V_t)dt + \xi\,\eta(V_t)dW_t\,.$$

As we have seen from Chapter 1, options are priced under a risk-adjusted process $\tilde{P}$. With this parameterization, the risk-adjusted volatility process is given by

$$\tilde{P}: \quad dV_t = \left[b(V_t) - \lambda^{v} \xi\,\eta(V_t) \right] dt + \xi\,\eta(V_t) d\tilde{W}_t\,, \tag{1.1}$$

where λ^v is the market price of volatility risk. In this chapter, we show how to develop option pricing formulas as a power series in ξ (the $\xi-$ *expansion*)[1].

The risk-adjusted volatility drift has a $\xi-$ dependence not only through the explicit appearance of ξ, but through $\lambda^v = \lambda^v(V_t, \xi)$ as well. But in this chapter, for simplicity, we shall assume that $\lambda^v = 0$. This will allow us to introduce the expansion method in the simplest case. Then, in Chapter 7 on the equilibrium theory, we show the modifications to this chapter's results for the general $\lambda^v \neq 0$ case. We also assume that the interest rate and dividend yield are constants, and the stock and volatility processes have correlation $\rho(V_t)$. We collect our assumptions together into

Assumption 1. *The risk-adjusted pricing process is given by*

$$(1.2) \qquad \tilde{P}: \begin{cases} dS = (r-\delta)S dt + \sigma S\, d\tilde{B} \\ dV = \tilde{b}(V)dt + \xi\,\eta(V)d\tilde{W} \end{cases},$$

where $d\tilde{B}$ and $d\tilde{W}$ are correlated Brownian motions with correlation $\rho(V)$, r, δ, and ξ *are constants, and $\tilde{b}(V)$ and $\eta(V)$ are independent of ξ.*

2 General Steps in the $\xi-$ expansion

The expansion begins with the fundamental transform solution for the option price. For a call option, and taking the k-plane integration along $\operatorname{Im} k = 1/2$ for simplicity, this solution representation is

$$(2.1) \qquad C(S,V,\tau) = Se^{-\delta\tau} - \frac{Ke^{-r\tau}}{2\pi} \int_{i/2-\infty}^{i/2+\infty} \exp(-ikX) \frac{H(k,V,\tau)}{k^2 - ik} dk,$$

where $X = \ln(S/K) + (r-\delta)\tau$. Under Assumption 1, then $H(k,V,\tau)$ is the solution with $H(k,V,\tau=0) = 1$ to the PDE from (**2**.2.19):

$$(2.2) \qquad \frac{\partial H}{\partial \tau} = \tfrac{1}{2}\xi^2\eta^2(V)\frac{\partial^2 H}{\partial V^2} + \left[\tilde{b}(V) + \xi d(k)\rho(V)\eta(V)V^{1/2}\right]\frac{\partial H}{\partial V} - c(k)VH$$

---

[1] Other approaches to treating the volatility process as a perturbation include Fournie, Lebuchoux and Touzi (1997) and Sircar and Papanicolaou (1999).

using $c(k)=(k^2-ik)/2$ and $d(k)=-ik$. Conceptually, the series is straightforward. The general steps are as follows:

General steps in the $\xi-$expansion:

- Look for solutions to (2.2) of the form

$$H(k,V,\tau)=H^{(0)}(k,V,\tau)+\xi H^{(1)}(k,V,\tau)+\xi^2 H^{(2)}(k,V,\tau)+\cdots.$$

- Calculate the call option price from (2.1). As it turns out, the k-plane integrals *can all be done analytically* because they reduce to derivatives of the B-S formula. Doing those integrals yields a result of the form

Series I:

$$C(S,V,\tau)=C^{(0)}(S,V,\tau)+\xi C^{(1)}(S,V,\tau)+\xi^2 C^{(2)}(S,V,\tau)+\cdots.$$

Then calculate the implied volatility from $C(S,V,\tau)=c(S,V^{imp},\tau)$, where $c(S,V,\tau)$ is the B-S formula.

- *Alternatively,* one can expand

Series II: $$V_{imp}=V_{imp}^{(0)}+\xi V_{imp}^{(1)}+\xi^2 V_{imp}^{(2)}+\cdots$$

Then calculate the option price from $C(S,V,\tau)=c(S,V^{imp},\tau)$.

Two series methods. As one sees from the last step, there are really two different ways to calculate at a given $\xi-$order, which we call Series I and Series II. For Series I, you expand the option price in a series and then use $C(S,V,\tau)=c(S,V^{imp},\tau)$ to obtain the implied volatility. For Series II, you expand V^{imp} in a series and then use $C(S,V,\tau)=c(S,V^{imp},\tau)$ to obtain the option price. The results from the two methods are often very close and they use almost the same formulas. The two series differ the most from each other for relatively far out-of-the money options. (We show some examples just below). Numerical comparisons with exact results suggest that, when the results differ, Series II is usually (but not always) more accurate. When Series I is more accurate, the practical differences between the results are negligible. So our recommendation is to use Series II, since it's really no more work.

Some examples. Before presenting the formulas for the expansion, we show some results. The examples illustrate both the general accuracy of the two series, and the somewhat greater accuracy of Series II.

Example 1. The square-root model. The advantage of the square-root model is that it can be solved exactly and evaluated rapidly. Specifically, we take the risk-adjusted process to be

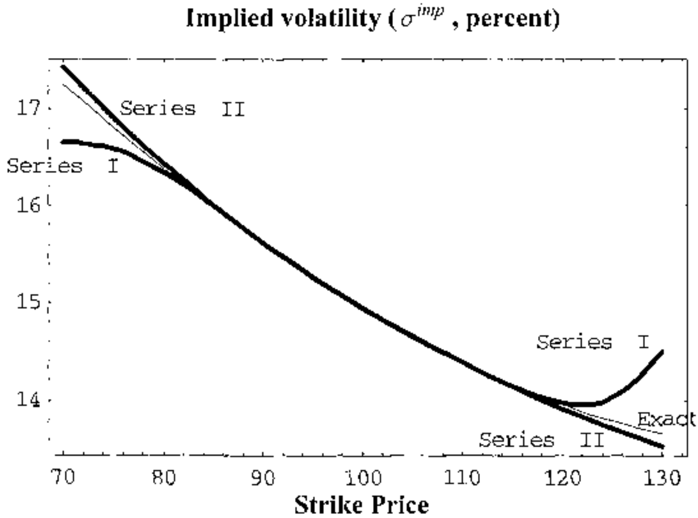
$$\tilde{P}: \quad \begin{cases} dS = (r - \delta)S\,dt + \sigma S\, d\tilde{B} \\ dV = (\omega - \theta V)\,dt + \xi\sqrt{V}\, d\tilde{W} \end{cases}, \tag{2.3}$$

where $d\tilde{B}$ and $d\tilde{W}$ have constant correlation ρ. Specifically, we take the case of a 3-month option, with an initial volatility of 15% $(V_0 = 0.0225)$. The long-run expected volatility is also 15% because we take $\omega_a = 0.09$, $\theta_a = 4$ in annualized units. The long-run expected volatility is given by $\langle V \rangle = \omega / \theta$. Finally, we take $r = \delta = 0$, $\xi = 0.1$ and $\rho = -0.5$.

Fig. 3.1 plots the annualized implied volatility σ^{imp}, in percent, for the three cases: (i) the exact results, (ii) the Series I expansion, (iii) the Series II expansion. The plot shows σ^{imp} vs. the strike price K when $S = 100$; this is called a smile or volatility skew plot. The figure shows the implied volatility for the two series at $O(\xi^2)$. (We discuss the smile in greater detail in Chapters 5 and 6) If volatility were constant and so the B-S model was valid, the smile would be a horizontal line at $\sigma^{imp} = 15\%$, which is the initial volatility. Instead, one sees in the figure a typical downward sloping graph which is characteristic of the negative correlation case.

As one can see from the figure, the two series and the exact results are essentially identical, close-to-the-money. But, the Series II results are more accurate when the strike price is relatively far from the money. The Series I smile plots tend to have an artificially exaggerated curvature. Numerical results corresponding to Fig 3.1 are given in Table 3.1. From the table, note that where the two series differ the most, the option price is very close to zero or parity. Very small price differences in these regimes can produce relatively large differences in the implied volatility. Since options this far from the money are dominated by transaction costs (the bid-ask spread), it may not make much practical difference which series is employed. Nevertheless, as we indicated above, it's no more effort to use Series II.

Fig 3.1 Implied Volatility vs. Strike Price:
Exact and Volatility of Volatility Series Expansion Results



Notes. Exact and two series results for the square-root model. The series results are through $O(\xi^2)$. The series results are drawn as bold lines and the exact result is drawn as the thin line. The plotted data is given in Table 3.1. The figure shows that Series II is typically more accurate.

Example 2. The GARCH diffusion. The GARCH diffusion is the continuous-time limit of many discrete-time GARCH models. Specifically, we take the risk-adjusted process to be

(2.4) $$\tilde{P}: \begin{cases} dS = (r-\delta)Sdt + \sigma S\,d\tilde{B} \\ dV = (\omega - \theta V)\,dt + \xi V d\tilde{W} \end{cases},$$

where the correlation is ρ. Fig 3.2 plots the smile for various correlations in the case where all the drift parameters (both stock and volatility) are zero, the time to expiration is 0.5 years, $\xi_a = 1$, and the initial volatility is 10% annualized $(V_0 = 0.01)$.

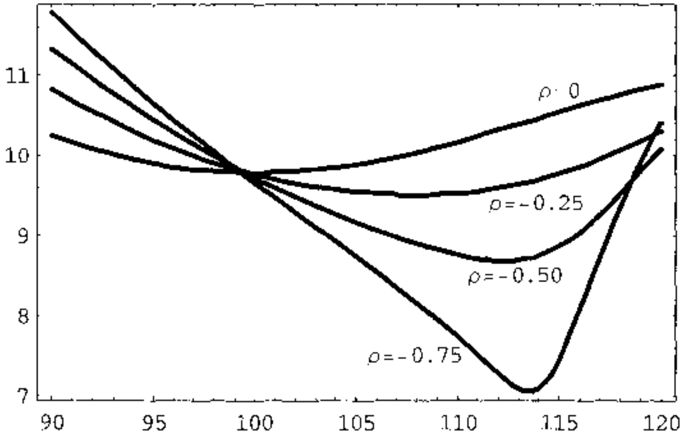
Table 3.1 Option Prices and Implied Volatility: Exact versus Volatility of Volatility Series Expansions

	Strike Price						
Model	**70**	**80**	**90**	**100**	**110**	**120**	**130**
Exact:	30.000028	20.00718	10.30318	2.98087	0.32593	0.01077	0.000110
	17.25	16.39	15.62	14.95	14.39	13.96	13.65
Series I:	30.000014	20.00702	10.30366	2.98086	0.32533	0.01087	0.000299
	16.65	16.35	15.62	14.95	14.38	13.98	14.50
Series II:	30.000034	20.00740	10.30362	2.98084	0.32539	0.01047	0.000094
	17.42	16.43	15.62	14.95	14.39	13.92	13.53
Black-	30.000001	20.00280	10.25802	2.99137	0.38076	0.02034	0.000503
Scholes:	15.00	15.00	15.00	15.00	15.00	15.00	15.00

Notes. The table shows exact and series results for call option prices under the square-root model: $dS = (r - \delta)S dt + \sigma S\, dB(t)$, $dV = (\omega - \theta V)\,dt + \xi\sqrt{V}\,dW(t)$, where $dB(t)$ and $dW(t)$ are Brownian motions with correlation ρ. The calculations are carried out through $O(\xi^2)$. Model parameters are $\rho = -0.5$, $V_a(0) = 0.0225$, $\omega_a = 0.09$, $\theta_a = 4$, $\xi_a = 0.10$, $\tau = 0.25$ years, $r = \delta = 0$, $S = 100$. The first entry is the price and the second entry is the implied volatility (percent, annualized). The Series I results are based on an expansion of the option price in a series through $O(\xi^2)$; then the Series I value for σ^{imp} is determined from the price in a standard manner. The Series II results are based on an expansion of V^{imp} in a series through $O(\xi^2)$; then the Series II value for the price is determined by inserting $(V^{imp})^{1/2}$ into the B-S equation. Both series are highly accurate, but the Series II results are generally better than the Series I results at larger values of the moneyness $X = \ln(S/K)$.

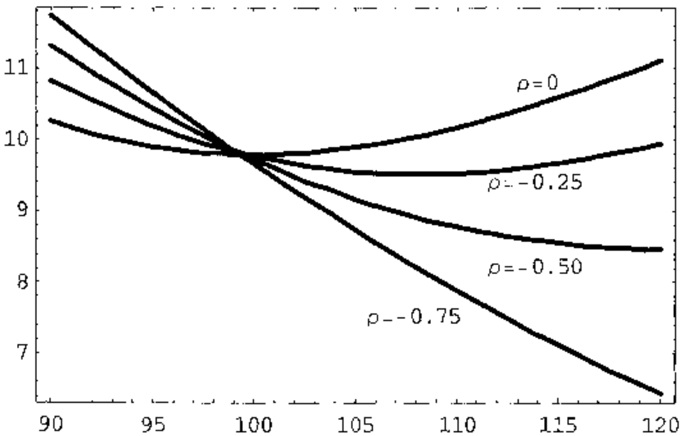
Fig 3.2 Implied Volatility vs. Strike Price:

Series I **Implied volatility (σ^{imp}, percent)**



Strike Price

Series II **Implied volatility (σ^{imp}, percent)**



Strike Price

Notes for Fig. 3.2 (previous page). Series I and Series II smiles plots for the GARCH diffusion process $dV = (\omega - \theta V)\,dt + \xi V dW(t)$. Series I is a price expansion and Series II is an implied volatility expansion. The calculations are carried out through $O(\xi^2)$. The parameters are $V_0 = .01$, $\xi_a = 1$, $\tau = 0.5$ years, $r = \delta = \omega = \theta = 0$. The greatest differences are at larger correlations and relatively far out of the money. Comparisons with exactly solvable models suggests that the Series II plots are the more accurate.

The figure shows the implied volatility for the two series at $O(\xi^2)$. As one can see, the greatest differences are at relatively large correlations and far from the money. Again, the option prices are very small in the region of the graph where the two series differ the most[2].

3 The Two Series for a Parameterized Model

Both of the numerical examples in the previous section are special cases of the following parameterized model for the risk-adjusted pricing process:

$$\text{(3.1)} \qquad \tilde{P}: \begin{cases} dS = (r-\delta)S\,dt + \sigma S\, d\tilde{B} \\ dV = (\omega - \theta V)\,dt + \xi V^{\varphi} d\tilde{W} \end{cases},$$

where $d\tilde{B}$ and $d\tilde{W}$ are correlated Brownian motions with correlation ρ. At time-t, an option expiring at time-T has a price which depends upon (i) the usual variables: S, K, V and $\tau = T - t$, and (ii) the parameters of the model: $r, \delta, \omega, \theta, \xi, \rho, \varphi$, which are all taken to be constants. In this section, we give the Series I and Series II formulas for the European-style call option through $O(\xi^2)$ for this particular model. The formulas are easily implemented in Mathematica or any computer language and can be evaluated very quickly. The put option price is obtained from put/call parity.

We use the B-S formula $c(S,V,t)$ and its first derivative with respect to V. That is, using $\tau = T - t$, recall that

---

[2] With no drift, this model can also be solved exactly. But the results cannot be evaluated rapidly and I didn't have the patience to obtain the exact results, which would require many hours in Mathematica. However, the exact formulas are given in Chapter 11.

$$c(S,V,t) = S e^{-\delta\tau}\Phi(d_+) - Ke^{-r\tau}\Phi(d_-)\,,$$

where $$d_\pm = \frac{1}{\sqrt{V\tau}}\left[X \pm \tfrac{1}{2}V\tau\right], \qquad X = \ln\left[\frac{Se^{-\delta\tau}}{Ke^{-r\tau}}\right],$$

and $$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-z^2/2}dz\,.$$

Then, the derivative we need is

$$c_V(S,V,\tau) = \frac{\partial}{\partial V}c(S,V,\tau) = \sqrt{\frac{\tau}{8\pi V}}\, Se^{-\delta\tau}\exp\left(-\tfrac{1}{2}d_+^2\right). \tag{3.2}$$

When we use these functions, they are not evaluated at the initial volatility V. Instead, V is replaced by $v = v(V,\tau)$, which is the time-average of the deterministic volatility and is given in this model by

$$v(V,\tau) = \frac{\omega}{\theta} + \left(V - \frac{\omega}{\theta}\right)\left(\frac{1-e^{-\theta\tau}}{\theta\tau}\right) \tag{3.3}$$

We use 4 integrals, labeled $J^{(i)} = J^{(i)}(V,\tau)$, $i = 1,2,3,4$. We also use certain ratios of derivatives of the B-S formula, which we write as $\tilde{R}^{(p,q)}$. In terms of these expressions, which are explained below, the two series are given by

Series I:

$$C(S,V,\tau) = c(S,v,\tau) + \xi\,\tau^{-1}J^{(1)}\,\tilde{R}^{(1,1)}c_V(S,v,\tau) \tag{3.4}$$
$$+\xi^2\left\{\tau^{-1}J^{(2)} + \tau^{-2}J^{(3)}\tilde{R}^{(2,0)} + \tau^{-1}J^{(4)}\tilde{R}^{(1,2)} + \frac{\tau^{-2}}{2}(J^{(1)})^2\tilde{R}^{(2,2)}\right\}c_V(S,v,\tau) + O(\xi^3$$

Series II:

$$V^{imp} = v(V,\tau) + \xi\,\tau^{-1}J^{(1)}\tilde{R}^{(1,1)} \tag{3.5}$$
$$+\xi^2\left\{\tau^{-1}J^{(2)} + \tau^{-2}J^{(3)}\tilde{R}^{(2,0)} + \tau^{-1}J^{(4)}\tilde{R}^{(1,2)}\right.$$
$$\left. + \tfrac{1}{2}\tau^{-2}(J^{(1)})^2\left[\tilde{R}^{(2,2)} - \left(\tilde{R}^{(1,1)}\right)^2\tilde{R}^{(2,0)}\right]\right\} + O(\xi^3)\,.$$

Note that the two series are very similar and use the same integrals and the same derivative ratios. The derivative ratios are first defined in terms of $R^{(p,q)}(X,V,\tau)$, and the tilde on $\tilde{R}$ means replace the argument V by v. The ratios needed are found in the following table, which uses $Z = V\tau$. In other words, in the series (3.4) and (3.5) use $\tilde{R}^{(p,q)} = R^{(p,q)}(X, v(V,\tau),\tau)$, where $R^{(p,q)}$ is given in Table 3.2.

Table 3.2 Normalized Derivatives of the Black-Scholes Formula

p,q	$R^{(p,q)}(X,V,\tau) = \left(\frac{\partial}{\partial V}\right)^p \left(S\frac{\partial}{\partial S}\right)^q c \,/\, \frac{\partial c}{\partial V}$
2,0	$\tau\left[\frac{1}{2}\frac{X^2}{Z^2} - \frac{1}{2Z} - \frac{1}{8}\right]$
1,1	$\left[-\frac{X}{Z} + \frac{1}{2}\right]$
1,2	$\left[\frac{X^2}{Z^2} - \frac{X}{Z} - \frac{1}{4Z}(4 - Z)\right]$
2,2	$\tau\left[\frac{1}{2}\frac{X^4}{Z^4} - \frac{1}{2}\frac{X^3}{Z^3} - 3\frac{X^2}{Z^3} + \frac{1}{8}\frac{X}{Z^2}(12 + Z) + \frac{1}{32}\frac{1}{Z^2}(48 - Z^2)\right]$

In Series II, the last expression in brackets is the source of an important cancellation. Note from Table 3.2 that $R^{(2,2)}$ contains powers of X through X^4. But the particular combination in brackets causes both the X^4 and X^3 terms to cancel, i.e.,

$$(3.6) \qquad R^{(2,2)} - \left(R^{(1,1)}\right)^2 R^{(2,0)} = \tau\left[-\frac{5}{2}\frac{X^2}{Z^3} + \frac{X}{Z^2} + \frac{1}{8}\frac{(12 - Z)}{Z^2}\right].$$

This leaves only a constant, order X and order X^2 term in V^{imp} from (3.5) through $O(\xi^2)$. I believe it's the mixing in of these higher order X^3 and X^4 terms, still present in (3.4), which causes the two series to differ at relatively large values of X.

Finally, here are the integrals needed. Actually, it turns out that $J^{(2)}$ vanishes because the drift is linear. But the series was shown with that term because for other volatility processes that term will be present. The non-vanishing integrals are

$$J^{(1)}(V,\tau)=\frac{\rho}{\theta}\int_0^\tau \left(1-e^{-\theta(\tau-s)}\right)\left[\frac{\omega}{\theta}+e^{-\theta s}\left(V-\frac{\omega}{\theta}\right)\right]^{\varphi+\frac{1}{2}} ds\,, \tag{3.7}$$

$$J^{(3)}(V,\tau)=\frac{1}{2\theta^2}\int_0^\tau \left(1-e^{-\theta(\tau-s)}\right)^2\left[\frac{\omega}{\theta}+e^{-\theta s}\left(V-\frac{\omega}{\theta}\right)\right]^{2\varphi} ds\,, \tag{3.8}$$

$$J^{(4)}(V,\tau)=\left(\varphi+\tfrac{1}{2}\right)\frac{\rho^2}{\theta}\int_0^\tau \left[\frac{\omega}{\theta}+e^{-\theta(\tau-s)}\left(V-\frac{\omega}{\theta}\right)\right]^{\varphi+\frac{1}{2}} J^{(6)}(V,\tau,s)\,ds\,, \tag{3.9}$$

using

$$J^{(6)}(V,\tau,s)=\int_0^s \left(e^{-\theta(s-u)}-e^{-\theta s}\right)\left[\frac{\omega}{\theta}+e^{-\theta(\tau-u)}\left(V-\frac{\omega}{\theta}\right)\right]^{\varphi-\frac{1}{2}} du\,. \tag{3.10}$$

Note that the integrals $J^{(i)}(V,\tau)$ are used in the series (3.4) and (3.5) exactly with the arguments as shown—there is no replacement of V by anything else.

Other volatility processes. The equations (3.4) and (3.5) for Series I and Series II are still valid for other volatility processes. The only thing that changes is the time-average volatility $v(V,\tau)$ and all the integrals. For example, as we indicated, if the volatility drift is not linear in V, then $J^{(2)}$ does not vanish. This is shown in the Appendix to this chapter where we derive the two series results for general stationary processes.

Appendix 3.1
Details of the Volatility of Volatility Expansion

In this Appendix, we develop the volatility of volatility series solution for option prices. There are two basic steps. In the first step, we develop a series for the fundamental transform $H(k,V,\tau)$. In the second step, we do the k-plane integrals that convert this series into option prices. We carry this out explicitly through $O(\xi^2)$ for a general risk-adjusted pricing process given by Assumption 1 of Chapter 3. Then we develop some integrals that appear for the parameterized model $dV_t = (\omega - \theta V_t)dt + \xi V_t^{\varphi} dW_t$, where ρ is a constant.

A caution. The development assumes that the risk-adjusted volatility drift is independent of ξ, which strictly holds only under risk-neutrality. For a more general treatment where the drift depends on ξ, which does occur in an equilibrium setting under power utility, see Chapter 7.

Step I. We want a series solution to

$$\text{(A.1)} \qquad \frac{\partial \hat{H}}{\partial \tau} = \tfrac{1}{2}\xi^2\eta^2(V)\frac{\partial^2 \hat{H}}{\partial V^2} + \left[\tilde{b}(V) + \xi d(k)\chi(V)\right]\frac{\partial \hat{H}}{\partial V} - c(k)V\hat{H},$$

where $\hat{H}(k,V,\tau=0)=1$ and using $\chi(V)=\rho(V)\eta(V)V^{1/2}$.

First, consider the solution when $\xi=0$: (A.1) becomes a first order PDE, discussed in Chapter 1, Appendix 2. This deterministic solution is built up from the *deterministic volatility* $Y(u,V)$, which is defined as the solution to $dY/du = \tilde{b}(Y)$, $Y(0)=V$. The deterministic solution, is given by

$$\text{(A.2)} \qquad \hat{H}^{(0)}(k,V,\tau) = \exp[-c(k)U(V,\tau)]$$

where $$U(V,\tau) = \int_0^{\tau} Y(u,V)\,du\,.$$

Next, consider V-derivatives of $\hat{H}^{(0)}$. The first V-derivative requires

$$\int_0^{\tau} \frac{\partial}{\partial V} Y(u,V)\,du = \int_0^{\tau} \frac{\tilde{b}(Y(u,V))}{\tilde{b}(V)}du = \frac{1}{\tilde{b}(V)}\int_0^{\tau}\frac{dY}{du}du = \frac{Y(\tau,V)-V}{\tilde{b}(V)}\,.$$

This shows that if we define a new function $\zeta(V,\tau)$ by

(A.3) $$\zeta(V,\tau) = \frac{Y(\tau,V) - V}{\tilde{b}(V)}, \quad \text{then}$$

$$\frac{\partial \hat{H}^{(0)}}{\partial V} = -c(k)\zeta(V,\tau)\hat{H}^{(0)} \quad \text{and} \quad \frac{\partial^2 \hat{H}^{(0)}}{\partial V^2} = \left[c^2(k)\zeta^2(V,\tau) - c(k)\frac{\partial \zeta}{\partial V}\right]\hat{H}^{(0)}.$$

These two relations are used below. Now we implement the expansion by looking for solutions to (A.1) of the form

(A.4) $$\hat{H}(k,V,\tau) = \hat{H}^{(0)}(k,V,\tau)\, h(k,V,\tau)$$

where $h(k,V,\tau)$ has the formal power series expansion

$$h(k,V,\tau) = 1 + \xi h^{(1)}(k,V,\tau) + \xi^2 h^{(2)}(k,V,\tau) + \cdots.$$

If you substitute (A.4) into (A.1) and use the relations above for the V-derivatives of $H^{(0)}$, it's easy to find the recursion system:

(A.5) $$\frac{\partial h^{(m)}}{\partial \tau} - \tilde{b}(V)\frac{\partial h^{(m)}}{\partial V} = f^{(m)}(V,\tau), \qquad \text{where}$$

(A.6) $$f^{(m)}(V,\tau) = \tfrac{1}{2}\eta^2(V)\left(\frac{\partial}{\partial V} - c(k)\,\zeta(V,\tau)\right)^2 h^{(m-2)}$$

$$+ d(k)\chi(V)\left(\frac{\partial}{\partial V} - c(k)\,\zeta(V,\tau)\right)h^{(m-1)}.$$

The system holds for all $m \geq 1$ with the additional conventions that $h^{(m)} = 0$, $m < 0$ and $h^{(0)} = 1$. The system is to be solved subject to the initial condition $h^{(m)}(k,V,\tau = 0) = 0$. From (A2.10) of Ch.1, Appendix 2, the solution to (A.5) is

(A.7) $$h^{(m)}(k,V,\tau) = \int_0^\tau f^{(m)}(Y(s,V), \tau - s)\,ds.$$

We now develop this last relation more explicitly for $m = 1,2$.

First order. When $m = 1$, $f^{(1)}(V,\tau) = -cd\chi(V)\zeta(V,\tau)$, so that (A.7) reads

(A.8) $$h^{(1)}(k,V,\tau) = -c(k)d(k)\,J^{(1)}(V,\tau),$$

(A.9) where $$J^{(1)}(V,\tau) = \int_0^\tau \chi(Y(s,V))\zeta(Y(s,V), \tau - s)\,ds.$$

An example. Consider the process $dV = (\omega - \theta V)\,dt + \xi V dW(t)$, where $\rho(V) = \rho$, a constant. Then some simple algebra yields

$$Y(s,V) = \frac{\omega}{\theta} + e^{-\theta s}\left(V - \frac{\omega}{\theta}\right), \qquad \chi(Y(s,V)) = \rho\left[\frac{\omega}{\theta} + e^{-\theta s}\left(V - \frac{\omega}{\theta}\right)\right]^{3/2}$$

and $$\zeta(Y(s,V),\tau-s)=\frac{1}{\theta}\left(1-e^{-\theta(\tau-s)}\right).$$

With these expressions, the integral in (A.9) is

$$J^{(1)}(V,\tau)=\frac{\rho}{\theta}\int_0^\tau\left(1-e^{-\theta(\tau-s)}\right)\left[\frac{\omega}{\theta}+e^{-\theta s}\left(V-\frac{\omega}{\theta}\right)\right]^{3/2}ds\,.$$

Mathematica can analytically integrate this, but the result is not particularly enlightening, so we will not report it. The important point is that it's a simple integral and can be quickly and easily evaluated numerically. This holds true throughout the expansion and for very general volatility processes.

Second order. Using (A.6) and subscripts for partial derivatives, we obtain

$$f^{(2)}(V,\tau)=-\tfrac{1}{2}c\,\eta^2(V)\,\zeta_V(V,\tau)+\tfrac{1}{2}c^2\eta^2(V)\,\zeta^2(V,\tau)$$
$$-cd^2\chi(V)J_V^{(1)}(V,\tau)+c^2d^2\chi(V)\zeta(V,\tau)J^{(1)}(V,\tau)\,.$$

This means that

(A.10) $$h^{(2)}(V,\tau)=-c\,J^{(2)}+c^2J^{(3)}-cd^2J^{(4)}+c^2d^2J^{(5)}\,,$$

(A.11) where $$J^{(i)}=J^{(i)}(V,\tau)=\int_0^\tau K^{(i)}(Y(s,V),\tau-s)\,ds\,.$$

Table 3A.1 Integrals needed for the $\xi-$ expansion through 2nd order

Integral	Integrand $K(V,\tau)$	Small-τ behavior of integral
$J^{(1)}(V,\tau)$	$\chi(V)\zeta(V,\tau)$	$\frac{1}{2}\chi\tau^2+\frac{1}{6}\left(\tilde{b}\chi\right)'\tau^3+O(\tau^4)$
$J^{(2)}(V,\tau)$	$\frac{1}{2}\eta^2(V)\zeta_V(V,\tau)$	$\frac{1}{12}\eta^2\tilde{b}''\tau^3+O(\tau^4)$
$J^{(3)}(V,\tau)$	$\frac{1}{2}\eta^2(V)\zeta^2(V,\tau)$	$\frac{1}{6}\eta^2\tau^3+\left(\frac{1}{8}\eta^2\tilde{b}'+\frac{1}{12}\eta'\eta\tilde{b}\right)\tau^4+O(\tau^5)$
$J^{(4)}(V,\tau)$	$\chi(V)J_V^{(1)}(V,\tau)$	$\frac{1}{6}\chi'\chi\tau^3+O(\tau^4)$
$J^{(5)}(V,\tau)$	$\chi(V)\zeta(V,\tau)J^{(1)}(V,\tau)$	Integral is exactly $\frac{1}{2}\left(J^{(1)}(V,\tau)\right)^2$

These various integrals that appear through second order are summarized in the above Table 3A.1, with some discussion below.

Small-τ behavior of the integrals. First, we want to emphasize again that the $J^{(i)}$ integrals are easily evaluated numerically for any τ. The purpose of considering the small-τ limit is to provide a more explicit picture of the various integrals, which may seem rather abstractly defined. In fact, in the special case where the volatility drift vanishes, the integrals are given *exactly* by the leading small-τ term shown in the table. The small-τ expressions shown in Table 3A.1 are obtained by developing a power series solution to $dY/d\tau = \tilde{b}(Y)$, $Y(0) = V$. Letting $Y(\tau,V) = V + y_1\tau + y_2\tau^2 + \cdots$, the coefficients are easily developed and the first few are:

$$y_1 = \tilde{b}(V), \quad y_2 = \tfrac{1}{2}\tilde{b}'(V)\tilde{b}(V), \quad y_3 = \tfrac{1}{6}\Big[(\tilde{b}')^2\tilde{b} + 4\tilde{b}''\tilde{b}^2\Big],$$

$$y_4 = \tfrac{1}{24}\Big[(\tilde{b}')^3\tilde{b} + 4\tilde{b}''\tilde{b}'\tilde{b}^2 + \tilde{b}'''\tilde{b}^3\Big],$$

where primes indicate differentiation. We used these terms as follows. For example, consider the integrand for $J^{(1)}$; we need

$$\chi(Y(s,V)) = \chi(V + y_1 s + \cdots) = \chi(V) + s\tilde{b}(V)\chi'(V) + O(s^2).$$

Also, since $\quad \zeta(V,\tau) = \tau + \tfrac{1}{2}\tilde{b}'(V)\tau + O(\tau^2), \quad$ we have

$$\begin{aligned}\zeta(Y(s,V),\tau - s) &= (\tau - s) + \tfrac{1}{2}\tilde{b}'(V + y_1 s + \cdots)(\tau - s)^2 + \cdots \\ &= (\tau - s) + \tfrac{1}{2}\tilde{b}'(V)(\tau - s)^2 + O\Big((\tau - s)^3, s(\tau - s)^2\Big).\end{aligned}$$

With these expansions, the integral becomes

$$\begin{aligned}J^{(1)}(V,\tau) &= \int_0^\tau \Big\{\chi(V)\Big[(\tau - s) + \tfrac{1}{2}\tilde{b}'(V)(\tau - s)^2\Big] + \tilde{b}(V)\chi'(V)s(\tau - s)\Big\}\,ds \\ &= \tfrac{1}{2}\chi(V)\tau^2 + \tfrac{1}{6}\Big[\tilde{b}(V)\chi(V)\Big]'\tau^3 + O(\tau^4).\end{aligned}$$

The other entries in Table A.1 are obtained in the same way.

An identity. An important cancellation occurs because of the identity

$$J^{(5)}(V,\tau) = \tfrac{1}{2}\Big(J^{(1)}(V,\tau)\Big)^2. \tag{A.12}$$

To prove this identity, note that $f(V,\tau) \equiv J^{(1)}(V,\tau)$ is the solution to

$$\frac{\partial f}{\partial \tau} - \tilde{b}(V)\frac{\partial f}{\partial V} = \chi(V)\zeta(V,\tau), \qquad \text{where } f(V,\tau = 0) = 0.$$

But $g(V,\tau) \equiv J^{(5)}(V,\tau)$ is the solution to

$$\frac{\partial g}{\partial \tau} - \tilde{b}(V)\frac{\partial g}{\partial V} = \chi(V)\zeta(V,\tau)f(V,\tau), \quad \text{where } g(V,\tau=0)=0 .$$

It's easy to see that this last equation has the solution $g = f^2/2$. ■

Step II. Collecting together our results so far, at this stage we have the series

$$H(k,V,\tau) = \exp[-c(k)U(V,\tau)]\{1-\xi c(k)d(k)J^{(1)}(V,\tau)$$
$$+\xi^2\left[-cJ^{(2)}+c^2J^{(3)}-cd^2J^{(4)}+\tfrac{1}{2}c^2d^2(J^{(1)})^2\right]+O(\xi^3)\} .$$

The call option price is given by

$$\text{(A.13)} \qquad C(S,V,\tau) = Se^{-\delta\tau} - \frac{Ke^{-r\tau}}{2\pi}\int_{i/2-\infty}^{i/2+\infty} \exp(-ikX)\frac{H(k,V,\tau)}{k^2-ik}dk ,$$

so we need to evaluate expressions of the form

$$I(p,q) = -\frac{Ke^{-r\tau}}{2\pi}\int_{i/2-\infty}^{i/2+\infty} \exp[-ikX - c(k)U(V,\tau)]\frac{c^p(k)d^q(k)}{k^2-ik}dk$$

The trick is to temporarily ignore the specific definitions of X and U and just consider the function $f(X,U)$ of two dummy variables defined by

$$\text{(A.14)} \qquad f(X,U) = e^X - \frac{1}{2\pi}\int_{i/2-\infty}^{i/2+\infty} \exp[-ikX - c(k)U]\frac{1}{k^2-ik}dk$$

$$= e^X\Phi\left(\frac{X}{\sqrt{U}}+\tfrac{1}{2}\sqrt{U}\right)-\Phi\left(\frac{X}{\sqrt{U}}-\tfrac{1}{2}\sqrt{U}\right),$$

where $\Phi(\bullet)$ is the cumulative normal. Using $f(X,U)$, then $I(p,q)$ is given by

$$\text{(A.15)} \qquad I(p,q) = Ke^{-r\tau}(-1)^p\left(\frac{\partial^{p+q}}{\partial U^p \partial X^q}\right)f(X,U), \qquad p \geq 1 .$$

Restoring X to its special value, the B-S formula is given in terms of $f(X,U)$ by

$$\text{(A.16)} \qquad c(S,V,\tau) = Ke^{-r\tau}f\left(X = \ln\left[\frac{Se^{-\delta\tau}}{Ke^{-r\tau}}\right], U = V\tau\right).$$

In the case of deterministic volatility, the deterministic B-S formula is given by

$$\text{(A.17)} \qquad c_b(S,V,\tau) = Ke^{-r\tau}f\left(X = \ln\left[\frac{Se^{-\delta\tau}}{Ke^{-r\tau}}\right], U = v(V,\tau)\tau\right),$$

$$\text{(A.18) using} \qquad v(V,\tau) = \frac{1}{\tau}U(V,\tau) = \frac{1}{\tau}\int_0^\tau Y(s,V)\,ds .$$

The function $v(V,\tau)$ is the time-average of the deterministic volatility. So we can also write $I(p,q)$ as

(A.19) $$I(p,q) = (-\tau)^{-p}\left(\frac{\partial^p}{\partial V^p}\right)\left(S\frac{\partial}{\partial S}\right)^q c(S,V,\tau)\Big|_{V=v(V,\tau)}, \quad p \geq 1.$$

Note that differentiated in (A.19) is the *constant volatility* formula $c(S,V,\tau)$, *not* the deterministic volatility formula $c_b(S,V,\tau)$. *After* the differentiation, you substitute the time-average volatility. Finally, it proves convenient to factor out an overall factor $\partial c/\partial V$, so we define the derivative ratios

(A.20) $$\begin{cases} R^{(p,q)}(X,V,\tau) = \left[\left(\dfrac{\partial}{\partial V}\right)^p \left(S\dfrac{\partial}{\partial S}\right)^q c(S,V,\tau)\right] / \dfrac{\partial c}{\partial V}, \\ \tilde{R}^{(p,q)}(X,V,\tau) = R^{(p,q)}(X, V = v(V,\tau), \tau) \end{cases}$$

With this notation, it is straightforward to show that the call option formula (A.13) is given by

Series I:

(A.21) $$C(S,V,\tau) = c(S,v,\tau) + \xi\tau^{-1}J^{(1)}\tilde{R}^{(1,1)}c_V(S,v,\tau)$$
$$+\xi^2\left\{\tau^{-1}J^{(2)} + \tau^{-2}J^{(3)}\tilde{R}^{(2,0)} + \tau^{-1}J^{(4)}\tilde{R}^{(1,2)} + \frac{\tau^{-2}}{2}(J^{(1)})^2\tilde{R}^{(2,2)}\right\}c_V(S,v,\tau) + O(\xi^3)$$

Then, a simple Taylor series expansion (see below) shows that this is equivalent to $C(S,V,\tau) = c(S,V^{imp},\tau)$, where

Series II:

(A.22) $$V^{imp} = v(V,\tau) + \xi\tau^{-1}J^{(1)}\tilde{R}^{(1,1)}$$
$$+\xi^2\left\{\tau^{-1}J^{(2)} + \tau^{-2}J^{(3)}\tilde{R}^{(2,0)} + \tau^{-1}J^{(4)}\tilde{R}^{(1,2)}\right.$$
$$\left.+\tfrac{1}{2}\tau^{-2}(J^{(1)})^2\left[\tilde{R}^{(2,2)} - \left(\tilde{R}^{(1,1)}\right)^2\tilde{R}^{(2,0)}\right]\right\} + O(\xi^3).$$

To see that Series II is equivalent to Series I quickly, think of Series II as $V^{imp} = v + \xi g_1 + \xi^2 g_2$. Then through $O(\xi^2)$,

(A.23) $$c(v + \xi g_1 + \xi^2 g_2, \tau) = c(v) + (\xi g_1 + \xi^2 g_2)c_V(v) + \tfrac{1}{2}\xi^2 g_1^2\left(\frac{c_{VV}}{c_V}\right)c_V(v).$$

But $$\left(\frac{c_{VV}}{c_V}\right) = \tilde{R}^{(2,0)} \qquad \text{and} \qquad g_1^2 = \tau^{-2}(J^{(1)})^2(\tilde{R}^{(1,1)})^2,$$

so the last term in (A.23) cancels the second term in the brackets in (A.22), leaving (A.21).

The parameterized model. So far the results apply to general stationary volatility processes. Now we specialize to the particular model $dV = (\omega - \theta V)\,dt + \xi V^{\varphi} d\tilde{W}$ with constant correlation ρ with the stock-price process. In this case, since the drift is a linear function of volatility, then

$$Y(s,V) = \frac{\omega}{\theta} + e^{-\theta s}\left(V - \frac{\omega}{\theta}\right) \quad \text{and} \quad \zeta\left(Y(s,V), \tau - s\right) = \frac{1}{\theta}\left(1 - e^{-\theta(\tau - s)}\right).$$

Since $\eta(V) = V^{\varphi}$, then $\chi(V) = \rho V^{\varphi + (1/2)}$ and

$$\chi\left(Y(s,V)\right) = \rho\left[\frac{\omega}{\theta} + e^{-\theta s}\left(V - \frac{\omega}{\theta}\right)\right]^{\varphi + \frac{1}{2}}.$$

Using Table 3A.1, these relations yield immediately

$$\text{(A.24)} \qquad J^{(1)}(V,\tau) = \frac{\rho}{\theta}\int_0^{\tau}\left(1 - e^{-\theta(\tau - s)}\right)\left[\frac{\omega}{\theta} + e^{-\theta s}\left(V - \frac{\omega}{\theta}\right)\right]^{\varphi + \frac{1}{2}} ds\,,$$

Since $\zeta_V = 0$, then $J^{(2)}(V,\tau) = 0$. Again, Table A.1 gives

$$\text{(A.25)} \qquad J^{(3)}(V,\tau) = \frac{1}{2\theta^2}\int_0^{\tau}\left(1 - e^{-\theta(\tau - s)}\right)^2\left[\frac{\omega}{\theta} + e^{-\theta s}\left(V - \frac{\omega}{\theta}\right)\right]^{2\varphi} ds\,.$$

A usable expression for $J^{(4)}$ takes a little work to develop. Table 3A.1 yields

$$J^{(4)}(V,\tau) = \rho\int_0^{\tau}\left[\frac{\omega}{\theta} + e^{-\theta s}\left(V - \frac{\omega}{\theta}\right)\right]^{\frac{1}{2} + \varphi} J_V^{(1)}\left(Y(s,V), \tau - s\right) ds\,.$$

Changing integration variable to $s' = \tau - s$ gives

$$J^{(4)}(V,\tau) = \rho\int_0^{\tau}\left[\frac{\omega}{\theta} + e^{-\theta(\tau - s)}\left(V - \frac{\omega}{\theta}\right)\right]^{\frac{1}{2} + \varphi} J_V^{(1)}\left(Y(\tau - s, V), s\right) ds\,.$$

Also the same transformation on $J^{(1)}$, but in the form $u = \tau - s$, yields

$$J^{(1)}(V,\tau) = \frac{\rho}{\theta}\int_0^{\tau}\left(1 - e^{-\theta u}\right)\left[\frac{\omega}{\theta} + e^{-\theta(\tau - u)}\left(V - \frac{\omega}{\theta}\right)\right]^{\varphi + \frac{1}{2}} du\,,$$

so that

$$J_V^{(1)}(V,\tau) = \left(\varphi + \tfrac{1}{2}\right)\frac{\rho}{\theta}\int_0^{\tau}\left(e^{-\theta(\tau - u)} - e^{-\theta\tau}\right)\left[\frac{\omega}{\theta} + e^{-\theta(\tau - u)}\left(V - \frac{\omega}{\theta}\right)\right]^{\varphi - \frac{1}{2}} du\,.$$

Then we have $J_V^{(1)}\left(Y(\tau - s, V), s\right) =$

$$\left(\varphi + \tfrac{1}{2}\right)\frac{\rho}{\theta}\int_0^{s}\left(e^{-\theta(s - u)} - e^{-\theta s}\right)\left[\frac{\omega}{\theta} + e^{-\theta(\tau - u)}\left(V - \frac{\omega}{\theta}\right)\right]^{\varphi - \frac{1}{2}} du\,.$$

This gives us the final result:

$$J^{(4)}(V,\tau) = \left(\varphi + \tfrac{1}{2}\right)\frac{\rho^2}{\theta}\int_0^\tau \left[\frac{\omega}{\theta} + e^{-\theta(\tau-s)}\left(V - \frac{\omega}{\theta}\right)\right]^{\varphi+\frac{1}{2}} J^{(6)}(V,\tau,s)\,ds \tag{A.26}$$

where we introduced the additional integral

$$J^{(6)}(V,\tau,s) = \int_0^s \left(e^{-\theta(s-u)} - e^{-\theta s}\right)\left[\frac{\omega}{\theta} + e^{-\theta(\tau-u)}\left(V - \frac{\omega}{\theta}\right)\right]^{\varphi-\frac{1}{2}} du\,. \tag{A.27}$$

4 Mixing Solutions and Applications

In this chapter, we show that option prices under stochastic volatility are a weighted sum of constant volatility prices. For put and call options, the result is a weighted sum of B-S prices. This mixing idea was first demonstrated, in the special case of no correlation between the stock price and volatility changes, by Hull and White (1987). Then, Romano and Touzi (1997) extended this to the case of a correlated process for put and call options. Our new results include further extensions to (i) generalized payoff functions, and (ii) generalized stock price volatility coefficients, but with no correlation.

There is a strong connection between the fundamental transform $\hat{H}$ and the probability densities used for mixing. For example, when the correlation vanishes, mixing is based upon the density P of the integrated variance $\int_0^T V(s)\,ds$. We show that P and $\hat{H}$ are Laplace transform-inversion pairs.

Mixing theorems are very attractive because of the variety of applications. Some examples from different categories: *Numerical*: we show a new Monte Carlo method. *Analytical*: sometimes one can calculate explicitly, or easily evaluate numerically, all of the steps of the mixing procedure. We illustrate this with the examples of the square-root and the geometric volatility processes. *Series*: The mixing representation for an option price can also be used to develop the volatility of volatility series expansion. This is shown in Chapter 5.

1 The Basic Mixing Solution

Recall the martingale pricing formula for a call option striking at K that expires in T periods from today, where today's stock price is S_0 and today's volatility is V_0. The option price is given by

$$C(S_0,V_0,T) = e^{-rT}\tilde{\mathbb{E}}_0[(S_T - K)^+], \tag{1.1}$$

where $\tilde{\mathbb{E}}_0[\cdots]$ is today's expectation under the risk-adjusted process $\tilde{P}$. The mixing idea is based upon the following procedure:

- Separate the stock price evolution into two independent degrees of freedom, one independent of the volatility process.
- Integrate out the independent variables in (1.1)
- What remains is a mixing formula

In this chapter, with the exception of Sec. 5, we assume that the risk-adjusted process $\tilde{P}$ has the general form

$$\tilde{P}: \begin{cases} dS_t = (r-\delta)S_t dt + \sigma_t S_t d\tilde{B}_t \\ dV_t = \tilde{b}(V_t)\,dt + a(V_t)\,d\tilde{W}_t \end{cases}, \tag{1.2}$$

where $d\tilde{B}_t$ and $d\tilde{W}_t$ are correlated Brownian motions, with correlation $\rho_t = \rho(V_t)$. We also assume that the interest rate r and dividend yield δ are constants.

To decompose the stock price process, let $d\tilde{Z}_t$ be a Brownian motion independent of $d\tilde{W}_t$, and write

$$d\tilde{B}_t = \rho_t\, d\tilde{W}_t + (1-\rho_t{}^2)^{1/2} d\tilde{Z}_t\,.$$

At the same time, introduce $X_t = \ln S_t$. From Ito's formula and some rearrangement, it's easy to see that $\tilde{P}$ is equivalent to the system:

$$\tilde{P}: \begin{cases} dX_t = (r-\delta)dt + dY_t - \frac{1}{2}(1-\rho_t^2)\sigma_t^2 dt + (1-\rho_t^2)^{1/2}\sigma_t d\tilde{Z}_t \\ dY_t = -\frac{1}{2}\rho_t^2\sigma_t^2 dt + \rho_t\,\sigma_t d\tilde{W}_t \\ dV_t = \tilde{b}_t dt + a_t d\tilde{W}_t \end{cases} \tag{1.3}$$

where $Y_0 = 0$. Note that Y_t is affected by the values σ_t and $d\tilde{W}_t$ innovations, but is independent of the $d\tilde{Z}_t$ innovations. Also $\varphi_t \equiv e^{Y_t}$ satisfies $d\varphi_t = \rho_t \sigma_t \varphi_t \, d\tilde{W}_t$; hence φ_t is a local martingale under the risk-adjusted volatility process alone.

Notation. We use $\langle \cdots \rangle$ to denote expectations under the risk-adjusted volatility process. For example, if φ_t is a martingale, then $\langle \varphi_t \rangle = 1$.

Since we want to create a Monte Carlo procedure, its helpful to think of (1.3) as the $\Delta t \to 0$ limit of a discrete-time process, where $t = 0, \Delta t, \cdots, T$. In discrete-time, draw two independent standard normal variates $\hat{W}_t, \hat{Z}_t$ at each time step $t \le T - \Delta t$. (The hat distinguishes these discrete-time random variables from the Brownian motions. For the other variables, it should not cause confusion if we use the same notation for their discrete-time counterparts). Then, the terminal log-stock price becomes, from integrating (1.3)

(1.4)

$$X_T = \ln\left[S_t e^{(r-\delta)T} \right] + Y_T - \tfrac{1}{2} \sum_{t=0}^{T-\Delta t} (1-\rho_t^2)\sigma_t^2 \Delta t + \sum_{t=0}^{T-\Delta t} (1-\rho_t^2)^{1/2} \sigma_t \hat{Z}_t \sqrt{\Delta t},$$

(1.5) where
$$Y_T = -\tfrac{1}{2} \sum_{t=0}^{T-\Delta t} \rho_t^2 \, \sigma_t^2 \Delta t + \sum_{t=0}^{T-\Delta t} \rho_t \sigma_t \hat{W}_t \sqrt{\Delta t}, \quad \text{and}$$

(1.6)
$$V_t = V_0 + \sum_{s=0}^{t-\Delta t} \tilde{b}(V_s) \Delta t + \sum_{s=0}^{t-\Delta t} a(V_s) \hat{W}_s \sqrt{\Delta t}, \qquad t = \Delta t, 2\Delta t, \cdots, T.$$

The martingale pricing solution is the $\Delta t \to 0$ limit of the multiple integrals over the normal variates:

(1.7)
$$C(S_0, V_0, T) = e^{-rT} \tilde{\mathrm{E}}_0[(e^{X_T} - K)^+]$$
$$= \lim_{\Delta t \to 0} e^{-rT} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} (e^{X_T} - K)^+ \prod_{t=0}^{T-\Delta t} \exp\left[-\tfrac{1}{2}(\hat{Z}_t^2 + \hat{W}_t^2) \right] \frac{d\hat{Z}_t d\hat{W}_t}{2\pi}.$$

Now, the mixing theorem follows by performing the integration over the entire set of variables $\{\hat{Z}_t\}$ in (1.7). While integrating over the $\{\hat{Z}_t\}$, you can treat Y_T and the set $\{\sigma_t\}$ as constants. That's because, as one can also see from (1.5) and (1.6), these variables are independent of $\{\hat{Z}_t\}$.

When you treat those variables as constants, (1.4) can be interpreted as a standard stock price evolution under *deterministic* volatility plus a stock price adjustment. Under that interpretation, (i) the deterministic volatility at time-t is given by $(1-\rho_t^2)^{1/2}\sigma_t$ and (ii) the stock price adjustment is that you also add a constant Y_T to the terminal log-stock price.

But, the expectation of $(S_T - K)^+$ has a well-known solution under deterministic volatility; namely, the expectation is given by the B-S formula, where the variance parameter is replaced by an effective variance $V \to V^{eff} = \int_0^T V_t\, dt / T$. (See Appendix 2.1 for a proof). Moreover, the stock price adjustment is just a multiplicative adjustment to today's stock price in the same B-S formula. To record these results we need some notations.

Notations. Define the effective stock price and effective volatility by

$$S_T^{eff} = \lim_{\Delta t \to 0} S_0 e^{Y_T} = S_0 \exp\left(-\tfrac{1}{2}\int_0^T \rho_t^2 \sigma_t^2 dt + \int_0^T \rho_t \sigma_t d\bar{W}_t\right), \tag{1.8}$$

$$V_T^{eff} = \lim_{\Delta t \to 0} \frac{1}{T}\sum_{t=0}^{T-\Delta t} (1-\rho_t^2)\sigma_t^2 \Delta t = \frac{1}{T}\int_0^T (1-\rho_t^2)\sigma_t^2 dt\,. \tag{1.9}$$

Also let $c(S_0, V_0, T)$ be the B-S price formula; that is,

$$c(S,V,T) = Se^{-\delta T}\Phi(d_+) - Ke^{-rT}\Phi(d_-),$$

where

$$d_{\pm} = \frac{1}{\sqrt{VT}}\left[\ln\left(\frac{Se^{-\delta T}}{Ke^{-rT}}\right) \pm \tfrac{1}{2}VT\right] \text{ and } \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-z^2/2} dz\,.$$

Using the bracket notation for expectation, we have

$$\left\langle c(S_T^{eff}, V_T^{eff}, T)\right\rangle = \lim_{\Delta t \to 0} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} c(S_T^{eff}, V_T^{eff}, T) \prod_{t=0}^{T-\Delta t} \exp\left(-\tfrac{1}{2}\hat{W}_t^2\right)\frac{d\hat{W}_t}{(2\pi)^{1/2}}$$

In summary, we have argued for the validity of the following theorem:

THEOREM 1 (Romano and Touzi, 1997): *Let $C(S_0, V_0, T)$ be the call option price under the stochastic volatility process $\tilde{P}$ of (1.2). Then*

$$\boxed{C(S_0, V_0, T) = \left\langle c(S_T^{eff}, V_T^{eff}, T)\right\rangle} \tag{1.10}$$

In words again, the option value under stochastic volatility is a weighted sum or mixture of the Black-Scholes values with an effective stock price and effective volatility. The effective variables depend *only* upon the volatility process. Hence, the problem reduces to a martingale pricing expectation over the volatility process alone.

Zero correlation. In general $\langle S^{eff} \rangle = S_0$, but when $\rho_t = 0$, then $S^{eff} = S_0$. The volatility process determines the option price through the probability distribution $P(U_T;V_0,T)$ of the integrated volatility: $U_T = \int_0^T V_t dt$. In that case, (1.10) can be rephrased as

$$\boxed{C(S_0,V_0,T) = \int_0^\infty c\left(S_0, \frac{U_T}{T}, T\right) P(U_T;V_0,T)\, dU_T} \tag{1.11}$$

Hull and White (1987) established this case.

The smile is symmetric under zero correlation. One application of (1.11), due to Renault and Touzi (1996, Proposition 3.1), establishes that the smile is symmetric as a function of $X = \ln(S/K) + (r - \delta)\tau$. The argument is the following. Written out more explicitly, (1.11) reads

$$C(S,V_0,T) = Se^{-\delta\tau} \int_0^\infty g(X,U_T) P(U_T;V_0,T)\, dU_T\,, \tag{1.12}$$

where $$g(X,U_T) = \Phi\left(\frac{X}{\sqrt{U_T}} + \frac{1}{2}\sqrt{U_T}\right) - e^{-X}\Phi\left(\frac{X}{\sqrt{U_T}} - \frac{1}{2}\sqrt{U_T}\right).$$

It's easy to verify the property:

$$g(-X,U) = e^X g(X,U) + 1 - e^X\,. \tag{1.13}$$

Hence, if you define $f(X,V,\tau)$ by $C(S,V,T) = Se^{-\delta\tau} f(X,V,\tau)$, then (1.12) implies that $f(X,V,\tau)$ inherits this same property:

$$f(-X,V,\tau) = e^X f(X,V,\tau) + 1 - e^X\,. \tag{1.14}$$

In particular, (1.14) holds under constant volatility, in which case we write $f_{BS}(X,V,\tau)$. The implied volatility $V^{imp}(X,V,\tau)$ is the solution to $f(X,V,\tau) = f_{BS}\left(X, V^{imp}(X,V,\tau), \tau\right)$. Hence using (1.14) twice:

$$
\begin{aligned}
(1.15) \qquad f(-X,V,\tau) &= f_{BS}\left(-X,V^{imp}(-X,V,\tau),\tau\right) \\
&= e^X f_{BS}\left(X,V^{imp}(-X,V,\tau),\tau\right)+1-e^X \\
&= e^X f(X,V,\tau)+1-e^X .
\end{aligned}
$$

The last two equations imply that

$$
(1.16) \quad f_{BS}\left(X,V^{imp}(-X,V,\tau),\tau\right) = f(X,V,\tau) = f_{BS}\left(X,V^{imp}(X,V,\tau),\tau\right).
$$

Since $f_{BS}(X,V,\tau)$ is single-valued as a function of V, the two expressions using f_{BS} in (1.16) can only be equal if

$$
(1.17) \qquad V^{imp}(-X,V,\tau) = V^{imp}(X,V,\tau) \quad (\rho = 0) \qquad \blacksquare
$$

2 The Connection between Mixing Densities and the Fundamental Transform

There is an close relationship between the fundamental transform $\hat{H}(k,V,\tau)$ and the probability distribution of the variables that appear in the mixing theorem. Essentially, they are transforms of each other, as we will show below.

One application of mixing theorems is (perturbation) series solutions for option prices, implied volatilities, and smile patterns. For example, when $\rho_t = 0$, you can expand the B-S integrand of (1.11) about the mean of U_T/T. This leads to a volatility moment expansion. But when $\rho \neq 0$, this idea doesn't really work unless you re-state the mixing theorem in terms of our fundamental transform. How to accomplish this is another subject of this section. The resulting formula will probably seem fairly abstract at this stage, but it will be made operational in Chapter 5

Recall Solution II given at (2.2.10) for call option prices in terms of the fundamental transform:

$$
(2.1) \qquad C(S,V,T) = Se^{-\delta T} - \frac{Ke^{-rT}}{2\pi} \int_{ik_i-\infty}^{ik_i+\infty} \exp(-ikX) \frac{\hat{H}(k,V,T)}{k^2-ik} dk ,
$$

where $X = \log[S/K] + (r-\delta)T$ and $\operatorname{Im} k$ suitably restricted. When volatility is constant, (2.1) reduces to the B-S solution:

$$(2.2)\qquad c(S,V,T) = Se^{-\delta T} - \frac{Ke^{-rT}}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty} \exp(-ikX)\frac{e^{-c(k)VT}}{k^2-ik}\,dk\,,$$

where $c(k) = (k^2 - ik)/2$. Now just insert the representation (2.2) into the right-hand-side of (1.10); after a little rearrangement, the result is

$$(2.3)\qquad C(S,V,T) = Se^{-\delta T}$$
$$-\frac{Ke^{-rT}}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty} e^{-ikX}\left\langle \exp\left(-ik\int_0^T \rho_s\sigma_s d\tilde{W}_s - \int_0^T \tilde{c}_s(k)V_s ds\right)\right\rangle\frac{dk}{k^2-ik}\,,$$

now using $\qquad \tilde{c}_t(k) = \tilde{c}(k,V_t) \equiv c(k) - \frac{1}{2}k^2\rho^2(V_t)$

Comparing (2.3) and (2.1), we obtain the mixing representation for the fundamental transform

$$(2.4)\qquad \hat{H}(k,V,T) = \left\langle \exp\left(-ik\int_0^T \rho_t\sigma_t d\tilde{W}_t - \int_0^T \tilde{c}_t(k)V_t dt\right)\right\rangle.$$

This Feynman-Kac style formula will prove to be a good starting point for perturbation theory. If the correlation is a constant, then (2.4) reads

$$(2.5)\qquad \boxed{\begin{aligned} &\hat{H}(k,V,T) = \left\langle \exp\left(-ik\rho\int_0^T \sigma_t dW_t - \tilde{c}(k)\int_0^T V_t dt\right)\right\rangle,\\ &\text{where}\qquad \tilde{c}(k) = \tfrac{1}{2}[k^2(1-\rho^2) - ik]. \end{aligned}}$$

Recall from Chapter 2 that $\hat{H}(k,V,T)$ of (2.4) is the solution to the PDE

$$(2.6)\qquad \frac{\partial \hat{H}}{\partial T} = \frac{1}{2}a^2(V)\frac{\partial^2 \hat{H}}{\partial V^2} + [\tilde{b}(V) - ik\rho(V)V^{1/2}a(V)]\frac{\partial \hat{H}}{\partial V} - c(k)V\hat{H}\,,$$

with the initial condition $\hat{H}(k,V,T=0) = 1$. It's not obvious that (2.6) has a probabilistic solution given by (2.4) or, with constant correlation, (2.5). Equation (2.5) is applied in Chapter 5 to smile calculations.

One way to think about (2.5) is to define two random variables $M_T = \int_0^T \sigma_t dW_t$ and, as before, $U_T = \int_0^T V_t dt$. Note that M_T is a local martingale under $\langle\cdots\rangle$. And $U_T \geq 0$ is not only the integrated volatility, but the *quadratic variation* process of M_T. In principle, there is a joint probability density $P(M_T, U_T; V_0, T)$ for the risk-adjusted volatility process. That is,

$P(M_T, U_T; V_0, T)\, dM_T\, dU_T$ is the probability that the values in the 2D interval close to (M_T, U_T) occur, after the elapse of time-T, and starting with V_0. Then (2.5) can be written

$$(2.7) \qquad \hat{H}(k,V,T) = \int_{\infty}^{\infty} \int_{0}^{\infty} \exp(-ik\rho M \quad \tilde{c}(k)U)\, P(M,U;V,T)\, dM\, dU\,.$$

The transform connection. Note that (2.7) is a characteristic function: a two-dimensional (Fourier-Laplace) transform of the joint probability density. To see the concept more clearly, consider the case where the correlation vanishes. In that case, the joint density becomes $P(U_T)$, the density of the integrated volatility, where we suppress the dependence on V_0 and T. Consider the Laplace transform of $P(U_T)$ using the scalar c as the transform variable. Then (2.7) becomes the statement of the following theorem

THEOREM 2. *When $\rho = 0$, the integrated volatility density and the fundamental transform are Laplace transform-inversion pairs. That is,*

$$(2.8) \qquad \boxed{\begin{array}{l} \text{If } \hat{P}(c) = \displaystyle\int_0^{\infty} e^{-cU} P(U)\, dU\,, \quad \text{where } U = \int_0^T V_t\, dt\,, \\ \text{then } \hat{H}(k,V_0,T) = \hat{P}(c(k))\,, \quad \text{where } c(k) = \tfrac{1}{2}(k^2 - ik) \end{array}}$$

Theorem 2 provides a practical prescription for obtaining $P(U)$: first develop the fundamental transform, then $P(U)$ can be obtained by inverting a Laplace transform. We show two examples of this in Secs. 6 and 7. In the first example, it's only one line of Mathematica code!

3 A Monte Carlo Application

The discrete-time version of the mixing theorem yields a simple Monte Carlo procedure, requiring only the draw of a single normal variate at each time step. We discuss some of the details in this section.

In this section let $\langle \cdots \rangle$ denote an average over N Monte Carlo (MC) simulations with time-step Δt. Then, we have the MC pricing formula for a put option:

$$P(S_0, V_0, T) = \lim_{\substack{N \to \infty \\ \Delta t \to 0}} \left\langle p(S^{eff}, V^{eff}, T) \right\rangle. \tag{3.1}$$

We emphasize the put option to stress that the MC statistics are often much better for a put option than a call. This is especially true in large volatility limits or in cases where the volatility process can explode (more below). In those cases, the Monte Carlo standard error is generally *much* smaller for put option price estimates. You can always recover call option prices from the put-call parity formula, rather than direct MC averaging.

To implement (3.1), draw a single standard normal variate $\hat{Z}_t$ at each time step, $t = 0, \Delta t, \ldots, T - \Delta t$. Except at the boundaries, this random draw is used to update the sequences

$$Y_{t+\Delta t} = Y_t - \tfrac{1}{2}\rho_t^2 \sigma_t^2 \Delta t + \rho_t \sigma_t \hat{Z}_t \sqrt{\Delta t}\ , \tag{3.2}$$

$$V_{t+\Delta t} = V_t \exp\left[\left(\frac{\tilde{b}(V_t)}{V_t} - \frac{1}{2}\frac{a^2(V_t)}{V_t^2}\right)\Delta t + \frac{a(V_t)}{V_t}\hat{Z}_t\sqrt{\Delta t}\right], \tag{3.3}$$

We start with $Y_0 = 0$ and the given V_0. The exponent in (3.3) is just the discretized version of the evolution of $d \ln V_t$, obtained from the Ito change-of-variable formula. The advantage of this transformation is that it makes its harder for the discretized process to reach the singular points $V_t = 0$ and $V_t = \infty$. Nevertheless, we still have to modify (3.3) at the boundaries for certain processes.

Boundary behavior. The possible exceptions to (3.3) occur at large and small volatility.

(i) *Small volatility.* It's possible that $V_t = 0$ may be reached to machine precision or V_t may be so small that an underflow/overflow occurs.

For example, consider option valuation where the continuous-time process is a GARCH diffusion process in a risk-neutral world. In a risk-neutral world there is no adjustment to the volatility drift. That is, $dV_t = (\omega - \theta V_t)dt + \xi V_t dW_t$. At $V_t = 0$, the SDE reduces to $dV_t = \omega\, dt$ and (3.3) should be replaced by $V_{t+\Delta t} = \omega \Delta t$ at $V_t = 0$. Other processes may require similar modifications.

(ii) *Large volatility.* If the continuous-time volatility process can explode (reaching $V = \infty$ in finite time), you will get overflows in the discrete-time process (3.3). To handle this, introduce a maximum value for $\ln V_t$; if the process reaches this cutoff value, no further increases are allowed. Having a cutoff like this is consistent with the Feller classification scheme in the following sense: if an explosion can occur, the boundary at infinity is an exit boundary, which is a trap or absorbing state. So, once reached, the process cannot leave; the effect of the cutoff is to simply turn a large value of volatility into the trap state.

Then, the result of a single simulation run is calculated from the B-S formula with the arguments

$$S^{eff} = S_0 \exp(Y_T) \quad \text{and} \quad V^{eff} = \frac{1}{T} \sum_{t=0}^{T-\Delta t} (1 - \rho_t^2) V_t \Delta t \, .$$

The exact continuous-time result is the limiting average (3.1).

An example. To see the performance of the method, we created a short program **mcmix1.c**, which implements this procedure. For variance reduction, the program uses both $\hat{Z}_t$ and $-\hat{Z}_t$ for each single simulation; this is the well-known antithetic technique.

The volatility process was taken to be the GARCH diffusion in a risk-neutral world. In Table 5.1, we show the smile effect for the risk-neutral model at $T = 20$ days to expiration. The table entries are the estimated put option price, MC standard error below in parenthesis, and BS implied sigma (annualized, in percent). Estimates are based upon 10^5 simulation runs, $T = 20$ days, with $\Delta t = 1$ day. The stock price $S = 100$ and $r = \delta = 0$.

With a running time about 30 seconds for each entry, one can see that the MC standard errors are all less than 1 penny, and some significantly less (see the row with $\rho = 0$). Since it is an easily coded procedure, the method is a way to get fairly accurate prices for short-term options without much hassle.

We also considered the case of a GARCH diffusion under a representative investor with log-utility. These results, which test both volatility explosions and a duality relation are reported in Chapter 9.

Of course, any Monte Carlo technique, including the current one, is expected to be slow compared to finite difference methods. To illustrate just how slow, we show in Table 4.2 estimates and running times versus an explicit lattice algorithm. Also shown for comparison is the volatility of volatility series expansion (explained in Chapter 3), which provides a third value for accuracy comparisons. The running times can be sensibly compared for the cases I and III in the table because they are both C-code programs running on the same machine. The series expansion of case II was evaluated in Mathematica, which is largely an interpreted language. So this case is not directly comparable and running times are not shown. As one can see from the table, the Monte Carlo method is about 100 times slower than the lattice algorithm.

Table 4.1 Put Prices and Implied volatilities.
Monte Carlo evaluation using the Mixing Theorem.
Risk-adjusted volatility process:
$dV_t = (\omega - \theta V_t)dt + \xi V_t\, dW_t$ **(GARCH diffusion)**

Corr.	**Strike Price**				
ρ	**90**	**95**	**100**	**105**	**110**
-1.0	0.0245	0.285	1.689	5.207	10.006
	(0.0006)	(0.002)	(0.004)	(0.002)	(0.0009)
	17.16	17.03	14.97	13.89	13.06
-0.50	0.0161	0.257	1.688	5.239	10.012
	(0.0001)	(0.0007)	(0.001)	(0.0008)	(0.0005)
	16.22	15.54	14.96	14.47	14.15
0.0	0.0095	0.229	1.688	5.272	10.021
	($7\text{x}10^{-6}$)	($3\text{x}10^{-5}$)	($3\text{x}10^{-5}$)	($3\text{x}10^{-5}$)	($1\text{x}10^{-5}$)
	15.19	15.02	14.96	15.01	15.15
0.50	0.0046	0.200	1.689	5.302	10.031
	($1\text{x}10^{-5}$)	(0.0003)	(0.0006)	(0.0002)	(0.0005)
	14.02	14.46	14.97	15.52	17.05
1.0	0.0015	0.170	1.692	5.332	10.043
	(0.0001)	(0.0014)	(0.003)	(0.001)	(0.0012)
	12.63	13.85	15.00	15.99	17.89
BS	0.0085	0.228	1.692	5.270	10.019
value	15.00	15.00	15.00	15.00	15.00

Notes: Result from the program **mcmix1.c**. Table entries show the MC put price, MC standard error in parenthesis, and the Black-Scholes implied volatility (σ^{imp} , in percent, annualized). Entries are for various strike prices and stock-volatility correlations ρ. Entries are based upon 10^5 simulation runs; run time was about 30 seconds per entry. The model parameters are $S = 100$, $\omega_a = 0.09$, $\theta_a = 4$, $V_0 = 0.0225$, $\xi_a = 1$. With 250 days-per-year, we took $\Delta t_a = 1/250$ and $T_a = 20/250$ years (20 days to expiration). The model also assumes $r = \delta = 0$ and $\gamma = 1$ (risk-neutral preferences). The results show typical smile patterns, in which, for example, out-of-the-money put prices are higher than B-S prices under a negative correlation. This data is plotted in Fig. 5.1

Table 4.2 Monte Carlo versus Two other Numerical Methods

I: An explicit lattice algorithm (program: lattct.c)

Time steps per day: m	**Days to Option Expiration 20**	**60**	**125**	**250**	**500**
2	1.6859	2.905	4.181	5.911	8.366
5	1.6858	2.906	4.182	5.911	8.365
10	1.6858	2.906	4.182	5.911	8.365
Run times:	0.06 sec	0.30	1.3	4.6	9.1

II: Series expansion in powers of the volatility of volatility ξ

ξ – series order: n	**Days to Option Expiration 20**	**60**	**125**	**250**	**500**
0	1.6924	2.931	4.229	5.979	8.447
2	1.6858	2.906	4.182	5.911	8.368

III: Monte Carlo (program: mcmix1.c)

	Days to Option Expiration 20	**60**	**125**	**250**	**500**
Put value	1.688	2.904	4.179	5.912	8.374
Stand.err	0.001	0.002	0.003	0.004	0.006
Run times:	31 sec	86	175	345	687

Notes: The Monte Carlo estimates are based upon 10^5 drawings, with $\rho = -0.5$. The series expansion is explained in Chapter 3. Programs with the .c suffix are written in C-code. The Monte Carlo method's virtue is its simplicity, but as the table shows, it can be 100 times slower than a direct lattice approach.

4 Arbitrary Payoff Functions

In this section, we maintain the process $\tilde{P}$ of (1.2), but generalize to a European-style financial claim with an arbitrary (volatility independent) payoff $g(S_T)$. In this case, the inner integral in (1.4) becomes

$$(4.1) \qquad e^{-rT} \int_{-\infty}^{\infty} \ldots \int_{-\infty}^{\infty} g(e^{X_T}) \prod_{t=0}^{T-\Delta t} \exp\left[-\tfrac{1}{2}(\hat{Z}_t^2)\right] \frac{d\hat{Z}_t}{(2\pi)^{1/2}}$$

Using an orthogonal change of variable, (4.1) can be reduced to the *one-dimensional* integral

$$(4.2) \quad e^{-rT} \int_{-\infty}^{\infty} g\left(S_0 \exp\left[(r-\delta)T + Y_T + x\sqrt{V_T^{eff}\, T} - \tfrac{1}{2} V_T^{eff}\, T \right] \right) \exp(-\tfrac{1}{2}x^2) \frac{dx}{\sqrt{2\pi}} .$$

Lemma 1 *Let* $f(S_0, V_0, T)$ *denote the fair value of an option with payoff* $g(S_T)$ *under the risk-adjusted process* $\tilde{P}_0 :\ dS_t = (r-\delta)S_t dt + \sigma_0 S_t dB_t$ *with constants* r, δ, *and* σ_0. *Then*

$$(4.3) \qquad f(S_0, V_0, T) = e^{-rT} J\left(S_0 \exp\left[(r - \delta - \tfrac{1}{2} V_0)T \right],\ (V_0 T)^{1/2} \right),$$

where $$J(a,b) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(a e^{bx}) \exp(-\tfrac{1}{2}x^2) dx .$$

Then we have immediately

Theorem 3 (Generalized mixing): *Let* $F(S_0, V_0, T)$ *denote the fair value, under* $\tilde{P}$ *of* (1.1), *of a European-style financial claim with payoff* $g(S_T)$. *Then,*

$$(4.4) \qquad F(S_0, V_0, T) = \left\langle f(S_T^{eff}, V_T^{eff}, T) \right\rangle ,$$

where f *is given by* (4.3)

Corollary 1 (Convexity): *If* $g(S)$ *is convex (concave), then* $F(S,V,t)$ *is convex (concave) as a function of* S. Proof : differentiate (4.4) with respect to S.

Here are some consistency checks:

(i) *A forward stock purchase*: $g(S_T) = S_T$. That is, $g(x) = x$ and so $J(a,b) = a \exp(b^2/2) \Rightarrow f(S,V,T) = S \exp(-\delta T)$. So (4.4) reduces to

$$F(S_0, V_0, T) = e^{-\delta T} \left\langle S_T^{eff} \right\rangle = e^{-\delta T} S_0 \left\langle e^{Y_T} \right\rangle = e^{-\delta T} S_0$$

which is, of course, correct for this claim.

(ii) *The call option*: $g(S) = (S-K)^+$. We recover the Romano-Touzi result.

(iii) *Perfect correlation*: $\rho = \pm 1$. Using (1.9), you can see that $\rho = \pm 1$ implies that $V^{eff} = 0$ and so $J(a,0) = g(a)$. Hence (4.4) becomes

$$F(S_0, V_0, T) = e^{-rT} \left\langle g\left(S_0 e^{(r-\delta)T + Y_T} \right) \right\rangle,$$

where

$$Y_T = -\tfrac{1}{2} \int_0^T \sigma_t^2 dt \pm \int_0^T \sigma_t \, dW_t .$$

As is should be, this is just the martingale pricing formula for the special case where the stock price innovations and the volatility innovations are identical up to a sign.

5 A General Model without Correlation

In this section, we again compare option values under two different risk-adjusted worlds. In the first world, $r = \delta$, volatility is a constant $V_0 = \sigma_0^2$ and the stock price follows the risk-adjusted process

(5.1) $$\tilde{P}_0 : \; dS_t = \sigma_0 \eta(S_t) d\tilde{B}_t .$$

In the second world, $r = \delta$, volatility is stochastic and the risk-adjusted process has the form

(5.2) $$\tilde{P} : \; \begin{cases} dS_t = \sigma_t \eta(S_t) \, d\tilde{B}_t^1 \\ dV_t = b(V_t) dt + a(V_t) \, d\tilde{B}_t^2 \end{cases},$$

where the $d\tilde{B}_t^i$ are standard (uncorrelated) Brownian motions. Note that we have generalized our usual stock price process of (1.2) to now include a volatility that is dependent upon the stock price level. But, weIn both worlds, consider a European-style claim with payoff $g(S_T)$ at time T. Using $\tau = T - t$, let the valuation formulas in the two worlds be

(5.3) $$f(S_t, V_0, \tau) = e^{-r\tau} \, \mathbb{E}^{\tilde{P}_0}[g(S_T) \mid S_t],$$

(5.4) $$h(S_t, V_t, \tau) = e^{-r\tau} \, \mathbb{E}^{\tilde{P}}[g(S_T) \mid S_t, V_t].$$

Before the main result, we need an assumption and a lemma.

Assumption. We assume that $f(S_t, V_0, \tau)$ is a bounded function of V_0. For call and put options, or linear combinations of them, this assumption follows from arbitrage bounds.

With τ as the time, recall the integrated volatility density $P(U, V_0, \tau)$. It satisfies a PDE:

LEMMA 2. *The integrated volatility density* $P(U, V, \tau)$ *satisfies the PDE*

$$P_\tau = \tfrac{1}{2} a^2(V)\, P_{VV} + b(V) P_V - V P_U \,. \tag{5.5}$$

PROOF: From Theorem 2, the Laplace transform

$$\hat{H}(c, V, \tau) = \int_0^\infty e^{-cU} P(U, V, \tau)\, dU \,. \tag{5.6}$$

satisfies the PDE

$$\hat{H}_\tau = \tfrac{1}{2} a^2(V)\, \hat{H}_{VV} + b(V) \hat{H}_V - cV\hat{H} \,. \tag{5.7}$$

with $\hat{H}(c, V, \tau = 0) = 1$. Applying the differential operator in (5.7) to both sides of (5.6) and integrating by parts yields

$$\int_0^\infty e^{-cU} \left[\tfrac{1}{2} a^2(V)\, P_{VV} + b(V) P_V - V P_U - P_\tau \right] dU = V P(U = 0, V, \tau) \,. \tag{5.8}$$

We show as part of Theorem 3 that $VP(U = 0, V, \tau) = 0$ either because (i) $V > 0$ and $P(U = 0, V, \tau) = 0$, or (ii) $V = 0$. Since the right-hand-side of (5.8) vanishes for any value of c, so must the expression in brackets. ■

Here is our general result for this section:

THEOREM 4. *The two valuation formulas (5.3) and (5.4) are related by*

(I) $f(S,V_0,\tau) = e^{-r\tau} F(S, V_0\tau)$,
for a function $F(S,\cdot)$ *satisfying* $F(S,0) = g(S)$.

(II) $h(S,V,\tau) = e^{-r\tau} \int_0^{\infty} F(S,U)\, P(U,V,\tau)\, dU$.

PROOF: Define two differential generators by their actions:

$$A^{\mathrm{I}} f = \tfrac{1}{2} V_0 \eta^2 f_{SS}$$

and
$$A^{\mathrm{II}} h = b\, h_V + \tfrac{1}{2} V \eta^2 h_{SS} + \tfrac{1}{2} a^2 h_{VV} .$$

Let $f(S,V_0,\tau) = \exp(-r\tau)\tilde{f}(S,V_0,\tau)$. Since $\tilde{f}$ satisfies $\tilde{f}_\tau = A^{\mathrm{I}} \tilde{f}$, divide both sides of this last equation by V_0 and define $\tau' = V_0 \tau$. This implies

$$\tilde{f}(S,V_0,\tau) = G(S, V_0\tau) = F(S,V_0\tau)$$

for some function $F(S,U)$ satisfying $F(S,0) = g(S)$, which establishes statement (I). For (II), let $h(S,V,\tau) = \exp(-r\tau)\tilde{h}(S,V,\tau)$. To prove statement (II), we must show $\tilde{h}_\tau = A^{\mathrm{II}}\tilde{h}$ and $h(S,V,0) = g(S)$. But $F(S,U)$ satisfies

$$F_U = \tfrac{1}{2}\eta^2 F_{SS} .$$

Using both this last equation and Lemma 2, we find that

$$\tilde{h}_\tau - A^{\mathrm{II}}\tilde{h} = -V \int_0^{\infty} (FP)_U\, dU = -V\,(FP)\Big|_{U=0}^{U=\infty} ,$$

where $P = P(U,V,\tau)$. So the theorem follows if we can show that $V \times F \times P$ vanishes at $U = 0$ and $U = \infty$.

Now P must vanish at $U = \infty$ because P is a probability density with a finite integral over $0 \le U < \infty$. And, since we have assumed that F is bounded, FP must vanish at $U = \infty$.

Intuitively, one also expects P to vanish at $U = 0$ because, if $V_0 > 0$, the event $U = \int_0^T V_t dt = 0$ will not occur. (See the next section for a graph confirming this in an example). To prove it, take $\rho = 0$ in the first mixing theorem, Theorem 1. That theorem assumes $\eta(S) = S$, and implies statement II in this theorem and $\tilde{h}_\tau = A^{\mathrm{II}}\tilde{h}$. Hence, the boundary term must vanish if $\eta(S) = S$; but, the behavior of P is independent of $\eta(S)$ since it only depends upon the variance process. So, FP must vanish at $U = 0$, assuming $V_0 > 0$. If $V_0 = 0$, then $V_0 FP$ also vanishes.

Finally, $h(S,V,0) = g(S)$ follows from the initial condition $P(U,V,0) = \delta^+(U)$, where $\delta^+(U) = 0$, $U > 0$ and $\int_0^\infty \delta^+(U)dU = 1$. ■

6 Example I: the Square Root Model (Mathematica)

One of the simplest mixing examples is the square-root model with no drift. In other words, take $dV = \xi\sqrt{V}d\tilde{W}(t)$. With that, the fundamental transform $H(c,V,\tau)$ satisfies the PDE $H_\tau = (1/2)\xi^2 V H_{VV} - cVH$. Take $\xi = 1$; it can be shown that the solution to the PDE with $H(c,V,\tau = 0) = 1$ is $H(c,V,\tau) = \exp(-V(2c)^{1/2}\tanh[(2c)^{1/2}\tau/2])$. The proposed solution can easily be checked in Mathematica; first define

```
H[V,t] = E^(-V Sqrt[2 c] Tanh[Sqrt[2 c]t/2]);
```

Then applying the PDE operator confirms that this is a solution:

```
In[7]:=
        Simplify[D[H[V, t], t] - (1/2) V D[H[V, t], {V, 2}] + c V H[V, t]]

Out[7]= 0
```

Given $\hat{H}(c,V,\tau)$, we can obtain $P(U,V,\tau)$ from the Laplace inversion formula

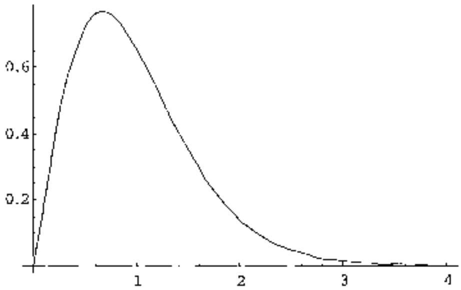
$$P(U,V,\tau) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{cU}\hat{H}(c,V,\tau)\,dc\,,$$

where the integral runs along a vertical line in the complex c-plane to the right of any singularities. In our case, we can take $\gamma = 0$; i.e. integrate along the imaginary c-axis . We'll do this by letting $c = iy$ and for definiteness take $V = \tau = 1$, letting $P(U) = P(U,V = 1,\tau = 1)$. The code is essentially one line:

```
P[U_,ymax_]:=
  Module[{ans},
   Off[NIntegrate::slwcon];
   ans = N[1/( Pi )*
      NIntegrate[Re[E^(-Sqrt[2 I y]Tanh[Sqrt[2 I y]/2])*
       E^(I y U)], {y,0,ymax}, MaxRecursion->20]];
   On[NIntegrate::slwcon];
  Return[ans]]
```

In the code above, we suppressed a "slow convergence" warning message. Here's a plot of the result, showing what $P(U)$ vs. U looks like:

```
In[93]:= Timing[Plot[P[U, 300], {U, 0, 4}, PlotDivision -> 1]];
```




```
Out[93]= {12.14 Second, - Graphics -}
```

Remarks. Note that the density vanishes as $U \to 0$; this should be a general feature of *any* stochastic volatility model. As $U \to \infty$, the density for the example vanishes faster than any power of U; this is a consequence of the fact that the fundamental transform $H(c,V,\tau) = \exp(-V(2c)^{1/2} \tanh[(2c)^{1/2} \tau/2])$ is analytic in c near $c = 0$. Here's a short proof: the Taylor series for the hyperbolic tangent function, $\tanh x$, about $x = 0$, only contains positive odd powers of x. Hence, $H = 1 + a_1 c + a_2 c^2 + \cdots$ with finite coefficients a_i. So every c-derivatives of $H(c,V,\tau)$ exists at $c = 0$. It's a well-known fact and you can see it from (5.6), that derivatives of the characteristic function of a density, in our case $H(c)$, generate moments $\langle U^m \rangle$, $m = 0,1,2,\cdots$ Each moment is finite, which can be true only if $P(U)$ vanishes faster than any power of U as $U \to \infty$.

We can check that the area under the curve above is 1:

```
Timing[NIntegrate[P[U, 300], {U, 0, 5}]]

{34.17 Second, 0.999985}
```

Close enough (and not bad timing for a double integration).

The mixing theorems, combined with the well-known B-S solution, yield the formula for call options with strike price K. In our example, we assume that $r = \delta = 0$, so the mixing solution is

(6.1) $$C(S,V,\tau) = \int_0^\infty b(S,U)\, P(U,V,\tau)\, dU\,,$$

where $$b(S,U) = S\,\Phi(d_1) - K\,\Phi(d_2)\,,$$

using $$d_{1,2} = U^{-1/2}\left[\ln\left(\frac{S}{K}\right) \pm \tfrac{1}{2}U\right].$$

First, we need the cumulative normal function

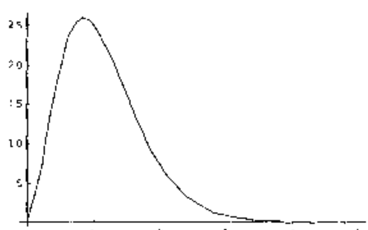
```
Cumnormal[xx_] := (1+Erf[xx/Sqrt[2]])/2
CN = Cumnormal;
```

Then, (6.1) is directly coded (I added the B-S price and a plot of the integrand):

```
MixingCval[S_,K_,ymax_,Umax_]:=
  Module[{X,d1,d2,bs},
    X = Log[S/K];
    d1[U_] = (X + U/2)/Sqrt[U];
    d2[U_] = (X - U/2)/Sqrt[U];
    bs[U_] = S CN[d1[U]] - K CN[d2[U]];
    Print["BS call option value = ",N[bs[1]]];
    Plot[bs[U] P[U,ymax],{U,0,Umax},PlotDivision->1];
    ans = NIntegrate[bs[U] P[U,ymax],{U,0,Umax}];
    Print["Square-root model call option value = ",ans]]
```

Here are the results, taking $S = K = 100$:

```
In[71]:=
        Timing[MixingCval[100, 100, 300, 5]]

     BS call option value = 38.2925
```




```
     Square-root model call option value = 36.478

Out[71]= {47.79 Second, Null}
```

We can check this result using the general purpose routine from Chapter 2 that integrates the fundamental transform directly:

```
cval[S_,K_,r_,V0_,tau_,ki_,omega_,theta_,ksi_,
          rho_,gam_,days_,pflag_,WP_,AG_,rflag_]
```

The result is

```
Timing[cval[100, 100, 0, 1, 1, 1/2, 0, 0, 1,
                              0, 1, 1, 1, 16, 6, 1]]

kmax=1000. tau=1. S=100 K=100. Call=36.4791 B-S=38.2925
  Imp sig=94.881723 %

{0.22 Second, 36.4791449}
```

The mixing theorem result of 36.48 is confirmed. Of course, if you begin the way we did with a known expression for the fundamental transform, it's a lot faster just to integrate that known expression directly to get an option value. This is shown by the relative timings of 0.22 sec vs. 48 secs for the two evaluations. But the example has shown that everything works out correctly and gives you a general picture of what the density for the integrated volatility looks like.

7 Example II: Options under Geometric Volatility

In general, it's difficult to carry out the mixing theorem prescriptions analytically, even when $\rho = 0$. That's because the integrated volatility density is usually difficult to compute in closed-form. Even in the last section, we relied upon a numerical Laplace transform inversion. But one interesting model where a closed-form result can be obtained is the geometric volatility model. Under that model, the risk-adjusted process $\tilde{P}$ is given by

$$\text{(7.1)} \qquad \tilde{P}: \begin{cases} dS_t = r\,S_t\,dt + \sigma_t S_t\,d\tilde{B}_t^1 \\ dV_t = \alpha V_t\,dt + \xi V_t\,d\tilde{B}_t^2 \end{cases}$$

where (α, ξ) are both constants, and the Brownian motions are independent. For simplicity, we assume no dividends. We simply show some results in this section to illustrate what explicit mixing formulas can look like. For proofs and more about the geometric volatility model, see Chapter 11. The call option value is given by mixing and the integrated volatility density is given by the following theorem:

THEOREM 5. *Assume the volatility process* $dV_t = \alpha V_t\, dt + \xi V_t dW_t$, *where* α *and* ξ *are constants. Let* $\mu = 1-(2\alpha/\xi^2)$ *and* $\lambda_j = -j^2 + \mu j$. *Also let* $1_{\mu>0} = 1$ *when* $\mu > 0$ *and* 0 *otherwise, and* $[x] =$ *integer part. Then, the integrated volatility density is given by*

$$(7.2)\quad Z\,P(Z, V_0, \tau) = 1_{\mu>0} \sum_{j=0}^{[\mu/2]} \frac{(\mu-2j)y^{\mu-j}e^{-y}}{\Gamma(1+\mu-j)} L_j^{(\mu-2j)}(y) \exp(-\tfrac{1}{2}\lambda_j \xi^2 \tau)$$

$$+\frac{e^{-y}}{4\pi^2}\int_0^\infty \left|\Gamma\left(-\frac{\mu}{2}+i\frac{\nu}{2}\right)\right|^2 (\nu \sinh \nu\pi)\, y^{\mu/2-i\nu/2}$$

$$\times\, U\left(-\frac{\mu}{2}+i\frac{\nu}{2},\, 1+i\nu, y\right)\exp\left[-\tfrac{1}{8}(\mu^2+\nu^2)\xi^2\tau\right] d\nu$$

where $y = 2V_0/Z\,\xi^2$, $\Gamma(z)$ *is the Gamma function,* $L_n^{(a)}(y)$ *is an associated Laguerre polynomial, and* $U(a,b,y)$ *is a confluent hypergeometric function.*

Notation: we used Z as the first argument of the density $P(Z, V_0, \tau)$ to avoid confusion with the special function $U(a,b,y)$.

Somewhat simpler formulas follow when the volatility drift vanishes completely ($\alpha = 0$). In that case, (7.2) reduces to

(7.3)

$$Z\,P(Z,V_0,\tau) = y e^{-y} - \frac{2}{\pi^{3/2}}\int_0^\infty \frac{\nu \sinh\left(\frac{1}{2}\nu\pi\right)}{1+\nu^2} f(y,\nu)\exp\left[-\tfrac{1}{8}(1+\nu^2)\xi^2\tau\right] d\nu$$

using $$f(y,\nu) = \frac{d}{dy}\left[y^{1/2}\exp\left(-\tfrac{1}{2}y\right)K_{i\nu/2}\left(\tfrac{1}{2}y\right)\right].$$

where $K_\nu(y)$ is a modified Bessel function.

In certain parameter ranges, the formulas above lead to option values that can differ significantly from B-S values. For example, consider the behavior of the integrated volatility density in (7.3) at large times. The integral vanishes in this limit, leaving, as $\tau \to \infty$,

$$P(Z,V_0,\tau) \approx \frac{2V_0}{Z^2\xi^2}\exp\left(-\frac{2V_0}{Z\xi^2}\right) = P_\infty(Z,V_0) \tag{7.4}$$

Substituting (7.4) into (6.1) provides, as we now show, an example of a completely explicit mixing theorem calculation:

Mixing theorem example (explicit). Assume the risk-adjusted process

$$\tilde{P}: \quad \begin{cases} dS_t = \sigma_t S_t \, d\tilde{B}_t^1 \\ dV_t = \xi V_t \, d\tilde{B}_t^2 \end{cases},$$

where $V_t = \sigma_t^2$, ξ is a constant, and the Brownian motions are independent. Let $C(S,K,V,\tau)$ be the value of a call option striking at K with τ periods to expiration. Then, from mixing, as $\tau \to \infty$,

(7.5) $\quad C(S,K,V,\tau) \to C_\infty(S,K,V) =$

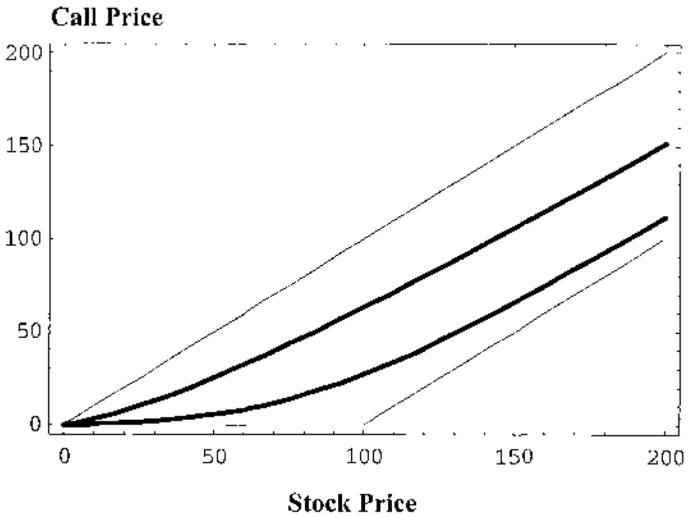
$$\frac{2V}{\xi^2}\int_0^\infty \left\{ S\Phi\left(\frac{[Log(S/K)+Z/2]}{\sqrt{Z}}\right) - K\Phi\left(\frac{[Log(S/K)-Z/2]}{\sqrt{Z}}\right)\right\} \exp\left(-\frac{2V}{Z\xi^2}\right)\frac{dZ}{Z^2}$$

$$= S - \sqrt{SK}\exp\left(-\left[\frac{1}{4}Log^2\left(\frac{S}{K}\right) + \frac{V}{\xi^2}\right]^{1/2}\right).$$

Note that $C_\infty(S,K,V)$ is strictly less than the stock price. In the B-S model, as the time to expiration grows large, the option price becomes the stock price for any non-negative interest rate and positive volatility. This new behavior is caused by the volatility drift toward the origin. Fig. 4.1 below plots the call value in (7.5) versus the stock price with $K = 100$ and $V/\xi^2 = 0.1$ (lower bold curve) and $V/\xi^2 = 1$ (upper bold curve).

In the example, the B-S implied volatility at $\tau = \infty$ is zero. If you developed the B-S implied volatility versus the time to expiration τ, you would find that it would decrease versus τ. This effect was seen in Monte Carlo studies of this model by Hull and White (1987) and explains results they found surprising.

Fig. 4.1 Call Price under Geometric Brownian Motion as $\tau \to \infty$



Notes. The figure shows the call option price under a stochastic volatility model (geometric Brownian motion) as the time to expiration tends to infinity. In the B-S model, the call price tends to the stock price (upper thin line) in this limit. When the volatility follows geometric Brownian motion, it can ultimately drift toward zero, causing the asymptotic call value to fall below the stock price, shown as the bold lines above. The result was derived from the mixing theorem.

5 The Smile

The B-S implied volatility σ^{imp} is the value of σ that equates a B-S option price to a market price (or a model price). The *smile* is the relation between σ^{imp} and the strike price K. But, we shall loosely let the smile mean any relationship between an implied volatility measure (such as V^{imp}) and the strike price. Sometimes the smile is called the volatility *skew*. Sometimes, instead of K, the independent variable is a dimensionless ratio. Often this ratio is the *moneyness* S/K, where S is the underlying security price. Another dimensionless ratio that is natural, and we use in this chapter, turns out to be

$$X = \ln\left[\frac{Se^{-\delta\tau}}{Ke^{-r\tau}}\right] ,$$

where δ is the dividend yield on the underlying security, and r is the short-term interest rate.

When the volatility process and the stock price process are uncorrelated, the smile typically looks like a parabola. In the general case, the processes are correlated, and the implied volatility can be expressed

$$V^{imp} = \beta_0 + \beta_1 X + \beta_2 X^2 + \cdots .$$

In this chapter, we explain how to calculate the β_i coefficients for general volatility processes, using results from Chapters 3 and 4.

1 Introduction and Summary of Results

Some smile plots. Previously in Chapter 4, Table 4.1 we estimated option prices under the GARCH diffusion model $dV_t = (\omega - \theta V_t)dt + \xi V_t dW_t$. Shown in Fig. 5.1 are some plots from Table 4.1: σ^{imp} versus the strike price. These graphs, due to the look of the $\rho = 0$ case, have become known as smile patterns[1]. If volatility were constant, the smile pattern for these data would be a horizontal line with $\sigma^{imp} = 15\%$ for every ρ.

The smile graphs shown are typical of the general pattern: with $\rho < 0$, and assuming the strike price and stock price are close ("near-the-money"), the smile has a negative slope. When $\rho = 0$, it looks like a parabola near-the-money. When $\rho > 0$, the near-the-money plot is approximately linear with a positive slope.

Shown in Fig. 5.2 are the corresponding smile plots based upon the volatility of volatility series—specifically, the V^{imp} Series II formula developed in Chapter 3. As one can see from the figure, the theoretical results are very close to the simulated results. To see how close, in Fig. 5.3 we have superimposed the expansion formula and simulated results for two cases; as one sees, the graphs are virtually identical.

In the next two figures, we show the typical time dependence of the smile based upon the same theoretical formulas. In Fig. 5.4 (top), we show the smile for the $\rho = 0$ case plotted for $\tau =$ 1 month, 1 quarter, 1 year, and 5 years. As one can see, the smile *flattens* as τ increases. Another view of the flattening is Fig. 5.4 (bottom) which shows a 3D Plot of the continuous variation from $\tau = 0.1$ years to $\tau = 2$ years. Again the flattening is evident. The flattening occurs for $\rho \neq 0$ also. In fact, the general situation is that, for fixed moneyness X, then $V^{imp}(X, V, \tau) \to V_\infty^{imp}$ as $\tau \to \infty$, where the asymptotic volatility V_∞^{imp} is independent of *both* the moneyness X and the initial volatility V. This is

---

[1] Frequently, smile plots based on market prices will use exclusively out-of-the-money options: put prices for strikes below the market ($K < S$) and call prices for strikes above the market. When put/call parity holds, which it does for the theoretical models in this book, it doesn't matter if puts or calls are used for the smile.

Figure 5.1 Smile Patterns for the GARCH Diffusion Monte Carlo method using the mixing theorem

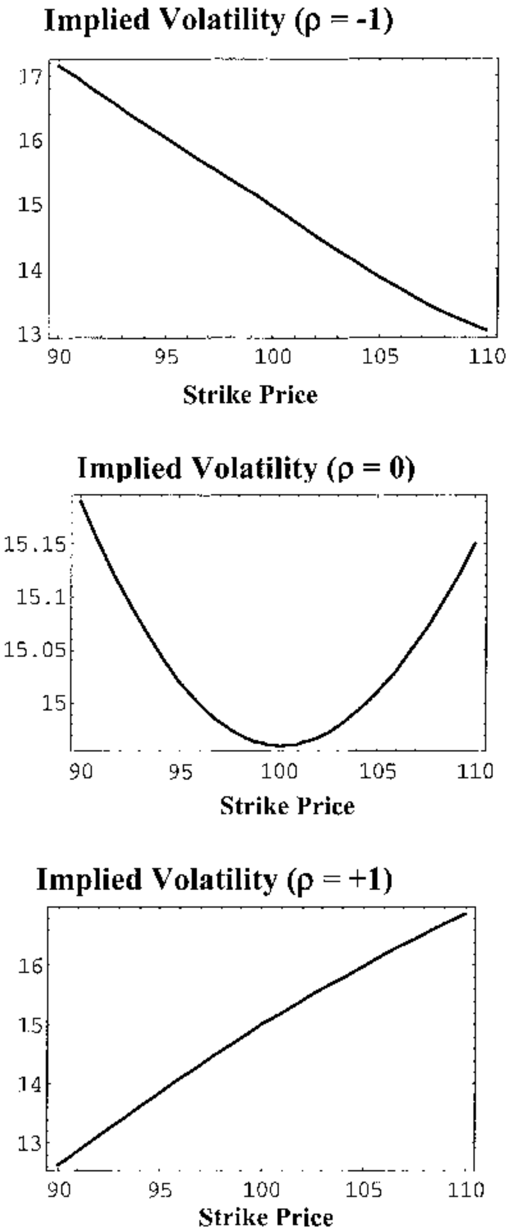


Figure 5.2 Smile Patterns for the GARCH Diffusion: Volatility of volatility expansion at 2nd order

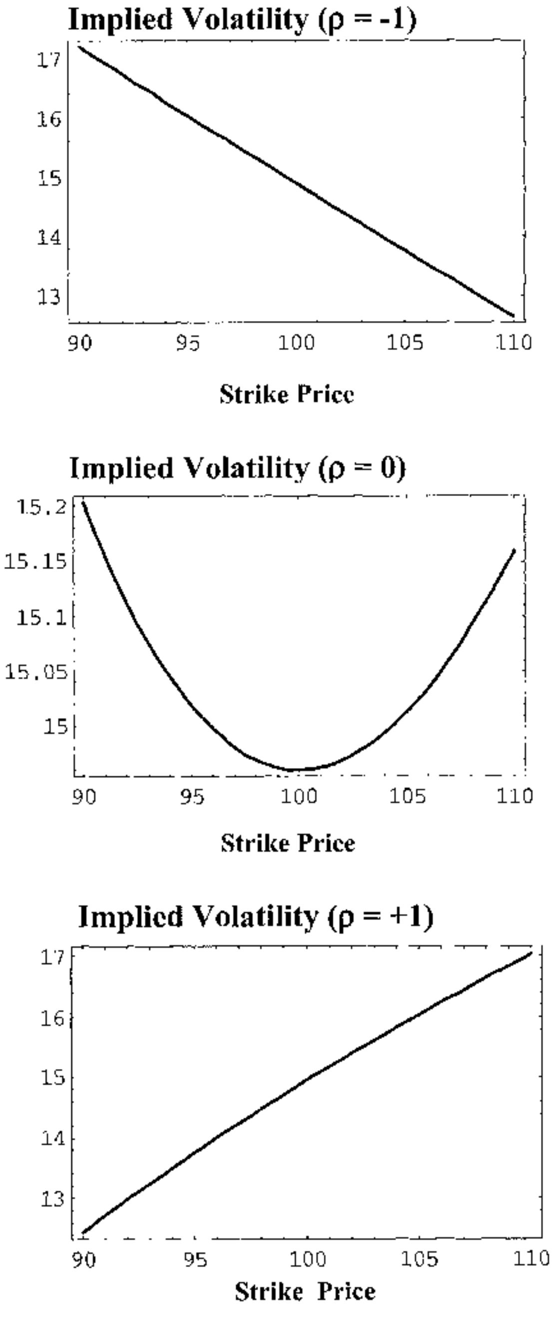


Figure 5.3 Smile Patterns for the GARCH Diffusion: Monte Carlo and volatility of volatility expansion superimposed

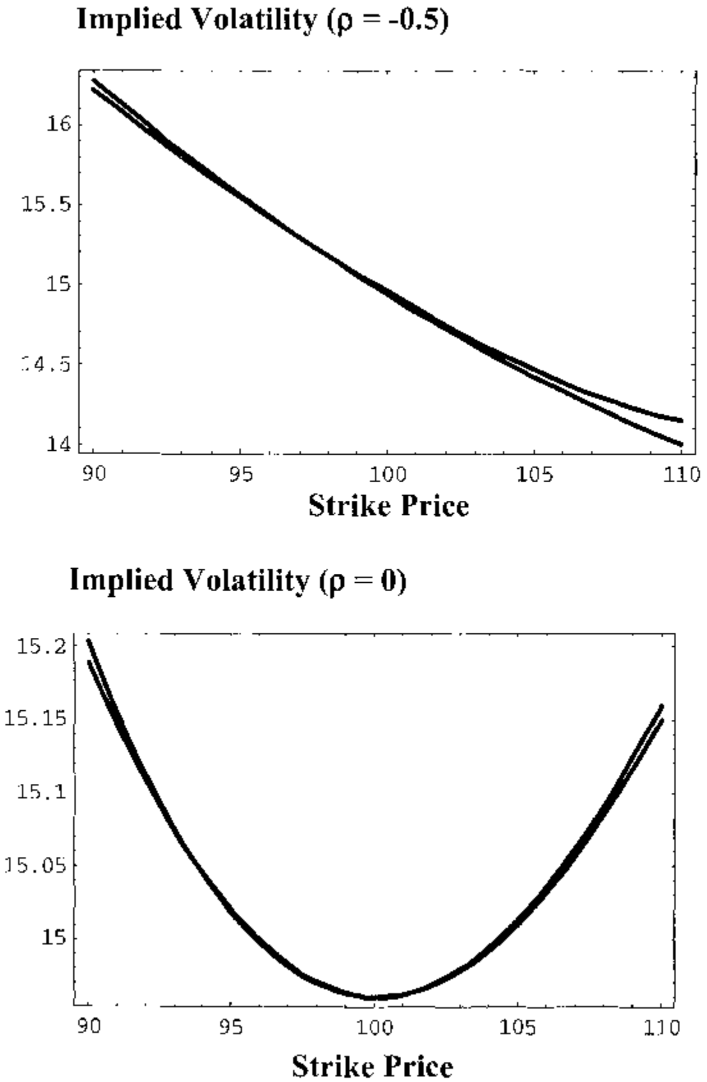
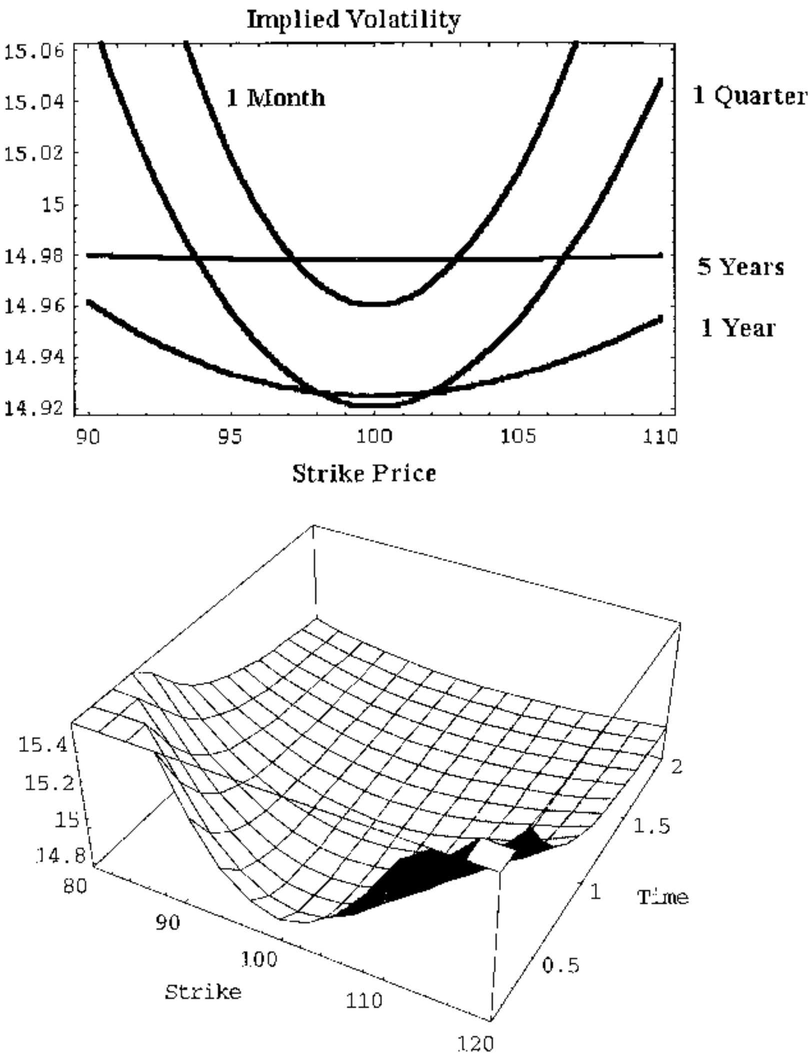


Fig. 5.4 Flattening of the Smile vs Time to Expiration



Notes. Both figures illustrate the flattening of the smile versus time to expiration. The top figure plots the implied volatility for the GARCH diffusion model described in Table 4.1. The volatility of volatility expansion is used. The bottom figure is the same, except the parameter $\xi = 1.5$.

explained briefly in Section 2.1 and proved in the next chapter, "The Term Structure of Implied Volatility".

The smile is determined by the risk-adjusted or martingale pricing process, so we need a model of that. Through this chapter, we employ our usual model:

$$\tilde{P}: \begin{cases} dS_t = (r-\delta)S_t dt + \sigma_t S_t d\tilde{B}_t \\ dV_t = \tilde{b}(V_t)dt + a(V_t)d\tilde{W}_t \end{cases} \tag{1.1}$$

where $d\tilde{B}_t$ and $d\tilde{W}_t$ are correlated Brownian motions, with correlation $\rho(V_t)$.

Under process $\tilde{P}$ of (1.1), a European-style call option expiring in τ periods with strike price K has the fair value $C(S,V,\tau)$. As usual, we denote the B-S formula with the same arguments by a small letter: $c(S,V,\tau)$. Then, the implied volatility V^{imp} is defined as the solution to the equation $c(S,V^{imp},\tau) = C(S,V,\tau)$. The implied volatility is a function not only of the explicit parameters we have displayed, namely S, V, and τ, but is also a function of all the implicit ones: the strike K, the interest rate r, and the dividend yield δ. In addition, if the risk-adjusted volatility process is parameterized, then the implied volatility depends upon those parameters also.

Summary of results. So the reader can see where we are headed, we first state our central results, which are all dependent upon Assumption 1. Many of the results are merely applications of Series II for V^{imp} developed in Chapter 3.

(I) The implied volatility V^{imp} depends upon S, K, r, and δ only through $X = \ln(S/K) + (r-\delta)\tau$.

(II) Parameterize the volatility process by a constant scale parameter ξ, letting $dV_t = \tilde{b}(V_t)dt + \xi\eta(V_t)dW_t$. Then, the implied volatility has the formal power series

$$V^{imp}(X,V,\tau) = v(V,\tau) + g_1(X,V,\tau)\xi + g_2(X,V,\tau)\xi^2 + \cdots,$$

where $v(V,\tau)$ is the time-average of the deterministic volatility (the volatility when $\xi = 0$).

(III) We have calculated the coefficients in (II) through $O(\xi^2)$ for a general stationary process. The result is

$$V^{imp}(X,V,\tau) = \beta_0(V,\tau) + \beta_1(V,\tau)X + \beta_2(V,\tau)X^2 + O(\xi^3).$$

Explicit expressions for the β_i are given in Section 3.2. We consider the formulas for the β_i our most useful result for three reasons. First, the results are very general, since they apply to any time-homogeneous volatility process that is stock-price independent. Second, they are simple to evaluate numerically as we will show. Finally, the results through $O(\xi^2)$ are probably sufficient for most trading applications, at least for stationary models. That's because further $O(\xi^3)$ corrections are typically very small for option expirations under a couple of years and $|X|$ not too large.

(IV) If $\rho = 0$, the coefficients of odd powers of X vanish. That is, V^{imp} is symmetric about $X = 0$.

(V) The expiration date limit defined by $V_e^{imp}(X,V) = \lim_{\tau\to 0} V^{imp}(X,\tau,V)$ exists and is a non-trivial functional of the volatility process coefficients. In particular, for a general process $dV_t = \tilde{b}(V_t)dt + \xi\eta(V_t)dW_t$ then

$$V_e^{imp}(X,V) = V - \tfrac{1}{2}\rho\xi\,\frac{\eta}{\sqrt{V}}X + \xi^2\left[\left(\tfrac{1}{12} - \tfrac{11}{48}\rho^2\right)\frac{\eta^2}{V^2} + \tfrac{1}{6}\frac{1}{V}(\rho\eta)\frac{d}{dV}(\rho\eta)\right]X^2 + O(\xi^3),$$

where $\eta = \eta(V)$ and $\rho = \rho(V)$. In principle, this formula provides a non-parametric method of estimating $\eta(V)$ and $\rho(V)$ from market prices of options (see V)

(VI) Some conjectures. It seems likely that the explicit terms in (V) are not corrected at $O(\xi^3)$. In other words, the coefficients of X and X^2 that are shown in the last equation are exact: the $O(\xi^3)$ terms contribute X^3 and higher corrections, but do not alter the ones shown. If this conjecture is true, then it implies that the at-the-money slope and curvature of the expiration date smile are *preference-free;* i.e. independent of the drift which is where preference effects appear. For example, with $dV_t = \tilde{b}(V_t)dt + a(V_t)dW_t$, the at-the-money, expiration date slope is given by

$$\left.\frac{\partial V_e^{imp}}{\partial X}\right|_{X=0,V=V_t} = -\tfrac{1}{2}\rho(V_t)\,\frac{a(V_t)}{\sqrt{V_t}}.$$

This provides a preference-free and simple direct relation between option prices and the volatility process coefficients

These results are discussed in Sections 2 and 3. Section 2 is devoted to the symmetric case ($\rho = 0$). In that section, especially Sec 2.2, the focus is on the expiration date smile. Section 3 is more general; we handle the general correlated case for any time. Our method in that section is to apply the volatility of volatility expansion, which was already developed in Chapter 3. Appendix 2 includes selected Mathematica code used in this chapter, including the code used to produce Figs 5.1, 5.2, and 5.3.

2 The Symmetric Case

When $\rho = 0$, why does the near-the-money smile look like a parabola? Although we already have an expression for the smile from Chapter 3, it's interesting to see that the mixing theorems of Chapter 4 also provide an answer. The first mixing theorem, (4.1.5), can be used to develop an approximate option price under stochastic volatility by a Taylor series expansion.[2]

2.1 A simple mixing theorem computation.

For simplicity, denote expectations under the risk-adjusted volatility process by $\langle \cdots \rangle$, and define a time-average volatility $\tilde{v} = \int_0^\tau V(s)ds/\tau$, where τ is the time to expiration. The tilde on $\tilde{v}$ is used to emphasize that $\tilde{v}$ is a random variable, to be distinguished from the time-average of the deterministic volatility v, which is not a random variable. With this notation, and freely dropping the τ -- dependence, the mixing solution when $\rho = 0$ is $C(S,V) = \langle c(S,\tilde{v}) \rangle$.

Denote the mean and variance, respectively, of the time-average volatility by $\mu_v = \mu_v(\tau, V_0) = \langle \tilde{v} \rangle$ and $\sigma_{\tilde{v}}^2 = \sigma_v^2(\tau, V_0) = \langle (\tilde{v} - \mu_v)^2 \rangle$. The mixing solution can be expanded:

$$C(S,V) = \langle c(S,\tilde{v}) \rangle = c(S,\mu_v) + \tfrac{1}{2}\sigma_v^2 \frac{\partial^2 c}{\partial V^2} + \cdots \tag{2.1}$$

[2] The volatility moment expansion for the symmetric case was developed by Hull and White (1987). The mixing theorem computations of (2.1)-(2.5) and the idea of the normalized smile height in Sec. 2.3 is due to Taylor and Xu (1994).

Now by definition, the B-S implied volatility is given by $C(S,V)=c(S,V^{imp})$ and this formula can also be expanded:

(2.2) $$C(S,V)=c(S,V^{imp})=c(S,\mu_v)+(V^{imp}-\mu_v)\frac{\partial c}{\partial V}+\cdots$$

Equating these last two expressions, truncated as shown, yields an approximate expression for the implied volatility:

(2.3) $$V^{imp}\approx\mu_v+\tfrac{1}{2}\sigma_v^2\left(\frac{\partial^2 c}{\partial V^2}/\frac{\partial c}{\partial V}\right).$$

The BS formula derivatives in (2.3) are to be evaluated at (S,μ_v). A straightforward calculation with the B-S formula, using $X=\ln(S/K)+(r-\delta)\tau$, yields

(2.4) $$\frac{\partial^2 c}{\partial V^2}/\frac{\partial c}{\partial V}=\tfrac{1}{2}\frac{X^2}{V^2\tau}-\tfrac{1}{2}\frac{1}{V}-\tfrac{1}{8}\tau\,.$$

That is, (2.3) is a quadratic approximation for the smile:

Quadratic Approximation

(2.5) $$V^{imp}(X,\tau,V_0)\approx\mu_v+\tfrac{1}{4}\frac{\sigma_v^2}{\mu_v^2}\left(\frac{X^2}{\tau}-\mu_v-\tfrac{1}{4}\mu_v^2\tau\right).$$

Equation (2.5) shows that both V^{imp} and σ^{imp} have a parabolic shape vs. the moneyness X for small enough X. The source of the X^2 dependence is (2.4), a formula involving only the B-S formula. There are two interesting limits to this formula: $\tau\to 0$ and $\tau\to\infty$. We will consider each in turn.

The $\tau\to 0$ limit. This limit is interesting because it is not trivial and, in practice, the smile is often reported relatively close to the option expiration. As $\tau\to 0$, we show in Appendix 5.1 that for general processes, $\mu_v=V_0+O(\tau)$ and $\sigma_v^2=a^2(V_0)\tau/3+O(\tau^2)$, where $a(V_t)$ is the volatility process diffusion coefficient. So the second and third terms in the parentheses in (2.5) vanish in this limit. Switching to σ^{imp}, which is usually plotted, then both close-to-expiration and close-to-the-money, we have:

$$\sigma^{imp}\underset{\tau\to 0}{\approx}\sigma_0\left(1+\tfrac{1}{12}\frac{a^2(V_0)}{V_0^3}X^2\right)^{1/2}\underset{\tau,X\to 0}{\approx}\sigma_0\left[1+\tfrac{1}{24}\frac{a^2(V_0)}{V_0^3}X^2+O(X^4)\right].$$

The $\tau\to\infty$ limit. In this limit, we also show in Appendix 5.1 that for general processes, typically (i) $\mu_v(\tau,V_0)\to A$ and (ii) $\sigma_v^2(\tau,V_0)\sim B/\tau$, where A

and B are two constants *independent of* τ *and* V_0. (And, of course, independent of X also, since the volatility process is independent of the stock price). So in this limit, (2.5) becomes

$$V^{imp}(X,\tau,V_0) \underset{\tau\to\infty}{\approx} A - \tfrac{1}{16}B + O(\tau^{-1}) = V_\infty^{imp} + O(\tau^{-1}). \tag{2.6}$$

In other words, for fixed X, the smile flattens to a constant V_∞^{imp} that is *independent of* X, τ, *and* V_0. For example, in Appendix 5.1, we consider the specific model $dV = (\omega - \theta V)dt + \xi V^\varphi dW(t)$, where ω, θ, ξ, and φ are constant parameters, and recall that $\rho = 0$. Then we show that

$$A = \frac{\omega}{\theta} \quad \text{and} \quad B = \left(\frac{\omega}{\theta}\right)^{2\varphi} \frac{\xi^2}{\theta^2} + O(\xi^4),$$

so (2.6) is equivalent to

$$\sigma^{imp}(X,\tau,V_0) \underset{\tau\to\infty}{\approx} \left[\frac{\omega}{\theta} - \frac{1}{16}\left(\frac{\omega}{\theta}\right)^{2\varphi} \frac{\xi^2}{\theta^2} + O(\xi^4)\right]^{1/2} + O(\tau^{-1})$$

With the exception of Appendix 5.1, the $\tau \to \infty$ limit is not discussed further in this chapter; it is discussed in detail in Chapter 6.

2.2 The smile is symmetric when $\rho = 0$

Here's a series expansion proof that the smile is symmetric in the moneyness X when $\rho = 0$. Consider all the terms of the formal expansion that results from equating (2.1) and (2.2):

$$(V^{imp} - \mu_v)\frac{\partial c}{\partial V} + \frac{1}{2!}(V^{imp} - \mu_v)^2 \frac{\partial^2 c}{\partial V^2} + \cdots$$
$$= \frac{1}{2!}\left\langle (\tilde{v} - \mu_v)^2 \right\rangle \frac{\partial^2 c}{\partial V^2} + \frac{1}{3!}\left\langle (\tilde{v} - \mu_v)^3 \right\rangle \frac{\partial^3 c}{\partial V^3} + \cdots,$$

where all the derivatives of the B-S formula are to be evaluated $V = \mu_v$. Divide both sides by $(\partial c / \partial V)$ and solve for $(V^{imp} - \mu_v)$. The resulting equation, although formal, has only two types of terms: (i) moments $\left\langle (\tilde{v} - \mu_v)^m \right\rangle$ and (ii) volatility derivative ratios $(\partial c^n / \partial V^n)/(\partial c / \partial V)$. Because the volatility process is independent of the stock price, the volatility moments do not depend upon the moneyness X. (Examples of volatility moments are shown in Table 5.2). The only dependence upon X is from the derivative ratios. Derivative ratios are shown in Table 5.3 as the $(n,0)$ entries of a more general expression. As one sees from the table, the $(n,0)$ entries are all even functions of X. This can be proved for all n by an induction argument. Hence the (formal expansion of the) smile must be an even function of X. ■

2.3 The expiration date smile (no drift)

In Sec. 2.1, we developed the quadratic approximation for the smile. How accurate is it? What are the corrections? The simplest way to answer these questions is to first consider the limit as $\tau \to 0$. We saw that the quadratic approximation for the smile had a $\tau \to 0$ limit, and it turns out that expressions for the smile that go beyond the quadratic approximation also have that limit.

I. The leading correction to the quadratic approximation. Assume the risk-adjusted volatility process has no drift, so that $dV_t = a(V_t)dW_t$ and is uncorrelated with the stock price process. Then, using the mixing theorem, we find that, for a general function $a(V_t)$, the expiration date smile is given by

Expiration date smile ($\rho = \bar{b} = 0$):

$$(2.7)\quad V_e^{imp}(X,V_0) = V_0 + \left\{\frac{1}{12}\frac{a_0^2}{V_0^2}\right\}X^2 + \left[\frac{1}{60}\left(\frac{a_0' a_0^3}{V_0^4}\right) - \frac{1}{48}\left(\frac{a_0^4}{V_0^5}\right)\right]X^4 + O(X^6),$$

using $a_0 = a(V_0)$ and $a_0' = da(V_0)/dV$.

Example 1. Consider the process $dV_t = \xi V_t^{\varphi} dW_t$. Then (2.7) becomes

$$(2.8)\quad V_e^{imp}(X,V_0) = V_0 + \frac{1}{12}\xi^2 V_0^{2\varphi-2}X^2 + \left(\frac{\varphi}{60} - \frac{1}{48}\right)\xi^4 V_0^{4\varphi-5}X^4 + O(X^6).$$

Calculation details are given in Appendix 5.4. As you can see, the next correction to the quadratic has a negative sign if $\varphi < 5/4$ and a positive sign if $\varphi > 5/4$ (The GARCH diffusion has $\varphi = 1$ and the square root model has $\varphi = 1/2$).

II. Further corrections to the quadratic approximation. To see what further corrections look like, we have automated the calculations for the model $dV = \xi V^{\varphi} dW$, with constant correlation ρ with the stock price process. The Mathematica code for this is given in Appendix 5.2. For this model, we find that the expiration date smile has the following expression:

$$V_e^{imp} = V_0\left[1 + c_1 y + c_2 y^2 + c_3 y^3 + \cdots\right], \qquad y = \xi X V_0^{\varphi-(3/2)},$$

where the coefficients depend upon ρ and φ: $c_i = c_i(\rho,\varphi)$. Dimensional considerations alone would lead to this form except that the c_i could depend upon X also, and this turns out not to be the case. When $\rho = 0$, which is the only case we consider in this section, only the even powers are present, and one can use $z = y^2 = \xi^2 X^2 V_0^{2\varphi-3}$. (For results when $\rho \neq 0$, see Sec. 3) .

Example 1. Consider the GARCH diffusion, where $dV_t = \xi V_t \, dW_t$ and $\rho = 0$. Then $\varphi = 1$ and we use $z = (\xi^2 X^2 / V_0)$. We find the series:

$$(2.9) \quad V_e^{imp}(X,V_0) = V_0 \times\left(1 + \tfrac{1}{12} z - \tfrac{1}{240} z^2 + \tfrac{31}{60480} z^3 - \tfrac{289}{3628800} z^4 + \tfrac{317}{22809600} z^5 - \tfrac{6803477}{2615348736000} z^6 + O(z^7)\right)$$

Example 2. Consider the square root model, where $dV_t = \xi \sqrt{V_t} \, dW_t$ and $\rho = 0$. Then $\varphi = 1/2$ and we use $z = (\xi^2 X^2 / V_0^2)$. We find the series:

$$(2.10) \quad V_e^{imp}(X,V_0) = V_0 \times\left(1 + \tfrac{1}{12} z - \tfrac{1}{80} z^2 + \tfrac{113}{30240} z^3 - \tfrac{5171}{3628800} z^4 + \tfrac{7057}{11404800} z^5 - \tfrac{381323707}{1307674368000} z^6 + O(z^7)\right)$$

Example 1 (continued) Consider options striking 10% away from the money on an underlying security with an annualized 10% volatility. That is, $\sigma_0 = X = 0.10$. A normalized smile height is then given by the ratio

$$\frac{\sigma_e^{imp}(X=0.1)}{\sigma_e^{imp}(X=0)} = \left(1 + \tfrac{1}{12} z - \tfrac{1}{240} z^2 + \cdots\right)^{1/2}, \text{ where } z = \xi^2 .$$

For broad-based equity indices, such as the S&P 500 index, typical values for ξ are in the range 1-2 on an annualized basis. Roughly, this means that the instantaneous variance a year from today, given today's variance, is uncertain by 100%-200% . (We are ignoring the mean reverting drift, which in reality acts to reduce this uncertainty). If $\xi = 1$ then $z = 1$, but if $\xi = 2$ then $z = 4$. Using (2.9) but truncating the series at $O(z), O(z^2)$, etc., the normalized smile height has the partial sums shown in Table 5.1

Table 5.1 Partial sums for $\sigma_e^{imp}(X=0.1)/\sigma_e^{imp}(X=0)$ through $O(z^n)$ in the GARCH diffusion model (no drift).

ξ	n: 1(quad)	2	3	4	5	6
1	1.0408	1.0388	1.0391	1.0390	1.0390	1.0390
1.5	1.0897	1.0800	1.0827	1.0818	1.0821	1.0820
2	1.1547	1.1255	1.1399	1.1310	1.1372	1.1326

The series seems quite well-behaved for these typical parameter values. Notice that for $z=1$ $(\xi=1)$, the quadratic approximation (the $n=1$ entry) is extremely accurate.

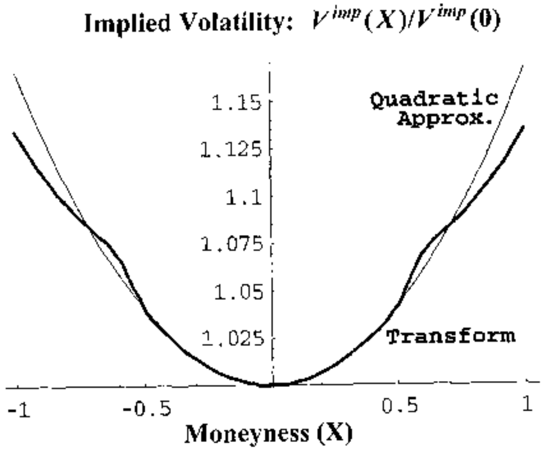
Example 2 (continued). For the square root model, we can solve for the smile "exactly", at least up the errors of a numerical k-plane integration. A problem with that calculation is that you can't really take $\tau=0$, but only a small value. For a numerical example, take $\xi^2=2$, $V_0=1$, and moneyness $|X|\leq 1$. Fig. 5.5 (top) compares the quadratic approximation to the "exact" transform-based results, where the transform is based on $\tau=1/50$. The bottom panel shows the quadratic approximation versus the series (2.10). As you can see the quadratic approximation is very accurate unless you are considering unusually large values for the moneyness. That is, remember $X=\ln(S/K)$, so that $X=\pm 1$ means $S/K=e\simeq 2.7$ or $S/K=1/e\simeq 0.37$.

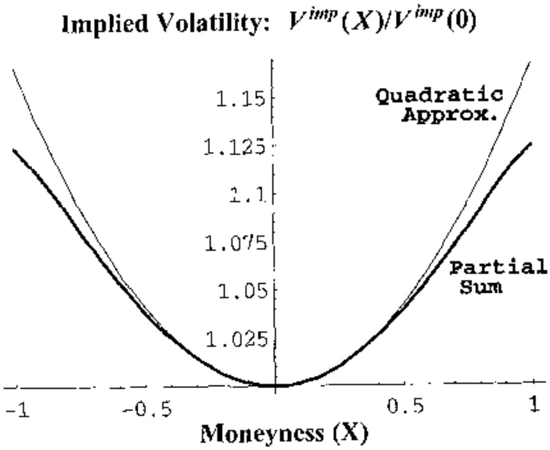
Conclusions. The results in this section suggest that, when the correlation is zero, and the time to expiration is small, the quadratic approximation is a very good one for practical values of the moneyness.[3] The controlling parameter, for models of the form $dV=\xi V^{\varphi}dW$ is actually $z=\xi^2 X^2 V^{2\varphi-3}$. The quadratic approximation can probably be used without any practical error as long as $z\leq 1$, again with $\rho=0$ and τ very small. For example, the value $z=1$ corresponds to the first line of Table 5.1 and $X\cong 0.7$ in Fig. 5.5. You can see from the table and the figure how accurate the quadratic approximation is for those values.

---

[3] For an example of a far-from-the-money analysis, where $|X|\to\infty$, see Zhu and Avellaneda(1998).

Fig. 5.5 Close-to-expiration Normalized Smiles: Square Root Model





Notes. Normalized smiles in the square root model. Parameters: no drift, no correlation, $V_0 = 1$, $\xi^2 = 2$. The "Partial Sum" is 6 terms of an expiration date smile series in powers of $z = (\xi X / V_0)^2$. The "Quadratic Approx." is the same series through order X^2. The "Transform" is the exact k-plane inversion at $\tau = 1/50$. The figures show the quadratic approximation is very accurate unless $z \geq 2$.

Table 5.2 Moments of the time-average volatility

m	$\left\langle (v - V_0)^m \right\rangle$
2	$\frac{1}{3} a_0^2 \tau + O(\tau^2)$
3	$\frac{2}{5} a_0' a_0^3 \tau^2 + O(\tau^3)$
4	$\frac{1}{3} a_0^4 \tau^2 + O(\tau^3)$
m (even)	$\left(\frac{1}{3}\right)^{m/2} (m-1)!!\, a_0^m\, \tau^{m/2}\, [1+O(\tau)]$
m (odd)	$\frac{1}{5}\left(\frac{1}{3}\right)^{(m-1)/2} (m-1)(m!!)\, a_0' a_0^m\, \tau^{(m+1)/2}\, [1+O(\tau)]$

Notes. The table shows the leading terms, as $\tau \to 0$, for the moments of the time-average volatility $v = \tau^{-1} \int_0^\tau V(s) ds$ under the process $dV_t = a(V_t) dW_t$. The notation $a_0 = a(V_0)$, $a_0' = da(V_0)/dV$, etc. is used. Also for m odd, $m!! = m(m-2)\cdots 1$. See Appendix 5.1 for some calculation details.

Table 5.3 Normalized derivatives of the Black-Scholes formula

n,m	$R^{(n,m)} = \left[\left(\frac{\partial}{\partial V} \right)^n \left(S \frac{\partial}{\partial S} \right)^m c \right] / \frac{\partial c}{\partial V}$
2,0	$\tau \left[\frac{1}{2} \frac{X^2}{Z^2} - \frac{1}{2Z} - \frac{1}{8} \right]$
3,0	$\tau^2 \left[\frac{1}{4} \frac{X^4}{Z^4} - \frac{1}{8} \frac{X^2}{Z^3} (12 + Z) + \frac{1}{64} \frac{1}{Z^2} (48 + 8Z + Z^2) \right]$
4,0	$\tau^3 \left[\frac{1}{8} \frac{X^6}{Z^6} - \frac{3}{32} \frac{X^4}{Z^5} (20 + Z) + \frac{3}{128} \frac{X^2}{Z^4} (240 + 24Z + Z^2) - \frac{1}{512} \frac{1}{Z^3} (960 + 144Z + 12Z^2 + Z^3) \right]$
n,0	$\tau^{n-1} \left\{ \frac{X^{2n-2}}{Z^{2n-2}} A_{2n-2}^{(n)} + \cdots + \frac{X^{2j}}{Z^{n-1+j}} \left[A_{2j}^{(n)} + O(Z) \right] + \cdots + \frac{X^2}{Z^n} \left[A_2^{(n)} + O(Z) \right] + \frac{X^0}{Z^{n-1}} \left[A_0^{(n)} + O(Z) \right] \right\}$
1,1	$\left[-\frac{X}{Z} + \frac{1}{2} \right]$
1,2	$\left[\frac{X^2}{Z^2} - \frac{X}{Z} - \frac{1}{4Z} (4 - Z) \right]$
2,1	$\tau \left[-\frac{1}{2} \frac{X^3}{Z^3} + \frac{1}{4} \frac{X^2}{Z^2} + \frac{1}{8} \frac{X}{Z^2} (12 + Z) - \frac{1}{16} \frac{1}{Z} (4 + Z) \right]$
2,2	$\tau \left[\frac{1}{2} \frac{X^4}{Z^4} - \frac{1}{2} \frac{X^3}{Z^3} - 3 \frac{X^2}{Z^3} + \frac{1}{8} \frac{X}{Z^2} (12 + Z) + \frac{1}{32} \frac{1}{Z^2} (48 - Z^2) \right]$

Notes. The table shows the indicated derivative ratios of the B-S formula, where the volatility $V = \sigma^2$ and $X = \ln[S/K] + (r - \delta)\tau$. When the volatility drift is zero, these expressions are used with the substitution $Z = V\tau$. But when a volatility drift is present, then $Z \to v(V,\tau)\tau$, where $v(V,\tau)$ is the time-average of the deterministic volatility. Other notation used: S is the stock price, K is the strike price, r is the interest rate, δ is the dividend yield, and τ is the time to expiration. The general coefficients of the (n,0) entry are determined by the recursion relation $A_{2j}^{(n+1)} = -[n + j - (1/2)] A_{2j}^{(n)} - (1/2) A_{2j-2}^{(n)}$. This relation and the general form of the entry can be established by an induction argument using $R^{(n+1,0)} = \tau(\partial R^{(n,0)} / \partial Z) + R^{(n,0)} R^{(2,0)}$. In particular, the recursion implies that $A_{2n}^{(n+1)} = 2^{-n}$. See Appendix 5.2 for calculation details of particular cases.

3 The Correlated Case

3.1 Non-zero correlation but no drift

In this subsection, we calculate the smile under the risk-adjusted volatility process $dV_t = a(V_t)dW_t = \xi\,\eta(V_t)dW$. For simplicity, we will assume that the process has constant correlation ρ with the stock price process. The more general case where $\rho = \rho(V_t)$ and a volatility drift term is present is treated in the following Section 3.2

What doesn't work (very well). Chapter 4 (Theorem I) is a generalized mixing theorem for the correlated case. The theorem uses an average of the B-S formula with an effective stock price and effective volatility. This suggests the question: can one use the mixing theorem to calculate the smile, when $\rho \neq 0$, by a two-dimensional Taylor series about $\langle S^{eff} \rangle = S_0$ and $\langle V^{eff} \rangle = (1-\rho^2)V_0$?

The answer is that it's very complicated do the expansion in this way because (i) $\langle V^{eff} \rangle \neq V_0$, yet (ii) $\lim_{\tau \to 0} V^{imp}(X=0) = V_0$. The Taylor series indeed generates the last relation (ii) but to do so, you have to sum an infinite subset of terms. The net effect is that, if you try this approach (and I have spent many hours in the attempt) you will need to develop many more terms to ultimately generate the same results that are achieved more easily with the alternative methods used below.

How to use the mixing theorem when $\rho \neq 0$. The generalized mixing theorem of Chapter 4 *can* be used to develop the smile when $\rho \neq 0$, but it needs to be in the form presented in Chapter 4, Sec. 4. That is, begin with the transform representation for the call option price:

$$(3.1) \qquad C(S,V,\tau) = Se^{-\delta\tau} - \frac{Ke^{-r\tau}}{2\pi} \int_{i/2-\infty}^{i/2+\infty} \exp(-ikX)\frac{\hat{H}(k,V,\tau)}{k^2 - ik}\,dk\,.$$

Then apply the mixing theorem. We will show that this method reproduces the results previously derived by the P.D.E. approach in Chapter 3.

An exact relation. Before beginning the series expansions, we first note that (3.1) provides an exact representation for the smile. Temporarily, ignore the definition of X already given and consider the function of two dummy variables $f(X,U)$ which we introduced in Chapter 3:

$$(3.2) \qquad f(X,U) = e^X - \frac{1}{2\pi}\int_{i/2-\infty}^{i/2+\infty} \exp[-ikX - c(k)U]\frac{1}{k^2 - ik}dk$$
$$= e^X \Phi\left(\frac{X}{\sqrt{U}} + \tfrac{1}{2}\sqrt{U}\right) - \Phi\left(\frac{X}{\sqrt{U}} - \tfrac{1}{2}\sqrt{U}\right),$$

where $\Phi(\bullet)$ is the cumulative normal function. Now, restoring X to its special value, the B-S formula is given in terms of this function by $c(S,V,\tau) = Ke^{-r\tau} f(X,V\tau)$. In principle you could solve for U in the equation

$$(3.3) \qquad f(X,U) = e^X - \frac{1}{2\pi}\int_{i/2-\infty}^{i/2+\infty} \exp(-ikX)\frac{H(k,V,\tau)}{k^2 - ik}dk\,.$$

Denote the solution for U in (3.3) by $U = G(X,V,\tau)$. Then, the implied volatility $V^{imp} = \tau^{-1}G(X,V,\tau)$. This exact result shows that the potential dependencies of the smile upon $S, K, r,$ and δ are all subsumed under a single dependence upon X as defined above. In addition, $V^{imp}(X,V,\tau)$ will also depend upon the parameters of the volatility process. This establishes Statement (I) in our summary of results in the Introduction section.

Method I: mixing theorem approach. From Chapter 4, Sec 4, the mixing Theorem (or Feynman-Kac style) representation for $H(k,V_0,\tau)$ is

$$(3.4) \qquad H(k,V_0,\tau) = \left\langle \exp\left(-ik\rho\int_0^\tau \sigma_s\, dW_s - \tilde{c}(k)\int_0^\tau V_s\, ds\right)\right\rangle$$

where $\qquad \tilde{c}(k) = \frac{1}{2}[k^2(1-\rho^2) - ik] = c(k) - \frac{1}{2}\rho^2 k^2\,.$

The reason that (3.4) is the more effective starting point for the mixing theorem is that the term $\int_0^\tau V(s)ds$ in the effective stock price combines with the same term in the effective volatility. If, instead, you applying mixing with the B-S formula, you have to deal with two terms of this same form throughout the entire expansion.

After a somewhat lengthy computation explained in Appendix 5.4, and using $d(k) = -ik$, we find

$$(3.5)\qquad H(k,V_0,\tau) = \exp\left[-c(k)V_0\tau\right]\Big\{1 - \tfrac{1}{2}\rho\, a_0\sigma_0\, c(k)d(k)\tau^2$$
$$+a_0^2\left[\tfrac{1}{6}c^2(k)\tau^3 + \tfrac{1}{8}\rho^2 c^2(k)d^2(k)\tau^4 V_0 - \tfrac{1}{12}\rho^2 c(k)d^2(k)\tau^3\right] - \tfrac{1}{6}\rho^2 a_0' a_0\, c(k)d^2(k)\tau^3 V_0\Big\}$$
$$+\, O(\xi^3)\,.$$

Notice that in the leading exponential, the $\tilde{c}(k)$ term in (3.4) has becomes $c(k)$ in (3.5). In addition, notice that the form of the answer is the B-S term $\exp\left[-c(k)V_0\tau\right]$ times a sum of polynomials in $c(k)$ and $d(k)$. This form allows the k-integrations to be performed analytically, as we have noted before. Next, we review the same computation via the P.D.E approach.

Method II: PDE approach. This section repeats the $\xi-$expansion development of Chapter 3, but for the special case of zero volatility drift. Rather than just take a vanishing drift in previous general results, it's instructive to see how the expansion works term-by-term in this simpler case.

Using the abbreviations $d(k) = -ik$ and $\chi(V) = \rho\,\eta(V)V^{1/2}$, we want to solve

$$(3.6)\qquad \frac{\partial H}{\partial \tau} = \tfrac{1}{2}\xi^2\eta^2(V)\frac{\partial^2 H}{\partial V^2} + \xi d(k)\chi(V)\frac{\partial H}{\partial V} - c(k)V H$$

The previous result (3.5) and our work in Chapter 3 all suggest that we should look for solutions of the form

$$(3.7)\qquad H(k,V,\tau) = e^{-c(k)V\tau}h(k,V,\tau)\,,$$

where we call $h(k,V,\tau)$ the *reduced fundamental transform*[4]. It has a formal power series expansion

$$(3.8)\qquad h(k,V,\tau) = \left[1 + \xi h^{(1)}(k,V,\tau) + \xi^2 h^{(2)}(k,V,\tau) + \cdots\right],$$

subject to $h^{(i)}(k,V,\tau=0) = 0$. Substituting (3.8) into (3.6) and matching the coefficients of ξ yields

$$\frac{\partial h^{(1)}}{\partial \tau} = -c(k)d(k)\chi(V)\,\tau\,.$$

This equation has the general solution

$$(3.9)\qquad h^{(1)}(k,V,\tau) = -\tfrac{1}{2}c(k)d(k)\chi(V)\,\tau^2 + g(k,V)\,.$$

---

[4] The reduced fundamental transform is also the generator of the moments of the time-average volatility: see Appendix 5.1

But $g(k,V)$ must vanish because of the initial condition, leaving

$$h^{(1)}(k,V,\tau) = -\tfrac{1}{2}c(k)d(k)\chi(V)\tau^2 .$$

Matching the coefficients of ξ^2 determines $h^{(2)}$ as the solution to

$$\frac{\partial h^{(2)}}{\partial \tau} = \tfrac{1}{2}\eta^2 c^2 \tau^2 + d\chi\left[\frac{\partial h^{(1)}}{\partial V} - c\tau h^{(1)}\right].$$

Substituting from (3.9) yields

$$\frac{\partial h^{(2)}}{\partial \tau} = \tfrac{1}{2}\eta^2 c^2 \tau^2 - \tfrac{1}{2}\chi'\chi\, c\, d^2 \tau^2 + \tfrac{1}{2}\chi^2 c^2 d^2 \tau^3 .$$

This is immediately integrated to give

$$h^{(2)}(k,V,\tau) = \tfrac{1}{6}\eta^2 c^2 \tau^3 - \tfrac{1}{6}\chi'\chi\, c\, d^2 \tau^3 + \tfrac{1}{8}\chi^2 c^2 d^2 \tau^4 .$$

Combining the two terms confirms (3.5). So, the mixing theorem and the PDE approach are complementary ways of obtaining the same ξ – expansion results. But, if you examine the calculation in Appendix 5.4, you'll see that the PDE approach is a lot less work!

Note that the general recursion, for $m \geq 2$, is

$$\frac{\partial h^{(m)}}{\partial \tau} = \tfrac{1}{2}\eta^2(V)\left(\frac{\partial}{\partial V} - c\tau\right)^2 h^{(m-2)} + d\chi(V)\left(\frac{\partial}{\partial V} - c\tau\right)h^{(m-1)} .$$

This last relation can be used to establish that

$$h^{(m)} = \begin{cases} O(\tau^{\frac{m}{2}+2}) & m \geq 2 \text{ and even} \\ O(\tau^{\frac{m}{2}+\frac{3}{2}}) & m \geq 1 \text{ and odd} \end{cases},$$

which is useful in selecting a power of τ to start a solution.

Evaluating the *k*-plane integrals. We have already discussed how to evaluate these integrals in Chapter 3. When the volatility drift vanishes, the integrals are all derivatives of the B-S formula:

$$\begin{aligned} I(p,q) &= -\frac{Ke^{-r\tau}}{2\pi}\int_{i/2-\infty}^{i/2+\infty} \exp[-ikX - c(k)V\tau]\frac{c^p(k)d^q(k)}{k^2 - ik}\,dk \\ &= (-\tau)^{-p}\left(\frac{\partial^p}{\partial V^p}\right)\left(S\frac{\partial}{\partial S}\right)^q c(S,V,\tau), \end{aligned}$$

where $p \geq 1$ and $q \geq 0$ are integers. Applying this to (3.1) with (3.8) yields

$$
\begin{aligned}
(3.10)\quad C(S,V,\tau) = {} & c(S,V,\tau) + \tfrac{1}{2}\xi\,\chi(V)\tau\left(\frac{\partial}{\partial V}\right)\left(S\frac{\partial}{\partial S}\right)c(S,V,\tau) \\
& + \xi^2\left[\tfrac{1}{6}\eta^2(V)\tau\left(\frac{\partial^2}{\partial V^2}\right) + \tfrac{1}{8}\chi^2(V)V\tau^2\left(\frac{\partial^2}{\partial V^2}\right)\left(S\frac{\partial}{\partial S}\right)^2\right. \\
& \left. + \tfrac{1}{6}\xi^2\chi'(V)\chi(V)\tau^2\left(\frac{\partial}{\partial V}\right)\left(S\frac{\partial}{\partial S}\right)^2\right]c(S,V,\tau) + O(\xi^3).
\end{aligned}
$$

To solve for the smile, equate (3.10) to

$$
c(S,V^{imp},\tau) = c(S,V,\tau) + (V^{imp} - V_0)\frac{\partial c}{\partial V} + \frac{1}{2!}(V^{imp} - V_0)^2\frac{\partial^2 c}{\partial V^2} + \cdots,
$$

where $V^{imp} = V_0 + \xi g_1 + \xi^2 g_2 + \cdots$. Matching coefficients of powers of ξ yields $g_1 = \tfrac{1}{2}\chi(V)\tau R^{(1,1)}$ and

$$
g_2 = \tfrac{1}{6}\eta^2\tau R^{(2,0)} + \tfrac{1}{6}\chi'\chi\,\tau^2 R^{(1,2)} + \tfrac{1}{8}\chi^2\tau^2\left[R^{(2,2)} - \left(R^{(1,1)}\right)^2 R^{(2,0)}\right].
$$

Collecting terms, after some rearrangement, we have

$$
\begin{aligned}
(3.11)\quad V^{imp}(X,V,\tau) = {} & V + \rho\,\xi\,\eta(V)\left(-\tfrac{1}{2}\frac{X}{\sqrt{V}} + \tfrac{1}{4}\sqrt{V}\tau\right) \\
& + \xi^2\left\{\left[\left(\tfrac{1}{12} - \tfrac{11}{48}\rho^2\right)\frac{\eta^2}{V^2} + \tfrac{1}{6}\rho^2\frac{\eta\,\eta'}{V}\right]X^2 + \rho^2\left(\tfrac{1}{24}\frac{\eta^2\tau}{V} - \tfrac{1}{6}\eta\,\eta'\tau\right)X\right. \\
& \left. + \frac{\eta^2\tau}{V}\left[-\left(\tfrac{1}{12} + \tfrac{1}{48}Z\right) + \rho^2\left(\tfrac{5}{48} + \tfrac{7}{192}Z\right)\right] - \rho^2\eta\,\eta'\tau\left(\tfrac{1}{6} - \tfrac{1}{24}Z\right)\right\} + O(\xi^3)
\end{aligned}
$$

The expiration date smile is then given by taking $\tau \to 0$:

$$
\begin{aligned}
(3.12)\quad V_e^{imp}(X,V) = {} & V - \tfrac{1}{2}\rho\,\xi\,\eta(V)\frac{X}{\sqrt{V}} \\
& + \xi^2\left[\left(\tfrac{1}{12} - \tfrac{11}{48}\rho^2\right)\frac{\eta^2}{V^2} + \tfrac{1}{6}\rho^2\frac{\eta\,\eta'}{V}\right]X^2 + O(\xi^3)
\end{aligned}
$$

Automation. We discussed in Sec. 2 the automation of the calculation of V^{imp} for the case where the volatility process is $dV = \xi V^{\varphi}d\bar{W}$ and has constant correlation ρ with the stock-price process. Here are some additional results from that program, where now $\rho \neq 0$ and $\tau > 0$. Based on dimensional considerations, the answer must be of the general form

$$
(3.13)\qquad V^{imp}(S,K,V,\tau) = \sum_{n\geq 0}\xi^n V^{1+(\varphi-3/2)n}c_n,
$$

where the coefficients $c_n = c_n(X,Z;\rho)$ are dimensionless. For the Mathematica code that generates the c_n coefficients, see Appendix 5.2. Through $O(\xi^2)$, the program reproduces (3.11) above. When $\varphi = 1$, and again using $Z = V\tau$, the computer algebra results for the coefficients through $O(\xi^4)$ are:

$$c_1 = \left(-\tfrac{1}{2}X + \tfrac{1}{4}Z\right)\rho\,,$$

$$c_2 = \left[\left(\tfrac{1}{12} - \tfrac{1}{16}\rho^2\right)X^2 - \left(\tfrac{1}{8}\rho^2 Z\right)X - \left(\tfrac{1}{12}Z + \tfrac{1}{48}Z^2 + \rho^2(\tfrac{1}{16}Z - \tfrac{5}{64}Z^2)\right)\right]$$

$$\begin{aligned} c_3 = &\left[\left(\tfrac{1}{32}\rho - \tfrac{1}{32}\rho^3\right)X^3 + \left(\tfrac{5}{192}\rho Z - \tfrac{1}{64}\rho^3 Z\right)X^2 + \right. \\ &\left. + \left(\tfrac{5}{96}\rho Z + \tfrac{5}{384}\rho Z^2 - \tfrac{5}{128}\rho^3 Z^2\right)X - \left(\tfrac{1}{64}\rho Z^2 + \tfrac{1}{32}\rho^3 Z^2 + \tfrac{13}{768}\rho Z^3 - \tfrac{7}{256}\rho^3 Z^3\right)\right] \end{aligned}$$

$$\begin{aligned} c_4 = &\left\{\left(-\tfrac{1}{240} + \tfrac{3}{128}\rho^2 - \tfrac{5}{256}\rho^4\right)X^4 + \left(\tfrac{1}{128}\rho^2 Z - \tfrac{1}{128}\rho^4 Z\right)X^3 + \right. \\ &\left(-\tfrac{1}{160}Z + \tfrac{1}{128}\rho^2 Z - \tfrac{1}{512}\rho^4 Z - \tfrac{1}{320}Z^2 + \tfrac{1}{96}\rho^2 Z^2 - \tfrac{5}{1024}\rho^4 Z^2\right)X^2 \\ &+ \left(\tfrac{1}{64}\rho^2 Z^2 + \tfrac{17}{1536}\rho^2 Z^3 - \tfrac{7}{512}\rho^4 Z^3\right)X + \left(\tfrac{7}{1440}Z^2 + \tfrac{1}{96}\rho^2 Z^2 - \tfrac{1}{512}\rho^4 Z^2\right. \\ &\left.\left. + \tfrac{1}{1920}Z^3 + \tfrac{3}{512}\rho^2 Z^3 - \tfrac{29}{2048}\rho^4 Z^3 + \tfrac{1}{960}Z^4 - \tfrac{65}{6144}\rho^2 Z^4 + \tfrac{21}{2048}\rho^4 Z^4\right)\right\}. \end{aligned}$$

As we expect, the $\tau \to 0$ limit $(Z \to 0)$ continues to exist. Notice what happens in that limit: each of the c_i reduces to a single power, namely X^i.

3.2 Including the volatility drift

We continue with the general correlated case, but now assume that the risk-adjusted volatility process is given by $dV_t = \tilde{b}(V_t)dt + \xi\,\eta(V_t)dW_t$. This process has correlation $\rho(V_t)$ with the stock price process. In Chapter 3, we constructed the ξ – expansion for V^{imp} for this general case. The result was $V^{imp} = v(V,\tau) + \xi g_1 + \xi^2 g_2 + \cdots$, where

$$g_1 = \tau^{-1}J^{(1)}\tilde{R}^{(1,1)} \quad \text{and}$$

$$\begin{aligned} g_2 = \; &\tau^{-1}J^{(2)} + \tau^{-2}J^{(3)}\tilde{R}^{(2,0)} + \tau^{-1}J^{(4)}\tilde{R}^{(1,2)} \\ &+ \tfrac{1}{2}\tau^{-2}(J^{(1)})^2\left[\tilde{R}^{(2,2)} - \left(\tilde{R}^{(1,1)}\right)^2\tilde{R}^{(2,0)}\right]. \end{aligned}$$

Recall that $J^{(i)} = J^{(i)}(V,\tau) = \int_0^\tau K^{(i)}(Y(s,V),\tau - s)\,ds$, and $Y(s,V)$ is the deterministic volatility. The integrands $K^{(i)}(V,\tau)$ are given in Table A.1 of the Appendix to Chapter 3. Also recall that the $\tilde{R}^{(n,m)}$ are the derivative ratios given explicitly in Table 5.3, but where the substitution $Z = v(V,\tau)\tau$ is made. After substituting the derivative ratios from Table 5.3, we collect together terms with the same powers of X, and we find:

The smile (general stationary, correlated processes with drift):

$$\text{(3.14)} \qquad V^{imp}(X,V,\tau) = \beta_0(V,\tau) + \beta_1(V,\tau)X + \beta_2(V,\tau)X^2 + O(\xi^3),$$

$$\text{(3.15)} \quad \text{where} \qquad \beta_0(V,\tau) = v + \tfrac{1}{2}\frac{\xi}{\tau}J^{(1)}$$

$$+\,\xi^2\left[\frac{J^{(2)}}{\tau} - \tfrac{1}{2}\frac{J^{(3)}}{v\tau^2}(1+\tfrac{1}{4}v\tau) - \frac{J^{(4)}}{v\tau^2}\left(1-\tfrac{1}{4}v\tau\right) - \frac{\left(J^{(1)}\right)^2}{v^2\tau^3}\left(\tfrac{3}{4}+\tfrac{1}{16}v\tau\right)\right],$$

$$\text{(3.16)} \qquad \beta_1(V,\tau) = -\frac{\xi}{v\tau^2}J^{(1)} + \xi^2\left[-\frac{J^{(4)}}{v\tau^2} + \frac{\left(J^{(1)}\right)^2}{2v^2\tau^3}\right],$$

$$\text{(3.17)} \quad \text{and} \qquad \beta_2(V,\tau) = \xi^2\left[\tfrac{1}{2}\frac{J^{(3)}}{v^2\tau^3} + \frac{J^{(4)}}{v^2\tau^3} - \tfrac{5}{4}\frac{\left(J^{(1)}\right)^2}{v^3\tau^4}\right].$$

Remember that the dependence upon V and τ in (3.14) occurs through both $v = v(V,\tau)$ and the $J^{(i)} = J^{(i)}(V,\tau)$. This establishes Result (III) in the Introduction. Equation (3.14) is probably the most useful formula in this chapter because (i) it's very general; (ii) it's easily evaluated numerically, and (iii) as a practical matter, the corrections beyond $O(\xi^2)$ may be negligible for most options of interest (say, strikes within 20% of the money and expirations under 2 years).

(II) *Expiration.* Taking $\tau \to 0$ in (3.14), there are no drift contributions, and we obtain the expiration date smile

$$\text{(3.18)} \quad V_e^{imp}(X,V) = V - \tfrac{1}{2}\rho\xi\,\frac{\eta}{\sqrt{V}}X$$

$$+\,\xi^2\left[\left(\tfrac{1}{12} - \tfrac{11}{48}\rho^2\right)\frac{\eta^2}{V^2} + \tfrac{1}{6}\frac{1}{V}(\rho\eta)(\rho\eta)'\right]X^2 + O(\xi^3),$$

where $\eta = \eta(V)$, $\rho = \rho(V)$, and the primes denote derivatives. This establishes result (IV) of the introduction.

Example 3: *the process* $dV = (\omega - \theta V)dt + \xi V^{\varphi}dW(t)$, *with constant correlation* ρ. The integrals $J^{(i)}$ for this case were given in Chapter 3. These integrals are used to implement the expansion in Mathematica code given in

Appendix 5.2. For the example of a GARCH diffusion model $(\varphi = 1)$, we compared the series with Monte Carlo results in Figs. 5.1, 5.2, and 5.3. The specific numerical values for this comparison are shown in Table 5.4 below, which again shows the good fit seen in the figures. Not surprisingly, the largest differences are at extreme correlation values or furthest from the money strikes. We also used these integrals to plot Fig 5.4, which shows the flattening of the smile as τ increases.

Example 4: *the process* $dV = (\omega - \theta V)\,dt + \xi V^{\varphi} dW(t)$, *with constant correlation* ρ *and with the initial volatility* $V = \omega/\theta$. This is a further specialization of the last example. For numerical work, the last example can be coded and then checked with this case. We still have $J^{(2)} = 0$ and, in addition

$$
\text{(3.19)} \qquad
\begin{aligned}
J^{(1)}\left(V = \frac{\omega}{\theta}, \tau\right) &= \left(\frac{\omega}{\theta}\right)^{1/2-\varphi} \frac{\rho}{\theta^2}\left[-1 + \theta\tau + e^{-\theta\tau}\right], \\
J^{(3)}\left(V = \frac{\omega}{\theta}, \tau\right) &= \left(\frac{\omega}{\theta}\right)^{2\varphi} \frac{1}{2\theta^3}\left[-\tfrac{3}{2} + \theta\tau + 2e^{-\theta\tau} - \tfrac{1}{2}e^{-2\theta\tau}\right], \\
J^{(4)}\left(V = \frac{\omega}{\theta}, \tau\right) &= \left(\frac{\omega}{\theta}\right)^{2\varphi} \frac{\rho^2}{\theta^3}\left(\tfrac{1}{2} + \varphi\right)\left[-2 + \theta\tau + (2 + \theta\tau)e^{-\theta\tau}\right].
\end{aligned}
$$

Table 5.4 Implied volatilities: (σ^{imp}, percent)
Monte Carlo evaluation versus
the volatility of volatility expansion

Corr.	Strike Price				
ρ	90	95	100	105	110
-1.0	17.16	16.03	14.97	13.89	13.06
	17.20	16.04	14.90	13.78	12.66
-0.50	16.22	15.54	14.96	14.47	14.15
	16.28	15.56	14.94	14.43	14.01
0.0	15.19	15.02	14.96	15.01	15.15
	15.20	15.02	14.96	15.01	15.16
0.50	14.02	14.46	14.97	15.52	16.05
	13.94	14.42	14.96	15.55	16.16
1.0	12.63	13.85	15.00	15.99	16.89
	12.44	13.76	14.94	16.03	17.04

Notes: Table entries are the Black-Scholes implied volatilities (σ^{imp}, in percent, annualized) for a GARCH diffusion model. $dV_t = (\omega - \theta V_t)dt + \xi V_t dW_t$. The first entry repeats the Monte Carlo estimates from Table 4.1, where parameter are given. The second entry is the square root of V^{imp} based on equation (3.14). The table shows the expansion fairly accurately reproduces the model. The largest differences are at extreme correlation values and farther from the money.

4 Deducing the Risk-adjusted Volatility Process from Option Prices

Implied Parameters. An advantage of relatively simple and explicit smile formulas, such as the ones in this chapter, is that they can be used to get a rough estimate of the market's value for the parameters, such as V_0, ξ, and ρ. When $\varphi = 1$, all three can also be estimated from GARCH models (for example, the GJR-GARCH model). So there are obviously various combinations of what to estimate directly and what to imply. For example, one could assume $\varphi = 1$, estimate V_0 from a basic GARCH model and then use the market smile to deduce two implied parameters ρ^{imp} and ξ^{imp}.

Non-parametric estimation. A more general idea than just parameter estimation is deducing the entire functional form of the volatility process from market prices. Option prices provide an *indirect* method for this.

Equation (3.18) shows that by measuring the slope of the smile (based on market prices) close to expiration, you obtain a direct measurement of (an implied) volatility of volatility coefficient. It's a preference-free result, since preferences only affect the volatility drift term. Specifically, under the conjecture that the $O(\xi^3)$ corrections do not have a term linear in X, then regardless of preferences,

$$(4.1) \qquad \left.\frac{\partial V_e^{imp}}{\partial X}\right|_{X=0} = -\tfrac{1}{2}\rho(V)\,\frac{a(V)}{\sqrt{V}}\,,$$

which was Result (V) of the Introduction. If the risk-adjusted volatility process is stationary and you measure (4.1) repeatedly over time, then V will range over all the accessible values to the volatility process and the product $\rho(V)a(V)$ will be mapped out.

Now $a(V)$ and $\rho(V)$ have the same form in both the risk-adjusted and the actual volatility process. In contrast to the indirect approach through options, an alternative *direct* approach would use the underlying security price series to estimate these coefficients. The general problem of recovering a stationary scalar diffusion from discrete-time data has been considered by a number of researchers[5]. Coefficient estimates based on these direct methods would provide a way to cross-check any results obtained from option prices.

Of course, economic time series are rarely truly stationary. If the volatility process is not stationary, it's a reasonable conjecture that (4.1) would still be valid under the substitution $a(V_t) \to a(V_t,t), \rho(V_t) \to \rho(V_t,t)$. Again, any estimates could be compared with direct methods.

By including the next-to-leading smile terms as $\tau \to 0$, at least in principle, an (implied) risk-adjusted volatility drift function could also be deduced from the

---

[5] See, for example, Duffie and Glynn (1996) or Hansen, Scheinkman, and Touzi (1998) and references therein.

market smile. This function would not be obtainable directly from the time series of the underlying security.

Remember that the risk-adjusted volatility drift consists of two pieces: (i) an actual, typically mean-reverting component and (ii) a risk-adjustment, due to preferences. If you measure the smile at a time of very high or very low volatility relative to the long run average volatility, there will be strong mean-reversion effects in the drift. On the other hand, the risk preference effects may be strongest in that environment, too. (See Chapter 7). For example, one could estimate an implied value for the CPRA risk aversion parameter from index option prices.

Some caveats: learning in the options markets. We have emphasized "in principle", particularly in regard to the drift measurements, because there are some caveats. One problem may be that our premises (mainly Assumption 1) are too strong. For example, maybe the actual and/or the risk-adjusted volatility process really does depend upon the stock price (although there is no really strong current evidence for this). Or the volatility process may be stock-price independent, but the drift may not be stationary enough for reliable measurement. (A truism: stationarity is always rejected statistically for any economic process if enough measurements are taken).

A more serious problem, I believe, that precedes the issues just raised above is that the options markets may not have "learned" to include certain effects that we have described. For example, it has been noted by various researchers that the characteristic downward-sloping smile for equity index options has not always been present. Yet, this negative slope has been a persistent feature of the markets since the October 1987 stock market crash.

In this chapter, the downward slope of the at-the-money, expiration date smile has followed from two assumptions: (i) volatility follows (1.1) and (ii) the stock-volatility correlation ρ is significantly negative. But a significantly negative correlation was present in the pre-1987 US equity markets and well-known to market participants in a qualitative way. One well-known example: the large increase in volatility accompanying the 1973-74 US bear markets. This suggests that it took the more sudden nature of the '87 crash for the markets to learn that they should price options with that skew pattern.

Option traders did not need to employ the advanced calculations of this book just to get the signs correct. For example, a simple argument for a put option buyer is the reasoning: "if the market drops significantly, I will win in two ways: my option will go in-the-money and volatility will rise. Therefore, I will pay more for this option than I would if volatility stayed the same or fell". Apparently, put option buyers did not think this way until Q4 1987. In the same way, it may be that the grossest effects of stochastic volatility are currently priced in the marketplace but more subtle ones, such as the volatility risk premium, are priced only crudely or not at all. Having said that, in recent empirical studies of both equity and currency options, evidence is emerging for a volatility risk premium being priced: see Guo (1998), Fleming (1998), and Kapadia (1999).

Another example of learning in the options markets was the widespread adoption of the B-S formula in the early days of the listed options markets at the CBOE. This formula is certainly not perfect and its imperfections are the subject of this book. Nevertheless, its discovery and widespread use may have removed some grosser mispricings present in the pre-CBOE OTC warrant markets[6]. In the same way, option formulas that account for stochastic volatility can be expected to play an increasing role in the marketplace in the future.

I am a strong believer in the general notion that security markets are highly efficient, but I also believe that market efficiency is a process. That is, the idea that prices "fully reflect" available information is not a static concept. For example, individual investors today, largely because of the Internet, have easy access to information previously worth acquiring only by Wall Street professionals. And those same individuals are now paying transactions costs that were once offered only to the largest institutional money pools. These trends are persistent and they act in the options markets too. In particular, as transaction costs in all their forms are lowered and markets become deeper and more active, it becomes worthwhile for traders to pay attention to details previously ignored.

---

[6] See Thorp and Kassouf (1967); also Kassouf (1969).

Appendix 5.1
Calculating Volatility Moments

In this Appendix, we discuss the calculation of moments (under the risk-adjusted process) of the time-average volatility $v = \tau^{-1}\int_0^\tau V(s)ds$. These moments play an important role in mixing theorem expansions for options under stochastic volatility. They are especially important for the symmetric $(\rho = 0)$ case. We show two calculation methods, one we call the SDE method and one we call the PDE method.

The PDE method is well-known for the first moment, but not so well-known for the higher moments. For general stationary volatility processes, we derive asymptotic solutions to the PDEs with expansions in either the time to expiration τ or the volatility of volatility scale parameter ξ. Examples below show that formal power series expansions in τ can be (i) exact and convergent or (ii) asymptotic and divergent. Of course, once you have a PDE, a finite difference code can be implemented for general cases. So our main result is the PDE for the generator of the moments, rather than its asymptotic solution, which may be ill-behaved.

Throughout this Appendix, the volatility process is the risk-adjusted process. But, since we are just discussing mathematical methods, we will simply write the process as $dV_t = b(V_t)dt + a(V_t)dW_t$, dropping the usual tilde over the drift coefficient. We continue to use the notation $\langle \cdots \rangle$ for time-*0* expectations under the volatility process and $\mathbb{E}[\cdots]$ for a general time-*0* expectation.

The mean of the time-average volatility. The definition is

(A1.1) $$\mu_v(\tau, V_0) = \frac{1}{\tau}\int_0^\tau \langle V_s \rangle \, ds \,,$$

so it's just an integration if you have $\langle V_s \rangle$. We now give the correspondence of (A1.1) to a PDE It is well-known [see Karlin and Taylor (1981), Ch. 15, Sec. 5] that, given a general time-homogenous stochastic process $dX_t = b(X_t)dt + a(X_t)dW_t$ and a function $g(x)$, then

(A1.2) $$f(x,t) = \mathbb{E}[\, g(X(t)) \,|\, X(0) = x \,] \qquad \text{satisfies}$$

(A1.3) $$\frac{\partial f}{\partial t} = \mathcal{A} f \equiv \tfrac{1}{2} a^2(x) \frac{\partial^2 f}{\partial x^2} + b(x) \frac{\partial f}{\partial x}, \qquad f(x,0) = g(x).$$

Our application takes the coefficients $b(x)$ and $a(x)$ from the volatility process and $g(x) = x$; then if we can solve (A1.3), we obtain $\langle V_t \rangle = f(V_0, t)$.

Example 1. (Linear drift models); let $dV = (\omega - \theta V)dt + a(V)dW(t)$, where $a(V) > 0$ is arbitrary. Then (A1.3) becomes

(A1.4) $$\frac{\partial f}{\partial t} = \tfrac{1}{2} a^2(x) \frac{\partial^2 f}{\partial x^2} + (\omega - \theta\, x) \frac{\partial f}{\partial x}, \quad f(x,0) = x.$$

Problem (A1.4) has the (not necessarily unique) solution

(A1.5) $$f(x,t) = \frac{\omega}{\theta} + (x - \frac{\omega}{\theta}) e^{-\theta t} \quad .$$

Subject to some growth restrictions on $a(V)$, this solves (A1.1):

(A1.6) $$\mu_v(\tau, V_0) = \frac{1}{\tau} \int_0^\tau \left[\frac{\omega}{\theta} + \left(V_0 - \frac{\omega}{\theta} \right) e^{-\theta s} \right] ds = \frac{\omega}{\theta} + \left(V_0 - \frac{\omega}{\theta} \right) \left(\frac{1 - e^{-\theta \tau}}{\theta \tau} \right)$$

Notice that $\lim_{\tau \to 0} \mu_v(\tau, V_0) = V_0$. In fact, $\mu_v(\tau, V_0)$ has a power series expansion about $\tau = 0$. In addition, $\lim_{\tau \to \infty} \mu_v(\tau, V_0) = \omega / \theta$, a constant independent of the initial volatility V_0. The linear drift models are special because the solution can be independent of the diffusion coefficient $a(\cdot)$.

Example 2. Let $dV = (\omega V - \theta V^2)dt + \xi V^{3/2} dW(t)$. Assume $\theta > -\xi^2 / 2$, which is necessary for the long run expected volatility to be finite. To solve (A1.3), make the transformation $f(x,t) = e^{\omega t} x g(x,t)$. Then (A1.3) becomes the problem for $g(x,t)$:

(A1.7) $$\frac{\partial g}{\partial t} = \tfrac{1}{2} \xi^2 x^3 \frac{\partial^2 g}{\partial x^2} + \left[\omega x - (\theta - \xi^2) x^2 \right] \frac{\partial g}{\partial x} - \theta\, x\, g, \qquad g(x,0) = 1,$$

We show how to solve problem (A1.7) in Chapter11; the solution is

(A1.8) $$g(x,t) = \frac{1}{\beta} X\left(\frac{2\omega}{\xi^2 x}, \omega t \right) M\left[1, 1 + \beta, \; -X\left(\frac{2\omega}{\xi^2 x}, \omega t \right) \right], \quad X(y,t) = \frac{y}{e^t - 1},$$

where $\beta = 1 + (2\theta / \xi^2)$ and $M(\alpha, \beta, z)$ is a confluent hypergeometric function. We will consider the leading behavior in two limits.

Limit $\tau \to 0$. In this limit, $X(y,t) \approx (y/t) \to \infty$. As $X \to \infty$, it's known that $M(1,1+\beta,-X) = \beta/X + O(1/X^2)$. So, indeed, $g(x,0) = 1$ which gives us $f(x,0) = x$ as desired.

Limit $\tau \to \infty$. With $g(x,t)$ defined by (A1.8), the first moment is given by

$$\mu_v(\tau, V_0) = \frac{V_0}{\tau} \int_0^\tau e^{\omega s} g(V_0, s)\, ds\,.$$

As the time to expiration becomes large, this integral is determined by the behavior of the integrand as $s \to \infty$. In this limit, $X \to 0$. Since $M(\alpha,\beta,z=0)=1$, it's easy to see that, as $s \to \infty$, $g(V_0,s) \approx [2\omega/(\beta\xi^2 V_0)]e^{-\omega s}$. Since the integrand becomes a constant at large time, so does μ_v. Specifically, $\mu_v(\tau) \to \mu_v(\infty) = 2\omega/(2\theta + \xi^2)$. This last result can be double-checked using the stationary distribution of the volatility process $P(V)$, which is easily developed. The verification uses the formula $\mu_v(\infty) = \int_0^\infty V P(V)\, dV$

The general case. The general case requires approximate methods, either a numerical solution or series expansions. For series expansions, there are two potentially useful formal power series: the volatility of volatility expansion and a small-τ expansion. We will discuss the latter.

For a general process with (arbitrarily) differentiable coefficients, (A1.3) can be formally integrated as follows:

(A1.9) $$f(x,t) = \exp(t\mathcal{A})g(x) = \exp(t\mathcal{A})x$$

$$= \left[1 + t\mathcal{A} + \frac{t^2}{2!}\mathcal{A}^2 + \cdots + \frac{t^N}{N!}\mathcal{A}^N\right]x + O(t^{N+1})$$

The individual terms in this expansion are very simple to develop. For example,

$$\mathcal{A}x = \left(\tfrac{1}{2}a^2(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x}\right)x = b(x)\,, \qquad \text{and}$$

$$\mathcal{A}^2 x = \mathcal{A}(\mathcal{A}x) = \left(\tfrac{1}{2}a^2(x)\frac{\partial^2}{\partial x^2} + b(x)\frac{\partial}{\partial x}\right)b(x) = \left[\tfrac{1}{2}a^2(x)b''(x) + b(x)b'(x)\right],$$

where primes denote differentiation . The series can be automated in Mathematica. See Appendix 5.2 for details.

Asymptotic nature of the series. The first line of (A1.9) is an exact operator relation, where $\exp(t\mathcal{A})$ is usually interpreted as an integral operator. The second line is an expression of an asymptotic series equality. That series is not necessarily convergent. Nevertheless, summing the first few terms can provide a good numerical approximation. But, if there are terms in the exact solution $f(x,t)$ to (A1.3) that vanish faster than any power of t, such as $\exp(-c/t)$, you will never see them in this expansion. See Example 2 (continued) below for an illustration of this behavior.

If you suspect that the series is divergent in a particular case, I would suggest restricting τ to less than a month and truncating the series after 3-6 terms. If you need values for τ in the, say 1-3 month range, try the volatility of volatility expansion (see below) or a Monte Carlo method. For longer times, or if greater accuracy is important, try a finite difference method on the PDE

General result. Using (A1.9) in (A1.1), we have the asymptotic result for a general process:

(A1.10)

$$\mu_v(\tau,V_0) = V_0 + \tfrac{1}{2}\tau\, b(V_0) + \tau^2\left[\tfrac{1}{12}a^2(V_0)b''(V_0) + \tfrac{1}{6}b(V_0)b'(V_0)\right] + O(\tau^3)$$

Example 1 (continued) For this example, $b(V) = \omega - \theta V$, $a(V)$ almost arbitrary (subject to some growth restrictions). Since $b'' = 0$, then (A1.10) reads

$$\mu_v(\tau,V_0) = V_0 + \tfrac{1}{2}\tau(\omega - \theta V_0) - \tfrac{1}{6}\tau^2\left[\theta(\omega - \theta V_0)\right] + O(\tau^3).$$

This is easily shown to be in agreement with exact result (A1.6). In this example, the series $\{1 + tA + (t^2/2)A^2 + \ldots\}$ is exact and convergent for all $t < \infty$.

Example 2 (continued) For this example, $b(V) = \omega V - \theta V^2$, $a(V) = \xi V^{3/2}$, so (A1.10) becomes

(A1.11) $$\mu_v(\tau,V_0) = V_0 + \tfrac{1}{2}\tau(\omega V_0 - \theta V_0^2) + \tau^2\left[-\tfrac{1}{6}\xi^2\theta V_0^3 + \tfrac{1}{6}(\omega V_0 - \theta V_0^2)(\omega - 2\theta V_0)\right] + O(\tau^3)$$

In the first part of this example, we wrote the leading term in the asymptotic expansion for $M(1,1+\beta,-X)$ as $X \to +\infty$. In fact, the complete expansion is known: as $X \to +\infty$, for $J,K = 1,2,3,\ldots$

(A1.12) $$M(1,1+\beta,-X) = \beta X^{-1}\sum_{n=0}^{J-1}(1-\beta)_n X^{-n} + O(|X|^{-1-J})$$

$$+\Gamma(1+\beta)(-X)^{-\beta}e^{-X}\sum_{n=0}^{K-1}\frac{(\beta)_n}{n!}(-X)^{-n}+O(|e^{-X}X^{-\beta-K}|).$$

The sums use Pochhammer's symbol $(\alpha)_n$, defined by $(\alpha)_0=1$ and $(\alpha)_n=\alpha(\alpha+1)\cdots(\alpha+n-1)$ for $n=1,2,3,\ldots$ Since

$$X\left(\frac{2\omega}{\xi^2 x},\omega\tau\right)=\left(\frac{2\omega}{\xi^2 x}\right)\frac{1}{e^{\omega\tau}-1},$$

the X^{-n} terms in the first line of (A1.12) have easily developed power series in t. From (A1.8) and (A1.12), we have for $J=1,2,\ldots$

$$\langle V_\tau\rangle = V_0\,e^{\omega\tau}\frac{1}{\beta}X\left(\frac{2\omega}{\xi^2 V_0},\omega\tau\right)M\left[1\,,1+\beta,\,-X\left(\frac{2\omega}{\xi^2 V_0},\omega\tau\right)\right]$$

$$=V_0\,e^{\omega\tau}\sum_{n=0}^{J-1}\left(-\frac{2\theta}{\xi^2}\right)_n X^{-n}+O(|X|^{-1-J})+O(|e^{-X}X^{1-\beta}|)$$

Next, let $\tau\to 0$ and only keep terms through $O(\tau^2)$:

(A1.13) $\quad\langle V_\tau\rangle = V_0\left(1+\omega\tau+\frac{1}{2}\omega^2\tau^2\right)$

$$\times\left[1-\left(\frac{2\theta}{\xi^2}\right)\left(\frac{\xi^2 V_0}{2\omega}\right)\left(\omega\tau+\tfrac{1}{2}\omega^2\tau^2\right)+\left(\frac{2\theta}{\xi^2}\right)\left(\frac{2\theta}{\xi^2}-1\right)\left(\frac{\xi^2 V_0}{2\omega}\right)^2\omega^2\tau^2\right]+O(\tau^3).$$

$$=V_0+\tau(\omega V_0-\theta V_0^2)+\tau^2\left[\tfrac{1}{2}\omega^2 V_0-\tfrac{3}{2}\omega\theta V_0^2+(\theta^2-\tfrac{1}{2}\theta\xi^2)V_0^3\right]+O(\tau^3)$$

Integrating (A1.13) over τ and dividing by τ yields agreement with (A1.11). This example again illustrates a general principle: the formal expansion power series expansion for $\exp(t\mathcal{A})$ correctly generates all terms of the type τ^n. But you will never see any of the terms from the second line of (A1.12) since they are all of the form $e^{-1/(\omega\tau)}\tau^{\beta+n}$. For small τ, this may not matter; but for large τ the neglected terms will be important.

Higher moments. First consider the second moment, defined by

(A1.14) $$\sigma_v^2(\tau,V_0) = \left\langle(v-\mu_v)^2\right\rangle = \left\langle v^2\right\rangle-\mu_v^2$$

So we need

(A1.15) $$\left\langle v^2\right\rangle = \frac{1}{\tau^2}\int_0^\tau\int_0^\tau\langle V_u V_s\rangle\,du\,ds\,.$$

We show two methods: first, a SDE iterative method, and second, a PDE method.

The SDE method. The SDE for volatility $dV_t = b(V_t)dt + a(V_t)dW_t$ is equivalent to the integral equation

$$V_u = V_0 + \int_0^u b(V_t)dt + \int_0^u a(V_t)dW_t \, .$$

As $u \to 0$, we can approximate the solution to the integral equation by

$$V_u \approx V_0 + b(V_0)u + a(V_0)W_u \, .$$

Under this approximation, and using $\langle W_u W_s \rangle = \min[u,s]$, we obtain

$$\langle V_u V_s \rangle \;=\; V_0^2 + V_0 b(V_0)(u+s) + a^2(V_0)\min[u,s] + O(u^2,s^2,us) \, .$$

This is easily integrated in (A1.15), and yields the leading term:

(A1.16) $$\sigma_v^2(\tau,V_0) \;=\; \tfrac{1}{3}a^2(V_0)\tau + O(\tau^2)$$

Generalizing this same development to higher moments gives the other entries in Table 5.1. The method can be made quite systematic, using "Often used formulas" from Appendix 5.4.

The PDE method. A generator of all of the $m \geq 2$ moments is given by

(A1.17) $$g(V_0,\tau;c) = \left\langle \exp\left[-c\textstyle\int_0^\tau (V_s - \langle V_s \rangle)\, ds\right]\right\rangle .$$

To use (A1.17), take derivatives:

(A1.18) $$\left\langle (v - \mu_v)^m \right\rangle = (-\tau)^{-m} \left.\frac{\partial^m g}{\partial c^m}\right|_{c=0} .$$

The generator $g(V_0,\tau;c)$ is closely related to the reduced fundamental transform (where the correlation is set to zero). To see this, first consider the function

(A1.19) $$H(V_0,\tau;c) = \left\langle \exp\left[-c\textstyle\int_0^\tau V_s \, ds\right]\right\rangle ,$$

where c is simply a constant (and *not* the function $c(k)$, although it plays a similar role). The function $H(V,\tau;c)$ is the solution to the PDE problem

(A1.20) $$\begin{cases} \dfrac{\partial H}{\partial \tau} = \tfrac{1}{2}a^2(V)\dfrac{\partial^2 H}{\partial V^2} + b(V)\dfrac{\partial H}{\partial V} - cVH \\ H(V,\tau=0;c) = 1 \end{cases} ,$$

That is, $H(V,\tau;c)$ is our fundamental transform for the case of zero correlation. Using this connection with our previous development and recalling the PDE for the first moment, (A1.17) can be written

(A1.21) $$H(V,\tau;c) = \exp\left(-c\int_0^\tau f(V,s)ds\right)g(V,\tau;c),$$

where $f(V,\tau)$ is the solution to the PDE problem

(A1.22) $$\begin{cases} \dfrac{\partial f}{\partial \tau} = \frac{1}{2}a^2(V)\dfrac{\partial^2 f}{\partial V^2} + b(V)\dfrac{\partial f}{\partial V} \\ f(V,\tau=0) = V \end{cases}$$

By substituting into (A1.21) into (A1.20) and using (A1.22), we find that $g(V,\tau;c)$ satisfies the PDE problem

(A1.23) $$\begin{cases} \dfrac{\partial g}{\partial \tau} - b(V)\dfrac{\partial g}{\partial V} = \frac{1}{2}a^2(V)\left(\dfrac{\partial^2 g}{\partial V^2} - 2c\tilde{\zeta}(V,\tau)\dfrac{\partial g}{\partial V} + c^2\tilde{\zeta}^2(V,\tau)g\right) \\ g(V,\tau=0;c) = 1 \end{cases}$$

(A1.24) where $$\tilde{\zeta}(V,\tau) = \int_0^\tau \frac{\partial f(V,s)}{\partial V}ds.$$

Linear drift case. For the class of models where $dV = (\omega - \theta V)dt + a(V)dW(t)$, then $f(V,t)$ is given by (A1.5); so $\partial f(V,s)/\partial V = e^{-s\theta}$ and $\tilde{\zeta}(V,\tau) = (1-e^{-\theta\tau})/\theta$. In this special case, $\tilde{\zeta} = \zeta$, where ζ is defined at (3.38) and $g = h$, the reduced fundamental transform defined at (3.35). Also, since in this case $\partial\zeta(V,\tau)/\partial V = 0$, (A1.23) can also be written

(A1.25) $$\begin{cases} \dfrac{\partial g}{\partial \tau} - b(V)\dfrac{\partial g}{\partial V} = \frac{1}{2}a^2(V)\left(\dfrac{\partial}{\partial V} - c\zeta(V,\tau)\right)^2 g \\ g(V,\tau=0;c) = 1 \end{cases}$$

Recursion systems. (A1.23) can be solved with a formal power series in τ, or by letting $a(V) = \xi\eta(V)$, a formal power series in ξ. Again, these should be thought of as asymptotic solutions. For the τ - expansion, let

$$g(V,\tau,c) = 1 + \sum_{m\geq 1} g^{(m)}(V,c)\tau^m,$$

$$\tilde{\zeta}(V,\tau) = \sum_{m\geq 1} p_m\tau^m \qquad \text{and} \qquad \tilde{\zeta}^2(V,\tau) = \sum_{m\geq 2} q_m\tau^m.$$

Then (A1.23) is equivalent to the recursion system

(A1.26) $$(m+1)g^{(m+1)} = b(V)g_V^{(m)} + \tfrac{1}{2}a^2(V)\left[g_{VV}^{(m)} - 2c\sum_{j=1}^{m} p_j\, g_V^{(m-j)} + c^2\sum_{j=2}^{m} q_j\, g^{(m-j)} \right].$$

For the $\xi-$ expansion, consider the linear drift models for simplicity and let

(A1.27) $$g(V,\tau,c) = 1 + \sum_{m\geq 2} \tilde{g}^{(2m)}(V,\tau,c)\xi^{2m}\,.$$

Then (A1.25) is equivalent to the recursion system

(A1.28) $$\begin{cases} \dfrac{\partial \tilde{g}^{(2m)}}{\partial \tau} - b(V)\dfrac{\partial \tilde{g}^{(2m)}}{\partial V} = \tfrac{1}{2}\eta^2(V)\left(\dfrac{\partial}{\partial V} - c\zeta(V,\tau)\right)^2 \tilde{g}^{(2m-2)} \\ \tilde{g}^{(0)}(V,\tau) = 1 \quad \text{and} \quad \tilde{g}^{(2m)}(V,\tau=0) = 0, \ m \geq 1 \\ \zeta(V,\tau) = \dfrac{1}{\theta}(1 - e^{-\theta\tau}) \end{cases}$$

This system is just (3.36) with no correlation term ($\psi \to 0$) .

Example 4. Let $dV = (\omega - \theta V)dt + \xi V^{\varphi} dW(t)$. We will solve for the second moment. From (A1.27) we have $g = 1 + \xi^2 \tilde{g}^{(2)} + O(\xi^4)$, where $\tilde{g}^{(2)}$ is the solution to

$$\frac{\partial \tilde{g}^{(2)}}{\partial \tau} - b(V)\frac{\partial \tilde{g}^{(2)}}{\partial V} = \tfrac{1}{2}c^2 V^{2\varphi}\frac{(1-e^{-\theta\tau})^2}{\theta^2} \quad \text{and} \quad \tilde{g}^{(2)}(V,\tau=0) = 0$$

We have already solved this last equation in Section 3; in the notation of Table 5.4, the solution is $\tilde{g}^{(2)}(V,\tau) = c^2 J^{(3)}(V,\tau)$, or $g(V,\tau,c) = 1 + \xi^2 c^2 J^{(3)}(V,\tau)$ $+O(\xi^4)$. From (A1.18)

$$\sigma_v^2(\tau,V_0) = \left\langle (v-\mu_v)^2 \right\rangle = \tau^{-2}\frac{\partial^2 g}{\partial c^2}\bigg|_{c=0} = 2\xi^2\tau^{-2}J^{(3)}(V,\tau) + O(\xi^4)\,.$$

Temporarily, consider the particular case where $V_0 = \omega/\theta$, then $J^{(3)}$ is given at (3.55), and so we have the result

(A1.29) $$\sigma_v^2\left(\tau,\, V_0 = \frac{\omega}{\theta}\right) = \left(\frac{\omega}{\theta}\right)^{2\varphi}\frac{\xi^2}{\theta^3\tau^2}\left[\tfrac{3}{2} + \theta\tau + 2e^{-\theta\tau} - \tfrac{1}{2}e^{-2\theta\tau}\right] + O(\xi^4)\,.$$

Notice that $\sigma_v^2(\tau,V_0)$ vanishes at both $\tau \to 0$ and $\tau \to \infty$; in fact, through $O(\xi^2)$

$$\sigma_v^2\left(\tau,\, V_0 = \frac{\omega}{\theta}\right) \approx \left(\frac{\omega}{\theta}\right)^{2\varphi} \times \begin{cases} \frac{1}{3}\tau & \text{as} \quad \tau \to 0 \\ \dfrac{\xi^2}{\theta^2 \tau} & \text{as} \quad \tau \to \infty \end{cases}$$

Now consider an arbitrary value for the starting volatility V_0 in this model. In the $\tau \to 0$ limit, we already have a general expression at (A1.16). In the $\tau \to \infty$ limit, we appeal to a general result proved in Chapter 6 that the asymptotic behavior of $J^{(3)}(V_0, \tau)$ is, in fact independent of V_0. With these facts, we can write for arbitrary V_0,

(A1.30)
$$\sigma_v^2(\tau, V_0) \approx \begin{cases} \frac{1}{3}\tau V_0^{2\varphi} & \text{as} \quad \tau \to 0 \\ \left(\dfrac{\omega}{\theta}\right)^{2\varphi} \left(\dfrac{\xi^2}{\theta^2 \tau} + O(\xi^4)\right) & \text{as} \quad \tau \to \infty \end{cases}$$

These two τ – dependencies (τ and τ^{-1}) are typical of the behavior in general models (with a mean reverting drift). In other words, for general models, we have $\sigma_v^2(\tau, V_0) \approx a^2(V_0)\tau/3$ as $\tau \to 0$ and $\sigma_v^2(\tau, V_0) \approx B/\tau$ as $\tau \to \infty$, where B is a constant independent of V_0.

Appendix 5.2 Working with Differential Operators in Mathematica

In this Appendix, we discuss the general subject of working with differential operators in Mathematica. For example, how does one implement the formal power series in t for the expected volatility? We showed in Appendix 5.1 that this series is given by

$$f(x,t) = \left[1 + t\mathcal{A} + \frac{t^2}{2!}\mathcal{A}^2 + \cdots + \frac{t^M}{M!}\mathcal{A}^M\right] x + O(t^{M+1}), \tag{A2.1}$$

where the differential operator $\mathcal{A}$ is defined by

$$\mathcal{A}f \equiv \tfrac{1}{2}a^2(x)\frac{\partial^2 f}{\partial x^2} + b(x)\frac{\partial f}{\partial x}$$

A general piece of code for handling operator sums, products, and powers has been written by Daniel Lichtblau[1] of Wolfram Reserach, Inc. (Lichtblau, 1999). The general functional form is **Op[L,expr]**, where **L** is an operator, **expr** is the function on which the operator acts, and **Op** returns the result of the operation. First we show the code block and then how to use it:

```
Op[a_, expr_] /; FreeQ[a,D] := a*expr
Op[L1_+ L2_, expr_] :=  Op[L1,expr] + Op[L2,expr]
Op[a*L_, expr_] /; FreeQ[a,D] :=  a*Op[L,expr]
Op[a : HoldPattern[D[__] &], expr_] := a[expr]
Op[L1__ ** L2_, expr_] := Expand[Op[L1,Op[L2,expr]]]
Commutator[L1_, L2_] := L1 ** L2 - L2 ** L1
Op[L1_^n_Integer, expr_] /; n>1 :=
  Nest[Expand[Op[L1,#]]&, expr, n]
```

[1] Reprinted by permission. Daniel reports that the code went through a few iterations between himself and Andrzej Kozlowski of Toyama University of International Studies, Japan.

This code uses a lot of advanced features in Mathematica; critical is the use of `**`, which is Mathematica's symbol for non-commutative multiplication. The code can be used without explaining all of the details of why it works. For example, to generate the (normal part of) the series defined in (A2.1), and the first moment of the time-average volatility μ_{ii} (**mu**), use:

```
ddx = (D[#,x] &);
A = (1/2) a[x]^2 ** ddx^2 + b[x] ** ddx;
f[x_,t_,M_]:= x + Sum[t^n/n! Op[A^n,x],{n,1,M}]
mu[x_,t_,M_]:= x + Sum[t^n/(n+1)! Op[A^n,x],{n,1,M}]
```

This defines **ddx** as an operator that takes x-derivatives and then builds up the operator $\mathcal{A}$ from those. For example, $\mathcal{A}^2$ behaves as we expect from the equations following (A1.9):

```
In[42]:= Op[A^2, x]
```

$$\text{Out[42]= } b[x]\, b'[x] + \frac{1}{2} a[x]^2\, b''[x]$$

The general second order expression for the first moment of the time-average volatility, (A1.10) is reproduced by

```
In[26]:= mu[x, t, 2]
```

$$\text{Out[26]= } x + \frac{1}{2} t\, b[x] + \frac{1}{6} t^2 \left(b[x]\, b'[x] + \frac{1}{2} a[x]^2\, b''[x]\right)$$

The series for the expected volatility in the model $dV = (\omega V - \theta V^2)dt + \xi V^{3/2} dW(t)$ is obtained by defining

```
b[x] = omega x - theta x^2;
a[x] = ksi x^(3/2);
```

Now the first moment for this model, (A1.11) is generated:

```
In[29]:= mu[x, t, 2]
```

$$\text{Out[29]= } x + \frac{1}{2} t\,(\text{omega}\, x - \text{theta}\, x^2) + \frac{1}{6} t^2\,(\text{omega}^2\, x - 3\,\text{omega}\,\text{theta}\, x^2 - \text{ksi}^2\,\text{theta}\, x^3 + 2\,\text{theta}^2\, x^3)$$

Appendix 5.3
Additional Mathematica Code for Chapter 5

This Appendix contains some additional Mathematica code used in Chapter 5.

Table 5.2 This table contains derivatives ratios $R^{(p,q)}$ of the B-S formula. These ratios are generated by `R[p_,q_,X_,Z_,t_]` , which is defined by

```
Cumnormal[xx_] := (1+Erf[xx/Sqrt[2]])/2
CN = Cumnormal;
d1[X_,Z_] := X/Sqrt[Z]+(1/2)Sqrt[Z]
d2[X_,Z_] := X/Sqrt[Z]-(1/2)Sqrt[Z]
f[X_,Z_] = E^X CN[d1[X,Z]]- CN[d2[X,Z]];
R[p_,q_,X_,Z_,t_] := t^(p-1)*
Collect[Simplify[D[f[X,Z],{Z,p},{X,q}]/D[f[X,Z],Z]],X]
```

For example, here is a table entry:

In[22]:= `R[4, 0, X, Z, t]`

Out[22]= $$t^3\left(\frac{X^6}{8\,Z^6}-\frac{3\,X^4\,(20+Z)}{32\,Z^5}+\frac{3\,X^2\,(240+24\,Z+Z^2)}{128\,Z^4}-\frac{960+144\,Z+12\,Z^2+Z^3}{512\,Z^3}\right)$$

Fig. 5.1 Each graph was created using only the five data points in Table 4.1 using the **`Interpolation`** function. For example, to plot the first graph, apart from some plot options, we used :

```
tab = {{90,17.16},{95,16.03},{100,14.97},{105,13.89},
      {110,13.0}};
smile = Interpolation[tab];
Plot[smile[K],{K,90,110}];
```

Figs 5.2, 5.3, 5.4, 5.5 and Table 5.5 These figures and table are based upon (3.3.5), where the integrals $J^{(i)}$, $i = 1-6$ are given by (3.3.7-3.3.10). Also needed is the time-average of the deterministic volatility $v(V,\tau)$ defined by (3.3.3). These functions are evaluated with

```
xpr1[omega_,theta_,V_,t_] :=
  omega/theta + E^(-theta t)(V-omega/theta)
xpr2[theta_,t_] := 1 - E^(-theta t)
xpr3[theta_,s_,u_] := E^(-theta(s -u))- E^(-theta s)
TimeAverageVdet[omega_,theta_,V_,t_] =
 Simplify[(1/t)Integrate[xpr1[omega,theta,V,s],
        {s,0,t}]];
```

```
J[1,0,0,rho_,phi_,V_,t_] :=
     (1/2) rho V^(phi+1/2) t^2
J[1,omega_,theta_,rho_,phi_,V_,t_] := N[(rho/theta)*
      NIntegrate[xpr2[theta,t-s]*
   xpr1[omega,theta,V,s]^(phi+1/2),{s,0,t}]]
J[2,omega_,theta_,rho_,phi_,V_,t_] :=0
J[3,0,0,rho_,phi_,V_,t_] := (1/6) V^(2 phi) t^3
J[3,omega_,theta_,rho_,phi_,V_,t_] :=
   N[(.5/theta^2) NIntegrate[xpr2[theta,t-s]^2*
   xpr1[omega,theta,V,s]^(2 phi),{s,0,t}]]
J[6,omega_,theta_,rho_,phi_,V_,t_,s_] :=
   NIntegrate[xpr3[theta,s,u]*
   xpr1[omega,theta,V,t-u]^(phi-1/2),{u,0,s}]
J[4,0,0,rho_,phi_,V_,t_] :=
      (1/6) rho^2 (phi+1/2) V^(2 phi) t^3
J[4,omega_,theta_,rho_,phi_,V_,t_] :=
   N[(rho^2 (phi+1/2)/theta)*
   NIntegrate[ xpr1[omega,theta,V,t-s]^( phi+1/2)*
   J[6,omega,theta,rho,phi,V,t,s],{s,0,t}]]
```

Note that **TimeAverageVdet**, which is the time-average of the deterministic volatility is evaluated immediately (`=` instead of `:=`), so there is no performance penalty from the **Integrate** when this function is used repeatedly in other functions.

The smile, which is the square root of V^{imp} is given by

```
Smile[K_,omega_,theta_,rho_,phi_,V_,t_,ksi_,ksiorder_]:=
      Module[{X,v,Z,c,beta,Vimp,ksiscaled},
        X = Log[100/K];
        v = TimeAverageVdet[omega,theta,V,t];
        ksiscaled = ksi  V^(1-phi);
        Z = v t;
        For[i=1, i<=4, i++,
            c[i] = J[i,omega,theta,rho,phi,V,t]];
(* beta[i,j] is the coefficient of X^i at order ksi^j *)
        beta[0,0] = v;
        beta[0,1] = (1/2)c[1]/t;
        beta[0,2] = c[2]/t -(1/2) c[3](1+ Z/4)/(Z t)-
         c[4](1- Z/4)/(Z t)+ c[1]^2(3/4+1/16 Z)/(Z^2 t);
        beta[1,0] = 0;
        beta[1,1] = -c[1]/(Z t);
        beta[1,2] = -c[4]/(Z t)+c[1]^2/(2 Z^2 t);
        beta[2,0] = 0;
        beta[2,1] = 0;
        beta[2,2] = (1/2) c[3]/(Z^2 t)+c[4]/(Z^2 t)-
                  5/4 c[1]^2/(Z^3 t);
        Vimp = Sum[beta[0,j]ksiscaled^j,{j,0,ksiorder}]+
        Sum[beta[i,j] X^i *
            ksiscaled^j,{i,1,2},{j,0,ksiorder}];
        Return[100.0 Sqrt[Vimp]]]
```

The **Smile** function assumes the volatility of volatility ξ (ksi) is the value appropriate for the process $dV = (\omega - \theta V)dt + \xi V dW(t)$. These values are typically in the range of 1-2 (annualized) for the S&P 500 index, for example. Then, these values are scaled (ksiscaled) to approximately what would be measured if, instead, the volatility process was $dV = (\omega - \theta V)dt + \xi V^{\varphi} dW(t)$. The advantage of this scaling is that it prevents comparing apples and oranges if you investigate the smile with two different values of φ (phi). In other words, consider a *fixed* data set (a time series of price returns), fitted to two different volatility processes. The first process has $\varphi = 1$ and the second one has a different φ. Then, $\xi -$ scaled represents approximately what the estimated ξ would be for the second fit, given the value for the first fit.

Then the second entries in Table 5.5, for example are given by

```
Table55 := Table[Smile[K,.09,4,rho,1,.0225,20./250.,1,2],
           {rho,-1,1,1/2},{K,90,110,5}]
```

Evaluating this gives the results:

```
Table55 // MatrixForm
```

$$\begin{pmatrix} 17.2037 & 16.0421 & 14.9031 & 13.7779 & 12.657 \\ 16.2788 & 15.5568 & 14.941 & 14.4256 & 14.0052 \\ 15.2032 & 15.0183 & 14.9603 & 15.0127 & 15.1594 \\ 13.9422 & 14.4204 & 14.9613 & 15.5463 & 16.1612 \\ 12.4393 & 13.7556 & 14.9439 & 16.0316 & 17.0375 \end{pmatrix}$$

The remaining 2D graphics, namely Figs 5.2, 5.3, and 5.4 are based on the following function, where we suppress some plot options:

```
SmilePlot[omega_,theta_,rho_,phi_,V_,t_,ksi_] :=
  Plot[Smile[K,omega,theta,rho,phi,V,t,ksi,2],{K,90,110}]
```

For Fig. 5.3, we used the **Show** command to display a smile plot of the actual data (the Fig. 5.1 **smile** `function` defined above) combined with the theoretical **SmilePlot**. Similarly, for Fig. 5.4 we created 4 smile plots and merged them with **Show.**

The 3D Plot, Fig 5.5 is created with the function

```
SmilePlot3D[omega_,theta_,rho_,phi_,V_,ksi_] :=
  Plot3D[Smile[K,omega,theta,rho,phi,V,t,ksi,2],
        {K,80,120},{t,.1,2}]
```

Specifically, Fig 5.5 was created with the following command (graphic output suppressed), which shows that it takes a little while:

```
Timing[SmilePlot3D[.09, 4, 0, 1, .0225, 1.5]]

{30.05 Second, - SurfaceGraphics -}
```

Equations (2.9) and (3.33). (Note: the routines use the function **f[X_,Z_]** from the Table 5.2 block above). The method is based upon the ξ – expansion for the fundamental transform $H(k,V,\tau)$. We use the abbreviations $c(k)=(k^2-ik)/2$ and $d(k)=-ik\rho$. Suppressing the k-dependence, one version of the expansion is $H(V,\tau)=\sum_{n\geq 0}\xi^n H^{(n)}(y,\tau)$, using $y=c(k)V$. The expansion coefficients $H^{(n)}(y,\tau)$, with a volatility process of the form $dV=\xi V^{\varphi}dW$, satisfy the recursion system

(A2.1) $$H^{(n)}(y,\tau)=\int_0^{\tau} K^{(n)}(y,\tau-s)\,e^{-ys}ds\,,$$

where

(A2.2) $$K^{(n)}(y,\tau)=\tfrac{1}{2}c^{2-2\varphi}y^{2\varphi}\frac{\partial^2 H^{(n-2)}}{\partial y^2}+dc^{-\varphi+1/2}y^{\varphi+1/2}\frac{\partial H^{(n-1)}}{\partial y}\,,$$

(A2.3) $$H^{(0)}(y,\tau)=e^{-y\tau}\,,\quad\text{and}\quad H^{(n)}(y,\tau)=0,\ n<0\,.$$

These three equations are implemented in a direct manner:

```
H[0,y_,t_] = E^(-y t);
H[n_,y_,t_] := 0/;n<0
H[n_,y_,t_] := H[n,y,t] =
  Module[{ans,f},
      f[y,s]=Simplify[(K[n,y,t]/.t->(t-s))E^(-y s)];
      ans = Simplify[Integrate[f[y,s],{s,0,t}]];
      Return[ans]]
```

```
K[1,y_,t_] := K[1,y,t] =
  Module[{ans,x},
    ans = Simplify[d y^(phi+1/2) c^(-phi+1/2)*
          D[H[0,x,t],x]/.x->y];
      Return[ans]]

K[n_,y_,t_] := K[n,y,t] =
  Module[{ans,arg1,arg2,x},
    arg1 = Simplify[d y^(phi+1/2) c^(-phi+1/2)*
     D[H[n-1,x,t],x]/.x->y];
    arg2 = Simplify[(1/2)  y^(2 phi) c^(2 - 2phi)*
     D[H[n-2,x,t],{x,2}]/.x->y];
    Return[arg1+arg2]]
```

The next step is to use the fundamental transform in a k-plane integration to get the option price. Each term of the expansion is $e^{-cV\tau}$ times polynomials in $c(k)$ and $(-ik)$. We just need the polynomial part, so we remove the $e^{-cV\tau}$ factor and make the substitution for $d(k)$:

```
Hf[n_,V_,t_] :=
  Collect[PowerExpand[H[n,c V,t]E^(c V t)/.
        d->(-I k rho)],{c,k}]//Expand
```

Then, as explained in Appendix 3.1 at (3.A.15) every factor of $c(k)$ and $(-ik)$ generates, by the k-plane integration, a V-derivative or an S-derivative of the B-S formula. The function **ID[X,Z,q,p]** serves as a placeholder for these derivatives initially; then in **VimpCoef** (see further below), the actual derivatives are substituted:

```
Hser[0,V_,t_] := ID[X,Z,0,0];
Hser[n_,V_,t_] :=
Module[{tmp,ans},
 tmp = Hf[n,V,t]/.(c^p_. k^q_.)->(-1)^p I^q ID[X,Z,q,p];
 ans = tmp/.(c^p_.)->(-1)^p ID[X,Z,0,p];
 Return[ans//Simplify]]
```

The final step uses the $\xi-$ expansion for V^{imp}. It's actually more convenient to define a new variable $Z^{imp} = V^{imp}\tau$, and then let $Z^{imp} = Z + a_1\xi + a_2\xi^2 + \cdots$, where $Z = V\tau$:

```
Zimp[ksi_,n_] :=
 Module[{j,ans},
  ans = Z + Sum[ksi^j a[j],{j,1,n}]+O[ksi]^(n+1);
  Return[ans]]
```

Then we write $c(S,V^{imp},\tau) = C(S,V,\tau)$, expand both sides in their $\xi-$ expansions and solve for the coefficients of corresponding powers of ξ. The left-hand-side of this equation is set up first:

```
Lhs[n_] :=
Coefficient[Series[ID[X,Zimp[ksi,n]],{ksi,0,n}],ksi^n]
```

Then we set the left-hand-side equal to the right-hand-side, solve for the unknown coefficients, make the derivative substitutions, and switch from Z^{imp} back to V^{imp}:

```
VimpCoef[ksiorder_,Zvalue_] :=
  Module[{j,ans},Clear[a,c];
      c[0] = 1;
      For[j = 1, j <= ksiorder, j++, Clear[ID];

      a[j]=a[j]/.Solve[Lhs[j]==Hser[j,V,t],a[j]][[1]];

      ID[X_,Z_] = -f[X,Z];
      ID[X_,Z_,n_,m_]= ID[X,Z,n,m]= D[ID[X,Z],{X,n},{Z,m}];
      a[j] = Simplify[a[j]];

 (* a[j] is the ksi-expansion coef for Zimp=Vimp t;
       to get the coef for Vimp, we must divide by t *)
      c[j] = Simplify[a[j]/t/V^(1+(phi-3/2)j)];
      c[j] = Simplify[c[j]/.V->(Z/t)];
      c[j] = Simplify[c[j]/.Z->Zvalue];
      Print["c[",j,"]=",c[j]];
      ]]
```

Once we have the coefficients of the expansion, then the smile is given by

$$V^{imp}(S,K,V,\tau)=\sum_{n\geq 0}\xi^n V^{1+(\varphi-3/2)n}c_n\,,$$

which is implemented with:

```
SmileNoDrift[t_,V_,ksi_,K_,rhox_,ksiorder_,phix_] :=
 Module[{ans,Zx = V t,Xx = Log[100/K]},
  ans = Sum[ksi^n  V^(1+(phi-3/2)n) c[n]/.phi->phix/.
               rho->rhox/.Z->Zx/.X->Xx,{n,0,ksiorder}];
  If[ans < 0, Return[0],Return[100.0 Sqrt[ans]]]]
```

Note that φ(phi), ρ(rho), Z and X are treated as global variables in these routines: not the best programming practice but it saved creating a lot of new arguments. That's why we substitute specific values for the global ones in the last code block. If you just want the general case, you can type, for example:

In[29]:=

```
VimpCoef[2, Z]
```

```
c[1] = 1/4 rho (-2 X + Z)

c[2] = 1/192
   (4 (4 + (-11 + 8 phi) rho^2) X^2 + 8 (1 - 4 phi) rho^2 X Z
     Z (-4 (4 + Z) + rho^2 (20 - 32 phi + 7 Z + 8 phi Z)))
```

Alternatively, another way to get a special case is to make a global assignment before running **VimpCoef.** This is faster. For example, for the special case $\varphi = 1$, you type:

In[30]:=

```
phi = 1;
VimpCoef[4, Z]
```

```
c[1] = 1/4 rho (-2 X + Z)

c[2] = 1/192 (-4 (-4 + 3 rho^2) X^2 - 24 rho^2 X Z +
     Z (-4 (4 + Z) + 3 rho^2 (-4 + 5 Z)))

c[3] = -1/768 rho (24 (-1 + rho^2) X^3 + 4 (-5 + 3 rho^2) X^2 Z +
     Z^2 (12 + rho^2 (24 - 21 Z) + 13 Z) + 10 X Z (-4 + (-1 + 3 rho^2) Z))

c[4] = 1/92160 (-24 (16 - 90 rho^2 + 75 rho^4) X^4 -
    720 rho^2 (-1 + rho^2) X^3 Z - 60 rho^2 X Z^2 (-24 - (-17 + 21 rho^2) Z)
    6 X^2 Z (48 (2 + Z) - 40 rho^2 (3 + 4 Z) + 15 rho^4 (2 + 5 Z)) +
    Z^2 (16 (28 + 3 Z + 6 Z^2) + 45 rho^4 (4 - 29 Z + 21 Z^2) -
      15 rho^2 (64 - 36 Z + 65 Z^2)))
```

Similarly, set $\rho = 0$ globally if that is your application. You just have to remember to **Clear** say **phi** or **rho** if you want the general case again.

Appendix 5.4
Calculating with the Mixing Theorem

In this Appendix, we give the details of the expansions in Chapter 5 that make use of the mixing theorem. When you calculate under the mixing theorem, you need to evaluate expectations of products of the same Brownian motion, but where the time index is different.

The basic relations are the following: with two factors, $\langle W_s W_\tau \rangle = \min[s,\tau]$; with four factors,

$$\left\langle W_{t_1} W_{t_2} W_{t_3} W_{t_4} \right\rangle = \left\langle W_{t_1} W_{t_2} \right\rangle \left\langle W_{t_3} W_{t_4} \right\rangle + \left\langle W_{t_1} W_{t_3} \right\rangle \left\langle W_{t_2} W_{t_4} \right\rangle + \left\langle W_{t_1} W_{t_4} \right\rangle \left\langle W_{t_2} W_{t_3} \right\rangle$$

With an odd number of factors, the expectation vanishes. With an even number of factors, the general answer is

$$\left\langle W_{t_1} W_{t_2} \cdots W_{t_n} \right\rangle = \sum_{pairs} \left\langle W_{p_1} W_{p_2} \right\rangle \cdots \left\langle W_{p_{n-1}} W_{p_n} \right\rangle$$

In other words, if you have an even number of Ws, the expectation is just the sum of the expectations of all pairwise combinations. If there are n terms, there are $(n-1)!! = (n-1)(n-3)\cdots 1$ such combinations. In addition, we need some stochastic integrals and expectations containing products of Brownian motions with an exponential. The formulas we used are given in the following table

Table 5A.1 Some often used formulas

$$\int_0^\tau W_s dW = \tfrac{1}{2}(W_\tau^2 - \tau)$$
$$\int_0^\tau (W_s^2 - s) dW = \tfrac{1}{3}W_\tau^3 - \tau W_\tau$$
$$\int_0^\tau s\, dW_s = \tau W_\tau - \int_0^\tau W_s ds$$
$$\langle \exp(xW_\tau)W_\tau^n \rangle = \left(\frac{d}{dx}\right)^n \exp\left(\tfrac{1}{2}x^2\tau\right)$$
$$\left\langle \exp(xW_\tau)\int_0^\tau W_s\, ds \right\rangle = \exp\left(\tfrac{1}{2}x^2\tau\right)\left(\tfrac{1}{2}x\tau^2\right)$$
$$\left\langle \exp(xW_\tau)\left(\int_0^\tau W_s\, ds\right)^2 \right\rangle = \exp\left(\tfrac{1}{2}x^2\tau\right)\left(\tfrac{1}{3}\tau^3 + \tfrac{1}{4}x^2\tau^4\right)$$
$$\left\langle \exp(xW_\tau)\int_0^\tau W_s^2\, ds \right\rangle = \exp\left(\tfrac{1}{2}x^2\tau\right)\left(\tfrac{1}{2}\tau^2 + \tfrac{1}{3}x^2\tau^3\right)$$

Notes. In the table, W_τ is a Brownian motion, x is a constant parameter, and $\langle \cdots \rangle$ is a time-0 expectation. The formulas in the table were obtained by repeated application of Ito's formula. For some examples of how this works, see Chapter 10 of Neftci (1996)

Calculation details for (2.7) This section describes the computation of (2.7), using the mixing theorem in the form (2.1). Since the volatility drift is zero, the volatility process is a martingale and $\langle V_t \rangle = V_0$, which also implies that $\mu_v = V_0$. Equating (2.1) to (2.2) again yields

$$\text{(A4.1)}\quad (V^{imp} - V_0)\frac{\partial c}{\partial V} + \frac{1}{2!}(V^{imp} - V_0)^2 \frac{\partial^2 c}{\partial V^2} + \cdots = \frac{1}{2!}\left\langle (v - V_0)^2 \right\rangle \frac{\partial^2 c}{\partial V^2} + \frac{1}{3!}\left\langle (v - V_0)^3 \right\rangle \frac{\partial^3 c}{\partial V^3} + \cdots,$$

where all the derivatives of the B-S formula are to be evaluated $V = V_0$. Divide both sides by $(\partial c / \partial V)$ and solve for $(V^{imp} - V_0)$.

Inserting our proposed solution into (A4.1) yields

$$\text{(A4.2)}\quad \left[(\beta_0 - V_0) + \beta_2 X^2 + \beta_4 X^4 + \cdots\right] + \frac{1}{2!}\left[(\beta_0 - V_0) + \beta_2 X^2 + \beta_4 X^4 + \cdots\right]^2 R^{(2,0)} + \cdots$$
$$= \frac{1}{2!}\left\langle (v - V_0)^2 \right\rangle R^{(2,0)} + \frac{1}{3!}\left\langle (v - V_0)^3 \right\rangle R^{(3,0)} + \frac{1}{4!}\left\langle (v - V_0)^4 \right\rangle R^{(4,0)} \cdots$$

Then, we match coefficients of corresponding powers X^{2m} on both sides of (A4.2). To denote these coefficients, let $R^{(n,m)}[X^q]$ = the coefficient of X^q in $R^{(n,m)}$. First, we match the coefficients of the constant term:

$$\text{(A4.3)} \qquad X^0: \quad (\beta_0 - V_0) + \varepsilon_1(\tau) = -\frac{1}{2!}\left\langle (v - V_0)^2 \right\rangle \left(\frac{1}{2Z} + \frac{1}{8} \right) \tau + \delta_1(\tau)$$

where
$$\delta_1(\tau) = \sum_{n \geq 3} \frac{1}{n!} \left\langle (v - V_0)^n \right\rangle R^{(n,0)}[X^0]$$

and
$$\varepsilon_1(\tau) = \sum_{n \geq 2} \frac{1}{n!} (\beta_0 - V_0)^n R^{(n,0)}[X^0].$$

The general $(n,0)$ entry of Table 5.2 shows that $R^{(n,0)}(X = 0) = O(\tau^0)$. And Table 5.1 shows that $\left\langle (v - V_0)^n \right\rangle = O(\tau^2)$ for $n \geq 3$. Hence $\delta_1(\tau) = O(\tau^2)$. If we also have $\varepsilon_1(\tau) = O(\tau^2)$, then (A4.3) has the solution

$$\text{(A4.4)} \qquad \beta_0(\tau, V_0) = V_0 - \frac{1}{12} \frac{a_0^2}{V_0} \tau + O(\tau^2).$$

The proposed solution is self-consistent because it indeed implies that $\varepsilon_1(\tau) = O(\tau^2)$. Next, we match the coefficients of X^2:

$$\text{(A4.5)} \qquad X^2: \quad \beta_2 + \varepsilon_2(\tau) = \frac{1}{4}\left\langle (v - V_0)^2 \right\rangle \frac{\tau}{Z^2} + \delta_2(\tau),$$

where
$$\delta_2(\tau) = \sum_{n \geq 3} \frac{1}{n!} \left\langle (v - V_0)^n \right\rangle R^{(n,0)}[X^2], \quad \text{and}$$

$$\varepsilon_2(\tau) = \sum_{n \geq 2} \frac{1}{n!} (\beta_0 - V_0)^{n-1} \left\{ (\beta_0 - V_0) R^{(n,0)}[X^2] - n\beta_2 R^{(n,0)}[X^0] \right\}$$

In this case, the general n entry for $R^{(n,0)}$ in Table 5.2 shows that the coefficient of X^2 is $O(\tau^{-1})$. Since each volatility moment that occurs in $\delta_2(\tau)$ is $O(\tau^2)$, this establishes that $\delta_2(\tau) = O(\tau)$. If $\varepsilon_2(\tau) = O(\tau)$, then (A4.5) has the solution

$$\text{(A4.6)} \qquad \beta_2(\tau, V_0) = \frac{1}{12} \frac{a_0^2}{V_0^2} + O(\tau).$$

Then (A4.6) is confirmed by using it along with (A4.4) and the indicated behaviors of the $R^{(n,0)}$ in the expression for $\varepsilon_2(\tau)$. Finally, we match the coefficients of X^4:

$$\text{(A4.7)} \quad X^4: \; \beta_4 + (\beta_0 - V_0)\beta_2 \frac{\tau}{2Z^2} = -\left[(\beta_0 - V_0)\beta_4 + \frac{1}{2}\beta_2^2 \right] \left(\frac{1}{2Z} + \frac{1}{8} \right) \tau + \varepsilon_3(\tau)$$

$$= \frac{1}{3!}\left\langle (v-V_0)^3 \right\rangle \frac{\tau^2}{4Z^4} - \frac{1}{4!}\left\langle (v-V_0)^4 \right\rangle \frac{3}{32}\frac{\tau^3}{Z^5}(20+Z) + \delta_3(\tau)\,,$$

where $$\delta_3(\tau) = \sum_{n\geq 5} \frac{1}{n!}\left\langle (v-V_0)^n \right\rangle R^{(n,0)}[X^4] \qquad \text{and}$$

$\varepsilon_3(\tau) =$

$$\sum_{n\geq 3} \frac{1}{n!}(\beta_0 - V_0)^{n-1}\left\{(\beta_0 - V_0)R^{(n,0)}[X^4] + n\beta_2 R^{(n,0)}[X^2] + n\beta_4 R^{(n,0)}[X^0]\right\}.$$

Taking $j=2$ in the general n entry for $R^{(n,0)}$ shows that the coefficient of X^4 is $O(\tau^{-2})$. Since the volatility moments of $\delta_3(\tau)$ are $O(\tau^3)$, then $\delta_3(\tau) = O(\tau)$. Tentatively assuming $\varepsilon_3(\tau) = O(\tau)$, then (A4.7) has the solution

$$\text{(A4.8)} \quad \beta_4 = \frac{1}{3!}\frac{2}{5}a_0' a_0^3 \frac{1}{4V_0^4} - \frac{1}{4!}\frac{1}{3}a_0^4 \frac{3}{32}\frac{1}{V_0^5}20$$

$$+\frac{1}{12}\frac{a_0^2}{V_0}\frac{1}{12}\frac{a_0^2}{V_0^2}\frac{1}{2V_0^2} + \frac{1}{2}\left(\frac{1}{12}\frac{a_0^2}{V_0^2}\right)^2\frac{1}{2V_0} + O(\tau)$$

$$= \frac{1}{60}\left(\frac{a_0' a_0^3}{V_0^4}\right) + \frac{1}{576}(-15+2+1)\left(\frac{a_0^4}{V_0^5}\right) = \frac{1}{60}\left(\frac{a_0' a_0^3}{V_0^4}\right) - \frac{1}{48}\left(\frac{a_0^4}{V_0^5}\right) + O(\tau)$$

Given β_4, it is straightforward to verify that, indeed, $\varepsilon_3(\tau) = O(\tau)$. Taking $\tau \to 0$ in the equations for β_0, β_2, and β_4 yields the result (2.7). ■

Calculation details for (3.5). The volatility follows the stochastic integral equation

$$\text{(A4.9)} \qquad V_s = V_0 + \int_0^s a(V_u)\,dW_u\,.$$

We assume $a(V)$ has all derivatives, so we can write

$$\text{(A4.10)} \qquad a(V_u) = a_0 + a_0'(V_u - V_0) + \tfrac{1}{2}a_0''(V_u - V_0)^2 + \cdots.$$

Now to construct a systematic expansion, parameterize the volatility diffusion coefficient by ξ, letting $a(V) = \xi\,\eta(V)$, where ξ is independent of V.

Then, as $\xi \to 0$, to a leading approximation, (A4.9) becomes $V_s = V_0 + a_0 W_s + O(\xi^2)$. Inserting this formula with $s \to u$ into (A4.10) yields $a(V_u) = a_0 + a_0' a_0 W_s + O(\xi^3)$. Then, using this in (A4.9) again gives the next approximation

(A4.11)
$$V_s = V_0 + a_0 W_s + a_0' a_0 \int_0^s W_u \, dW_u + O(\xi^3)$$
$$= V_0 + a_0 W_s + \tfrac{1}{2} a_0' a_0 (W_s^2 - s) + O(\xi^3)\,.$$

The second line uses a Table 5A.1 formula. This iteration can be performed systematically in this way to any ξ – order.

One way the volatility appears in (3.4) is through the integrated volatility:

(A4.12)
$$\int_0^\tau V_s ds = V_0 \tau + a_0 \int_0^\tau W_s ds + \tfrac{1}{2} a_0' a_0 \int_0^\tau (W_s^2 - s) ds + O(\xi^3)\,,$$

which we obtained by substitution of (A4.11).

The other way the volatility appears in (3.4) is through the expression

(A4.13)
$$\int_0^\tau \sigma_s dW_s - \int_0^\tau f(V_s) dW_s\,, \text{ where } f(V) = V^{1/2}\,.$$

Note that this is conceptually the same as (A4.9) except that instead of $a(V)$, we have $f(V)$. We can expand $f(V)$ as a Taylor series just as we did in (A4.10) and make the same substitutions as before. The result is, using Table 5A.1 formulas:

(A4.13)
$$\int_0^\tau \sigma_s dW_s = \sigma_0 W_\tau + \tfrac{1}{2} \frac{a_0}{\sigma_0} \int_0^\tau W_s dW_s$$
$$- \tfrac{1}{8} \frac{a_0^2}{\sigma_0^3} \int_0^\tau W_s^2 dW_s + \tfrac{1}{4} \frac{a_0' a_0}{\sigma_0} \int_0^\tau (W_s^2 - s) dW_s + O(\xi^3)\,.$$

$$= \sigma_0 W_\tau + \tfrac{1}{4} \frac{a_0}{\sigma_0} \left(W_\tau^2 - \tau \right)$$
$$+ \frac{a_0^2}{\sigma_0^3} \left(-\tfrac{1}{24} W_\tau^3 + \tfrac{1}{8} \int_0^\tau W_s ds \right) + \tfrac{1}{4} \frac{a_0' a_0}{\sigma_0} \left(\tfrac{1}{3} W_\tau^3 \quad \tau W_\tau \right) + O(\xi^3)$$

Standard forms. Our method for substitutions is to reach standard forms that are convenient for computations. Convenience means (i) explicit polynomials in τ and W_τ, (e.g., τW_τ^2), or (ii) integrals of the form $\int_0^\tau f(W_s, s) ds$ (e.g., $\int_0^\tau W_s ds$) where the integrand is a polynomial. Based on the low order terms, it seems likely that each expansion integral with respect to dW_s can be reduced to a standard form by using Ito's formula. The advantage of the standard forms is that they lead to expressions whose expectations can be readily computed, since as we shall show below, only powers (and exponentials) of Brownian motions are involved.

Putting it all together, we now have an expression for the fundamental transform as a power series in ξ :

$$\text{(A4.14)} \qquad \hat{H}(k,V_0,\tau) = \exp\left[-\bar{c}(k)V_0\tau\right]$$

$$\times\left\langle \exp\left(-ik\rho\sigma_0 W_\tau\right)\left\{1-a_0\left[\bar{c}(k)\int_0^\tau W_s ds + \frac{ik\rho}{4\sigma_0}\left(W_\tau^2-\tau\right)\right]+O(\xi^2)\right\}\right\rangle.$$

As we indicated, each of the expectations that must be computed is of the general form

$$f(x,\tau,t_1,t_2,\cdots,t_n) = \left\langle e^{xW_\tau} W_{t_1} W_{t_2} \cdots W_{t_n} \right\rangle,$$

where $x = -ik\rho\sigma_0$ is just a parameter. Any term of this type can be evaluated. For example, the specific forms that occur in the computation through $O(\xi^2)$ can be evaluated using the Table 5A.1 entries. . After a somewhat lengthy computation, and using $d(k) = -ik$, we obtain (3.5).

6 The Term Structure of Implied Volatility

The term structure of implied volatility is the relation between the option implied volatility and time to maturity: $\sigma^{imp}(\tau)$. Using our previous notation, it's the square root of $V^{imp}(X,V,\tau)$, holding the moneyness X and the initial volatility V fixed. In practice, the implied volatility is usually measured at a strike price close to the money. ($X=0$ is a natural choice). In fact, the qualitative behavior is the same at any strike: a graph of $\sigma^{imp}(\tau)$ vs. τ ultimately flattens to a limiting asymptotic value, $\sigma_{\infty}^{imp}=(V_{\infty}^{imp})^{1/2}$, that is independent of both X and V. This general behavior is analogous to the term structure of interest rates and the existence of a long-run rate of interest.

The asymptotic implied volatility depends only upon the parameters of the volatility process. It can be calculated from the simple relation

$$V_{\infty}^{imp}=8\lambda(k_0),$$

where λ is the first eigenvalue of a differential operator, and k_0 is a complex number. We illustrate 3 ways to calculate V_{∞}^{imp} for general models: a series method, a variational method, and a differential equation-based method. Computation times for the latter two methods are just a couple of seconds in Mathematica.

1 Deterministic Volatility

The volatility models that we consider in this book typically have a similar structure: $dV_t = b(V_t)dt + a(V_t)dW_t$, where the drift term $b(V_t)$ exhibits mean-reversion. For example, the GARCH diffusions and other models have the linear drift form $b(V_t) = \omega - \theta V_t$, where ω and θ are positive constants. If the volatility becomes small, then $b(V_t)$ is positive, causing the volatility to tend to grow larger. If the volatility is large, then $b(V_t)$ is negative, causing the volatility to tend to grow smaller.

To a first approximation, the term structure is explained by letting the Brownian noise term vanish[1]. For the linear drift models, we are left with the deterministic volatility evolution $\dot{V}_t = \omega - \theta V_t$, where the dot means a time derivative. The solution to the differential equation $\dot{y} = \omega - \theta y$, where $y(0) = V$ is given by

$$y(t,V) = \frac{\omega}{\theta} + \left(V - \frac{\omega}{\theta}\right)e^{-\theta t}. \tag{6.1}$$

In (6.1), the behavior is especially simple as $t \to \infty$; no matter what the starting value V, the volatility tends to the fixed point $V^* = \omega/\theta$. This value is called a fixed point because if the volatility starts there, it stays there. The fixed point is *attractive* or *stable* because small departures of the volatility from V^* are damped over time.

Option valuation under deterministic volatility is a well-known application of the B-S theory. Options are still priced by the B-S formula, but the volatility parameter in the formula is modified. The modified volatility is simply the time-average of the deterministic volatility. In other words, if $C(S,V,\tau)$ is the general call option value and $c(S,V,\tau)$ is the B-S value, then under deterministic volatility:

---

[1] For simplicity, we call the term structure of implied volatility just the term structure. With the exception of one subsection, in this chapter the risk-adjusted volatility process and the actual volatility process are assumed identical (a risk-neutral world). See Chapter 8 (Duality and Changes of Numeraire) to convert the results in this chapter to log-utility.

$$C(S,V,\tau) = c\left(S,\mu(V,\tau),\tau\right), \tag{6.2}$$

where $$\mu(\tau,V) = \frac{1}{\tau}\int_0^\tau y(s,V)\,ds = \frac{\omega}{\theta} + \left(V - \frac{\omega}{\theta}\right)\left(\frac{1-e^{-\theta\tau}}{\theta\tau}\right).$$

The B-S implied volatility is given by $\sigma^{imp}(\tau,V) = [\mu(\tau,V)]^{1/2}$.

Shown in Fig. 6.1 is a plot of both $\sigma^{imp}(\tau,V)$ and $[y(\tau,V)]^{1/2}$ versus τ, where $\omega_a = 0.02$ and $\theta_a = 2$, (annualized parameters). We show two cases: (i) initial volatility $\sigma_a = 8\%$ ($V_a = 0.0064$) and (ii) initial volatility $\sigma_a = 12\%$ ($V_a = 0.0144$). Notice that the implied volatility (the bold line) behaves a lot like the actual volatility $y(\tau,V)$ (the thin line); the only difference is that the implied volatility changes more slowly because it's a time-average. But both functions begin at V and evolve in a smooth monotonic fashion with a limiting asymptotic value $\sigma_\infty^{imp} = (\omega/\theta)^{1/2} = 10\%$. The asymptotic value is independent of the starting value V, as well as S, K, r, and δ.

The rate of convergence to the asymptotic value is determined by the parameter θ, which has the dimensions $[1/\tau]$. Since the "decay rate" is determined by the exponential term $\exp(-\theta\tau)$, this type of behavior is often described as having a "half-life" $\tau_{1/2} = 1/\theta$. In our example, $\tau_{1/2} = 0.5$ years and one can see from Fig 6.1 that both the actual and implied volatilities have moved, very roughly, about half-way toward their final asymptotic value at $\tau = 0.5$ years.

Many other models of interest to researchers have a deterministic limit that behaves in the same way as this example. In general, volatility evolution in the deterministic limit is $\dot{V}_t = b(V_t)$, where $b(\cdot)$ is the drift coefficient. If a model is mean-reverting, $b(V)$ will typically have a single zero at some $V = V^*$. The zero will be attractive, meaning not only $b(V^*) = 0$ but also $b'(V^*) < 0$, where the prime means a derivative. If you picture the graph of $b(V_t)$ you can see that the volatility evolution will be similar to Fig. 6.1. It follows from $\dot{V}_t = b(V_t)$ that $b(V)$ has the dimensions of $[V/t]$, so that $b'(V^*)$ has the dimensions $[1/t]$. This causes $|1/b'(V^*)|$ to play the role of the half-life parameter in general models, at least asymptotically.

Fig. 6.1 Term Structure of Implied Volatility (Deterministic Model)

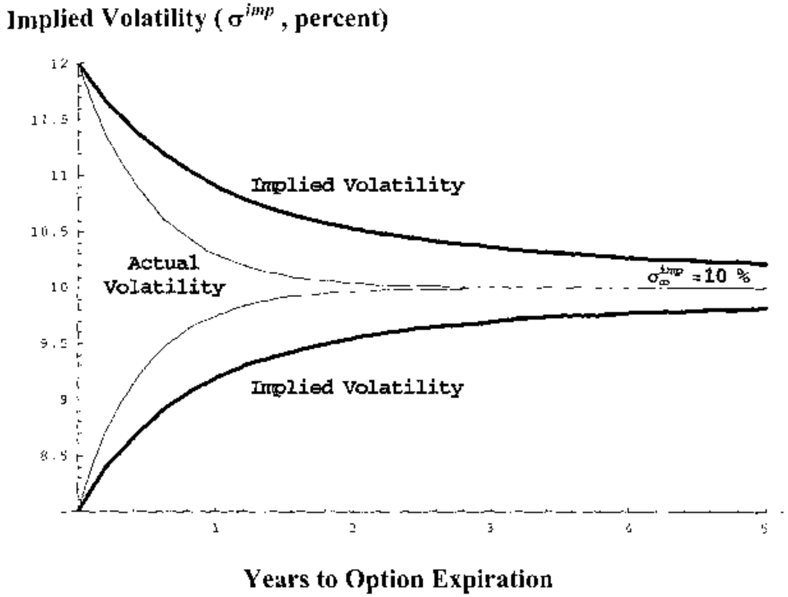
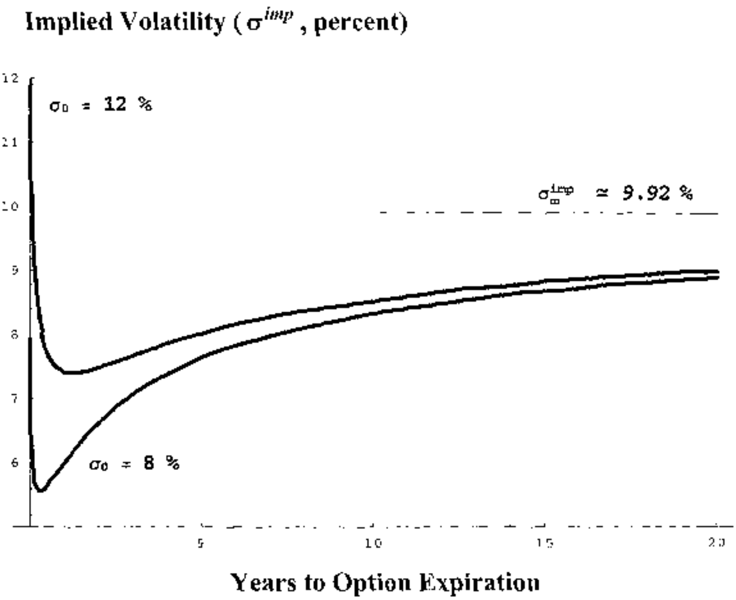


Fig. 6.2 Term Structure of Implied Volatility (Stochastic Model)



2 Deterministic Volatility II: a Transform Perspective

In the last section we showed that the deterministic volatility model $\dot{V}_t = \omega - \theta V_t$ has an asymptotic implied volatility $V_\infty^{imp} = \omega/\theta$. In this section we consider this same problem with the transform method. The advantage of the transform method is that it also solves the case we are really interested in—stochastic volatility.

Call option Solution II of (**2**.2.10) is:

$$(2.1)\quad C(S,V,\tau) = Se^{-\delta\tau} - Ke^{-r\tau}\frac{1}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty} e^{-ikX}\,\frac{\hat{H}(k,V,\tau)}{k^2-ik}\,dk\,,\quad 0<\operatorname{Im}k<1\,.$$

The natural strike price K at which to measure the term structure is given by $X=0$, which corresponds to $Ke^{-r\tau} = Se^{-\delta\tau}$. If $r\neq\delta$ and you measure at $K=S$, you are systematically moving to one side of the volatility smile pattern as the time to expiration increases. With the better choice $X=0$, (2.1) simplifies to:

$$(2.2)\qquad \frac{C(S,V,\tau)}{Ke^{-r\tau}} = 1-\frac{1}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty}\frac{\hat{H}(k,V,\tau)}{k^2-ik}\,dk$$

We established in Chapter 2 that, under constant volatility, this solution was valid for the entire strip $0<\operatorname{Im}k<1$. The same holds true under deterministic volatility because, as we will show, $\hat{H}(k,V,\tau)$ is an entire function under either constant or deterministic volatility.

We established the solution for the fundamental transform $\hat{H}(k,V,\tau)$ under deterministic volatility in Appendix 3.1 at (**3**.A.2). For the drift function $b(V)=\omega-\theta V$, that formula becomes

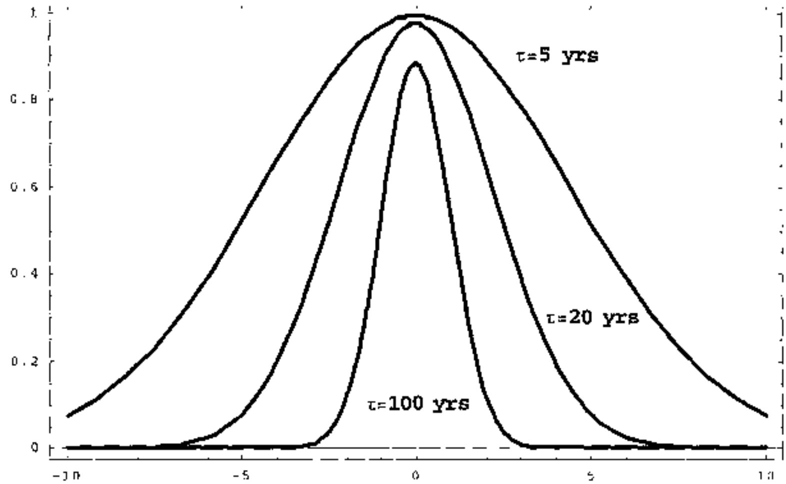
$$\hat{H}^{(0)}(k,V,\tau) = \exp[-c(k)U(V,\tau)]\,,\quad \text{where } c(k)=\tfrac{1}{2}(k^2-ik)\,,$$

and
$$U(V,\tau) = \int_0^{\tau} y(s,V)\,ds = \frac{\omega}{\theta}\tau + \left(V-\frac{\omega}{\theta}\right)\left(\frac{1-e^{-\theta\tau}}{\theta}\right).$$

This shows that $\hat{H}^{(0)}(k)$ is an entire function of k in the complex k-plane. A general plot of the modulus $|\hat{H}^{(0)}(k)|$ has already been given in Chapter 2, Fig. 2.1. The asymptotic theory considers $\tau\to\infty$. Suppose we are integrating in (2.1) along $\operatorname{Im}k = 1/2$. In Fig. 6.3, we plot $|\hat{H}^{(0)}(k_r+i/2)|$ versus k_r for

$\tau = 5$, 20, and 100 years, using the previous numerical example $\omega_a = 0.02$, $\theta_a = 2$, and $V_a = 0.0064$.

Fig. 6.3. $|\hat{H}(k)|$ along an Integration Contour (Im k = 1/2) Various Times to Maturity τ (Deterministic Model)



As you can see from Fig. 6.3, the fundamental transform becomes increasingly peaked about $k_r = 0$ as the time to maturity increases. For $\tau >> 1$, (2.2) becomes

$$\frac{C(S,V,\tau)}{Ke^{-r\tau}} \underset{\tau\to\infty}{\approx} 1 - \frac{1}{2\pi}\int_{ik_i-\infty}^{ik_i+\infty} \exp\left[-c(k)\frac{\omega}{\theta}\tau - c(k)\frac{1}{\theta}\left(V - \frac{\omega}{\theta}\right)\right]\frac{dk}{k^2 - ik}$$

Of course because this is the B-S theory, we could evaluate this integral exactly (see Chapter 2, Appendix 1). But an alternative method will also work in the stochastic volatility case: the asymptotic method of steepest descent.[2] As $\tau \to \infty$, Fig. 6.3 shows that the exponential factor in the integral damps the contribution everywhere except near $k_r = 0$, which is our integration origin. If

[2] For a nice discussion of the methods of steepest descent, saddle points, and the method of stationary phase, see Carrier, Krook, and Pearson (1966, Chapt. 6).

we didn't know this point, we could find it by looking for the stationary point k_0 determined by $c'(k_0)=0$, which has the solution $k_0 = i/2$. This solution k_0 is also a *saddle point* because, while the modulus $|\hat{H}|$ is decreasing in the real direction, it's increasing in the imaginary direction (see Fig. 2.1 in Chapter 2). Along this integration contour, $c(k) = 1/8 + k_r^2/2$. This is an exact relation, but in the stochastic case (see below), we will expand the integrand in a Taylor series about the saddle point. In this special case, the Taylor series only has the two terms. The leading asymptotic contribution to the integral is given by

$$\frac{C(S,V,\tau)}{Ke^{-r\tau}} \underset{\tau\to\infty}{\approx} 1 - \frac{4}{2\pi}\exp\left(-\frac{\omega}{8\theta}\tau\right)\exp\left[-\frac{1}{8\theta}\left(V-\frac{\omega}{\theta}\right)\right]\int_{-\infty}^{\infty}\exp\left(-k_r^2\frac{\omega}{2\theta}\tau\right)dk_r$$

The integral that remains is just a Gaussian

$$\int_{-\infty}^{\infty}\exp\left(-k_r^2\frac{\omega}{2\theta}\tau\right)dk_r = \sqrt{\frac{2\pi\theta}{\omega\tau}}.$$

So we obtain the result

$$\frac{C(S,V,\tau)}{Ke^{-r\tau}} \underset{\tau\to\infty}{\approx} 1 - \sqrt{\frac{8\theta}{\pi\omega\tau}}\exp\left[-\frac{1}{8\theta}\left(V-\frac{\omega}{\theta}\right)\right]\exp\left(-\frac{\omega}{8\theta}\tau\right).$$

This result can be compared with the Black-Scholes formula, which is easily shown to be, in this limit,

$$\text{(2.3)} \qquad \frac{c(S,V,\tau)}{Ke^{-r\tau}} \underset{\tau\to\infty}{\approx} 1 - \sqrt{\frac{8}{\pi V\tau}}\exp\left(-\frac{V}{8}\tau\right).$$

Comparing the last two equations implies that $V_\infty^{imp} = \omega/\theta$, just as we expected. The important idea is that we now have a method for the stochastic case.

3 Stochastic Volatility—The Eigenvalue Connection

Notice that as $\tau\to\infty$, the fundamental transform in the previous section had the following special form

$$\hat{H}^{(0)}(k,V,\tau) = \exp[-c(k)U(V,\tau)] \underset{\tau\to\infty}{\approx} \exp[-\lambda(k)\tau]u(k,V),$$

where

$$\lambda(k) = c(k)\frac{\omega}{\theta} \quad \text{and} \quad u(k,V) = \exp\left[-c(k)\frac{1}{\theta}\left(V-\frac{\omega}{\theta}\right)\right].$$

This form is special because, first of all, the dependence upon V and τ has *separated* into the product of two terms, one depending upon τ and one depending upon V. (Both terms depend upon k) . Suppose, that under stochastic volatility, the same form of solution holds:

$$\hat{H}(k,V,\tau) \underset{\tau\to\infty}{\approx} \exp[-\lambda(k)\tau]\,u(k,V) \tag{3.1}$$

with new functions $\lambda(k)$ and $u(k,V)$ to be determined. If we substitute this form into the PDE (2.2.19) satisfied by the fundamental transform, then we are left with the ordinary differential equation for $u(k,V)$:

$$\mathcal{L}_k\, u = \lambda(k)\, u\,, \tag{3.2}$$

where

$$\mathcal{L}_k u = -\tfrac{1}{2}a^2(V)\frac{d^2u}{dV^2} - \left[\tilde{b}(V) - ik\rho(V)a(V)V^{1/2}\right]\frac{du}{dV} + c(k)V\,u\,.$$

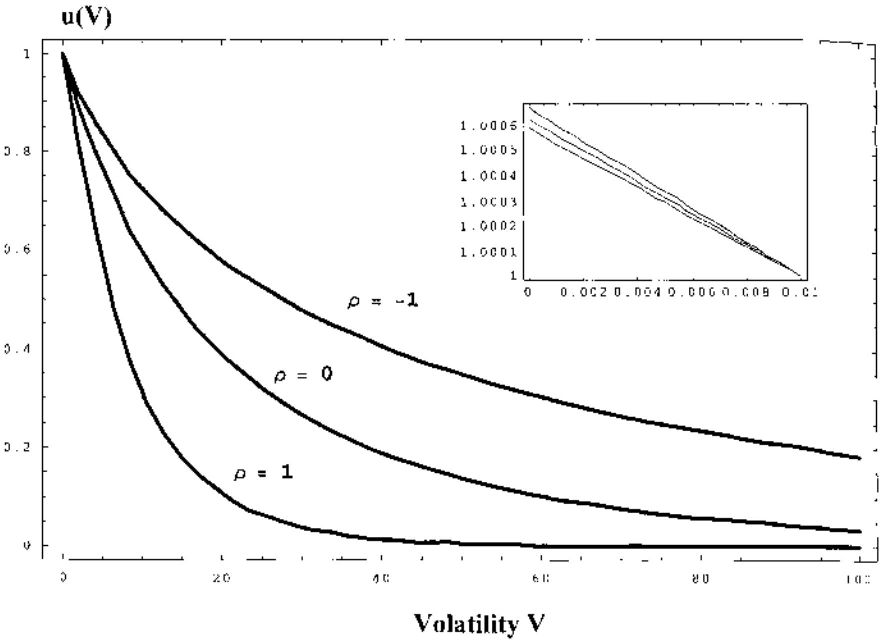
This is an *eigenvalue* equation, where $\lambda(k)$ is an eigenvalue of the differential operator $\mathcal{L}_k$, and u is the associated *eigenfunction*.[3] In general, there can be many solutions to (3.2). In fact, you may be able to develop the fundamental transform at all times τ (not just $\tau\to\infty$) as a sum over such solutions—this is called an *eigenfunction expansion*[4]. But, in the limit $\tau\to\infty$, the dominant term of such a sum uses the *smallest* or *first* eigenvalue. This may seem confusing at this point because there are a lot of complex numbers appearing in (3.2), so what do we mean by smallest? Below, we show that, in fact, everything we calculate is real-valued and the first or smallest eigenvalue is well-defined.

What does the first eigenfunction look like? In Fig. 6.4 we show plots of $u(k,V)$ vs. V with $k = i/2$. The model is the GARCH diffusion process $dV = (\omega - \theta V)\,dt + \xi V dW(t)$, with $\omega = 0.02$, $\theta = 2$, $\xi = 1.5$ and $\rho = -1,0,1$. How we calculated that function is explained in Sec. 8.

[3]Eigenvalue problems are not well-defined until we specify a class of admissible functions. This is discussed later in Sec. 7

[4] See my article (Lewis 1998).

Fig. 6.4 First Eigenfunction for the GARCH Diffusion Process



Notes. The figure shows a plot of the first eigenfunction $u(k,V)$ for the GARCH diffusion model, $dV = (\omega - \theta V)\,dt + \xi V dW(t)$, with $\omega = 0.02$, $\theta = 2$, $\xi = 1.5$ and $\rho = -1,0,1$. The parameter k is set to $i/2$ The function has been normalized so that $u(V = \omega/\theta) = 1$. Since $\omega/\theta = 0.01$, the range $V < \omega/\theta$ is difficult to resolve in the scale of the main plot and is shown in the inset. The Mathematica code for this plot is given in the Appendix to this chapter.

With this general form of solution, then (2.2) becomes in the stochastic case:

$$(3.3) \qquad \frac{C(S,V,\tau)}{Ke^{-r\tau}} \underset{\tau\to\infty}{\approx} 1 - \frac{1}{2\pi} \int_{ik_i-\infty}^{ik_i-\infty} \exp[-\lambda(k)\tau]u(k,V) \frac{dk}{k^2-ik},$$

The Ridge Property. Suppose that $\lambda(k)$ has a saddle point k_0 in the complex k-plane determined by the solution to $\lambda'(k_0)=0$. We showed in Chapter 2 that the fundamental transform is often an analytic characteristic function. As we explained in that chapter, analytic characteristic functions have the *ridge property,* which means that any saddle point must lie along the purely imaginary axis. In other words, $k_0 = iy_0$, where y_0 is a real number. This saddle point location will be confirmed in computational examples below.

The reality of the eigenvalue problem (3.2). Recall the reflection property from (**2**.2.20): $\hat{H}^*(k,V,\tau) = \hat{H}(-k^*,V,\tau)$. Combining this property with the ridge property, any saddle point must be found along $k = iy$, where $\hat{H}^*(iy,V,\tau) = \hat{H}(iy,V,\tau)$. That is: *the fundamental transform is real along the imaginary k-axis.* In turn, this shows that both the first eigenvalue and the associated eigenfunction are *real* along the imaginary axis. And finally, we can see from (3.2) that each of the coefficients of the equation will be real along that axis. In other words, to summarize: the asymptotic term structure is determined by the smallest solution to an eigenvalue problem, where the eigenvalue, eigenfunction, and associated PDE are all real-valued.[5]

An important element of the saddle point method is moving the integration contour so that it traverses the saddle point. Before we can do that, recall that (3.3) is a valid formula as long as the integration contour lies in the intersection of the fundamental strip $\alpha < \operatorname{Im} k < \beta$ with the strip $0 < \operatorname{Im} k < 1$; this is the strip of regularity. We now make the further assumption that the saddle point $k_i = \operatorname{Im} k = y_0$ lies within the strip of regularity[6]. If it does, then, by *Cauchy's theorem* (See Chapter 2, Appendix 1), we can move the integration contour to

---

[5] The complex-valued coefficients in (3.2) are needed for the full transform, but not for its asymptotic saddle point behavior.

[6] Practical numerical examples—see Table 6.1—show that y_0 is often close to 1/2, so this is not problematic in my experience.

$\mathrm{Im}\,k = y_0$ without changing the value of the integral. Next, expand $\lambda(k)$ in a Taylor series about k_0:

$$\lambda(k) = \lambda(k_0 + k_r) \simeq \lambda(k_0) + \tfrac{1}{2}k_r^2\lambda''(k_0)\,,$$

so (3.3) becomes

$$\frac{C(S,V,\tau)}{Ke^{-r\tau}} \underset{\tau\to\infty}{\approx} 1 - \frac{1}{2\pi}\exp[\,-\lambda(k_0)\tau\,]\frac{u(k_0,V)}{k_0^2 - ik_0}\int_{-\infty}^{\infty}\exp\left[-\tfrac{1}{2}k_r^2\lambda''(k_0)\tau\right]dk_r\,.$$

Note that this last integral is over a real integration variable. We know $\lambda''(k_0) \geq 0$ because of the ridge property. Performing the integral gives us

$$\text{(3.4)}\qquad \frac{C(S,V,\tau)}{Ke^{-r\tau}} \underset{\tau\to\infty}{\approx} 1 - \frac{u(k_0,V)}{k_0^2 - ik_0}\frac{1}{\sqrt{2\pi\lambda''(k_0)\tau}}\exp[-\lambda(k_0)\tau\,]\,.$$

Notice that the denominator term $k_0^2 - ik_0 = y_0(1-y_0) > 0$ since, by assumption $0 < y_0 < 1$. The arbitrage bound $C(S,V,\tau) \leq Se^{-\delta\tau}$ combined with $Ke^{-r\tau} = Se^{-\delta\tau}$, implies that in (3.4) we must have $C(S,V,\tau)/Ke^{-r\tau} \leq 1$. This implies that not only is $u(k_0,V)$ real, but it's non-negative as well. That same bound also strengthens the inequality $\lambda''(k_0) \geq 0$ to $\lambda''(k_0) > 0$. Finally, comparing (3.4) with (2.3) yields a simple result for the (at-the-money) asymptotic implied volatility:

$$V_\infty^{imp}(X=0) = 8\lambda(k_0)$$

Next, we repeat the calculation for an arbitrary value for the moneyness measure X. In that case, (3.4) becomes:

$$\text{(3.5)}\qquad \frac{C(S,V,\tau)}{Ke^{-r\tau}} \underset{\tau\to\infty}{\approx} e^X - \frac{u(k_0,V)}{k_0^2 - ik_0}\frac{1}{\sqrt{2\pi\lambda''(k_0)\tau}}\exp[-\lambda(k_0)\tau - ik_0X\,]\,.$$

But the B-S solution, for general X, has the asymptotic form:

$$\text{(3.6)}\qquad \frac{c(S,V,\tau)}{Ke^{-r\tau}} \underset{\tau\to\infty}{\approx} e^X - \sqrt{\frac{8}{\pi V\tau}}\exp\left[-\tfrac{1}{2}\left(\frac{X}{\sqrt{V\tau}} - \tfrac{1}{2}\sqrt{V\tau}\right)^2\right].$$

Comparing the two solutions (3.5) and (3.6) implies that

$$\text{(3.7)}\qquad -\tfrac{1}{2}\left(\frac{X}{\sqrt{V^{imp}\tau}} - \tfrac{1}{2}\sqrt{V^{imp}\tau}\right)^2 \underset{\tau\to\infty}{\approx} -\lambda(k_0)\tau - ik_0X$$

After some rearrangement, (3.7) is equivalent to

(3.8)
$$\boxed{V^{imp}(X,V,\tau) \underset{\tau\to\infty}{\approx} 8\lambda(k_0) + (4-8y_0)\frac{X}{\tau} - \frac{X^2}{2\lambda(k_0)\tau^2} + O(\tau^{-3})}$$

where recall that $k_0 = iy_0$. This last equation is important because it implies that, as $\tau \to \infty$, the smile *flattens* to a common asymptotic value regardless of the moneyness *X*. And that common value is

(3.9)
$$\boxed{V_\infty^{imp} = \lim_{\tau\to\infty} V^{imp}(X,V,\tau) = 8\lambda(k_0)}$$

We will see in examples below that, when $\rho = 0$ (the symmetric case), then $y_0 = 1/2$ and the linear term in (3.8) vanishes.

4 Example I: The Square Root Model

For this model, volatility process (under risk neutrality) is $dV = (\omega - \theta V)\,dt + \xi\sqrt{V}\,dW(t)$, where the Brownian motion has correlation ρ with the stock price process. In Fig. 6.1, we showed an example of the term structure with $\omega/\theta = 0.01$ and $\xi = 0$. Next, we keep the same parameters but turn on the volatility of volatility parameter to $\xi = 1$, keeping $\rho = 0$. (We chose a value for ξ larger than would typically be measured in order to emphasize the effects).

The term structure under stochastic volatility is shown in Fig. 6.2. Now there is more structure to the plot. Instead of a monotonic evolution in τ to 10%, there is a dip to a significantly lower value when τ less than a year. At large τ, there is a clear indication of a common asymptote, just as we would expect from the theory of the last section. The new asymptotic value is no longer 10% but lower at approximately 9.92%. We found this value by applying the general theory of the previous section, as we now show.

The formulas for the fundamental transform are given at (**2**.3.1) and (**2**.3.2), taking the parameter $\gamma = 1$. (We will refer to expressions used there). We showed in Chapter 2 that the fundamental strip for this model is at least as large as the unit strip $I_0 = \{k \mid 0 < \operatorname{Im} k < 1\}$.

With k in the unit strip, then $\mathrm{Re}\, d > 0$, which leads to the limiting behaviors $f_1(t) \approx \tilde{\omega}(g-d)t$ and $f_2(t) \approx g-d$ as $t \to \infty$. Being careful to note the time rescaling that occurred in (2.3.1), this means that, as $\tau \to \infty$,

$$\hat{H}(k,V,\tau) \underset{\tau\to\infty}{\approx} \exp[-\lambda(k)\tau]u(k,V),$$

where

(4.1) $$\lambda(k) = -\omega g(k) = \frac{\omega}{\xi^2}\left\{[(\theta + ik\rho\xi)^2 + (k^2 - ik)\xi^2]^{1/2} - (\theta + ik\rho\xi)\right\}$$

and $$u(k,V) = \exp[g(k)V].$$

The stationary point k_0 in the complex k-plane is the solution to $d\lambda(k)/dk = 0$. This equation has two solutions:

(4.2) $$k_0 = \frac{i}{1-\rho^2}\left\{\frac{1}{2} - \frac{\rho}{\xi}\left(\theta \pm \frac{1}{2}\left[4\theta^2 + \xi^2 - 4\rho\,\theta\,\xi\right]^{1/2}\right)\right\}.$$

As promised, it's purely imaginary. As $\xi \to 0$, we want $k_0 \to i/2$ in order to reproduce the B-S solution. This limit will be correct if we choose the minus sign in (4.2). Substituting that value for k_0 into (4.1) yields

(4.3) $$\lambda(k_0) = \frac{\omega}{2(1-\rho^2)\xi^2}\left\{\left[4\theta^2 + \xi^2 - 4\rho\,\theta\,\xi\right]^{1/2} - (2\theta - \rho\xi)\right\}$$

$$= \frac{\omega}{2(1-\rho^2)\xi^2}\left\{\left[(2\theta - \rho\xi)^2 + (1-\rho^2)\xi^2\right]^{1/2} - (2\theta - \rho\xi)\right\}.$$

In the second line of (4.3), the positivity of $\lambda(k_0)$ is manifest, assuming $\omega > 0$, $\xi^2 > 0$, and $|\rho| < 1$. In fact, the limit $|\rho| \to 1$ is well-defined, and is given by

(4.4) $$\lim_{|\rho|\to 1} \lambda(k_0) = \frac{\omega}{4[2\theta - sign(\rho)\xi]}.$$

A more practical limit is $\rho = 0$. When $\rho = 0$, then (4.2) shows that the stationary point sticks at $k_0 = i/2$. This happens in general models, as you will see several times in different examples below. It's only when $\rho \neq 0$ that the stationary point moves away from $k_0 = i/2$. Which direction it moves (north or south) depends upon the sign of ρ.

Finally, the asymptotic implied volatility is given by

$$(4.5)\qquad V_\infty^{imp} = \frac{4\omega}{(1-\rho^2)\xi^2}\left\{\left[4\theta^2+\xi^2-4\rho\,\theta\,\xi\right]^{1/2}-(2\theta-\rho\xi)\right\}$$

$$= \frac{\omega}{\theta}+\frac{\omega\rho}{2\theta^2}\xi+\frac{\omega(-1+5\rho^2)}{16\,\theta^3}\xi^2+\frac{\omega\rho(-3+7\rho^2)}{32\theta^4}\xi^3$$

$$+\frac{\omega(1-14\rho^2+21\rho^4)}{128\theta^5}\xi^4+O(\xi^5)\,.$$

For Fig. 6.2, the parameters are $\rho=0$, $\omega_a=0.02$, $\theta_a=2$, and $\xi_a=1$, which yields

$$V_\infty^{imp}(\rho=0)=\frac{4\omega}{\xi^2}\left\{\left[4\theta^2+\xi^2\right]^{1/2}-2\theta\right\}=\frac{1}{25}(2\sqrt{17}-8)\,.$$

Or, in others words $\sigma_\infty^{imp}\cong 9.92\%$.

The volatility of volatility expansion in the square root model. The second line of (4.5) shows that a volatility of volatility expansion for V_∞^{imp} exists, at least for $|\,\xi\,|$ inside a radius of convergence. Two terms of that expansion, when $\rho=0$, are $V_\infty^{imp}\cong(\omega/\theta)-\omega\xi^2/(16\theta^3)$, which yields $\sigma_\infty^{imp}\cong 9.93\%$ for the same example above. This suggests that, for models that cannot be solved exactly, the $\xi-$expansion can provide a good approximation for σ_∞^{imp}. See Sec. 6 and 7 for an example.

The convergence of the expansion in (4.5) is determined by the power series expansion of the square root term:

$$\left[1+\frac{(\xi^2-4\theta\rho\xi)}{4\theta^2}\right]^{1/2}.$$

This radius is determined by considering ξ as a complex parameter. In the complex $\xi-$plane, there are branch point singularities at $\xi=\xi^*$, where ξ^* solves $\xi^2-4\theta\rho\xi+4\theta^2=0$. If R is the distance to the solution closest to the origin; then the series will converge for $|\,\xi\,|<R$.

For example, when $\rho=0$, the branch points are at $\xi^*=\pm 2i\theta$, so the series will converge for $|\,\xi\,|<2\,|\,\theta\,|$. More generally, the branch points are found at $\xi^*=z(\rho)\theta$, where $z(\rho)$ is a solution to $z^2-4\rho\,z+4=0$. The solutions to this equation are given by $z=2\rho\pm 2i\,[1-\rho^2]^{1/2}$, which traces out a circle of radius 2 as ρ ranges from -1 to 1. Hence $|\xi| < 2|\theta|$ is the radius of convergence in the square root model for all $|\rho| \le 1$.

In Sec. 6, we develop the ξ – expansion for V_∞^{imp} for the GARCH diffusion—in that case, we don't know if the series converges in any radius.

5 Example II: The 3/2 Model

The fundamental solution is given at (2.3.3). In the limit $\tau \to \infty$, we have

$$X\left(\frac{2\omega}{\xi^2 V}, \omega\tau\right) \approx \frac{2\omega}{\xi^2 V}\exp(-\omega\tau).$$

This implies that $\hat{H}(k,V,\tau)$ again separates to the eigenfunction form:

(5.1) $$\hat{H}(k,V,\tau) \approx \exp[-\lambda(k)\tau]\, u(k,V), \quad \text{where now}$$

(5.2) $$\lambda(k) = \omega\alpha(k)$$

$$= \frac{\omega}{\xi^2}\left\{\left[(\theta + ik\rho\xi + \tfrac{1}{2}\xi^2)^2 + (k^2 - ik)\xi^2\right]^{1/2} - (\theta + ik\rho\xi + \tfrac{1}{2}\xi^2)\right\},$$

and $$u(k,V) = \frac{\Gamma(\beta(k) - \alpha(k))}{\Gamma(\beta(k))}\left(\frac{2\omega}{\xi^2 V}\right)^{\alpha(k)}.$$

The stationary point k_0 is given by

$$k_0 = \frac{i}{1-\rho^2}\left\{\frac{1}{2} - \frac{\rho}{\xi}(\theta + \tfrac{1}{2}\xi^2) + \frac{\rho}{2\xi}\left[(2\theta + \xi^2)^2 + \xi^2 - 2\rho\xi\,(2\theta + \xi^2)\right]^{1/2}\right\}.$$

Again, the stationary point resides on the imaginary axis. The asymptotic implied volatility is given by

(5.3) $$V_\infty^{imp} = \frac{4\omega}{(1-\rho^2)\xi^2}\left\{\left[(2\theta + \xi^2 - \rho\xi)^2 + (1-\rho^2)\xi^2\right]^{1/2} - (2\theta + \xi^2 - \rho\xi)\right\}$$

$$= \frac{\omega}{\theta} + \frac{\omega\rho}{2\theta^2}\xi - \frac{\omega(1 + 8\theta - 5\rho^2)}{16\,\theta^3}\xi^2 - \frac{\omega\rho(3 + 16\theta - 7\rho^2)}{32\theta^4}\xi^3$$

$$+ \frac{\omega(1 - 14\rho^2 + 21\rho^4 + 32\theta^2 + 12\theta - 60\theta\rho^2)}{128\theta^5}\xi^4 + O(\xi^5).$$

It's interesting that (5.3) may be obtained from (4.5) by making the substitution $\theta \to \theta + \xi^2/2$ in (4.5). The radius of convergence of (5.3) again vanishes with θ, but that radius has a more complicated dependence on parameters now.

6 Example III: The GARCH Diffusion Model

The GARCH diffusion model, under risk neutrality, has the volatility process $dV = (\omega - \theta V)\,dt + \xi V dW(t)$, so that the eigenvalue problem (3.2) becomes

$$(6.1) \qquad -\tfrac{1}{2}\xi^2 V^2 \frac{d^2u}{dV^2} - \left[\omega - \theta V - ik\rho\xi V^{3/2}\right]\frac{du}{dV} + c(k)V\,u = \lambda(k)u$$

We don't have an exact solution, so we need approximate methods. In this section, we show one such method: the volatility of volatility series expansion. Previously, in Chapter 3, we showed how to use that expansion to develop the full time dependence for the fundamental transform. Now, we don't want the time dependence— only the first eigenvalue solution to (6.1). There are two unknowns: the eigenfunction $u(k,V)$ and the first eigenvalue $\lambda(k)$.

It's convenient to change variables from V to $x = c(k)V$. While this would generally make x complex-valued, the solution we need resides on the purely imaginary k-axis. That means it suffices to let k be purely imaginary and within the strip $0 < \operatorname{Im} k < 1$. With that restriction, $c(k)$ is a real, positive number and x is a real, positive number, just like V.

We let $u(k,V) = f(x)$, where we will suppress the explicit k-dependence. Finally, introduce the new parameters $A = c(k)\omega$, $B = \theta$, and $D = i\rho k/\sqrt{c(k)}$. All three parameters are real with k restricted as indicated. With these changes, (6.1) becomes

$$(6.2) \qquad \mathcal{L}_k f = \lambda f, \qquad (0 < \operatorname{Im} k < 1,\ \operatorname{Re} k = 0)$$

where $$\mathcal{L}_k f = -\tfrac{1}{2}\xi^2 x^2 \frac{d^2 f}{dx^2} - \left(A - Bx - \xi D\, x^{3/2}\right)\frac{df}{dx} + x\,f.$$

To create the series, substitute into (6.2) the formal expansions

$$\lambda = \sum \xi^j \lambda^{(j)}, \qquad f(x) = \sum \xi^j f^{(j)}(x)$$

For example, $f^{(1)}$ satisfies

$$(6.3)\qquad -(A - Bx)\frac{df^{(1)}}{dx} + \xi D x^{3/2}\frac{df^{(0)}}{dx} + x\, f^{(1)} = \lambda^{(0)} f^{(1)} + \lambda^{(1)} f^{(0)}.$$

When $\xi = 0$, we already have shown that

$$\lambda^{(0)} = c(k)\frac{\omega}{\theta} \qquad \text{and} \qquad u^{(0)} = \exp\left[-c(k)\frac{1}{\theta}\left(V - \frac{\omega}{\theta}\right)\right].$$

In terms of the new variables, this translates into

$$\lambda^{(0)} = \frac{A}{B} \;\text{ and }\; f^{(0)}(x) = \exp\left[-\frac{1}{B}(x - x^*)\right], \quad \text{where } x^* = \frac{A}{B}.$$

So (6.3) can be rewritten

$$(6.4)\qquad \frac{df^{(1)}}{dx} + \frac{1}{B} f^{(1)} = h^{(1)}(x) \quad ,$$

where
$$h^{(1)}(x) = -\frac{\left(\lambda^{(1)} + \frac{D}{B} x^{3/2}\right)}{A - Bx}\exp\left[-\frac{1}{B}(x - x^*)\right].$$

Now (6.4) is an ordinary differential equation with the general solution

$$(6.5)\qquad f^{(1)}(x) = Ce^{-x/B} + e^{-x/B}\int_{x_0}^{x} e^{y/B} h^{(1)}(y)dy\,,$$

where C and x_0 are constants. The solutions to an eigenvalue equation $\mathcal{L} f = \lambda f$ are clearly determined only up to some constant multiplier. So we need a normalization. Because $f^{(0)}(x = x^*) = 1$, we will enforce the normalization $f(x = x^*) = 1$. This means that $f^{(i)}(x = x^*) = 0$ for all $i \geq 1$. Potentially, $f^{(1)}(x = x^*) = 0$ can be achieved by choosing $C = 0$ and $x_0 = x^*$ in (6.5).

But (6.4) shows the integrand $h^{(1)}$ has a denominator term that vanishes at $x = x^*$, so we have to be careful. We need an assumption: suppose df/dx exists at $x = x^*$. Then, from (6.4) we see that $df^{(1)}/dx$ exists at $x = x^*$ if and only if $h^{(1)}(x = x^*)$ exists (since $f^{(1)}(x^*) = 0$ by the normalization condition). By L'Hospital's rule, $h^{(1)}(x = x^*)$ exists if the numerator expression for $h^{(1)}(x)$ also vanishes at $x = x^*$. This determines $\lambda^{(1)}$; we must have

$$(6.6)\qquad \lambda^{(1)} = -\frac{D}{B}\left(\frac{A}{B}\right)^{3/2}.$$

Then we can indeed take $C=0$, $x_0 = x^*$ and satisfy the normalization. Moreover, $f^{(1)}(x)$ has now been determined:

$$f^{(1)}(x) = e^{-x/B} \int_{x^*}^{x} e^{y/B} h^{(1)}(y) dy \tag{6.7}$$

This basic argument works to all orders in the expansion. The general recursion system is, for $n = 1, 2, \ldots$

$$\lambda^{(n)} = \left[D\, x^{3/2} \frac{d}{dx} f^{(n-1)} - \tfrac{1}{2} x^2 \frac{d^2}{dx^2} f^{(n-2)} \right]\Bigg|_{x=x^*}, \tag{6.8}$$

$$f^{(n)}(x) = e^{-x/B} \int_{x^*}^{x} e^{y/B}\, h^{(n)}(y)\, dy\,,$$

$$h^{(n)}(x) = \frac{-1}{(A-Bx)} \left[\sum_{j=1}^{n} \lambda^{(j)} f^{(n-j)} - Dx^{3/2} \frac{d}{dx} f^{(n-1)} + \tfrac{1}{2} x^2 \frac{d^2}{dx^2} f^{(n-2)} \right],$$

where terms with $(n-2)$ are omitted at $n=1$. Applying this algorithm, we find

$$\begin{aligned}
\lambda = \frac{A}{B} - \left(\frac{A}{B}\right)^{3/2} \frac{D}{B}\xi - \frac{A^2}{2B^4}(1-3D^2)\xi^2 \\
+ \left(\frac{A}{B}\right)^{3/2} \frac{D}{16B^4}(-3B^2 + 40A - 42AD^2)\xi^3 \\
+ \frac{A^2}{32B^7}\left[4A(8-73D^2+40D^4) - B^2(8-27D^2)\right]\xi^4 + O(\xi^5)
\end{aligned} \tag{6.9}$$

The stationary point must also be determined order by order in ξ. We find

$$k_0 = i\left\{ \frac{1}{2} + \left(\frac{\omega}{\theta}\right)^{1/2} \frac{\rho}{8\theta}\xi + \frac{\rho^2\omega}{8\theta^3}\xi^2 + \left(\frac{\omega}{\theta}\right)^{1/2} \frac{\rho\left[\omega(-6+41\rho^2)+6\theta^2\right]}{256\theta^4}\xi^3 + O(\xi^4) \right\}$$

As expected, k_0 is pure imaginary. The stationary point sticks at $k_0 = i/2$ if $\rho = 0$. Finally, the asymptotic implied volatility is given by

$$\begin{aligned}
V_\infty^{imp} &= 8\lambda(k_0) \\
&= \frac{\omega}{\theta} + \left(\frac{\omega}{\theta}\right)^{3/2} \frac{\rho}{2\theta}\xi + \frac{\omega^2(-1+7\rho^2)}{16\,\theta^4}\xi^2 + \left(\frac{\omega}{\theta}\right)^{3/2} \frac{\rho[\omega(-10+31\rho^2)+6\theta^2]}{64\theta^4}\xi^3 \\
&\quad + \frac{1}{256\theta^7}[\omega^3(4-81\rho^2-157\rho^4)+4\omega^2\theta^2(-2+15\rho^2)]\,\xi^4 + O(\xi^5)\,.
\end{aligned} \tag{6.10}$$

A dimensionality check. Recall the time dimensions for the GARCH diffusion parameters: $[V]=1/[t]$, $[\theta]=1/[t]$, $[\omega]=1/[t]^2$, $[\xi^2]=1/[t]$. So if we write $V_\infty^{imp}=\theta\, g(\bullet)$, then $g(\bullet)$ must be a function of dimensionless ratios. With only 3 parameters with dimensions, there are only two independent ratios, so we must have

$$V_\infty^{imp}=\theta\, g\left(\frac{\omega}{\theta^2},\frac{\xi}{\sqrt{\theta}}\right).$$

Indeed, one can check that (6.10) is equivalent to

$$\begin{aligned} g(x,z) = x &+ \tfrac{1}{2}\rho\, x^{3/2}z + \tfrac{1}{16}(-1+7\rho^2)x^2z^2 \\ &+ \tfrac{1}{64}\rho\,[(-10+31\rho^2)x^{5/2}+6x^{3/2}]\,z^3 \\ &+ \tfrac{1}{256}[(4-81\rho^2+157\rho^4)x^3+(-8+60\rho^2)x^2]\,z^4+O(z^5)\,. \end{aligned}$$

Numerical examples. We have extended (6.10) through $O(\xi^{10})$, although the expressions are too lengthy to report here. However, numerical examples showing the behavior of the partial sums through $O(\xi^{10})$ are given in Table 6.1. As one sees, the series is fairly well-behaved for typical parameter values and the partial sums tend to stabilize at higher order if θ is not too small. The series results are consistent with variational estimates, which are explained in the next section.

Table 6.1 Asymptotic Implied Volatility σ_{∞}^{imp} GARCH Diffusion Model: Series and Variational Methods

Model Parameters: $\omega_a = 0.02$, $\theta_a = 2$, $\xi_a = 1.5$.

	Correlation, ρ, between stock prices and volatility				
Series Order	**-1**	**-0.5**	**0.0**	**0.5**	**1.0**
ξ^0	10.0	10.0	10.0	10.0	10.0
ξ^2	9.8215	9.9071	9.9982	10.0946	10.1961
ξ^4	9.7870	9.8881	9.9973	10.1150	10.2418
ξ^6	9.7783	9.8828	9.9967	10.1212	10.2577
ξ^8	9.7764	9.8814	9.9964	10.1232	10.2640
ξ^{10}	9.7762	9.8811	9.9962	10.1238	10.2669
Variational	9.7759	9.8812	9.9961	10.1238	10.2708
Stationary Pt.	0.489 i	0.494 i	0.5 i	0.507 i	0.515 i

Notes for Tables 6.1 and 6.2 The tables show the asymptotic ($\tau \to \infty$) implied volatility for the GARCH diffusion model: $dV = (\omega - \theta V)\,dt + \xi V dW(t)$ versus the correlation ρ. Parameters are annualized. Two methods of calculation are shown; (i) a series expansion in powers of ξ and (ii) a variational method. The series results are the partial sums. Generally, there is good agreement between the two sets of results. The agreement is better in Table 6.1 than 6.2 because the series performs better at larger θ.

Table 6.2 Asymptotic Implied Volatility σ_∞^{imp} GARCH Diffusion Model: Series and Variational Methods

Model Parameters: $\omega_u = 0.01$, $\theta_u = 1$, $\xi_u = 1.5$.

	Correlation, ρ, between stock prices and volatility				
Series Order	**-1**	**-0.5**	**0.0**	**0.5**	**1.0**
ξ^0	10.0	10.0	10.0	10.0	10.0
ξ^2	9.6615	9.8161	9.9930	10.1910	10.4088
ξ^4	9.5453	9.7425	9.9851	10.2747	10.6143
ξ^6	9.5015	9.7041	9.9763	10.3271	10.7722
ξ^8	9.5001	9.6868	9.9666	10.3615	10.9163
ξ^{10}	9.5242	9.6834	9.9562	10.3833	11.0712
Variational	9.4789	9.6924	9.9577	10.3253	10.9712
Stationary Pt.	0.478 i	0.486 i	0.5 i	0.522 i	0.578 i

7 A Variational Principle Method

There is a deep connection between eigenvalue problems and variational principles[7]. In this section, we make that connection for our application. Very briefly, the first eigenvalue is a minimum of a certain functional. This extremal property can be exploited as a calculation tool, enabling the first eigenvalue (and hence the asymptotic implied volatility) to be estimated to high accuracy. What makes our application special is the presence of the complex-valued parameter k, a complication that requires careful handling.

We begin with the GARCH diffusion process of Sec. 6. After completing a full treatment including an example, we then extend the development to general processes.

We gave the full time-development equation for the fundamental transform at (**2**.2.19). With the volatility process given by the GARCH diffusion, we make the same change of variables as in Sec. 6, letting $x = c(k)V$, $A = c(k)\omega$, $B = \theta$, and $D = i\rho k / \sqrt{c(k)}$. In addition, we let $\hat{H}(k,V,\tau) = f(x,\tau)$, where we imply the k-dependence. Then (**2**.2.19) becomes

$$(7.1) \qquad -\mathcal{L}_k f = \tfrac{1}{2}\xi^2 x^2 \frac{\partial^2 f}{\partial x^2} + \left(A - Bx - \xi D\, x^{3/2} \right) \frac{\partial f}{\partial x} - x\, f = \frac{\partial f}{\partial \tau}.$$

The k-plane restriction. Throughout this section, we take the parameter k to be purely imaginary and restricted to the interval $0 < \operatorname{Im} k < 1$. Because of that restriction, the new variable x is real and positive, and the coefficients in (7.1) are all real. Because of the ridge property and the martingale property, that restricted interval in the complex k-plane suffices to determine the asymptotic implied volatility for the option problem.

An auxiliary stochastic process. With our k-plane restrictions, (7.1) can be associated with the *real-valued*, auxiliary stochastic process

$$(7.2) \qquad dx = \left(A - Bx - \xi D\, x^{3/2} \right) dt + \xi x\, dB(t), \qquad 0 < x < \infty,$$

---

[7] A classical and extensive reference is Courant and Hilbert (1989), Chapts IV and VI.

where $dB(t)$ is a Brownian motion. We use "auxiliary" because (7.2) is *not* where we started. We began with the GARCH diffusion under risk neutrality, which is $dV = (\omega - \theta V)\,dt + \xi V dB(t)$ —the auxiliary process has an extra term with the D coefficient.

The forward equation. Next, consider the so-called "forward equation" for the auxiliary process:

$$\frac{\partial p}{\partial \tau} = \mathcal{A}^{\dagger} p = \tfrac{1}{2}\xi^2 \frac{\partial^2}{\partial x^2}(x^2 p) - \frac{\partial}{\partial x}\left\{(A - Bx - \xi D\, x^{3/2})p\right\}. \tag{7.3}$$

Our notation is that $\mathcal{A}$ is the generator for the stochastic process (7.2) and $\mathcal{A}^{\dagger}$ is the formal adjoint. A time-independent solution to (7.3) is

$$p(x) = x^{-2-2B/\xi^2} \exp\left\{-\frac{2A}{\xi^2 x} - \frac{4D\sqrt{x}}{\xi}\right\} \tag{7.4}$$

We use the notation $p(x)$ to stress the positivity of the solution. When $p(x)$ can be normalized, ($\int_0^\infty p(x)dx < \infty$), it may be interpreted as the long-run stationary probability distribution for the auxiliary process[8]. But we want to emphasize that the variational theory of this section does not require that $p(x)$ be integrable. The properties that are important are (i) $\mathcal{A}^{\dagger} p(x) = 0$ and (ii) $p(x) > 0$.

The variational principle. Recall the eigenvalue problem $\mathcal{L}_k u = \lambda(k) u$ defined at (6.2), where λ is the first eigenvalue and u is the first eigenfunction. Multiply both sides by $u(x)p(x)$ and integrate by parts. Using $\mathcal{A}^{\dagger} p(x) = 0$ and some algebra, you can establish the formula:

$$\lambda = \frac{\int_0^\infty p(x)\left\{\frac{1}{2}\xi^2 x^2 [u'(x)]^2 + x[u(x)]^2\right\}dx}{\int_0^\infty p(x)[u(x)]^2\,dx} \tag{7.5}$$

if $\quad a$(i) the boundary terms from the parts integrations vanish:

$$\lim_{x \to 0,\infty} x^2 p(x) u(x) u'(x) = 0,$$

and $\quad a$(ii) all the integrals in (7.5) exist.

---

[8] See Karlin and Taylor (1981, Chapter 15)

These are typical conditions associated with a variational method—let's call $\mathcal{A}$(i) the *endpoint conditions.* We pointed out previously that $\mathcal{L}_k u = \lambda(k)u$ is not well-defined until we specify a class of functions on which $\mathcal{L}_k$ acts. Different classes can give different eigenvalues. One natural class of functions for our problem is seen to be all twice differentiable functions $f(x)$ such that the integrals in (7.5) exist and the endpoint conditions $\mathcal{A}$(i) hold. We call such functions *admissible* and denote the set of all such function by $\mathcal{A}$, so (7.5) holds if $u \in \mathcal{A}$.

The variational principle asserts that, for all $f(x) \subset \mathcal{A}$, then

$$(7.6) \qquad \lambda = \min_{f(x)\in\mathcal{A}} \left\{ \frac{\int_0^\infty p(x)\left\{\frac{1}{2}\xi^2 x^2 [f'(x)]^2 + x[f(x)]^2\right\}dx}{\int_0^\infty p(x)[f(x)]^2\,dx} \right\}$$

Specifically, a function $f(x)$ is admissible if

$$\mathcal{A}(\mathrm{i}) \quad \lim_{x\to 0,\infty} x^2 p(x) f(x) f'(x) = 0\,,$$

and the integrals

$$\mathcal{A}(\mathrm{ii}) \quad \int p(x) f^2 dx\,, \qquad \int p(x) x f^2 dx\,, \qquad \int p(x) x^2 (f')^2 dx$$

are convergent. Note that the endpoint conditions do not require that either $f(x)$ or $f'(x)$ individually exist at $x = 0, \infty$. As we stressed before, the integrability conditions do not require that $p(x)$ itself be integrable. For example, when $D < 0$, then $p(x)$ is not integrable, but $f(x) = \exp(-\alpha x)$ for $\alpha > 0$ is admissible. We use exactly this form in our computational example below.

The variational principle (7.6) follows from the *Euler-Lagrange* equations of the theory of the calculus of variations. It's a powerful tool that may be used to estimate λ to high accuracy by selecting suitable *trial* functions $f(x)$. Of course, a trial function should be admissible at the very least. In fact, for admissible $f(x)$, the inequality

$$(7.7) \qquad \lambda \le \frac{\int_0^\infty p(x)\left\{\frac{1}{2}\xi^2 x^2 [f'(x)]^2 + x[f(x)]^2\right\}dx}{\int_0^\infty p(x)[f(x)]^2\,dx}$$

is the tightest possible upper bound because it will be realized as an equality if $f(x)$ is chosen to be the first eigenfunction.

It's interesting that in (7.7), the only explicit parameter that appears is ξ^2. Of course, we know that the eigenvalue λ depends upon the *four* parameters of the problem: A, B, D, and ξ^2 or equivalently ω, θ, ρ, and ξ^2. The other three parameters have not disappeared, but are contained in $p(x)$.

The case $\rho = 0$ for the GARCH diffusion. Fig. 6.2 shows an example term structure when $\rho = 0$. Note how the asymptotic implied volatility, 9.92%, is less than the deterministic value, 10%. While Fig. 6.2 is a plot of the square root model, it suggests a result for the GARCH diffusion because the two models share the same linear drift form. Indeed, the variational principle implies that, when $\rho = 0$, then σ_∞^{imp} never exceeds $(\omega/\theta)^{1/2}$ in the GARCH diffusion. Let's see why.

We assume that $\omega > 0$ and $\theta > 0$. If $\rho = 0$, then $D = 0$ and $p(x)$ is normalizable. In that case $f(x) = 1$ is admissible and (7.7) implies that

$$\lambda(k) \leq \frac{\int p(x) x\, dx}{\int p(x) dx} = \frac{A}{B} = c(k)\frac{\omega}{\theta}.$$

The stationary point for $c(k)$ is $k_0 = i/2$, so we obtain $\lambda(k_0) \leq c(i/2)\omega/\theta = \omega/(8\theta)$. In other words, when $\rho = 0$, then $V_\infty^{imp} \leq \omega/\theta$. ■

When $\rho \neq 0$, then $V_\infty^{imp} > \omega/\theta$ is possible. For example, the first two terms of the ξ - expansion for the GARCH diffusion at (6.10) are

$$V_\infty^{imp} = \frac{\omega}{\theta} + \left(\frac{\omega}{\theta}\right)^{3/2} \frac{\rho}{2\theta}\xi + O(\xi^2),$$

and this will be larger than ω/θ for small ξ and positive ρ. See Tables 6.1, 6.2 or Fig. 6.5 for more examples of $V_\infty^{imp} > \omega/\theta$.

Numerical example. We continue with the GARCH diffusion for a numerical example using the variational principle. Although we have suppressed the k-dependence in many formulas, to actually calculate, we need to reinstate it. More explicitly, (7.7) is a bound for $\lambda(k)$, where $k = iy$, y is real and in the interval $0 < y < 1$. The weight function is more explicitly $p(k, x)$, where

$$p(k,x) = x^{-2-2\theta/\xi^2} \exp\left\{ -\frac{\omega\,(k^2 - ik)}{\xi^2 x} - \frac{4\,i\rho\,k}{\xi}\left[\frac{2x}{(k^2 - ik)}\right]^{1/2} \right\}.$$

Let $\tilde{p}(y,x) = p(iy,x)$, a real positive function, given by

$$\tilde{p}(y,x) = x^{-2-2\theta/\xi^2} \exp\left\{ -\frac{\omega\, y(1-y)}{\xi^2 x} + \frac{4\rho}{\xi}\left[\frac{2xy}{(1-y)}\right]^{1/2} \right\}. \tag{7.8}$$

Then, we can calculate the asymptotic implied volatility from

$$V_\infty^{imp} \le \max_{0<y<1} \min_{f(x)\in\mathcal{A}} \left\{ \frac{8\int_0^\infty p(y,x)\left\{\frac{1}{2}\xi^2 x^2 [f'(x)]^2 + x[f(x)]^2\right\}dx}{\int_0^\infty p(y,x)[f(x)]^2\,dx} \right\} \tag{7.9}$$

Note that its a *maximum* over y because the fundamental transform has a saddle point in the k-plane, which happens to have a maximum in the real direction and a minimum in the imaginary direction. So the fundamental transform has a minimum as a function of y at the saddle point. But the eigenvalue affects the fundamental transform through a multiplicative term $\exp(-\lambda\tau)$; that means we need a maximum in the eigenvalue as a function of y.

Let's check the consistency of these new ideas with the series solution of Sec. 6. Choosing a suitable trial function is something of an art. Your goal is to select a function that is admissible, produces integrals that can be calculated, and captures the qualitative behavior of the first eigenfunction. For example, for the GARCH diffusion, we choose the trial function $f(x) = \exp(-\alpha x)$, where α is a parameter which is optimized. This choice for the trial function is motivated by the series solution $u(x) = \exp(-x/\theta)[1 + O(\xi)]$. The integrals in (7.9) may be computed by using

$$\int_0^\infty \tau^{\mu-1} \exp\left(-s\tau - \frac{1}{\tau}\right) d\tau = 2\, s^{-\mu/2} K_\mu(2s^{1/2}), \tag{7.10}$$

where $K_\mu(\cdot)$ is the modified Bessel function of the second kind of order μ. In this example, we find that (7.9) becomes

$$(7.11)\ V_\infty^{imp} \le \max_{0<y<1} \min_{a>0} \frac{\xi^2}{g_\mu(y,a,b)} \left\{ g_{\mu-2}(y,a,b) + \frac{\kappa y(1-y)}{a^2} g_{\mu-1}(y,a,b) \right\},$$

using $$g_\mu(y,a,b) = \left(\frac{2}{a}\right)^\mu \sum_{n=0}^{\infty} \left(\frac{by}{\sqrt{a}}\right)^n \frac{1}{n!} K_{\mu-n/2}(a), \quad \text{and}$$

$$\kappa = \frac{32\omega}{\xi^4}, \quad b = \frac{8\rho}{\xi^2}\sqrt{\omega}, \text{ and } \mu = 1 + \frac{2\theta}{\xi^2}.$$

In (7.11), the minimization over the original parameter α has been replaced by a minimization over a new parameter a. The relationship between the two is that $a = 4\sqrt{\alpha A}/\xi$. The optimization (7.11) is very straightforward to implement in Mathematica: see Appendix 1 to this Chapter.

Numerical results from computing the right-hand-side of (7.11) are given in Tables 6.1 and 6.2. The implied volatility estimate (an upper bound) is given in the table under "Variational". And "Stationary Point" reports $k_0 = iy_0$, where y_0 is the maximizing value in (7.11). The table shows a good match to the series results, which helps support the consistency and assumptions of both approaches.

Again using the bounds from (7.11), in Fig. 6.5 we plot σ_∞^{imp} versus the correlation ρ. The other parameters are $\omega_a = 0.01$, $\theta_a = 1$, and $\xi_a = 1.5$. Since $\omega/\theta = 0.01$, the deterministic volatility value for σ_∞^{imp} is 10%. The figure illustrates the fact that σ_∞^{imp} can be higher or lower than the deterministic value, depending upon the correlation.

In Fig. 6.6 we plot σ_∞^{imp} versus the volatility of volatility ξ for $\rho = 0$ and $\rho = -0.5$. The other parameters are the same as Fig. 6.4. This figure shows that σ_∞^{imp} stays quite close to the deterministic value when $\rho = 0$, even for relatively large ξ. But, for $\rho = -0.5$, σ_∞^{imp} drops off from the deterministic value much more rapidly with ξ.

General processes. We now extend the variational principle to general risk-adjusted processes of the form $dV = \tilde{b}(V)dt + a(V)dW$, with correlation $\rho(V)$. This is the one subsection in this chapter where we assume a genuine risk-adjustment may be present. Under this general process, the evolution equation for the fundamental transform has been given at (**2**.219).

It's not useful to make exactly the same change of variable that we made for the GARCH diffusion. But we can make (2.2.19) similar by letting $x = V$ and $\hat{H}(k,V,\tau) = f(x,\tau)$. Then (2.2.19) becomes

$$\frac{\partial f}{\partial \tau} = -\mathcal{L}_k f \,, \tag{7.12}$$

where $\quad -\mathcal{L}_k f = \frac{1}{2} a^2(x)\frac{\partial^2 f}{\partial x^2} + \left[\tilde{b}(x) - ik\rho(x)a(x)x^{1/2}\right]\frac{\partial f}{\partial x} - c(k)xf \,.$

Just as in the GARCH diffusion, we assume that $k = iy$, where y is real and in the interval $0 < y < 1$. Then, all of the coefficients in (7.12) are real-valued and we can associate it with the auxiliary process:

$$dx = \left[\tilde{b}(x) - ik\rho(x)a(x)x^{1/2}\right]dt + a(x)dB(t)\,, \tag{7.13}$$
$$\text{where } 0 < x < \infty\,,\;\; k = iy\,,\;\; 0 < y < 1\,.$$

This can be written more compactly by defining the (real-valued) auxiliary drift coefficient

$$\beta_k(x) = \tilde{b}(x) - ik\rho(x)a(x)x^{1/2}\,, \tag{7.14}$$

so that (7.13) reads

$$dx = \beta_k(x)dt + a(x)dB(t)\,,\;\; (0 < x < \infty\,,\;\; k = iy\,,\;\; 0 < y < 1). \tag{7.15}$$

The forward equation for the auxiliary process is

$$\frac{\partial p_k}{\partial \tau} = A^{\dagger} p_k = \frac{1}{2}\frac{\partial^2}{\partial x^2}\left[a^2(x)p_k(x)\right] - \frac{\partial}{\partial x}\left[\beta_k(x)p_k(x)\right]. \tag{7.16}$$

And there is a (time-independent) solution to $A^{\dagger} p_k = 0$ given by

$$p_k(x) = \frac{1}{a^2(x)}\exp\left[\int^x \frac{2\beta_k(y)}{a^2(y)}dy\right]\,, \tag{7.17}$$

which is the analog of (7.4). Note that $p_k(x)$ is a positive real number for all (x,k) such that $0 < x < \infty$, $k = iy$, $(0 < y < 1)$. But $p_k(x)$ is not necessarily integrable with respect to x.

The variational principle for the first eigenvalue then becomes

$$(7.18)\qquad \lambda(k) = \min_{f(x)\in \mathcal{A}(k)} \left\{ \frac{\int_0^\infty p_k(x)\left\{\frac{1}{2}a^2(x)[f'(x)]^2 + c(k)x[f(x)]^2\right\}dx}{\int_0^\infty p_k(x)[f(x)]^2\,dx} \right\}.$$

The space $\mathcal{A}(k)$ of admissible functions consists of all real-valued functions that satisfy, for $k = iy$, $(0 < y < 1)$, the conditions:

(i) $\lim_{x\to 0,\infty} a^2(x)p_k(x)f(x)f'(x) = 0$,

(ii) $\int p_k(x)[f(x)]^2\,dx < \infty$, $\quad \int p_k(x)x[f(x)]^2\,dx < \infty$,

$$\int p_k(x)a^2(x)[f'(x)]^2 dx < \infty .$$

Note that if a first eigenvalue exists for $k = iy$, $(0 < y < 1)$, then it must be strictly positive since every term in (7.18) is positive. So we really have a two-sided bound: $\lambda > 0$ and $\lambda <$ the upper bound of (7.18).

General process (zero correlation). Let's apply the general process variational principle to the case $\rho = 0$. In that case, the auxiliary process (7.14) is *independent of* k and coincides with the risk-adjusted volatility process. So $p_k(x)$ is independent of k. Let's assume that the risk-adjusted volatility process has a long run stationary distribution $\tilde{p}(V)$ and a finite first moment. Then $p_k(x) = \tilde{p}(V)$, $p(x)$ is integrable and $f = 1$ is admissible. If we choose $f = 1$ for a trial function, then (7.18) implies that

$$(7.19)\qquad \lambda(k) \le c(k)\frac{\int_0^\infty V\,\tilde{p}(V)\,dV}{\int_0^\infty \tilde{p}(V)dV},$$

The stationary point for the right-hand-side of (7.19) is, as we expect, at $k_0 = i/2$, where $c(i/2) = 1/8$. Moreover, since $\tilde{p}(V)$ is integrable, let's normalize the denominator to 1. Then, we have the bound:

$$(7.20)\qquad V_\infty^{imp} \le \int_0^\infty V\,\tilde{p}(V)\,dV. \qquad (\rho = 0) \qquad \blacksquare$$

Let's check this result for the exactly solvable models. For example, for the 3/2 model, it's easy to find the stationary distribution

$$\tilde{p}(V) = CV^{\,3-2\theta/\xi^2}\exp\left[-2\omega/(V\xi^2)\right],$$

where C is the normalization constant. For simplicity, assume $\omega, \theta > 0$. Then, (7.20) reads

$$V_\infty^{imp} \le \frac{2\theta}{2\theta + \xi^2} \frac{\omega}{\theta} . \quad \text{(3/2 model)}$$

This can be proven correct using the exact solution (5.3). It's a tighter bound than simply the deterministic limit ($\xi^2 = 0$).

A second check is models with a linear drift: $dV = (\omega - \theta V)\,dt + a(V)\,dW(t)$. In the case of linear drift models with some growth restrictions on $a(V)$, we showed in Appendix 5.1 (Example 1) that the long-run expected volatility is always ω/θ. So for all such 'qualified' linear drift models, (7.20) reads $V_\infty^{imp} \le \omega/\theta$, which was our previous result under the GARCH diffusion alone.

Finally, we could relax the assumption that $\tilde{p}(V)$ have a first moment, since the inequality (7.20) also makes sense if the right-hand-side is $+\infty$.

An open issue. Suppose you've solved the PDE problem (**2**.2.19) for the fundamental transform $\hat{H}(k,V,\tau)$. Your solution turns out to be a regular function in the complex k-plane in the strip $\alpha < \operatorname{Im} k < \beta$. Next, you let $k = iy$, where $\max[0,\alpha] < y < \min[1,\beta]$. As $\tau \to \infty$, you find that $\hat{H}(k,V,\tau) \approx \exp[-\tilde{\lambda}(k)\tau]\,\varphi(k,V)$.

Separately, with k in the same interval, you've found $\lambda(k)$, the first eigenvalue solution to $\mathcal{L}_k u = \lambda(k) u$, $u \in \mathcal{A}(k)$, where $\mathcal{A}(k)$ is the class of admissible functions defined in this section.

The open issue: is it always true that $\lambda(k) = \tilde{\lambda}(k)$? In other words, we've really just summarized the developments in Secs. 3 and 6, and are asking if they always lead to the same value for the asymptotic implied volatility. If they don't, then the conditions that define the function space $\mathcal{A}(k)$ must be revised.

8 A Differential Equation (DSolve) Method

In this section, we explain how to find the asymptotic implied volatility by solving a differential equation[9]. Numerically, we do this with Mathematica's **DSolve** function (actually **NDSolve,** to be precise)—hence the reference in the section title.

The method is very fast and produces values in just a couple seconds of desktop computer time. The variational method can be fast, too, with only a single parameter being optimized. But if you want higher accuracy in the variational method, you have to develop more complex trial functions, with more parameters. As we indicated, this is something of an art. In contrast, the method in this section, if you can set it up, can be made arbitrarily accurate just by adjusting function arguments.

A tradeoff is that the method in this section requires a certain asymptotic analysis, which is explained below. The method works for the GARCH diffusion, which is one of our main interests, because we can perform that analysis. For other models, you have to investigate.

Consider again the eigenvalue problem under GARCH diffusion process, given at (6.1) and we repeat here for convenience

$$(8.1) \qquad -\tfrac{1}{2}\xi^2 V^2 \frac{d^2u}{dV^2} - \left[\omega - \theta V - ik\rho\,\xi V^{3/2}\right]\frac{du}{dV} + c(k)V\,u = \lambda(k)u\,.$$

As before, consider k a purely imaginary parameter: $k = iy$ and y is in the interval $0 < y < 1$. So all of the coefficients in (8.1) are real numbers.

It makes the discussion simpler if we do a rescaling first, so multiply both sides of (8.1) by $2/\xi^2$, and define new (real) parameters

$$\tilde{\omega} = \frac{2}{\xi^2}\omega\,, \quad \tilde{\theta} = \frac{2}{\xi^2}\theta\,, \quad \tilde{d} = \frac{2ik\rho}{\xi}\,, \quad \tilde{c} = \frac{2}{\xi^2}c\,, \quad z = \frac{2}{\xi^2}\lambda\,.$$

Then (8.1) becomes

---

[9] The method in this section is adapted from a similar procedure in Aslanyan and Davies (1998)

$$V^2u'' + \left(\tilde{\omega} - \tilde{\theta}V - \tilde{d}V^{3/2}\right)u' - \tilde{c}V\,u + z\,u = 0\,. \tag{8.2}$$

We can find the first eigenvalue by the following procedure. First, forget that z in (8.2) is related to an eigenvalue and just think of it a parameter that is fixed at some real value, say 6. Then, it's possible to develop asymptotic solutions for (8.2) both as $V \to 0$ and $V \to \infty$, which are singular points. Because (8.2) is a second order equation, there are two such solutions in any regime. But the one we report is the smaller one. Exactly how to do this is explained in Chapter 10; here we merely quote the results:

First, as $V \to 0$, we find that the "well-behaved" solution has the form:

$$u \approx a_0\left[1 - \frac{z}{\tilde{\omega}}V\right] + O(V^{3/2})\,, \tag{8.3}$$

where a_0 is arbitrary. At the other extreme, as $V \to \infty$, we find that the well-behaved solution has the form:

$$u \approx \exp\left[\left(\beta - \tilde{d}\right)\sqrt{V}\right]V^{\left(\frac{1}{4}-\frac{\alpha}{2}+\frac{\tilde{\theta}}{2}\right)}\left[b_0 + \frac{b_1}{\sqrt{V}} + O\left(\frac{1}{V}\right)\right], \tag{8.4}$$

where $\quad \alpha = \dfrac{\tilde{d}(1+2\tilde{\theta})}{2\beta} \quad$ and $\quad \beta = -\dfrac{2}{\xi}\eta(k) = -\dfrac{2}{\xi}\sqrt{k^2(1-\rho^2) - ik}$

and b_0 and b_1 are constants that play no role. In general, the method of this section "works" whenever you can develop asymptotic solutions to the eigenvalue equation. This will be true in many models of interest. Next, consider the function

$$g(V) = \frac{u'(V)}{u(V)}. \tag{8.5}$$

This function satisfies the *first order* (non-linear) differential equation, called the *Riccati equation*:

$$V^2\frac{dg}{dV} + V^2g^2 + \left(\tilde{\omega} - \tilde{\theta}V - \tilde{d}V^{3/2}\right)g - \tilde{c}V + z = 0 \tag{8.6}$$

Now pick a small value $V_{\min}$, a large value $V_{\max}$ and an arbitrary point a in between: $V_{\min} < a < V_{\max}$.

We can solve (8.6) in the interval $V_{\min} \le V \le a$ by starting the solution at $V_{\min}$. We start the solution by using (8.3), which implies that for small enough $V_{\min}$,

$$g(V_{\min}) \approx -\frac{z}{\tilde{\omega}}. \tag{8.7}$$

Call this solution $g_0(V,z)$. Similarly, we can solve (8.6) in the interval $a \le V \le V_{\max}$ by using (8.4), which implies that for large enough $V_{\max}$.

$$g(V_{\max}) \approx \frac{(\beta - \tilde{d})}{2\sqrt{V_{\max}}} + \frac{(1 - 2\alpha + 2\tilde{\theta})}{4V_{\max}}. \tag{8.8}$$

Call this solution $g_1(V,z)$. Finally, define the function

$$F_a(z) = g_1(a,z) - g_0(a,z) = \frac{u_1'(a,z)u_0(a,z) - u_0'(a,z)u_1(a,z)}{u_0(a,z)u_1(a,z)}. \tag{8.9}$$

Now for a general value of z, the solution which behaves like (8.3) as $V \to 0$, if continued beyond the point $V = a$, will *not* behave like (8.4) as $V \to \infty$. There's a second solution that grows much more rapidly than (8.4) as $V \to \infty$; call that one the "ill-behaved" solution (see Chapter 10 for its form). For an arbitrarily chosen value of z, if you start the solution with (8.3), and continue that solution beyond the point $V = a$, you'll get a mixture of the well-behaved solution and the ill-behaved solution as $V \to \infty$.

But, for $z = z_0 \equiv 2\lambda/\xi^2$, where λ is the first eigenvalue, the solution $u_0(V,z_0)$, if it was continued beyond $V = a$, would be found proportional to the well-behaved solution $u_1(V,z_0)$, at least in the limit where $V_{\min} \to 0$ and $V_{\max} \to +\infty$. If $u_0(V,z_0) = m\, u_1(V,z_0)$, with m a constant, then the numerator in (8.9) vanishes and the denominator is proportional to $u^2(a,z_0)$, the square of the first eigenfunction.

If you increase z from zero, the first value at which $F_a(z)$ vanishes must then be the first (smallest) eigenvalue. Moreover, since we know the first eigenfunction $u(a,z_0)$ is positive for all V, the denominator in (8.9) will not vanish when the numerator does. To summarize, in the limit where $V_{\min} \to 0$ and $V_{\max} \to +\infty$, the first eigenvalue $\lambda = \xi^2 z_0/2$, where z_0 is the first (smallest) zero of $F_a(z)$ on the real positive z-axis.

In Mathematica, the **`NDSolve`** function easily finds numerical solutions to (8.6), creating $F_a(z)$. Then, **`FindRoot`** finds the zero z_0 of $F_a(z)$. All this happens when the parameter k is fixed at a pure imaginary value in the vicinity of $k = i/2$. That is $z_0 = (2/\xi^2)\lambda(k)$ and finally we use **`FindMinimum`** to find the stationary value k_0. The code is in the Appendix to this chapter.

A by-product of the calculation of λ is that you then have available the full function $g(V, z_0)$, which is defined over the entire range $V_{\min} \le V \le V_{\max}$ by

$$g(V, z_0) = \begin{cases} g_0(V, z_0) & V_{\min} \le V \le a \\ g_1(V, z_0) & a \le V \le V_{\max} \end{cases}$$

Note that $g(V, z_0)$ is continuous at $V = a$ for any values of $V_{\min}$ and $V_{\max}$ because $g_0(a, z_0) = g_1(a, z_0)$. Then, the first eigenfunction is given by the limiting value, as the boundaries become exact, of

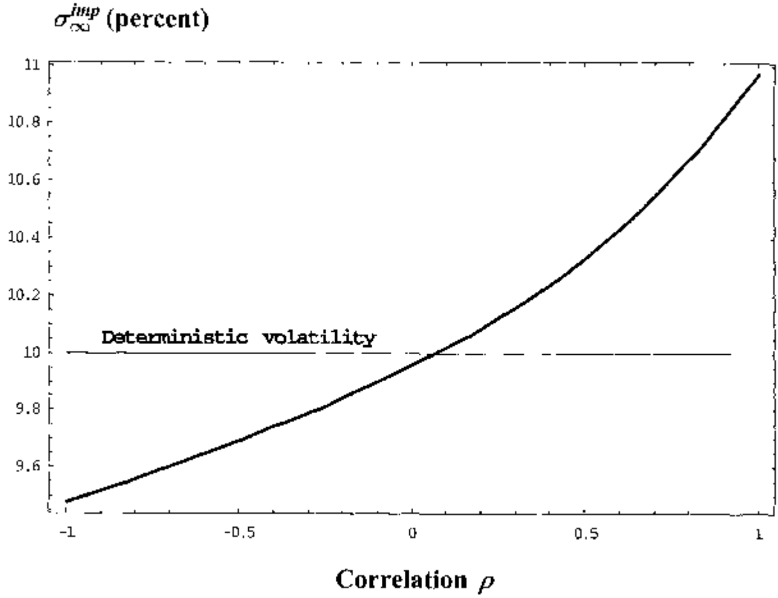
$$u(V) = \exp\left[\int_a^V g(x, z_0)\, dx \right] \tag{8.10}$$

In Mathematica, the expression (8.10) evaluates extremely rapidly because $g(x, z_0)$, being the result of a solution to a differential equation is an "interpolating function" and such functions are rapidly integrated. We used (8.10) to produce the plots shown in Fig. 6.4. This code is also given in the Appendix.

The eigenvalues are independent of a in the limit that $V_{\min} \to 0$ and $V_{\max} \to \infty$. In practice, there's a very weak dependence with finite endpoints. Since a is a volatility value, a natural choice for the GARCH diffusion and the one we selected was $a = \omega/\theta$. A brief sensitivity analysis showed very little difference in results if the value was 50% higher or lower.

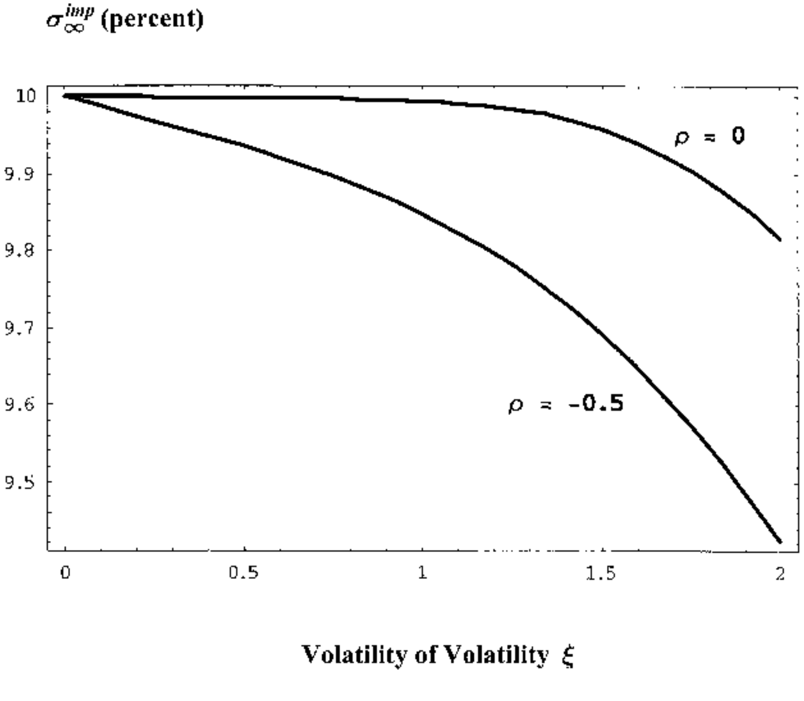
Numerical results are shown in Table 6.3 below. As you can see, the values for the asymptotic implied volatility are virtually indistinguishable from the variational method results. The method is very straightforward, fast, and should be easy to adapt to many models.

Fig. 6.5 Asymptotic Implied Volatility vs Correlation: GARCH Diffusion Model



Notes. The figure shows the asymptotic implied volatility for the GARCH diffusion model $dV = (\omega - \theta V)\,dt + \xi V dW(t)$ versus the correlation parameter ρ. The other parameters are $\omega_a = 0.01$, $\theta_a = 1$, and $\xi_a = 1.5$. Since $\omega/\theta = 0.01$, the deterministic volatility value for σ_∞^{imp} is 10%. The figure illustrates the fact that σ_∞^{imp} can be higher or lower than the deterministic value, depending upon the correlation. The values are upper bounds calculated from a variational method, but both a series and a differential equation method produce the same plot.

Fig. 6.6 Asymptotic Implied Volatility vs. Volatility of Volatility: GARCH Diffusion Model



Notes. The figure shows the asymptotic implied volatility for the GARCH diffusion model $dV = (\omega - \theta V)\,dt + \xi V dW(t)$ versus the volatility of volatility parameter ξ. The other parameters are $\omega_a = 0.01$, $\theta_a = 1$. This figure shows that σ_{∞}^{imp} stays quite close to the deterministic value (10%) when $\rho = 0$, even for relatively large ξ. But, for $\rho = -0.5$, σ_{∞}^{imp} drops off from the deterministic value much more rapidly with ξ. The values are upper bounds calculated from a variational method.

Table 6.3 Asymptotic Implied Volatility σ_∞^{imp} (GARCH Diffusion) Variational and Differential Equation (DSolve) Methods

I. Model Parameters: $\omega_a = 0.02$, $\theta_a = 2$, $\xi_a = 1.5$.

Method	Correlation, ρ, between stock prices and volatility				
	-1	**-0.5**	**0.0**	**0.5**	**1.0**
(i) Variational					
σ_∞^{imp}	9.7759	9.8812	9.9961	10.1238	10.2708
Stationary Point	0.489 i	0.494 i	0.5 i	0.507 i	0.515 i
(ii) DSolve					
σ_∞^{imp}	9.7759	9.8812	9.9961	10.1237	10.2701
Stationary Point	0.490 i	0.494 i	0.499 i	0.507 i	0.517 i

II. Model Parameters: $\omega_a = 0.01$, $\theta_a = 1$, $\xi_a = 1.5$.

Method	Correlation, ρ, between stock prices and volatility				
	-1	**-0.5**	**0.0**	**0.5**	**1.0**
(i) Variational					
σ_∞^{imp}	9.4789	9.6924	9.9577	10.3253	10.9712
Stationary Point	0.478 i	0.486 i	0.5 i	0.522 i	0.578 i
(ii) DSolve					
σ_∞^{imp}	9.4785	9.6919	9.9570	10.3243	10.9649
Stationary Point	0.478 i	0.488 i	0.5 i	0.521 i	0.577 i

Notes. The panels show the asymptotic ($\tau \to \infty$) implied volatility for the GARCH diffusion model: $dV = (\omega - \theta V)\,dt + \xi V dW(t)$ versus the correlation ρ. Parameters are annualized. Two methods of calculation are shown: (i) a variational method and (ii) a differential equation method (DSolve). The results are extremely close. While both methods produce values in just a couple of seconds in Mathematica, the DSolve method ran the fastest and can be made arbitrarily accurate.

Appendix 6.1
Mathematica Code for Chapter 6

1. (Sec. 7) A Variational Principle Method. Equation (7.11) can be implemented as follows. First, define the function $g_\mu(y,a,b)$ with the rules:

```
g[mu_,y_,a_,0,nmax_] := (2/a)^mu  BesselK[mu,a]
g[mu_,y_,a_, b_,nmax_] := (2/a)^mu *
 NSum[(b y/Sqrt[a])^n/n! BesselK[mu-n/2,a],{n,0,nmax}]
```

Next, the variational inequality itself is written in the form

$$V_\infty^{imp} \le \max_{0<y<1} \min_a \{\xi^2 \, \mathrm{Obj}(y,a)\},$$

where the objective function $\mathrm{Obj}(y,a)$ is coded:

```
Obj[mu_,y_,a_,b_,kappa_,nmax_]:=
 Module[{num,den},
    num = g[mu-2,y,a,b,nmax]+
       kappa y (1-y)/a^2 g[mu-1,y,a,b,nmax];
    den = g[mu,y,a,b,nmax];
 Return[num/den]]
```

Finally, the search for the extreme values is performed. This is done with the code block on the next page:

```
AsymImpSigma[omega_,theta_,ksi_,rho_,nmax_]:=
 Module[{mux,kappa,b,first,ans },
   If[ksi == 0,
      Return[{100. Sqrt[omega/theta],1/2}],Null];
   mux = 1 + 2 theta/ksi^2;
   kappa = 32 omega/ksi^4;
   b = Sqrt[omega] 8 rho /ksi^2;
   first[y_] :=
    FindMinimum[Obj[mux,y,a,b,kappa,nmax],
    {a,{.1,.11}}][[1]];
   ans = FindMinimum[-first[y],{y,{.5,.51}}];
   Return[{100.Sqrt[-ksi^2 ans[[1]]],y/.ans[[2]]}]]
```

The function **AsymImpSigma** takes as arguments the four parameters for the GARCH diffusion process ω, θ, ξ, and ρ. In addition, **nmax** tells how many terms to keep in the summation for $g_\mu(y,a,b)$. It returns a two element list $\{\sigma_\infty^{imp}, y_0\}$. The first element is an upper bound for the asymptotic implied volatility, in percent. The second element is the imaginary part of the value of the stationary point in the complex k-plane, which is given by $k_0 = i y_0$.

For example, with $\omega = 0.02$, $\theta = 2$, $\xi = 1.5$, and $\rho = -0.5$, experimentation showed **nmax = 20** was more than sufficient for convergence. In the code block, the function **first** is the first optimization (over **a**). We plotted **first** and saw it had a minimum in the vicinity of **a = .1**; that's why we start the search for a minimum there. Here is a typical result with the **Timing**:

```
In[458]:= Timing[AsymImpSigma[.02, 2, 1.5, -.5, 20]]

Out[458]= {3.57 Second, {9.88117, 0.494382}}
```

The value for σ_∞^{imp} was approximately 9.88% and the stationary point was at $k_0 \simeq 0.494\,i$. We then used this function to plot the asymptotic implied volatility versus parameters: Figs 6.3 and 6.4.

2. (Sec. 8) A Differential Equation (Dsolve) Method

Solving the nonlinear differential equation (8.6) for $g_0(V,z)$ and $g_1(V,z)$ is the essence of the algorithm. For example, $g_0(V,z)$ is obtained from:

```
g0[xmin_,a_,z_,k_]:=
  Module[{c = cf[k],d = df[k],a1,g},
    Clear[g0f];
    a1 = -z/omega;
    g0f = g/.
     NDSolve[{x^2 g'[x] + x^2 g[x]^2 +
     (omega - theta x - d x^(3/2))g[x] - c x + z == 0,
       g[xmin] == a1},g,{x,xmin,a},
       AccuracyGoal->10][[1]];
  Return[g0f[a]]]
```

Most of the notations follow the text in Sec. 8, except that we use x instead of V The parameter k is the purely imaginary complex value. The functions **`cf`** and **`df`** are given below. The parameters **`omega`** (ω) and **`theta`** (θ) are globals, and created in a function given below.

The function **`g0[xmin_,a_,z_,k_]`** takes as arguments the left endpoint of the volatility interval $V_{\min} = x_{\min}$, the right end point **`a`**, the parameter **`z`** and the complex number **`k`**. The code block solves the differential equation (8.2) and returns the value of the function at the right end point $g_0(a,z)$.

Also the solution to the equation **`g0f`** is made a global variable that is available for other purposes, such as plotting the solution. The object **`g0f`** is what Mathematica calls an interpolating function object—it behaves just like a function and the solution to the differential equation at an arbitrary point V in $V_{\min} \le V \le a$ is given by **`g0f[V],`** once the code block above has been executed.

The function $g_1(V,z)$, defined for $a \le V \le V_{\max}$ is very similar:

```
g1[a_,xmax_,z_,k_]:=
 Module[{a1,b1,c = cf[k],d = df[k],g},
 Clear[g1f];
 a1 = -(etaf[k]+.5 d);
 b1 = (1/4 - alpha/2 + theta/2);
 g1f=g/.
  NDSolve[{x^2 g'[x] + x^2 g[x]^2 +
  (omega - theta x - d x^(3/2))g[x] - c x + z == 0,
   g[xmax] == a1 Sqrt[1/xmax] + b1/xmax},
  g,{x,a,xmax},AccuracyGoal->10][[1]];
 Return[g1f[a]]]
```

The parameter **alpha** above is another global. The function $F_a(z)$ defined at (8.9) and the first eigenvalue $\lambda(k)$ are given by

```
F[z_,a_,xmin_,xmax_,k_] :=
    g1[a,xmax,z,k] - g0[xmin,a,z,k]

FirstEigenvalue[k_,vmin_,vmax_,a_,
                omegax_,thetax_,ksix_,rho_,gam_]:=
    Module[{lam,z0,z1},
     Clear[omega,theta,cf,df,etaf]; (* globals *)
     omega = 2.0 omegax/ksix^2;
     theta = 2.0 thetax/ksix^2;
     cf[kx_] := Re[(kx^2 - I kx)]/ksix^2;
     df[kx_] := 2 rho/ksix Re[(1 - gam + I kx)];
     etaf[kx_] :=
      Sqrt[Re[kx^2(1 - rho^2) - I kx]/ksix^2];
     alpha = -df[k] (1 + 2 theta)/(4 etaf[k]);
     z0 = .99 omega/theta cf[k]; z1 = 1.02z0;
     lam = ksix^2/2.0 *
      (z/.FindRoot[F[z,a,vmin,vmax,k] == 0,{z,z0,z1},
        MaxIterations->1000]);
    Return[lam]]
```

The function **FirstEigenvalue** takes as arguments the parameters of the GARCH diffusion process, the value of k, and the volatility integration interval. One argument which is not mentioned in Sec. 8 is **gam** (γ), the risk adjustement parameter, which has legal values $\gamma = 1$ (risk-neutrality) or $\gamma = 0$ (log-utility). All of the numerical results presented in Chapter 6 are obtained with $\gamma = 1$. The function returns the numerical approximation to the first eigenvalue $\lambda(k)$. In theory, this value can be made arbitrarily accurate by taking **vmin** to 0 and **vmax** to $+\infty$ and invoking Mathematica's ability to use arbitrary precision numbers.

The asymptotic implied volatility σ_∞^{imp} is then computed from

```
AsymImpSigmaD[vmin_,vmax_,a_,
              omegax_,thetax_,ksix_,rho_,gam_] :=
 Module[{ans,val,statpoint},
   ans =
    FindMinimum[-FirstEigenvalue[ki I, vmin, vmax, a,
     omegax,thetax,ksix,rho,gam],{ki,{.49,.51}},
     MaxIterations->300,AccuracyGoal->5];
    val = 100.0 Sqrt[-8 ans[[1]]];
    statpoint = ki/.ans[[2]];
 Return[{val,statpoint}]]
```

The function behaves like **AsymImpSigma** defined earlier—it returns a two element list $\{\sigma_\infty^{imp}, |k_0|\}$, where k_0 is the saddle point. For example, the data in the first panel of Table 6.3 is found from:

```
In[76]:=
   table63a =
    Table[AsymImpSigmaD[.00001, 2000, .01, .02, 2, 1.5, rho, 1],
    {rho, -1, 1, 1/2}] // TableForm

Out[76]//TableForm=
   9.77592     0.489521
   9.88115     0.494352
   9.99608     0.499474
   10.1237     0.507449
   10.2701     0.517394
```

Here is the second panel of the Table 6.3 results, showing the timing:

In[77]:=

```
Timing[table63b =
  Table[AsymImpSigmaD[.00001, 2000, .01, .01, 1, 1.5, rho, 1],
 {rho, -1, 1, 1/2}]]
```

Out[77]=

```
{6.65 Second, {{9.47852, 0.478093}, {9.69188, 0.48778},
  {9.95699, 0.499995}, {10.3243, 0.521471}, {10.9649, 0.577458}}}
```

As one sees, the routine is very fast, averaging about 1 second per value. So we haven't spent any time on performance issues.

The first eigenfunction, equation (8.10), is obtained from

```
FirstEigenfunction[V_,a_]:=
    Module[{arg},
      If[V < a, arg = -NIntegrate[g0f[y],{y,V,a}],
                arg = NIntegrate[g1f[y],{y,a,V}]];
    Return[E^arg]]
```

And the plots in Fig. 6.3 were made with the following:

```
FirstEigenPlot[k_,vmin_,vmax_,pvmin_,pvmax_,a_,
               omegax_,thetax_,ksix_,rho_,gam_] :=
    Module[{lam,z0,z1},
    Clear[omega,theta,cf,df,etaf];
    omega = 2.0 omegax/ksix^2;
    theta = 2.0 thetax/ksix^2;
    cf[kx_] := Re[(kx^2 - I kx)]/ksix^2;
    df[kx_] := 2 rho/ksix Re[(1 - gam + I kx)];
    etaf[kx_] :=
      Sqrt[Re[kx^2(1-rho^2)-I kx]/ksix^2];
    alpha = -df[k] (1+2 theta)/(4 etaf[k]);
    z0 = .99 omega/theta cf[k]; z1 = 1.02 z0;
    lam = ksix^2/2.0 *
     (z/.FindRoot[F[z,a,vmin,vmax,k] == 0,{z,z0,z1},
         MaxIterations->1000 ]);
    Plot[FirstEigenfunction[V,a],{V,pvmin,pvmax},
      PlotDivision->1,PlotRange->All,
      PlotStyle->{Thickness[0.006]},
      ImageSize->{500,300}]]
```

We're repeating some code constructs here, which is a signal that the entire set of routines should be made more modular. This function displays the plot and returns the associated graphics object.

7 Utility-based Equilibrium Models

Equilibrium models describe how investors' endowments, probability beliefs, and preferences are aggregated to determine security prices. The general problem of aggregation is a difficult one in finance. A relatively simple approach, which avoids some aggregation issues, postulates a representative consumer-investor whose behavior serves as a proxy for the totality of all investors.

In this chapter, we apply the representative agent model to our problem. The representative has a power utility function with a risk-aversion parameter γ, where $\gamma = 1$ corresponds to risk-neutrality. We find a simple pattern for the effect of risk attitudes on option prices: with $\rho \le 0$, call or put option prices are raised in an interval $\gamma_1(\rho) < \gamma < 1$ and lowered in an interval $\gamma < \gamma_1(\rho)$ relative to their risk-neutral values.

Another key result is the following. When the representative agent is a pure investor with a distant horizon, then the risk-adjustment problem has a simple solution. The volatility risk premium is given by

$$\lambda^v(V) = (1-\gamma)\rho(V)\sqrt{V} - a(V)\psi(V), \qquad \psi(V) = u'(V)/u(V),$$

where $u(V)$ is the first eigenfunction of a linear differential operator well-known from previous chapters. The eigenfunction depends upon γ. For a pure investor with any horizon, and $\gamma < 1$, we prove that when $\psi(V,t)$ exists, it has the same sign as γ.

1 A Representative Agent Economy

To determine the risk premiums introduced in the first chapter, we develop a representative agent equilibrium model[1]. Even with a representative consumer, equilibrium models can rapidly become intractable if they involve too many "state variables". State variables can be thought of as all of factors that are necessary to fully determine a security price.

For example, if you introduce an independent volatility process for every individual stock in a marketplace, you may find that to price an option on one stock, you have to use as state variables the volatilities of *every* stock in the market. Not only is this computationally hopeless, it's a pretty good bet that market participants don't value options this way.

To avoid that dead-end, an implicit assumption in this chapter is that, to price an option, the price and volatility of the underlying security serve as a sufficient set of state variables. We consider only equilibrium models that are consistent with that assumption. This assumption turns out to be consistent with a representative agent equilibrium models when the underlying security is a *market index*—that's our primary application. The same setup can handle options on individual stocks under certain additional assumptions.[2] Our main goal is to establish some general connections between the risk aversion of the representative investor and the volatility risk premium.

With a market security, our investment universe consisting of only three types of securities:

---

[1] An important treatment of the problem of equilibrium and option pricing under stochastic volatility was given by Pham and Touzi (1996). The models in this chapter are generally consistent with their framework, but more oriented toward computations. One difference is that we allow the dividend yield to be exogenous and hence, match the market's actual yield.

[2] The development in this chapter can be generalized as follows. Modify the well-known Index model for individual stock returns to include stochastic volatility. Each individual security volatility is given by $\sigma_i^2 = \beta_i^2\sigma_m^2 + u_i^2$, where $V_m = \sigma_m^2$ is the market volatility. The market volatility is stochastic but u_i^2 is a security-specific *constant*. Then, the risk premium coefficient $g(V_m, t)$, which is the key function defined in this chapter, remains the same.

(i) market index shares (the stock) with price S_t and volatility V_t,
(ii) generalized (European-style) index options with price $F(S_t,V_t,t)$,
(iii) money market instruments, with price $B_t = e^{r(t-T)}$, where r is constant.

An important assumption is the following:

Assumption A1. *Borrowing and lending only occurs between individual investors. Every option is also a contract between individuals, one the buyer and one the seller.*

In other words, in this world both generalized options and money market instruments exist only in short/long pairs. Securities of this type are called financial claims. It's the way the listed and OTC options markets work in practice. But the assumption excludes certain types of derivative securities, such as warrants or convertible bonds. The money market security, the only bond in the model, is just a riskless financial claim.

An implication of Assumption A1 is that the market index security, which aggregates every security with capitalization weights, contains no bonds or options—no financial claims. The market index is purely *equity*, consisting of shares of common stock. The evolution of the market index security is described by our standard process:

Assumption A2. *The equity price process is given by:*

(1.1)
$$P: \begin{cases} dS_t = (\alpha_t^e S_t - D_t)dt + \sigma_t S_t dB_t \\ dV_t = b(V_t)dt + a(V_t)dZ_t \end{cases},$$

where dB_t and dZ_t are Brownian motions with correlation $\rho(V_t)$, α_t^e is the expected total return of the market, and D_t is the dollar dividend rate of the market. The money market security follows $dB_t = rB_t dt$, where r is a constant.

By Ito's formula, the option price $F(S_t,V_t,t)$ must have also have a price evolution of the general form

(1.2)
$$dF_t = \alpha_t^F F_t dt + \kappa_t F_t dB_t + \eta_t F_t dZ_t,$$

which *defines* the expected return of the option α_t^F, and the two functions κ_t and η_t. As usual, we assume that options do not pay dividends.

The representative agent derives income only from investments, and decides at each point in time and simultaneously: (i) how to invest society's wealth W_t and (ii) at what rate C_t to consume from wealth. The investment allocation is a 3-component vector $\boldsymbol{x}_t = (x_t, y_t, z_t)$. The components are the fractions of wealth allocated at time t to the stock, option, and money market security respectively, where $z_t = 1 - x_t - y_t$. Negative wealth fractions mean short sales of stock, or sales of a financial claim. With a general allocation $\boldsymbol{x}_t$, not necessarily optimal or in equilibrium, the wealth evolution is given by

$$(1.3) \qquad dW_t = [x_t W_t(\alpha^e - r) + y_t W_t(\alpha^F - r) + (rW_t - C_t)]\,dt \\ + x_t W_t \sigma_t dB_t + y_t W_t(\kappa_t dB_t + \eta_t dZ_t)\,.$$

The representative agent determines the 3 optimal values $\{\hat{\boldsymbol{x}}_t, \hat{C}_t\}$ at each point in time by maximizing an expected utility function. (Hats denote optimal values). Specifically, we adopt the following model

Assumption A3. (Time-additive utility). *The risk preferences of investors may be described by a single representative consumer-investor. The representative acts at each point in time to maximize her remaining lifetime expected utility over consumption and terminal wealth. Specifically, the representative pursues the program*

$$(1.4) \qquad \max_{\{\boldsymbol{x}_t, C_t\}} \mathbb{E}_t \left\{ \int_t^{\mathbb{T}} U(C_s, s)ds + B(W_{\mathbb{T}}, \mathbb{T}) \right\} \equiv J(W_t, V_t, t)$$

The time $\mathbb{T}$ is a lifetime planning horizon, assumed to be known and fixed but perhaps at infinity. At the very least, if we have an option expiring at time T, we need $\mathbb{T} \geq T$. The function $U(C_t, t)$ is the rate of consumption utility, or for brevity, just the consumption utility. The function $B(W_{\mathbb{T}}, \mathbb{T})$ is the bequest function, a utility function for all the consumption that occurs after time $\mathbb{T}$. Finally, $J(W_t, V_t, t)$ is called the derived utility function of wealth and has the form shown because W_t and V_t are our only state variables.

In fact, the solution to (1.4) is trivial for the investments weights, under the following assumption.

Assumption A4. (Market clearing) *The market clears so that in equilibrium, the optimal wealth fraction allocated to the stock is unity* ($\hat{x}_t = 1$) *and there is no net demand for any financial claim* ($\hat{y}_t = \hat{z}_t = 0$).

The absence of financial claims in the market index doesn't mean that these securities play no role. To the contrary, the idea that the market-level investment weights for financial claims is *optimally* zero helps determine the valuation equation for every possible claim[3].

Two special cases are simpler and still interesting:

(I) **The infinite horizon consumption-investment problem**:

(1.5) $$J(W_t, V_t, t) = \max_{\{x_t, C_t\}} \mathbb{E}_t \left\{ \int_t^\infty U(C_s, s)\, ds \right\}.$$

(II) **The pure investment problem** (consumption at a final date):

(1.6) $$J(W_t, V_t, t) = \max_{\{x_t\}} \mathbb{E}_t \{ B(W_{\mathbb{T}}, \mathbb{T}) \}.$$

We show in Sec. 3 that there is a very compelling $\mathbb{T} \to \infty$ limit for the pure investment problem, also—a limit closely connected with our previous development. All of the risk adjustments shown in the examples in Chapter 2 are based on the $\mathbb{T} \to \infty$ limit of (1.6), which we call the "pure investment problem with a distant planning horizon".

Power utility. We specialize exclusively in power utility, also called constant proportional risk aversion (CPRA), which means:

(1.7) $$U(C_t, t) = e^{-Rt} \frac{C_t^\gamma}{\gamma} \quad \text{and} \quad B(W_{\mathbb{T}}, \mathbb{T}) = e^{-R\mathbb{T}} \frac{W_{\mathbb{T}}^\gamma}{\gamma},$$

where γ is the CPRA parameter and $R \geq 0$ is an impatience parameter, both constants. We assume that the representative agent is either risk neutral ($\gamma = 1$) or risk-averse ($\gamma < 1$). For any γ, $U(C, t)$ is an increasing function of C, ($U_c > 0$); hence, more is preferred to less. With $\gamma < 1$, U has a negative second derivative ($U_{cc} < 0$). This means the representative is risk-averse, and

[3] An early application of the idea that an optimal weight of zero could price options was Samuelson and Merton (1969). In fact, they were actually valuing warrants, which are excluded here because their market weight is not zero.

prefers a certain outcome to an uncertain one with the same mathematical expectation. Clearly, $B(W_{\mathbb{T}},\mathbb{T})$ has the same properties. The CPRA parameter is related to Pratt's (1964) proportional risk aversion measure δ by $\delta = -U_{CC}/U_C = 1-\gamma$.

If we modify $C^\gamma \to C^\gamma - 1$ in U, then the limit $\gamma \to 0$ exists and is given by logarithmic utility: $U(C_t,t) = e^{-Rt}\log C_t$, where log is the natural logarithm.

The risk neutral case $(\gamma = 1)$ is often ill-defined in intermediate formulas, but the final result is that the volatility risk premium is zero. The martingale pricing process, in a risk neutral world, is identical to (1.1) except α_t^e is replaced by the riskless rate r, because all securities must have the same expected return. The volatility process is left unchanged.

We assume that consumption and wealth never become negative: this constraint is satisfied automatically for power utility functions and need not be imposed separately. For infinite horizon problems, there will typically be additional restrictions on the CPRA parameters to ensure that the problem is well-defined and the derived utility function is finite. These typically occur when the CPRA parameter is in the range $0 < \gamma < 1$ and will be derived in due course.

Share Issuance, Repurchase and Dividend Policies. The market index security has a certain number of shares outstanding. There are two ways to approach this. In a *constant shares model*, the number of equity shares outstanding is fixed, so that in equilibrium $\hat{W}_t = N_0 S_t$, where N_0 is a constant. In a *variable shares model*, $\hat{W}_t = N_t S_t$ where obviously N_t can vary. Which model you select is determined by how you want to describe the dividend policy, as we now explain.

Consider the general case, where the shares can vary. Then $d\hat{W}_t = N_t dS_t + dN_t S_t = (\alpha^e \hat{W}_t - \hat{C}_t)\,dt + \sigma_t \hat{W}_t dB_t$, using (1.3) under market clearing[4] . We can always define a *spending fraction* $f_t = \hat{C}_t / \hat{W}_t$. Then, the optimal wealth evolution becomes $d\hat{W}_t = (\alpha^e{}_t - f_t)\hat{W} dt + \sigma_t \hat{W}_t dB_t$.

---

[4] It turns out [see (1.8)] that dN_t is proportional to dt. That's why we omitted a potential cross-product term $dN_t dS_t$.

But (1.1) implies $N_t dS_t = N_t\left[(\alpha_t^e - \delta_t)S_t dt + \sigma_t S_t dB_t\right]$, using $D_t = \delta_t S_t$, where δ_t is the dividend yield. In other words, $N_t dS_t = (\alpha_t^e - \delta_t)W_t dt + \sigma_t W_t dB_t$. Solving the two relations for $dN_t S_t$ yields

$$dN_t S_t = (\delta_t - f_t)N_t S_t dt \quad \text{or} \quad dN_t = (\delta_t - f_t)N_t dt\,. \tag{1.8}$$

This last equation has the following interpretation. In a constant shares model, $dN_t = 0$, which means the dividend yield is equal to the spending fraction. Or, what is the same, $N_t D_t = \hat{C}_t$ from the first equation at (1.8). In other words, the aggregate dividend equals the aggregate spending.

For example, under the CPRA equilibrium model, it will turn out that $\hat{C}_t = f(V_t,t)\hat{W}_t$, so that $\delta_t = f(V_t,t)$. The constant shares model implies an explicit equilibrium dividend policy as a function of the time and instantaneous volatility. The dividend policy cannot be arbitrarily specified, but instead is determined endogenously in a constant shares model.

In the real world, dividends on stocks are set periodically by boards of directors. In addition, these same boards often decide from time to time to issue new shares and retire old ones by repurchase activity. These decisions are business judgments and certainly not mechanical, based upon a formula. Factors frequently cited as influencing such decisions are investment opportunities (or lack of the same), market interest rates, and differential taxation of dividends versus capital gains. This suggests a world with an exogenous dividend policy.

If the dividend yield δ_t is set exogenously, and $f_t > \delta_t$, then (1.8) shows that $dN_t < 0$. That's because consumers desire to spend more than they are receiving in dividends, so to make up the difference, they sell some shares. The firms must repurchase these shares and retire them, so the number of shares outstanding will decrease. Alternatively, if $f_t < \delta_t$, then consumers don't want to spend all of the income they have received in dividends. They reinvest the excess income in shares, which can only happen if the firms issue new shares. That's why $dN_t > 0$ in that case. So, in the aggregate, firms in this model can set whatever dividend yield they like, but then must respond to the spending preferences of the representative consumer anyway.

In this book, we treat dividends as exogenous, since we want to use the observed dividend yields in formulas. This also makes for a sensible deterministic limit.

In summary, we adopt the variable shares model and the reader can treat the dividend yield in formulas as exogenous, just like the interest rate.

The Bellman equation. The problem (1.4) can be reduced to solving differential equations. There are a couple of approaches to this that are standard in finance. The older approach, which we follow, is dynamic stochastic programming [see Merton (1990)]. In dynamic programming, the differential equations follow from an intermediate equation: the Bellman equation of optimality. A solution to (1.4) necessarily satisfies the optimality equation:

$$0 = \max_{\{x_t, C_t\}} \left\{ U(C_t, t) + \frac{\partial J}{\partial t} + \mathcal{L}(J) \right\}, \tag{1.9}$$

where $\mathcal{L}(J)$ is the differential generator of J:

$$\mathcal{L}(J)dt = \mathbb{E}_t \left[J_W \, dW_t + \tfrac{1}{2} J_{WW} dW_t^2 + J_V dV_t + \tfrac{1}{2} J_{VV} \, dV_t^2 + J_{WV} \, dW_t dV_t \right],$$

using subscripts for partial derivatives. Using (1.1) and (1.3) and taking expectations in the differential generator, (1.9) becomes

(1.10)
$$\begin{aligned} 0 = \max_{\{x_t, C_t\}} \Big\{ & U(C,t) + \frac{\partial J}{\partial t} + \left[(\alpha^e - r)xW + (\alpha^F - r)yW + (rW - C) \right] J_W \\ & + \tfrac{1}{2} \left[x^2 V + y^2(\kappa^2 + \eta^2 + 2\rho\kappa\eta) + 2xy\sigma(\kappa + \rho\eta) \right] W^2 J_{WW} \\ & + bJ_V + \tfrac{1}{2} a^2 J_{VV} + \left[x\rho\sigma a + ya(\rho\kappa + \eta) \right] W J_{WV} \Big\}. \end{aligned}$$

The equilibrium risk premiums and consumption. The risk premiums and optimal consumption follow by differentiating (1.10) with respect to C, x and y. Then substitute the market clearing values $\hat{x} = 1$ and $\hat{y} = 0$. In addition, substitute $\kappa F = \sigma S F_S$ and $\eta F = a F_V$, which follow from Ito's formula. The results are relations for optimal consumption and the two expected returns:

$$U_C(\hat{C}, t) = e^{-Rt} \hat{C}^{\gamma - 1} = J_W(W, V, t), \tag{1.11}$$

$$\alpha^e - r = AV + B\rho\sigma a, \tag{1.12}$$

$$(\alpha^F - r)F = (\alpha^e - r)S \, F_S + \varphi \, F_V, \tag{1.13}$$

using

$$\varphi = A\rho\sigma a + Ba^2, \quad A = -W \frac{J_{WW}}{J_W}, \quad \text{and } B = -\frac{J_{WV}}{J_W} \tag{1.14}$$

Substituting $\hat{C}$ and α^e from these formulas back into (1.10) yields

$$(1.15)\quad 0 = U(\hat{C},t) + \frac{\partial J}{\partial t} + (rW - \hat{C})J_W - \tfrac{1}{2}VW^2 J_{WW} + bJ_V + \tfrac{1}{2}a^2 J_{VV}\,.$$

Because of the choice of power utility functions, there is a solution to (1.10) and (1.15) of the form

$$(1.16)\qquad J(W,V,t) = \exp(-Rt)g(V,t)\frac{W^\gamma}{\gamma}\,.$$

We call $g(V,t)$ the *risk premium coefficient*, since we will show that all of the risk premium functions follow from it. Then (1.14) yields $A = 1-\gamma$ and $B = -\psi(V,t)$, where we introduce the notation:

$$(1.17)\qquad \psi(V,t) = \frac{J_{WV}}{J_W} = \frac{g_V(V,t)}{g(V,t)}\,.$$

In terms of this notation, the expected excess return on the market is given by[5]

$$(1.18)\qquad \alpha^e - r = (1-\gamma)V_t - \rho(V_t)\sigma_t a(V_t)\psi(V_t,t)\,.$$

And, the expected excess return on the option is given by

$$(1.19)\qquad (\alpha^F - r)F = (\alpha^e - r)SF_S + \varphi(V,t)\,F_V\,,$$

where

$$(1.20)\qquad \varphi(V,t) = (1-\gamma)\,\rho(V)\,V^{1/2}a(V) - a^2(V)\psi(V,t)$$

Optimal consumption can be written in terms of a spending fraction:

$$(1.21)\qquad \hat{C}_t = f(V_t,t)\hat{W}_t\,, \text{ where } f(V_t,t) = [g(V_t,t)]^{1/(\gamma-1)}\,.$$

From (1.15), the risk premium coefficient satisfies the non-linear PDE:

(1.22)

$$\frac{\partial g}{\partial t} = -(1-\gamma)g^{-\gamma/(1-\gamma)} + [(R-r\gamma) - \tfrac{1}{2}\gamma(1-\gamma)V]\,g - b(V)\frac{\partial g}{\partial V} - \tfrac{1}{2}a^2(V)\frac{\partial^2 g}{\partial V^2}$$

where, from (1.16), the boundary condition is $g(V,T) = 1$.

---

[5] An alternative interpretation of (1.18) takes α^e as an exogenous constant; then (1.18) determines the equilibrium interest rate $r = r(V_t,t)$. The alternative interpretation was used in the influential equilibrium theory of Cox, Ingersoll, and Ross (1985) and applied to options in Bailey and Stulz (1989).

The equilibrium option valuation PDE. Finally, the system closes by combining (1.19) with Ito's lemma:

$$(\alpha^e - r)SF_S + \varphi(V,t)\,F_V = -rF + \frac{\partial F}{\partial t} + \mathcal{A}F\,, \tag{1.23}$$

where the differential generator $\mathcal{A}$ was originally defined in Ch.1, Sec.4 and is repeated here for convenience:

$$\mathcal{A}F = (\alpha^e S - D)F_S + bF_V + \tfrac{1}{2}\sigma^2 S^2 F_{SS} + \tfrac{1}{2}a^2 F_{VV} + \rho\sigma a S F_{SV}\,.$$

In (1.23), the expected return of the equity on the left-hand-side cancels with a similar term on the right. The result is a PDE that values any European-style generalized option:

$$-\frac{\partial F}{\partial t} = -rF + \tilde{\mathcal{A}}F\,, \tag{1.24}$$

where

$$\begin{aligned} \tilde{\mathcal{A}}F &= (r-\delta)S\,\frac{\partial F}{\partial S} + \tfrac{1}{2}VS^2\,\frac{\partial^2 F}{\partial S^2} \\ &+ [\,b(V) - \varphi(V,t)\,]\frac{\partial F}{\partial V} + \tfrac{1}{2}a^2(V)\frac{\partial^2 F}{\partial V^2} + \rho(V)a(V)V^{1/2}S\,\frac{\partial^2 F}{\partial S \partial V}\,. \end{aligned} \tag{1.25}$$

The implication of (1.24) is that the martingale pricing process must be:

$$\tilde{P}:\ \begin{cases} dS_t = (rS_t - D_t)dt + \sigma_t S_t\, d\tilde{B}_t \\ dV_t = [\,b(V) - \varphi(V,t)\,]dt + a(V_t)d\tilde{Z}_t \end{cases}, \tag{1.26}$$

To summarize at this point, to obtain the volatility drift adjustment $\varphi(V,t)$ carry out the following procedure:

- Solve (1.22) for the risk premium coefficient $g(V,t)$;
- Compute $\psi(V,t) = g_V / g$;
- Obtain $\varphi(V,t) = (1-\gamma)\rho(V)\sqrt{V}a(V) - a^2(V)\psi(V,t)$

The risk premiums. In Chapter 1, we introduced three related risk premiums: the equity risk premium λ^e, the volatility risk premium λ^v, and the hedging portfolio risk premium λ^h. Those market prices of risk are now determined:

(1.27) $$\lambda_t^e = \frac{\alpha_t^e - r}{\sigma_t} = (1-\gamma)\sigma_t - \rho(V_t)a(V_t)\psi(V_t,t),$$

(1.28) $$\lambda_t^v = \frac{\varphi(V_t,t)}{a(V_t)} = (1-\gamma)\rho(V_t)\sigma_t - a(V_t)\psi(V_t,t),$$

(1.29) $$\lambda_t^h = \frac{\alpha_t^h - r}{\sigma_t^h} = \pm\sqrt{1-\rho^2(V_t)}\,a(V_t)\psi(V_t,t) \; .$$

The sign in the hedging risk premium was ambiguous because it was dependent upon the payoff function of the claim used for hedging. As we promised, the three equations above show that every risk premium is determined by the risk premium coefficient $g(V_t,t)$.

The infinite horizon problem. This special case was defined at (1.5) and is obtained by taking the limit $\mathbb{T} \to \infty$. In that limit, the risk coefficient becomes time-independent $[g(V,t) \to g(V)]$ and satisfies the ordinary (non-linear) differential equation

(1.30) $$(1-\gamma)g^{-\gamma/(1-\gamma)} = [(R-r\gamma) - \tfrac{1}{2}\gamma(1-\gamma)V]\,g - b(V)\frac{dg}{dV} - \tfrac{1}{2}a^2(V)\frac{d^2g}{dV^2}\,.$$

The limit $\mathbb{T} \to \infty$ seems possible for the representative agent, who serves as a proxy for the totality of all investors, with their many diverse individual horizons. For example, if the "individual investor" pool was dominated by investing institutions such as pension funds, college endowments, corporations, and so on– long-lived entities—then $\mathbb{T}$ could be large.

The pure investment problem. The same form of solution (1.16) solves this problem. The development is very similar to the consumption-based case, only the consumption utility function does not appear. The result is that $J(W,V,t) = \exp(-R\,t)g(V,t)W^{\gamma}/\gamma$, where the risk premium coefficient now satisfies the *linear* PDE

(1.31) $$\frac{\partial g}{\partial t} = [(R-r\gamma) - \tfrac{1}{2}\gamma(1-\gamma)V]\,g - b(V)\frac{\partial g}{\partial V} - \tfrac{1}{2}a^2(V)\frac{\partial^2 g}{\partial V^2},$$

again with $g(V,\mathbb{T}) = 1$.

We will show in Sec. 2 that the equations for the risk premium function are typically well-defined when $\gamma < 0$. We will show solutions for some processes.

This will establish that the fundamental PDE, (1.24) is indeed consistent with some equilibrium, namely constant proportional risk aversion and some non-trivial stochastic volatility processes. In the deterministic limit, the consistency of the B-S theory with a CPRA equilibrium was proved by Bick (1987).

What is the value of the CPRA parameter γ? When we calculate $g(V,t)$, we find an important qualitative distinction between two cases (i) $0<\gamma<1$ versus (ii) $\gamma<0$. As a general rule, $g(V,t)$ is well-behaved for all t when $\gamma<0$. But when $0<\gamma<1$, $g(V,t)$ can diverge at finite t and the limit $\mathbb{T}-t\to\infty$ may not exist. So it's quite important to know which of these regimes is more plausible, empirically.

A classic study by Friend and Blume (1975) estimated this risk aversion parameter by two methods. First, they examined the proportion of risky assets held by households of different wealth levels. Secondly, they considered the equity premium (1.27) under the assumption of constant volatility, where it becomes $(\alpha^e - r)/V = 1-\gamma$. They estimated the left-hand side for some U.S. stock market composites by using the average realized returns over long time periods (up to 100 years) as a proxy for the expected return. We quote from their main conclusions:

> *First, regardless of the wealth level, the coefficients of proportional risk aversion $(1-\gamma)$ are on average well in excess of one and probably in excess of two. Thus, investors require a substantially larger premium to hold equities or other risky assets than they would if their attitudes toward risk were described by logarithmic utility functions.*

For example, assume equities are expected to return 4-6% per year more than T-bills, with a standard deviation of 12%. Then $1-\gamma$ would range from about 3 to 4, so that γ would range from -2 to -3. Other more recent research has not altered these general conclusions, although there are some arguments for values of $1-\gamma$ much higher than these. So, this is a regime where, in our experience, the limit $\mathbb{T}\to\infty$ is not problematic.

2 Examples

Two special cases are considered first: $\gamma = 0$ (log-utility) and $\gamma = 1$ (risk-neutrality)

Example 1. Log-utility. If $\gamma = 0$, (1.22) becomes

$$\frac{\partial g}{\partial t} = -1 + R\, g - b(V)\frac{\partial g}{\partial V} - \tfrac{1}{2} a^2(V)\frac{\partial^2 g}{\partial V^2}. \tag{2.1}$$

With the terminal condition $g(V,\mathcal{T}) = 1$, (2.1) has a solution independent of V:

$$g(V,t) = g(t) = \frac{1}{R} + (1 - \frac{1}{R})\exp[-R(\mathcal{T} - t)]. \tag{2.2}$$

Optimal consumption is given by $\hat{C} = \hat{W}_t / g(t)$. For a very distant horizon, $g \to 1/R$ and so $\hat{C} = R\hat{W}_t$. Regardless of the horizon, g is independent of V, and so $\psi = 0$ and the expected equity return is $\alpha_t^e - r = V_t$. The volatility risk premium is given by $\lambda_t^v = \rho(V_t)\sqrt{V_t}$. The hedging portfolio risk premium, given by (1.29), vanishes[6]: $\lambda_t^h = 0$. When the actual volatility process is $dV_t = b(V_t)dt + a(V_t)dW_t$, then the martingale pricing process for any option is:

$$\tilde{P}: \begin{cases} dS_t = (rS_t - D_t)dt + \sigma_t S_t d\tilde{B}_t \\ dV_t = \left[b(V_t) - \rho(V_t)\sqrt{V_t}\, a(V_t) \right] dt + a(V_t) d\tilde{W}_t \end{cases} \tag{2.3}$$

where $d\tilde{B}_t$ and $d\tilde{W}_t$ have correlation $\rho(V_t)$. In a variable shares model, the dividend rate D_t is arbitrary. In a constant shares model, the equilibrium dividend policy is $\delta_t = 1/g(t)$ for a finite horizon and $\delta_t = R$ for an infinite horizon. In the latter case, you would make the substitution $D_t \to R\, S_t$ in (2.3).

Example 2. Risk-neutrality. The case $\gamma = 1$ is a degenerate one and technically (1.22) is ill-defined, especially if $R \neq r$. With $R = r$, a natural

---

[6] The vanishing of the hedging portfolio risk premium, which represents an "orthogonal" risk, is a key property of an alternative approach to option pricing due to Föllmer and Schweizer (1991). The notion that the Föllmer and Schweizer approach implies a CPRA equilibrium with logarithmic utility was first established by Pham and Touzi (1996). The GARCH option pricing models of Duan (1995,1996) and Amin and Ng (1993,1994), which are discrete-time models, also require a logarithmic utility in an equilibrium framework based upon a representative agent.

guess is that $g(V,t)$ sticks at its boundary value $g(V,t)=1$. This can be justified with a discrete-time analysis. It implies everything that you would expect under risk neutrality: $\alpha_t^e = r$, $\varphi = \psi = 0$ and there is no adjustment to the volatility drift in the martingale pricing process:

$$\tilde{P}: \begin{cases} dS_t = (rS_t - D_t)dt + \sigma_t S_t d\tilde{B}_t \\ dV_t = b(V_t)dt + a(V_t)d\tilde{W}_t \end{cases}. \tag{2.4}$$

Example 3. The square root model. The volatility process is $dV = (\omega - \theta V)\,dt + \xi\sqrt{V}dW$, with constant correlation ρ with the stock price process. We consider the pure investment problem (1.31). The risk premium coefficient satisfies

$$\frac{\partial g}{\partial t} = [(R - r\gamma) - \tfrac{1}{2}\gamma(1-\gamma)V]\,g - (\omega - \theta V)\frac{\partial g}{\partial V} - \tfrac{1}{2}\xi^2 V\frac{\partial^2 g}{\partial V^2}, \tag{2.5}$$

with $g(V,\mathbb{T}) = 1$. Use $\tau = \mathbb{T} - t$, and let $g(V,t) = \exp[-(R - r\gamma)\tau]h(V,\tau)$. Then $h(V,\tau)$ satisfies

$$\frac{\partial h}{\partial \tau} = \tfrac{1}{2}\xi^2 V\frac{\partial^2 h}{\partial V^2} + (\omega - \theta V)\frac{\partial h}{\partial V} - c_\gamma V\,h, \qquad c_\gamma = -\tfrac{1}{2}\gamma(1-\gamma). \tag{2.6}$$

We have encountered this equation before. It's the equation satisfied by the fundamental transform at (**2**.2.19) with $\rho = 0$ and with the substitution $c(k) \to c_\gamma$. So, the solution is given by (**2**.3.2). Translating that solution into our notation here, we have

$$g(V,t,\mathbb{T}) = \exp[a_1(\tau) + a_2(\tau)V], \tag{2.7}$$

where

$$a_1(\tau) = \left[\tfrac{1}{2}\omega(\tilde{\theta} + d) - R + r\gamma\right]\tau - \frac{2\omega}{\xi^2}\ln\left|\frac{1 - \kappa\exp\left(\frac{1}{2}\xi^2\tau d\right)}{1-\kappa}\right|, \tag{2.8}$$

$$a_2(\tau) = \tfrac{1}{2}(\tilde{\theta} + d)\left[\frac{1 - \exp\left(\frac{1}{2}\xi^2\tau d\right)}{1 - \kappa\exp\left(\frac{1}{2}\xi^2\tau d\right)}\right], \tag{2.9}$$

using $\quad \tilde{\theta} = \dfrac{2\theta}{\xi^2}, \quad \tilde{c} = \dfrac{2}{\xi^2}c_\gamma, \quad d = \sqrt{\tilde{\theta}^2 + 4\tilde{c}}, \quad \kappa = \dfrac{\tilde{\theta} + d}{\tilde{\theta} - d}.$

The solution (2.7) holds for all $\tau < \tau^*$, where τ^* is a divergence time. If $c_\gamma > 0$, then there is no divergence and the solution is valid for all τ. This correspond to the "good" case $\gamma < 0$. But when $c_\gamma < 0$, which corresponds to $0 < \gamma < 1$ a divergence is possible but not mandatory—it depends upon the other parameters. For example, we previously worked out an example of this divergence at (**2**.2.11) and refer the reader to that discussion. To apply it to the discussion here, just make the translation $c(k) \to c_\gamma = -\gamma(1-\gamma)/2$ in (**2**.2.11).

So, to proceed we will assume that we are working with a pure investment problem where the time horizon is such that $\tau < \tau^*$. With that restriction, then (1.17) becomes

$$\psi(V,t) = \frac{g_V(V,t)}{g(V,t)} = a_2(\mathbb{T}-t) \,, \tag{2.10}$$

which turns out to be independent of V. The equity expected return is given by (1.18), which becomes

$$\alpha_t^e - r = \left[(1-\gamma) - \rho\xi a_2(\mathbb{T}-t)\right] V_t$$

What does the risk premium coefficient function look like? In Fig. 7.1, we show a plot of $g(V,t)$ vs. V for the parameters $\omega_a = 0.02$, $\theta_a = 2$, $\xi_a = 0.2$, and $\tau = 1$ year. We also take $R_a = r_a = 0.05$. Then, we show the function for the cases $\gamma = 1/2$, $\gamma = 0$, and $\gamma = -2$.

We can take the limit of a very distant planning horizon $\mathbb{T} \to \infty$, and find $\psi \to \psi_\infty$, $\varphi \to \varphi_\infty$, and $(\alpha_t^e - r) \to (\alpha^e - r)$, where

$$\psi_\infty = \frac{1}{\xi^2}\left(\theta - \sqrt{\theta^2 - \gamma(1-\gamma)\xi^2}\right), \tag{2.11}$$

$$\varphi_\infty = \left[(1-\gamma)\rho\xi - \xi^2\psi_\infty\right] V \,, \quad \text{and} \quad \alpha^e - r = (1-\gamma-\rho\xi\psi_\infty)\, V \tag{2.12}$$

These equations makes sense as long as ψ_∞ is real. From (2.11), ψ_∞ is real for any $\gamma < 0$. When $0 < \gamma < 1$, ψ_∞ is real if $\theta^2 \geq \gamma(1-\gamma)\xi^2$. Otherwise, the limit $\mathbb{T} \to \infty$ does not exist. In this same limit, the martingale pricing process becomes

$$(\mathbb{T} \to \infty) \quad \tilde{P}: \begin{cases} dS_t = (rS_t - D_t)\,dt + \sigma_t S_t\, dB_t \\ dV = (\omega - \tilde{\theta} V)\,dt - \xi\sqrt{V}\,dW_t \end{cases}, \tag{2.13}$$

where $\quad \tilde{\theta} = (1-\gamma)\rho\xi + \sqrt{\theta^2 - \gamma(1-\gamma)\xi^2}\,, \qquad \theta^2 \geq \gamma(1-\gamma)\xi^2 .$

When $\gamma = 0$ or $\gamma = 1$ in (2.13), we of course recover our previous results. So the effect of society's risk aversion, in this model, is to cause the mean reversion parameter θ to be altered. When $\gamma < 0$ and $\rho = 0$, then $\tilde{\theta} > \theta$. But when $\rho < 0$, then $\tilde{\theta}$ can be larger or smaller than θ. For example, suppose $\theta_a = 4$, which corresponds to a half-life of 3-months. With $\xi_a = 0.2$, $\rho = -0.7$, and $\gamma = -2$, then $\tilde{\theta}_a \cong 3.5$, a modest change.

Example 4. The 3/2 model. The volatility process is $dV = (\omega V - \theta V^2)\,dt + \xi V^{3/2} dW$. We consider the pure investment problem with a distant horizon. The risk adjustment is determined by the first eigenfunction of (1.31). We calculated this eigenfunction previously at (6.5.2). Using that formula and (2.10), the result is the martingale pricing process:

$$(2.14) \qquad (\mathbb{T} \to \infty) \quad \tilde{P}: \begin{cases} dS_t = (rS_t - D_t)\,dt - \sigma_t S_t \, d\tilde{B}_t \\ dV = (\omega V - \tilde{\theta} V^2)\,dt + \xi V^{3/2} d\tilde{W}_t \end{cases},$$

where
$$\tilde{\theta} := -\tfrac{1}{2}\xi^2 + (1-\gamma)\rho\xi + \sqrt{(\theta + \tfrac{1}{2}\xi^2)^2 - \gamma(1-\gamma)\xi^2}\,,$$

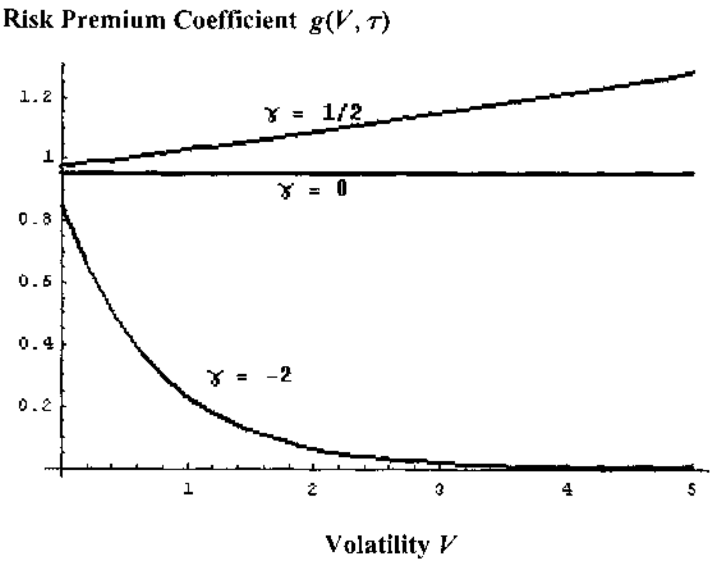
with the restriction on the parameters $(\theta + \frac{1}{2}\xi^2)^2 \geq \gamma(1-\gamma)\xi^2$.

The sign of ψ. Why is the sign of $\psi(V)$ important? It's important because it's the sign of ψ or the sign of $\rho\psi$ that determines how each of the risk premiums are changed by the presence of stochastic volatility. For example, the equity expected return is given by $\alpha^e - r = (1-\gamma)V - \rho\sigma_t a(V_t)\psi(V_t)$. Since $\sigma_t a(V_t)$ is positive, it's the sign of $\rho\psi$ that determines how stochastic volatility changes the classical equilibrium relation $\alpha^e - r = (1-\gamma)V$.

We have our first result about the sign of ψ from the square root model of Example 3. In (2.11) it's easy to see that $\gamma < 0$ implies that $\psi_\infty < 0$ and $0 < \gamma < 1$ implies that $\psi_\infty > 0$. In other words, for $\gamma < 1$, then ψ_∞ and γ have the *same* sign.

This sign result for ψ is very typical. Fig 7.1 shows it holds at intermediate times in the square root model. Another type of example: the full non-linear PDE for the consumption-investment problem (1.22) has the solution, as $\tau \to 0$, $g(V,\tau) \approx 1 + [(1-R) - \gamma(1-r) + \gamma(1-\gamma)V/2]\tau + O(\tau^2)$. This solution is obtained by simply substituting the initial condition $g(V,\tau = 0) = 1$ into the right-hand-side of (1.22). So, as $\tau \to 0$, for *any* volatility process

Fig 7.1 Risk Premium Coefficient $g(V, \tau)$ in the Square Root Model with a Pure Investment Program



Notes. The figure shows a plot of the risk premium coefficient function in the square root model for various CPRA risk preferences, indicated by the values of γ. The length of the investment horizon τ is one year. The slope takes the same sign as γ, which may be a very general property of the risk adjustment under many volatility processes.

(2.15) $$\psi(V,\tau) = \gamma(1-\gamma)\tau/2 + O(\tau^2) = -c_\gamma \tau + O(\tau^2)\,.$$

Again, for $\gamma < 1$, we see that ψ and γ have the same sign[7]. Equation (2.15) also follows if you take $\tau \to 0$ in the pure investment problem (1.31) for a general process.

[7] The opposite sign relation was argued in Wiggins (1987).

For the pure investment problem, this sign relation follows simply from Theorem 2.4 of chapter 2 as we now show.

THEOREM 7.1. *Assume $\gamma < 1$ and let $g(V,t)$ be the solution, if any, to the pure investment problem with horizon $\mathbb{T}$. That is, when it exists, $g(V,t)$ satisfies the PDE (1.31) with the terminal condition $g(V,\mathbb{T}) = 1$. Then $\psi(V,t) \equiv g_V(V,t)/g(V,t)$ has the same sign as γ.*

PROOF: To solve the PDE (1.31) for the pure investment theory, let $g(V,t) = \exp[-(R - r\gamma)\tau]h(V,\tau)$, where $\tau = \mathbb{T} - t$. Then $h(V,\tau)$ satisfies

$$\text{(2.15)} \qquad \frac{\partial h}{\partial \tau} = \tfrac{1}{2}a^2(V)\frac{\partial^2 h}{\partial V^2} + b(V)\frac{\partial h}{\partial V} - c_\gamma V h, \qquad c_\gamma = -\tfrac{1}{2}\gamma(1-\gamma),$$

and $h(V,\tau = 0) = 1$. Hence $\psi(V,t) = h_V(V,\tau)/h(V,\tau)$. By Theorem 2.4 of Chapter 2, when it exists, $h(V,\tau)$ is positive and $h_V(V,\tau)$ has the opposite sign of c_γ. For $\gamma < 1$, γ and c_γ have the opposite sign. Hence $\psi(V,t)$ has the opposite sign of c_γ and the same sign as γ. ■

3 The Pure Investment Problem with a Distant Planning Horizon

The key to the proof just given is that (2.15) is essentially the evolution equation for the fundamental transform, which we developed extensively in prior chapters. There are two differences now. First, ρ does not appear. Second, the variable $c(k) = (k^2 - ik)/2$ becomes $c_\gamma = -\gamma(1-\gamma)/2$.

The eigenvalue connection, again. Our development of the asymptotic $\tau \to \infty$ theory for the fundamental transform in Chapter 6 can be applied directly to develop the asymptotic theory of the pure investment problem. That theory suggests that as $\tau \to \infty$, the solutions to (2.15) typically separate into the eigenvalue form $h(V,\tau) \approx \exp[-\lambda\tau]u(V)$. The exponent λ is the first eigenvalue and $u(V)$ is associated eigenfunction solution to the equation:

$$\text{(2.16)} \qquad \mathcal{L}u = \lambda u,$$

where $$\mathcal{L}u = -\tfrac{1}{2}a^2(V)\frac{d^2 u}{\partial V^2} - b(V)\frac{du}{dV} + c_\gamma V u$$

In order to make the eigenvalue problem (2.16) well-defined, you need what amounts to boundary conditions at $V = 0$ and $V = \infty$. We treated this issue in a couple of different ways previously. In Chapter 6, Sec. 7, the boundary conditions were that functions had to be "admissible" to a variational problem. Admissibility was a collection of conditions, all of which restricted the behavior at the boundaries. In Chapter 6, Sec 8, boundary conditions were imposed. The boundary conditions were that the solutions had match the "relatively small" or well-behaved solutions of $(\mathcal{L} - z)u = 0$ as $V \to 0$ and $V \to \infty$, treating z as a parameter.

The risk premium coefficient. In summary, we expect to find solutions for the risk premium coefficient, as $\mathbb{T} \to \infty$, of the form

(2.17) $$g(V,t) \approx \exp\{-[R - r\gamma + \lambda(\gamma)]\tau\}u(V).$$

Then (2.17) gives us the very simple asymptotic relation

(2.18) $$\psi_\infty(V) = \lim_{\mathbb{T}\to\infty} \psi(V,t) = \lim_{\mathbb{T}\to\infty} \frac{g_V(V,t)}{g(V,t)} = \frac{u'(V)}{u(V)}.$$

Example 4. The GARCH diffusion model. With $dV = (\omega - \theta V)dt + \xi V dW$, then $u(V)$ is the first eigenfunction solution to

(2.19) $$\mathcal{L}u = -\tfrac{1}{2}\xi^2 V^2 \frac{d^2u}{dV^2} - (\omega - \theta V)\frac{du}{dV} + c_\gamma V u = \lambda(\gamma)u.$$

We showed how to compute $u(V)$ in Chapter 6, Sec. 8. For example, a graph of $u(V)$ is shown in Fig. 6.4. The function that corresponds to the solution to (2.19) is the one labeled $\rho = 0$ in the figure. For the figure, the parameter k was set equal to $i/2$, which meant that $c(k) = 1/8$. This corresponds to $c_\gamma = 1/8$, which means $\gamma = (1 - \sqrt{2})/2 \cong -0.207$. Since $\gamma < 0$, we expect $u' < 0$ and $\psi < 0$, and Fig. 6.4 confirms this. To compute ψ for other values of γ, just follow the method described in Ch.6, Sec. 8 (Mathematica code was given in Appendix 6.1).

Special case: geometric Brownian motion. Geometric Brownian motion is the special case of (2.19) when $\omega = 0$. In that special case, the eigenvalue problem can be solved exactly. See Chapter 11 for the derivation. For $-2\theta < \xi^2$, the result is that $\lambda = 0$ and

(2.20) $$u(V) = \frac{2}{\Gamma(\mu)}\left(\frac{y}{2}\right)^{\mu} K_\mu(y), \quad \text{where } y = \frac{2}{\xi}\sqrt{2c_\gamma V} \text{ and } \mu = 1 + \frac{2\theta}{\xi^2}.$$

In (2.20), K_μ is a modified Bessel function of the second kind. This yields

$$(2.21) \qquad \psi_\infty(V) = \frac{u'(V)}{u(V)} = \frac{1}{2V}\left[\mu + y\frac{K'_\mu(y)}{K_\mu(y)}\right]$$

and so the risk-adjusted volatility process is given by

$$(2.22) \qquad dV = \left[-\theta V - (1-\gamma)\rho(V)\xi V^{3/2} - \tfrac{1}{2}\xi^2 V \psi_\infty(V)\right]dt + \xi V\, dW$$

$$= \left\{-(1-\gamma)\rho(V)\xi V^{3/2} + \tfrac{1}{2}\xi^2 V\left[1 + y\frac{K'_\mu(y)}{K_\mu(y)}\right]\right\}dt + \xi V\, dW\,.$$

As a check, note that, as $y \to 0$, then $K'_\mu(y)/K_\mu(y) \approx -\mu/y$. This will let you recover the actual volatility process in (2.22) when $\gamma = 1$.

The risk-adjusted pricing process. Using (2.18), we now know the martingale pricing process under the pure investment problem with a distant horizon. It's given by:

$$(2.23) \quad \tilde{P}: \begin{cases} dS = (rS - D)dt + \sigma S\, dB \\ dV = \left[b(V) - (1-\gamma)\rho(V)\sqrt{V}a(V) + a^2(V)\psi_\infty(V)\right]dt + a(V)dW \end{cases}$$

where $\psi_\infty(V) = u'(V)/u(V)$ and $u(V)$ is the first eigenfunction of (2.16). There are two correction terms to the drift. For $\gamma < 1$, the first correction term is $-(1-\gamma)\rho\sqrt{V}a$ and has the sign opposite of ρ. The second correction term is $a^2\psi_\infty$ and has the sign opposite of γ.

Note that the impatience parameter R does not appear in (2.23) since it doesn't appear in ψ_∞. In a pure investment problem like (1.6), a discounting factor is irrelevant and can be dropped. Utility maximization problems are unaffected by overall multiplicative constants

An important issue. We briefly remark on an important issue connected with (2.23). We know $b(V)$ is mean-reverting, but what about $\tilde{b}(V)$? Put another way, the issue is the behavior of $\tilde{b}(V)$ for large V and the implications of that.

For example, consider the GARCH diffusion, where $a(V) = \xi V$. We show in Chapter 10 that, as $V \to \infty$, then $\psi_\infty(V) \approx -(2c_\gamma/\xi^2 V)^{1/2} + O(1/V)$. This asymptotic relation holds for any $\omega > 0$ and so also holds for (2.21) where it can be easily checked. This relation makes sense as long as $c_\gamma \ge 0$, which means $\gamma \le 0$. Hence, the risk-adjusted drift $\tilde{b}(V)$, under this preference model, behaves as

$$(2.24) \qquad \tilde{b}(V) \underset{V\to\infty}{\approx} \omega - \theta V - \left[(1-\gamma)\rho + \sqrt{-\gamma(1-\gamma)}\right]\xi V^{3/2} + O(V).$$

Instead of always being mean-reverting, (2.24) is sometimes large and positive at large V. For example, if $\gamma = 0$ and $\rho < 0$, then the risk-adjusted volatility drift behaves like $-\rho\xi V^{3/2}$ at large V.

This causes the risk-adjusted volatility process to "explode", in a precise technical sense. An explosion means that the volatility, under this process, can reach $V \to \infty$ in finite expected time, starting at a finite value. (Remember, this is the risk-adjusted volatility we are talking about; the actual volatility remains mean-reverting and does not explode). The effects of explosions in the risk-adjusted volatility process are discussed in Chapter 9.

4 Preference Adjustments to the Volatility of Volatility Series Expansion

In Chapter 3, we developed the volatility of volatility series solution for options. We noted that in general the risk-adjusted drift had a dependence upon ξ, the expansion parameter, but for simplicity we neglected it. In this section, we show how to adjust that series to include the preference effects developed in this chapter.

Our preference model will be the infinite horizon consumption-investment problem (1.5). This implies that we can write $\psi = \psi(V)$, dropping the time dependence.

In Appendix 3.1 to Chapter 3, we developed a series solution for the fundamental transform. That PDE satisfied by that transform was given at (3.2.2), where the actual volatility process is parameterized $dV = b(V)dt + \xi\,\eta(V)dW$, with correlation $\rho(V)$. Putting in what we've learned about the volatility drift adjustment in this chapter, that equation now reads

$$(4.1) \quad \frac{\partial \hat{H}}{\partial \tau} = \tfrac{1}{2}\xi^2\eta^2(V)\frac{\partial^2 \hat{H}}{\partial V^2}$$

$$+\left\{b(V) + \xi\left[d(k) - 1 + \gamma\right]\rho(V)\sqrt{V}\eta(V) + \xi^2\eta(V)\psi(V)\right\}\frac{\partial \hat{H}}{\partial V} - c(k)V\hat{H}\,.$$

As in Ch. 3, we are using $c(k) = (k^2 - ik)/2$ and $d(k) = -ik$. The initial condition is $\hat{H}(k, V, \tau = 0) = 1$. Now (4.1) can still be solved with an expansion of the form

$$\hat{H}(k, V, \tau) = \hat{H}^{(0)}(k, V, \tau) + \xi \hat{H}^{(1)}(k, V, \tau) + \xi^2 \hat{H}^{(2)}(k, V, \tau) + \cdots$$

But, the new ingredient is that you have to also develop the same expansion for $\psi(V)$ in order to have a consistent ξ – order:

$$\psi(V) = \psi^{(0)}(V) + \xi^2 \psi^{(2)}(V) + \cdots.$$

Assume that you're in possession of the series for $\psi(V)$; then the first three terms for the fundamental transform satisfy, by power matching,

(4.2) $$\mathcal{L}\hat{H}^{(0)} = 0\,,$$

(4.3) $$\mathcal{L}\hat{H}^{(1)} = (d - 1 + \gamma)\rho\sqrt{V}\eta\frac{\partial \hat{H}^{(0)}}{\partial V}\,,$$

(4.4) $$\mathcal{L}\hat{H}^{(2)} = (d - 1 + \gamma)\rho\sqrt{V}\eta\frac{\partial \hat{H}^{(1)}}{\partial V} + \tfrac{1}{2}\eta^2\frac{\partial^2 \hat{H}^{(0)}}{\partial V^2} + \eta\,\psi^{(0)}\frac{\partial \hat{H}^{(0)}}{\partial V}\,,$$

using $$\mathcal{L}f = \left(\frac{\partial}{\partial \tau} - b(V)\frac{\partial}{\partial V} + c(k)V\right)f\,.$$

So, the modifications due to the preference model through $O(\xi^2)$ are relatively minor. First of all, the factor $d(k)$ should be replaced by $d(k) - 1 + \gamma$ in all formulas. Lastly, there's one extra term in the equation for $\hat{H}^{(2)}$ involving $\psi^{(0)} = g_0'/g_0$. In turn, $g_0(V)$ is the solution to (1.30) with $\xi = 0$, which is the first order non-linear equation

(4.5) $$(1-\gamma)g_0^{-\gamma/(1-\gamma)} = \left[(R - r\gamma) - \tfrac{1}{2}\gamma(1-\gamma)V\right]g_0 - b(V)\frac{dg_0}{dV}\,.$$

For the remainder of this section, we specialize to the model for which the ξ – expansion was applied in the most detail in Chapt 3, Sec 3, namely

(4.6) $$\tilde{P}:\ \begin{cases} dS = (r - \delta)S\,dt + \sigma S\,dB \\ dV = (\omega - \theta V)\,dt + \xi V^{\varphi}dW \end{cases},$$

where dB and dW have constant correlation ρ. In this case, $g_0(V)$ satisfies (4.5) with $b(V) = \omega - \theta V$. Solutions to (4.5) involve one arbitrary constant; we choose the constant to keep g_0 real when $V > V^* = \omega/\theta$. With this choice, we find

$$g_0(V) = [Q(V)]^{\gamma-1}\ , \tag{4.7}$$

using $\quad \mu = -\dfrac{\gamma}{2\theta^2}, \quad x = \mu(\omega - \theta\, V), \quad \alpha = \dfrac{R - r\gamma}{\theta(1-\gamma)} - \tfrac{1}{2}\gamma\dfrac{\omega}{\theta^2} > 0,$

$$Q(V) = \frac{\theta x^\alpha e^{-x}}{\bar{\Gamma}(\alpha, x)}, \quad \text{and} \quad \bar{\Gamma}(\alpha, x) = \int_0^x e^{-t} t^{\alpha-1} dt\ , \ (\operatorname{Re}\alpha > 0)\,.$$

The solution makes sense as long as $\alpha > 0$ and $\theta > 0$. The restriction $\alpha > 0$ is only active if γ is in the "dangerous" interval $0 < \gamma < 1$. Both restriction together imply that $Q(V^*) = Q^* = \alpha\theta > 0$. Differentiating (4.7) leads to

$$\psi^{(0)}(V) = \frac{d \ln g_0(V)}{dV} = (1-\gamma)\left[\frac{\gamma}{2\theta} - \frac{Q(V) - Q^*}{\omega - \theta V}\right] = \frac{\gamma(1-\gamma)}{2(R+\theta)} + O(\gamma^2), \tag{4.8}$$

where the last expression follows by expanding the middle one about $\gamma = 0$. At $V = V^*$, $\psi^{(0)}(V^*) = \gamma(1-\gamma)/[2\,(Q^* + \theta)]$. As expected, (4.8) vanishes for the log-utility investor ($\gamma = 0$). As usual, $\psi^{(0)}(V)$ changes sign as γ passes from the range $0 < \gamma < 1$ to $\gamma < 0$, taking the same sign as γ.

The expected equity return, in excess of the riskless rate, is given to this approximation by

$$\alpha^e - r = (1-\gamma)V - \rho\xi V^{3/2}\psi^{(0)}(V) + O(\xi^2) \tag{4.9}$$

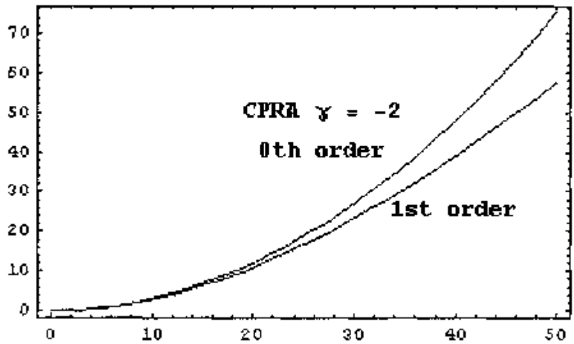
The effect of stochastic volatility on the market risk premium will depend upon the sign of the stock-volatility correlation, ρ. For a negative (positive) correlation , a representative agent more risk-averse than a logarithmic investor will expect a lower (higher) return on the market asset.

For a numerical example, suppose $\rho = -0.5$, $\gamma = -2$, and the additional parameters: $r_a = R_a = 0.05$, $\omega_a = 0.0225$, $\theta_a = 1$, $\xi_a = 1$. Then, if the market volatility (σ) is 15 % annualized, the excess return $\alpha^e - r$ is 6.75 % per year at leading order (deterministic volatility). The stochastic volatility correction, at $O(\xi)$ is -47 basis points per year, for a total market excess return of 6.28 % per year.

Plots of the market excess return as a function of the market volatility are shown in Fig. 7.2 for these parameters and $\sigma_a \le 50\%$. The excess return $\alpha^e - r$ under constant volatility is shown as the "0th order" plot and (4.9) is shown as the "1st order" plot. As one sees, the 1st order corrections are fairly modest, on the order of 10% of constant volatility results.

Fig. 7.2 Market Expected Return Premium versus Volatility

Expected Return Premium ($\alpha^e - r$, Percent per year)



Market Annualized Volatility (σ , Percent)

Notes. Stochastic volatility corrections to the market expected return in the volatility of volatility expansion. The volatility process is the linear drift form $dV = (\omega - \theta V)\,dt + \xi\,\eta(V)dW(t)$ and the preference model is CPRA (power utility) with risk aversion parameter $\gamma = -2$. The graph labeled "0th order" is the classical relation $\alpha^e - r = (1-\gamma)V$ and the graph labeled "1st order" is the $O(\xi)$ correction. The 1st order corrections are modest, roughly about -10% of the leading value.

Series results with preference corrections. It's straightforward but tedious to work out the first three terms in the expansion for the fundamental transform with the preference corrections shown in (4.2) to (4.4). Recall that the volatility process is given by (4.6), so everything depends upon the parameter φ. The results use the notations $c = (k^2 - ik)/2$ and $D = -(1 - \gamma + ik)\rho$. We also use α defined at (4.7) and

$$\hat{H}^{(0)}(k,V,\tau) = \exp\Big[-\frac{c\omega\tau}{\theta} + \frac{c}{\theta}(V - \frac{\omega}{\theta})(e^{-\theta\tau} - 1)\Big] \,, \tag{4.10}$$

$$\hat{H}^{(1)}(k,V,\tau) = -i\frac{cD}{\theta}J_1(V,\tau,\tau)\hat{H}^{(0)}(k,V,\tau)\,,$$

$$\hat{H}^{(2)}(k,V,\tau) = \frac{1}{2\theta^2}\big[c^2 J_2(V,\tau) - 2\theta c D^2 J_4(V,\tau) + \gamma(1-\gamma)cJ_5(V,\tau)\big]\hat{H}^{(0)}(k,V,\tau)$$

where

$$J_1(V,\tau,s) = \int_0^s\Big\{\frac{\omega}{\theta} + (V - \frac{\omega}{\theta})e^{-\theta(\tau-u)}\Big\}^{\varphi+1/2}(e^{-\theta u} - 1)\,du \,,$$

$$J_2(V,\tau) = \int_0^\tau\Big[\frac{\omega}{\theta} + (V - \frac{\omega}{\theta})e^{-\theta(\tau-s)}\Big]^{2\varphi}(e^{-\theta s} - 1)^2\,ds\,,$$

$$J_3(V,\tau,s) = \int_0^s\Big\{\frac{\omega}{\theta} + (V - \frac{\omega}{\theta})e^{-\theta(\tau-u)}\Big\}^{\varphi-1/2}(e^{-\theta s} - e^{-\theta(s-u)})\,du\,,$$

$$J_4(V,\tau) = \int_0^\tau\Big[\frac{\omega}{\theta} + (V - \frac{\omega}{\theta})e^{-\theta(\tau-s)}\Big]^{\varphi+1/2}$$
$$\times\Big\{\frac{(e^{-\theta s} - 1)}{\theta}cJ_1(V,\tau,s) + (\varphi + \tfrac{1}{2})J_3(V,\tau,s)\Big\}\,ds\,,$$

$$J_5(V,\tau) = \int_0^\tau\Big[\frac{\omega}{\theta} + (V - \frac{\omega}{\theta})e^{-\theta(\tau-s)}\Big]^{2\varphi}(e^{-\theta s} - 1)\Big[1 + \frac{z^\alpha(\tau,s)e^{-z(\tau,s)}}{\Gamma(\alpha, z(\tau,s))} - \frac{\alpha}{z(\tau,s)}\Big]ds$$

and

$$z(\tau,s) = \frac{\gamma}{2\theta c}(y - y_0)e^{-\theta(\tau-s)}$$

Only integer powers of $c(k)$ and (ik) appear, so every term can be integrated in the k-plane using the methods discussed in Chapter 3.

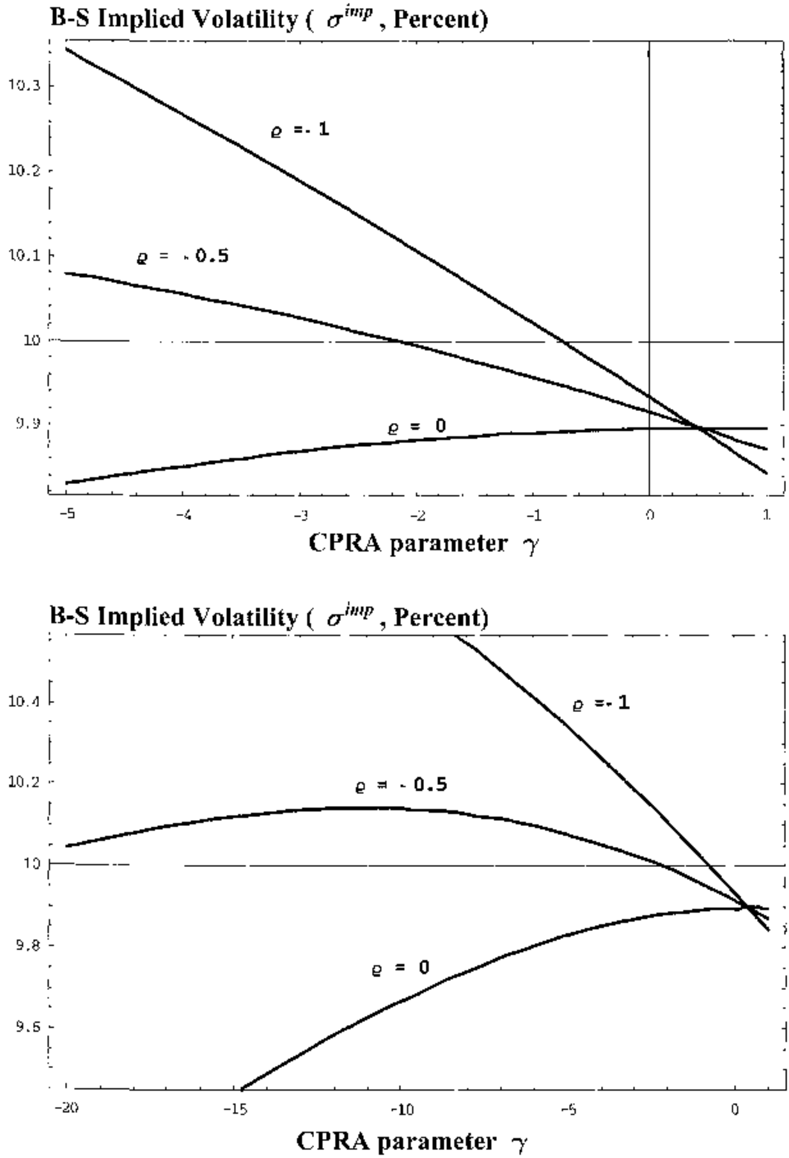
5 The Effect of Risk Attitudes on Option Prices

A relatively simple case in which to see the preference effects in option prices is the square root model of Example 3, Sec. 2. The preference model is taken to be a pure investor with a distant horizon. The effect of the risk attitudes of the representative agent, in that case, was shown to be an alteration of the drift parameter from θ to $\tilde{\theta} = (1-\gamma)\rho\xi + [\theta^2 - \gamma(1-\gamma)\xi^2]^{1/2}$. This adjustment is already reflected in the Mathematica code for this model we gave in Appendix 2.2.

For a numerical example, we take an at-the-money option with $\omega_a = 0.02$, $\theta_a = 2$, $\xi_a = 0.1$, and $\tau = 1/2$ years. The interest rate and dividend yield are set to zero. The at-the-money implied volatility, for the risk-neutral case is slightly less than 10% . In Fig. 7.3 we plot the (at-the-money) implied volatility as a function of the CPRA parameter : $\sigma^{imp}(\gamma)$. Three plots are shown, for the cases $\rho = 0, -1/2, -1$. The top panel shows the interval $(-5 \le \gamma \le 1)$ and the bottom panel shows same case but with the larger interval $(-20 \le \gamma \le 1)$

The figure shows that, with a zero stock-volatility correlation, risk aversion lowers option prices relative to risk neutrality when $\gamma < 0$ (The graph actually has a maximum at $\gamma = 1/2$). But, with a significantly negative correlation, option prices are raised unless the risk aversion is extreme. This pattern is probably quite general. For example, I have also taken the formulas (4.10) for the volatility of volatility expansion from the last section and plotted the same graphs when the volatility model is the GARCH diffusion. The plots have the same pattern as Fig. 7.3. So in one case, we have the square root volatility process with a pure investment utility model. And in the other case we have the GARCH diffusion volatility process with a consumption-investment utility model.

Fig. 7.3 The Effect of Risk Attitudes on Option Prices



Notes. The preference model is a representative agent who is a pure investor with power utility W^{γ} and a distant planning horizon. The figures are based on at-the-money options in a square root model. With a zero or negative correlation, prices are raised in the interval $\gamma_1(\rho) < \gamma < 1$ and lowered in the interval $\gamma < \gamma_1(\rho)$.

A qualitative way to understand why this pattern should be very general is to consider the volatility process under general preferences from (1.26)

$$dV = \left[b(V) - \varphi(V,t) \right] dt + a(V)\, dW$$

where $$\varphi(V,t) = (1-\gamma)\rho(V)\sqrt{V}a(V) - a^2(V)\psi(V,t)\,.$$

First, assume that $\rho = 0$. Then, the risk-adjusted process is

$$dV = \left[b(V) + a^2(V)\psi(V,t) \right] dt + a(V)\, dW\,.$$

We have seen that ψ takes the same sign as γ; so when $\gamma < 0$, the risk-adjusted volatility is lower than under risk-neutrality. From the B-S theory, we know that the general effect of lower volatility is lower put and call option prices. And, when $0 < \gamma < 1$, then $\psi > 0$, and that's why the prices are (slightly) higher in that regime.

Now consider the general case when $\rho \neq 0$. The term $-(1-\gamma)\rho(V)\sqrt{V}a(V)$ acts to raise the volatility for all $\rho < 0$ and $\gamma < 1$. This term dominates if ρ is significantly negative and $|\gamma|$ not too large. But, if you take an extremely large negative value for γ, then the $a^2\psi$ term will win out because ψ has some quadratic dependencies on γ. [Very crudely, $\psi \approx \gamma(1-\gamma) f(V,t)$]. Since these qualitative aspects hold for every model, that's why I believe Fig 7.3 shows the general pattern. Briefly, we can summarize this pattern by saying that the effect of CPRA preferences, with $\rho \leq 0$, is that call or put option prices are raised in an interval $\gamma_1(\rho) < \gamma < 1$ and lowered in an interval $\gamma < \gamma_1(\rho)$.

8 Duality and Changes of Numeraire

Duality relates option prices (or general financial claim prices) between two points of view. In one view, the stock price S, which can be thought of an exchange rate of so many dollars per share, is perceived as the risky asset. The numeraire, which is dollars, is perceived as the riskless asset. This is the point of view which we have adopted so far in the exposition. But, you could do your banking and conduct your transactions in shares instead, especially in a simple economy of only two assets: shares and dollars. In the new point of view, the shares become your numeraire and the fluctuating exchange rate makes dollars seem risky.

This change-of-numeraire transformation, or duality for short, has a number of interesting consequences. First, put and call options prices become exchanged. Second, the volatility process becomes transformed into the auxiliary volatility process. These results are known. A new observation is that, under our CPRA equilibrium model (with consumption at a final date), the entire transformation can be interpreted as merely a change of parameters in the original model. That is, the parameters of the model are the set $\{r, \delta, \rho, \gamma\}$, where r is the interest rate, δ is the dividend yield, ρ is the stock/volatility correlation and γ is the CPRA risk-aversion parameter. Under a change of numeraire, the (risk-adjusted) stochastic processes keep the *same form*, but we have $\{r, \delta, \rho, \gamma\} \rightarrow \{\delta, r, -\rho, 1-\gamma\}$. Duality converts a risk neutral representative investor into a log-utility investor. An investor more risk-averse than log-utility becomes risk-loving.

Finally, we note in these introductory remarks that the change-of-numeraire transformation is very important in understanding the failure of the martingale pricing formula (and its corrected version). This failure, which occurs in some very typical volatility process models, is discussed in the next chapter.

1 Put-Call Duality

For orientation, let's begin with the simple case of the Black-Scholes' PDE for financial claim prices. We consider a world with a constant interest rate r, constant dividend yield δ, and constant stock price volatility σ. So, the risk-adjusted stock price process is given by $dS_t = (r-\delta)S_t dt + \sigma S_t dB_t$. The PDE for a general European-style claim price $F(S,t)$ is given by

$$(1.1) \qquad -\frac{\partial F}{\partial t} = -rF + (r-\delta)S\,\frac{\partial F}{\partial S} + \tfrac{1}{2}\sigma^2 S^2\,\frac{\partial^2 F}{\partial S^2}.$$

Suppose we have a solution $F(S,t) = C(S,t)$ to (1.1) for a particular payoff function, say the call option. That is, $C(S,T) = \max(S-K,0) = (S-K)^+$. The duality transformation can be "discovered" by considering a simple change of variable to $X = 1/S$, and looking for a new solution of the form $F(S,t) = S\,f(X,t)$. Then, simple manipulations show that (1.1) becomes the new equation

$$(1.2) \qquad -\frac{\partial f}{\partial t} = -\delta f + (\delta - r)X\,\frac{\partial f}{\partial X} + \tfrac{1}{2}\sigma^2 X^2\,\frac{\partial^2 f}{\partial X^2}$$

That is, the PDE retains the same form but the role of δ and r have been exchanged. Moreover, we have $f(X,T) = (S-K)^+/S = K(k-X)^+$, where $k = 1/K$. This payoff is K times the payoff for a put option with strike price k. Hence, we have found that

$$(1.3) \qquad C(S,K,t) = S\,K\,\hat{P}\left(\frac{1}{S},\frac{1}{K},t\right) = \hat{P}(K,S,t),$$

where $P = P(S,K,t)$ is the put option solution and $\hat{P}(S,K,t)$ is the put option solution when the (implied) parameters r and δ have become exchanged.

Equation (1.3) is a duality relation, also called "put-call symmetry". It was discovered by Grabbe (1983) in the case of foreign exchange options, where it has a natural interpretation. A review is offered in Carr and Chesney (1996).

Suppose, instead, that the payoff was simply S, so that $F(S,t) = Se^{-\delta(T-t)}$. Then, $f(X,t)$ is the solution to (1.2) with the constant payoff $f(X,T)=1$. That solution is $f(X,t) = e^{-\delta(T-t)}$, which is the value of the money market security with interest rate δ. If one had started with the money market security, then $f(X,t) = Xe^{-r(T-t)}$, which is the risky security with dividend yield r.

Now let's make the same transformation under stochastic volatility. We assume that the risk-adjusted pricing process now has the general form, with ρ a constant,

$$(1.4) \qquad \tilde{P}: \begin{cases} dS_t = (r-\delta)S_t\,dt + \sigma_t S_t\,d\tilde{B}_t \\ dV_t = [b(V) - \varphi(V,t)]dt + a(V_t)\,d\tilde{W}_t \end{cases}, \qquad d\tilde{B}_t d\tilde{W}_t = \rho\,dt\,.$$

That is, the original volatility process is stationary with volatility drift coefficient $b(V)$, but we allow a time-dependence in the volatility drift to arise from the risk-adjustment. The time-dependence occurs because we consider equilibrium under consumption at a final date $\mathbb{T}$, which generates a time-dependent drift for all $\mathbb{T} < \infty$. The option price $F(S,V,t)$ must satisfy the PDE:

$$(1.5) \qquad -\frac{\partial F}{\partial t} = -rF + \tilde{\mathcal{A}}\,F\,,$$

where
$$\tilde{\mathcal{A}}F := (r-\delta)S\,\frac{\partial F}{\partial S} + \tfrac{1}{2}V\,S^2\,\frac{\partial^2 F}{\partial S^2} + \tilde{b}(V,t)\,\frac{\partial F}{\partial V} + \tfrac{1}{2}a^2(V)\frac{\partial^2 F}{\partial V^2} + \rho\,a(V)\sqrt{V}S\,\frac{\partial^2 F}{\partial S \partial V}\,,$$

using $\tilde{b}(V,t) = b(V) - \varphi(V,t)$. Again let $F(S,V,t) = S\,f(X,V,t)$, where $X = 1/S$. Now we find the new PDE:

$$(1.6) \qquad -\frac{\partial f}{\partial t} = -\delta f + \hat{\mathcal{A}}f\,,$$

where
$$\hat{\mathcal{A}}f = (\delta - r)X\,\frac{\partial f}{\partial X} + \tfrac{1}{2}V\,X^2\,\frac{\partial^2 f}{\partial X^2} + \hat{b}(V,t)\frac{\partial f}{\partial V} + \tfrac{1}{2}a^2(V)\,\frac{\partial^2 f}{\partial V^2} - \rho\,a(V)\sqrt{V}X\,\frac{\partial^2 f}{\partial X \partial V}\,,$$

and using $\hat{b}(V,t) := \tilde{b}(V,t) + \rho\,a(V)\sqrt{V}$. In addition to the swapping of r and δ, we have two new changes: the correlation term (the mixed partial derivative) has switched sign and the volatility drift coefficient has changed. The correlation term change can be interpreted as a change in the sign of ρ, where the *meaning* of ρ is that it describes the correlation between the volatility and the "risky security". As before, we still have:

(1.7) $$C(S,K,V,t) = S\,K\,\hat{P}\left(\frac{1}{S},\frac{1}{K},V,t\right) = \hat{P}(K,S,V,t)\,,$$

but the meaning of $\hat{P}$ has changed. Now $\hat{P}$ is the put option solution to (1.6). In general, the *form* of the PDE has now changed because the volatility drift is now a different function of V.

But, if we also adopt the CPRA equilibrium model for the pure investor (no intermediate consumption), then the *form* of the PDE is also preserved. To see this, note that under the CPRA equilibrium (even with intermediate consumption), we have the general expression from (7.1.20)

(1.8) $$\tilde{b}(V,t) = b(V) - (1-\gamma)\rho\sqrt{V}a(V) + a^2(V)\psi(V,t)\,, \text{ where } \psi = g_V\,/\,g\,.$$

And, if we take the pure investment equilibrium with planning horizon $\mathbb{T}$, then $g(V,t)$, the risk premium coefficient, is determined as the solution to (7.1.31). We repeat that equation here for convenience:

(1.9) $$\frac{\partial g}{\partial t} = [(R - r\gamma) - \tfrac{1}{2}\gamma(1-\gamma)V]\,g - b(V)\frac{\partial g}{\partial V} - \tfrac{1}{2}a^2(V)\frac{\partial^2 g}{\partial V^2}\,,$$

with $g(V,\mathbb{T}) = 1$. Now, under the duality transformation, the new volatility drift is given by the *auxiliary process*:

(1.10) $$\hat{b}(V,t) = \tilde{b}(V,t) + \rho\,a(V)\sqrt{V} = b(V) + \gamma\rho\sqrt{V}a(V) + a^2(V)\psi(V,t)\,.$$

But, this equation is the same as (1.8) under the parameter changes

(1.11) $$\{\rho,\gamma\} \to \{\hat{\rho},\hat{\gamma}\} = \{-\rho, 1-\gamma\}$$

That's because $\psi(V,t) = \psi(V,t;\gamma) = \psi(V,t;1-\gamma)$. *Proof*: $g(V,t;\gamma)$ doesn't depend upon ρ at all. The term $\gamma(1-\gamma)V$ in (1.9) is invariant under $\gamma \to (1-\gamma)$. The only other term in (1.9) that depends upon γ generates an overall multiplicative time-dependence $\exp[-(R-r\gamma)(T-t)]$ for $g(V,t;\gamma)$ that drops out in the computation of $\psi(V,t;\gamma)$. ■

This establishes our introductory statement about the CPRA equilibrium with consumption at a final date. Namely, under that equilibrium, the duality transformation for a financial claim is equivalent to (i) changing the payoff function, and (ii) changing the parameters of the risk-adjusted model as follows[1]:

(1.12) $$\{r,\delta,\rho,\gamma\} \to \{\hat{r},\hat{\delta},\hat{\rho},\hat{\gamma}\} = \{\delta,r,-\rho,1-\gamma\}$$

The dual of the dual, that is making the same interchanges again in (1.12) takes one back to the original model. If we write out all the parameter dependencies, then (1.7) reads explicitly

(1.13) $$P(S,K,V,\tau;r,\delta,\rho,\gamma) = C(K,S,V,\tau;\delta,r,-\rho,1-\gamma).$$

American-style options. By letting $y = S/K$, (1.5) provides an equation of the same form, except with the substitution $S \to y$. For American-style call options, this standardized equation holds for a continuation region $0 < y < \overline{y}_{call}(\tau,V)$, where $\overline{y}_{call}(\tau,V)$ is a *free boundary* which must be determined. In the same way, one can let $x = XK$ in (1.6) and get an equation for a standardized put in the region $\hat{\underline{x}}_{put}(\tau,V) < x < \infty$, where the underbar denotes that it's a lower bound and the hat means that you need to make the duality transformation (1.12) in the parameters. By duality, $\overline{y}_{call}(\tau,V) = \hat{\underline{x}}_{put}(\tau,V)$, or in other words,

(1.14) $$\overline{S}_{call}(\tau,V)\hat{\underline{S}}_{put}(\tau,V) = K^2,$$

which is a relation known to hold under constant volatility. See Carr and Chesney (1996) for generalizations and further discussion of this relation under constant volatility.

The problem of the exercise boundary for American-style options under stochastic volatility has been studied by Touzi (1999). His assumption for the form of the martingale pricing process is generally compatible with (1.4) except that he assumes that the volatility process is bounded below away from zero and bounded above. Technically, this rules out most of the models we study in this book but his conclusions probably apply to those models anyway.

Touzi proves that the "smooth fit" conditions still hold, which are that $C_S = 1$ and $P_S = -1$ at the early exercise points. In addition, he proves that the put option price is still convex in S and increasing in V in the continuation region, just as in the European case (see Chapter 2, Appendix 3). If you

---

[1] The duality relation between $\gamma = 0$ and $\gamma = 1$ is valid even with intermediate consumption, since $\psi = 0$ in those cases.

visualize a graph of the put price versus S, then increasing the volatility must raise the curve, since $P_V \geq 0$. Clearly, this will lower the critical point where the graph intersects the parity line $(K-S)^+$. So the critical exercise boundary $\underline{S}_{put}(\tau,V)$ must be a decreasing function of the volatility, which Touzi also proves. Then, (1.13) implies that $\overline{S}_{call}(\tau,V)$ must be an increasing function of the volatility.

We summarize the results in the following table.

Table 8.1 Some Relationships between Points of View: Dollars as the Numeraire vs. Shares as the Numeraire Model: CPRA Equilibrium with no Consumption

	View I	View II
Numeraire	Dollars	Equity Shares
Risky security	Equity shares	Dollars
Risky security price	S	$X = 1/S$
Interest rate on numeraire	r	δ
Yield on risky security	δ	r
Volatility-risky correlation	ρ	$-\rho$
CPRA parameter	γ	$1-\gamma$

2 Introduction to the Change of Numeraire

The modern point of view about the transformation in Sec. 1 is that it represents a change of numeraire from the implicit one (say dollars) to the risky asset. Moreover, this change is closely associated with a change of probability measure, just as we have a change in measure in moving from the actual price process to the martingale pricing process. This point of view has been emphasized by Geman, El Karoui, and Rochet (1995) and extended by Schroder (1999). In this section, we motivate the mathematics behind these transformations. In the next section, we show how the formalism reproduces the results in Sec. 1.

By "numeraire", we just mean the currency base in which financial transactions are expressed. For example, we have often just written, say $S = 30$ in numerical examples but what we really meant and left implied was $S = 30$ dollars per (equity) share. This can be thought of as an exchange rate between dollars and shares.

In a simple economy with only dollars and equity shares, which is our equilibrium model, you could imagine that, in fact, a dual banking system would arise. That is, for simplicity, imagine that everyone has to choose a bank but can choose from either of two types, both operating in the same economy. In the "traditional" bank, deposits and loans are recorded in dollars. Also, since the bank is "full service", it keeps track of security holding as well, including equity shares and any financial claims such as options; they are always marked to market in dollars.

But, people realize that so much of their wealth is tied up in shares and they often just do transactions in shares, so the demand arises for a new-style bank in which everything is just recorded in shares. The current exchange rate $S(t)$, equal to so many dollars per share, is always available to translate any dollar amount to a share amount and vice-versa.

The traditional bank posts the continuous interest rate r on all dollar deposits and charges the same rate for dollar loans. The new-style bank posts the interest rate δ on all shares and charges δ for loans of shares. Of course, this "interest rate" is posted on deposits by recording an increase in the number of shares; these new shares are issued by firms, as we discussed in Chapter 7, to satisfy the aggregate consumption preferences of consumers.

For a new-style consumer-investor, when they look at their wealth X on deposit in the new-style bank, say 10,000 shares, things look pretty quiet. There is a steady but usually pretty small rate of increase of their balance due to the dividend/interest rate postings. This is, of course, the solution $X(t) = X(0)e^{\delta t}$ we found above. But otherwise, not much is going on, even though the stock market is fluctuating wildly as usual. New-style investors don't have to pay any attention to stock market, at least until they have to do a transaction in dollars; then the exchange rate matters to them. If they don't do any dollar transactions, then life is really (financially) riskless!

If a new-style investor does want to keep a dollar deposit of $B(0)$ dollars posted on their account, then the new-style bank will dutifully mark it to market at so many shares per dollar, or $B(t) = B(0)e^{rt} / S(t)$. This entry on their account will fluctuate a lot.

The people who create accounting software for banks only want to support one version, which they then sell to both types of banks. They realized that they could accomplish this by careful labeling. For example, the monthly statements that the banks mail to their customers have a column heading called "numeraire balance" and one called "risky security balance". They sell this software to both types of banks. The new-style bank records the slowly varying amount $X(0)e^{\delta t}$ in the numeraire column and the rapidly varying $B(0)e^{rt} / S(t)$ in the security column.

In our equilibrium model, aggregate wealth is 100% invested in the stock market, dollar holdings are always two-way borrowing/lending transactions and they net to zero in the aggregate. In such an economy, dollar holdings or obligations might often be short term, entered into for temporary purposes and closed down later. In that case, the new-style banks might become very popular because they dramatically lower the perceived fluctuations in long-term wealth.

The representative investor who serves as a proxy for all investors, could be viewed as either a traditional investor or a new-style investor. As a traditional investor, the representative has CPRA risk-aversion parameter γ and places all of society's wealth in the risky asset. As a new-style investor, she has CPRA parameter $1-\gamma$ and places all wealth (measured in shares) in the numeraire. The qualitative change in risk attitude depends upon the starting point. For example, the representative becomes more risk-averse if γ (which we have previously assumed is ≤ 1) starts in the range $(1/2,1)$. Otherwise, the representative either becomes less risk-averse, with starting $\gamma \in (0,1/2)$, or risk-loving, with starting $\gamma < 0$. Of course, it's not just the change in risk-attitude driving the new investment decision to place all wealth in the numeraire: the expected rates of return of the two basic securities (under the real-world probability measure) have also changed.

3 Mathematics of the Change of Numeraire

In Section 1, we performed the duality transformation by making two changes: (i) we let $F = Sf$, where F was a financial claim price, and (ii) we expressed the result in terms of the new variable $X = 1/S$. Mathematically, the change of numeraire process is very similar. Instead of starting from the PDE, we start from the assumption that all reasonable financial claims are priced as discounted expectations under a risk-adjusted process $\tilde{P}$. Under the stochastic volatility models discussed in this book, this assumption sometimes fails and, when it does, the failure is closely linked to the change-of-numeraire process. This is discussed in Chapter 9.

Under the assumption, we have

$$F = e^{-rT}\mathbb{E}^{\tilde{P}}[F_T] = e^{-rT}\int F_T d\tilde{P}_T, \tag{3.1}$$

where $F \equiv F(S,V,t=0)$), is the current price of a financial claim under stochastic volatility expiring at time T. To relate this notation to our previous discussions, under volatility-independent payoffs, then $d\tilde{P}_T \equiv \tilde{p}(S,V,S_T,T)dS_T$, the martingale transition density of Chapter 2. As in Section 1, we let $F_T = S_T f_T$, and note that S_T is given formally by the expression $S_T = Se^{(r-\delta)T}G_T$, where

$$G_T = \exp\left\{\int_0^T \sigma_t d\tilde{B}_t - \frac{1}{2}\int_0^T \sigma_t^2 dt\right\}. \tag{3.2}$$

We also assume that G_T is a martingale under $\tilde{P}$, ($\mathbb{E}^{\tilde{P}}[G_T]=1$), and the Girsanov theorem may be applied[2]. That means that $G_T d\tilde{P}_T$ may be interpreted as another probability measure, call it dQ_T, so that

$$F = Se^{-\delta T}\int f_T G_T d\tilde{P}_T = Se^{-\delta T}\int f_T dQ_T = Se^{-\delta T}\mathbb{E}^Q[f_T]. \tag{3.3}$$

Or, dividing both sides by S, and noting that $F/S = f$, we have

$$f = e^{-\delta T}\mathbb{E}^Q[f_T]. \tag{3.4}$$

What makes this operationally useful is that the Girsanov theorem also states that $d\hat{B}_t \equiv d\tilde{B}_t - \sigma_t dt$ is a Brownian motion under the Q-process. Actually, it's more convenient to use $-d\hat{B}_t$, which is also a Brownian motion under Q, so

---

[2] A sufficient condition for G_T to be a martingale and the Girsanov theorem to hold is the Novikov condition $\mathbb{E}^{\tilde{P}}\exp\left\{\frac{1}{2}\int_0^T \sigma_t^2 dt\right\} < \infty$. See, for example, Øksendal (1998). This condition is problematic under the unbounded volatility models we study. As we shall see in Chapter 9, a more refined criterion involves the existence, or not, of explosions to $+\infty$ in the auxiliary volatility process.

the Girsanov transformation from $\tilde{P}$ to Q can also be performed with the substitution $d\tilde{B}_t = -d\hat{B}_t + \sigma_t dt$. To apply this, note that by Ito's formula, the evolution of $X_t = 1/S_t$ under $\tilde{P}$ is given by

$$(3.5) \qquad \tilde{P}: \begin{cases} dX_t = (\delta - r + \sigma_t^2)X_t dt - \sigma_t X_t d\tilde{B}_t \\ dV_t = \tilde{b}(V_t)dt + a(V_t)\left[\rho d\tilde{B}_t + \sqrt{1-\rho^2}d\tilde{Z}_t\right] \end{cases},$$

where we have written $d\tilde{W}_t = \rho d\tilde{B}_t + \sqrt{1-\rho^2}d\tilde{Z}_t$, breaking up $d\tilde{W}$ into the sum of two independent Brownian motions. Then, making the substitution: $d\tilde{B}_t = -d\hat{B}_t + \sigma_t dt$, we have under the $Q-$process, the evolution

$$(3.6) \qquad Q: \begin{cases} dX_t = (\delta - r)X_t dt + \sigma_t X_t d\hat{B}_t \\ dV_t = \left[\tilde{b}(V,t) + \rho\, a(V_t)\sigma_t\right]dt + a(V_t)\left[-\rho d\hat{B}_t + \sqrt{1-\rho^2}d\tilde{Z}_t\right] \end{cases}.$$

Of course, now we can write $d\hat{W}_t = -\rho d\tilde{B}_t + \sqrt{1-\rho^2}d\tilde{Z}_t$, and so we have the final result that, under this change of numeraire, the evolution of the new "basic risky security" X_t is now given by

$$(3.7) \qquad Q: \begin{cases} dX_t = (\delta - r)X_t dt + \sigma_t X_t d\hat{B}_t \\ dV_t = \left[\tilde{b}(V,t) + \rho\, a(V_t)\sigma_t\right]dt + a(V_t)d\hat{W}_t \end{cases}, \quad d\hat{B}_t d\hat{W}_t = -\rho dt.$$

Hence, any financial claim with price $f(X,V,t)$, satisfies the PDE (1.6), reproducing our previous result. This shows that the duality transformation is mathematically the same as the combination of (i) a change of variable from S to $X = 1/S$ and (ii) a change of measure, using the Girsanov transformation. We stress again that the new volatility process, under Q, is what we have previously termed the auxiliary volatility process.

Applications. Duality is an exact relationship with various applications. For example, a computer code that includes the preference models of Chapter 7 could be partly tested by setting γ to 0 or 1 and checking (1.13) or the equivalent (4.1) below.

Another application of duality is understanding the volatility explosions that sometimes occur in either the risk-adjusted volatility process or the auxiliary volatility process. These ideas are discussed in the next chapter.

A third application makes the connection between the implied volatility in the two dual worlds. This is discussed in the next section.

4 Implications for the Term Structure

Under the class of models studied in this book, put-call *parity* holds. This in turn means that the term structure of implied volatility may be determined from either put or call option prices; both yield the same term structure. Combining this idea with duality implies that, with consumption at a final date:

$$V^{imp}(S,K,\tau;r,\delta,\rho,\gamma)=V^{imp}(K,S,\tau;\delta,r,-\rho,1-\gamma)\,. \tag{4.1}$$

For example, suppose $r=\delta=0$. Consider the implied volatility of an option under log-utility with a "moneyness" $S/K=1.25$ and stock-volatility correlation -0.50. Then (4.1) says that this implied volatility should be identical to an option under risk-neutral preferences, moneyness of 0.8, and correlation +0.50.

The symmetric case. Note that when $\rho=0$, then the risk-adjusted volatility process with a particular γ is identical to one where γ is replaced by $1-\gamma$, since ψ is invariant. In that case, the duality switches in (4.1) take the moneyness variable $X\to -X$, where $X=\ln(S/K)+(r-\delta)\tau$. That is, (4.1) becomes $V^{imp}(X,\tau)=V^{imp}(-X,\tau)$. This is a third proof that, under zero correlation, the smile is symmetric in the moneyness X.

As indicated previously, the at-the-money term structure is naturally measured where $Se^{-\delta\tau}=Ke^{-r\tau}$. The at-the-money implied volatility then satisfies the duality relation:

$$V^{imp}_{risk-neutral}(\rho,\tau)=V^{imp}_{log-utility}(-\rho,\tau)\,. \tag{4.2}$$

Example application of (4.2). Consider the GARCH diffusion process $dV_t=(\omega-\theta V_t)\,dt+\xi V_t\,dB_t$. We calculated the asymptotic implied volatility under risk-neutrality at (6.6.10); for the reader's convenience, it was:

(i) Risk-neutrality

$$V_\infty^{imp} = \frac{\omega}{\theta} + \left(\frac{\omega}{\theta}\right)^{3/2} \frac{\rho}{2\theta}\xi + \frac{\omega^2(-1+7\rho^2)}{16\,\theta^4}\xi^2 + \left(\frac{\omega}{\theta}\right)^{3/2} \frac{\rho[\omega(-10+31\rho^2)+6\theta^2]}{64\theta^4}\xi^3 + \frac{1}{256\theta^7}[\omega^3(4-81\rho^2+157\rho^4)+4\omega^2\theta^2(-2+15\rho^2)]\,\xi^4 + O(\xi^5)\,. \tag{4.3}$$

Then duality (4.2) tells us immediately that the series under log-utility must be[3]

(ii) Log-utility

$$V_\infty^{imp} = \frac{\omega}{\theta} - \left(\frac{\omega}{\theta}\right)^{3/2} \frac{\rho}{2\theta}\xi + \frac{\omega^2(-1+7\rho^2)}{16\,\theta^4}\xi^2 + \left(\frac{\omega}{\theta}\right)^{3/2} \frac{\rho[\omega(10-31\rho^2)-6\theta^2]}{64\,\theta^4}\xi^3 + \frac{1}{256\,\theta^7}\Big[\omega^3(4-81\rho^2+157\rho^4)+4\omega^2\theta^2(-2+15\rho^2)\Big]\xi^4 + O(\xi^5)\,. \tag{4.4}$$

[3] In fact, I had previously calculated the log-utility series (4.4) by another method. It was the close relationship between these two series that suggested the duality relationship in the first place.

9 Volatility Explosions and the Failure of the Martingale Pricing Formula

In economically reasonable stochastic volatility models, the actual volatility should be a recurrent process. If the volatility process is modeled as a stationary, but unbounded process, then the singular boundary at infinity is unreachable (natural or entrance-type). But very typically, the risk-adjusted volatility process is qualitatively different. In very simple cases, such as the GARCH diffusion under log-utility, the risk-adjusted volatility process can explode, running off to infinity in finite expected time. The boundary at infinity can become an exit or trap state. Alternatively, the auxiliary volatility process can explode.

An explosion in the auxiliary volatility process causes some (discounted) financial claim prices to fail to be martingales; instead, they are only local martingales. For those claims, we propose that arbitrage-free fair values are given by the (discounted) expected terminal value (martingale pricing) plus an explosion 'correction' term. Our primary and new result, is that the Solution II formulas presented in Chapter 2 can fully account for the necessary modifications to the martingale pricing formulas.

1 Introduction

The representation of financial claim prices as expected terminal values under some stochastic process is a fundamental result in finance, closely connected with the absence of arbitrage opportunities in reasonable models. Equivalently, the notion is that reasonable financial claim prices, relative to some numeraire, should be martingales under some risk-adjusted pricing process. This idea is one of the strongest generalizations flowing from Black and Scholes' option theory.

However, it is now appreciated that the above idea overstates the general situation—the absence of arbitrage under unbounded processes only implies a weaker result, namely that financial claim prices are only local martingales under a pricing process. [See Delbaen and Schachermayer (1994), Corollary 1.2]. As we show in this chapter, stochastic volatility models provide many explicit examples of this more general phenomenon. Prices are only local martingales here because of volatility explosions. This means the volatility can reach infinity in finite expected time. Even though the explosion does not occur in the real-world process, it can occur in either the risk-adjusted process or the auxiliary process. Explosion examples are quite typical and occur in many of our running examples: for example, the GARCH diffusion process under log-utility preferences [see (1.2) below].

If a financial claim price is not a true martingale under the pricing process, then its fair value is *not* given by the expected terminal value—so what is the fair value? That is the question we consider in this chapter. A very important step was achieved by Sin (1998), which we discuss in Sec. 4. He proved the failure of the martingale pricing formula for the stock price in the GARCH diffusion case noted above. In Secs. 5-6, we take up the question about fair values. For put and call options, our answer is: use the Solution II formulas given in Chapter 2. We show that the Solution II formulas prevent certain arbitrage opportunities.

Three models are treated as examples. We discuss explosions in the GARCH diffusion model in Sec. 3. Then, in Sec. 7, we discuss the 3/2 model as an example of the application of the generalized pricing formulas. In Sec. 8, we discuss the constant elasticity of variance (CEV) model, which is a stochastic volatility model with perfect correlation between the stock price and option price. The CEV model provides a relatively simple illustration of many of the effects of volatility explosions. Numerical examples with the various models show that in the face of volatility explosions, option prices remain quite well-behaved.

For one dimensional (1D) stationary diffusions, we need the general classification theory of boundary behavior developed by Feller and Russian schools. This theory is standard, and is reviewed in Sec. 2. Readers already familiar with it may just want to examine our notation and skip immediately to Sec. 3, which begins the new results.

At the outset, our case is more difficult than the 1D theory because we have a two dimensional correlated process. The 2D case has not been classified in the systematic manner of 1D diffusions, although some partial results are known[1].

The GARCH diffusion under two risk preference models. One working example in this chapter is the GARCH diffusion process under the two representative agent models: (i) risk neutrality and (ii) logarithmic utility. As explained in Chapter 8 , these two cases are duals to each other. The two risk-adjusted pricing processes are:

Utility model: risk neutrality

(1.1)
$$P_1: \begin{cases} dS_t = (r-\delta)S_t\,dt + \sigma_t S_t\,d\bar{B}_t \\ dV_t = (\omega - \theta V_t)dt + \xi V_t\,d\tilde{W}_t \end{cases}$$

Utility model: logarithmic-utility:

(1.2)
$$P_2: \begin{cases} dS_t = (r-\delta)S_t\,dt + \sigma_t S_t\,d\bar{B}_t \\ dV_t = (\omega - \theta V_t - \rho\xi V_t^{3/2})dt + \xi V_t\,d\tilde{W}_t \end{cases}$$

In both cases, $d\tilde{B}_t$ and $d\tilde{W}_t$ are correlated Brownian motions with constant correlation ρ. The parameters $\{r,\delta,\omega,\theta,\xi\}$ are constants.

2 The Feller Boundary Classifications

The Feller boundary classification theory is a well-established approach to 1D time-independent diffusions. We can apply that theory to the volatility process, which is a "stand-alone" time-homogeneous process with one variable.

[1] See Khas'minskii (1960).

Boundaries are classified as either: (i) *regular*, (ii) *natural*, (iii) *entrance*, or (iv) *exit*. Some meanings of each of these categories are summarized in Table 9.1. If a boundary is not regular, then it's singular.

Table 9.1 Feller Boundary Classifications

Boundary Type	Some Meanings
regular	The process can both enter (in finite time) and leave. PDE boundary conditions must be specified (such as reflection or absorption).
natural	Unreachable from the interior in finite time. Process cannot be started there. No PDE boundary conditions can be specified.
entrance	Unreachable from the interior in finite time, but the process may be started there.
exit	Reachable from the interior in finite time, but once reached the process is stuck there. Also called an *absorbing* or *trap* boundary.

To determine which of these categories applies in specific cases, we need to introduce some standard terminology. First, we write a general time-homogeneous volatility process as

(2.1) $$dV_t = b(V_t)dt + a(V_t)dW_t,$$

where $b(V)$ is the drift coefficient and $a(V)$ is the volatility (of volatility) coefficient; we always assume that $a(V) > 0$ for $0 < V < \infty$. In addition, the coefficients are typically smooth bounded functions for $V < \infty$, so that all the expressions below exist away from either $V = 0$ or $V = \infty$.

The boundary classifications depend upon the finiteness, or not, of certain functionals. To motivate their introduction, we review the problem of determining a long-run stationary distribution for (2.1) when there is one. Almost all of the needed functionals occur in the solution to this problem. First, we define the *scale density:*

(2.2) $$s(V) = \exp\left\{-\int^V \frac{2b(x)}{a^2(x)}dx\right\} \qquad \text{for} \quad 0 < V < \infty,$$

and the *scale measure:*

(2.3) $$S(c,d) = \int_c^d s(x)dx \qquad \text{for } c < d .$$

As it turns out, the lower integration limit in (2.2) is arbitrary for the theory. These functions occur repeatedly in computations involving the PDEs that are associated with the stochastic process (2.1). When (2.1) represents the *actual* volatility process, we expect a long-run stationary probability distribution for volatility $\Psi(V)$ to exist. The alternative, that over the long run, the actual volatility runs off to zero or infinity doesn't make much sense economically and is generally rejected by the statistical evidence.

Is volatility really a stationary process? The assumption that the volatility process itself is time-independent is an idealization that is easily rejected statistically for specific models. For example, if one estimates a GARCH-type volatility model over two non-overlapping time periods on the same security, then some of the estimated coefficients will usually be statistically different from each other. This does not mitigate the fact that stationary volatility models are a highly significant improvement over the original Black-Scholes assumption of constant volatility. And, within the class of time-independent diffusions, one expects the reasonable models to exhibit stationarity or recurrence *before* risk-adjustment.

The stationary volatility distribution. A general setup considers the volatility process (2.1) defined on an interval $V \in (V_l, V_r)$, where $V_l \geq 0$ and $V_r \leq \infty$. We assume that the coefficients of the process $a(V), b(V)$ are such that all the *interior* points $\{V \mid V_l < V < V_r\}$ are regular points as described in Table 9.1. Let $p(V,Y,t)dY$ denote the probability that the volatility, beginning at V, ends up in the dY neighborhood of Y after the elapse of time t. It is well-known that $p(V,Y,t)$ is a solution to the backward equation

(2.4) $$\tfrac{1}{2} a^2(V) \frac{\partial^2 p}{\partial V^2} + b(V) \frac{\partial p}{\partial V} = \frac{\partial p}{\partial t} .$$

The solution is subject to the initial condition that $p(V,Y,t=0) = \delta(V-Y)$, where $\delta(V-Y)$ is the Dirac delta function representing all the mass concentrated at $Y = V$. If the boundary points are *regular* points, then there are many solutions to (2.4), and the process is not well-defined until the boundary behavior is defined. For example, the classical boundary conditions represent *absorption* or *reflection* at the boundary.

Absorption. For absorption at a regular point, the appropriate boundary condition is $p(V_b, Y, t) = 0$ using V_b to indicate either boundary value. This has the obvious interpretation of a trap state, i.e., if the volatility reaches $V = V_b$ then it remains at that value for all subsequent times. Consequently, there is zero probability of reaching some other value $Y \neq V_b$ and so $p(V_b, Y, t) = 0$ as indicated.

Reflection. For reflection at a regular point, the associated condition is $p_V(V_b, Y, t) = 0$, where the subscript indicates the partial derivative with respect to V. To understand why a vanishing derivative implies reflection, consider for example, the left boundary and suppose that the volatility process has reached a value V in the vicinity of $V_l > 0$ at time t. If the boundary were not present, then the probability density for jumping to Y in the next instant Δt would be given approximately by the normal distribution

$$p_0(V, Y, t + \Delta t) \approx \frac{1}{\sqrt{2\pi a^2(V_l)\Delta t}} \exp\left[\frac{-(Y - V - b(V_l)\Delta t)^2}{2a^2(V_l)\Delta t}\right].$$

But, reflection means that if, in fact, the process 'jumps' to a value Y to the left of V_l ($Y < V_l$), then we instead perform a mirror reflection of the process, identifying this as a jump to $Y' = V_l + (V_l - Y) = 2V_l - Y$. This last expression just moves the original jump the same amount to the right of the boundary V_l that the unmodified jump was to the left of V_l. That means that there are now two ways for the process to reach a legal value $Y > V_l$ from the vicinity of V_l: (i) a natural jump to the right, and (ii) a natural jump to the left, which is then mirror reflected. In other words, the modified approximate probability $p(V, Y, t + \Delta t)$ of a jump to a value $Y > V_l$, which accounts for the reflection, is $p(V, Y, t + \Delta t) \approx p_0(V, Y, t + \Delta t) + p_0(V, 2V_l - Y, t + \Delta t)$. Differentiating this last expression yields

$$p_V(V, Y, t + \Delta t) \approx \frac{2(Y - V - b(V_l)\Delta t)}{\sqrt{2\pi a^2(V_l)\Delta t}} \exp\left[\frac{-(Y - V - b(V_l)\Delta t)^2}{2a^2(V_l)\Delta t}\right]$$

$$+ \frac{2(2V_l - Y - V - b(V_l)\Delta t)}{\sqrt{2\pi a^2(V_l)\Delta t}} \exp\left[\frac{-(2V_l - Y - V - b(V_l)\Delta t)^2}{2a^2(V_l)\Delta t}\right].$$

Letting $V \to V_l$ and $\Delta t \to 0$, we have

$$p_V(V_l,Y,t+\Delta t) \to \frac{2(Y-V_l)+2(V_l-Y)}{\sqrt{2\pi a^2(V_l)\Delta t}}\exp\left[-\frac{(Y-V_l)^2}{2a^2(V_l)\Delta t}\right] = 0 = p_V(V_l,Y,t)$$

which was the assertion.

Now there are a couple of ways to think about the stationary distribution; one is to consider it as the ultimate limit of $p(V,Y,t)$ as $t \to \infty$. That is, if a stationary distribution $\Psi(Y)$ exists, then $\Psi(Y) = \lim_{t\to\infty} p(V,Y,t)$. All memory of the starting value V is lost. Closely related is the notion that $\Psi(Y)$ is an invariant distribution, since once it is achieved, no further evolution can change it:

(2.5)
$$\Psi(Y) = \int_{V_l}^{V_r} \Psi(V)\,p(V,Y,t)\,dV\,.$$

Equation (2.5) determines an ordinary differential equation followed by $\Psi(Y)$ as follows. Take the time-derivative of both sides and on the right-hand-side substitute from (2.4)

$$0 = \int_{V_l}^{V_r} \Psi(V)\left[\tfrac{1}{2}a^2(V)\frac{\partial^2 p}{\partial V^2} + b(V)\frac{\partial p}{\partial V}\right]dV\,.$$

After 2 parts integrations, we have

(2.6)
$$0 = \int_{V_l}^{V_r} p(V,Y,t)\left\{\tfrac{1}{2}\frac{d^2}{dV^2}\left[a^2(V)\Psi(V)\right] - \frac{d}{dV}\left[b(V)\Psi(V)\right]\right\}dV$$

$$\tfrac{1}{2}a^2(V)\frac{\partial p}{\partial V}\Psi(V) + p(V,Y,t)\left\{\tfrac{1}{2}\frac{d}{dV}\left[a^2(V)\Psi(V)\right] - \left[b(V)\Psi(V)\right]\right\}\Bigg|_{V_l}^{V_r}$$

If the process is absorbed at the boundaries, then ultimately there will be no mass in the interior: i.e., $\Psi(V) = 0$ for $V_l < V < V_r$. But if the process is reflected at the boundaries, then we expect a non-trivial limiting stationary distribution to exist. From (2.6), this can occur if both the integrand and the parts terms separately vanish. The parts term with $\partial p / \partial V$ already vanishes because of the reflection condition. So, in the case of reflection from two regular boundaries, the stationary distribution $\Psi(V)$ must satisfy

(2.7)
$$\tfrac{1}{2}\frac{d^2}{dV^2}\left[a^2(V)\Psi(V)\right] - \frac{d}{dV}\left[b(V)\Psi(V)\right] = 0$$

and the two boundary conditions

(2.8)
$$\tfrac{1}{2}\frac{d}{dV}\left[a^2(V)\Psi(V)\right] - \left[b(V)\Psi(V)\right]\Big|_{V=V_{r,l}} = 0$$

In the case of singular boundaries, it turns out that, if $\Psi(V)$ exists, then (2.7) still holds and (2.8) still holds in the sense of a limit. To avoid immediately worrying about (2.8), let's determine the most general solution to (2.7). Integrating (2.7) once yields

$$f' = 2\frac{b(V)}{a^2(V)} f + C_1,$$

where $f(V) = a^2(V)\Psi(V)$ and C_1 is a constant. Hence $s(V)$ of (2.2) is an integrating factor, and one more integration yields

$$f(V) = \frac{C_2}{s(V)} + \frac{C_1}{s(V)} \int_{V_0}^{V} s(x)dx ,$$

where C_2 and V_0 are also constants. We see that the general solution to (2.7), using the scale functions, is given by

(2.9) $$\Psi(V) = \frac{C_2}{a^2(V)s(V)} + \frac{C_1}{a^2(V)s(V)} S(V_0,V) .$$

For example, for the GARCH diffusion under risk-neutrality (1.1), we have $s(V) = V^{\bar\theta} \exp(\bar\omega / V)$, using $\bar\omega = 2\omega / \xi^2$ and $\bar\theta = 2\theta / \xi^2$. Hence (2.9) reads

(2.10) $$\Psi(V) = c_2 V^{-\bar\theta-2} e^{-\bar\omega/V} + c_1 V^{-\bar\theta-2} e^{-\bar\omega/V} \int_{V_0}^{V} x^{\bar\theta} e^{\bar\omega/x} \, dx ,$$

with two different constants $c_{1,2}$. If a stationary probability distribution exists, it must be both non-negative everywhere and normalizable (integrable), so that we can achieve $\int \Psi(V)dV = 1$. Let's consider the two cases: (i) two regular reflecting boundaries, and (ii) two singular boundaries. We are not trying to be systematic, since both a mixed case and other more exotic boundary behavior is possible.

(i) two regular reflecting boundaries. With regular left and right boundaries $0 < V_l < V_r < \infty$, the boundary condition for solutions to (2.7) associated with reflecting boundaries is given by (2.8). (Empirically, one might estimate that reflecting boundaries lie at the minimum and maximum values of the observed volatility process over a long time period). Since (2.8) is just the first integral of (2.7), the condition says that $C_1 = 0$ in (2.9). That is,

(2.11) $$\Psi(V) = \frac{C_2}{a^2(V)s(V)} = c_2 V^{-\bar\theta-2} e^{-\bar\omega/V} , \quad \text{GARCH diffusion (1.1)}$$

and the remaining constant is determined by the normalization $\int_{V_L}^{V_H} \Psi(V)\,dV = 1$.

(ii) two singular boundaries. In this case, the volatility ranges over the entire half line. A characteristic of singular boundary problems is that we have *no* boundary conditions beyond the normalization condition. However, this still suffices to determine the solution.

To see how, consider that the first term of (2.10) is integrable over $[0,\infty]$ if $\omega > 0$ and $\tilde{\theta} > -1$. Since both of these parameter conditions are satisfied empirically, assume that these conditions hold. Then, the integral in the second term, because $\tilde{\theta} > -1$, will diverge as $V \to \infty$, behaving like $V^{\tilde{\theta}+1}$. Hence, the entire second term in (2.10), as $V \to \infty$, will behave as $1/V$, which is not integrable.

This is helpful, because if both terms were integrable, then how would we fix the constants? Because the second term must be dropped ($c_1 = 0$), we are left with the *same* functional solution as under reflecting boundary conditions, namely (2.11). Of course, the constant c_2 takes on a different numerical value in each instance. In summary, the singular case behaves "as if" there was a reflection at the boundaries.

We now continue with the classifications for the singular case. What were the new functionals that appeared in this computation? First, we see from (2.9) that the stationary density $\Psi(V)$ is proportional to the so-called *speed density*

(2.12) $$m(V) = \frac{1}{a^2(V)s(V)} \quad \text{for } 0 < V < \infty.$$

The normalization integral for $\Psi(V)$ is the boundary limit of the *speed measure*

(2.13) $$M(c,d) = \int_c^d m(x)dx \qquad \text{for } c < d.$$

Next, recall the second term in (2.9), which we needed to *not* be integrable at either boundary. The integral of this term is now formally promoted to a functional $\Sigma(b)$, defined in reference to a particular boundary $b = 0$ or $b = \infty$. Specifically, we have

(2.14) $$\Sigma(0) = \lim_{c \downarrow 0} \int_c^d S(c,x)m(x)dx, \text{ and } \quad \Sigma(\infty) = \lim_{d \uparrow \infty} \int_c^d S(x,d)m(x)dx.$$

Finally, we need one further functional $N(b)$, which differs from (2.14) in two respects: (i) integrate with respect to the other argument of $S(\cdot,\cdot)$, and (ii) take the fixed argument of $S(\cdot,\cdot)$ to be the opposite boundary. That is,

$$(2.15)\quad N(0)=\lim_{c\downarrow 0}\int_c^d S(x,d)m(x)dx\,,\qquad N(\infty)=\lim_{d\uparrow\infty}\int_c^d S(c,x)m(x)dx\,.$$

The boundary classifications, at $b=0$ for example, are determined by whether or not the four functionals $S(0,d)$, $M(0,d)$, $\Sigma(0)$, and $N(0)$ are finite or infinite. Note that since it is just the finiteness of the quantities that is in question, the dependence upon the fixed boundary for $\Sigma(0)$ and $N(0)$ can be ignored. Actually not all four functionals need be computed, since the finiteness or not of some imply the behavior of others. The specific criteria are shown in Table 9.2, adapted from Karlin and Taylor[2] (1981) and using the left boundary at $b=0$ as the example.

Table 9.2 Boundary Classification Criteria

	Criteria			
Boundary type	$S(0,d)$	$M(0,d)$	$\Sigma(0)$	$N(0)$
Regular	$<\infty$ *	$<\infty$ *	$<\infty$	$<\infty$
Exit	$<\infty$	$=\infty$ *	$<\infty$ *	$=\infty$
Entrance	$=\infty$ *	$<\infty$	$=\infty$	$<\infty$ *
	$<\infty$ *	$=\infty$ *	$=\infty$ *	$=\infty$
Natural	$=\infty$ *	$<\infty$ *	$=\infty$	$=\infty$ *
	$=\infty$ *	$=\infty$ *	$=\infty$	$=\infty$

Notes: The minimal sufficient criteria for each row are indicated by an asterisk, so that the other row relations are automatically satisfied.

We saw from the example computation of the stationary density (2.11) that we needed $\Sigma(b)=\infty$ at $b=0$ or ∞ to have a stationary density. From Table 9.2, $V=0$ and $V=\infty$ must be either entrance or natural boundaries. This is exactly what one would expect based upon the rough meanings of Table 9.1, since these boundaries are unreachable in finite time. And, it's what we expect for any

[2] Adapted from Table 7.2, p. 234 of *A Second Course in Stochastic Processes*, by Samuel Karlin and Howard Taylor. Copyright © 1981 by Academic Press. Reprinted with permission of the publisher.

mean-reverting *actual* volatility process defined on the entire half-line: a stationary process will exist if $V = 0$ and $V = \infty$ are either natural boundaries or entrance boundaries. As we have already seen, another stationary alternative is to introduce reflection at regular boundaries. More exotic behaviors are also possible theoretically (see Karlin and Taylor).

Next, we turn to the case of the *risk-adjusted* volatility process and see that this tidy state of affairs can change completely.

3 Volatility Explosions I

In this section, we turn to the case of the GARCH diffusion under log-utility; that is (1.2). From (2.2), we have $s(x) = x^{\tilde{\theta}} \exp(2\tilde{\rho}x^{1/2} + \tilde{\omega}/x)$, using $\tilde{\rho} = 2\rho/\xi$ and $(\tilde{\omega},\tilde{\theta})$ defined above. Hence the scale measure (2.3) becomes

$$S(c,d) = \int_c^d x^{\tilde{\theta}} \exp\left(2\tilde{\rho}\sqrt{x} + \frac{\tilde{\omega}}{x}\right)dx \, .$$

In particular, we see that

$$S(0,d) = \begin{cases} \infty & \text{for (i) } \omega>0 \text{ or (ii) } \omega=0 \text{ and } \tilde{\theta} \le -1 \\ <\infty & \text{for } \omega=0 \text{ and } \tilde{\theta} > -1 \end{cases} \qquad (3.1)$$

$$S(c,\infty) = \begin{cases} \infty & \text{for (i) } \rho>0 \text{ or (ii) } \rho=0 \text{ and } \tilde{\theta} \ge -1 \\ <\infty & \text{for } \rho<0 \end{cases} \qquad (3.2)$$

(Recall that the case $\omega < 0$ is ignored.) For real-world volatility processes on market equity indices, the empirical evidence is that ρ is significantly and strongly negative. For example, see the typical estimates in Appendix 1 of Chapter 1. Most researchers find ρ significantly negative on equity index data when estimating GARCH-type models that allow for it. And, of course, this sign for the correlation is the one consistent with the typical post-1987 volatility smile pattern for equity indices. A closely related notion is the observation that the equity market is often more volatile on the downside than the upside. This is very noticeable in long-term charts of broad-based equity indices, especially around major bear markets. And the statistical tests are a confirmation of the idea on a shorter time scale.

From (3.2), the fact that $S(V,\infty) < \infty$ for $\rho < 0$ tells us that we have very different behavior than the stationary case. (Recall from above that we found

$S(V,\infty)=\infty$ in creating a stationary volatility distribution). Of course, the fact that this particular functional is finite is admittedly an obscure notion. Fortunately, it is easy to see directly from the evolution equation $dV_t = (\omega - \theta V_t - \rho\xi V_t^{3/2})dt + \xi V_t dW_t$ why this case is very different.

Suppose we have a random jump to a large value of V_t; then, from this point the process will behave approximately like the deterministic process $dV_t \approx -\rho\xi V_t^{3/2}dt$. This equation is easily integrated, with the result :

(3.3) $$V_t = \frac{V_0}{\left[1+\frac{1}{2}V_0^{1/2}\rho\xi t\right]^2} .$$

Hence, if $\rho < 0$, the denominator can vanish and so $V_t \to \infty$ at the *finite* time $t^* = 2V_0^{-1/2}/(|\rho|\xi)$. Of course, we have neglected the random noise term in (1.2); but the rough calculation suggests that if we restored the noise, then $V_t \to \infty$ in finite *expected* time.

The standard terminology for this behavior is, somewhat over-dramatically, an *explosion.* With less flourish, what we have is the non-zero probability of an escape to the boundary in finite time. When the boundary is "close", the phenomenon is common. For example, if the volatility process was just simple Brownian motion, $dV_t = dB_t$ on our same half-line $V_t \geq 0$, we would certainly expect that the process can (probability > 0) reach the boundary $V=0$ in finite time. An explosion is conceptually the same as this case, but to a right boundary at infinity instead. Moreover, $V=\infty$ can itself be rendered "close" by a change of variable; this is shown below in an example.

Let's continue with our characterization of the boundaries for the GARCH diffusion under log-utility. For the speed measure, we have

(3.4) $$M(c,d) = \int_c^d x^{-\tilde{\theta}-2}\exp\left(-2\tilde{\rho}\sqrt{x} - \frac{\tilde{\omega}}{x}\right)dx .$$

Hence

(3.5) $$M(0,d) = \begin{cases} \infty & \text{for } \omega = 0 \text{ and } \tilde{\theta} \geq \text{-}1 \\ <\infty & \text{for (i) } \omega > 0\text{, or (ii) } \omega = 0 \text{ and } \tilde{\theta} < \text{-}1 \end{cases}$$

(3.6) $$M(c,\infty) = \begin{cases} \infty & \text{for (i) } \rho < 0 \text{ or (ii) } \rho = 0 \text{ and } \tilde{\theta} \leq \text{-}1 \\ <\infty & \text{for (i) } \rho > 0 \text{ or (ii) } \rho = 0 \text{ and } \tilde{\theta} > \text{-}1 \end{cases}$$

Next we have

$$(3.7)\qquad \Sigma(\infty) = \lim_{d\uparrow\infty} \int_c^d S(x,d) x^{-\tilde{\theta}-2} \exp\left(-2\tilde{\rho}\sqrt{x} - \frac{\tilde{\omega}}{x}\right) dx$$

As both $(x,d) \to \infty$, and assuming $\rho < 0$, then

$$(3.8)\quad S(x,d) \approx \int_x^d y^{\tilde{\theta}} \exp\left(2\tilde{\rho}\sqrt{y}\right) dy \approx x^{\tilde{\theta}+1/2} \exp(2\tilde{\rho}\sqrt{x})[1 + O(x^{-1/2})]$$

Inserting (3.8) into (3.7) yields an integrand that behaves as $x^{-3/2}$ for large x, which means the integral converges as $d \to \infty$. So if, $\rho < 0$ we have just shown that $\Sigma(\infty) < \infty$. Also from (3.6), we have $M(c,\infty) = \infty$ with negative ρ. Table 9.2 shows that $\Sigma(\infty) < \infty$ and $M(c,\infty) = \infty$ suffice to classify $V = \infty$ as an *exit* boundary under a negative correlation coefficient.

In the alternative, assume that $\rho > 0$. In this case (3.8) reads

$$(3.9)\qquad S(x,d) \approx d^{\tilde{\theta}+1/2} \exp(2\tilde{\rho}\sqrt{d}) - x^{\tilde{\theta}+1/2} \exp(2\tilde{\rho}\sqrt{x})$$

and so

$$(3.10)\qquad \Sigma(\infty) \approx \lim_{d\uparrow\infty} d^{\tilde{\theta}+1/2} \exp\left(2\tilde{\rho}\sqrt{d}\right) \int_c^d x^{-\tilde{\theta}-2} \exp\left(-2\tilde{\rho}\sqrt{x} - \frac{\tilde{\omega}}{x}\right) dx$$

plus a finite term, which we can ignore. The integral in (3.10) converges, but the term in front of it diverges, so we have $\Sigma(\infty) = \infty$. Next, consider

$$(3.11)\qquad N(\infty) = \lim_{d\uparrow\infty} \int_c^d S(c,x) x^{-\tilde{\theta}-2} \exp\left(-2\tilde{\rho}\sqrt{x} - \frac{\tilde{\omega}}{x}\right) dx$$

Now as $x \to \infty$,

$$S(c,x) \approx \int_c^x y^{\tilde{\theta}} \exp\left(2\tilde{\rho}\sqrt{y}\right) dy \approx x^{\tilde{\theta}+1/2} \exp\left(2\tilde{\rho}\sqrt{x}\right);$$

hence
$$N(\infty) \approx \lim_{d\uparrow\infty} \int_c^d x^{-3/2} dx < \infty.$$

Collecting our results, we have established that (ii) for $\rho > 0$, $S(c,\infty) = \infty$, $M(c,\infty) < \infty$, $\Sigma(\infty) = \infty$, and $N(\infty) < \infty$. Reading from Table 9.2, we see that $V = \infty$ is an *entrance* boundary when the correlation is positive under log-utility. There is no explosion in this case, since the boundary is unreachable in finite time. Continuing in this way, we characterize the boundaries for the two GARCH diffusions under most cases of interest in Table 9.3

Table 9.3 Boundary Classifications for the GARCH Diffusion Process

Risk-adjusted Volatility Process	$V = 0$		$V = \infty$	
	Condition	**Type**	**Condition**	**Type**
Risk neutral: $dV = (\omega - \theta V)dt + \xi V dW$	$\omega > 0$ any θ	Entrance	any ω, θ	Natural
	$\omega = 0$, any θ	Natural		
Log-utility: $dV = (\omega - \theta V - \rho \xi V^{3/2})dt + \xi V dW$	$\omega > 0$ any ρ, θ	Entrance	$\rho > 0$ any ω, θ	Entrance
	$\omega = 0$ any ρ, θ	Natural	$\rho < 0$ any ω, θ	Exit (by explosion)

The Feller explosion test. As we have just discussed, an explosion of the volatility process to $V = \infty$ means that $V = \infty$ can be reached in finite time. Since the boundary is reachable in finite time, it must be either a regular boundary or an exit boundary. From Table 9.2, but interpreted for $V = \infty$ instead of $V = 0$, you can see that a sufficient condition for the boundary at infinity to be either a regular or an exit boundary is that $S(V,\infty) < \infty$ and $\Sigma(\infty) < \infty$. In fact, the conditions are both necessary and sufficient. This assertion is proven as a theorem in, for example Durrett (1996, Ch. 6, p. 214): *Feller's test: A (volatility) process explodes to* $+\infty$ *if, and only if, both* $S(V,\infty) < \infty$ *and* $\Sigma(\infty) < \infty$.

Of course, the theorem can be applied at the lower boundary, too, to check if $V = 0$ can be reached in finite time; this occurs if and only if both $S(0,V) < \infty$ and $\Sigma(0) < \infty$.

Calculating explosion probabilities. We assume the volatility process is defined on the entire half-line $V_t \in (0,\infty)$. First, we need a definition.

Definition. The probability of a volatility explosion from V by time τ is given by

$$P_e(V,\tau) = \lim_{x \to \infty} \Pr\{Y(t_x) = x, t_x \le \tau \mid Y(0) = V\}, \quad (3.12)$$

where t_x is the moment when the volatility process $dY_t = b(Y_t)dt + a(Y_t)dW_t$ first reaches the arbitrary point x, starting from $Y(0) = V$. In (3.12) both Pr and

P_e are probabilities. The probability of *ultimate* explosion (see below) is the $\tau \to \infty$ limit of (3.12)

The PDE problem for explosions. It can be shown that the explosion probability satisfies the backward Kolmogorov equation for the process. In the general case, this is

$$\tfrac{1}{2} a^2(V) \frac{\partial^2 P_e}{\partial V^2} + b(V) \frac{\partial P_e}{\partial V} = \frac{\partial P_e}{\partial \tau}, \tag{3.13}$$

subject to the initial condition $P_e(V, \tau = 0) = 0$ and the boundary condition $P_e(V = \infty, \tau) = 1$, $(\tau > 0)$. When the explosion probability is zero, there is no solution to (3.13) that satisfies these conditions. One example follows immediately; a second example may be found in Sec. 7.

Example 1. We continue with the GARCH diffusion process after risk-adjustment under the CPRA equilibrium. In particular, consider the risk-adjusted process under log-utility $(\gamma = 0)$. We can calculate the explosion probability in the case where $\omega = \theta = 0$. For this example, (3.13) becomes,

$$\tfrac{1}{2} \xi^2 V^2 \frac{\partial^2 P_e}{\partial V^2} - \rho \xi V^{3/2} \frac{\partial P_e}{\partial V} = \frac{\partial P_e}{\partial \tau}. \tag{3.14}$$

Note that we could have also started with the model under risk neutrality $(\gamma = 1)$; in that case, (3.14) represents the *auxiliary* volatility process if we switch the sign on ρ.

Since the process takes a strictly positive time to explode from finite V_t, we have the initial condition (i) $P_e(V, \tau = 0) = 0$, $(V < \infty)$ and the boundary condition (ii) $P_e(V = \infty, \tau) = 1$. In addition, for our example, the volatility process is

$$dV_t = -\rho \xi V_t^{3/2} dt + \xi V_t \, dB_t. \tag{3.15}$$

This particular process clearly becomes stuck at $V_t = 0$ should it start there. (By Table 9.3, $V_t = 0$ is a natural boundary, which won't be reached in finite time if it doesn't start there.) Hence, in this particular case, but not in general, we also expect the solution to satisfy the condition (iii) $P_e(V = 0, \tau) = 0$.

The explosion test. The first step in explosion problems should always be to apply the Feller explosion test. In this case, we already know the answer from the boundary classifications in Table 9.3 ('explosion' if and only if the boundary is exit or regular). But let's just double-check to illustrate the use of the specific test. For the process in (3.15), the scale density and the scale measure are given by

$$s(V) = \exp\left(\frac{4\rho}{\xi}\sqrt{V}\right) \quad \text{and} \quad S(c,d) = \int_c^d \exp\left(\frac{4\rho}{\xi}\sqrt{y}\right)dy .$$

Clearly, $S(c,\infty) < \infty$ if, and only if, $\rho < 0$. Then, the speed density and Σ are given by

$$m(V) = V^{-2}\exp\left(-\frac{4\rho}{\xi}\sqrt{V}\right) \quad \text{and} \quad \Sigma(c,d) = \int_c^d S(x,d)\exp\left(-\frac{4\rho}{\xi}\sqrt{x}\right)x^{-2}dx .$$

By taking x and d both large, it's easy to develop the leading behavior of $S(x,d)$ and $\Sigma(c,d)$ with parts integrations. The result is that, for any ρ, $\Sigma(c,d)$ vanishes at least as fast as $O(1/\sqrt{d})$ as $d \to \infty$. That is, both $S(V,\infty) < \infty$ and $\Sigma(\infty) < \infty$ if and only if $\rho < 0$. In other words, we have re-confirmed that there is a strictly positive probability of a volatility explosion to $+\infty$ in (3.15) if, and only if, $\rho < 0$.

PDE solution. To solve the problem, consider the change of variable from V_t to X_t, where $X_t = f(V_t) = V_t^{-1/2}$. By the Ito change-of-variable formula, the process X_t satisfies the SDE

$$dX_t = \left(\tfrac{3}{8}\xi^2 X_t + \tfrac{1}{2}\rho\xi\right)dt + \tfrac{1}{2}\xi X_t dB_t , \tag{3.16}$$

where dB_t is a Brownian motion. Equation (3.16) can be interpreted as the risk-adjusted process followed by a stock (with *constant* volatility) with price $S_t = X_t$ which pays dividends at a constant dollar rate. That is, in more traditional notation, the stock price follows the process

$$dS_t = (rS_t - D)dt + \sigma S_t dB_t , \tag{3.17}$$

where the interest rate r, the dividend rate D, and the stock price volatility σ are all constants. By comparing (3.16) and (3.17), the mapping between the two problems is that (i) $r = (3/8)\xi^2$, (ii) $D = |\rho|\xi/2$, $(\rho \le 0)$, and (iii) $\sigma = \xi/2$.

Of course, we also have $S_t = V_t^{-1/2}$. The event "V_t reaches infinity" becomes the event "S_t reaches zero". When V_t does reach infinity, and $\rho < 0$, we already know that the process is stuck there because that boundary is an exit boundary (see Table 9.3). The equivalent stock price process is absorbed at $S = 0$. Hence, the explosion probability $P_e(V,\tau) = A(S,\tau)$, where $A(S,\tau)$ is the absorption probability of process (3.17). That is,

$$A(S,\tau) = \Pr\{ X(t_0) = 0, t_0 \le \tau \mid X(0) = S \} \tag{3.18}$$

where t_0 is the moment when the process $dX_t = (rX_t - D)dt + \sigma X_t dB_t$ first reaches the origin, starting from $X(0) = S$.

I have previously solved for the absorption probabilities $A(S,\tau)$ of (3.17)[3]. First, note that the ultimate explosion/absorption probability $P_\infty(V) = P_e(V, \tau = \infty)$ is very easily calculated. That function is the solutions to the ordinary differential equation

$$\tfrac{1}{2}\xi^2 V^2 \frac{d^2 P_\infty}{dV^2} = \rho \xi V^{3/2} \frac{dP_\infty}{dV}, \tag{3.19}$$

which satisfies $P_\infty(V = 0) = 0$ and $P_\infty(V = \infty) = 1$. Equation (3.19) is easily integrated and, indeed, a solution satisfying the conditions is

$$P_\infty(V) = 1 - \exp\left(-\alpha\sqrt{V}\right)\left(1 + \alpha\sqrt{V}\right), \quad \text{using } \alpha = 4\frac{|\rho|}{\xi} \ge 0 . \tag{3.20}$$

When $\tau < \infty$, and then applying the Proposition cited in footnote 3, one obtains

$$P_e(V,\tau) = P_\infty(V) - \int_0^\infty \overline{g}(\mu)\,\eta\left(\alpha\sqrt{V},\mu\right)\exp\left[-\tfrac{1}{8}(\mu^2+1)\,\xi^2\tau\right]d\mu . \tag{3.21}$$

$$\text{using } \eta(x,\mu) = e^{-x}x^{1+i\mu}\, U(i\mu, 1+2i\mu, x) \text{ and } \overline{g}(\mu) = \frac{2}{\pi}\frac{\cosh(\pi\mu)}{\mu^2+1}, \tag{3.22}$$

where $U(a,b,z)$ is a confluent hypergeometric function[4].

A Monte Carlo Check. We can check these calculations with Monte Carlo calculations; as it turns out, the explosion probabilities are easily estimated to good accuracy with a relatively small number of random drawings. The process we have simulated is the log-volatility $y_t = \ln V_t$. Using Ito's formula, the GARCH diffusion process under log-utility may be simulated as

---

[3] See Proposition 2.3 in Lewis (1998).

[4] See Abramowitz and Stegun (1970). $U(i\mu, 1+2i\mu, x)$ can also be written in terms of Bessel functions, but with no real gain in notational simplicity; however, this change may be useful for computations.

$$(3.23) \qquad y_{t+\Delta t} = y_t + \left[\frac{\omega}{V(y_t)} - \theta^* - \rho\,\xi\,V^{1/2}(y_t)\right]\Delta t + \xi\,Z_t\sqrt{\Delta t}\ .$$

In (3.23) $V(y_t) = \exp y_t$, $\theta^* = \theta + (1/2)\xi^2$, and Z_t is a unit normal variate, drawn independently at each time step $t = 0, \Delta t, \cdots, \tau$. Each simulation begins with $y_0 = \ln(V_0)$, where V_0 is the initial volatility, i.e., the value being tested in (3.21).

In addition, the following modifications to (3.23) are employed to handle the boundary behavior. First, should $V(y_t) = 0$ to machine precision, then the next Monte Carlo step is $V_{t+\Delta t} = \omega\,\Delta t$ and then $y_{t+\Delta t}$ is determined from $y_{t+\Delta t} = \ln(V_{t+\Delta t})$. Next, $y_{t+\Delta t}$ in (3.23) was restricted to a maximum value, $y_{\max} = 100$, so that $V_{t+\Delta t} \le V_{\max} = e^{100} \approx 2.69\times 10^{43}$. During one simulation run, for which $t = 0, \Delta t, \cdots, \tau$, an explosion was counted if the final value V_τ exceeded another large number, namely $V_0 \exp(10\,\xi\,\tau^{1/2})$ (or $V_{\max}$ if this was smaller), where V_0 was the starting volatility. Essentially, explosions were defined to be a "10-sigma" upside event; under a log-normal distribution, for example, they would occur with negligible frequency.

With 250 days per year, the time step was taken to be Δt = 1 day = 1/250 year. With all parameter values on an annual basis, Table 9.4 reports the Monte Carlo (MC) results, MC standard error, and the exact results (3.21). The parameter values are $\omega = \theta = 0$, $V_0 = \xi = 1$, and various values of the correlation ρ and the time period τ. If one thinks of τ as a time to option expiration, the results illustrate the probability that the risk-adjusted security price process undergoes a volatility explosion under logarithmic utility starting from $V_0 = 1$ (100% per year) prior to expiration. The results were found to be quite insensitive to the particular large numbers chosen. Although only 10,000 replications were used (with an antithetic technique), one can see from the table that MC results are always within at least two standard errors of the analytical calculations.

Table 9.4 Volatility Explosion Probabilities P_e for the GARCH Diffusion Process under Log-utility. Exact and Monte Carlo (MC) Estimates.

	Volatility explosion probability for various correlations ρ:								
	$\rho = -0.50$			$\rho = -0.75$			$\rho = -1$		
Time (yrs.)	MC $\overline{P}_e$	MC std. error	Exact P_e	MC $\overline{P}_e$	MC std. error	Exact P_e	MC $\overline{P}_e$	MC std. error	Exact P_e
1	$<10^{-4}$	0	4.10^{-6}	.0002	.0001	.0002	.0035	.0006	.0039
2	0.014	0.001	0.015	0.103	0.003	0.107	0.282	0.005	0.292
3	0.106	0.003	0.107	0.329	0.005	0.336	0.574	0.005	0.580
4	0.219	0.004	0.221	0.494	0.005	0.496	0.718	0.005	0.717
5	0.310	0.005	0.313	0.594	0.005	0.593	0.784	0.004	0.786
10	0.512	0.005	0.512	0.751	0.004	0.750	0.881	0.003	0.882
20	0.582	0.005	0.581	0.794	0.004	0.794	0.902	0.003	0.905
40	0.595	0.005	0.593	0.799	0.004	0.801	0.904	0.003	0.908
∞			0.594			0.801			0.908

Notes. The volatility process is $dV_t = (\omega - \theta V_t - \rho\xi V_t^{3/2})dt + \xi V_t dW_t$ and the model parameters are $\omega = \theta = 0$, $V_0 = \xi = 1$

Table 9.5 shows the same calculations for the case $\omega = \theta = 1$; in this case there is no analytical solution. In the prior Table 9.4 $\omega = 0$ and the origin was a natural boundary; although inaccessible in finite time, there could be a concentration of probability near $V = 0$. So, as $\tau \to \infty$, the volatility could end up either concentrated near the origin or exploding to infinity. The net effect was the probability of ultimate explosion $P_\infty(V) < 1$ for $\omega = 0$. When $\omega > 0$, things are different. While $V = 0$ is still inaccessible, the origin becomes an entrance boundary (see Table 9.3). The turning on of the "repulsive force" associated with $\omega > 0$ is apparently able to prevent any ultimate concentration of probability near $V = 0$. Hence the process has nowhere else to escape to but $+\infty$, which sends the probability of ultimate explosion $P_\infty(V) \to 1$. This phenomenon is clearly seen in Table 9.5 for the cases $\rho = -.75$ and $\rho = -1$, and would also be seen at less negative values of ρ if the time was lengthened.

Table 9.5 Volatility Explosion Probabilities P_e for the GARCH Diffusion Process under Log-utility. Monte Carlo (MC) Estimates.

	Volatility explosion probability for various correlations ρ:							
	$\rho = -0.25$		$\rho = -0.50$		$\rho = -0.75$		$\rho = -1$	
Time (yrs.)	**MC** $\bar{P}_e$	**MC std. error**	**MC** $\bar{P}_e$	**MC std. error**	**MC** $\bar{P}_e$	**MC std. error**	**MC** $\bar{P}_e$	**MC std. error**
1	$<10^{-4}$	0	$<10^{-4}$	0	$<10^{-4}$	0	.0006	.0002
2	$<10^{-4}$	0	0.001	.0003	0.021	0.001	0.109	0.003
3	.0001	.0001	0.015	0.001	0.121	0.003	0.356	0.005
4	.0006	.0002	0.044	0.002	0.245	0.004	0.556	0.005
5	.0019	.0004	0.081	0.003	0.365	0.005	0.700	0.005
10	0.014	0.001	0.275	0.004	0.741	0.004	0.958	0.002
20	0.039	0.002	0.556	0.005	0.957	0.002	0.999	.0003
40	0.091	0.003	0.836	0.004	0.999	.0004	1.000	0

Notes. The volatility process is $dV_t = (\omega - \theta V_t - \rho \xi V_t^{3/2})dt + \xi V_t dW_t$ and the model parameters are $\omega_a = \theta_a = 1$, $V_0 = \xi_a = 1$.

4 Volatility Explosions II Failure of the Martingale Pricing Formula

Suppose that the risk-adjusted volatility process is the GARCH diffusion under risk-neutrality, given at (1.1). We assume the correlation between the stock and volatility process is ρ, a constant. It's easier to focus on the main issue if we also assume that $\delta = r = 0$ and this involves no loss of generality. Then, the risk-adjusted pricing process $\tilde{P}$ is given by

$$\tilde{P}: \begin{cases} dS_t = \sigma_t S_t d\tilde{B}_t \\ dV_t = (\omega - \theta V_t)dt + \xi V_t d\tilde{W}_t \end{cases}, \qquad d\tilde{B}_t d\tilde{W}_t = \rho dt \tag{4.1}$$

The martingale pricing formula for the stock, when it's valid, is of course $S_t = \mathbb{E}_t[S_T]$. Remarkably, Sin (1998) proves that, in fact, this formula fails in this model if $\rho > 0$.

THEOREM 9.1 (Sin 1998)[5] *Let* $\tau = T - t > 0$ *and suppose that the risk-adjusted stock and volatility process* $\tilde{P}$ *is given by (4.1) with* $\omega > 0$. *Under that process and if* $\rho \le 0$, *then* $S_t = \mathbb{E}_t[S_T] = \mathbb{E}_t[S_T \mid V_t]$, *where* $\mathbb{E}_t$ *is a time-t expectation under* $\tilde{P}$. *But if* $\rho > 0$, *then*

$$S_t = \mathbb{E}_t[S_T] + S_t \hat{P}_{\exp}(V_t, \tau) > \mathbb{E}_t[S_T], \tag{4.2}$$

where $\hat{P}_{\exp}(V_t, \tau) = \hat{P}_{\exp}(V_t, T-t) > 0$ *is the probability of an explosion to* $+\infty$ *before time T, starting at* $V_t \ge 0$, *in the auxiliary volatility process:*

$$dV_t = (\omega - \theta V_t + \rho \xi V_t^{3/2})dt + \xi V_t d\hat{W}_t. \tag{4.3}$$

The ideas behind the proof. Sin's proof, while short, is fairly technical for the level of this book. Because of its importance to our subject, instead of reproducing it, we will try to motivate the arguments using more informal (and less rigorous) reasoning. Essentially, the proof is based upon the same change of measure used in Chapter 8 which transformed $\tilde{P}$ to the *auxiliary* process Q, but *without* the corresponding change in viewpoint to using shares as numeraire. This is discussed further below. When that particular Girsanov transformation is valid, we showed in Chapter 8, Sec. 3 that for any reasonable financial claim (or random variable) X_T, then the change of measure is defined by

$$\mathbb{E}^{\tilde{P}}[S_T X_T] = S_0 \, \mathbb{E}^{Q}[X_T]. \tag{4.4}$$

Moreover, if one starts with the general model (with zero dividends and interest):

$$\tilde{P}: \begin{cases} dS_t = \sigma_t S_t d\tilde{B}_t \\ dV_t = b(V_t)dt + a(V_t)d\tilde{W}_t \end{cases}, \quad d\tilde{B}_t d\tilde{W}_t = \rho dt \tag{4.5}$$

then the change of measure is effected by $d\tilde{B}_t = d\hat{B}_t + \sigma_t dt$, where $d\hat{B}_t$ is a Brownian motions under Q. Using this, we obtain the evolutions under Q after the change of measure:

[5] Theorem 9.1 combines results from Sin's Theorem 3.3, Lemma 4.2 and Lemma 4.3.

$$(4.6) \qquad Q: \begin{cases} dS_t = V_t S_t\,dt + \sigma_t S_t\,d\hat{B}_t \\ dV_t = [\,b(V_t) + \rho\sigma_t a(V_t)\,]dt + a(V_t)\,d\hat{W}_t \end{cases}, \quad d\hat{B}_t d\hat{W}_t = \rho\,dt\,.$$

The theorem, of course, makes reference to a particular volatility process. But it will become clear from the discussion that the only property of the volatility process that is relevant is that, under $\tilde{P}$, the volatility process does not explode, while it may explode under Q.

Relationship to the change-of-numeraire. Note that this transformation, in relation to the previous chapter, is simply a change of measure *without* a change of numeraire. This is analogous to the usual change of measure in finance in moving from the actual price process to the risk-adjusted process. The stock price drift coefficient changes, but the volatility coefficient of the stock price remains the same. The value of one share (in dollars) is still risky under Q, although we know that the value of one share (in shares) is constant (since the dividend yield is zero). Also, under the change to Q, the stock price-volatility process retains the same variance-covariance structure, with the same correlation. In the last chapter, we made this same measure change simultaneously with the variable change to $X_t = 1/S_t$, which has the opposite sign for the covariance.

Proof ideas (continued). Choosing $X_T = 1$ in (4.4) then implies that, *if* the Girsanov transformation were valid, then $\mathbb{E}^{\tilde{P}}[\,S_T\,] = S_0\,\mathbb{E}^{Q}[\,1\,] = S_0$. What goes wrong to invalidate this? One implication of a valid Girsanov transformation is that the two measures must be *equivalent* , which means that the sets of events which have zero probability (called the null sets) are the same under $\tilde{P}$ and Q . But, the event E: "V_t explodes prior to T", i.e. "V_t reaches $+\infty$ prior to T" is a zero probability event under $\tilde{P}$ but has a strictly positive probability under Q for some models. For example, we know the explosion probability is strictly positive under Q for the example model under discussion: the GARCH diffusion with $\rho > 0$ and $T > t$. So, one thing that is going wrong is that the two probability measures are not equivalent.

Closely related is the problem that the Girsanov transformation is based on the ratio of two probability densities

$$(4.7) \qquad \frac{dQ_T}{d\tilde{P}_T} = \frac{S_T}{S_0} = \exp\left\{ \int_0^T \sigma_t\,d\tilde{B}_t - \tfrac{1}{2}\int_0^T \sigma_t^2\,dt \right\},$$

which is quite ill-defined when $\sigma_t \to \infty$. If the volatility were bounded, then everything would be ok. So, to create a valid Girsanov transformation, Sin defines a new process where the stock price stops when the *volatility* exceeds an arbitrary bound, call it n. That is, $S_t^{(n)}$ follows (4.4) as long as $V_t < n$. But the first time that $V_t \geq n$, call this (stopping) time τ_n, then $S_t^{(n)} = S_{\tau_n}$ and the stock price stays at this level for all subsequent times $t > \tau_n$.

Now for finite n, everything is bounded and we should have a valid martingale $S_T^{(n)} / S_0$ and a valid Girsanov transformation to Q_n defined by

$$\mathbb{E}^{\tilde{P}}\left[S_T^{(n)} X_T \right] = S_0 \, \mathbb{E}^{Q_n}[X_T] \tag{4.8}$$

The new stock price behavior is equivalent to (4.6) under a new volatility process $\sigma_t^{(n)}$ that drops to zero once it reaches n. This means that the measure change to Q_n is now accomplished by $d\tilde{B}_t = d\hat{B}_t + \sigma_t^{(n)} dt$, where $d\hat{B}_t$ is a Brownian motion under Q_n.

Now take a particular X_T, namely $X_T = 1_{\{\tau_n > T\}}$, which means that $X_T = 1$ if the volatility first reaches level n after the (expiration) time T. Otherwise, V_t reaches n before T, and $X_T = 0$. With this random variable, (4.8) becomes

$$\mathbb{E}^{\tilde{P}}\left[S_T^{(n)} 1_{\{\tau_n > T\}} \right] = S_0 \, \mathbb{E}^{Q_n}\left[1_{\{\tau_n > T\}} \right]. \tag{4.9}$$

Finally, Sin proves that (4.9) is valid in the limit that $n \to \infty$, where the event $1_{\{\tau_n > T\}}$ becomes the event $\bar{E}$: "V_t does not explode prior to T". In this limit, the left-hand-side of (4.9) becomes $\mathbb{E}^{\tilde{P}}[S_T]$ since V_t does not explode under $\tilde{P}$. But the right-hand-side becomes S_0 times the probability that V_t does not explode prior to T under Q. In other words, the right-hand-side becomes 1 minus the probability that V_t does explode prior to T under the auxiliary process: $1 - \hat{P}_e(V_t, \tau)$ in our previous notation.

Remarks. If, as Sin does, you restrict ω to be strictly positive, ($\omega > 0$), then an explosion to $+\infty$ will occur starting at any $V_t \geq 0$. But, you can allow $\omega = 0$ the theorem will be remain valid for any $V_t > 0$. That's because, as we showed in the previous section, $\hat{P}_{\exp}(V_t, \tau) > 0$ for all $V_t > 0$ and assuming that $\tau > 0$.

Not only can the martingale pricing formula fail, but it can fail quite badly. For example, we saw that if $\omega > 0$ and you wait long enough, then $\hat{P}_e(V_t, \tau) \to 1$. That means that not only do we have $S_t > \mathbb{E}_t[\,S_T\,]$, but in fact $\mathbb{E}_t[\,S_T\,] \to 0$ as $T \to \infty$. On the other hand, when $\omega = 0$, then the probability of ultimate explosion is less than 1, so that $\mathbb{E}_t[\,S_T\,] > 0$ as $T \to \infty$.

Nomenclature. In probability theory language, a process that satisfies $S_t \geq \mathbb{E}_t[\,S_T\,]$ is called a *supermartingale*. When $dS_t = \sigma_t S_t \, d\tilde{B}_t$, then S_t is a called a local martingale. When $dS_t = \sigma_t S_t \, d\tilde{B}_t$ but $S_t \neq \mathbb{E}_t[\,S_T\,]$, then the stock price S_t is not a martingale, but instead, is called a *strictly local martingale.* So Sin's theorem tells us that, under very common stochastic volatility models, the stock price, and by implication other financial claim prices are sometimes only strictly local martingales.

Generalization. As we discussed, the specific properties of the volatility process play no role in Sin's argument; the only issue is explosions to $+\infty$. And, of course, it's easy to restore the interest rate and dividend yield. This means we have the immediate generalization:

THEOREM 9.2. *Let $\tau = T - t > 0$ and suppose that the risk-adjusted stock and volatility process $\tilde{P}$ has the form:*

$$\tilde{P}: \begin{cases} dS_t = (r-\delta)S_t + \sigma_t S_t \, d\tilde{B}_t \\ dV_t = b(V_t)dt + a(V_t)d\tilde{W}_t \end{cases}, \qquad 0 \leq (S_t, V_t) < \infty$$

where $d\tilde{B}_t$ and $d\tilde{W}_t$ are Brownian motions under $\tilde{P}$, $d\tilde{B}_t d\tilde{W}_t = \rho dt$, and each of $\{r, \delta, \rho\}$ are constants. In addition, suppose that the volatility under $\tilde{P}$ cannot explode to $+\infty$. Then,

$$S_t e^{-\delta\tau} = e^{-r\tau}\mathbb{E}_t[\,S_T \mid V_t\,] + S_t e^{-\delta\tau}\hat{P}_{\exp}(V_t, \tau) \;\geq\; e^{-r\tau}\mathbb{E}_t[\,S_T\,], \tag{4.10}$$

where $\hat{P}_{\exp}(V_t, \tau)$ is the probability of an explosion to $+\infty$ before time $T = t + \tau$, starting at V_t, in the auxiliary volatility process:

$$dV_t = [\,b(V_t) + \rho\sigma_t a(V_t)\,]dt + a(V_t)d\hat{W}_t\,. \tag{4.11}$$

5 When Martingale Pricing Fails: Generalized Pricing Formulas

In the last section, we showed that the martingale pricing formula for the stock price sometimes fails. If the formula fails for the stock price, it must fail for other simple claims[6]. When it does fail, how can it be corrected? First, a remark about the sign of the correction, and then a specific proposal, which is new.

Financial claim prices are typically supermartingales. Typically, financial claim prices $F(t)$ are bounded below. For example, the stock price, the call option price, and the put option price are all bounded below at zero. In addition, we have a general statement, the "first fundamental theorem of asset pricing"; which is actually a fairly large collection of related theorems. One qualitative version: a price process $G(t)$ does not admit arbitrage opportunities (under various definitions) if and only there exists at least one probability measure $\tilde{P}$ under which $G(t)$ is a local martingale. (In some, but not all versions of the theorems, $\tilde{P}$ must be *equivalent* to the real-world measure P). The discounted claim price, which is the price relative to the numeraire, $G(t) = e^{-rt}F(t)$ is a local martingale in our setting here because it satisfies the valuation PDE (**2**.1.1). As we showed at (**1**.5.5) , this implies that we have $dG = \sigma_t S G_S d\tilde{B}_t + a(V_t) G_V d\tilde{Z}_t$, where $d\tilde{B}_t$ and $d\tilde{Z}_t$ are Brownian motions under some measure $\tilde{P}$.

Without being precise about the exact meaning of "arbitrage-free", let's accept that arbitrage-free financial claim prices must be local martingales under some probability measure $\tilde{P}$. Then, there is a general theorem from the theory of stochastic differential equations that says that, if a local martingale is bounded below, then it's a supermartingale[7]. Hence, the "first fundamental theorem" combined with the "bounded local martingale theorem", tells us that any (discounted) arbitrage-free financial claim price, if it is bounded below, will be a supermartingale: $G_t \geq \mathbb{E}_t[G_T]$. So, if the martingale pricing formula $G_t = \mathbb{E}_t[G_T]$ must be "corrected", it will be done by adding a positive quantity to the right-hand-side for claim prices that are bounded below. Our specific proposal for correcting the martingale formula for put and call options is given by the following theorem:

---

[6] Sin (1998) noted a call option at a very low strike.

[7] For example, see Øksendal (1998), exercise 7.12(c).

THEOREM 9.3 (Generalized Pricing Formulas). *Suppose that the risk-adjusted stock and volatility process $\tilde{P}$ has the form:*

$$\tilde{P}: \begin{cases} dS_t = (r-\delta)S_t + \sigma_t S_t d\tilde{B}_t \\ dV_t = b(V_t)dt + a(V_t)d\tilde{W}_t \end{cases}, \quad 0 \le (S_t, V_t) < \infty$$

where $d\tilde{B}_t$ and $d\tilde{W}_t$ are Brownian motions under $\tilde{P}$, $d\tilde{B}_t d\tilde{W}_t = \rho dt$, and each of $\{r, \delta, \rho\}$ are constants. In addition, suppose that the volatility under $\tilde{P}$ cannot not explode to $+\infty$. Then, arbitrage-free fair values for European-style put and call options on the stock price are given by:

$$C_t = e^{-r\tau}\mathbb{E}_t\left[(S_T - K)^+\right] + S_t e^{-\delta\tau}\hat{P}_{\exp}(V_t, \tau) \ , \tag{5.1}$$

$$P_t = e^{-r\tau}\mathbb{E}_t\left[(K - S_T)^+\right], \tag{5.2}$$

where $\hat{P}_{\exp}(V_t, \tau)$ is the probability of an explosion to $+\infty$ before time $T = t + \tau$, starting at V_t, in the auxiliary volatility process:

$$dV_t = \left[b(V_t) + \rho a(V_t)V_t^{1/2}\right]dt + a(V_t)d\hat{W}_t \, .$$

Note that we are using the short-hand notation $C_t \equiv C(S_t, V_t, t)$ and similarly for the put option price.

PROOF : Both claim prices, after discounting, are local martingales: $dG = \sigma_t S G_S d\tilde{B}_t + a(V_t) G_V d\tilde{Z}_t$, where $G_t = e^{-rt}C_t$ or $G_t = e^{-rt}P_t$. The put price payoff on expiration is bounded above : $(K - S_T)^+ \le K$. There would be an arbitrage opportunity if the put price were not bounded above by $P_t \le Ke^{-r(T-t)}$ at all times. (Sell the put, place the proceeds in a money market security earning the rate r; close at expiration). Hence, the discounted put price process is a bounded local martingale. But a bounded local martingale must be a martingale (Durrett, 1996, Corollary 2.6, p. 41), and so (5.2) must hold. Similarly, there would be an arbitrage opportunity if put-call parity did not hold at all times. Put-call parity is the relation $C_t - P_t = S_t e^{-\delta\tau} - Ke^{-r\tau}$. Using (4.10) and the result for the put option, we have

$$\begin{aligned} C_t &= e^{-r\tau}\mathbb{E}_t\left[(K - S_T)^+\right] + e^{-r\tau}\mathbb{E}_t\left[(S_T - K)\right] + S_t e^{-\delta\tau}\hat{P}_{\exp}(V_t, \tau) \\ &= e^{-r\tau}\mathbb{E}_t\left[(S_T - K)^+\right] + S_t e^{-\delta\tau}\hat{P}_{\exp}(V_t, \tau) \ , \end{aligned}$$

which is (5.1) ■

Remarks. We have a number of comments before relating these generalized formulas to our development in Chapter 2.

(i) Note that we have only established that these option prices are "arbitrage-free" with respect to the two arbitrage opportunities mentioned in the proof. It's only a conjecture that the prices are arbitrage-free in the more general sense in finance, namely that no "reasonable" trading strategy can be constructed, based on those prices, that produces arbitrage profits. Arbitrage profits occur in a trading strategy if, starting with zero investment, your return under all possible outcomes is either (i) zero or (ii) a positive amount with a strictly positive probability. Reasonable trading strategies do not include, for example, unlimited borrowing which otherwise would allow repeated doubling of a loosing bet until a winner is achieved. This is a very important but very technical subject, well beyond our scope[8].

(ii) The same argument can be applied to any European-style "call-like" claim G_t. We say a claim is call-like if there exist constants $K_1 > 0$ and $K_2 < \infty$ such that the payoff function has the bounds (i) $G(S_T) \le (S_T - K_1)^+$, and (ii) $(S_T - K_1)^+ - G(S_T) \le K_2$. In other words, the payoff is bounded above by some call option, but not too far above. That tells us that the payoff $h(S_T) \equiv (S_T - K_1)^+ - G(S_T)$ is "put-like", since it lies in the range $0 \le h(S_T) < K_2$. To prevent arbitrage, the fair value of the claim G_t at any time $t \le T$ must equal the fair value of a call option striking at K_1 minus the fair value of one h-claim. The h-claim is bounded and so must be a martingale. This gives us the generalized result for any call-like claim:

$$G_t = e^{-r\tau}\mathbb{E}_t[G(S_T)] + S_t e^{-\delta\tau}\hat{P}_{\exp}(V_t,\tau) \tag{5.5}$$

(iii) A necessary, but not sufficient, condition, for the auxiliary process to explode and yield $\hat{P}_{\exp}(V_t,\tau) > 0$ is $\rho > 0$. That's because, otherwise, the volatility drift in the auxiliary process is bounded above by the drift in the risk-adjusted process $\hat{b} \le \tilde{b}$. Moreover, since the two processes have the same diffusion terms and the risk-adjusted process does not explode, then by *comparison theorems*[9] the auxiliary process cannot explode either.

---

[8] A good treatment is Duffie (1992), which has comprehensive references. Additional relevant work includes Delbaen and Schachermayer (1994,1995a,b). A nice introduction is offered by Protter (1999).

[9] For example, see Karatzas and Shreve (1991) Ch. 5, 2.18 Proposition, p. 293.

(iv) The generalized formulas are solutions to the valuation PDE problem, which is to find a solution to

$$\frac{\partial F}{\partial t} = -rF + \tilde{\mathcal{A}}\, F\,, \tag{5.3}$$

where $$\tilde{\mathcal{A}} F = (r-\delta)S\,\frac{\partial F}{\partial S} + \tfrac{1}{2} V S^2\,\frac{\partial^2 F}{\partial S^2} + \tilde{b}(V)\,\frac{\partial F}{\partial V} + \tfrac{1}{2} a^2(V)\,\frac{\partial^2 F}{\partial V^2} + \rho(V)a(V)V^{1/2} S\,\frac{\partial^2 F}{\partial S \partial V},$$

such that $F(S,V,T) = g(S_T)$, the given payoff function. The expectation terms satisfy both the PDE and the initial condition because you can interpret them as having the form

$$e^{-r\tau}\mathbb{E}_t\left[g(S_T)\right] = e^{-r\tau}\int \tilde{p}(S_t,V_t,S_T,T-t)g(S_T)dS_T\,.$$

Here $\tilde{p}(S,V,Z,T-t)$ is a fundamental solution to the PDE, in the variables (S,V,t) satisfying $p(S,V,Z,0) = \delta(S-Z)$ and $\delta(\bullet)$ is the Dirac delta function. That is, starting in the state (S,V) at time-t, then $\tilde{p}(S,V,S_T,\tau)dS_T \geq 0$ is the probability of arriving at S_T with any value for the volatility. But a solution of the form (5.1) is *also* a solution to (5.3) with the same initial condition. That is, by direct substitution of (5.1) into (5.3), and using the fact that the expectation term already satisfies the initial condition, one sees that $\hat{P}_e(V,\tau) \equiv \hat{P}_{\exp}(V,T-t)$ must be a solution to

$$\frac{\partial \hat{P}_e}{\partial \tau} = \left[\tilde{b}(V) + \rho(V)a(V)V^{1/2}\right]\frac{\partial \hat{P}_e}{\partial V} + \tfrac{1}{2} a^2(V)\,\frac{\partial^2 \hat{P}_e}{\partial V^2}, \tag{5.4}$$

which vanishes at $\tau = 0$. Indeed, this is exactly the PDE satisfied by the explosion probability, which also has the boundary conditions $\hat{P}_e(V,\tau=0) = 0$ and $\hat{P}_e(V=\infty,\tau>0) = 1$. If no explosion occurs in the auxiliary process, then there is no solution to (5.4) that satisfies these two conditions, and so we take $\hat{P}_e = 0$.

It's not completely surprising that volatility explosions are associated with multiple solutions to the evolution PDE. It's typical to have multiple solutions, all with the same initial conditions, when a boundary can be reached by a diffusion process in finite time. For example, consider simple Brownian motion on the positive axis. Since the origin can be reached in finite time, there are multiple solutions to the heat equation with the same initial condition: the probability density is a delta function at time-*0*. Then, a unique solution is selected by specifying what the process does when it reaches zero: absorption or reflection, say. What's subtly different here is that it's not, in fact, the original process that is reaching the boundary (at infinity). Instead, the boundaries are being reached in the dual (auxiliary) process. Not only is the volatility reaching infinity in the dual process, but there is some evidence that the stock price is reaching zero simultaneously (see Sec. 6).

6 Generalized Pricing Formulas and the Transform-based Solutions

Introduction. The only difference between the usual martingale-style pricing formulas and the generalized formulas are terms containing explosion probabilities. In this section, we show that if you have obtained a fundamental transform solution of the valuation PDE, then there is typically no need to separately calculate explosion probabilities. Explosion probabilities are already contained in the fundamental transform.

In addition, we show that the generalized option fair values, given at (5.1), do not represent a new class of valuation formulas, distinct from our previous development—they are, in fact, the Solution II formulas already given in Chapter 2.

The model. Throughout this section, we assume that the risk-adjusted stock price and volatility process, on $(S_t, V_t) \geq 0$, has the general form:

$$(6.1) \qquad \tilde{P}: \begin{cases} dS_t = (r-\delta)S_t\,dt + \sigma_t S_t\,d\tilde{B}_t \\ dV_t = \tilde{b}(V_t)dt + a(V_t)d\tilde{W}_t \end{cases}, \qquad d\tilde{B}_t d\tilde{W}_t = \rho\,dt\,,$$

where $\{r, \delta, \rho\}$ are constants. We will also refer to the process Q, which is dual to (6.1) on $(X_t, V_t) \geq 0$, and is given by

$$(6.2) \quad Q: \begin{cases} dX_t = (\delta - r)X_t\,dt + \sigma_t X_t\,d\hat{B}_t \\ dV_t = \left[\tilde{b}(V_t) + \rho a(V_t)\sigma_t\right]dt + a(V_t)d\hat{W}_t \end{cases}, \quad d\hat{B}_t d\hat{W}_t = -\rho\,dt\,.$$

The PDE problem for fundamental transforms under $\tilde{P}$, given at (2.2.19), is to find solutions to

$$\frac{\partial \hat{H}}{\partial \tau} = \tfrac{1}{2} a^2(V) \frac{\partial^2 \hat{H}}{\partial V^2} + \left[\tilde{b}(V) - ik\rho(V)a(V)V^{1/2} \right] \frac{\partial \hat{H}}{\partial V} - c(k)V\hat{H}\,, \tag{6.3}$$

subject to $\hat{H}(k,V,\tau=0)=1$. Recall our definition that a fundamental transform is *any* solution of (6.3) with $\hat{H}(k,V,\tau=0)=1$. Throughout this section, we are using $\tau = T-t$, where t is the time and $T>t$ is an expiration time.

In (6.3), we also use $c(k)=(k^2-ik)/2$; the important fact is that $c(k)$ vanishes at both $k=0$ and $k=i$. Suppose that we've found a fundamental transform. With that solution, form the function $f(V,\tau)=1-\hat{H}(k=i,V,\tau)$. We see that $f(V,\tau)$ solves the PDE

$$\frac{\partial f}{\partial \tau} = \tfrac{1}{2} a^2(V) \frac{\partial^2 f}{\partial V^2} + \left[\tilde{b}(V) + \rho(V)a(V)V^{1/2} \right] \frac{\partial f}{\partial V}\,, \tag{6.4}$$

with the initial condition $f(V,\tau=0)=0$. If $\hat{H}(k=i,V,\tau)=1$, then $f=0$, a trivial solution. But if $\hat{H}(k=i,V,\tau)<1$, then we've found a non-trivial solution to the explosion PDE for the auxiliary process, with vanishing initial condition. To know if this solution is actually the explosion probability, we have to look at $V\to\infty$.

Typically, but not always, one finds that, as $V\to\infty$, then $C(S,V,\tau)\to Se^{-\delta\tau}$ for $\tau>0$. This behavior is associated with $\hat{H}(k,V,\tau)\to 0$, as $V\to\infty$. To see why, consider that the general form call option solution, $C_H(S,V,\tau)$, is equal to $Se^{-\delta\tau}$ plus an integral that vanishes with $\hat{H}$. We will show below that $C_H=C$, the generalized fair value. Moreover, when $\hat{P}_{\exp}>0$, we know of no counter-example to the relation that $\hat{H}(k,V,\tau)\to 0$, as $V\to\infty$[10]. When $\hat{H}(k,V,\tau)\to 0$, then $f(V=\infty,\tau)=1$. This shows that, at least very typically, $f(V,\tau)=\hat{P}_{\exp}(V,\tau)$ since the two functions both satisfy the same PDE with the same boundary conditions. In other words, the fundamental transform already contains the explosion probability for the auxiliary process; just take $\hat{P}_{\exp}(V,\tau)=1-\hat{H}(k=i,V,\tau)$.

---

[10] We do show a counter-example in Chapter 10 where $\hat{H}(k,V=\infty,\tau)\neq 0$, but the counter-example model has $\hat{P}_{\exp}=0$. See that chapter for much more discussion of the large-V behavior of the fundamental transform.

The properties we have just described are more than just typical—they are guaranteed—*if* the fundamental transform is the (Fourier) transform of a norm-preserving probability density for the problem. First, we make an assumption that such a transition density exists.

Assumption (A1) *There exists a probability transition density* $\tilde{p}(S,V,S_T,\tau)$ *for the risk-adjusted process (6.1), norm-preserving as a density for* S_T .

In other words, there exists a solution $\tilde{p}(S,V,S_T,\tau)$ to the backward Kolmogorov equation for the $\tilde{P}$ process, namely

$$\frac{\partial \tilde{p}}{\partial \tau} = \tilde{\mathcal{A}}\, \tilde{p}\,, \text{ where} \tag{6.5}$$

$$\tilde{\mathcal{A}}\, \tilde{p} = (r-\delta)S\,\frac{\partial \tilde{p}}{\partial S} + \tfrac{1}{2} V S^2 \frac{\partial^2 \tilde{p}}{\partial S^2} + \tilde{b}(V)\,\frac{\partial \tilde{p}}{\partial V} + \tfrac{1}{2} a^2(V)\frac{\partial^2 \tilde{p}}{\partial V^2} + \rho\, a(V)\sqrt{V} S\,\frac{\partial^2 \tilde{p}}{\partial S \partial V}$$

and where the solution has the two properties:

$$\begin{cases} \tilde{p}(S,V,S_T,\tau=0) = \delta(S-S_T) \\ \int_0^\infty \tilde{p}(S,V,S_T,\tau)dS_T = 1 \end{cases}.$$

Then, we can make the following assertions:

THEOREM 9.4. *When both the density described in (A1) and its transform* $\hat{H}$ *exist, then* $\hat{H}$ *has following properties :*

(i) *it's norm-preserving:* $\hat{H}(k=0,V,\tau)=1$,

(ii) *when neither the risk-adjusted volatility process nor the auxiliary volatility process can explode, it's martingale-preserving, i.e.,*

$$\hat{H}(k=i,V,\tau)=1\,,$$

(iii) *when the auxiliary volatility process can explode, but the risk-adjusted process cannot, it's martingale-defective, i.e.,*

$$\hat{H}(k=i,V,\tau)=1-\hat{P}_{exp}(V,\tau)<1\,,$$

where $\hat{P}_{exp}(V,\tau)$ *is the explosion probability of the auxiliary process.*

PROOF. By assumption, both $\tilde{p}$ and the following $\hat{H}$ exist:

$$\hat{H}(k,V,\tau) \equiv \int_0^\infty e^{ik\tilde{X}}\,\tilde{p}(S,V,S_T,\tau)\,dS_T\,,\ \text{ where } \tilde{X} = \ln\left[\frac{S}{S_T}\right] + (r-\delta)\tau\,.$$

Since $\tilde{p}(S,V,S_T,\tau)$ solves (6.5), we showed in Chapter (2) that $\hat{H}(k,V,\tau)$ solves (6.3). Moreover, $\tilde{p}(S,V,S_T,0) = \delta(S-S_T)$ implies that $\hat{H}(k,V,\tau=0)=1$, which confirms that $\hat{H}$ is indeed a fundamental transform. The normalization of $\tilde{p}$ implies that $\hat{H}(k=0,V,\tau)=1$, which establishes statement (i). Next, let $k=i$, which produces:

$$\hat{H}(k=i,V,\tau) = \frac{e^{-(r-\delta)\tau}}{S}\int_0^\infty S_T\,\tilde{p}(S,V,S_T,\tau)\,dS_T = \frac{e^{-(r-\delta)\tau}}{S}\,\mathbb{E}[S_T\,]\,.$$

Now the premise of statements (ii) and (iii) is that the risk-adjusted volatility process cannot explode; that allows us to apply Theorem (9.2) to the last expression, yielding

$$\hat{H}(k=i,V,\tau) = 1 - \hat{P}_{\text{exp}}(V,\tau)\,.$$

Then, either the auxiliary process cannot explode and $\hat{P}_{\text{exp}}=0$, which establishes statement (ii). Or, the auxiliary process can explode and $\hat{P}_{\text{exp}}>0$, which establishes statement (iii). ∎

Now, let's add a regularity property which is frequently satisfied:

Property (P1) *A fundamental transform $\hat{H}(k,V,\tau)$ has property (P1) if it's regular in the complex k-plane within a strip: $-\varepsilon_1 < \operatorname{Im} k < 1+\varepsilon_2$, where $\varepsilon_i > 0$ are any two strictly positive constants.*

Then, we can connect the generalized option formulas of Theorem (9.3) to the previous valuation formulas that we developed in Chapter 2:

THEOREM 9.5. *Assume that the transform for the density in (A1)* also has *property (P1). Then, the arbitrage-free fair value for the option prices C_t and P_t, given at (5.1), are related in the following way to the solutions C_I, C_{II}, and P_{II} given in Chapter 2:*

(6.6)
$$\boxed{\begin{aligned} &(i) \quad C_I = e^{-r\tau}\mathbb{E}_t\left[(S_T-K)^+\right], \\ &(ii) \quad C_{II} = C_t = C_I + Se^{-\delta\tau}\hat{P}_{\text{exp}}(V,\tau)\,, \\ &(iii) \quad P_{II} = P_I = P_t = e^{-r\tau}\mathbb{E}_t\left[(K-S_T)^+\right] \end{aligned}}$$

PROOF. The fundamental transform $\hat{H}(k,V,\tau)$ is norm-preserving and is the transform of the risk-adjusted transition density $\tilde{p}(S,V,S_T,T)$. Both the call option payoff function and the fundamental transform have valid Fourier transforms in the region $1 < \operatorname{Im} k < \varepsilon_2$. (see Table 2.1 in Chapter 2). Hence they can both be inverted in that strip. Inverting the call option Solution I, you obtain the martingale-style solution to the valuation PDE, $C_I = e^{-r\tau}\mathbb{E}_t\left[(K - S_T)^+\right]$, which is statement (i).

Since the strip of regularity for the fundamental transform extends (at least slightly) beyond $\operatorname{Im} k > 1$, we can compute $C_{II} - C_I$ from the Residue Theorem (Ch. 2, Appendix 2.1). We integrate along a rectangular contour with the bottom side in the strip $0 < \operatorname{Im} k < 1$ and the top side in the strip $1 < \operatorname{Im} k < \varepsilon_2$. By assumption, $\hat{H}$ is regular throughout this rectangle, the only singularity in the integrand is a simple pole at $k = i$. The integration contributions from the vertical sides vanish as these sides are extended to $\pm\infty$. The Residue Theorem yields $C_{II} - C_I = Se^{-\delta\tau}(1 - \hat{H}(k = i,V,\tau))$, or in other words,

(6.7) $$C_{II} = e^{-r\tau}\mathbb{E}_t\left[(S_T - K)^+\right] + Se^{-\delta\tau}\left[1 - \hat{H}(k = i,V,\tau)\right].$$

Since the fundamental transform satisfies the premises of Theorem (9.4), we know that $\hat{P}_{exp}(V,\tau) = 1 - \hat{H}(k = i,V,\tau)$. With this substitution, then (6.7) can be compared with (5.1), and the comparison establishes the identity $C_{II} = C_I$. Finally, P_{II} is obtained from C_{II} by put-call parity, which also holds for C_I and P_I. In other words, $C_{II} - P_{II} = C_I - P_I$, which establishes that $P_{II} = P_I$. In addition, $P_I = P_{II}$ follows from the same Residue Theorem argument, but applied to a contour enclosing $k = 0$, and using $1 = \hat{H}(k = 0,V,\tau)$. ■

Now we want to consider the case where the risk-adjusted volatility process can explode, but is martingale-preserving. Note that this case is not covered under Theorem 9.4. We will consider certain defective transition densities not equal to the density described in (A1). We stress that, even though there is an explosion in (6.1), the conservation of probability still holds (see below). Our approach is the duality transformation.

Duality. Let $\tilde{p}^*(S,V,S_T,\tau)$ denote a (possibly defective) transition probability density for (6.1). That is, $\tilde{p}^*$ is a solution to (6.5) with $\tilde{p}^*(S,V,S_T,\tau=0)=\delta(S-S_T)$, but is not necessarily norm-preserving as a density for S_T (except at $\tau=0$). Throughout the remaining discussion in this section, asterisks serve as warning labels that a density may be defective. The interpretation is that $\tilde{p}^*$ is a conditional transition density, conditional on no volatility explosion in (6.1) prior to τ. Let $\tilde{q}^*(X,V,X_T,\tau)$ be a transition probability density for the dual process (6.2), possibly defective as a density for X_T. That is, $\tilde{q}^*$ has the initial condition $\tilde{q}^*(X,V,X_T,\tau=0)=\delta(X-X_T)$ and solves the PDE

$$\text{(6.8)} \qquad \frac{\partial \tilde{q}}{\partial \tau} = \tilde{\mathcal{B}}\,\tilde{q}\,, \text{ where}$$

$$\tilde{\mathcal{B}}\,\tilde{q} = (\delta - r)X\,\frac{\partial \tilde{q}}{\partial X} + \tfrac{1}{2}VX^2\,\frac{\partial^2 \tilde{q}}{\partial X^2} + \hat{b}(V)\,\frac{\partial \tilde{q}}{\partial V} + \tfrac{1}{2}a^2(V)\,\frac{\partial^2 \tilde{q}}{\partial V^2} - \rho\, a(V)\sqrt{V}S\,\frac{\partial^2 \tilde{q}}{\partial X \partial V}$$

using $\hat{b}(V)=\tilde{b}(V)+\rho(V)a(V)V^{1/2}$. Then, we have the following relationships:

THEOREM 9.6. *Under the duality transformation:*

$$\text{(6.9)} \qquad \tilde{p}^*(S,V,S_T,\tau) = \frac{S}{S_T^3}\,\tilde{q}^*\left(\frac{1}{S},V,\frac{1}{S_T},\tau\right).$$

PROOF. Make the substitution and change of variable $\tilde{p}^*(S,V,S_T,\tau)=S\,f(X,V,S_T,\tau)$, where $X=1/S$, in (6.5). Since p^* solves (6.5), then $f(X,V,S_T,\tau)$ solves (6.8). Moreover, using $X_T \equiv 1/S_T$, we have

$$\text{(6.10)} \quad f(X,V,S_T,0) = X\delta(S-S_T) = X\delta\left(\frac{1}{X}-\frac{1}{X_T}\right) = X_T^3\,\delta(X-X_T)\,.$$

This shows that $f(X,V,S_T,\tau)$ is X_T^3 times a (possibly defective) transition density for (6.8), so we can write $f(X,V,S_T,\tau)=X_T^3\,\tilde{q}^*(X,V,X_T,\tau)$, which yields (6.9). ■

Then, we can relate the normalization and martingale integrals of the two solutions:

THEOREM 9.7. *Assume the two (possibly defective) transition densities* $\tilde{p}^*$ *and* $\tilde{q}^*$ *are duals in the sense of Theorem (9.6). Then,*

(i) $\bar{p}^*$ *is martingale-defective if, and only if,* $\bar{q}^*$ *is norm-defective.*
(ii) $\tilde{p}^*$ *is norm-defective if, and only if,* $\tilde{q}^*$ *is martingale-defective.*

PROOF. A simple change of integration variable from S_T to $X_T = 1/S_T$ yields

$$\int_0^\infty \tilde{p}^*(S,V,S_T,\tau)\,S_T\,dS_T = \int_0^\infty \tilde{q}^*(X,V,X_T,\tau)\,dX_T$$

and
$$\int_0^\infty \tilde{p}^*(S,V,S_T,\tau)\,dS_T = \int_0^\infty \tilde{q}^*(X,V,X_T,\tau)\,X_T\,dX_T\ ,$$

from which statements (i) and (ii) follow immediately. ■

Of course, we could have stated the theorem in the positive sense: $\tilde{q}^*$ is norm-preserving if and only if $\tilde{p}^*$ is martingale-preserving.

The dual transition densities $\tilde{p}^*$ and $\tilde{q}^*$, by Fourier inversion, have corresponding fundamental transforms, which we write as $\hat{H}_{\tilde{p}}$ and $\hat{H}_{\tilde{q}}$. The last integral in the proof of Theorem (9.7) implies that $\hat{H}_{\tilde{p}}(k=0,V,\tau) = \hat{H}_{\tilde{q}}(k=i,V,\tau)$. Now assume that the *dual* volatility process cannot explode, so that $\tilde{q}$ is norm-preserving but possibly martingale-defective in X_T. We can apply Theorem (9.4)(iii) to this process and conclude that there exists a fundamental transform such that $\hat{H}_{\tilde{q}}(k-i,V,\tau) = 1 - P_e(V,\tau)$, where $P_e(V,\tau)$ is the explosion probability of the volatility under the *original* process (6.1) [since, of course, the auxiliary process for the dual (6.2) is just (6.1) again]. In other words, we have learned that if the original volatility process (6.1) can explode, but is martingale-preserving, and the dual process admits a norm-preserving density, then there exists a fundamental transform such that $\hat{H}_{\tilde{p}}(k=0,V,\tau) - 1 - P_{\exp}(V,\tau)$. This lets us finish the option story, because we now have a formula that relates P_I and P_{II}.

THEOREM 9.8. *Assume that the risk-adjusted volatility process under (6.1) can explode, but is martingale-preserving. Assume that the dual process admits a norm-preserving density, whose fundamental transform has property* (P1). *Then, there exists a fundamental transform for (6.1) such that the two put option formulas of Chapter 2 are related by*

(6.10)

$$\boxed{\begin{array}{ll}\text{(i)} & P_I = P_{II} - K\,e^{-r\tau} P_{\exp}(V,\tau)\,, \\ \text{(ii)} & P_{II} = e^{-r\tau}\mathbb{E}_t\left[(K - S_T)^+\right],\end{array}}$$

where $P_{\exp}(V,\tau)$ *is the explosion probability of the risk-adjusted volatility process.*

PROOF. Let $\hat{H}$ be the transform for $\tilde{p}^*$, where $\tilde{p}^*$ and $\tilde{q}$ are duals, and $\tilde{q}$ is the *norm-preserving* density for the dual process described in the theorem. Apply the same Residue Theorem argument above to a rectangular contour enclosing $k=0$; this yields $P_{II} - P_I = Ke^{-r\tau}\left[1 - \hat{H}(k=0,V,\tau)\right]$. We showed above that this $\hat{H}$ satisfies $\hat{H}_{\tilde{p}}(k=0,V,\tau) = 1 - P_{\exp}(V,\tau)$. This establishes statement (i). Since $\hat{H}$ is also martingale-preserving and regular, the Residue Theorem also implies that $C_I = C_{II} = e^{-r\tau}\mathbb{E}_t\left[(S_T - K)^+\right]$. But P_{II} is determined from put-call parity; that is $C_{II} - P_{II} = S\,e^{-\delta\tau} - K\,e^{-r\tau}$. Again using the fact that (6.1) is martingale-preserving, one can write $S\,e^{-\delta\tau} - K\,e^{-r\tau} = e^{-r\tau}\mathbb{E}_t\left[S_T - K\right]$, or

$$P_{II} = e^{-r\tau}\mathbb{E}_t\left[(S_T - K)^+\right] - e^{-r\tau}\mathbb{E}_t\left[S_T - K\right],$$

and the terms can be combined to yield statement (ii) ■

Remark. Finally, to complete the story, we note that if the risk-adjusted volatility process can explode, but is martingale-preserving, then the arbitrage-free fair values for the put option and the call option are given by the usual formulas $P_t = e^{-r\tau}\mathbb{E}_t\left[(K - S_T)^+\right]$ and $C_t = e^{-r\tau}\mathbb{E}_t\left[(S_T - K)^+\right]$. That's because the put option price is still a local martingale, bounded above, and hence a martingale. And, the call option price is determined from put-call parity and the fact that the process is martingale-preserving, which yields the expectation formula again. So, once again, the Solution II formulas continue to give the fair values in the presence of an explosion possibility.

Additional interpretation when the risk-adjusted volatility can explode. It's important to stress that, even though $\tilde{p}^*$ may be defective, we still have conservation of total probability. It's expressed by the relation

$$1 = \int_0^\infty \tilde{p}^*(S,V,S_T,\tau)\, dS_T + P_{\exp}(V,\tau) \tag{6.11}$$

Now in (6.10), P_I is an expectation over the defective density $\tilde{p}^*$ and P_{II} is the expectation over some norm-preserving density $\tilde{p}$. How does the norm-preserving density $\tilde{p}$ relate to (6.11)? The results in (6.10-6.11) suggest the following. Suppose the explosion $V_T \to \infty$ in the risk-adjusted process also drives the stock price to zero: $S_T \to 0$. Then, we can also interpret the explosion probability as an absorption probability. Specifically, define the absorption probability $A(S,V,\tau)$ by

$$A(S,V,\tau) = \Pr[\, S(t_0) = 0,\ t_0 \le \tau \mid S(0) = S,\, V(0) = V\,]\,, \tag{6.12}$$

where t_0 is the moment when the process (6.1) first reaches $S(t) = 0$, starting at (S,V). In words, $A(S,V,\tau)$ is the probability of absorption at a zero stock price prior to time τ, starting from the values (S,V). From (6.1), we see that $S = 0$ is a trap state, just as $V = \infty$ is a trap state when $P_{\exp} > 0$. So *if* $S = 0$ is reached, the stock price will remain there. We can always decompose the put option price into two pieces, an expectation conditional on no absorption, call that density p^{**}, and an absorption piece:

$$\begin{aligned} P(t) = P_{II} &= e^{-r\tau}\mathbb{E}_t\left[(K - S_T)^+\right] \\ &= e^{-r\tau}\int_0^\infty \tilde{p}^{**}(S,V,S_T,\tau)(K - S_T)^+ dS_T + A(S,V,\tau)Ke^{-r\tau}\,. \end{aligned} \tag{6.12}$$

Of course, (6.12) is valid even if absorption cannot occur. Comparing (6.12) with (6.10)(i) then suggests that, at least when the auxiliary volatility process cannot explode, we have the correspondences

$$\tilde{p}^{**}(S,V,S_T,\tau) = \tilde{p}^*(S,V,S_T,\tau) \quad \text{and} \tag{6.13}$$

$$A(S,V,\tau) = P_{\exp}(V,\tau)\,. \tag{6.14}$$

In other words, we are suggesting that the event $\{V_T \to \infty\}$ happens if and only if the event $\{S_T \to 0\}$ happens. This interpretation is supported by numerical examples (see Table 9.8 in the next section), in which the risk-adjusted volatility process, but not the auxiliary volatility process, can explode.

A notable aspect of (6.14) is that the right-hand-side has no S-dependence. This is quite plausible; recall the formal solution to the stock price evolution in (6.1), namely

$$S_T = S_0 \exp\left\{\int_0^T \sigma_t d\tilde{B}_t - \frac{1}{2}\int_0^T \sigma_t^2 dt\right\}.$$

The exponential is quite improper as $\sigma_t \to \infty$, but if it indeed can be interpreted as having the value zero, then we see that this value forces $S_T \to 0$, *independent of the starting value* S_0, which agrees with (6.14).

Summary. At this point, we have now established the relationships between the Chapter 2 solutions and explosions for 3 cases, namely either (i) neither the risk-adjusted nor the auxiliary volatility process can explode, or the two cases where one, but not the other can explode. In principle, there is a fourth alternative, namely (iv) both volatility processes can explode. We didn't make any equilibrium assumptions for the above results, other than the implicit idea that the fair values for option prices should be arbitrage-free. However, under the CPRA equilibrium, it's possible that case (iv) may not occur. In any event, we leave case (iv) as an open issue. Numerical examples now follow.

7 Generalized Pricing Formulas Example I: the 3/2 model

The 3/2 model (actual) volatility process is $dV = (\omega V - \theta V^2)dt + \xi V^{3/2} dW_t$. The risk-adjusted volatility process, under a CPRA equilibrium with CPRA parameter γ has the same form $dV = (\omega V - \tilde{\theta} V^2)dt + \xi V^{3/2} dW_t$. More specifically, the process maintains the same form under a risk-adjustment where the representative is a pure investor (consumption at a final date) with a distant planning horizon. The risk-adjusted parameter $\tilde{\theta}$ was given previously at (7.2.14), which we repeat here:

(7.1) $$\tilde{\theta} = -\tfrac{1}{2}\xi^2 + (1-\gamma)\rho\xi + \sqrt{(\theta + \tfrac{1}{2}\xi^2)^2 - \gamma(1-\gamma)\xi^2}\,.$$

For the example, we take the risk-neutrality case: $\gamma = 1$, so that $\tilde{\theta} = \theta$. In addition, we take $\omega = 1$, $\theta = 0$, and $\xi = 1$, so that the risk-adjusted process becomes $dV = Vdt + V^{3/2} dW_t$, and the auxiliary volatility process becomes

(7.2) $$dV = Vdt + V^{3/2}dW_t + \rho\xi\sigma V^{3/2}dt = (V + \rho V^2)dt + V^{3/2}dW_t .$$

First, we apply the Feller explosion test, which was explained in Sec. 3. The scale density $s(V) = V^{-2\rho}e^{2/V}$ and so the scale measure is given by

(7.3) $$S(c,d) = \int_c^d V^{-2\rho}e^{2/V}\,dV .$$

Clearly $S(c,\infty) < \infty$ if and only if $\rho > 1/2$. Also, it's easy to show that, as $d \to \infty$, then $\Sigma(c,d) = O(1/d)$. So, $\Sigma(\infty) = \lim_{d\to\infty}\Sigma(c,d) < \infty$. That is, $\Sigma(\infty) < \infty$. Hence, by the Feller test, we have an explosion to $+\infty$ in the auxiliary volatility process if and only if $\rho > 1/2$. (This result also proves that the actual volatility process, with $\theta = 0$, does not explode). The probability of an explosion $\hat{P}_e \equiv \hat{P}_{\exp}(V,\tau)$ is given by a solution to (5.4):

(7.4) $$\frac{\partial \hat{P}_e}{\partial \tau} = \tfrac{1}{2}V^3\frac{\partial^2 \hat{P}_e}{\partial V^2} + \left(V + \rho V^2\right)\frac{\partial \hat{P}_e}{\partial V} ,$$

which must also satisfy $\hat{P}_e(V,\tau = 0) = 0$ and $\hat{P}_e(V = \infty, \tau > 0) = 1$. This problem, with these boundary conditions, only has a solution when $\alpha \equiv 2\rho - 1 > 0$. When that condition holds, then the solution is given by

(7.5) $$\hat{P}_e(V,\tau) = \frac{\Gamma(\alpha, X)}{\Gamma(\alpha)}, \text{ using } X = \frac{2}{V}\left(\frac{1}{e^\tau - 1}\right),$$

employing the standard Gamma functions:

$$\Gamma(\alpha, x) = \int_x^\infty e^{-z}z^{\alpha-1}dz \quad \text{and} \quad \Gamma(\alpha) = \int_0^\infty e^{-z}z^{\alpha-1}dz, \quad (\mathrm{Re}\,\alpha > 0).$$

In fact, the limit $\alpha \to 0$ $(\rho \to 1/2)$ also exists: in that limit, $\hat{P}_e(V,\tau) \to 0$ as long as $V > 0$. For $\rho < 1/2$, there is no solution and $\hat{P}_e(V,\tau) = 0$. We have previously confirmed, in Chapter 2, that this explosion probability is also given by $\hat{P}_e(V,\tau) = 1 - \hat{H}(k = i, V, \tau)$. Here $\hat{H}(k,V,\tau)$ is the fundamental transform for the model given at (2.3.3) and derived in Chapter 11. Even without any specific knowledge of the Gamma function, it's easy to see that the numerator in (7.5) is just some positive fraction of the denominator, assuring that $0 \le \hat{P}_e(V,\tau) \le 1$ for $\rho \ge 1/2$. And the boundary conditions are easily seen to be satisfied. A plot of the explosion probability is shown below in Fig. 9.1 for $\tau = 2$.

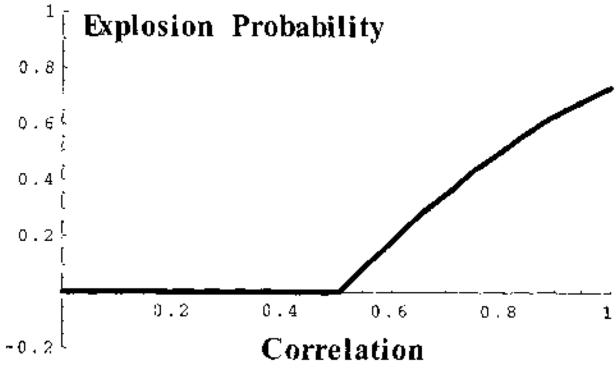


Fig. 9.1 Explosion Probability in the Volatility Process $(V+\rho V^2)dt + V^{3/2}dW_t$ vs. ρ

We have numerically evaluated the fundamental density over the final stock price S_T for this example, given by $\tilde{p}(S,V,S_T,T)$. The density over S_T is obtained by calculating the k-plane inversion of the fundamental transform using expression (**3**.2.12). This k-plane integration was performed along the contour $\operatorname{Im} k = 1$ and taking $|k_r| \le k_{\max} \cong 35$. This last value was determined, after some sensitivity analysis, to be large enough so that the table entries below do not change if $k_{\max}$ is made larger. The inversion produces the (complete) transition density for the risk-adjusted process because the fundamental transform is norm-preserving for the example model.

We then used this density in a further integration, calculating the expected values (martingale values) of 1_T (a normalization check), the stock price S_T, and the call option price C_T. That is, we numerically calculated

$$\text{(i)}\quad \mathbb{E}[1_T] = \int_0^\infty \tilde{p}(S,V,S_T,T)\,dS_T\,,$$

$$\text{(ii)}\quad \mathbb{E}[S_T] = \int_0^\infty \tilde{p}(S,V,S_T,T)S_T\,dS_T\,,$$

and

$$\text{(iii)}\quad \mathbb{E}[C_T] = \int_0^\infty \tilde{p}(S,V,S_T,T)(S_T-K)^+\,dS_T\,.$$

These integrals were actually performed over the integration variable $y = \ln S_T$, using a finite range $(y_{\min}, y_{\max}) = (-10, 40)$. All computations, including the integrations, were done at machine precision in Mathematica. The code is very similar to the code already given in Chapter 2, Appendix 2.2 for the square root model, except that the $\hat{H}$ function is given by (**2**.3.3) .

We took initial values $S = V = 1$, $T = 2$, and strike price $K = 1$. This was done for various values of ρ. In addition, the fair value for the call option C_0 was computed from the Solution II formula, given at (**2**.2.10), by integrating along $\operatorname{Im} k = 1/2$. The results are shown in Table 9.6 below.

Table 9.6 Numerical Tests of the Generalized Pricing Formula
Equilibrium Model: CPRA risk-neutral representative ($\gamma = 1$)
Risk-adjusted Volatility model: $dV = (\omega V - \theta V^2)dt + \xi V^{3/2} dW_t$
Auxiliary model: $dV = (\omega V - \theta V^2 + \rho \xi V^2)dt + \xi V^{3/2} dW_t$

ρ	**Norm** $\mathbb{E}[1_T]$	**Explosion Prob.** $\hat{P}_e(V,T)$	$\mathbb{E}[S_T]$	$\hat{P}_e +$ $\mathbb{E}[S_T]$	$\mathbb{E}[C_T]$	$\hat{P}_e +$ $\mathbb{E}[C_T]$	**Solution II** C_{II}
0	0.992	0	1.00	1.00	0.61	0.61	0.61
0.2	0.995	0	1.00	1.00	0.63	0.63	0.63
0.4	0.998	0	0.98	0.98	0.63	0.63	0.65
0.6	0.999	0.18	0.80	0.98	0.48	0.66	0.68
0.8	1.000	0.50	0.50	1.00	0.21	0.72	0.72
1.0	1.000	0.73	0.27	1.00	0.03	0.76	0.76

Notes. The table shows numerical tests of the generalized pricing formulas for financial claims that replace the expected value formula. These generalized formulas account for explosions in the auxiliary volatility process, which occur with probability $\hat{P}_{\exp}$ and only when $\rho > 1/2$. The volatility model parameters were $\omega_a = 1$, $\theta = 0$, $\xi_a = 1$. Additional parameters were $S = K = V = 1$, and $T = 2$ (years). Also $r - \delta = 0$. The table demonstrates very good agreement with the generalized formulas (4.1) and (5.2) The comparison between the last two columns show that the Call option Solution II formula is valid even when there are explosions in the auxiliary volatility process and the fundamental transform is not martingale-preserving: $\hat{H}(k = i, V, \tau) < 1$.

Discussion of Results. The numerical results show very good agreement with the proposed modifications of the expected value formula for fair values. In the table, the results above the dotted line correspond to the case of no volatility explosion in the auxiliary process. In those cases, one expects fair values to be given by expectations, and the numerical results agree with this. Below the dotted line, the auxiliary process can explode. The numerical results show that one clearly needs to add the explosion probability to both $\mathbb{E}[S_T]$ and $\mathbb{E}[C_T]$ to recover the fair value, which is what the theory of the last two sections showed.

We continue to stress that these results support the result that the transform-based formulas in Chapter 2, particularly Solution type II, are completely general. The comparison between the last two columns in Table 9.6 illustrate the fact that that call option solution II remains valid even when there are explosions in the auxiliary volatility process and, as a by-product, the fundamental transform is not martingale-preserving: $\hat{H}(k=i,V,\tau)<1$. That is to say, Solution II already fully accounts for the circumstances under which option fair values are not martingales, but only strictly local martingales.

Next, we examined the model which is the dual to the one in Table 9.6. The dual model has log-utility for the representative investor and a change in sign of the correlation. In the dual model, the risk-adjusted volatility process and the auxiliary process of the original model switch roles. Now, under log-utility, the risk-adjusted volatility process can explode. Hence, the fundamental transform is defective, with a norm strictly less than one. That means that put option solution I is an expectation over a defective distribution; the correct fair values are given by solution II.

These expected results are clearly confirmed in Table 9.7, where we compare theoretical solutions I and II to a Monte Carlo evaluation of the expected value. Since the put option price is a martingale, the expected value gives its fair value. As the table shows, the solution II is the correct one in the case where there are explosions in the risk-adjusted volatility process. The results in the table agree with (6.10).

In addition, we confirm the notion that volatility explosion probabilities may be interpreted as stock price absorption probabilities, with absorption at $S_T = 0$. This is shown in Table 9.8. In general, this interpretation may require the condition that the auxiliary volatility process cannot explode, which is the case here.

Table 9.7 Numerical Tests of Generalized Pricing Formulas
Equilibrium Model: CPRA representative with log-utility ($\gamma = 0$)
Risk-adjusted Volatility model: $dV = (\omega V - \theta V^2 - \rho \xi V^2)dt + \xi V^{3/2} dW_t$
Auxiliary model: $dV = (\omega V - \theta V^2)dt + \xi V^{3/2} dW_t$

ρ	Explosion Probability: Exact	Explosion Probability: M. Carlo	Put Option Solution I $P_I(S,V,\tau)$	Put Option Solution II $P_{II}(S,V,\tau)$	Monte Carlo $\mathbb{E}_t[P_T]$	Monte Carlo Std. Error
0	0	0	0.6050	0.6050	0.6054	0.0010
-0.2	0	0	0.6272	0.6272	0.6278	0.0011
-0.4	0	0	0.6515	0.6519	0.6532	0.0012
-0.6	0.1781	0.20±.01	0.5023	0.6804	0.6824	0.0014
-0.8	0.5021	0.50±.01	0.2129	0.7150	0.7182	0.0017
-1.0	0.7312	0.73±.01	0.0329	0.7641	0.7674	0.0024

Notes. The table shows numerical tests of the generalized pricing formulas for financial claims. These generalized formulas account for explosions in the risk-adjusted volatility process, which occur with probability $P_{\exp} = 1 - \hat{H}(k = 0)$ and only when $\rho < -1/2$. The model is the equilibrium dual to the model tested in Table 9.6. That's why the explosion probabilities are identical, as well as the at-the-money Solution II option values. The volatility model parameters were $\omega_0 = 1$, $\theta = 0$, $\xi_a = 1$. Additional parameters were $S = K = V = 1$, and $T = 2$ (years). Also $r = \delta = 0$. The Put option Solution I is the k-plane inversion in the region $\operatorname{Im} k < 0$ of the (defective) fundamental transform given at (**2**.3.3). The Put option Solution II, P_{II}, is the k-plane inversion in the region $0 < \operatorname{Im} k < 1$. The table entries confirm $P_{II} - P_I = 1 - \hat{H}(k = 0) = P_{\exp}$. The close agreement between the Put option solutions and Monte Carlo results (based on 10,000 drawings) provide evidence for the notion that the Put option Solution II formula is generally valid, even when there are explosions in the risk-adjusted volatility process and the fundamental transform is defective.

Table 9.8 Explosion Probability Interpreted as an Absorption Probability

Correlation ρ	Exact Explosion Probability $(V_T \to \infty)$	Monte Carlo Explosion Probability $(V_T \to \infty)$	Monte Carlo Absorption Probability $(S_T \to 0)$
0	0	0	0
-0.2	0	0	0
-0.4	0	0	0.001 ± 0.0003
-0.6	0.1781	0.202 ± 0.004	0.201 ± 0.004
-0.8	0.5021	0.504 ± 0.005	0.503 ± 0.005
-1.0	0.7312	0.733 ± 0.004	0.732 ± 0.004

Notes. The model and parameter values are identical to those in Table 9.8. This table just repeats the Table 9.7 entries for the explosion probabilities (adding the next significant digit) and also reports the Monte Carlo absorption probability for the absorption of the stock price at zero. The volatility explosion probability is estimated by counting an explosion event if $V_T \geq 1.4 \times 10^6$. The stock price absorption probability is estimated by counting an absorption event if $\ln S_T \leq -1.0 \times 10^6$. Note that this is a very small value for S_T . The results in the table support the interpretation that volatility explosions in the risk-adjusted process drive the stock price to absorption at zero (at least in cases where the auxiliary volatility process cannot explode).

8 Generalized Pricing Formulas Example II: the CEV model

The constant elasticity of variance (CEV) model, is a 1D model for the risk-adjusted stock price evolution. With zero interest rate, and unit volatility scale, the risk-adjusted process $\tilde{P}$ is given by

$$\tilde{P}: \quad dS_t = S_t^{\varphi} d\tilde{B}_t, \tag{8.1}$$

where φ is a constant. With φ in the range $0 \le \varphi \le 1$, this model for stock options was presented as an alternative to the B-S geometric Brownian motion process by Cox and Ross (1976). It has been applied or extended by many authors subsequently.[11] Our interest is that it provides an illustration of volatility explosions and generalized pricing formulas, which leads us to consider $\varphi > 1$, and more generally, $0 \le \varphi < \infty$. Since $\varphi > 1$ has rarely been considered, the results for that case may be new.

Relationship to 3/2 model. The model (8.1) can be written in the form of a stochastic volatility model, $dS_t = \sigma_t S_t d\tilde{B}_t$, where $\sigma_t = S_t^{\varphi-1}$. Using Ito's formula, the SDE for $V_t = \sigma_t^2 = S_t^{2\varphi-2}$ is easily developed. The result is that (8.1) can be rewritten as the (degenerate) stochastic volatility system:

$$\tilde{P}: \begin{cases} dS_t = \sigma_t S_t d\tilde{B}_t \\ dV_t = (\varphi-1)(2\varphi-3)V_t^2 dt + 2(\varphi-1)V_t^{3/2} d\tilde{B}_t \end{cases} \tag{8.2}$$

This is a special case of the 3/2 model where the stock price and volatility process are perfectly correlated[12]. The system (8.2) makes sense for any value of φ; we consider $\varphi \ge 0$.

The auxiliary volatility process. The auxiliary volatility process for (8.2) is

$$\begin{aligned} dV_t &= \left\{(\varphi-1)(2\varphi-3)V_t^2 + 2(\varphi-1)V_t^2\right\}dt + 2(\varphi-1)V_t^{3/2} d\hat{B}_t \\ &= (\varphi-1)(2\varphi-1)V_t^2 dt + 2(\varphi-1)V_t^{3/2} d\hat{B}_t . \end{aligned} \tag{8.3}$$

The Feller explosion test. Let's apply this test to both the volatility process under $\tilde{P}$ and the auxiliary process.

---

[11] Examples are Beckers (1980), Schroder (1989), Lo, Hui, and Yuen (1999).

[12] The fact that the CEV model is embedded in the 3/2 model was noted by Heston (1997).

For the volatility process, the scale density $s(V) = V^{-p}$, where $p = (\varphi - 3/2)/(\varphi - 1)$. So, the scale measure $S(c,d) = d^{1-p} - c^{1-p}$. Hence, $S(c,\infty) < \infty$ if and only if $p > 1$, which implies $-\infty < \varphi < 1$ or simply $\varphi < 1$ for brevity. Similarly, one can show that $\Sigma(\infty) < \infty$ if and only if $\varphi < 1$ also. By the Feller test, the volatility explodes to $+\infty$ if and only if $\varphi < 1$. Since $S = V^{1/(2\varphi-2)}$, then if $\varphi < 1$ and $V \to \infty$, we have $S \to 0$. That is, volatility explosions in the risk-adjusted process, which occur only when $\varphi < 1$, drive the stock price to zero. When $S = 0$ is reached, we stop the process (absorption). Absorption is compatible with $S = 0$ classified as either an exit or regular boundary. Stopping the process is required when φ is in the range $1/2 \le \varphi < 1$, since the boundary classification theory shows that $S = 0$ is an exit (absorbing or trap point) in that case. For $\varphi < 1/2$, other boundary behaviors could be specified, but absorption is a simple choice, and may be interpreted as a bankruptcy event for the firm.

For the auxiliary volatility process, the scale density $s(V) = V^{-q}$, where $q = (\varphi - 1/2)/(\varphi - 1)$. The explosion condition is $q > 1$, which implies $1 < \varphi < \infty$ or simply $\varphi > 1$ for brevity. So, there are explosions in the auxiliary volatility process if and only if $\varphi > 1$.

The martingale pricing failure. Since the auxiliary volatility process can explode, Theorem 9.2 tells us that the stock price fails to be a martingale when $\varphi > 1$. Instead of being a martingale, the stock price satisfies (4.10):

(8.4)
$$\mathbb{E}_0[S_T] = S_0\left[1 - \hat{P}_{\exp}(V_0,T)\right].$$

In this model, the explosion probability $\hat{P}_e(V,\tau) \equiv \hat{P}_{\exp}(V,T-t)$ is the solution to the PDE problem

(8.5)
$$\frac{\partial \hat{P}_e}{\partial \tau} = 2(\varphi-1)^2 V^3 \frac{\partial^2 \hat{P}_e}{\partial V^2} + (\varphi-1)(2\varphi-1)V^2 \frac{\partial \hat{P}_e}{\partial V},$$

subject to $\hat{P}_e(V,0) = 0$ and $\hat{P}_e(\infty,\tau) = 1$, $(\tau > 0)$. There is a solution to this explosion problem if and only if $\varphi > 1$, and in that case, one can verify the solution:

(8.6)
$$\hat{P}_e(V,\tau) = \frac{\Gamma\left(\nu, \dfrac{2\nu^2}{V\tau}\right)}{\Gamma(\nu)}, \quad \text{using} \quad \nu = \frac{1}{2(\varphi-1)} > 0.$$

Since $V = S^{(2\varphi-2)} = S^{1/\nu}$, then (8.4) reads

$$\frac{\mathbb{E}_0[S_T]}{S_0} = 1 - \frac{\Gamma\left(\nu, \frac{2\nu^2}{T} S_0^{-1/\nu}\right)}{\Gamma(\nu)}. \tag{8.7}$$

Next, we will verify (8.7) by an alternative approach. Then, we will discuss the fair value for the call option, which should have the same volatility explosion correction term according to Theorem 9.5.

Verification—Bessel processes. There are a number of ways to verify (8.7) through formulas already in the literature. One approach is to map the problem (8.1) into the evolution equations for two well-known processes: the Bessel process and the Bessel-squared process[13]. The Bessel process arises naturally in the study of Brownian motion—specifically, it is based upon the evolution equation for the radial distance from the origin of a standard Brownian motion in d-dimensions. Specifically, if $\vec{W}_t = (W_t^1, W_t^2, \cdots, W_t^d)$ is a d-dimensional Brownian motion process, then it's easy to show that its radial distance from the origin,

$$R_t = |\vec{W}_t| = \left[\sum_{i=1}^{d} (W_t^i)^2\right]^{1/2}, \quad \text{satisfies the SDE} \qquad dR_t = \frac{(d-1)}{2R_t} dt + dB_t,$$

where B_t is a *one-dimensional* Brownian motion[14]. The SDE for R_t defines the *d-dimensional Bessel process*. Next, consider $X_t = (R_t)^2$. From Ito's formula, it's easy to see that X_t follows the SDE $dX_t = d\,dt + 2\sqrt{X_t}dB_t$; this is the evolution equation for the *d-dimensional Bessel-squared process*. In fact, these SDEs make sense for values of d more general than positive integers. To emphasize that, let $d \to \delta \geq 0$. Then, the following definition is standard:

Definition. *For any* $\delta \geq 0$, *the Bessel-squared process with dimension* δ, *abbreviated* BESQ$^\delta$, *is the diffusion process* $X_t \geq 0$ *with the SDE*

$$dX_t = \delta\, dt + 2\sqrt{X_t}\, dB_t \tag{8.8}$$

where B_t *is a one-dimensional Brownian motion. The real number* $\nu = -1 + \delta/2$ *is called the index of the process* BESQ$^\delta$.

---

[13] See Revuz and Yor (1999, Ch.6, Sec.3 and Ch.11) for a general treatment of Bessel processes. For applications to finance, see Geman and Yor (1993).

[14] See Durrett (1996, Ch.5, Ex. 1.2, p 179)

Boundary classification. We can apply the Feller explosion test to (8.8), where by "explosion", we mean hitting the origin in finite time. The scale density $s(x) = x^{-\delta/2}$ and so the scale measure $S(b,c) = c^{1-\delta/2} - b^{1-\delta/2}$. Hence, $S(0,c) < \infty$ if and only if $\delta < 2$. Similarly, $\Sigma(0) < \infty$ if and only if $\delta < 2$. With these results, the Feller test says that the origin $X = 0$ can be reached in finite time by the BESQ^δ if and only if $\delta < 2$ $(\varphi < 1)$ This corresponds to the well-known fact that d-dimensional Brownian motion cannot hit the origin if $d \geq 2$ $(\varphi > 1)$. The boundary $X = 0$ can be completely classified using Table 9.2. The results are shown in the following Table 9.9. Note that the boundary can certainly be classified for $\delta \leq 0$, although the BESQ^δ process is only defined for $\delta \geq 0$. The relation to the CEV model parameter is developed below.

Table 9.9 Boundary classification at $X = 0$ for the $BESQ^\delta$ process (8.8)

Boundary type	CEV Parameter φ	Dimension δ	Bessel Index ν
Entrance	$1 < \varphi < \infty$	$\delta \geq 2$	$\nu \geq 0$
Exit	$1/2 \leq \varphi < 1$	$\delta \leq 0$	$\nu \leq -1$
Regular	$-\infty < \varphi < 1/2$	$0 < \delta < 2$	$-1 < \nu < 0$

A Green function. A transition density $q_\nu(x,y,t)$ in y is known for the BESQ^δ process. This function is the density for the process to start at $X_0 = x$ and then arrive at $X_t = y$ after the elapse of time t. Since the process starts at $X_0 = x$, the density satisfies the initial condition $q_\nu(x,y,0) = \delta(x-y)$. For $t > 0$, the density is given by[15]

$$q_\nu(x,y,t) = \frac{1}{2t}\left(\frac{y}{x}\right)^{\nu/2} \exp\left[-\left(\frac{x+y}{2t}\right)\right] I_\nu\left(\frac{\sqrt{xy}}{t}\right), \tag{8.9}$$

where I_ν is the modified Bessel function with index $\nu = -1 + \delta/2$. You can anticipate that we have already chosen the parameter ν in (8.6) to agree with this notation.

[15] See Geman and Yor (1993, Proposition 2.2). The formula is a special case of a result by Feller (1951, eq. 6.2). But there is a typo in Feller and the term $4b^2$ in his eq. 6.2 should be replaced by 1; then take $b \to 0$.

The transition density is norm-preserving in y for all $\delta > 0$; that is, $\int_0^\infty q_\nu(x,y,t)dy = 1$. Table 9.9 shows that, for δ in the range $0 < \delta < 2$, the origin is a regular boundary, and so admits reflecting or absorbing behavior. The fact that the density is norm-preserving suggests that the process must reflect off the origin in this case, and indeed, that is correct (see Revuz and Yor, Ch.11, 1.5 Proposition). For $\delta = 0$, the process is absorbed if it hits the origin and $\int_0^\infty q_{-1}(x,y,t)dy < 1$.

To map (8.1) into the Bessel process, let $x = S^\psi$. Then, Ito's formula yields

(8.10) $$dx = \tfrac{1}{2}\psi(\psi-1)x^{1-(2\varphi-2)/\psi}dt + \psi x^{1+(\varphi-1)/\psi}dB_t\,,$$

which is close to the correct form if we choose $\psi = 1-\varphi$. With that choice, (8.10) becomes

$$dx_t = \tfrac{1}{2}\varphi(\varphi-1)\frac{dt}{x_t} + (1-\varphi)\,dB_t\,.$$

This can be put in the correct form by introducing a new time variable $s = (1-\varphi)^2 t$, and then let $x(t) = R(s)$, so that $R(s)$ satisfies the SDE

(8.11) $$dR_s = \frac{(\delta-1)}{2R_s}ds + dB_s\,, \quad \text{where } \delta = \frac{2\varphi-1}{\varphi-1}\,.$$

As we promised, the index of the process $\nu = 1/(2\varphi-2)$ as in (8.6). Then, the mapping from the stock price to the Bessel process is given by

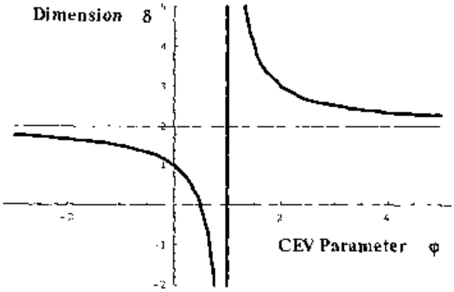
(8.12) $$S_t = R_s^{2-\delta} = |\vec{W}_s|^{2-\delta}, \qquad s = (1-\varphi)^2 t\,.$$

We have already shown that for $\varphi > 1$, then S_t is a local martingale, but not a martingale. As it turns out, $|\vec{W}_t|^{2-d}$, where $\vec{W}_t$ is a d- dimensional Brownian motion (integer dimensions), and $d \geq 3$ is a famous counter-example of a local martingale which is not a martingale[16]. The relationship between the Bessel process dimension and the CEV model parameter is plotted in Fig. 9.2 below.

---

[16] See Durrett (1996), Ch.2, Example 2.1, p.40. The counter-example was discovered by Johnson and Helms (1963)—a reference kindly supplied by Philip Protter.

Fig. 9.2 Bessel Process Dimension δ vs. CEV Model Parameter φ



The mapping from the stock price to the Bessel-squared process is given by

(8.13) $$S_t = X_s^{1-\delta/2} = X_s^{-\nu} \ .$$

This last relation and (8.9) allows us to explicitly compute $\mathbb{E}_0[S_T]$ as follows, where we use $T' = (1-\varphi)^2 T$:

$$\mathbb{E}_0[S_T] = \mathbb{E}_0[S_T \mid S_0] = \mathbb{E}_0\left[X_{T'}^{-\nu} \mid x = S_0^{-1/\nu}\right] = \int_0^\infty y^{-\nu} q(x,y,T')\,dy$$

$$= \frac{1}{2T'}\int_0^\infty y^{-\nu}\left(\frac{y}{x}\right)^{\nu/2} \exp\left[-\left(\frac{x+y}{2T'}\right)\right] I_\nu\left(\frac{\sqrt{xy}}{T'}\right) dy = x^{-\nu}\int_0^\infty q_\nu(y,x,T')\,dy \ .$$

Note that in the last expression we are integrating with respect to the first argument of the density instead of the second. This last integral was obtained by Feller (1951, eqn. 6.1), who found, in our notation:

$$\int_0^\infty q_\nu(y,x,T')\,dy = 1 - \frac{\Gamma\left(\nu, \dfrac{x}{2T'}\right)}{\Gamma(\nu)} = 1 - \frac{1}{\Gamma(\nu)}\Gamma\left(\nu, \frac{S_0^{-1/\nu}}{2(1-\varphi)^2 T}\right).$$

Since the integral is multiplied by $x^{-\nu} = S_0$, and $(1-\varphi)^2 = \nu^2/4$, we have confirmed (8.7).

Denote a transition density for (8.1) in S_T by $G_\varphi(S, S_T, T)$; we have at this point established that a (fundamental) solution to the PDE problem

(8.14) $$\frac{\partial G}{\partial \tau} = \tfrac{1}{2} S^{2\varphi} \frac{\partial^2 G}{\partial S^2} \ , \qquad G(S, S_T, 0) = \delta(S - S_T)\ , \quad \varphi > 1\ ,$$

is given by

$$G_{\varphi}(S,S_T,\tau)=\frac{1}{\nu}S_T^{-1-1/\nu}q_{\nu}\left(S^{-1/\nu},S_T^{-1/\nu},(1-\varphi)^2\tau\right), \tag{8.15}$$

using
$$\nu=\nu(\varphi)=\frac{1}{2(\varphi-1)}>0\,.$$

We have also noted that this transition density is norm-preserving but martingale-defective in S_T for all $\varphi>1/2$ $(\delta>0)$.

Duality. To develop a duality transformation, let $G(S,\tau)=S\,f(X,\tau)$ in (8.14), where $X=1/S$. Then, one finds that $f(X,\tau)$ satisfies

$$\frac{\partial f}{\partial\tau}=\tfrac{1}{2}\sigma^2X^{2\tilde{\varphi}}\frac{\partial^2 f}{\partial X^2}\quad,\quad\text{with}\quad\tilde{\varphi}=2-\varphi\,. \tag{8.16}$$

Moreover, we showed in Theorem 9.6 that $f(X,\tau)$ is X_T^3 $(=S_T^{-3})$ times a (possibly defective) transition density for (8.16), which is $G_{\tilde{\varphi}}(X,X_T,\tau)$. In other words, the duality relation (6.9) reads, in this case

$$G_{\tilde{\varphi}}(S,S_T,\tau)=\frac{S}{S_T^3}G_{2-\tilde{\varphi}}\left(\frac{1}{S},\frac{1}{S_T},\tau\right). \tag{8.17}$$

From Theorem (9.7), we know that since G_{φ} is norm-preserving and martingale-defective for a range that includes $\varphi>1$, then $G_{\tilde{\varphi}}$ is norm-defective and martingale-preserving for $\tilde{\varphi}<1$. Note that $\tilde{\nu}=1/(2-2\tilde{\varphi})$, so that (8.17) combined with (8.15) yields, for $\tilde{\varphi}<1$

$$G_{\tilde{\varphi}}(S,S_T,\tau)=\frac{1}{\tilde{\nu}}S\,S_T^{-2+1/\tilde{\nu}}q_{\tilde{\nu}}\left(S^{1/\tilde{\nu}},S_T^{1/\tilde{\nu}},(1-\tilde{\varphi})^2\tau\right).$$

Or, in other words, if $\varphi<1$, then a fundamental, norm-defective, martingale-preserving transition density for (8.1) is given by

$$G_{\varphi}(S,S_T,\tau)=\frac{1}{\tilde{\nu}}S\,S_T^{-2+1/\tilde{\nu}}q_{\tilde{\nu}}\left(S^{1/\tilde{\nu}},S_T^{1/\tilde{\nu}},(1-\tilde{\varphi})^2\tau\right),\quad\tilde{\nu}=\frac{1}{2-2\varphi}>0\,. \tag{8.18}$$

Relation to Cox's CEV model solution. As it turns out, the transition density in (8.18) is the norm-defective density used by Cox[17] in unpublished notes to value options in (8.1) for $0<\varphi<1$. As we have shown in Table 9.9, the boundary point $S=0$ is regular or absorbing for the process (8.1) when $\varphi<1$. For φ in the range $0<\varphi<1/2$, one could choose to have the process be absorbed

[17] To prove the assertion, just compare (8.18) to the Cox solution as reported by Schroder (1989), eqn. (1), for example.

instead of reflected. The BESQ^{δ} process is reflected at the origin, but (8.18) can be interpreted as the density for a process that is absorbed at $S=0$. That is the route taken in the Cox solution. So the interpretation of (8.18) is that it is the transition density conditional on no absorption at the origin prior to τ.

Duality again. With the possibility of absorption, put options, for example, are valued by taking expectations with respect to (8.18) and then adding an absorption term. We want to show this solution and then invoke duality again. The advantage of this method is that the dual to the put option solution, in the well-known regime $\varphi<1$, is a call option solution in the less well-known regime $\varphi>1$. And, in the latter regime, the call option price is *not* a martingale and requires an explosion term to value it—just like the stock price in (8.4). So, beginning with a familiar starting point, this will provide another illustration of the generalized pricing formulas.

To recapitulate, with $0\leq\varphi<1$ and absorption at the origin, put option fair values are given by

$$P_{\varphi}(S,K,\tau)=\int_{0}^{\infty}G_{\varphi}(S,S_T,\tau)(K-S_T)^{+}\,dS_T+K\,P_{abs}(S,\tau;\varphi)\,. \tag{8.19}$$

In (8.19) $P_{abs}(S,\tau;\varphi)$ is the probability that the process in (8.1), starting at S, first hits the origin prior to τ. The absorption probability $P_a(S,\tau)\equiv P_{abs}(S,\tau;\varphi)$ satisfies the PDE problem

$$\begin{cases}\dfrac{\partial P_a}{\partial\tau}=\tfrac{1}{2}S^{2\varphi}\dfrac{\partial^2 P_a}{\partial S^2}\\ P_a(S,0)=0,\\ P_a(0,\tau)=1,\;\;(\tau>0)\end{cases}. \tag{8.20}$$

The solution to (8.20) is readily found to be

$$P_{abs}(S,\tau;\varphi)=\frac{\Gamma\left(\tilde{\nu},\dfrac{2\tilde{\nu}^2}{\tau}\cdot S^{1/\tilde{\nu}}\right)}{\Gamma(\tilde{\nu})},\quad\text{where}\quad\tilde{\nu}(\varphi)=\frac{1}{2-2\varphi}>0\,. \tag{8.21}$$

Then, the duality transformation for stochastic volatility models at (8.1.7) implies that the put option price and the call option price in the dual model are related by

(8.22) $$C_\varphi(S,K,\tau) = S\,K\,P_{2-\varphi}\left(\frac{1}{S},\frac{1}{K},\tau\right).$$

Assuming that $\varphi > 1$, then the right-hand-side of (8.22) is given by (8.19)

$$C_\varphi(S,K,\tau) = S\,K\int_0^\infty G_{2-\varphi}\left(\frac{1}{S},S_T,\tau\right)\left(\frac{1}{K}-S_T\right)^+ dS_T + S\,P_{abs}\left(\frac{1}{S},\tau;2-\varphi\right).$$

But, using (8.17), we can write

$$G_{2-\varphi}\left(\frac{1}{S},S_T,\tau\right) = \frac{1}{SS_T^3}G_\varphi\left(S,\frac{1}{S_T},\tau\right)$$

and the integral above becomes

$$S\,K\int_0^\infty G_{2-\varphi}\left(\frac{1}{S},S_T,\tau\right)\left(\frac{1}{K}-S_T\right)^+ dS_T = \int_0^\infty G_\varphi\,(S,X_T,\tau)(X_T - K)^+\,dX_T\,,$$

after a change of integration variable to $X_T = 1/S_T$. Also, (8.21) gives

$$P_{abs}\left(\frac{1}{S},\tau;2-\varphi\right) = \frac{\Gamma\left(\dfrac{1}{2\varphi-2},\dfrac{S^{2-2\varphi}}{2(\varphi-1)^2\tau}\right)}{\Gamma\left(\dfrac{1}{2\varphi-2}\right)} = \frac{\Gamma\left(\nu,\dfrac{2\nu^2}{\tau}S^{-1/\nu}\right)}{\Gamma(\nu)},$$

again using $\nu = 1/(2\varphi-2)$ from (8.6). We collect terms and relabel the dummy integration variable $X_T \to S_T$. Then, to summarize: the duality transformation generates, for $\varphi > 1$, the call option value formula:

(8.23) $$C_\varphi(S,K,\tau) = \int_0^\infty G_\varphi\,(S,S_T,\tau)(S_T-K)^+\,dS_T + S\frac{\Gamma\left(\nu,\dfrac{2\nu^2}{\tau}S^{-1/\nu}\right)}{\Gamma(\nu)}$$

$$= \mathbb{E}\left[(S_T-K)^+\right] + S\frac{\Gamma\left(\nu,\dfrac{2\nu^2}{\tau}S^{-1/\nu}\right)}{\Gamma(\nu)}, \qquad (\varphi > 1).$$

which is the martingale value of the call option plus the same explosion correction to the stock price in (8.7). This is another illustration of the Theorem 9.5(ii) formula.

10 Option Prices at Large Volatility

In this chapter, we develop the large volatility behavior of the fundamental transform and option prices. Two volatility processes with the same boundary classifications at $V=\infty$ of Chapter 9 can generate different results for option values. Large volatility asymptotica is important for building both PDE solvers for the option price and eigenfunction solvers for the fundamental transform and risk premiums.

1 Introduction

Consider the call option price $C(S,V,\tau)$ under some stochastic volatility process. A natural question: what happens to the price when $S\to\infty$ and when $V\to\infty$? For example, if you try to build a solver for the option valuation PDE, then you need to know these things.

To narrow our problem slightly, let's suppose that the martingale pricing process is determined under either risk-neutrality or log-utility preferences and the correlation is a constant. Also, it's more convenient to work in an environment where the dividend yield is zero, so we specialize to that. If the actual volatility process is of the form $dV_t = b(V_t)dt + a(V_t)dW_t$, then from the theory of Chapter 7, we know the risk-adjusted process must be

$$\tilde{P}: \begin{cases} dS_t = rS_t\,dt + \sigma_t S_t\,d\tilde{B}_t \\ dV_t = \left[b(V_t) - (1-\gamma)\rho\sigma_t a(V_t)\right]dt + a(V_t)\,d\tilde{W}_t \end{cases}, \tag{1.1}$$

where r and the correlation ρ are constants.

We have certain expectations about the $S \to \infty$ and $V \to \infty$ limits from the B-S theory, where $C(S,V,\tau) \to S$ in both cases[1]. Under stochastic volatility, suppose the following limits exist:

$$f_1(X,\tau) = \lim_{V \to \infty} \frac{C(S,V,\tau)}{S} \quad \text{and} \quad f_2(V,\tau) = \lim_{S \to \infty} \frac{C(S,V,\tau)}{S}, \tag{1.2}$$

where $X = \ln(S/K) + r\tau$, and τ is the time to expiration. Since call option prices are non-negative and satisfy the arbitrage bound $C(S,V,\tau) \le S$, we must have $0 \le f_i \le 1$.

The case $S \to \infty$ seems to be settled by the following non-rigorous argument. If you substitute $C(S,V,\tau) \approx f_2(V,\tau)S$ into the valuation PDE (7.1.24), then $f_2(V,\tau)$ must satisfy

$$\frac{\partial f_2}{\partial \tau} = \left[b(V) - \varphi(V,t) + \rho(V)a(V)V^{1/2} \right] \frac{\partial f_2}{\partial V} + \tfrac{1}{2} a^2(V) \frac{\partial^2 f_2}{\partial V^2} \tag{1.3}$$

Since we also have the initial condition $f_2(V,\tau = 0) = 1$, then the solution will "stick" at the value $f_2(V,\tau) = 1$ for all times and we find the B-S theory result again. Note that this argument, if it's valid at all, works under any risk adjustment—not just logarithmic utility or risk neutrality.

But the large volatility case seems to have some surprises. If the B-S theory result held, we would have $f_1 = 1$ also. Indeed, for many volatility process (such as the GARCH diffusion or the square root model), we either know from the exact solution or have strong suspicions that $f_1 = 1$.

But we also supply a *counter-example* where $f_1 < 1$. In the deterministic limit, where $a(V) = 0$, the counter-example model can be solved exactly and the assertion demonstrated. When the counter-example is stochastic, we do not have a rigorous proof but, in our opinion, strong plausibility arguments. The arguments are based partly upon certain asymptotic expansions and partly upon Monte Carlo estimates. The same approaches can be used to investigate the question for other models without exact solutions.

---

[1] In this chapter we use both "$\approx$" and "$\to$" interchangeably to mean "is asymptotically equal to".

We illustrate our method with three running examples. One of them, an important case of practical interest, is the GARCH diffusion. For the GARCH diffusion, we argue for $C(S,V,\tau) \to S$ as $V \to \infty$. For this model, (1.1) becomes

$$dV_t = \left[\omega - \theta V_t - (1-\gamma)\rho \xi V^{3/2}\right]dt + \xi V_t d\tilde{W}_t .$$

Our second model is the 3/2 model, and (1.1) becomes

$$dV_t = (\omega V_t - \tilde{\theta} V_t^2)dt + \xi V_t^{3/2} d\tilde{W}_t ,$$

where $\tilde{\theta} = \theta + (1-\gamma)\rho\xi$. For the 3/2 model, we have an exact solution, which indeed behaves as $C(S,V,\tau) \to S$. Below, we take $V >> 1$ in the exact solution and develop the next-to-leading terms. This provides a consistency check on the asymptotic expansion method we are using for the models we cannot solve exactly.

Our final model and proposed counter-example is the volatility process $dV_t = -V_t^{5/2}dt + \xi V_t^{3/2} d\tilde{W}_t$, which we call the *modified 3/2* model. For simplicity we take $\rho = 0$. Note that at large volatility, the noise term is identical and the drift term only differs by a half-integer power from the original 3/2 model. But, we argue that the modified model is completely different. Both our asymptotic expansion and numerical work suggest that for this model, $C(S,V,\tau) \to f_1(\tau)S$, where $f_1(\tau) < 1$, as $V \to \infty$. As we mentioned, in the special case where $\xi = 0$, we can confirm that $f_1(\tau)$ is strictly less than one from an exact solution.

2 Asymptotica for the Fundamental Transform

Recall the relationship between the call option price and the fundamental transform $H(k,V,\tau)$, where $\tau = T - t$, the time to expiration. From Chapter 2, and with no dividends, we have

$$C(S,V,\tau) = S - \frac{Ke^{-r\tau}}{2\pi} \int_{ik_i-\infty}^{ik_i+\infty} \exp(-ikX) \frac{H(k,V,\tau)}{k^2 - ik} dk , \tag{2.1}$$

where K is the strike and $X = \ln(S/Ke^{-r\tau})$. In this chapter, we shall assume that the integration is along the contour $\mathrm{Im}(k) = k_i = 1/2$.

Equation (2.1) suggests that if $C(S,V,\tau) \to S$ as $V \to \infty$, then $H(k,V,\tau)$ will get small unless something very unusual happens. So, we are led to examine under what conditions $H(k,V,\tau) \to 0$ in specific models as $V \to \infty$. From (**2**.2.19), $\hat{H}(k,V,\tau)$ satisfies the PDE

$$(2.2) \qquad -\frac{\partial \hat{H}}{\partial \tau} = \mathcal{L}_k \hat{H} = -\tfrac{1}{2} a^2(V) \frac{\partial^2 \hat{H}}{\partial V^2} - \beta(k,V) \frac{\partial \hat{H}}{\partial V} + c(k) V \hat{H}$$

with the initial condition $H(k,V,\tau = 0) = 1$. In (2.2), under our preference model assumptions, $\beta(k,V) = b(V) - (1 - \gamma + ik)\rho \sigma a(V)$. And we also use $c(k) = (k^2 - ik)/2$.

Until further notice, we assume that $a(V) > 0$ when $V > 0$. This strictly excludes the deterministic volatility case, where $a(V) = 0$. The deterministic case is developed separately below. We shall illustrate a procedure for examining if there are solutions to (2.2) that behave as $H(k,V,\tau) \to 0$ at large-V. The procedure allows one to investigate the $V \to \infty$ regime even when the model cannot be solved exactly.

We make the assumption that the fundamental transform can be built up from solutions to the equation $\mathcal{L}_k u = \lambda u$, with u in a suitable domain and $\lambda = \lambda(k)$ is complex-valued. In Chapter 6, we found that indeed, as $\tau \to \infty$, then $H(k,V,\tau) \approx u_\lambda(V) \exp[-\lambda(k)\tau]$ where in this case the solution was the first eigenvalue. Now we are not restricting the eigenvalue to be the first one, nor the parameter k to be pure imaginary.

So our method relies upon the following two assumptions.

Assumption 1. The fundamental transform has a representation of the form:

$$(2.3) \qquad H(k,V,\tau) = \int_{\lambda \in Sp(\mathcal{L}_k)} g(\lambda) u_\lambda(V) e^{-\lambda\tau} d\lambda .$$

Assumption 2. It's permissible to take $V \to \infty$ inside the integral.

The totality of all of the points in the complex $\lambda-$plane that contributes to the integral in (2.3) is called the spectrum of $\mathcal{L}_k$, written $Sp(\mathcal{L}_k)$. The spectrum may consist of a finite or infinite number of discrete eigenvalues λ_n, $n = 0,1,\ldots,n_{\max} \le \infty$, plus a continuous component. The spectrum may be purely discrete, purely continuous, or a mixture of the two. Each of the terms in the integrand in (2.3) depends parametrically on the transform variable k also.

We can divide both sides of $\mathcal{L}_k u = \lambda u$ by $a(V) > 0$ for any $V > 0$. This produces the form

(2.4) $$u'' + p(V)u' + \bar{q}(V)u = 0,$$

where $p(V) = 2\beta(V)/a^2(V)$ and $\bar{q}(V) = 2[-c(k)V + \lambda]/a^2(V)$ and the primes denote V-derivatives. We can always eliminate the first derivative term in (2.4) by letting

(2.5) $$u(V) = f(V)\exp\left(-\tfrac{1}{2}\int p(V)dV\right),$$

where for simplicity we are suppressing the dependencies on k and λ. With that substitution, $f(V)$ satisfies the equation

(2.6) $$f'' + q(V)f = 0, \text{ where } \quad q(V) = \bar{q}(V) - \tfrac{1}{2}p'(V) - \tfrac{1}{4}p^2(V).$$

Example 1. The 3/2 model. In this case, the eigenvalue equation is

(2.7) $$\tfrac{1}{2}\xi^2 V^3 u'' + \left[\omega V - \theta V^2 - (1-\gamma+ik)\rho\xi V^2\right]u' - c(k)Vu = -\lambda u$$

and so $$p(V) = -\frac{\tilde{\theta}}{V} + \frac{\tilde{\omega}}{V^2}, \qquad \bar{q}(V) = -\frac{\tilde{c}}{V^2} + \frac{\tilde{\lambda}}{V^3},$$

$$q(V) = -\frac{(\tilde{c} + \frac{1}{2}\tilde{\theta} + \frac{1}{4}\tilde{\theta}^2)}{V^2} + \frac{(\tilde{\lambda} + \tilde{\omega} + \frac{1}{2}\tilde{\omega}\tilde{\theta})}{V^3} - \frac{1}{4}\frac{\tilde{\omega}^2}{V^4},$$

using $\tilde{\omega} = \dfrac{2}{\xi^2}\omega, \quad \tilde{\theta} = \dfrac{2}{\xi^2}[\theta + (1-\gamma+ik)\rho\xi], \quad \tilde{c} = \dfrac{2}{\xi^2}c, \quad \tilde{\lambda} = \dfrac{2}{\xi^2}\lambda.$

Example 2. The GARCH diffusion. In this case, the eigenvalue equation is

$$(2.8)\qquad \tfrac{1}{2}\xi^2 V^2 u'' + \left[\omega - \theta V - (1-\gamma+ik)\rho\,\xi\, V^{3/2}\right]u' - c(k)Vu = -\lambda\, u\,,$$

and so
$$p(V) = \frac{\tilde{d}}{V^{1/2}} - \frac{\tilde{\theta}}{V} + \frac{\tilde{\omega}}{V^2}\,, \qquad \tilde{q}(V) = -\frac{\tilde{c}}{V} + \frac{\tilde{\lambda}}{V^2}\,,$$

$$q(V) = -\frac{(\tilde{c} + \frac{1}{4}\tilde{d}^2)}{V} + \frac{1}{4}\frac{\tilde{d}(1+2\tilde{\theta})}{V^{3/2}} + \frac{(\tilde{\lambda} - \frac{1}{2}\tilde{\theta} - \frac{1}{4}\tilde{\theta}^2)}{V^2} + O\left(\frac{1}{V^{5/2}}\right),$$

using
$$\tilde{d} = \frac{2\rho}{\xi}(1-\gamma+ik)\ ,\ \tilde{\theta} = \frac{2}{\xi^2}\theta\,,\ \text{and}\ \ \tilde{\omega}, \tilde{c}, \tilde{\lambda}\ \text{as in (2.7).}$$

Example 3. The modified 3/2 model. For simplicity, we assume that $\xi^2 = 2$ and $\theta = 1$. In this case, the eigenvalue equation is

$$(2.9)\qquad V^3 u'' - V^{5/2}u' - c(k)Vu = -\lambda\, u\,,$$

and so
$$\tilde{q}(V) = -\frac{c}{V^2} + \frac{\lambda}{V^3} \qquad \text{and} \qquad q(V) = -\frac{1}{4}\frac{1}{V} - \frac{1}{4}\frac{1}{V^{3/2}} - \frac{c}{V^2} + \frac{\lambda}{V^3}\,.$$

Types of singularities. The behavior of $q(V)$ in (2.6), as $V \to \infty$, can be used to classify the nature of the singularity at $V = \infty$[2]. A sufficient condition for $V = \infty$ to be an *ordinary point* is $q(V) = O(V^{-4})$ as $V \to \infty$. A sufficient condition for $V = \infty$ to be a *regular singularity* is $q(V) = O(V^{-2})$ as $V \to \infty$. In the case of an *irregular singularity*, $q(V)$ may have an essential singularity. But, if $q(V)$ has at most a pole, then we can speak of an irregular singularity of finite rank. Specifically, the least integer r for which $q(V) = O(V^{2r-2})$ as $V \to \infty$ is called the *rank* of the irregular singularity. Hence, at $V = \infty$, the 3/2 model has a regular singularity. Both the GARCH diffusion and the modified 3/2 model have an irregular singularity of rank one.

In the case of a regular singularity, there is a convergent power series expansion for solutions $f(V)$ of (2.6) in powers of $(1/V)$. In the case of an irregular singularity of rank one, there is an asymptotic power series expansion for $f(V)$, generally divergent. We will establish these facts by developing the series for the three examples, which will also illustrate the general procedure for these two types of singularities. In the first two models, but not the third, we shall see that there are solutions with leading terms "small enough" as $V \to \infty$ to establish that there is a solution to (2.2) that vanishes as $V \to \infty$.

---

[2] This material is based upon Erdélyi (1956, Chapt. III).

(i) *The 3/2 model (regular singularity case)*. In general, with a regular singularity, there exists an expansion $q(V) = \sum_{n=2}^{\infty} q_n V^{-n}$; in our particular case we see from (2.7) that only $q_2, q_3,$ and q_4 are non-vanishing. For any model with a regular singularity, there exists a solution to (2.7) of the form $f(V) = \sum_{n=0}^{\infty} a_n V^{-\alpha-n}$. Substituting this into (2.7) and matching coefficients, the vanishing of the leading term requires

(2.10) $$\alpha(\alpha+1) + q_2 = 0 .$$

This is a quadratic equation that determines α. The vanishing of the next leading term requires that

$$2(\alpha+1)a_1 + q_3 a_0 = 0 .$$

And the vanishing of the general term for $n \geq 4$ requires that

(2.11) $$(\alpha+n-2)(\alpha+n-1)a_{n-2} + q_4 a_{n-4} + q_3 a_{n-3} + q_2 a_{n-2} = 0 .$$

For other models with a regular singularity, there will always be a quadratic equation that determines α and then recurrence relations for the remaining coefficients a_n. Solving (2.10) generates two values for α; in our example,

(2.12) $$\alpha = \alpha(k) = -\tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} - q_2} = -\tfrac{1}{2} \pm \sqrt{\tilde{c} + \tfrac{1}{4}(1+\tilde{\theta})^2} .$$

We perform the k-plane integration in (2.1) along $k = k_r + i/2$. Along this path $\tilde{c} = (k_r^2 + 1/4)/\xi^2$, a real and strictly positive expression everywhere along the contour. So choosing the + sign in (2.12) produces the *smallest* solution as $V \to \infty$. Once α is fixed, then a_1 is determined in terms of an arbitrary a_0. And then (2.10) determines successively $a_2, a_3, \ldots$. As we indicated, this series is convergent within some radius $(1/V) < R$. We won't develop it beyond the leading term because the leading term is sufficient to show that $H(k,V,\tau) \to 0$. (And, we have an exact solution for this case anyway). The leading term is, of course, $f(V) \approx V^{-\alpha}[1 + O(V^{-1})]$. We also have

(2.13) $$\exp\left(-\tfrac{1}{2}\int p(V)dV\right) = V^{\tilde{\theta}/2}\exp\left(\frac{\tilde{\omega}}{2V}\right) = V^{\tilde{\theta}/2}\left[1 + O(V^{-1})\right]$$

Hence, from (2.5):

(2.14) $$u(V) \approx a_0 V^{\tilde{\theta}/2-\alpha} = a_0 V^{-\tilde{\alpha}}\left[1 + O(V^{-1})\right], \text{ using}$$

(2.15) $$\tilde{\alpha} = -\tfrac{1}{2}(1+\tilde{\theta}) + \sqrt{\tilde{c} + \tfrac{1}{4}(1+\tilde{\theta})^2} .$$

Since $\tilde{c} > 0$, we see that the exponent $\tilde{a}$ in (2.15) is strictly positive, and so $u(V) \to 0$ as $V \to \infty$. Also, note that there is *no* $\lambda -$dependence at leading order; the first term with a $\lambda -$dependence in this model is the next-to-leading correction. Hence from (2.3), and using Assumption 2, as $V \to \infty$,

$$(2.15) \qquad \hat{H}(k,V,\tau) = \int_{\lambda \in Sp(\mathcal{L}_k)} g(\lambda) u(V) e^{-\lambda\tau} d\lambda \;\approx\; V^{-\tilde{a}} \varphi(k,\tau) \;,$$

where
$$\varphi(k,\tau) = a_0 \int_{\lambda \in Sp(\mathcal{L}_k)} g(\lambda) e^{-\lambda\tau} d\lambda \,.$$

If our manipulations are valid, then $H(k,V,\tau) \to 0$ as $V \to \infty$. In this case, we can check (2.15) against the exact solution, which is repeated from (2.3.3):

$$(2.16) \qquad \hat{H}(k,V,\tau) = \frac{\Gamma(b-a)}{\Gamma(b)} \left[X\left(\frac{\tilde{\omega}}{V}, \omega\,\tau\right) \right]^a M\left[a,b,-X\left(\frac{\tilde{\omega}}{V}, \omega\,\tau\right)\right],$$

using
$$X(x,t) = \frac{x}{\exp(t)-1}, \quad \mu = \tfrac{1}{2}(1-\hat{\theta}), \quad \delta = \left[\mu^2 + \tilde{c}\right]^{1/2},$$

$$a = -\mu + \delta\,, \qquad b = 1 + 2\delta\,, \qquad \text{and} \qquad \tilde{\omega} = \frac{2}{\xi^2}\omega\,.$$

Taking $V \to \infty$ in (2.16) drives $X \to 0$. Since the confluent hypergeometric function satisfies $M(a,b,z=0) = 1$, the large-V behavior of (2.16) is

$$(2.17) \qquad \hat{H}(k,V,\tau) \approx \frac{\Gamma(b-a)}{\Gamma(b)} \left(\frac{\tilde{\omega}}{\exp(\omega\tau)-1} \right)^a V^{-a} \quad \text{as} \quad V \to \infty\,.$$

It is easy to see that $a = \tilde{a}$, so that (2.17) agrees with (2.15). One can also read off from (2.17) the unknown time dependence $\varphi(k,\tau)$ in (2.15).

(ii) *The GARCH diffusion model (irregular rank one singularity)*. According to Erdélyi, it was discovered by Thomé that a formal asymptotic series solution exists to (2.6) when the singularity at infinity is irregular of finite rank. For the method to work in the case of an irregular rank one singularity, it is necessary that $q_0 \neq 0$ in the expansion $q(V) = \sum_{n=0}^{\infty} q_n V^{-n}$. Instead, from (2.8), you can see that $q_0 = 0$ for the GARCH diffusion. However, one more transformation will bring the problem into the required form, namely

$$(2.18) \qquad x = \sqrt{V} \quad \text{and} \quad f(V) = x^{1/2} y(x)\,.$$

This transforms (2.6) into

$$(2.19) \qquad y'' + \overline{q}(x) y = 0 \qquad \text{where} \qquad \overline{q}(x) = 4x^2 q(x^2) - \tfrac{3}{4}\frac{1}{x^2}\,.$$

A rank one singularity, if it doesn't have $q_0 \neq 0$, will always have $q_1 \neq 0$. Thus, this transformation will always result in a new potential $\overline{q}(V) = \sum_{n=0}^{\infty} \overline{q}_n V^{-n}$, where $\overline{q}_0 \neq 0$. In our example,

$$(2.20) \qquad \bar{q}(x) = -(4\tilde{c} + \tilde{d}^2) + \frac{\tilde{d}(1+2\tilde{\theta})}{x} + \frac{(4\tilde{\lambda} - \frac{3}{4} - 2\tilde{\theta} - \tilde{\theta}^2)}{x^2} + O\left(\frac{1}{x^3}\right).$$

Then, the formal expansion formula is

$$(2.21) \qquad y(x) = e^{\beta x} \sum_{n=0}^{\infty} a_n x^{-\alpha-n}, \qquad a_0 \neq 0$$

Substitution of (2.21) into (2.19) yields the recurrence formula
(2.22)

$$\beta^2 a_n - 2\beta(\alpha + n - 1)a_{n-1} + (\alpha + n - 2)(\alpha + n - 1)a_{n-2} + \sum_{j=0}^{n} \overline{q}_j a_{n-j} = 0,$$

which is valid for all $n \geq 0$ by taking $a_{-m} = 0$, $m = 1, 2, \ldots$. The first condition arises when $n = 0$. Since $a_0 \neq 0$, we have

$$(2.23) \qquad \beta^2 + \overline{q}_0 = 0.$$

So $\beta^2 = -\bar{q}_0 = (4\tilde{c} + \tilde{d}^2)$. In our case, the smaller solution is given by

$$\beta = -\frac{2}{\xi}\eta(k), \text{ defining } \eta(k) = \sqrt{(k^2 - ik) + \rho^2(1 - \gamma + ik)^2}.$$

The next condition, at $n = 1$, is given by

$$(2.24) \qquad -2\beta\alpha + \overline{q}_1 = 0 \Rightarrow \quad \alpha = \frac{\tilde{d}(1+2\tilde{\theta})}{2\beta}.$$

Using (2.23) and (2.24), then (2.22) may be simplified to

$$(2.25) \quad 2\beta n a_n = (\alpha + n)(\alpha + n - 1)a_{n-1} + \sum_{j=2}^{n+1} \overline{q}_j a_{n+1-j} = 0, \qquad n = 1, 2, \ldots$$

Taking $n = 1$ in (2.25) yields $2\beta a_1 = [\alpha(\alpha + 1) + \overline{q}_2] a_0$ or

$$(2.26) \qquad a_1 = \frac{1}{2\beta}\left[\alpha(\alpha+1) + (4\tilde{\lambda} - \tfrac{3}{4} - 2\tilde{\theta} - \tilde{\theta}^2)\right] a_0.$$

So we have the smaller solution

$$(2.27) \qquad y(x) \approx e^{\beta x}\left(a_0 x^{-\alpha} + a_1 x^{-\alpha-1} + \ldots\right),$$

which yields

$$(2.28)\qquad f(V) \approx \exp\left(\beta\sqrt{V}\right)V^{\frac{1}{4}-\frac{\alpha}{2}}\left(a_0 + \frac{a_1}{V^{1/2}} + \frac{a_2}{V} + \ldots\right).$$

Since

$$(2.29)\qquad \exp\left(-\tfrac{1}{2}\int p(V)dV\right) = V^{\tilde{\theta}/2}\exp\left(\frac{\tilde{\omega}}{2V} - \tilde{d}\sqrt{V}\right),$$

we have

$$(2.30)\qquad u(V) \approx \exp\left[(\beta - \tilde{d})\sqrt{V}\right]V^{\frac{1}{4}-\frac{\alpha}{2}+\frac{\tilde{\theta}}{2}}\left(a_0 + \frac{a_1}{V^{1/2}} + O(V^{-1})\right).$$

Again we find that the leading term in the large V expansion for $u(V)$ is independent of λ. The $\lambda-$dependence, as one sees from (2.26), does not appear until the next-to-leading correction. (The reader is warned that this does not always happen; in some models, there is λ- dependence in the leading term; an example is the square root model). Consequently, arguing as in the previous example, we expect the following large-V behavior:

(2.31)

<u>GARCH Diffusion model:</u>

$$\boxed{\hat{H}(k,V,\tau) \underset{V\to\infty}{\approx} \exp\left[(\beta - \tilde{d})\sqrt{V}\right]V^{\frac{1}{4}-\frac{\alpha}{2}+\frac{\tilde{\theta}}{2}}\varphi(k,\tau)}$$

with some $\varphi(k,\tau)$ that is not determined by this procedure. Consequently, $H(k,V,\tau) \to 0$ *if* $\mathrm{Re}(\beta - \tilde{d}) < 0$. For example, take the case $\gamma = 1$. Then, the inequality reads

$$(?)\qquad \mathrm{Re}\sqrt{k^2(1-\rho^2) - ik} > \mathrm{Re}(ik\rho).$$

Along the k-plane integration contour, $k = i/2 + k_r$, and this reads

$$\mathrm{Re}\sqrt{(1+\rho^2) + 4k_r^2(1-\rho^2) - 4i\rho^2 k_r} > -\rho\,,$$

which is indeed true for all $|\rho| \leq 1$ and all real k_r. So we conclude that $C(S,V,\tau) \to S$ as $V \to \infty$ in the GARCH diffusion model.

Equation (2.31) is quite important. It was used in setting up the solver for the leading eigenvalue/eigenfunction of $H(k,V,\tau)$ in Chapter 6, Sec. 8. That eigenvalue determined the asymptotic implied volatility as $\tau \to \infty$. The leading eigenfunction also determines the risk adjustment in the case of a representative agent who is a pure investor with a distant horizon. We have confirmed that (2.31) also holds for the case of geometric Brownian motion. (See Chapter 11 for that). Geometric Brownian motion is the same as the GARCH diffusion, except that $\omega = 0$. But ω does not influence the leading behavior of the fundamental transform as $V \to \infty$

(iii) ***The modified 3/2 model (irregular rank one singularity).*** The computations are similar to the GARCH diffusion case; one makes the same transformation (2.18), with the result

$$\overline{q}(x) = -1 - \frac{1}{x} - \frac{(4c + \frac{3}{4})}{x^2} + \frac{4\lambda}{x^4}.$$

Solving (2.23) yields the smaller solution $\beta = -1$ and solving (2.24) yields $\alpha = 1/2$. The first coefficient recursion (2.26) yields $a_1 = 2ca_0$. Again there is no λ- dependence through this order. But now

$$\exp\left(-\tfrac{1}{2}\int p(V)dV\right) = \exp\left(\sqrt{V}\right),$$

which cancels the exponential decay in $y(x)$. So now we have

$$f(V) = x^{1/2}y(x) \approx a_0\left(1 + \frac{2c}{\sqrt{V}} + O(V^{-1})\right),$$

or, in other words, as $V \to \infty$,

(2.33) $$H(k,V,\tau) \approx \left(1 + \frac{2c}{\sqrt{V}}\right)\varphi(k,\tau) \ .$$

So, what is qualitatively different in this case is that the fundamental transform does not die off with large volatility. From equation (2.1), we then expect the call option price to tend to a value strictly less than the stock price as $V \to \infty$. More specifically, from the homogeneity of degree 1 in prices of (2.1), we have $C(S,V,\tau) \approx S\, f_1(X,\tau)$, where $f_1 < 1$ for $\tau > 0$.

To further test the conjecture that $C/S < 1$ as $V \to \infty$ for the modified 3/2 model, we have performed Monte Carlo simulations to estimate call option prices at large volatility. The program was based upon the Monte Carlo method that followed from the mixing theorem, as explained in Chapter 4. We have taken $T = 10$ days to option expiration, where there are 250 days per year, and the volatilities are measured on an annual basis. Note that (2.33), when inserted into (2.1), implies that $C(S,V,\tau) \approx A + B/\sqrt{V}$, as $V \to \infty$. Applying this to two different large values V_n and V_{n-1}, where $V_n = 10V_{n-1}$, yields the extrapolation formula

(2.34) $$C_\infty = 1.4625\, C_n - .4625\, C_{n-1}$$

Results are shown in Table 10.1 below. The table supports the conjecture with an extrapolated final estimate approximately at 33.26, well below 100.

Table 10.1 Monte Carlo Estimates for Call Option Prices as $V \to \infty$. Volatility process: $dV_t = -V_t^{5/2}dt + \xi V_t^{3/2}dW_t$

Volatility V	Number of MC Runs	Steps per day	MC Call Price $C(S,V,T)$	MC Stand. Error	B-S Call Price	Run-time (secs)
1	1000	10	7.88	0.001	7.97	2
10	1000	10^2	20.08	0.008	24.82	18
10^2	1000	10^3	28.95	0.006	68.27	176
10^3	100	10^4	31.93	0.016	99.84	169
10^4	10	10^5	32.84	0.05	100.00	169
10^5	10	10^6	33.13	0.03	100.00	1728
∞	Extrapolated :		33.26			

Notes. Model parameters: $T = 10/250$ years, $\xi_a = 1$, $S = K = 100$, $r = \delta = 0$. With 250 days per year, the number of time steps per day is always $10 \times V$. The $V = \infty$ entry is an extrapolation based upon the assumed asymptotic form $C \approx a + b/\sqrt{V}$. The table entries are for the parameter $\xi_a = 1$; however, the $V \to \infty$ call value can be calculated exactly in the $\xi = 0$ limit (see text), with the result $C \cong 34.18$. The results support the conjecture that, as $V \to \infty$, $C(S,V,T) = f_1(T)S$, where $f_1(T) < 1$ in this model. Hence the model provides a counter-example to the more common limit $C(S,V,T) \to S$.

Note that, as we increase the volatility in the Monte Carlo simulations, we also decrease the time step, in order to preserve $V_0\Delta t$. For very large volatilities, the time step becomes very small and we can only do a small number of simulations. One might think that this would lead to very large Monte Carlo errors. But in fact, the observed errors, at a fixed number of drawings , become smaller as $V \to \infty$. Our interpretation of this behavior is that this process is becoming dominated by the deterministic limit as $V \to \infty$. In the deterministic limit, only a *single* Monte Carlo run is needed because each run is identical. As further evidence for this idea, we note that the final estimated call prices is close to, but apparently below, the deterministic limit value which can be calculated exactly. The deterministic limit of all three examples is discussed below.

The boundary classification is not determinative. We have also calculated the boundary classification for the modified 3/2 process. In the notation of chapter 9, we find that $S(c,\infty)=\infty$, $M(c,\infty)<\infty$, $\Sigma(\infty)=\infty$, and $N(\infty)<\infty$. Hence, from Table 9.2, $V=\infty$ is an *entrance* boundary. That is, $V=\infty$ may not be reached in finite time, but the process may be started there. Another case of an entrance boundary, from Table 9.3 is the GARCH diffusion under log-utility with a positive stock-volatility correlation. But for the GARCH diffusion, we expect that $C\to S$ and $P\to Ke^{-rT}$. To confirm this numerically, we have estimated put prices for the GARCH diffusion in Table 10.2 below. We estimate put prices because their Monte Carlo errors are much smaller for the reasons discussed in Chapter 9. Both positive and negative correlations are reported. We take $r=0$, $S=K=100$; as one sees, there is every indication that $P\to 100$ as $V\to\infty$ for either sign of the correlation. Since, $P\to 100$ implies $C\to 100$ by put-call parity, we see that the boundary classification per se does not determine whether $C<S$ or $C=S$ for large volatility.

Table 10.2 Monte Carlo Estimates for Put Option Prices as $V\to\infty$. Model: GARCH Diffusion under Log-utility.

Volatility V	B-S Price	Put option prices for various correlations ρ		
		$\rho=-0.5$	$\rho=0$	$\rho=0.5$
1	7.97	7.98 (.04)	7.95 (.001)	7.94 (.03)
10	24.82	24.78 (.07)	24.56 (.003)	24.39 (.04)
10^2	68.27	68.87 (.11)	67.63 (.002)	66.53 (.07)
10^3	99.84	99.92 (.01)	99.80 (.003)	99.62 (.01)

Notes. The volatility process is $dV_t=(\omega-\theta V_t-\rho\xi V_t^{3/2})dt+\xi V_t dW_t$. Model parameters are $T=10/250$ years, $\omega_a=\theta_a=\xi_a=1$, $S=K=100$, $r=\delta=0$. Results are from 1000 Monte Carlo (MC) runs. With 250 days per year, the number of time steps per day is always $10\times V$. MC standard errors are in parenthesis. The results support the conjecture that $P(S,V,T)\to K$ as $V\to\infty$ for this model.

Asymptotica for the deterministic case. Previously, in Appendix 3.1, we presented the solution for the fundamental transform in the deterministic limit. For convenience, that solution is

$$(2.35) \qquad H(k,V,\tau) = \exp\left(-c(k)\int_0^\tau Y(u,V)du\right) = \exp[-c(k)v(V,\tau)\tau],$$

where $Y(t,V)$ is the solution to the ordinary differential equation $dY = b(Y)dt$, starting at $Y(0) = V$. The effective volatility $v(V,\tau) = (1/\tau)\int_0^\tau Y(u,V)du$ is the time average of the deterministic volatility. We also know that the call option formula under deterministic volatility is the B-S formula with the effective volatility substituted for the constant volatility. We can summarize the implications of (2.35) with the

THEOREM 10.1. *With* $0 < \tau < \infty$, *the following three statements are equivalent under deterministic volatility::*

(2.36)
(i) $v(V,\tau) \to \infty$ *as* $V \to \infty$.
(ii) $H(k,V,\tau) \to 0$ as $V \to \infty$, *where* $\mathrm{Im}\, k = 1/2$.
(iii) $C(S = Ke^{-r\tau}, V, \tau) \to S$ *as* $V \to \infty$ *(no dividend case).*

PROOF: (i) $\Leftrightarrow$ (ii) because $0 < c(k) < \infty$ when $\mathrm{Im}\, k = 1/2$ in (2.34). (i) $\Leftrightarrow$ (iii) because taking the volatility parameter to infinity in the B-S formula yields the stock price when $\tau > 0$. But if the volatility and time to expiration are finite then $C_{BS} < S$. Finally (iii) $\Leftrightarrow$ (ii) because (iii) tells us that the integral in (2.1) must be zero. From (2.35), since $c > 0$ and $Y(u,V) \geq 0$, then $H(k,V,\tau)$ is strictly real and non-negative along the integration contour. When $S = Ke^{-r\tau}$, then $X = 0$ and the denominator $k^2 - ik$ is real and strictly positive. Hence, the entire integrand is non-negative everywhere along the contour. That means the only way the integral can vanish is for $H(k,V,\tau)$ to vanish for all k along the contour.

Example 2 (continued). For the GARCH diffusion $b(V) = \omega - \theta V$,

$$Y(\tau,V) = (V - \frac{\omega}{\theta})e^{-\theta\tau} + \frac{\omega}{\theta} \quad \text{and} \quad v(V,\tau) = \frac{1}{\theta\tau}(V - \frac{\omega}{\theta})(1 - e^{-\theta\tau}) + \frac{\omega}{\theta}.$$

Hence as $V \to \infty$, both $Y(\tau,V) \to \infty$ and $v(V,\tau) \to \infty$. So $C(S,V,\tau) \to S$.

Example 1 (continued). For the 3/2 model, $b(V) = \omega V - \theta V^2$,

$$Y(\tau,V) = \frac{\omega}{\theta + \left(\frac{\omega}{V} - \theta\right)e^{-\omega\tau}} \quad \text{and} \quad v(V,\tau) = \frac{1}{\theta\tau}\ln\left[1 + \frac{V\theta}{\omega}(e^{\omega\tau} - 1)\right].$$

Hence as $V \to \infty$, $Y(\tau,V) < \infty$, but $v(V,\tau) \to \infty$. So $C(S,V,\tau) \to S$.

Example 3 (continued) For the modified 3/2 model, $b(V) = -V^{5/2}$,

$$Y(\tau,V) = \frac{V}{\left[1 + \frac{3}{2}V^{3/2}\tau\right]^{2/3}} \quad \text{and} \quad v(V,\tau) = \frac{2}{\tau V^{1/2}}\left[\left(1 + \frac{3}{2}V^{3/2}\tau\right)^{1/3} - 1\right].$$

Hence as $V \to \infty$,

$$Y(\tau,V) \approx (2/3\tau)^{2/3} < \infty \qquad \text{and} \qquad v(V,\tau) \approx 3(2/3)^{2/3}\tau^{-2/3} < \infty$$

Since $v(V,\tau) < \infty$, then $C(S,V,\tau) < S$. In fact, by substituting $v(V,\tau)$ into the B-S formula, we have the exact result

(2.37) $$\lim_{V\to\infty} C(S = K,V,\tau) = S[\Phi(d_1) - \Phi(-d_1)],$$

where $$d_1 = \tfrac{1}{2}3^{1/2}(\tfrac{2}{3})^{1/3}\tau^{1/6}.$$

For the computations in Table 10.1, (2.37) yields

(2.38) $$\lim_{V\to\infty} C(S = K = 100, \tau = 10/250) \cong 34.18$$

It's interesting that this is so close to the final extrapolated value in the table, which is based upon $\xi = 1$ instead of $\xi = 0$.

From Theorem 10.1, $C \to S$ or $C < S$ depending on whether $v(V,\tau) \to \infty$ or $v(V,\tau) < \infty$. We can settle the deterministic case for all possible power law behaviors by an integration:

THEOREM 10.2. *With* $0<\tau<\infty$, *suppose the volatility process is deterministic and given by* $dV=-V^{p}dt$, *with* p *a constant independent of* V. *Then, as* $V\to\infty$, $C\to f_1(\tau)S$, *where*

(*i*) $f_1(\tau)=1$ *when* $p\le 2$;
(*ii*) $f_1(\tau)<1$ *when* $p>2$.

PROOF: Integrating $dY/d\tau=-Y^{p}$, $Y(0)=V$, yields

$$Y(\tau,V)=\begin{cases}\dfrac{V}{\left[1+(p-1)V^{p-1}\tau\right]^{1/(p-1)}}, & p\neq 1\\ Ve^{-\tau}, & p=1\end{cases}.$$

Integrating again yields

$$v(V,\tau)=\begin{cases}\dfrac{1}{\tau}\dfrac{V^{2-p}}{p-2}\left\{\left[1+(p-1)V^{p-1}\tau\right]^{\frac{p-2}{p-1}}-1\right\}, & p\neq 1,2\\[2ex] \dfrac{1}{\tau}\ln[1+V\tau], & p=2\\[2ex] V(1-e^{-\tau}), & p=1\end{cases}$$

By examining the various cases, one can show that $v(V,\tau)\to\infty$ with V for $p\le 2$ and $v(V,\tau)\to g(\tau)<\infty$ for $p>2$. Here $g(\tau)$ is just some finite function whose precise form is not important. Then, substitution of $v(V,\tau)$ into the B-S formula yields the assertion.

11 Solutions to Models

Three models have served as working examples throughout the book: the square root model, the 3/2 model and the GARCH diffusion model. This chapter presents the solutions for the fundamental transform for the first two. We show a closed-form solution for the GARCH diffusion process only in a special case: when the volatility process is geometric Brownian motion. Once you have the fundamental transform, then option prices are obtained by an integration.

1 The Square Root Model

This model has played an important role in finance in many applications. It was originally developed by Feller. It was applied to the problem of options under stochastic volatility by Heston (1993). He postulated the variance process:

(1.1) $$P: \; dV = (\omega - \theta\, V)\, dt + \xi \sqrt{V} dW(t),$$

where ω, θ, ξ and the correlation ρ with the stock-price process are constants. We have shown in Chapter 7 that the model can survive in the same form after risk adjustment by a representative investor. Specifically, if the representative is a "pure investor" with a distant horizon, and has power utility with CPRA parameter γ, then the martingale pricing process for (1.1) becomes

(1.2) $$\tilde{P}: \; dV = (\omega - \tilde{\theta} V)\, dt + \xi \sqrt{V} d\tilde{W}(t),$$

where $$\tilde{\theta} = (1-\gamma)\rho\xi + \sqrt{\theta^2 - \gamma(1-\gamma)\xi^2}\;.$$

Hence, the PDE for the fundamental transform (**2**.2.19) for this model is

(1.3) $$\frac{\partial \hat{H}}{\partial \tau} = \tfrac{1}{2}\xi^2 V \frac{\partial^2 \hat{H}}{\partial V^2} + [(\omega - \tilde{\theta} V) - ik\rho\xi V]\frac{\partial \hat{H}}{\partial V} - c(k) V \hat{H}$$

where $c(k) = (k^2 - ik)/2$ and we need $\hat{H}(k,V,\tau = 0) = 1$.

It is convenient to employ the reduced variables $t = \xi^2\tau/2$, $\tilde{\omega} = 2\omega/\xi^2$, $\hat{\theta}(k) = 2(\theta + ik\rho\xi)/\xi^2$ and $\tilde{c}(k) = 2c(k)/\xi^2$. Also, we can write $\hat{H}(k,V,\tau) = f(V,t)$, where the parametric k-dependence of $f(V,t)$ is understood and not indicated explicitly. With this notation, (1.3) becomes

(1.4) $$\frac{\partial f}{\partial t} = V\frac{\partial^2 f}{\partial V^2} + (\tilde{\omega} - \hat{\theta} V)\frac{\partial f}{\partial V} - \tilde{c}\, V f$$

Parabolic differential equations of this form where (i) all the "space" coefficients are at most linear in the space variable V, and (ii) a constant initial value may be solved by a solution of the form:

(1.5) $$f(V,t) = \exp[A(t) + B(t)\, V]$$

Substituting (1.5) into (1.4) and matching coefficients of the constant term and the term linear in V leads to a pair of differential equations

(1.6) $$\dot{A}\,(t) = \tilde{\omega} B(t), \quad \text{and} \quad \dot{B} = B^2 - \hat{\theta} B - \tilde{c}\,,$$

using a dot for the t-derivative. The initial condition is $A(0) = B(0) = 0$. These equations are easily integrated and the result was given at (**2**.3.2).

2 The 3/2 Model

Under the same utility model for the square root model, the 3/2 model has the risk-adjusted variance process

(2.1) $$\tilde{P}:\ dV = (\omega V - \tilde{\theta} V^2)\,dt + \xi V^{3/2} d\tilde{W}(t)\,,$$

where $$\tilde{\theta} = -\tfrac{1}{2}\xi^2 + (1-\gamma)\rho\xi + \sqrt{(\theta + \tfrac{1}{2}\xi^2)^2 - \gamma(1-\gamma)\xi^2}\,,$$

and ω, θ, ξ, and ρ are constants. As usual, ρ is the correlation with the stock price process. Hence, the PDE for the fundamental transform for this model is

(2.2) $$\frac{\partial \hat{H}}{\partial \tau} = \tfrac{1}{2}\xi^2 V^3 \frac{\partial^2 \hat{H}}{\partial V^2} + \left[(\omega V - \tilde{\theta} V^2) - ik\rho\xi V^2\right]\frac{\partial \hat{H}}{\partial V} - c(k) V \hat{H}$$

Introduce $s = \xi^2\tau/2$, $\tilde{\omega} = 2\omega/\xi^2$, $\tilde{c}(k) = 2c(k)/\xi^2$, $\hat{\theta}(k) = 2(\theta + ik\rho\xi)/\xi^2$.

Also let $H(k,V,\tau) = f(V,s)$. Then, (2.2) is equivalent to

$$\frac{\partial f}{\partial s} = V^3 \frac{\partial^2 f}{\partial V^2} + (\tilde{\omega} V - \hat{\theta} V^2)\frac{\partial f}{\partial V} - \tilde{c}\, V f\,.$$

Unfortunately a couple more variable changes are needed. We could do them all at once, but it would be harder to follow. First, change from (V,s) to (y,t) using $y = \tilde{\omega}/V$ and $t = \tilde{\omega}\, s$. In this notation, $f(V,s) = u(y,t)$, where $u(y,t)$ is the solution to

(2.3) $$u_t = y\, u_{yy} + (2 + \hat{\theta} - y)u_y - \frac{\tilde{c}}{y} u\,, \qquad u(y,0) = 1\,.$$

We are using subscripts for derivatives. For the second change, let $u(y,t) = y^{\nu} \exp[(1-\nu)t] g(y,t)$, choosing ν to satisfy the equation $\nu^2 + (1+\hat{\theta})\nu - \tilde{c} = 0$. The solution with the negative root is chosen; it will have an advantageous property shown below. Taking

$$\nu = -\mu - \delta\,, \text{ where } \mu = \tfrac{1}{2}(1+\hat{\theta}) \text{ and } \delta = \sqrt{\mu^2 + \tilde{c}}\,,$$

then $g(y,t)$ satisfies

(2.4) $$g_t = (yg)_{yy} - ((y+\beta)g)_y\,, \qquad g(y,0) = y^{-\nu}\,, \qquad \beta = 1 + 2\delta\,.$$

As shown by Feller (1951), (2.4) may be solved by a method that works because of the linearity of the coefficients, but does not work in more general cases. The method is to apply the Laplace transform with respect to y:

(2.5) $$G(x,t) = \mathcal{L}[g] = \int_0^{\infty} e^{-xy} g(y,t)\,dy\,.$$

When you apply the Laplace transform, there's a parts integration term, which is called by Feller the *boundary flux* at the origin:

$$F_0(t) = \lim_{y \to 0} \left[(yg)_y - (y+\beta)g\right]\,.$$

If the boundary flux vanishes (and it does for our solution— see below), then taking the Laplace transform of (2.4), implies that $G(x,t)$ satisfies

(2.6) $$G_t = x(1-x)G_x - \beta x G\,.$$

And we have $$G(x,0) = \int_0^{\infty} e^{-xy} y^{-\nu} dx = \Gamma(1-\nu)x^{\nu-1}\,,$$

using $\Gamma(z)$, the Gamma function. Now (2.6) is a first-order partial differential equation. It may be solved by the method of characteristics explained in Chapter 1, Appendix 2. For that method, one first solves

$$\frac{dX}{dt} = X(1-X), \qquad X(t=0) = x.$$

which is easy to integrate and has the solution

$$X(x,t) = \frac{xe^t}{1+x(e^t-1)}.$$

Also, it is easy to show that

$$\exp\left[-\beta\int_0^t X(x,s)ds\right] = \left[1+x(e^t-1)\right]^{-\beta}.$$

Hence, the solution to (2.6), using (1.A2.10), is given by

$$G(x,t) = \Gamma(1-\nu)\left[X(x,t)\right]^{\nu-1}\left[1+x(e^t-1)\right]^{-\beta} \tag{2.7}$$

The conventional notation for a Laplace transform/inversion pair is:

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty \exp(-st)f(t)dt \quad \text{and} \quad f(t) = \mathcal{L}^{-1}[F(s)].$$

Using that notation, the Laplace inversion of (2.7) may be accomplished by the inversion formula

$$\mathcal{L}^{-1}\left[s^{-a}(s-c)^{-b}\right] = \frac{t^{a+b-1}}{\Gamma(a+b)}M(b,a+b,ct), \qquad \text{Re}(a+b)>0, \tag{2.8}$$

where $M(a,b,z)$ is a confluent hypergeometric function. In our application of (2.8), we have $a=1-\nu$, $b=\beta+\nu-1=\alpha$, and $a+b=\beta$. We show below that Re $\beta>1$, so that the condition in (2.8) is valid. These formulas lead to

$$u(y,t) = \frac{\Gamma(\beta-\alpha)}{\Gamma(\beta)}[X(y,t)]^{\alpha}M[\alpha,\beta,-X(y,t)], \tag{2.9}$$

which is easily transformed into the result given at (**2**.3.3).

The boundary flux. Now (2.9) implies that, as $y\to 0$, then $g(y,t)\approx y^{\beta-1}$. Now $\beta-1 = 2[\mu^2+\tilde{c}]^{1/2}$. It is conventional to define the square root of a complex number z such that $\text{Re}\sqrt{z}>0$ everywhere in the z-plane excluding a branch cut along the negative real axis. With this same convention, Re $\beta>1$, which is sufficient to guarantee that the boundary flux at the origin vanishes.

3 Geometric Brownian Motion

Geometric Brownian motion has the actual volatility process:

(3.1) $$P:\ dV = \alpha V dt + \xi V dW(t),$$

where α, ξ, and the correlation ρ are constants. We will not make a risk adjustment, although the solution given can be adapted to log-utility with just changes in notation. Our results provide a closed-form solution and explanation of results by Hull and White (1987). The fundamental transform is the solution to

(3.2) $$\frac{\partial \hat{H}}{\partial \tau} = \tfrac{1}{2}\xi^2 V^2 \frac{\partial^2 \hat{H}}{\partial V^2} + (\alpha V - ik\rho\xi V^{3/2})\frac{\partial \hat{H}}{\partial V} - c(k) V \hat{H}$$

with the usual initial condition $\hat{H}(k,V,\tau=0)=1$. First we give the solution and then sketch very briefly how it was obtained.

Notation. We use the Pochammer symbol $(a)_n$, where $(a)_0 = 1$, $(a)_1 = a$, $(a)_2 = a(a+1)$, etc. Also appearing is the special function ${}_2F_1(a,b;c;z)$, the Gauss hypergeometric function, which is defined by the series

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \qquad |z|<1,$$

and can be analytically extended to the entire complex z-plane. See the Bateman Manuscript Project (1953) for more properties. We find a solution of the eigenfunction expansion type:

(3.3) $$H(k,V,\tau) = g_1(k,V,\tau) + g_2(k,V,\tau),$$

where g_1 is a discrete spectrum contribution and g_2 is a continuous spectrum contribution. There is a another discrete spectrum contribution that is omitted because it frequently vanishes. In particular, if $\rho = 0$, the omitted term always vanishes. Below the solution is Fig. 11.1, a graph of the parameter region in which the given solution is valid.

Both terms use $\beta = 2\alpha/\xi^2$, $\mu = \beta - 1$, and $x = \sqrt{V}/\xi$. We also use $[\cdots]$ in a summation for integer part. The discrete spectrum term vanishes if $\beta \geq 1$ $(2\alpha \geq \xi^2)$. Otherwise, it's given by

(3.4) $$g_1(k,V,\tau) = \delta_{(\beta<1)} \sum_{n=0}^{[1-\beta]} f_n(k)\, x^{r_n}\, \gamma^{b_n - 1}\, U(a_n, b_n, \gamma\, x) \exp\left(wx + \tfrac{1}{2} s_n \xi^2 \tau \right)$$

using $\eta(k) = [k^2(1-\rho^2) - ik]^{1/2}$, $\gamma(k) = 4\eta(k)$, $w(k) = -2\eta(k) + 2ik\rho$,

$$a_n(k) = \tfrac{3}{2} - \beta - n - \frac{ik\rho}{\eta(k)}(\beta - \tfrac{1}{2}), \qquad b_n = 3 - 2\beta - 2n,$$

$$s_n = \tfrac{1}{4} n^2 + \tfrac{1}{2} n\mu, \qquad r_n = -\mu + [\mu^2 + 4 s_n]^{1/2},$$

$$f_n(k) = -2(\mu + n) \frac{\Gamma(a_n)}{\Gamma(b_n)} \sum_{i=0}^{n} \frac{(b_n - a_n)_{n-i}}{(b_n)_{n-i}} \frac{(-\gamma)^{n-i}}{(n-i)!} \frac{(-w)^i}{i!}.$$

The continuous spectrum term is given by

(3.5)

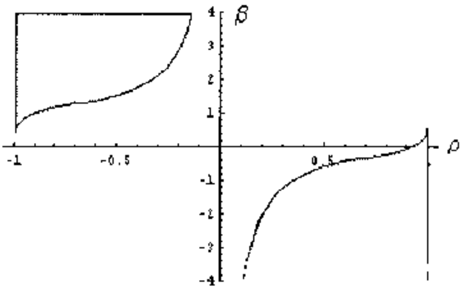
$$g_2(k,V,\tau) = \int_0^\infty f(\nu,k)\, (\gamma\, x)^{\mu + i\nu}\, U(a_\nu, b_\nu, \gamma\, x)\, \exp\left[wx - \tfrac{1}{8}(\mu^2 + \nu^2)\xi^2 \tau \right] d\nu$$

using

$$a_\nu(k) = \tfrac{1}{2} + i\nu - \frac{ik\rho}{\eta(k)}(\beta - \tfrac{1}{2}), \quad b_\nu = 1 + 2i\nu, \quad \delta(k) = (\beta - \tfrac{1}{2})\left(1 - \frac{ik\rho}{\eta(k)}\right),$$

$$f(\nu,k) = \frac{\nu \sinh(2\pi\nu)}{\pi^2} \frac{|\Gamma(\mu + i\nu)|^2}{\Gamma(\delta)} \Gamma(a_\nu)\, \Gamma(a_{-\nu})\ {}_2F_1\left(\mu + i\nu, \mu - i\nu;\ \delta\,; -\frac{w}{\gamma} \right)$$

Fig. 11.1 Valid Solution Region (white areas)



Zero Correlation. The solution is obviously pretty unwieldy and its hard to work with, even in Mathematica where all the functions are built in. It becomes simpler when the correlation vanishes. In that case, the PDE becomes

$$\frac{\partial \hat{H}}{\partial \tau} = \tfrac{1}{2}\xi^2 V^2 \frac{\partial^2 \hat{H}}{\partial V^2} + \alpha V \frac{\partial \hat{H}}{\partial V} - c(k)\, V \hat{H}\,, \tag{3.6}$$

We still use $\beta = 2\alpha / \xi^2$ and $\mu = \beta - 1$. But now we use $y = 2\sqrt{2c(k)V} / \xi$ and $\lambda_j = -\mu j - j^2$. Then, the solution becomes

(3.7)

$$\hat{H}(c,V,\tau) = 2\left(\frac{y}{2}\right)^{-\mu} \left\{ \delta_{(\beta<1)} \sum_{j=0}^{[(1-\beta)/2]} \frac{-\mu - 2j}{j!\,\Gamma(2-\beta-j)} K_{-\mu-2j}(y) \exp\left(-\tfrac{1}{2}\lambda_j \xi^2 \tau\right) \right.$$
$$\left. + \frac{1}{4\pi^2}\int_0^\infty \left|\Gamma\left(\frac{\mu+i\nu}{2}\right)\right|^2 (\nu \sinh \nu\pi)\, K_{i\nu}(y)\, \exp\left[-\tfrac{1}{8}(\mu^2+\nu^2)\,\xi^2\tau\right] d\nu \right\}.$$

The confluent hypergeometric functions of the general case have turned into Bessel functions.

Other applications of (3.7)

1. Mixing . It's possible to take the inverse Laplace transform of $\hat{H}(c,V,\tau)$ with respect to c. That's how we obtained the mixing solution example in Chapter 4, Sec. 7. With the condition $\mathrm{Re}\,\alpha > 0$, the inversion formula we used was[1]:

$$\mathcal{L}^{-1}\left\{2\alpha^{1/2}\, s^{\mu-1} K_{2\nu}\left[2(\alpha\, s)^{1/2}\right]\right\} = t^{-\mu+1/2} \exp\left(-\frac{\alpha}{2t}\right) W_{\mu-1/2,\nu}\left(\frac{\alpha}{t}\right).$$

Here $W_{k,\nu}(z)$ is a Whittaker function:

$$W_{k,\nu}(z) = \exp\left(-\frac{z}{2}\right) z^{\nu+1/2}\, U\left(\tfrac{1}{2}+\nu-k,\ 1+2\,\nu,\ z\right).$$

So you end up with the U functions again.

2. Interest rates. When $c = 1$, (3.6) is mathematically equivalent to an interest rate term structure model studied by Dothan (1978). The solution (3.7) agrees with Dothan's when $\alpha = 0$, but apparently differs when $2\alpha < -\xi^2$.

How to derive the solution. This is a very rough outline. The method is an eigenfunction expansion technique and follows the general steps explained in

---

[1] See Bateman Manuscript (1954) , page 283, equation (5.16)

Lewis (1998). After the change of variable: $\hat{H}(V,\tau) = g(x,t)$, where $x = \sqrt{V}/\xi$ and $t = \xi^2\tau/2$, introduce the Laplace transform pair:

$$G(x,s) = \int_0^\infty e^{-st} g(x,t)dt \quad \text{and} \quad g(x,t) = \frac{1}{2\pi i}\int_{s_r - i\infty}^{s_r + i\infty} e^{st} G(x,s)ds \, .$$

In the second integral, s_r is a real constant chosen so that the inversion path lies to the right of singularities in the complex s-plane. Then, (3.2) becomes the ordinary differential equation

$$\text{(3.8)} \qquad \mathcal{L}\,G = \tfrac{1}{4}x^2 G'' + \left[\left(\tfrac{\beta}{2} - \tfrac{1}{4}\right)x - ik\rho x^2\right]G' - \left[s + 2c\,x^2\right]G = -g_0(x)\,,$$

where the primes are x-derivatives. From the initial condition, one has $g_0(x) = 1$, but that is substituted later. A Green function-style solution to (3.8) is given by

$$\text{(3.9)} \qquad G(x,s) = G_2(x,s)\int_0^x p(y,s)G_1(y,s)g_0(y)dy$$
$$+ G_1(x,s)\int_x^\infty p(y,s)G_2(y,s)g_0(y)dy \, .$$

The expression uses $p(x,s) = 4/[x^2 W(x,s)]$, where $W(x,s)$ is the Wronskian of two independent solutions $G_1(x,s), G_2(x,s)$ of the homogeneous equation $\mathcal{L}\,G = 0$. The integrals in (3.9) do not exist for an arbitrary choice of solutions. Roughly speaking, one wants G_1, G_2 to be the solutions which are relatively small near $x = 0, \infty$ respectively.

More precisely, consider the function spaces $L_p(0,X)$ and $L_p(X,\infty)$, where $f \in L_p(X,Y)$ means $\int_X^Y p(y)\,|\,f(y)\,|^2\,dy < \infty$. Choose $G_1 \in L_p(0,X)$, and $G_2 \in L_p(X,\infty)$. Such solutions always exist by Weyl's limit point/limit circle theory for complex s, except for isolated singularities. When the integrable solutions are unique, which is the case here, no boundary conditions may be specified at $x = 0$ or $x = \infty$. For a brief review of this theory, see Weyl (1950); for a more thorough discussion, see Hille (1969). The integrals in (3.9) exist for almost all complex s. When $g_0 \notin L_p(0,\infty)$, there arise new poles in the complex s-plane in addition to the singularities that occur when $g_0 \in L_p(0,\infty)$.

Proceeding in this manner, we reduce $\mathcal{L}\,G = 0$ to the confluent hypergeometric equation and find solutions G_1 and G_2 of the form

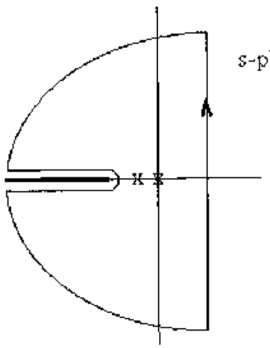
$$G_1 = \exp(wx)(\gamma x)^r M(a,b,\gamma x) \in L_p(0,X)\,, \qquad \mathrm{Im}\, s \nless 0\,,$$
$$G_2 = \exp(wx)(\gamma x)^r U(a,b,\gamma x) \in L_p(X,\infty)\,,$$

When $g_0 = 1$ and $\mathrm{Im}\, k > 0$, then $g_0 \in L_p(0,\infty)$ requires $\beta > 1$ and $\rho < 0$. When $g_0 \in L_p(0,\infty)$, the only singularities in the complex s-plane are poles due to the vanishing of the Wronskian and a branch-cut singularity on the real s-axis. For example, the treatise of Titchmarsh (1962) deals exclusively with this case.

When $g_0 = 1$ and $\beta < 1$ we find that the integrals above have additional singularities in the s-plane. Mathematically, this behavior is known as enlargement of the point spectrum when an operator is extended.

We find three types of singularities in the s-plane: (i) poles due to the behavior of the integral $I(x,s) = \int_0^x p(y,s)G_1(y,s)dy$; (ii) a branch cut singularity on the negative real s-axis at $s = -(1-\beta)^2/4$; (iii) poles due to the vanishing of the Wronskian. The poles due to the vanishing of the Wronskian don't appear if the parameters are in the white regions of Fig. 11.1. The Laplace inversion contour is extended to the contour shown in Fig.11.2. The poles from $I(x,s)$ of case (i) generate the term $g_1(k,V,\tau)$ given at (3.4) and the integrals along the branch cut generate $g_2(k,V,\tau)$ given at (3.5).

Fig. 11.2 Contour for the Laplace Transform Inversion



References

Abramowitz, M. and I. A. Stegun (1970), Eds., *Handbook of Mathematical Functions*, Dover Publications, New York

Amin, K. I., and V.K. Ng (1993): *Option Valuation with Systematic Stochastic Volatility*, Journal of Finance, 48, July, 881-910.

Amin, K. I., and V.K. Ng (1994): *A Comparison of Predictable Volatility Models Using Option Data*, preprint, Research Dept., Int'l Monetary Fund, Sept.

Aslanyan, A. and E.B. Davies (1998): *Spectral Instability for Some Schrodinger Operators*, [Online] Los Alamos National Laboratory e-print archive. Available: http://xxx.lanl.gov/abs/math/?9810063, Oct.

Avellaneda, M., A. Levy, and A. Par's (1995): *Pricing and Hedging Derivative Securities in Markets with Uncertain Volatilities*, Applied Math. Finance, 2, 73-88.

Bailey, W. and R.M. Stulz (1989): *The Pricing of Stock Index Options in a General Equilibrium Model*, Journal of Financial and Quantitative Analysis, 24, Mar. 1-11.

Ball, C.A., and A. Roma (1994): *Stochastic Volatility Option Pricing*, Journal of Financial and Quantitative Analysis, 29, no. 4, Dec., 589-607.

Bateman Manuscript Project, A. Erdelyi (Ed.) (1953) , *Higher Transcendental Functions, I*, McGraw-Hill.

Bateman Manuscript Project, A. Erdelyi (Ed.) (1954) , *Tables of Integral Transforms, I*, McGraw-Hill.

Baxter, Martin and Andrew Rennie (1996): *Financial Calculus: An Introduction to Derivative Pricing*, Cambridge University Press.

Beckers, S., (1980): *The Constant Elasticity of Variance Model and Its Implications for Option Pricing*, The Journal of Finance, 35, no.3, June, 661-673.

Bergman, Y.Z., B.D. Grundy, and Z. Wiener (1996): General Properties of Option Prices, The Journal of Finance, 51, No. 5, Dec., 1573-1610.

Black, F. and M. Scholes (1973): *The Pricing of Options and Corporate Liabilities*, Journal of Political Economy 81, May, 637-659.

Blair, B., S-H. Poon, and S.J. Taylor (1998): *Asymmetric and Crash Effects in Stock Volatility for the S&P 100 Index and its Constituents*, working paper, Dept. of Accounting and Finance, Lancaster University, U.K.

Bick, A. (1987): *On the Consistency of the Black-Scholes Model with a General Equilibrium Framework*, J. Financial and Quantitative Analysis, 22, Sept., 259-275.

Bollerslev, T., (1986): *Generalized Autoregressive Conditional Heteroskedasticity*, Journal of Econometrics 31, 307-327.

Bollerslev, T., R.Y. Chou and K.F. Kroner (1992): *ARCH Modeling in Finance. A Review of the Theory and Empirical Evidence*, J. of Econometrics, 52, 5-59.

Carr, P. (1999): *FAQ's in Option Pricing Theory* [Online], Available: http://www.math.nyu.edu/research/carrp/papers/faq2.ps

Carr, P. and M. Chesney (1996): *American Put Call Symmetry*, manuscript, Journal of Financial Engineering (forthcoming).

Carrier, G., M. Krook, and C. Pearson (1966): *Functions of a Complex Variable: Theory and Technique*, McGraw-Hill, New York.

Chou, R.Y. (1988): *Volatility Persistence and Stock Valuations: Some Empirical Evidence Using GARCH*, Journal of Applied Econometrics, 3, 279-294.

Cvitanić, J., H. Pham, and N. Touzi (1997): *Super-replication in Stochastic Volatility Models under Portfolio Constraints*, Journal of Applied Probability (to appear).

Courant, R. and D. Hilbert (1989): *Methods of Mathematical Physics, I*, John Wiley and Sons, New York.

Cox, J., J. Ingersoll, Jr. and S. Ross (1985): *An Intertemporal General Equilibrium Model of Asset Prices*, Econometrica, 53, 363-384.

Cox, J.C. and S.A Ross, (1976): *The Valuation of Options for Alternative Stochastic Processes,* Journal of Financial Economics, 3, Jan/March, 145-67.

Delbaen, F. and W. Schachermayer (1994): *A General Version of the Fundamental Theorem of Asset Pricing*, Mathematische Annalen, 300, 463-520.

Delbaen, F. and W. Schachermayer (1995a): *Arbitrage Possibilities in Bessel Processes and their Relations to Local Martingales*, Probability Theory and Related Fields, 102, 357-366.

Delbaen, F. and W. Schachermayer (1995b): *The No-arbitrage Property under a Change of Numéraire,* Stochastics, 53, 213-226.

Dothan, L.U. (1978): *On the Term Structure of Interest Rates*, Journal of Financial Economics, 6, 59-69.

Drost, F.C. and T.E. Nijman (1993): *Temporal Aggregation of GARCH Processes*, Econometrica, 61, No. 4, July, 909-927.

Duffie, Darrell (1992): *Dynamic Asset Pricing Theory*, Princeton University Press, Princeton, New Jersey.

Duffie, D. and P. Glynn (1996): *Estimation of Continuous-time Markov Processes Sampled at Random Time Intervals*, GSB, Stanford University, manuscript.

Dumas, B., J. Fleming, and R.E. Whaley (1998): *Implied Volatility Functions: Empirical Tests,* Journal of Finance, 53, 2059-2106.

Duan, Jin-Chuan (1995): *The GARCH Option Pricing Model*, Mathematical Finance, 5, Jan., 13-32.

Duan, J-C. (1996): *A Unified Theory of Option Pricing under Stochastic Volatility--from GARCH to Diffusion*, Hong University of Science and Technology, Oct, manuscript.

Dumas, B. (1989): *Two-Person Dynamic Equilibrium in the Capital Market*, The Review of Financial Studies, 2, no. 2, 157-188.

Dumas, B., J. Fleming, and R.E. Whaley (1998): *Implied Volatility Functions: Empirical Tests*, Journal of Finance, Dec..

Durrett, Richard (1984): *Brownian Motion and Martingales in Analysis*, Wadsworth Advanced Books, Belmont, California.

Durrett, Richard (1996): *Stochastic Calculus: A Practical Introduction*, CRC Press, Boca Raton, Florida.

Engle, R.F. (1982): *Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation*, Econometrica, 50, 987-1007.

Engle, R.F. and J.V. Rosenberg (1995): *GARCH Gamma*, The Journal of Derivatives, Summer, 47-59.

Erdelyi, A. (1956): *Asymptotic Expansions*, Dover Publications, New York.

Feller, W. (1951): *Two Singular Diffusion Problems*, Annals of Mathematics, 54, No. 1, July, 173-182.

Feller, W. (1952): *The Parabolic Differential Equations and the Associated Semi-groups of Transformations*, Annals of Mathematics, 55, No.3, May, 468-519.

Feynman, R.J. (1948): *Space-time Approach to Nonrelativistic Quantum Mechanics*. Reviews of Modern Physics, 20, 367-87.

Fleming, J. (1998): *The Economic Significance of the Forecast Bias of S&P 100 Index Option Implied Volatility*, Jones Graduate School of Management, Rice University, April 18, manuscript.

Föllmer, H. and M. Schweizer (1991): Hedging of Contingent Claims Under Incomplete Information, in *Applied Stochastic Analysis*, eds. M.H.A. Davis and R. J. Elliot, Gordon and Breach Science Publishers, New York.

Fournie, E., J. Lebuchoux, and N. Touzi (1997): *Small Noise Expansion and Importance Sampling*, Asymptotic Analysis, 14, 361-376.

Friedman, Avner (1975): *Stochastic Differential Equations and Applications*, 1, Chapt. 6, Academic Press, New York.

Friend, I. and M. Blume (1975): *The Demand for Risky Assets*, American Economic Review, 65, 900-922.

Frey, R. and C.A. Sin (1999): *Bounds on European Option Prices under Stochastic Volatility*, Mathematical Finance, 9, 97-116.

Garman, M. (1985): *Towards a Semigroup Pricing Theory*, The Journal of Finance, 40, No. 3, July, 847-862.

Geman, H. and M. Yor (1993): *Bessel Processes, Asian Options, and Perpetuities*, Mathematical Finance, 3, No.4, October, 349-375.

Geman, H., N. El Karoui, J.-C. Rochet (1995): *Changes of Numéraire, Changes of Probability Measure and Option Pricing*, J. of Applied Probability, 32, 443-458.

Glosten, L.R., R. Jagannathan, and D. Runkle (1993): *On the Relationship between the Expected Value and the Volatility of the Nominal Excess Return on Stocks*, Journal of Finance, 48, 1779-1801.

Gradshteyn, I.S. and I. M. Ryzhik (1994): *Tables of Integrals, Series, and Products*. Fifth Edition. Academic Press, San Diego, Ca.

Guo, D. (1998): *The Risk Premium of Volatility Implicit in Currency Options*, Journal of Business and Economic Statistics, 16, Oct., 498-507.

Hansen, L.P., J.A. Scheinkman, and N. Touzi (1998): *Spectral Methods for Identifying Scalar Diffusions, Journal of Econometrics*, 86, 1-32.

Heston, S.L., (1993): *A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options*, The Review of Financial Studies, 6, No. 2, 327-343.

Heston, S.L. and S. Nandi, (1997): *A Closed-Form GARCH Option Pricing Model*, Federal Reserve Bank of Atlanta, Working Paper 97-9.

Heston, S.L. (1997): *A Simple New Formula for Options with Stochastic Volatility*, John M. Olin School of Business, Washington University, Sept. 19, 1997 manuscript.

Hille, Einar (1969): *Lectures on Ordinary Differential Equations*, Chapt. 10, Addison-Wesley, Reading, Mass.

Hull, J. and A. White (1987): *The Pricing of Options on Assets with Stochastic Volatilities*, The Journal of Finance, June, 281-300.

Ingersoll, Jonathan E. Jr. (1987): *Theory of Financial Decision Making*, Rowman and Littlefield Publishers, Savage, Maryland.

Johnson, G. and L.L. Helms (1963): *Class D Supermartingales*, Bulletin of the American Mathematical Society, 69, 59-62.

Johnson, H. and D. Shanno (1987): *Option Pricing when the Variance is Changing*, Journal of Financial and Quantitative Analysis, June, 143-151.

Kac, M. (1949): *On Distributions of Certain Wiener Functionals*, Trans. of the Amer. Math. Soc., 56, . 1-13. Also reprinted in *Mark Kac: Probability, Number Theory, and Statistical Physics. Selected Papers*, The MIT Press, Cambridge, Mass. 1979.

Kamara, A. and T.W. Miller, Jr. (1995): *Daily and Intradaily Tests of European Put-Call Parity*, Journal of Financial and Quantitative Analysis, 30, No. 4, Dec, 519-539.

Kapadia, N. (1999): *Do Equity Options Price Volatility Risk?*, School of Management, University of Massachusetts, Amherst, May, manuscript.

Karatzas, Ioannis and Steven Shreve (1991): *Brownian Motion and Stochastic Calculus*, second edition, Springer.

Karlin, Samuel and Howard M. Taylor (1981): *A Second Course in Stochastic Processes*, Academic Press, New York.

Kassouf, S.T. (1969): *An Econometric Model for Option Price with Implications for Investors' Expectations and Audacity*, Econometrica, 37, No.4, Oct, 685-693.

Khas'minskii, R.Z. (1960): *Ergodic Properties of Recurrent Diffusion Processes and Stabilization of the Solution to the Cauchy Problem for Parabolic Equations*, Theory of Probability and Its Applications, V, No. 2, 179-196.

Kendall, M.G. (1953): *The Analysis of Economic Time Series: Part I, Prices*, Journal of the Royal Statistical Society, 96, 11-25 .

Lewis, A.L. (1998): *Applications of Eigenfunction Expansions in Continuous-Time Finance*, Mathematical Finance, 8, No.4, Oct, 349-383 .

Lichtblau, D. (1999, Apr. 1): *Commutators and Operator Powers in Mathematica*, [Online], Available email: USENET Newsgroup: comp.soft-sys.math.mathematica. Available: http://library.wolfram.com/mathgroup/archives/1999/Apr/msg00029.html

Lo, C.F., C.H. Hui, and P.H. Yuen (1999): *Constant Elasticity of Variance Option Pricing Model with Time-dependent Parameters*, The Chinese University of Hong Kong, manuscript.

Lukacs, Eugene. (1970): *Characteristic Functions*, Hafner Publishing, New York.

MacBeth, J.D. and L.J. Merville (1980): *Tests of the Black-Scholes and Cox Call Option Valuation Models*, The Journal of Finance, 35, no. 2, May, 285-303.

Mandelbrot, B. (1963): *The Variation of Certain Speculative Prices*, Journal of Business, 36, Oct, 394-419.

Mathews, Jon and R.L. Walker (1965): *Mathematical Methods of Physics*, W.A. Benjamin, New York.

Merton, R.C. (1969): *Lifetime Portfolio Selection Under Uncertainty: The Continuous-Time Case*, Review of Economics and Statistics, 51, Aug, 247-257. Also reprinted in Merton (1990).

Merton, R.C. (1973): *An Intertemporal Capital Asset Pricing Model*, Econometrica, 41, Sept, 867-87. Also reprinted in Merton (1990).

Merton, R.C. (1976): *Option Pricing When Underlying Stock Returns are Discontinuous*, Journal of Financial Economics, 3 , Jan-March, 125-144.

Merton, R.C.(1990): *Continuous-Time Finance*, Basil Blackwell, Cambridge, Mass.

Musiela, Marek and Marek Rutkowski (1997): *Martingale Methods in Financial Modelling*, Springer, Berlin.

Nandi, S. (1998): *How Important is the Correlation between Returns and Volatility in a Stochastic Volatility Model? Empirical Evidence from Pricing and Hedging in the S&P500 Index Options Market*, Journal of Banking and Finance, 22, 589-610.

Nelson, D.B. (1990): *ARCH Models as Diffusion Approximations*, Journal of Econometrics, 45, 7-38.

Neftci, S.N. (1996): *An Introduction to the Mathematics of Financial Derivatives*, Academic Press, San Diego, Ca.

Øksendal, Bernt (1998): *Stochastic Differential Equations*, Fifth edition, Springer.

Pham, H. and N. Touzi (1996): *Equilibrium Prices in a Stochastic Volatility Model*, Mathematical Finance, 6, No. 2, April, 215-236.

Pliska, Stanley R. (1997): *Introduction to Mathematical Finance. Discrete Time Models*, Blackwell.

Pratt, J.W. (1964): *Risk Aversion in the Small and in the Large*, Econometrica, 32 , Jan, 122-136.

Protter, P (1999): *A Partial Introduction to Finance*, Dept. of Mathematics, Purdue University, manuscript.

Renault, E. and N. Touzi (1996): *Option Hedging and Implied Volatilities in a Stochastic Volatility Model*, Mathematical Finance, 6, No. 3, July, 279-302.

Revuz, Daniel and Marc Yor (1999): *Continuous Martingales and Brownian Motion*, Third edition, Springer.

Romano, M. and Nizar Touzi (1997): *Contingent Claims and Market Completeness in a Stochastic Volatility Model*, Mathematical Finance, 7, No.4, Oct, 399-412.

Rubinstein, M. (1974): *An Aggregation Theorem for Securities Markets*, Journal of Financial Economics,1, 225-244.

Rubinstein, M. (1985): *Nonparametric Tests of Alternative Option Pricing Models Using All Reported Trades and Quotes on the 30 Most Active CBOE Option Classes from August 23, 1976 through August 31, 1978*, J.of Finance, 40, June, 455-480.

Samuelson, P.A. and R.C. Merton (1969): *A Complete Model of Warrant Pricing that Maximizes Utility*, Industrial Management Review,10, Winter, 17-46. Also, reprinted in Merton (1990) as Chapter 7.

Schroder, M. (1989); *Computing the Constant Elasticity of Variance Option Pricing Formula*, The Journal of Finance, 44, March , 211-219.

Schroder, M. (1999): *Changes of Numeraire for Pricing Futures, Forwards, and Options*, The Review of Financial Studies, Winter, 12, No. 5, 1143-1163.

Scott, L.O.(1987): *Option Pricing when the Variance Changes Randomly: Theory, Estimation, and an Application*, J. of Fin.and Quantitative Analysis, Dec, 419-438.

Sin, C. (1998): *Complications with Stochastic Volatility Models*, Advances in Applied Probability, 30, 256-268.

Sircar, K.R. and G.C. Papanicolaou (1999): *Stochastic Volatility, Smile and Asymptotics*, Applied Mathematical Finance, 6, 107-145.

Slater, L. J. (1960): *Confluent Hypergeometric Functions,* Cambridge University Press.

Stein E. M., and J.C. Stein (1991): *Stock Price Distributions with Stochastic Volatility: An Analytic Approach*, The Review of Financial Studies, Vol 4 Number 4, 727-752.

Taylor, Stephen J. (1994): *Modeling Stochastic Volatility: A Review and Comparative Study*, Mathematical Finance, 4, No. 2, Apr, 183-204.

Taylor, S.J. and X. Xu (1994): *The Magnitude of Implied Volatility Smiles: Theory and Empirical Evidence for Exchange Rates*, The Rev.of Futures Markets, 13, 355-380.

Thorp, Edward O. and Sheen T. Kassouf (1967): *Beat the Market: A Scientific Stock Market System*, Random House, New York.

Titchmarsh, E.C. (1962): *Eigenfunction Expansions Associated with Second-order Differential Equations*, Oxford University Press, London.

Titchmarsh, E.C. (1975): *Theory of Fourier Integrals,* Second edition, Oxford University Press, London.

Touzi, N. (1999): *American Options Exercise Boundary When the Volatility Changes Randomly*, Applied Mathematics and Optimization, 39, 411-422.

Weyl, Hermann (1950): *Ramifications, Old and New, of the Eigenvalue Problem*, Bulletin of the American Mathematical Society, Mar, 115-139.

Wiggins, J.B. , (1987): *Option Values Under Stochastic Volatility*, Journal of Financial Economics, 19, 351-372.

Zakoian, Jean-Michel (1990): *Threshold Heteroskedastic Model,* INSEE, Paris.

Zhu, Y. and M. Avellaneda (1998): *A Risk-Neutral Stochastic Volatility Model,* Int'l. Journal of Theoretical and Applied Finance, 1, No. 2, 289-310

Subject Index

Frequent Notations and Abbreviations

$\mathcal{A}$	Differential generator for a diffusion process.
$a(V)$, $b(V)$	Drift and diffusion coefficient for the volatility process, as in $dV = b(V)dt + a(V)dW$
$\tilde{b}(V)$	Drift coefficient for the risk-adjusted volatility process
B-S	Black and Scholes
$c(k)$	$= (k^2 - ik)/2$: Frequently occurring expression
$c(S,V,\tau)$	Call option price in Black and Scholes' model
$C(S,V,\tau)$	Call option price under stochastic volatility
CPRA	Constant proportional risk aversion—power utility function
dB, dW, dZ	Brownian motion processes, frequently correlated.
$d(k)$	$= -ik$: Frequently occurring expression
$\mathbb{E}_t$	An expectation taken at time t, conditional on knowledge of the random variables at that time.
$\hat{H}(k,V,\tau)$	Fundamental (Fourier) transform
Im	Imaginary part
k	Fourier transform variable a complex number $= k_r + ik_i$
$\mathcal{L}$	Linear differential operator. (Sometimes: Laplace transform)
$M(a,b,x)$	Confluent hypergeometric function
P	The actual (real-world) price evolution process (measure)
$\tilde{P}$	The martingale (or risk-adjusted) pricing process (measure)
P_e	An explosion probability; $\hat{P}_e(V,\tau)$ is the probability of an explosion to $+\infty$ in the auxiliary volatility process.
Q	The martingale pricing process (measure) using the stock price as numeraire, or, a label for auxiliary volatility process.
PDE	Partial differential equation
$\Pr(A)$	Probability of the event A occurring
r	Short-term interest rate, continuously compounded
R	Impatience rate in CPRA utility function, continuous basis.
S_t	Stock price or underlying security price at time t
SDE	Stochastic differential equation
T	Option expiration date or time
$\mathbb{T}$	Investment-consumption horizon date—the time of death
$U(V_0,\tau)$	Integrated volatility, as in $U(V_0,\tau) = \int_0^\tau V(s)ds$

(continued)

Frequent Notations and Abbreviations (continued)

$U(a,b,x)$	Confluent hypergeometric function
v	Time average volatility, as in $v = \int_0^\tau V(s)ds/\tau$
V_t	$=\sigma_t^2$: instantaneous variance rate at time-t
V_∞^{imp}	Asymptotic implied volatility as $\tau \to \infty$
$\hat{Z}_t$	Standard (mean zero, variance 1) normal variate, drawn at t

α, β	Strip of regularity for $\hat{H}(k,V,\tau)$: $\alpha < \operatorname{Im} k < \beta$
γ	CPRA risk-aversion parameter; Pratt's measure is $1-\gamma$
δ	Dividend yield, continuously compounded rate
$\Gamma(\alpha,x)$	Incomplete Gamma function = $\int_x^\infty e^{-t} t^{\alpha-1} dt$, $\operatorname{Re}\alpha > 0$.
$\eta(V)$	Volatility of volatility: as in $dV = b(V)dt + \xi\,\eta(V)dW$.
θ	Volatility drift parameter; θ_a indicates annualized units
λ	An eigenvalue, usually the smallest eigenvalue
ξ	Volatility of volatility scale parameter; as in $dV = (\omega - \theta V)\,dt + \xi\, V dW$
ρ	Correlation between price and volatility processes
σ_t	Instantaneous standard deviation of price return at time t
σ_∞^{imp}	Asymptotic implied volatility as $\tau \to \infty$
τ	Time to option expiration or an investment horizon
$\Phi(\cdot)$	Cumulative normal distribution
ω	Volatility drift parameter; ω_a indicates annualized units

$\approx$	Asymptotic equality
$\cong$	Approximate numerical equality
x^+	$\max[x,0]$
$\blacksquare$	End of proof mark
$\langle \cdots \rangle$	Expectation (same as $\mathbb{E}_t[\cdots]$) under the risk-adjusted volatility process $dV = \bar{b}(V)dt + a(V)d\tilde{W}$