Master 2 de Mathématiques Fondamentales de Paris Centre

Geometric Analysis on manifolds

Thibault Lefeuvre

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Université de Paris and Sorbonne Université, CNRS, IMJ-PRG,
F-75006 Paris, France.
https://thibaultlefeuvre.blog/
tlefeuvre@imj-prg.fr

« On affirme, en Orient, que le meilleur moyen pour traverser un carré est d'en parcourir trois côtés. »

Les Sept Piliers de la sagesse, Thomas Edward Lawrence

Foreword

These are a set of lecture notes on geometric and microlocal analysis given during the years 2022—2024 at the *Master 2 de Mathématiques fondamentales de Paris Centre* in thirty-six hours. These notes do not intend to cover the whole subject. On the contrary, they are far from being exhaustive but are intended to be self-contained: starting from scratch (or almost, i.e. from the basics of the theory of distributions), we define oscillatory integrals, pseudodifferential operators and show their main properties on closed manifolds, both from an analytic and spectral perspective. One of the main objectives is to give a short microlocal proof of the Hodge Theorem in de Rham (or Dolbeault) cohomology. In order to show the full strength of microlocal analysis, one would need to treat more ambitious topics such as semiclassical analysis, scattering theory, Pollicott-Ruelle resonances in hyperbolic dynamics, or the Atiyah-Singer index Theorem — but these would require much more time than thirty-six hours. I plan to enrich these notes from time to time: and maybe that I will end up writing some of these developments at some point.

Thibault Lefeuvre,
Paris, September 2021.

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Contents

Introduction

What is microlocal analysis?

The goal of microlocal analysis is to describe the properties of linear differential operators acting on functions (and/or distributions) on a differentiable manifold M. In these lecture notes, we will mostly stick to closed manifolds, namely, compact manifolds without boundary. Differential operators appear naturally in various situations: vector fields $Y \in C^\infty(M, TM)$ acting as derivations on functions $Y : C^\infty(M) \to C^\infty(M)$ are examples of differential operators of order 1; the Laplacian operator Δ_g induced by a metric g, acting on functions $C^\infty(M)$, is a differential operator of order 2; the exterior derivative $d : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k+1} T^*M)$; if M is a complex manifold, the Cauchy-Riemann operator $\overline{\partial}$ acting on a holomorphic vector bundle $E \to M$.

Given a general differential operator P on M, one can investigate the following questions:

(i) What is $\ker P$? Can one describe the regularity of solutions to $Pu = 0$? Are they smooth?

(ii) Given $f \in C^\infty(M)$, can one solve $Pu = f$? What if $f \in \mathcal{D}'(M)$ is merely a distribution?

(iii) Can one define a spectrum for the operator P? On which functional spaces?

(iv) If P has some good properties (if it is elliptic for example), what is the index of P, namely, the difference between the dimension of its kernel and the dimension of its coimage in appropriate spaces? Can this index be related to topological invariants of the manifold M?

The last question paves the way to the celebrated Atiyah-Singer index Theorem [BGV92], which is maybe one of the most striking achievements of mathematics in the XXth century. We will not attempt to give a proof of the index Theorem, which is really beyond the scope of this class. However, we will see at the very end one of its manifestations by showing that the index of the Dirac operator $d + d^*$ (where d^* is the formal adjoint of the exterior derivative acting on forms) is the Euler characteristic of the manifold, the latter being a topological invariant.

A physical example

A very basic example in $\mathbb{R}^n$ is the *wave operator* $\Box := \partial_t^2 - c^2\Delta$ acting on distributions defined on the space $\mathbb{R} \times \mathbb{R}^n$, for some $c > 0$. The solutions to the system

$$\begin{cases} \Box u = 0, \\ u|_{t=0} = 0, \partial_t u|_{t=0} = \delta_0, \end{cases}, \tag{0.0.1}$$

where δ_0 is the Dirac mass in $\mathbb{R}^n$ describe the behaviour of punctual source located at $x = 0$ emiting a wave at time $t = 0$ and propagating with speed $c > 0$, such as:

(i) a firework, then $n = 3$, u is the *surpressure* i.e. the difference between the original air pressure u_0 and the effective pressure at time t, and the sound speed is $c \approx 343\text{m/s}$ in the air;

(ii) or a stone thrown in the water, then $n = 2$ and u is now the height of the water from its initial level.

From a physical perspective, the solutions to (0.0.1) can be easily described: there is a *wavefront* localized on the sphere $\{x = ct\}$ such that any observer inside this sphere has already heard the noise once (in the firework case) whereas any observer outside this sphere has still not been reached by the sound. How can we explain the existence of this wavefront and its evolution in time? There is also a subtlety here: in the three-dimensional case (fireworks), an observer only hears the sound once, whereas in the two-dimensional case, there are multiple internal waves contained inside the wavefront sphere of radius ct. How to explain these different behaviours between odd and even dimensions?

Towards a microlocal theory

Microlocal analysis starts with the following basic observation in $\mathbb{R}^n$: given an operator $P = \sum_\alpha a_\alpha \partial_x^\alpha$ with constant coefficients $a_\alpha \in \mathbb{C}$ (such as the Laplacian operator $\Delta := \sum_i \partial_{x_i}^2$), the Fourier transform intertwines polynomial multiplication in the phase space (i.e. in the momentum variable $\xi \in \mathbb{R}^n$ which we see as being dual to x) and differentiation in the physical space (i.e. in the space variable $x \in \mathbb{R}^n$), namely,

$$Pf = \mathcal{F}^{-1}\left(p(\bullet)\mathcal{F}f\right), \tag{0.0.2}$$

where $p(\xi) := \sum_\alpha a_\alpha i^{|\alpha|}\partial_x^\alpha$. In particular, $p_\Delta(\xi) = -|\xi|^2$ if $\Delta := \sum_i \partial_{x_i}^2$ is the usual Laplacian on $\mathbb{R}^n$. Note that, without any consideration of convergence, (0.0.2) can be written as

$$Pf(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n_\xi}\int_{\mathbb{R}^n_y} e^{i(x-y)\cdot\xi}p(\xi)f(y)\mathrm{d}\xi\mathrm{d}y. \tag{0.0.3}$$

The polynomial p is called the *symbol* of the operator P and can be defined similarly for differential operators with non-constant coefficients (i.e. $a_\alpha \in C^\infty(\mathbb{R}^n)$). The symbol is thus a smooth function in $C^\infty(\mathbb{R}^{2n})$ with very specific behaviour in the ξ variable (namely, it is polynomial). It will be convenient from now on to identify $\mathbb{R}^{2n}$ with the cotangent of $\mathbb{R}^n$: the reason for this identification will become apparent later. The symbol of differential operators is therefore an injective arrow

$$\sigma : \mathrm{Diff}^k(\mathbb{R}^n) \longrightarrow \mathrm{Pol}^k(T^*\mathbb{R}^n) \hookrightarrow C^\infty(T^*\mathbb{R}^n), \tag{0.0.4}$$

mapping into the set of functions $\mathrm{Pol}^k(T^*\mathbb{R}^n)$ that are polynomial in the ξ-variable. A key idea in microlocal analysis is that **the main analytic properties of the differential operator P should be readable from the "algebraic" properties of its symbol**.

Conversely, given a function $p \in \mathrm{Pol}^k(T^*\mathbb{R}^n)$ that is polynomial in the ξ-variable, one can define an inverse to the symbol map (0.0.4), namely, (0.0.3) defines an operator

$$\mathrm{Op} : \mathrm{Pol}^k(T^*\mathbb{R}^n) \longrightarrow \mathrm{Diff}^k(\mathbb{R}^n),$$

such that $\mathrm{Op} \circ \sigma = \sigma \circ \mathrm{Op} = \mathbb{1}$. This is called a *quantization*. More generally, we will see that if we impose adequate growth conditions on functions in $C^\infty(T^*\mathbb{R}^n)$ (roughly speaking, it should behave like a polynomial as $|\xi| \to \infty$) and work with *the space of symbols* $S^k(T^*\mathbb{R}^n)$ of order k, then the quantization of this function, defined by (0.0.2) or (0.0.3), still makes sense and produces a class of well-behaved operators

$$\mathrm{Op} : S^k(T^*\mathbb{R}^n) \longrightarrow \Psi^k(\mathbb{R}^n).$$

The target space $\Psi^k(\mathbb{R}^n)$ is called the space of **pseudodifferential operators**.

In order to answer questions (A—D) of the first paragraph, we will construct on a closed manifold M a graded algebra

$$\Psi^\star(M) := \bigcup_{m=-\infty}^{+\infty} \Psi^m(M)$$

of *pseudodifferential* operators such that, whenever $k \in \mathbb{N}^*$ is a positive integer, the set $\mathrm{Diff}^k(M)$ of differential operators of order k on M is contained in $\Psi^k(M)$ and, when $P \in \mathrm{Diff}^k(M)$ is *invertible* (in an appropriate sense), its inverse $P^{-1} \in \Psi^{-k}(M)$ is pseudodifferential. The need to consider more general operators than just *differential* ones becomes indeed clearly apparent when trying to *invert* differential operators and this can be seen from a very simple example: on $\mathbb{R}$, the derivative ∂_x is a differential operator of order 1 but its inverse, the *integration* $\int_0^x$, is not.

As we shall see, determining whether a given (pseudo)differential operator is invertible will not be difficult. This will be achieved through the concept of *ellipticity*: an operator is said to be elliptic if its symbol does not vanish except possibly for $\xi = 0$. For instance, the Laplacian Δ_g (if g is an arbitrary metric on M) will have symbol $p(x, \xi) = |\xi|_g^2$ which never vanishes except at $\xi = 0$. Such operators will be invertible modulo compact (more precisely, *smoothing*) operators and the latter will be negligible in the microlocal theory that we will build. This is a typical example where an algebraic property (the non-vanishing of the symbol) connects to an analytic property of the operator. We shall study elliptic operators and show that, under some mild assumptions, such operators, when defined on a closed manifold, have a discrete L^2-spectrum in $\mathbb{C}$. This will eventually lead us to the Hodge theorem, showing that there exists a natural isomorphism between harmonics forms on a differentiable manifold and its (real) cohomology groups.

Chapter 1

Oscillatory integrals

Oscillatory integrals play a fundamental role in microlocal analysis. They are generalized integral, that is, integrals of symbols over $\mathbb{R}^N$ that do not converge absolutely but converge thanks to an oscillatory phase factor. These integrals appear everywhere. In particular, the Schwartz kernel of pseudodifferential and Fourier Integral operators is always an oscillatory integral.

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1.1 Definition. First properties

We let $X \subset \mathbb{R}^n$ be an open set. The goal of this paragraph is to study oscillatory integrals of the form

$$I(x) := \int_{\mathbb{R}^N} e^{i\phi(x,\theta)} a(x,\theta) \mathrm{d}\theta \in \mathcal{D}'(X).$$

for some well-chosen *phase function* $\phi \in C^\infty(X \times \mathbb{R}^N)$ and *amplitude* $a \in C^\infty(X \times \mathbb{R}^N)$.

1.1.1 Symbols

1.1.1.1 Definition

The amplitudes we will work with are called *symbols*: they can be thought of as "generalized polynomial functions in the ξ-variable". They are the smooth functions that we will quantize later in order to produce pseudodifferential operators.

Definition 1.1.1 (Symbol classes)**.** For $m \in \mathbb{R}, \rho, \delta \in [0,1]$, we introduce the class $S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ of smooth symbols $a \in C^\infty(X \times \mathbb{R}^N)$ of type (ρ, δ), namely such that: for all $K \subset X$, for all multi-indices $\alpha \in \mathbb{N}^N, \beta \in \mathbb{N}^n$, there exists $C := C(K, \alpha, \beta)$ such that the following inequality holds

$$|\partial_x^\beta \partial_\theta^\alpha a(x,\theta)| \leq C\langle\theta\rangle^{m-\rho|\alpha|+\delta|\beta|}, \quad \forall (x,\theta) \in K \times \mathbb{R}^N. \tag{1.1.1}$$

In the following, we will mostly consider the case where $N = n$ and $\theta = \xi$ is a covector, that is we will take $(x, \xi) \in T^*X$ and symbols will be seen as elements $a \in C^\infty(T^*X)$ satisfying the bounds (1.1.1). In this case, we will write the class $S^m_{\rho,\delta}(T^*X)$. Let us also make a few remarks:

- The case $\rho = 1, \delta = 0$ is most standard.
- It can be checked that $S^m_{\rho,\delta}$ is a Fréchet space with the semi-norms given by (1.1.1), namely

$$p_{m,K,k,k'}(a) := \sup_{|\beta| \leq k, |\alpha| \leq k'} \sup_{(x,\theta) \in K \times \mathbb{R}^N} \frac{|\partial_x^\beta \partial_\theta^\alpha a(x,\theta)|}{\langle\theta\rangle^{m-\rho|\alpha|+\delta|\beta|}}. \tag{1.1.2}$$

- We have the obvious inclusion $S^m_{\rho,\delta} \subset S^{m'}_{\rho',\delta'}$ for $m \leq m'$, $\delta \leq \delta'$ and $\rho \geq \rho'$.
- There is no point in introducing classes for $\rho > 1$ or $\delta < 0$. For instance, if $a \in S^m_{\rho,\delta}(X \times \mathbb{R})$ with $m < 0$ and $\rho > 1$, then applying $|\theta|\ \partial_\theta$ many times and integrating by parts to recover the expression of a, one gets that $a \in S^{-\infty}(X \times \mathbb{R})$ actually. Similar phenomenon occurs for $m \geq 0$ and $\delta < 0$.

We define $S^{-\infty}(X \times \mathbb{R}^N) := \cap_{m\in\mathbb{R}} S^m_{\rho,\delta}(X \times \mathbb{R}^N)$. It can be checked that this is indeed independent of ρ and δ and this corresponds to symbols decreasing faster than any polynomials, i.e. such that for all compact $K \subset X$, $\alpha \in \mathbb{N}^N, \beta \in \mathbb{N}^n, M \in \mathbb{N}$, there exists $C := C(K, \alpha, \beta, M) > 0$ such that:

$$|\partial_x^\beta \partial_\theta^\alpha a(x,\theta)| \ \leq C\langle\theta\rangle^{-M}, \quad \forall (x,\theta) \in K \times \mathbb{R}^N.$$

Exercise 1.1.2. Prove the previous claim.

Example 1.1.3. A typical example is any polynomial function of the form

$$a(x,\theta) = \sum_{|\alpha| \ \leq m} c_\alpha(x)\theta^\alpha,$$

or $a'(x,\theta) := \langle\theta\rangle^m$; both lie in $S^m_{1,0}$. Smooth functions that are not symbols are for instance $b(x,\theta) := e^{i|\theta|^3}$. If $N = n$, we can consider $c(x,\theta) := e^{ix\cdot\theta}$, this is a symbol in $S^0_{0,1}$. On the other hand, $c'(x,\theta) = e^{-|\theta|^2}$ is a symbol in $S^{-\infty}_{1,0}$. Symbols which are compactly supported in the θ-variable are also naturally in $S^{-\infty}_{1,0}$.

Exercise 1.1.4. Check that $a(x,\theta) := \langle\theta\rangle^m$ is indeed a symbol in $S^m_{1,0}(X \times \mathbb{R}^N)$.

Another important class is that of homogeneous functions:

Example 1.1.5. Let $a \in C^\infty(X \times \mathbb{R}^n)$ be positively homogeneous of order m for $|\theta| \ \geq 1$, namely $a(x,\lambda\theta) = \lambda^m a(x,\theta)$ for all $\lambda \geq 1, |\theta| \ \geq 1$. Then $a \in S^m_{1,0}(X \times \mathbb{R}^N)$.

Exercise 1.1.6. Prove the claim of the previous example.

We have the following properties:

Lemma 1.1.7. $S^m_{\rho,\delta} \cdot S^{m'}_{\rho,\delta} \subset S^{m+m'}_{\rho,\delta}$.

Proof. Let $a \in S^m_{\rho,\delta}, b \in S^{m'}_{\rho,\delta}$. Then by Leibniz' formula:

$$\begin{aligned}\left|\partial_\xi^\alpha \partial_x^\beta (ab)(x,\theta)\right| &= \left|\sum_{\alpha_1+\alpha_2=\alpha, \beta_1+\beta_2=\beta} C(\alpha_1,\alpha_2,\beta_1,\beta_2)\partial_\xi^{\alpha_1}\partial_x^{\beta_1}a(x,\theta)\partial_\xi^{\alpha_2}\partial_x^{\beta_2}b(x,\theta)\right| \\ &\lesssim \langle\theta\rangle^{m-|\alpha_1|\rho+|\beta_1|\delta}\langle\theta\rangle^{m'-|\alpha_2|\rho+|\beta_2|\delta} \lesssim \langle\theta\rangle^{m-|\alpha|\rho+|\beta|\delta},\end{aligned}$$

where $C(\alpha_1,\alpha_2,\beta_1,\beta_2) \geq 0$ are some coefficients. □

1.1.1.2 Asymptotic summation

The following result shows that one can always sum in the symbolic sense a sequence of of symbols with decaying order. The result is unique up to a negligible (i.e. $S^{-\infty}$) remainder.

Lemma 1.1.8 (Borel summation lemma)**.** *Let $a_j \in S^{m_j}_{\rho,\delta}(X \times \mathbb{R}^N)$, for $j \in \mathbb{N}$ where $(m_j)_{j\in\mathbb{N}}$ is a strictly decreasing sequence of real numbers diverging to $-\infty$. Then, there exists $a \in S^{m_0}_{\rho,\delta}(X \times \mathbb{R}^N)$, unique modulo $S^{-\infty}(X \times \mathbb{R}^N)$, such that $a \sim \sum_{j\geq 0} a_j$, that is for all $k \in \mathbb{N}$:*

$$a - \sum_{j=0}^{k-1} a_j \in S^{m_k}_{\rho,\delta}(X \times \ \mathbb{R}^N).$$

Of course, the lemma can be applied to a non-strictly decreasing sequence of real numbers; it suffices to gather the terms of same order. This result should be compared with the classical result asserting that one can construct a smooth function in a neighborhood of 0 in $\mathbb{R}^n$ with an arbitrary Taylor expansion at 0.

Proof. **Uniqueness:** Assume that two such symbols $a, a' \in S^{m_0}_{\rho,\delta}(X \times \mathbb{R}^N)$ exist. Then

$$a - a' = a - \sum_{j=0}^{k-1} a_j - \left(a' - \sum_{j=0}^{k-1} a_j \right) \in S^{m_k}_{\rho,\delta}(X \times \mathbb{R}^N),$$

for all $k \in \mathbb{N}$, that is $a - a' \in S^{-\infty}(X \times \mathbb{R}^N)$.

Existence: Let $\chi \in C^\infty(\mathbb{R}^N, [0,1])$ be a smooth cutoff function such that $\chi = 1$ for $|\theta| \geq 2$ and $\chi = 0$ for $|\theta| \leq 1$. We want to define

$$a(x,\theta) := \sum_{j=0}^{+\infty} a_j(x,\theta)\chi(\theta/\lambda_j), \tag{1.1.3}$$

for some good choice of $\lambda_j \geq 1$ converging to $+\infty$. We let $X = \cup_\ell K_\ell$ be an exhaustion of X by compact subsets. We want to achieve the bound: for all $j \geq 0$, for all $|\alpha| + |\beta| + \ell \leq j$,

$$\sup_{x \in K_\ell} \left| \partial_\theta^\alpha \partial_x^\beta \left(\chi(\theta/\lambda_j) a_j(x,\theta) \right) \right| \leq 2^{-j} \langle \theta \rangle^{m_{j-1} - \rho|\alpha| + \delta|\beta|}. \tag{1.1.4}$$

Indeed, if (1.1.4) holds, then for arbitrary α, β and ℓ, taking j large enough so that $|\alpha| + |\beta| + \ell \leq j$, we have

$$\partial_\theta^\alpha \partial_x^\beta a(x,\theta) = \sum_{k=0}^{j} \partial_\theta^\alpha \partial_x^\beta \left(a_k(x,\theta)\chi(\theta/\lambda_k) \right) + \sum_{k=j+1}^{+\infty} \partial_\theta^\alpha \partial_x^\beta \left(a_k(x,\theta)\chi(\theta/\lambda_k) \right),$$

and the first term is bounded for $(x,\theta) \in K_\ell \times \mathbb{R}^N$ by $\lesssim C_j \langle \theta \rangle^{m_0 - \rho|\alpha| + \delta|\beta|}$ while the second term is bounded for $(x,\theta) \in K_\ell \times \mathbb{R}^N$ by

$$\left| \sum_{k=j+1}^{+\infty} \partial_\theta^\alpha \partial_x^\beta \left(a_k(x,\theta)\chi(\theta/\lambda_k) \right) \right| \leq \sum_{k=j+1}^{+\infty} 2^{-k} \langle \theta \rangle^{m_{k-1} - \rho|\alpha| + \delta|\beta|} \leq 2^{-j} \langle \theta \rangle^{m_j - \rho|\alpha| + \delta|\beta|},$$

that is $a \in S^{m_0}_{\rho,\delta}(X \times \mathbb{R}^N)$. It remains to show (1.1.4). For that, we start by observing that

$$|\partial_\theta^\alpha \chi(\theta/\lambda)| = \lambda^{-|\alpha|} |(\partial_\theta^\alpha \chi)(\theta/\lambda)| \leq C_\alpha \langle \theta \rangle^{-|\alpha|},$$

uniformly in $\lambda \geq 1$, since $|\theta|/2 \leq \lambda \leq |\theta|$ by the support property of χ. As a consequence:

$$\left| \partial_\theta^\alpha \partial_x^\beta \left(\chi(\theta/\lambda) a_j(x,\theta) \right) \right| \leq C_j \langle \theta \rangle^{m_j - \rho|\alpha| + \delta|\beta|},$$

for all $x \in K_\ell, \theta \in \mathbb{R}^N$ and $|\alpha| + |\beta| + \ell \leq j$, and $\lambda \geq 1$. Observe that in the previous inequality, using $|\theta| \geq \lambda$:

$$C_j\langle\theta\rangle^{m_j} = C_j\langle\theta\rangle^{m_j-m_{j-1}}\langle\theta\rangle^{m_{j-1}} \leq C_j\langle\lambda\rangle^{m_j-m_{j-1}}\langle\theta\rangle^{m_{j-1}} \leq 2^{-j}\langle\theta\rangle^{m_{j-1}},$$

by taking $\lambda =: \lambda_j$ large enough so that $C_j\langle\lambda\rangle^{m_j-m_{j-1}} \leq 2^{-j}$. This eventually shows (1.1.4).

Eventually, it remains to show that $a \sim \sum_{j\geq 0} a_j$. But we have

$$a - \sum_{j=0}^{k} a_j = \sum_{j=0}^{k} a_j(\chi(\theta/\lambda_j) - 1) + \sum_{j=k+1}^{+\infty} a_j\chi(\theta/\lambda_j),$$

and the first term is clearly in $S^{-\infty}(X\times\mathbb{R}^N)$, while the second term is in $S^{m_{k+1}}_{\rho,\delta}(X\times\mathbb{R}^N)$ by our previous computation.

$\square$

Exercise 1.1.9. Taking inspiration from the previous proof, show the following result, known as Borel's Theorem: for every sequence $(a_\alpha)_{\alpha\in\mathbb{N}^n}$ such that $a_\alpha \in \mathbb{C}$, there exists $u \in C^\infty(\mathbb{R}^n)$ such that $a_\alpha = (\alpha!)^{-1}\partial_x^\alpha u(0)$.

We also give the following useful lemma:

Lemma 1.1.10. *Let $a \in C^\infty(X\times\mathbb{R}^N)$ and $a_j \in S^{m_j}_{\rho,\delta}(X\times\mathbb{R}^N)$, where $(m_j)_{j\in\mathbb{N}}$ is a decreasing sequence converging to $-\infty$ such that:*

1. *For any compact subset $K \subset X$ and $\alpha,\beta \in \mathbb{N}^n$, there exists $C := C(\alpha,\beta,K) > 0$ and $M := M(\alpha,\beta) > 0$ such that for all $(x,\theta) \in K\times\mathbb{R}^N$:*
$$|\partial_\theta^\alpha\partial_x^\beta a(x,\theta)| \leq C\langle\theta\rangle^M. \tag{1.1.5}$$
2. *For any compact subset $K \subset X$, $k \in \mathbb{N}$, there exists $C := C(k,K) > 0$ such that for all $(x,\theta) \in K\times\mathbb{R}^N$:*
$$\left| a(x,\theta) - \sum_{j=0}^{k-1} a_j(x,\theta)\right| \leq C\langle\theta\rangle^{m_k}.$$

Then $a \in S^m_{\rho,\delta}(X\times\mathbb{R}^N)$ and $a \sim \sum_{j\geq 0} a_j$.

In practice, this is often what one wants to use: one is given a smooth function a with tame growth as $|\theta| \to \infty$ (as in (1.1.5)) and one also knows that $a \sim_{C^0} \sum_j a_j$ (i.e. the asymptotic sum holds *a priori* only at the level of functions, but not for their derivatives).

Proof. Omitted, see [GS94, Proposition 1.9]. $\square$

1.1.2 Phase functions

We now introduce the second class of important functions, namely phase functions:

Definition 1.1.11. A function $\phi \in C^\infty(X \times \mathbb{R}^N)$ is called a *phase function* if:

1. ϕ is real-valued,
2. ϕ is 1-homogeneous in the second variable, namely $\phi(x, \lambda\theta) = \lambda\phi(x, \theta)$ for all $(x, \theta) \in X \times \mathbb{R}^N$,
3. $d\phi \neq 0$ on $X \times (\mathbb{R}^N \setminus 0)$.

One can actually generalize this approach and take *complex-valued* phase functions by requiring $\Im\phi \geq 0$.

Example 1.1.12. If $N = n$, a typical example is $\phi(x, \theta) := x \cdot \theta$. In fact, this is is the one we will mostly use.

1.1.3 Definition and regularization of an oscillatory integral

Given a symbol $a \in S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ and a phase function ϕ, we consider the integral:

$$I_a(x) := \int_{\mathbb{R}^N} e^{i\phi(x,\theta)} a(x, \theta) \mathrm{d}\theta, \tag{1.1.6}$$

and we want to see it as a function (or a distribution) in the x-variable. A priori, this might not be well-defined. Nevertheless, we can make the following observation:

Lemma 1.1.13. *Let $k \in \mathbb{N}$ and $m \in \mathbb{R}$ be such that $m + k < -N$. Then:*

$$S^m_{\rho,\delta}(X \times \mathbb{R}^N) \mapsto I_a \in C^k(X)$$

is continuous.

Continuity means the following: since both spaces are Fréchet[1], for every seminorm p on $C^k(X)$, there is seminorm p' on $S^m_{\rho,\delta}$ such that

$$p(I_a) \leq p'(a).$$

Proof. First of all, observe that for $x \in X$, $I_a(x)$ is well-defined as the integral converges absolutely. For $|\beta| \leq k$, one has:

$$\begin{aligned}
&\partial_x^\beta (e^{i\phi(\bullet,\theta)} a(\bullet, \theta))(x) \\
&= \sum_{\beta_1+\beta_2=\beta} C(\beta_1, \beta_2) \partial_x^{\beta_1} (e^{i\phi(\bullet,\theta)}) \partial_x^{\beta_2} (a(\bullet, \theta))(x) \\
&= e^{i\phi(x,\theta)} \sum_{\beta_1+\beta_2=\beta} C(\beta_1, \beta_2) P\left(\phi(x, \theta), \nabla_x\phi(x, \theta), ..., \nabla_x^{|\beta_1|}\phi(x, \theta)\right) \partial_x^{\beta_2} a(x, \theta),
\end{aligned}$$

where P is a polynomial expression in all its coordinates. More precisely, it is a sum of products of derivatives of ϕ such that the sum of the number derivatives of each factor is $|\beta_1|$. Indeed, we have:

$$\partial_x(e^{i\phi(x,\theta)}) = i\partial_x\phi e^{i\phi(x,\theta)}, \partial_x^2(e^{i\phi(x,\theta)}) = i\left(\partial_x^2\phi + i(\partial_x\phi)^2\right) e^{i\phi}, ...$$

[1] $C^k(X)$ is Fréchet (by taking an exhaustion of X by compact sets).

We now fix a compact $K \subset X$. We want to bound $\|I_a\|_{C^k(K)}$. We take $\beta \in \mathbb{N}^n$ such that $|\beta| \leq k$. Since ϕ is 1-homogeneous in the θ-variable, there is a constant $C > 0$ such that $|\phi(x, \theta)| \leq C\langle\theta\rangle$. The same occurs for the higher order derivatives $\partial_x^\beta \phi$ and thus, there is a constant $C > 0$ such that for all $(x, \theta) \in K \times \mathbb{R}^N$:

$$\left|P\left(\phi(x, \theta), \nabla_x \phi(x, \theta), ..., \nabla_x^{|\beta_1|}\phi(x, \theta)\right)\right| \leq C\langle\theta\rangle^{|\beta_1|}.$$

Moreover, for all $(x, \theta) \in X \times \mathbb{R}^N$:

$$|\partial_x^{\beta_2} a(x, \theta)| \lesssim \langle\theta\rangle^{m+|\beta_2|\delta} \lesssim \langle\theta\rangle^{m+|\beta_2|}.$$

Combining the two previous bounds, we see that:

$$|\partial_x^\beta (e^{i\phi(\bullet,\theta)} a(\bullet, \theta))(x)| \lesssim \langle\theta\rangle^{m+|\beta_1|+|\beta_2|} \lesssim \langle\theta\rangle^{m+k}.$$

Thus the function $\mathbb{R}^N \mapsto \partial_x^\beta (e^{i\phi(x,\theta)} a(x, \theta))$ is integrable for $m + k < -N$ and this shows that I_a is C^k. □

We now have the following technical lemma. It is of fundamental importance in the theory of pseudodifferential operators and Fourier Integral operators.

Theorem 1.1.14 (Existence and uniqueness of oscillatory integrals)**.** *Take $\rho > 0, \delta < 1$ and a phase function ϕ. For all $m \in \mathbb{R}$, there exists a unique continuous extension of*

$$I : S^{-\infty}(X \times \mathbb{R}^N) \ni a \mapsto I_a \in \mathcal{D}'(X)$$

to a map

$$S^m_{\rho,\delta}(X \times \mathbb{R}^N) \ni a \mapsto I_a \in \mathcal{D}'(X),$$

such that for all $m < -N, a \in S^m_{\rho,\delta}(X \times \mathbb{R}^N)$, I_a is given by (1.1.6).

Recall that continuity here means the following: for every compact subset $K \subset X$, there exists $k \in \mathbb{N}$, $C > 0$ and a seminorm p on $S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ (given by (1.1.1)) such that for every $\varphi \in C^\infty(X)$ with compact support in K the following holds:

$$(I_a, \varphi) \leq Cp(a)\|\varphi\|_{C^k(X)}.$$

We can even refine the conclusion of the theorem and say that for $k \in \mathbb{N}$ and $m - k\min(\rho, 1 - \delta) < -N$, the map

$$S^m_{\rho,\delta}(X, \times\mathbb{R}^N) \ni a \mapsto I_a \in \mathcal{D}'_{(k)}(X)$$

is continuous, where $\mathcal{D}'_{(k)}(X)$ denotes distributions of order k (i.e. topological dual of $C^k_{\mathrm{comp}}(X)$).

Proof. **Uniqueness:** Let I and I' be two such continuous extension and $J := I - I'$. It satisfies $J_a = 0$ for all $a \in S^{-\infty}$ and $J : S^m_{\rho,\delta} \to \mathcal{D}'(X)$ is continuous. So it suffices to show that if $S^m_{\rho,\delta}(X \times \mathbb{R}^N) \ni a \mapsto J_a \in \mathcal{D}'(X)$ is continuous and $J_a = 0$ for all $a \in S^{-\infty}(X \times \mathbb{R}^N)$, then $J_a = 0$ for all $a \in S^m_{\rho,\delta}(X \times \mathbb{R}^N)$. Consider $\chi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^N)$ and set $a_j(x, \theta) := \chi(\theta/j)a(x, \theta)$, where χ is a cutoff function ≤ 1 on $\mathbb{R}^N$ with support near 0 such that $\chi = 1$ on $B(0,1)$ and $\chi = 0$ outside $B(0,2)$. Observe that $a_j \in S^{-\infty}_{\rho,\delta}(X \times \mathbb{R}^N)$. We claim that the following holds true:

$$(J_{a_j}, \varphi) \to_{j\to\infty} (J_a, \varphi). \tag{1.1.7}$$

If the claim (1.1.7) holds true, then the proof of uniqueness is over since the left-hand side in (1.1.7) is always 0. So let us prove (1.1.7). By assumption J is a continuous extension from $S^{-\infty}$ to $S^m_{\rho,\delta}$ so we would be done if we could prove that $a_j \to a$ in $S^m_{\rho,\delta}$. Unfortunately, this is not quite exactly the case. Nevertheless, we claim that for $a \in S^m_{\rho,\delta}$, we have that $a_j \to a$ in the $S^{m'}_{\rho,\delta}$-topology, for any $m' > m$. In other words:

Lemma 1.1.15. *For any $m < m'$, $S^{-\infty}(X \times \mathbb{R}^N)$ is dense in $S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ for the $S^{m'}_{\rho,\delta}(X \times \mathbb{R}^N)$ topology.*

Proof. In order to simplify, take $m = 0$ and $m' > 0$. We define $b_j(x,\theta) := (a - a_j)(x,\theta) = (1 - \chi(\theta/j))a(x,\theta)$ which is supported in the complement of $B(0,j)$. Then, using the notation of (1.1.2):

$$\begin{aligned} p_{m',K,0,0}(b_j) &:= \sup_{(x,\theta)\in K\times\mathbb{R}^N} \frac{b_j(x,\theta)}{\langle\theta\rangle^{m'}} \\ &\leq 2p_{0,K,0,0}(a) \sup_{|\theta|\ \geq j} \langle\theta\rangle^{-m'} \leq 2p_{0,K,0,0}(a)\langle j\rangle^{-m'} \to 0. \end{aligned}$$

Higher order derivatives are treated in the same fashion and this is left as an exercise for the reader. □

Exercise 1.1.16. Complete the proof of Lemma 1.1.15.

We then use the continuity of $J : S^{m'}_{\rho,\delta} \to \mathcal{D}'(X)$ (since we now have $a_j \to a$ in $S^{m'}_{\rho,\delta}$) to conclude that $\langle J_{a_j}, \varphi\rangle \to_{j\to\infty} \langle J_a, \varphi\rangle$.

Existence: Existence is based on an integration by parts argument. First of all, we need the following:

Lemma 1.1.17. *Let ϕ be a phase function. There exists $a_j \in S^0(X \times \mathbb{R}^N), b_j, c \in S^{-1}(X \times \mathbb{R}^N)$ such that:*

$$L := \sum_{j=1}^{N} a_j\partial_{\theta_j} + \sum_{j=1}^{n} b_j\partial_{x_j} + c$$

satisfies ${}^tLe^{i\phi} = e^{i\phi}$. Moreover, if $a \in S^m_{\rho,\delta}(X \times \mathbb{R}^n)$ and $\varphi \in C^\infty(X)$ has support in $K \subset X$, then:

$$|L^k(a\varphi)(x,\theta)|\ \leq Cp(a) \sum_{|\beta|\ \leq k} \sup_{x\in K} |\partial_x^\beta\varphi(x)|\langle\theta\rangle^{m-kt}, \tag{1.1.8}$$

where $t := \min(\rho, 1-\delta), C := C(K,k) > 0$ *and* p *is some seminorm on* $S^m_{\rho,\delta}$ *as defined in* (1.1.2)*. The formal adjoint* ${}^tL$ *is given by the expression:*

$$ {}^tLu = -\sum_{j=1}^{N} \partial_{\theta_j}(a_j u) - \sum_{j=1}^{n} \partial_{x_j}(b_j u) + cu. $$

Proof. We set:

$$ \Phi(x,\theta) = \sum_j |\partial_{x_j}\phi(x,\theta)|^2 + |\theta|^2 \sum_j |\partial_{\theta_j}\phi(x,\theta)|^2. $$

Since $d\phi \neq 0$, we have $\Phi(x,\theta) \neq 0$ for $\theta \neq 0$ and Φ is 2-homogeneous in the θ-variable that is $\Phi(x,\lambda\theta) = \lambda^2\Phi(x,\theta)$ for all $\lambda > 0$.

Exercise 1.1.18. Check the 2-homogeneity.

Then define for $\chi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^N)$ a bump function with support near 0:

$$ \begin{aligned} Q &:= (1-\chi(\theta))\Phi(x,\theta)^{-1}\left(\sum_j |\theta|^2 \partial_{\theta_j}\phi(x,\theta) D_{\theta_j} + \sum_j \partial_{x_j}\phi(x,\theta) D_{x_j}\right) + \chi(\theta) \\ &= \sum_{j=1} a'_j \partial_{\theta_j} + \sum_{j=1} b'_j \partial_{x_j} + c', \end{aligned} $$

where $a'_j \in S^0, b'_j \in S^{-1}, c' \in S^{-\infty}$. The fact that these are indeed symbols in these classes follows from Example 1.1.5. Observe that $Qe^{i\phi} = e^{i\phi}$. We thus set $L := {}^tQ$. Moreover, using for instance:

$$ {}^t(a'_j\partial_{\theta_j}) = {}^t(\partial_{\theta_j}) \circ {}^t(a'_j) = -\partial_{\theta_j}(a'_j \bullet) = -\partial_{\theta_j} a'_j - a'_j \partial_{\theta_j}, $$

we get that L has the announced structure.

It remains to show (1.1.8). This can be achieved by iteration on k for instance. Using that $\partial_{\theta_j} a \in S^{m-\rho}_{\rho,\delta}$ and $\partial_{x_j} a \in S^{m+\delta}_{\rho,\delta}$, we have:

$$ \begin{aligned} L^k(a\varphi) &= L^{k-1}\left(\sum_j a_j \partial_{\theta_j}(a\varphi) + \sum_j b_j \partial_{x_j}(a\varphi) + ca\varphi\right) \\ &= L^{k-1}\left(\sum_j \underbrace{a_j(\partial_{\theta_j} a)}_{\in S^{m-\rho}_{\rho,\delta}}\varphi + \sum_j \underbrace{b_j(\partial_{x_j} a)}_{\in S^{m-(1-\delta)}_{\rho,\delta}}\varphi + \underbrace{b_j a}_{\in S^{m-1}_{\rho,\delta}}\partial_{x_j}\varphi + \underbrace{ca}_{\in S^{m-1}_{\rho,\delta}}\varphi\right) \end{aligned} $$

As a consequence, applying the iteration assumption for $k-1$, we get:

$$ |L^k(a\varphi)| \lesssim \langle\theta\rangle^{m-\rho-(k-1)t} + \langle\theta\rangle^{m-(1-\delta)-(k-1)t} + \langle\theta\rangle^{m-1-(k-1)t} \lesssim \langle\theta\rangle^{m-kt}, $$

and it is not hard to check the constant in front is correct. $\square$

Once we have this lemma, we observe that the following holds for $a \in S^{-\infty}(X \times$

$\mathbb{R}^N$):

$$(I_a, \varphi) = \int_X \int_{\mathbb{R}^N} e^{i\phi(x,\theta)} a(x,\theta)\varphi(x)\mathrm{d}\theta\mathrm{d}x = \int_X \int_{\mathbb{R}^N} e^{i\phi(x,\theta)} L^k(a\varphi)(x,\theta)\mathrm{d}\theta\mathrm{d}x. \quad (1.1.9)$$

Using (1.1.8), for $k \in \mathbb{N}$ such that $kt - m > N$, we get:

$$|(I_a, \varphi)| \lesssim p(a) \sum_{|\beta| \le k} \sup_{x \in K} |\partial_x^\beta \varphi(x)|.$$

As a consequence, for $k \in \mathbb{N}$ large enough (depending on m), we have a continuous extension $I^{(k)} : S^m_{\rho,\delta} \to \mathcal{D}'(X)$ such that $I^{(k)} = I$ on $S^{-\infty}$. Note that this extension could a priori depend on a choice of k but for any other choice of k' such that $k't - m > N$, (1.1.9) shows that $I^{(k)} = I^{(k')}$.

□

Even if it only makes sense through the extension procedure described above, we will still use the notation (1.1.6) for oscillatory integrals. In practice, the previous theorem is hard to manipulate and does not allow to make explicit computation. However, we have the following useful Lemma:

Lemma 1.1.19. *Let $a \in S^m_{\rho,\delta}(X \times \mathbb{R}^N)$. Let $\chi \in \mathcal{S}(\mathbb{R}^N)$ with $\chi(0) = 1$. Then:*

$$I_a^{(\varepsilon)} := \int_{\mathbb{R}^N} e^{i\phi(x,\theta)} a(x,\theta)\chi(\varepsilon\theta)\mathrm{d}\theta \to_{\varepsilon \to 0} I_a$$

in $\mathcal{D}'(X)$.

Note that the left-hand side is a smooth function and is therefore easier to manipulate.

Proof. We have:

$$\begin{aligned}(I_a^{(\varepsilon)}, \varphi) &= \int_{\mathbb{R}^N} e^{i\phi(x,\theta)} L^k(a(x,\theta)\chi(\varepsilon\theta)\varphi(x))\mathrm{d}\theta\mathrm{d}x \\ &= \int_{\mathbb{R}^N} e^{i\phi(x,\theta)} \chi(\varepsilon\theta) L^k(a(x,\theta)\varphi(x))\mathrm{d}\theta\mathrm{d}x + \mathcal{O}(\varepsilon).\end{aligned}$$

(Indeed, either the derivative in L is a D_{x_j} (in which case it does not affect the χ) or it is a D_{θ_j} and it affects the χ: it gives two terms, one with $\chi(\varepsilon\theta)$ and one with $\varepsilon\chi'(\varepsilon\theta)$; the first one contributes to the main term while the second contribute to a $\mathcal{O}(\varepsilon)$.) Now, this is an absolutely converging integral (since $(x,\theta) \mapsto L^k(a(x,\theta)\varphi(x))$ is integrable) and we can apply the Lebesgue dominated convergence Theorem A.3.13 to conclude. □

Remark 1.1.20. Note that the usual operations such as differentiation under the integral sign are still valid for oscillatory integrals. Indeed, these are ultimately defined as absolutely converging integrals and for these, differentiation under the integral sign is allowed.

Example 1.1.21. The Dirac mass is an oscillatory integral, that is:

$$T := \frac{1}{(2\pi)^n} \int e^{ix\cdot\xi} \mathrm{d}\xi$$

is equal to the Dirac mass $\delta_0(x)$ defined as $\langle \delta_0, \varphi \rangle := \varphi(0)$. This integral does not converge *a priori* but Theorem 1.1.14 shows that it is a well-defined oscillatory integral and it turns out to be equal to the Dirac mass. Indeed, using Lemma 1.1.19, we have by dominated convergence:

$$\begin{aligned}(T, \varphi) &= \lim_{\varepsilon \to 0} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_x} \int_{\mathbb{R}^n_\xi} e^{ix\xi} \varphi(x) \chi(\varepsilon\xi) \mathrm{d}x \mathrm{d}\xi \\ &= \lim_{\varepsilon \to 0} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\xi} \chi(\varepsilon\xi) \widehat{\varphi}(-\xi) \mathrm{d}\xi = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\xi} \widehat{\varphi}(-\xi) \mathrm{d}\xi = \varphi(0),\end{aligned}$$

by Fourier inversion formula.

1.1.4 Exercises

Exercise 1

Define $\xi = (\xi', \xi_n) \in \mathbb{R}^n$, $\xi'^2 = \sum_{j=1}^{n-1} \xi_j^2, \xi^2 = \xi'^2 + \xi_n^2$. To which symbol space do the following belong?

1. $(\xi'^2 + i\xi_n)^{-1}$,
2. $\chi(|\xi|)(\xi'^2 + i\xi_n)^{-1}$, where $\chi \in C^\infty(\mathbb{R})$ is such that $\chi = 1$ for $|x| \geq 1$ and $\chi = 0$ for $|x| \leq 1/2$,
3. $(\xi^2 + 1)^{-1}$,
4. $(\xi'^2 + 1)^{1}$,
5. $e^{i\xi^2}$,
6. $e^{ix\cdot\xi}$.

Exercise 2

1. Let $a \in C^\infty(X \times \mathbb{R}^N)$ such that for all $x \in X, a(x, \bullet)$ has compact support in $\mathbb{R}^N$. Show that $a \in S^{-\infty}(X \times \mathbb{R}^N)$.
2. Let $a \in C^\infty(X \times \mathbb{R}^N)$ be positively homogeneous of order m for $|\theta| \geq 1$, namely $a(x, \lambda\theta) = \lambda^m a(x, \theta)$ for all $\lambda \geq 1, |\theta| \geq 1$. Show that $a \in S^m_{1,0}(X \times \mathbb{R}^N)$.

Exercise 3

Assume that $a \in S^m_{\rho,0}(X \times \mathbb{R}^N)$ with $m < 0$ and $\rho > 1$. Show that $a \in S^{-\infty}(X \times \mathbb{R}^N)$.

Hint: Apply $|\theta|\partial_\theta$ many times and then integrate by parts to recover the expression of a.

Exercise 4: Borel's theorem

The goal of this exercise is to show the following:

Theorem 1.1.22 (Borel)**.** *For every sequence $(a_\alpha)_{\alpha\in\mathbb{N}^n}$ of complex numbers $a_\alpha \in \mathbb{C}$, there exists $u \in C^\infty(\mathbb{R}^n)$ such that $\partial^\alpha u(0) = a_\alpha$.*

Let $\chi \in C^\infty(\mathbb{R}^n, [0,1])$ be such that $\chi = 0$ for $|x| \geq 1$ and $\chi = 1$ for $|x| \leq 1$. Define for $\lambda > 0$:

$$u_N(x,\lambda) := \chi(\lambda x) \sum_{|\alpha|=N} \frac{a_\alpha}{\alpha!} x^\alpha.$$

1. Compute $\partial_x^\beta u_N(0)$.
2. Show that $\|u_N(x,\lambda)\|_{C^{N-1}} \leq 2^{-N}$ if $\lambda \geq \lambda_N$, where λ_N is chosen large enough.
3. Show that $u(x) := \sum_{N\geq 0} u_N(x,\lambda_N)$ solves the problem.

Exercise 5:

Show Lemma 1.1.10.

1.2 Schwartz kernel Theorem. Applications

We consider two open subsets $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ for $n, m \in \mathbb{N}$. In the following, we will say that a linear operator $A : C^\infty_{\mathrm{comp}}(X) \to \mathcal{D}'(Y)$ is *continuous* if it is sequentially continuous, in the sense that for all $\varphi, \psi \in C^\infty_{\mathrm{comp}}(X)$, if $\varphi_j \to \varphi$ in $C^\infty_{\mathrm{comp}}(X)$, then:

$$(A\varphi_j, \psi) \to_{j\to\infty} (A\varphi, \psi).$$

We refer to Appendix A.2 for the definition of convergence of smooth functions.

1.2.1 Main theorem

1.2.1.1 Structural result

We start with the following important structural result on linear operators:

Theorem 1.2.1 (Schwartz, 1952)**.** *Let $A : C^\infty_{\mathrm{comp}}(Y) \to \mathcal{D}'(X)$ be a continuous linear operator. Then, there exists a unique distribution $K_A \in \mathcal{D}'(X \times Y)$ such that for all $\varphi \in C^\infty_{\mathrm{comp}}(Y), \psi \in C^\infty_{\mathrm{comp}}(X)$:*

$$(A\varphi, \psi) = (K_A, \psi \otimes \varphi).$$

Conversely, such a distribution K_A defines uniquely a continuous linear operator $A : C^\infty_{\mathrm{comp}}(Y) \to \mathcal{D}'(X)$.

More generally, X and Y can be replaced smooth manifolds, see §A.4. The proof is given as an exercise below (and as a homework!).

Example 1.2.2. Consider the identity map $\mathbb{1} = C^\infty_{\text{comp}}(\mathbb{R}^n) \to C^\infty_{\text{comp}}(\mathbb{R}^n)$. Then

$$K_{\mathbb{1}}(x, y) = \delta_0(x - y),$$

where δ_0 is the Dirac mass. The last distribution has to be understood as the integration over the diagonal Δ in $\mathbb{R}^n \times \mathbb{R}^n$. Indeed:

$$(\mathbb{1}\varphi, \psi) = \int_{\mathbb{R}^n} \varphi(x)\psi(x)\mathrm{d}x = (\delta_0(x - y), \psi \otimes \varphi).$$

Lemma 1.2.3. *Let $A : C^\infty_{\text{comp}}(Y) \to \mathcal{D}'(X)$ be a continuous linear operator with Schwartz kernel $K_A \in \mathcal{D}'(X \times Y)$. Then the following are equivalent:*

(i) *$K_A \in C^\infty(X \times Y)$,*

(ii) *A admits a (unique) continuous extension to a map $A : \mathcal{E}'(Y) \to C^\infty(X)$.*

Before proving Lemma 1.2.3, let us give some examples.

Example 1.2.4. A trivial (but stupid!) example is the zero operator $A : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ such that $Af := 0$. A more interesting one is any multiplication in the Fourier variable by a compactly supported function. Indeed, taking $\chi \in C^\infty_{\text{comp}}(\mathbb{R}^n)$, we define $A_\chi : C^\infty_{\text{comp}}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$ as:

$$A_\chi f := \mathcal{F}^{-1}(\chi \mathcal{F} f).$$

This is easily seen to be equal to

$$A_\chi f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_y} \left(\int_{\mathbb{R}^n_\xi} e^{i\xi(x-y)} \chi(\xi)\mathrm{d}\xi \right) f(y)\mathrm{d}y,$$

and convergence of this integral is ensured by the fact that both f and χ have compact support. The kernel of A_χ is also directly given by

$$K_\chi(x, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\xi} e^{i\xi(x-y)} \chi(\xi)\mathrm{d}\xi = \frac{\widehat{\chi}(y - x)}{(2\pi)^n},$$

and this is clearly a smooth function of x and y. Given a distribution $f \in \mathcal{E}'(\mathbb{R}^n)$, $A_\chi f$ is a smooth function. The operator A_χ is called a *mollifier*. Letting the support of χ grow, we get a family of smoothing operators which approximate the identity.

Proof of Lemma 1.2.3. If (i) holds, then given $u \in \mathcal{E}'(Y)$, we claim that $Au(x) = (K_A(x, \bullet), u)_Y$. Indeed, first of all, this holds for all $u \in C^\infty_{\text{comp}}(Y)$ because if we pair against $\psi \in C^\infty_{\text{comp}}(X)$, we get $(Au, \psi) = (K_A, \psi \otimes u) = (x \mapsto K_A(x, \bullet), u)_Y, \psi)_X$. Now, the expression $\mathcal{E}'(Y) \ni u \mapsto (x \mapsto (K_A(x, \bullet), u)_Y) \in C^\infty(X)$ is a well-defined continuous extension of the map initially defined for $u \in C^\infty_{\text{comp}}(Y)$, that is, it is well-defined and if $u_j \to u$ in $\mathcal{E}'(Y)$, $(K_A(x, \bullet), u_j)_Y \to (K_A(x, \bullet), u)_Y$ in $C^\infty(X)$. Note that by Lemma A.3.7, this is a smooth function of x with derivatives $\partial_x^\alpha Au(x) = (\partial_x^\alpha K_A(x, \bullet), u)_Y$. The uniqueness of the extension follows from the density of smooth functions in compactly supported distributions.

Conversely, assume that (ii) holds. Observe that the map $X \ni x \mapsto \delta_x \in \mathcal{D}'(X)$ is continuous. Indeed: if $x_n \to x \in X$, then $\delta_{x_n} \to \delta_x$ in $\mathcal{D}'(X)$, that is for all $\varphi \in C^\infty_{\mathrm{comp}}(X)$, one has $(\delta_{x_n}, \varphi) = \varphi(x_n) \to \varphi(x) = (\delta_x, \varphi)$. As a consequence, $F(\bullet, y) := [A\delta_y](\bullet) \in C^\infty(X)$ and $y \mapsto K_A(\bullet, y) \in C^\infty(X)$ is continuous. In particular, $(x, y) \mapsto F(x, y)$ is continuous. Now, we claim that $F = K_A$. For that, it suffices to check that for all $\varphi \in C^\infty_{\mathrm{comp}}(Y), \psi \in C^\infty_{\mathrm{comp}}(X)$, $(A\varphi, \psi) = (F, \psi \otimes \varphi)$ since we can then conclude by uniqueness of the Schwartz kernel. Given $\varphi \in C^\infty_{\mathrm{comp}}(Y)$, we can approximate it by Dirac masses: indeed, take a Cartesian grid of $\mathbb{R}^n$ with rectangles of volume $1/N$ and call $y_N^{(i)}$ the vertices of the rectangles; then,

$$T_N := \sum_{i=1}^{K_N} \tfrac{1}{N}\varphi(y_N^{(i)})\delta_{y_N^{(i)}} \to \varphi, \qquad \text{in } \mathcal{E}'(Y),$$

since we have for $\psi \in C^\infty_{\mathrm{comp}}(Y)$, using Riemann integrals:

$$(T_N, \psi) = \sum_{i=1}^{K_N} \tfrac{1}{N}(\varphi\psi)(y_N^{(i)}) \to \int_Y \varphi(y)\psi(y)\mathrm{d}y.$$

As a consequence, we have the equality (as smooth function in $C^\infty(X)$):

$$AT_N = \sum_{i=1}^{K_N} \tfrac{1}{N}\varphi(y_N^{(i)})F(x, y_N^{(i)}).$$

Taking the limit as $N \to \infty$, and using the continuity of A, this equality yields:

$$A\varphi = \int_Y F(x, y)\varphi(y)\mathrm{d}y.$$

Pairing against $\psi \in C^\infty_{\mathrm{comp}}(X)$, we then obtain:

$$(A\varphi, \psi) = \int_{X\times Y} F(x, y)\psi(x)\varphi(y)\mathrm{d}x\mathrm{d}y = (K_A, \psi \otimes \varphi) = F(, \psi \otimes \varphi).$$

Hence $F = K_A$ is continuous (and smooth in the x-variable). Applying the same argument with $\partial^\alpha \delta_y$ instead of δ_y, we obtain that $\partial_y^\alpha \partial_x^\beta K \in C^0(X \times Y)$ is continuous and this eventually proves that K is smotoh. □

Exercise 1.2.5. As an exercise, verify that you know how to prove the last claim: if $u \in \mathcal{D}'(\mathbb{R}^n)$ is a distribution such that $\partial^\alpha u \in C^0(\mathbb{R}^n)$ (the derivative is understood in the distributional sense, and this means that this distributional derivative coincides with a continuous function on $\mathbb{R}^n$) for all $\alpha \in \mathbb{N}^n$, then u coincides with a smooth function. *Hint: First, localize in a compact set using a test function with compact support. Second, use the operator* $(1 + \Delta)^k$, *where* $\Delta = -\sum \partial_{x_i}^2$ *is the Laplacian, and the Fourier transform.*

1.2.1.2 Adjoint and transpose operators

We now define the *transpose* of an operator:

Lemma 1.2.6. *Given a continuous operator $A : C^\infty_{\mathrm{comp}}(Y) \to \mathcal{D}'(X)$, the* transpose operator ${}^tA : C^\infty_{\mathrm{comp}}(X) \to \mathcal{D}'(Y)$ *is defined as the unique continuous operator satisfying for all $\varphi \in C^\infty_{\mathrm{comp}}(Y), \psi \in C^\infty_{\mathrm{comp}}(X)$:*

$$(A\varphi, \psi) = ({}^tA\psi, \varphi).$$

Its Schwartz kernel is given by $K_{{}^tA}(y, x) = K_A(x, y)$.

Usually, we prefer to write the identity as $(A\varphi, \psi) = (\varphi, {}^tA\psi)$ and we think of this equality as

$$(A\varphi, \psi)_X = \int_X A\varphi(x)\psi(x)\mathrm{d}x = \int_Y \varphi(y){}^tA\psi(y)\mathrm{d}y = (\varphi, {}^tA\psi)_Y,$$

even though this only makes sense at the level of distributions. There is nothing to prove and the statement is almost tautological (check this).

The L^2-scalar product is defined by

$$\langle \varphi, \psi \rangle_{L^2(X)} := \int_X \varphi(x)\ \overline{\psi(x)}\mathrm{d}x.$$

Unlike the bilinear pairing $(\bullet, \bullet)_X$, the scalar product is Hermitian, that is $\mathbb{C}$-antilinear in the second variable. We will use the notation $\langle A\varphi, \psi \rangle = (A\varphi, \overline{\psi})$ (one has to think of this as the L^2-scalar product, but these are not L^2-functions here). We now introduce the formal adjoint of an operator. We will use the convention $\langle \varphi, A^*\psi \rangle := \overline{(A^*\psi, \overline{\varphi})}$.

Lemma 1.2.7. *Given a continuous operator $A : C^\infty_{\mathrm{comp}}(Y) \to \mathcal{D}'(X)$, the* formal adjoint operator $A^* : C^\infty_{\mathrm{comp}}(X) \to \mathcal{D}'(Y)$ *is defined as the unique continuous operator satisfying for all $\varphi \in C^\infty_{\mathrm{comp}}(Y), \psi \in C^\infty_{\mathrm{comp}}(X)$:*

$$\langle A\varphi, \psi \rangle = \langle \varphi, A^*\psi \rangle.$$

Its Schwartz kernel is given by $K_{A^*}(y, x) = \overline{K_A(x, y)}$.

Once again, we want to think of this as the L^2-adjoint of the operator A. For instance, if A had a smooth Schwartz kernel, then the following equality would hold: $\langle A\varphi, \psi \rangle_{L^2(X)} = \langle \varphi, A^*\psi \rangle_{L^2(Y)}$.

1.2.2 Operators with oscillatory kernels

The heart of the theory of pseudodifferential operators and Fourier Integral operators is to consider operators whose Schwartz kernel $K \in \mathcal{D}'(X \times Y)$ *is* an oscillatory integral. Given a phase function $\phi \in C^\infty(X \times Y \times \mathbb{R}^N)$ and $a \in S^m_{\rho,\delta}(X \times Y \times \mathbb{R}^N)$, we set:

$$K_A(x, y) := \int_{\mathbb{R}^N} e^{i\phi(x,y,\theta)} a(x, y, \theta)\mathrm{d}\theta \in \mathcal{D}'(X \times Y), \tag{1.2.1}$$

where this is understood as an oscillatory integral. This kernel defines a bounded operator $A : C^\infty_{\mathrm{comp}}(Y) \to \mathcal{D}'(X)$. The continuity properties can be refined under certain assumptions:

Theorem 1.2.8. *Assume that:*

(i) *for every $x \in X, (y,\theta) \mapsto \phi(x,y,\theta)$ is a phase function;*

(ii) *for every $y \in Y, (x,\theta) \mapsto \phi(x,y,\theta)$ is a phase function.*

Then $A : C^\infty_{\mathrm{comp}}(Y) \to C^\infty(X)$ is continuous and admits a continuous extension $A : \mathcal{E}'(Y) \to \mathcal{D}'(X)$.

Proof. (1) Following the existence part of the proof of Theorem 1.1.14, since $d_{(y,\theta)}\phi \neq 0$, we can find an operator

$$L = \sum_j a_j(x,y,\theta)D_{\theta_j} + \sum_j b_j(x,y,\theta)D_{y_j} + c(x,y,\theta),$$

where:

$$a_j \in S^0_{1,0}(X \times Y \times \mathbb{R}^N), b_j \in S^{-1}_{1,0}(X \times Y \times \mathbb{R}^N), c \in S^{-1}_{1,0}(X \times Y \times \mathbb{R}^N),$$

and such that ${}^tLe^{i\phi} = e^{i\phi}$. (This is a parametric version of Lemma 1.1.17 with parameter x.) Then, for $\varphi \in C^\infty_{\mathrm{comp}}(Y), \psi \in C^\infty_{\mathrm{comp}}(X)$, and $k \in \mathbb{N}$ large enough, we get:

$$\langle A\varphi, \psi\rangle = \langle K_A, \psi \otimes \varphi\rangle = \int_{\mathbb{R}^n_x}\int_{\mathbb{R}^m_y}\int_{\mathbb{R}^N_\theta} e^{i\phi(x,y,\theta)}\psi(x)L^k(a(x,y,\theta)\varphi(y))\mathrm{d}x\ \mathrm{d}y\mathrm{d}\theta,$$

that is

$$A\varphi(x) = \int_{\mathbb{R}^m_y}\int_{\mathbb{R}^N_\theta} e^{i\phi(x,y,\theta)}L^k(a(x,y,\theta)\varphi(y))\mathrm{d}y\mathrm{d}\theta$$

Following the proof of Lemma 1.1.13, we see that this is indeed a smooth function in $C^\infty(X)$.

(2) For the second part of the proof, the action of $A : \mathcal{E}'(Y) \to \mathcal{D}'(X)$ is defined by duality, namely given $u \in \mathcal{E}'(Y)$ and $\psi \in C^\infty_{\mathrm{comp}}(X)$, we have:

$$\langle Au, \psi\rangle := \langle u, {}^tA\psi\rangle.$$

Now, ${}^tA : C^\infty_{\mathrm{comp}}(X) \to \mathcal{D}'(Y)$ is the operator whose kernel is given by $K_{{}^tA}(y,x) = K_A(x,y)$. As a consequence, using the proof just above, the second condition ensures that ${}^tA : C^\infty_{\mathrm{comp}}(X) \to C^\infty(Y)$ is continuous, which is exactly the desired result. □

1.2.3 Pseudodifferential operators

They correspond to $X = Y$, $N = n$ and $\phi(x,y,\xi) := (x-y)\cdot\xi$, namely these are all the linear operators $A : C^\infty_{\mathrm{comp}}(X) \to \mathcal{D}'(X)$ with kernel given by

$$K_A(x,y) := \int_{\mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi}a(x,y,\xi)\mathrm{d}\xi,$$

for some symbol $a \in S^m_{\rho,\delta}(X \times X \times \mathbb{R}^n)$.

Definition 1.2.9 (Quantization)**.** Given a symbol $a \in S^m_{\rho,\delta}(X \times X \times \mathbb{R}^n)$, we define its *quantization* $\mathrm{Op}(a) : C^\infty_{\mathrm{comp}}(X) \to \mathcal{D}'(X)$ as the linear operator whose kernel is given by

$$K(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi} a(x,y,\xi) \mathrm{d}\xi.$$

We denote this class of operator by

$$\Psi^m_{\rho,\delta}(X) := \left\{ \mathrm{Op}(a) \mid a \in S^m_{\rho,\delta}(X \times X \times \mathbb{R}^n) \right\} \tag{1.2.2}$$

and we call it the class of pseudodifferential operators (ΨDOs).

The factor $(2\pi)^{-n}$ is there for normalizing reasons. Let us give some important examples:

Example 1.2.10. Take $X = Y = \mathbb{R}^n$ and $a(x,y,\xi) := 1$. Then:

$$K_{\mathrm{Op}(a)}(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi} \mathrm{d}\xi = \delta_0(x-y) = K_{\mathbb{1}}(x,y).$$

In other words, the quantization of the constant function is the identity. Phew!

Take $X = Y = \mathbb{R}$ and $a(x,y,\xi) := i\xi$. Then:

$$\mathrm{Op}(a)\varphi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}_y} \int_{\mathbb{R}_\xi} e^{i(x-y)\cdot\xi} i\xi\varphi(y) \mathrm{d}\xi \mathrm{d}y = \mathcal{F}^{-1}\left(i\xi\mathcal{F}(\varphi)\right)(x) = \partial_x\varphi(x).$$

More generally, the quantization of symbols that are polynomial in ξ yields *differential operators*, namely if $A = \sum_{|\alpha|\ \leq m} a_\alpha(x) D^\alpha_x$, then setting $a(x,\xi) := \sum_{|\alpha|\ \leq m} a_\alpha(x)\xi^\alpha$, one finds that

$$A\varphi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_y} \int_{\mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi} a(x,\xi)\varphi(y) \mathrm{d}\xi \mathrm{d}y = \mathrm{Op}(a)\varphi(x).$$

For the moment, let us just point out that the quantization map gives a surjective linear map

$$\mathrm{Op} : S^m_{\rho,\delta}(X \times X \times \mathbb{R}^n) \to \Psi^m_{\rho,\delta}(X),$$

but we have not said anything yet about the topology of the target space, nor about the injectivity of this map. The space $\Psi^{-\infty}(X \times \mathbb{R}^N)$ corresponds to the quantization of symbols in $S^{-\infty}(X \times \mathbb{R}^N)$. The Schwartz kernels of these operators are *smooth functions* in $C^\infty(X \times Y)$ as follows from Lemma 1.1.13.

It is straightforward to generalize the previous discussion to sections of the trivial bundle $\mathbb{C}^r \to X$. The symbols are then *matrix-valued* namely $a \in S^m_{\rho,\delta}(X \times X \times \mathbb{R}^n, \mathrm{Hom}(\mathbb{C}^p, \mathbb{C}^q))$, that is $a = (a_{ij})_{1\leq i\leq q, 1\leq j\leq p}$ and each $a_{ij} \in S^m_{\rho,\delta}(X \times X \times \mathbb{R}^n)$. They act as operators $A : C^\infty_{\mathrm{comp}}(X, \mathbb{C}^p) \to C^\infty(X, \mathbb{C}^q)$. (We use the notation $C^\infty(X)$ for $C^\infty(X, \mathbb{C})$.)

1.2.4 Exercises

Exercise 1

Consider the gradient operator $\nabla : C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n, \mathbb{C}^n)$ defined by $\nabla\varphi(x) := {}^t(\partial_{x_1}\varphi(x), ..., \partial_{x_n}\varphi(x))$. Show that it is obtained as the quantization of the symbol $a(x, y, \xi) := i\xi$.

Exercise 2: The Schwartz kernel Theorem

Let $X, Y \subset \mathbb{R}^n$ be open subsets. Recall that, given $\varphi \in C^\infty_{\mathrm{comp}}(X)$ and $(\varphi_n)_{n\geq 0}$, a sequence of functions in $C^\infty_{\mathrm{comp}}(X)$, $\varphi_n \to \varphi$ if for all $n \geq 0$ large enough, all functions φ_n have support in a fixed compact subset $K \subset X$ and $\|\varphi_n - \varphi\|_{C^m(K)} \to 0$ for all $m \geq 0$. Denote by $\mathcal{L}(C^\infty_{\mathrm{comp}}(Y), \mathcal{D}'(X))$ the space of continuous linear operators acting on compactly supported functions on Y, that is, $A \in \mathcal{L}(C^\infty_{\mathrm{comp}}(Y), \mathcal{D}'(X))$ if and only if $A\varphi_n \to A\varphi$ in $\mathcal{D}'(X)$ for all $\varphi_n \to \varphi$ in $C^\infty_{\mathrm{comp}}(Y)$. The aim of this homework is to prove the Schwartz kernel Theorem:

Theorem (Schwartz, 1952). *The map*

$$\Phi : \mathcal{D}'(X \times Y) \ni K \mapsto A_K \in \mathcal{L}(C^\infty_{\mathrm{comp}}(Y), \mathcal{D}'(X)),$$

defined by $(A_K(\varphi), \psi) := (K, \psi \otimes \varphi)$ *for* $\psi \in C^\infty_{\mathrm{comp}}(X)$ *and* $\varphi \in C^\infty_{\mathrm{comp}}(Y)$*, is an isomorphism.*

We will not care much about the continuity of Φ, so by isomorphism, we simply mean a bijective linear map.

We recall that Sobolev spaces $H^s(\mathbb{R}^n)$ are defined as the completion of $C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ with respect to the norm

$$\|f\|^2_{H^s} := \int_{\mathbb{R}^n} \langle\xi\rangle^{2s} |\widehat{f}(\xi)|^2 \, \mathrm{d}\xi.$$

For $\ell \in \mathbb{R}$, the space $\langle x\rangle^\ell H^s(\mathbb{R}^n)$ consists of all functions $f = \langle x\rangle^\ell \widetilde{f}$, for some $\widetilde{f} \in H^s(\mathbb{R}^n)$. The natural norm on this space is

$$\|f\|_{\langle x\rangle^\ell H^s(\mathbb{R}^n)} := \|\langle x\rangle^{-\ell} f\|_{H^s(\mathbb{R}^n)}.$$

1. **Definition of Φ.** Show that Φ is well-defined.
2. **Weighted Sobolev spaces.**
   (a) Show that the pairing given by
   $$C^\infty_{\mathrm{comp}}(\mathbb{R}^n) \times C^\infty_{\mathrm{comp}}(\mathbb{R}^n) \ni \varphi, \psi \mapsto (\varphi, \psi) := \int_{\mathbb{R}^n} \varphi(x)\psi(x)\mathrm{d}x \in \mathbb{C} \quad (1.2.3)$$
   extends continuously to a pairing $\langle x\rangle^\ell H^s(\mathbb{R}^n) \times \langle x\rangle^{-\ell} H^{-s}(\mathbb{R}^n) \to \mathbb{C}$, for all $s, \ell \in \mathbb{R}$.

(b) Show that for all $s,\ \ell \in \mathbb{R}$, there exists a natural identification $\langle x\rangle^{\ell}H^s(\mathbb{R}^n)$ with $\langle x\rangle^{-\ell}H^{-s}(\mathbb{R}^n)$ using the extension of the pairing (1.2.3).
In other words, show that there is a natural isometry $\Psi : \left(\langle x\rangle^{\ell}H^s(\mathbb{R}^n)\right)' \to \langle x\rangle^{-\ell}H^{-s}(\mathbb{R}^n)$ *such that for all* $T \in \left(\langle x\rangle^{\ell}H^s(\mathbb{R}^n)\right)'$, $T(\varphi) = (\Psi(T), \varphi)$, *where the last pairing is understood as the continuous extension obtained in* (a).

As a first step towards proving the general Schwartz kernel Theorem, we want to prove it on Schwartz spaces. Recall that $\varphi_n \to \varphi$ in $\mathscr{S}(\mathbb{R}^n)$ if all Schwartz semi-norms are convergent. We say that $A : \mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ is continuous if $A\varphi_n \to A\varphi$ for all $\varphi_n \to \varphi$ in $\mathscr{S}(\mathbb{R}^n)$, that is, for all $\psi \in \mathscr{S}(\mathbb{R}^n)$,

$$(A\varphi_n, \psi) \to (A\varphi, \psi).$$

For $M \geq 0$, we denote by $\|\cdot\|_{(M)}$ the norm $\|\cdot\|_{\langle x\rangle^{-M}H^{+M}(\mathbb{R}^n)}$.

2. **Schwartz kernel Theorem on Schwartz spaces.**

(a) Show that $A \in \mathcal{L}(\mathscr{S}(\mathbb{R}^n), \mathscr{S}'(\mathbb{R}^n))$ if and only if there exists $M \geq 0$ large enough and $C > 0$ such that for all $\varphi, \psi \in \mathscr{S}(\mathbb{R}^n)$:

$$|(A\varphi, \psi)| \ \leq C\|\varphi\|_{(M)}\|\psi\|_{(M)}.$$

(b) Deduce that $A : \langle x\rangle^{-M}H^{M}(\mathbb{R}^n) \to \langle x\rangle^{+M}H^{-M}(\mathbb{R}^n)$ is bounded.

(c) We now consider a continuous map $A : L^2(\mathbb{R}^n) \to H^m(\mathbb{R}^n)$. Show that for $m > 0$ large enough, and $x \in \mathbb{R}^n$, the map $L^2(\mathbb{R}^n) \ni \varphi \mapsto (A\varphi)(x) \in \mathbb{C}$ can be well-defined and is continuous.

(d) Deduce that that $A\varphi(x) = (T_x, \varphi) = \int_{\mathbb{R}^n} T_x(y)\varphi(y)\, \mathrm{d}y$ for a certain map $T : \mathbb{R}^n \ni x \mapsto T_x \in L^2(\mathbb{R}^n_y)$ such that $T \in C^0_{\text{bded}}(\mathbb{R}^n_x, L^2(\mathbb{R}^n_y))$. (The last space denotes continuous functions with values in L^2 such that the C^0 norm is uniformly bounded on $\mathbb{R}^n$.)

(e) Prove that $\mathscr{S}(\mathbb{R}^n_x) \otimes \mathscr{S}(\mathbb{R}^n_y)$ is dense in $\mathscr{S}(\mathbb{R}^n_x \times \mathbb{R}^n_y)$.

(f) Eventually, deduce a version of the Schwartz kernel Theorem for Schwartz spaces: the map $\mathscr{S}'(\mathbb{R}^n \times \mathbb{R}^n) \ni K \mapsto A_K \in \mathcal{L}(\mathscr{S}(\mathbb{R}^n), \mathscr{S}'(\mathbb{R}^n))$ is an isomorphism.

3. **Schwartz kernel Theorem.**

(a) Show that $C^\infty_{\text{comp}}(X) \otimes C^\infty_{\text{comp}}(Y)$ is dense in $C^\infty_{\text{comp}}(X \times Y)$.

(b) Prove the Schwartz kernel Theorem.

1.3 Stationary phase lemma

The goal of this paragraph is to investigate the asymptotic behaviour of oscillatory integrals of the form:

$$I(h) := \int_X e^{\frac{i}{h}\phi(x)}a(x)\mathrm{d}x,$$

as $h \to 0$, where $a \in C^\infty_{\text{comp}}(\mathbb{R}^n)$, and ϕ is a certain phase function.

1.3.1 Non-stationary phase lemma

We start with the most elementary case:

Lemma 1.3.1 (Non-stationary phase lemma)**.** *Let $a \in C^\infty_{\text{comp}}(\mathbb{R}^n)$, $\phi \in C^\infty(X)$ such that $d\phi \neq 0$ on the support of a. Then $I(h) = \mathcal{O}(h^\infty)$.*

By this notation, we mean that the following estimate holds: for all $N \geq 0$, there exists $C_N > 0$ such that for all $0 < h \leq 1$, one has $|I(h)| \leq C_N h^N$.

Proof. As most of the previous proofs, this is based on an repeated integration by parts by introducing L such that

$$^tL := h|d_x\phi(x)|^{-2}\sum_{j=1}^{n}\partial_{x_j}\phi(x)D_{x_j}$$

which satisfies $^tLe^{\frac{i}{h}\phi} = e^{\frac{i}{h}\phi}$. □

1.3.2 Stationary phase lemma

We now consider the case where $d\phi$ might cancel on isolated points. We will say that the phase function ϕ is *non-degenerate* (on X) if there is only a discrete set of points $x_1, ..., x_n, ... \in X$ on which $d\phi$ vanishes and the Hessian $d^2\phi(x_i)$ is non-degenerate (i.e. is an invertible matrix) when the gradient vanishes.

Theorem 1.3.2 (Stationary phase lemma)**.** *Let $\phi \in C^\infty(X), a \in C^\infty_{\text{comp}}(X)$ and assume that ϕ is non-degenerate with single critical point $x_0 \in \operatorname{supp}(a)$. Then, for all $k \in \mathbb{N}$, there exists differential operators $A_{2k}(x, D)$ of order $\leq 2k$ such that for all $N \in \mathbb{N}$:*

$$\begin{aligned} I(h) = &\frac{(2\pi h)^{n/2}}{|\det \nabla^2\phi(x_0)|^{1/2}}e^{\frac{i}{h}\phi(x_0)}e^{\frac{i\pi}{4}\operatorname{sgn}(\nabla^2\phi(x_0))}\sum_{k=0}^{N-1}h^k\left[A_{2k}(x, D)a\right](x_0) \\ &+ R(h), \end{aligned}$$

where:

$$R(h) \leq h^{n/2+N}\frac{\sum_{|\alpha| \ \leq n+1}\|\partial^\alpha\langle\nabla^2\phi(x_0)^{-1}D, D\rangle^N a'\|_{L^1(\mathbb{R}^n)}}{(2\pi)^{n/2}|\det \nabla^2\phi(x_0)|^{1/2}(N!)2^N}.$$

Moreover $A_0 = \mathbb{1}$ and thus:

$$I(h) = \frac{(2\pi h)^{n/2}}{|\det \nabla^2\phi(x_0)|^{1/2}}e^{\frac{i}{h}\phi(x_0)}e^{\frac{i\pi}{4}\operatorname{sgn}(\nabla^2\phi(x_0))}a(x_0) + \mathcal{O}(h^{n/2+1}).$$

In the general case, one has to sum over the finite number of critical points $x_1, ..., x_p \in X$ (note that a has compact support in X so this number is always finite if ϕ is non-degenerate). In order to do the proof, we will need the following lemma:

Lemma 1.3.3 (Morse Lemma)**.** *Let $\phi \in C^\infty(X)$. Assume that $\nabla\phi(x_0) = 0$ and that the Hessian $\nabla^2\phi(x_0)$ is non-degenerate (i.e. invertible). Then there exists a diffeomorphism $\kappa : V \to U$, where U is a small neighborhood of x_0 and V is a small neighborhood of $0 \in \mathbb{R}^n$ such that for all $y \in V$:*

$$\kappa^*\phi(y) = \phi(x_0) + \frac{1}{2}\left(y_1^2 + ... + y_r^2 - (y_{r+1}^2 + ... + y_n^2)\right).$$

The quantity $\operatorname{sgn}(\nabla\phi(x_0)) := r - (n - r)$ is called the signature of the Hessian. Moreover, $|\det d\kappa(0)| = |\det d^2\phi(x_0)|^{-1/2}$.

Proof. Without loss of generality, we can directly assume that $x_0 = 0$. We consider the following Taylor expansion:

$$\phi(x) = \phi(0) + \underbrace{d\phi(0)}_{=0}x + \int_0^1 (1-t)^2 d^2\phi(tx)(x,x)\mathrm{d}t = \phi(0) + \langle Q(x)x, x\rangle,$$

where

$$Q(x) = \int_0^1 (1-t)d^2\phi(tx)\mathrm{d}t, \qquad Q(0) = \frac{1}{2}d^2\phi(0).$$

We want to write $Q(x) = \frac{1}{2}A(x)^\top I_{r,n-r}A(x)$ for some matrix $A(x) \in \mathrm{GL}_n(\mathbb{R})$ and

$$I_{r,n-r} = \begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix},$$

with same sign as $Q(0)$. For $x = 0$, $Q(0)$ is a symmetric matrix with signature $(r, n-r)$ so it can be written $Q(0) = A(0)^\top I_{r,n-r}A(0)$ for some matrix $A(0) \in \mathrm{GL}_n(\mathbb{R})$. We then want to solve $F(x, A(x)) = 0$, where

$$F : \mathbb{R}^n \times \mathrm{GL}_n(\mathbb{R}) \to \mathrm{Sym}_n(\mathbb{R})$$

is defined by $F(x, A) := \frac{1}{2}A^\top I_{r,n-r}A - Q(x)$. We apply the implicit function theorem at $(0, A(0))$. We have:

$$\begin{aligned}\partial_A F(0, A(0))H &= \frac{1}{2}\left(H^\top I_{r,n-r}A(0) + A(0)^\top I_{p,q}H\right) \\ &= \frac{1}{2}\left((A_0^{-1}H)^\top Q(0) + Q(0)(A_0^{-1}H)\right).\end{aligned}$$

It suffices to show that this differential is surjective. Given $B \in \mathrm{Sym}_n(\mathbb{R})$, it suffices to set $H := \frac{1}{4}A(0)Q(0)^{-1}B$ to obtain $\partial_A F(0, A(0))H = B$. Hence, by the implicit function theorem, we can find an inverse depending smoothly on $x \in \mathbb{R}^n$ such that $F(x, G(x, \bullet)) = \mathbb{1}_{\mathrm{Sym}_n(\mathbb{R})}$. We then set $A(x) := G(x, 0_{\mathbb{R}^n})$, which solves $F(x, A(x)) = 0$. This proves that $\phi(x) = \phi(0) + \langle I_{p,q}A(x)x, A(x)x\rangle$. We then define the new coordinate as $y := A(x)x$. It remains to check that $x \mapsto A(x)x$ is a diffeomorphism for x close to 0. But looking at the differential at 0, we get $d(A(x)x)(x = 0) = A(0)$ and this is invertible so $x \mapsto A(x)x$ is a local diffeomorphism. $\square$

We will also need the following formula:

Lemma 1.3.4. *Given $Q \in \mathrm{Sym}_n(\mathbb{R})$, a symmetric positive definite matrix, one has:*

$$\mathcal{F}\left(e^{-\frac{1}{2}\langle Qx,x\rangle}\right) = \frac{(2\pi)^{n/2}}{(\det Q)^{1/2}} e^{-\frac{1}{2}\langle Q^{-1}\xi,\xi\rangle}. \tag{1.3.1}$$

Given $Q \in \mathrm{Sym}_n(\mathbb{R})$, one has:

$$\mathcal{F}\left(e^{\frac{i}{2}\langle Q\bullet,\bullet\rangle}\right)(\xi) = \frac{(2\pi)^{n/2}}{|\det Q|^{1/2}} e^{i\frac{\pi}{4}\mathrm{sgn}(Q)} e^{-\frac{i}{2}\langle Q^{-1}\xi,\xi\rangle}. \tag{1.3.2}$$

Proof. We start by proving (1.3.1). We have:

$$\begin{aligned}
\mathcal{F}\left(e^{-\frac{1}{2}\langle Qx,x\rangle}\right) &= \int_{\mathbb{R}^n} e^{-\frac{1}{2}\ \langle Qx,x\rangle - ix\cdot\xi} \mathrm{d}x \\
&= \int_{\mathbb{R}^n} e^{-\frac{1}{2}\langle Q(x+iQ^{-1}\xi),x+iQ^{-1}\xi\rangle - \frac{1}{2}\langle Q^{-1}\ \xi,\xi\rangle} \mathrm{d}x \\
&= e^{-\frac{1}{2}\langle Q^{-1}\ \xi,\xi\rangle} \int_{\Gamma} e^{-\frac{1}{2}\langle Qz,z\rangle} \mathrm{d}z,
\end{aligned}$$

where $z := x + iQ^{-1}\xi$. Note that this corresponds to a contour $\Gamma \subset \mathbb{C}^n$. Since $z \mapsto e^{-\frac{1}{2}\langle Qz,z\rangle}$ is holomorphic, we can shift this contour from Γ to $\mathbb{R}^n$ and thus

$$\mathcal{F}\left(e^{-\frac{1}{2}\langle Qx,x\rangle}\right) = e^{-\frac{1}{2}\langle Q^{-1}\ \xi,\xi\rangle} \int_{\mathbb{R}^n} e^{-\frac{1}{2}\langle Qz,z\rangle} \mathrm{d}z.$$

We can write $Q = P^\top DP$, where $P \in \mathrm{O}_n(\mathbb{R})$ and D is diagonal $D = \mathrm{diag}(\lambda_1, ..., \lambda_n)$. Hence:

$$\begin{aligned}
\mathcal{F}\left(e^{-\frac{1}{2}\langle Qx,x\rangle}\right) &= e^{-\frac{1}{2}\langle Q^{-1}\ \xi,\xi\rangle} \int_{\mathbb{R}^n} e^{-\frac{1}{2}\sum_i \lambda_i z_i^2}\ \mathrm{d}z \\
&= e^{-\frac{1}{2}\langle Q^{-1}\ \xi,\xi\rangle} \prod_i \int_{\mathbb{R}} e^{-\lambda_i z^2} 2\mathrm{d}z \\
&= e^{-\frac{1}{2}\langle Q^{-1}\ \xi,\xi\rangle} \prod_i (2/\lambda_i)^{1/2} \int_{\mathbb{R}} e^{-z^2} \mathrm{d}z \\
&= e^{-\frac{1}{2}\langle Q^{-1}\ \xi,\xi\rangle} \frac{(2\pi)^{n/2}}{(\det Q)^{1/2}}.
\end{aligned}$$

We now prove (1.3.2). We set $Q_\varepsilon := Q +\ i\varepsilon I_n$. Then $x \mapsto e^{\frac{i}{2}\langle Q_\varepsilon x,x\rangle}$ is Schwartz and we can make an explicit computation:

$$\begin{aligned}
\mathcal{F}\left(e^{\frac{i}{2}\langle Q_\varepsilon x,x\rangle}\right) &= \int_{\mathbb{R}^n} e^{\frac{i}{2}\langle Q_\varepsilon x,x\rangle - ix\cdot\xi} \mathrm{d}x \\
&= \int_{\mathbb{R}^n} e^{\frac{i}{2}\langle Q_\varepsilon(x - Q_\varepsilon^{-1}\xi),x-Q_\varepsilon^{-1}\xi\rangle - \frac{i}{2}\langle Q_\varepsilon^{-1}\xi,\xi\rangle} \mathrm{d}x \\
&= e^{-\frac{i}{2}\langle Q_\varepsilon^{-1}\xi,\xi\rangle} \int_{\mathbb{R}^n} e^{\frac{i}{2}\langle Q_\varepsilon z,z\rangle} \mathrm{d}z.
\end{aligned}$$

Since Q is symmetric, we can write $Q_\varepsilon = P^\top \mathrm{diag}(\lambda_1 + i\varepsilon, ..., \lambda_n + i\varepsilon)P$ for some $P \in \mathrm{O}_n(\mathbb{R})$. Hence:

$$\mathcal{F}\left(e^{\frac{i}{2}\langle Q_\varepsilon x,x\rangle}\right) = e^{-\frac{i}{2}\langle Q_\varepsilon^{-1}\xi,\xi\rangle} \prod_i \int_{\mathbb{R}} e^{\frac{1}{2}(i\lambda_k - \varepsilon)z^2} \mathrm{d}z.$$

We set $w := (\varepsilon - i\lambda_i)^{1/2} z$ where $(\varepsilon - i\lambda_i)^{1/2}$ is the square root with negative imaginary part if $\lambda_i > 0$ and positive imaginary part if $\lambda_i > 0$. We observe that

$$(\varepsilon - i\lambda_i)^{1/2} \;\to_{\varepsilon\to 0} \begin{cases} \sqrt{\lambda_i} e^{i\pi/4} & \lambda_i < 0 \\ \sqrt{\lambda_i} e^{-i\pi/4} & \lambda_i > 0. \end{cases}$$

Hence:

$$\mathcal{F}\left(e^{\frac{i}{2}\langle Q_\varepsilon x,x\rangle}\right) = e^{-\frac{i}{2}\langle Q_\varepsilon^{-1}\xi,\xi\rangle} \prod_i (\varepsilon - i\lambda_i)^{-1/2} \int_{\Gamma_i} e^{-z^2} \mathrm{d}z,$$

where $\Gamma_i \subset \mathbb{C}$ is a contour integral (on a line in $\mathbb{C}$ passing through 0). We can shift this integral back to $\mathbb{R}$ by holomorphicity of the integrand and the last integral is there equal to $(2\pi)^{1/2}$. Hence, we obtain by taking the limit as $\varepsilon \to 0$:

$$\mathcal{F}\left(e^{\frac{i}{2}\langle Qx,x\rangle}\right) = \frac{(2\pi)^{n/2}}{|\det Q|^{1/2}} e^{i\mathrm{sgn}(Q)\pi/4} e^{-\frac{i}{2}\ \langle Q^{-1}\xi,\xi\rangle}.$$

$\square$

Proof of Lemma 1.3.2. We can always assume that a has compact support near $x_0 \in X$. We then apply the change of variable provided by the Morse lemma, namely we write $x = \kappa(y)$, then:

$$\begin{aligned} I(h) &= \int_X e^{\frac{i}{h}\phi(x)} a(x) \mathrm{d}x \\ &= \int_V e^{\frac{i}{h}\left(\phi(x_0) + \frac{1}{2}\left(y_1^2 + ... + y_r^2 - (y_{r+1}^2 + ... + y_n^2)\right)\right)} a(\kappa(y)) |\det d\kappa(y)|\ \mathrm{d}y \\ &= e^{\frac{i}{h}\phi(x_0)} \int_{\mathbb{R}^n} e^{\frac{i}{2h}\langle Qy,y\rangle} a'(y)\ \mathrm{d}y, \end{aligned}$$

where $a'(y) := a(\kappa(y)) |\det d\kappa(y)|$ and Q is the matrix

$$\begin{pmatrix} I_r & 0 \\ 0 & -I_{n-r} \end{pmatrix}.$$

We then apply Parseval's identity (A.3.2) (be careful, there is a conjugate in Parseval's identity), which gives (this amounts to changing Q by Q/h):

$$\int_{\mathbb{R}^n} e^{\frac{i}{2h}\langle Qy,y\rangle} a'(y)\ \mathrm{d}y = (2\pi)^{-n/2} e^{i\frac{\pi}{4}\mathrm{sgn}(Q)} h^{n/2} |\det Q|^{-1/2} \int_{\mathbb{R}^n} e^{-\frac{ih}{2}\langle Q^{-1}\xi,\xi\rangle} \widehat{a'}(\xi) \mathrm{d}\xi$$

Applying Taylor's formula to the exponential map, we get that for all $z \in i\mathbb{R}$,

$$\left| e^z - \sum_{k=0}^{N-1} \frac{z^k}{k!} \right| \leq \frac{|z|^N}{N!}.$$

Exercise 1.3.5. Check this.

This gives:

$$e^{-\frac{ih}{2}\langle Q^{-1}\xi,\xi\rangle} = \sum_{k=0}^{N-1} h^k \frac{\langle Q^{-1}\xi,\xi\rangle^k}{k!(2i)^k} + R_N(h,\xi),$$

where:

$$|R_N(h,\xi)| \leq h^N \frac{\langle Q^{-1}\xi,\xi\rangle^N}{N!2^N}.$$

Also note that we have (see Example 1.2.10):

$$\int_{\mathbb{R}^n} \langle Q^{-1}\xi,\xi\rangle^k \widehat{a'}(\xi) \mathrm{d}\xi = (2\pi)^n \left(\langle Q^{-1}D_x, D_x\rangle^k a' \right)(0),$$

where we use the notation $D_x = i^{-1}\partial_x$ and thus $\langle Q^{-1}D_x, D_x\rangle = -\sum_{ij} Q^{ij}\partial_i\partial_j$. Combining everything, we get:

$$\begin{aligned} I(h) &= e^{\frac{i}{h}\phi(x_0)}(2\pi)^{-n/2} e^{i\frac{\pi}{4}\mathrm{sgn}(Q)} h^{n/2} \left(\sum_{k=0}^{N-1} \frac{h^k (2\pi)^n}{k!(2i)^k} \left(\langle Q^{-1}D_x, D_x\rangle^k a' \right)(0) + S_N(h) \right) \\ &= e^{\frac{i}{h}\phi(x_0)}(2\pi h)^{n/2} e^{i\frac{\pi}{4}\mathrm{sgn}(Q)} \sum_{k=0}^{N-1} \frac{h^k}{k!(2i)^k} \left(\langle Q^{-1}D_x, D_x\rangle^k a' \right)(0) + \mathcal{O}_{a,N,Q}(h^{n/2+N}). \end{aligned} \tag{1.3.3}$$

We observe that we can write

$$\frac{1}{k!(2i)^k} \left(\langle Q^{-1}D_x, D_x\rangle^k a' \right)(0) = \left[A_{2k}(x,D)a \right](x_0),$$

which is the expected form.

As to the remainder in (1.3.3), its expression is:

$$\mathcal{O}_{a,N,Q}(h^{n/2+N}) = e^{\frac{i}{h}\phi(x_0)}(2\pi)^{-n/2} e^{i\frac{\pi}{4}\mathrm{sgn}(Q)} h^{n/2} |\det Q|^{-1/2} \int_{\mathbb{R}^n} R_N(h,\xi)\widehat{a}'(\xi)\mathrm{d}\xi,$$

and this is bounded by

$$\begin{aligned} |\mathcal{O}_{a,N,Q}(h^{n/2+N})| &\leq (2\pi)^{-n/2} h^{n/2} |\det Q|^{-1/2} \int_{\mathbb{R}^n} h^N \frac{|\langle Q^{-1}\xi,\xi\rangle^N|}{N!2^N} |\widehat{a}'(\xi)| \mathrm{d}\xi \\ &= (2\pi)^{-n/2} h^{n/2+N} |\det Q|^{-1/2} (N!)^{-1} 2^{-N} \| \langle Q^{-1}\xi,\xi\rangle^N \widehat{a}'(\xi) \|_{L^1(\mathbb{R}^n)}. \end{aligned}$$

Observe that by Lemma A.3.10, one has:

$$\|\langle Q^{-1}\xi,\xi\rangle^N \widehat{a}'(\xi)\|_{L^1(\mathbb{R}^n)} = \|\mathcal{F}(\langle Q^{-1}D,D\rangle^N a')\|_{L^1(\mathbb{R}^n)} \leq \sum_{|\alpha|\ \leq n+1} \|\partial^\alpha \langle Q^{-1}D,D\rangle^N a'\|_{L^1}$$

This completes the proof. □

Example 1.3.6. Let us consider

$$Q := \begin{pmatrix} 0 & -I_n \\ -I_n & 0 \end{pmatrix},$$

which is a $2n\times 2n$-matrix so that $\frac{1}{2}\langle Q(x,y),(x,y)\rangle = -x\cdot y$, for $x,y\in\mathbb{R}^n$. The sign of Q is $0 = n-n$. Note that $Q^{-1} = Q$. The expansion is then given by:

$$\begin{aligned}\int_{\mathbb{R}^n_x}\int_{\mathbb{R}^n_y} e^{-\frac{i}{h}x\cdot y} a(x,y)\mathrm{d}x\mathrm{d}y &= (2\pi h)^n \sum_{k=0}^{N-1} \frac{h^k}{k! i^k}\left((-D_x\cdot D_y)^k a\right)(0) + S_N(h)\\ &= (2\pi h)^n \sum_{k=0}^{N-1} \frac{h^k}{k! i^k}\left((\partial_x\cdot \partial_y)^k a\right)(0) + S_N(h),\end{aligned}$$

where

$$|S_N(h)| \leq C_N h^{N+n} \sum_{|\alpha+\beta|\ \leq 2n+1} \|\partial_x^\alpha \partial_y^\beta (\partial_x\cdot\partial_y)^N a\|_{L^1(X)}.$$

Note that $\partial_x\cdot\partial_y = \sum_{j=1}^n \partial_{x_j}\partial_{y_j}$ and thus, using Newton's identity:

$$\begin{aligned}\sum_{k=0}^{N-1}\frac{h^k}{k! i^k}(\partial_x\cdot\partial_y)^k &= \sum_{k=0}^{N-1}\frac{h^k}{k! i^k}\sum_{k_1+\ldots+k_n=k}\frac{k!}{k_1!\ldots k_n!}\partial_{x_1}^{k_1}\partial_{y_1}^{k_1}\ldots\partial_{x_n}^{k_n}\partial_{y_n}^{k_n}\\ &= \sum_{|\alpha|\ \leq N-1}\frac{h^{|\alpha|}}{\alpha! i^{|\alpha|}}\partial_x^\alpha\partial_y^\alpha.\end{aligned}$$

This yields:

$$\int_{\mathbb{R}^n_x}\int_{\mathbb{R}^n_y} e^{-\frac{i}{h}x\cdot y} a(x,y)\mathrm{d}x\mathrm{d}y = (2\pi h)^n \sum_{|\alpha|\ \leq N-1}\frac{h^{|\alpha|}}{\alpha! i^{|\alpha|}}(\partial_x^\alpha\partial_y^\alpha a)(0,0) + S_N(h). \quad (1.3.4)$$

1.3.3 Exercises

Exercise 1

Study the convergence in $\mathcal{D}'(\mathbb{R}^n)$ of:

1. $u_h(x) := h^{-N}e^{-\frac{i}{h}x}, v_h(x) := h^{-1/2}e^{-\frac{i}{h}x^2/2}, w_h(x) := h^{-1/2}e^{+\frac{i}{h}x^2/2}$,
2. $u_h(x) := h^{-N}e^{-\frac{i}{h}f(x)}, v_h(x) := h^{-1/2}e^{-\frac{i}{h}(f(x))^2/2}$, where $f\in C^\infty(\mathbb{R})$ and $f'\neq 0$.

Exercise 2: Stirling's formula

Define
$$F(\lambda) := \Gamma(\lambda+1) = \int_0^{+\infty} e^{-t} t^{\lambda} \mathrm{d}t,$$
for $\lambda \geq 0$. Recall that $F(n) = \Gamma(n+1) = n!$ for all $n \geq 0$. We want to find an asymptotic of F as $\lambda \to \infty$.

1. Rewrite this integral by means of the change of variable $t = \lambda(1+s)$.
2. Show that
$$F(\lambda) = (\lambda e^{-1})^{\lambda} \sqrt{2\pi\lambda}(1 + a_1 \lambda^{-1} + a_2 \lambda^{-2} + ...),$$
and give a precise meaning to "...".
3. Deduce Stirling's formula.
4. Compute a_1, a_2.

Chapter 2

Wavefront set calculus

The wavefront set of a distribution describes the (co)-directions in the cotangent space in which a distribution is irregular and does not "look" locally like a smooth function. As we shall see, the wavefront set is a useful concept which allows to do many things: for instance, under a certain wavefront set condition, the product of distribution can be well-defined. This will allow us to extend certain linear operators, initially defined on smooth functions, to distributions as long as the wavefront set of their Schwartz kernel satisfies some good conditions.

Contents

2.1 The wavefront set

2.1.1 Definition. First properties

The notion of *wavefront set* is of fundamental importance in microlocal analysis. It describes the singularities of a distribution: not only in space (which is the purpose of the *singular support*, a *local* notion), but also the *direction* of irregularity of the distribution in phase space (hence the suffix *micro*). It translates this general important principle: **the regularity (resp. irregularity) of a (compactly supported) distribution is connected to the decay (resp. growth) as $|\xi| \to \infty$ of its Fourier transform.**

In the following, $X \subset \mathbb{R}^n$ is an open subset, $T^*X \simeq \mathbb{R}^n_x \times \mathbb{R}^n_\xi$ is the cotangent bundle of X and $T^*_0 X := T^*X \setminus \{0\}$. The norm $|\bullet|$ is the Euclidean norm in $\mathbb{R}^n$.

Definition 2.1.1 (Wavefront set of a distribution)**.** Let $u \in \mathcal{D}'(X)$. The *wavefront set* of u, denoted by $\mathrm{WF}(u) \subset T^*_0X$, is the *complement* of the set of points $(x_0, \xi_0) \in T^*_0X$ such that the following holds: there exists a smooth cutoff function $\varphi \in C^\infty_{\mathrm{comp}}(X)$ with $\varphi(x_0) \neq 0$ and compact support near x_0 and $\varepsilon > 0$ such that for all $N \in \mathbb{N}$, there exists $C_N > 0$ such that, if

$$|\xi/|\xi| - \xi_0/|\xi_0|| < \varepsilon, \tag{2.1.1}$$

then:

$$|\widehat{\varphi u}(\xi)| \leq C_N \langle \xi \rangle^{-N}. \tag{2.1.2}$$

We make several important comments:

(i) A function satisfying the bounds (2.1.2) will be said to decay *super-polynomially*.

(ii) As we shall see below in Proposition 2.1.6, Definition (2.1.1) is *independent* of the choice of cutoff functions φ as long as it is sufficiently localized near x_0. In other words, "there exists φ" could be replaced in Definition 2.1.1 by "for all φ with support sufficiently close to x_0".

(iii) It follows from its very definition that the wavefront set $\mathrm{WF}(u)$ is a *closed conic subset* of T^*X[1] in the sense that it is invariant by radial dilation: if $(x, \xi) \in \mathrm{WF}(u)$, then $(x, \lambda\xi) \in \mathrm{WF}(u)$ for all $\lambda > 0$. Note that (2.1.1) defines a distance on the sphere bundle $S^*X := T^*X/\sim$, where $\sim$ is the equivalence relation: $(x, \xi) \sim (x, \lambda\xi)$ for $\lambda > 0, (x, \xi) \in T^*X$. Given a covector $\xi_0 \neq 0$, the set of $\xi \in T^*_0X$ satisfying the inequality (2.1.1) is called a *conic neighborhood* of ξ_0. We will write $V_{\xi_0}(\varepsilon) \subset \mathbb{R}^n_\xi$ to denote this neighborhood. Also observe that (2.1.2) is trivially satisfied for ξ in a bounded domain of $\mathbb{R}^n$ so the statement is only meaningful when we allow $|\xi| \to \infty$.

(iv) The wavefront set $\mathrm{WF}(u)$ is understood as the set of points where the distribution is singular. In other words, this reflects this standard point of view that the more the Fourier transform decreases, the more regular the function is.

[1] Or equivalently a closed subset of the sphere bundle S^*X.

It is obvious that $\mathrm{WF}(u) \subset T^*X|_{\mathrm{supp}(u)}$. As a consequence, it is legitimate to introduce the following terminology:

Definition 2.1.2 (Singular support of a distribution)**.** Let $\pi : T^*X \to X$ be the projection. We define the *singular support* of a distribution as the projection of its wavefront set on the base, namely $\mathrm{suppsing}(u) := \pi(\mathrm{WF}(u))$.

We start with a first example:

Example 2.1.3. If $f \in C^\infty(X)$, then f defines a distribution $f|\mathrm{d}x| \in \mathcal{D}'(X)$ given by the usual formula

$$(f|\mathrm{d}x|, \varphi) := \int_X f(x)\varphi(x)\mathrm{d}x.$$

The notation $f|\mathrm{d}x|$ will become clear later when working on manifolds. Keep in mind that this embedding of $C^\infty(X)$ in distributions is **non-canonical**[2] and **depends on a choice of density**. Here, $|\mathrm{d}x|$ is the Lebesgue measure on $\mathbb{R}^n$ and is the natural choice. However, on manifolds, no such canonical choice is available.

It is immediate that $\mathrm{WF}(f|\mathrm{d}x|) = \emptyset$. Indeed, for any $(x_0, \xi_0) \in T_0^*X$, we can take any $\varphi \in C^\infty_{\mathrm{comp}}(X)$ and $\varphi f \in \mathcal{S}(\mathbb{R}^n)$ is a Schwartz function so its Fourier transform decreases faster than any polynomial in any direction.

Example 2.1.4. We now consider the Dirac mass δ_0 in $\mathbb{R}^n$. We identify $T^*\mathbb{R}^n \simeq \mathbb{R}^n_x \times \mathbb{R}^n_\xi$. We then have: $\mathrm{WF}(\delta_0) = \{0\} \times (\mathbb{R}^n_\xi \setminus \{0\})$. First of all, $\mathrm{supp}(u) = \{0\}$ so the inclusion $\mathrm{WF}(\delta_0) \subset \{0\} \times (\mathbb{R}^n_\xi \setminus \{0\})$ is obvious. For the converse, it suffices to observe that for any $\varphi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ with $\varphi(0) = 1$, one has:

$$\widehat{\varphi\delta_0}(\xi) = (\delta_0, \varphi e^{-i\xi\bullet}) = \varphi(0) = 1,$$

so this does not decrease, whatever the direction in ξ.

Exercise 2.1.5. Let $\delta_{\mathbb{R}^k}$ be the Dirac mass on the k-plane $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$, that is

$$(\delta_{\mathbb{R}^k}, \varphi) := \int_{\mathbb{R}^k} \varphi(x, 0)\mathrm{d}x.$$

Show that $\mathrm{WF}(\delta_{\mathbb{R}^k}) = N^*\mathbb{R}^k \setminus \{0\}$, the *conormal to* $\mathbb{R}^k$, that is $\mathbb{R}^k$ times the set of covectors vanishing on $\mathbb{R}^k$:

$$N^*\mathbb{R}^k := \left\{(x, \xi) \in T^*\mathbb{R}^n \mid x \in \mathbb{R}^k, \forall v \in \mathbb{R}^k, (\xi, v) = 0\right\}.$$

More generally, given $E \subset \mathbb{R}^n$ a vector subspace of dimension k, we can define integration on E with respect to an arbitrary smooth measure $\mathrm{d}\mu_E$ as follows

$$(\delta_{E,\mathrm{d}\mu_E}, \varphi) = \int_{x\in E} \varphi(x)\mathrm{d}\mu_E(x) := \int_{\mathbb{R}^k} \varphi(Ax)a(x)\mathrm{d}x,$$

where $A \in \mathrm{O}(\mathbb{R}^n)$ is some invertible matrix such that $A : \mathbb{R}^k \to E$ is an isometry and the function $a \in C^\infty(\mathbb{R}^k)$ defines the density. Show that $\delta_{E,\mathrm{d}\mu_E}$ has wavefront

[2] It could be made canonical though by declaring $\mathcal{D}'(X)$ to be the topological dual of $C^\infty(X, \Omega^1 X)$, where $\Omega^1 X$ denotes the bundle of 1-densities.

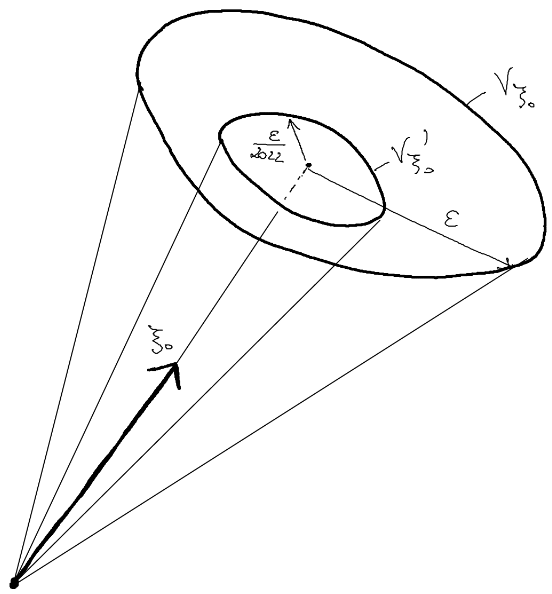

Figure 2.1: The conic neighborhoods V_{ξ_0} and $V_{\xi_0'}$.

set contained in the conormal N_0^*E of E (minus the 0 section), where

$$N^*E := \{(x,\xi) \in T^*X \mid x \in E, \forall v \in E, (\xi, v) = 0\}.$$

Proposition 2.1.6. *The following properties hold:*

(i) *Definition 2.1.1 is independent of the choice of cutoff function φ in the following sense: if $(x_0,\xi_0) \in T_0^*X$, $\varphi \in C^\infty_{\mathrm{comp}}(X)$ is such that $\xi \mapsto \widehat{\varphi u}(\xi)$ decays super-polynomially in a conic neighborhood of ξ_0, then for any $\delta > 0$ small enough and for any other function $\varphi' \in C^\infty_{\mathrm{comp}}(X)$ such that $\mathrm{supp}(\varphi') \subset \{|\varphi| > \delta\}$, the function $\xi \mapsto \widehat{\varphi' u}(\xi)$ decays super-polynomially in a conic neighborhood of ξ_0. Note that φ' could vanish at x_0.*

(ii) *If $u, v \in \mathcal{D}'(X)$, then $\mathrm{WF}(u+v) \subset \mathrm{WF}(u) \cup \mathrm{WF}(v)$.*

(iii) *If $\psi \in C^\infty(X)$, then $\mathrm{WF}(\psi u) \subset \mathrm{WF}(u)$. In particular, if $\psi \neq 0$ on X, then $\mathrm{WF}(\psi u) = \mathrm{WF}(u)$.*

However, note that the conic neighborhood of ξ_0 may shrink as we change the cutoff function φ. In particular, Proposition 2.1.6 shows that the notion of wavefront set does not depend on the choice of function φ in (2.1.2), as long as it is localized sufficiently close to x_0.

Proof. We start by proving that (i) implies both (ii) and (iii). Take $u, v \in \mathcal{D}'(X)$ and $(x_0,\xi_0) \notin \mathrm{WF}(u)$ and $\notin \mathrm{WF}(v)$. By assumption, we can find $\chi_u, \chi_v \in C^\infty_{\mathrm{comp}}(X)$ not vanishing near x_0 such that $\widehat{\chi_u u}, \widehat{\chi_v v}$ decay super-polynomially in a conic neighborhood of ξ_0. Take $\delta > 0$ small enough, $\chi \in C^\infty_{\mathrm{comp}}(X)$ not vanishing near x_0 such that $\mathrm{supp}(\chi) \subset \{|\chi_u| > \delta\} \cap \{|\chi_v| > \delta\}$. Then, by (i), $\widehat{\chi(u+v)} = \widehat{\chi u} + \widehat{\chi v}$ decay super-polynomially near ξ_0 and this proves (ii).

Let us now prove (iii). Let $(x_0, \xi_0) \notin \mathrm{WF}(u)$. We can find χ localized near x_0 such that $\widehat{\chi u}$ decays super-polynomially near ξ_0. Taking $\chi' \in C^\infty_{\mathrm{comp}}(X)$ such that $\chi'(x_0) \neq 0$ and $\mathrm{supp}(\chi') \subset \{|\chi| > \delta\}$, we have:

$$\widehat{\chi'(\psi u)} = \widehat{(\chi'\psi)u}$$

and this decays super-polynomially near ξ_0 by (i) since $\mathrm{supp}(\chi'\psi) \subset \{|\chi| > \delta\}$.

We now prove (i). Take $(x_0, \xi_0) \notin \mathrm{WF}(u)$ and ξ_0 unitary. Assume that $\xi \mapsto \widehat{\varphi u}(\xi)$ decays super-polynomially near ξ_0, i.e. that (2.1.2) holds in a conic neighborhood $V_{\xi_0}(\varepsilon)$ of ξ_0 for $|\xi| \gg 1$ large enough, as in (2.1.1). Take $V_{\xi_0}(\varepsilon/2022)$ the small conic neighborhood of ξ_0 contained inside $V_{\xi_0}(\varepsilon)$ and defined by (2.1.1) with ε replaced by $\varepsilon/2022$.

We claim that there exists $\delta > 0$ small enough such that for every $\xi \in V_{\xi_0}(\varepsilon/2022)$ and $\eta \in B(0, \delta|\xi|)$ the covector $\xi + \eta \in V_{\xi_0}(\varepsilon)$. Using $|\xi + \eta| \geq (1-\delta)|\xi|$:

$$\begin{aligned}
\left|\frac{\xi+\eta}{|\xi+\eta|} - \xi_0\right| &\leq \frac{|\eta|}{|\xi+\eta|} + \left|\frac{\xi}{|\xi+\eta} - \xi_0\right| \\
&\leq \frac{\delta|\xi|}{|\xi|(1-\delta)} + \left|\frac{|\xi|}{|\xi+\eta|}\left(\frac{\xi}{|\xi|} - \xi_0\right) + \xi_0\left(\frac{|\xi|}{|\xi+\eta|} - 1\right)\right| \\
&\leq \frac{\delta}{1-\delta} + \frac{1}{1-\delta}\frac{\varepsilon}{2022} + \frac{1}{1-\delta} - 1 < \varepsilon,
\end{aligned}$$

when $\delta > 0$ is chosen small enough (depending on $\varepsilon > 0$).

Take $\varphi' \in C^\infty_{\mathrm{comp}}(X)$ such that $\mathrm{supp}(\varphi') \subset \{|\varphi| > \delta'\}$. We can write

$$\varphi' u = \varphi\varphi^{-1}\varphi' u = \psi\varphi u,$$

where $\psi := \varphi^{-1}\varphi' \in C^\infty_{\mathrm{comp}}(X)$. We then get for $\xi \in V_{\xi_0}(\varepsilon/2022)$:

$$\begin{aligned}
\widehat{\psi\varphi u}(\xi) &= (2\pi)^n \widehat{\psi} \star \widehat{\varphi u}(\xi) = (2\pi)^n \int_{\mathbb{R}^n} \widehat{\psi}(\eta)\widehat{\varphi u}(\xi - \eta)\mathrm{d}\eta \\
&= (2\pi)^n \left(\int_{B(0,\delta|\xi|)} \widehat{\psi}(\eta)\widehat{\varphi u}(\xi-\eta)\mathrm{d}\eta + \int_{\mathbb{R}^n\setminus B(0,\delta|\xi|)} \widehat{\psi}(\eta)\widehat{\varphi u}(\xi-\eta)\mathrm{d}\eta\right)
\end{aligned} \tag{2.1.3}$$

The first term can be bounded using the decay of $\widehat{\varphi u}$. As to the second term, since φu is a distribution with compact support in some open subset $\Omega \subset X$, it has finite order, that is, there exists $C, M > 0$ such that for all $\psi \in C^\infty(\mathbb{R}^n)$,

$$|(\varphi u, \psi)| \leq C\|\psi\|_{C^M(\Omega)}.$$

Taking $\psi(x) := e^{-ix\cdot\xi}$ yields

$$|\widehat{\varphi u}(\eta)| \leq C\langle\eta\rangle^M. \tag{2.1.4}$$

Observe that in the first integral, $|\eta| \leq \delta|\xi|$, and we thus have

$$\langle \xi - \eta\rangle^{-1} = \sqrt{1+|\xi-\eta|^2}^{-1} \leq \sqrt{1+(1-\delta)|\xi|^2}^{-1} \leq C_\delta \langle \xi\rangle^{-1}.$$

Thus, taking $N \gg M$ large enough, and using that $\psi, \widehat{\psi} \in \mathcal{S}$ are Schwartz, and elementary inequalities on the Japanese bracket, we obtain:

$$\begin{aligned} |\widehat{\varphi\psi u}(\xi)| &\lesssim \int_{B(0,\delta|\xi|)} \langle \xi - \eta\rangle^{-N} \mathrm{d}\eta + \int_{\mathbb{R}^n \setminus B(0,\delta|\xi|)} \langle \eta\rangle^{-N} \langle \xi - \eta\rangle^{M} \mathrm{d}\eta \\ &\lesssim \langle \xi\rangle^n \langle \xi\rangle^{-N} + \int_{\mathbb{R}^n \setminus B(0,\delta|\xi|)} \frac{\langle \xi\rangle^M + \langle \eta\rangle^M}{\langle \eta\rangle^N} \mathrm{d}\eta \\ &\lesssim \langle \xi\rangle^n \langle \xi\rangle^{-N} + \int_{\mathbb{R}^n \setminus B(0,\delta|\xi|)} \frac{1}{\langle \eta\rangle^{N-M}} \mathrm{d}\eta, \end{aligned}$$

since $\langle \xi\rangle \lesssim \langle \eta\rangle$ on the support of the last integral. Now, take $N = M + n + 1 + N'$, and use that $\eta \mapsto \langle \eta\rangle^{-(n+1)}$ is integrable on $\mathbb{R}^n$, which gives:

$$|\widehat{\varphi\psi u}(\xi)| \lesssim \langle \xi\rangle^{-(M+1+N')} + \int_{\mathbb{R}^n} \langle \eta\rangle^{-(n+1)} \langle \xi\rangle^{-N'} \mathrm{d}\eta \lesssim \langle \xi\rangle^{-N'}.$$

As a consequence, up to taking another N larger than the first one, we get that for all $N \geq 0$, there exists $C_N > 0$ such that for all $\xi \in V_{\xi_0}(\varepsilon/2022)$,

$$|\widehat{\varphi\psi u}(\xi)| \leq C_N \langle \xi\rangle^{-N}.$$

This concludes the proof of (i). □

Let us state some elementary properties of the wavefront set:

Lemma 2.1.7. *If* $\mathrm{WF}(u) = \emptyset$*, then* $u = f\mathrm{d}x$ *for some* $f \in C^\infty(X)$*.*

This is the converse of Example 2.1.3.

Proof. Fix a point $x_0 \in X$ and take $\xi_0 \in S^*_{x_0}X$. By assumption, there is a smooth function $\varphi_{(x_0,\xi_0)} \in C^\infty_{\mathrm{comp}}(X)$ and a conic neighborhood $V_{\xi_0}(\varepsilon_0)$ of ξ_0 such that for all $N \geq 0$, there exists $C_N > 0$ such that for all $\xi \in V_{\xi_0}(\varepsilon_0)$:

$$|\widehat{\varphi_{(x_0,\xi_0)} u}(\xi)| \leq C_N \langle \xi\rangle^{-N}$$

By compactness of $S^*_{x_0}X$, there is a finite number $\xi_0, ..., \xi_N$ such that $\cup_i V_{\xi_i}(\varepsilon/2022)$ covers the whole set of direction $S^*_{x_0}X$. Then set $\varphi_{x_0} := \prod_i \varphi_{(x_0,\xi_i)}$ which is a legitimate cutoff function supported near x_0. We claim that for all $N \geq 0$, there exists $C_N > 0$ such that for all $\xi \in \mathbb{R}^n$ large enough:

$$|\widehat{\varphi_{x_0} u}(\xi)| \leq C_N \langle \xi\rangle^{-N} \tag{2.1.5}$$

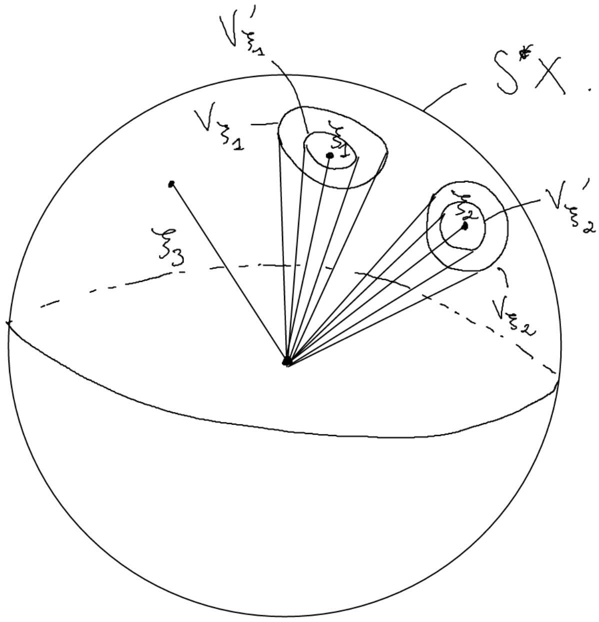

Figure 2.2: Covering the unit sphere in the cotangent bundle.

Indeed, for $1 \leq i \leq N$, we can write

$$\begin{aligned} \widehat{\varphi_{x_0} u}(\xi) &= (2\pi)^n \mathcal{F}(\varphi_{(x_0,\xi_i)} u) \star \mathcal{F}\left(\prod_{j \neq i} \varphi_{(x_0,\xi_j)}\right)(\xi) \\ &= \int_{\mathbb{R}^n} h_i(\eta) \mathcal{F}(\varphi_{(x_0,\xi_i)} u)(\xi - \eta) \mathrm{d}\eta, \end{aligned}$$

where $h_i(\eta) := (2\pi)^n \mathcal{F}\left(\prod_{j\neq i} \varphi_{(x_0,\xi_j)}\right)(\eta)$. This is obviously a Schwartz function so h_i decreases faster than any polynomial. Repeating the exact same proof as in Proposition 2.1.6, using that $|\widehat{\varphi_{x_0,\xi_i}}(\xi)|$ decays super-polynomially for all $\xi \in V_{\xi_i}(\varepsilon_i)$, we get that for ξ in the conic neighborhood $V_{\xi_i}(\varepsilon_i/2022)$ of ξ_i, the previous quantity $\widehat{\varphi_{x_0} u}(\xi)$ decreases faster than any polynomial $\lesssim \langle \xi \rangle^{-N}$. Since $\cup_i V_{\xi_i}(\varepsilon_i/2022)$ covers the whole set of directions $S^*_{x_0} X$, this proves that (2.1.5) holds.

As a consequence, $\widehat{\varphi_{x_0} u}$ decreases faster than any polynomial. Also note that

$$D^\alpha_\xi \widehat{\varphi_{x_0} u} = (-1)^{|\alpha|} \widehat{x^\alpha \varphi_{x_0} u} = (-1)^{|\alpha|} \widehat{\varphi_{x_0} \psi u}$$

where $\psi := x^\alpha \widetilde{\varphi_{x_0}}$ and $\varphi_{x_0} \prec \widetilde{\varphi_{x_0}}$, that is $\widetilde{\varphi_{x_0}} = 1$ on the support of φ_{x_0}. By Proposition 2.1.6, $\mathrm{WF}(\psi u) \subset \mathrm{WF}(u)$ so we can apply the same argument as before, and we obtain that $D^\alpha_\xi \widehat{\varphi_{x_0} u}$ decreases faster than any polynomial.

As a consequence, $\widehat{\varphi_{x_0} u}$ is a Schwartz function and we can apply the Fourier inversion formula to obtain:

$$\varphi_{x_0} u = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i \bullet \cdot \xi} \widehat{\varphi_{x_0} u}(\xi) \mathrm{d}\xi,$$

and this is a smooth function. Since $\varphi_{x_0} > 0$ near x_0, we thus obtain that $u = u_{f_{x_0}}$, for

$$f_{x_0} = \varphi_{x_0}^{-1} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{i\bullet\cdot\xi} \widehat{\varphi_{x_0} u}(\xi) \mathrm{d}\xi$$

on an open subset $U_{x_0} \subset X$ near x_0.

As a consequence, taking a cover $X = \cup_i U_{x_i}$, a partition of unity $\mathbf{1} = \sum_i \chi_i$ subordinated to that cover (i.e. such that $\mathrm{supp}(\chi_i) \subset U_i$) and setting

$$f = \sum_i \chi_i f_{x_i},$$

we see that $f \in C^\infty(X)$ and by construction $u = f|\mathrm{d}x|$: indeed for $\varphi \in C^\infty_{\mathrm{comp}}(X)$, we have

$$(f|dx|, \varphi) = \sum_i (f_{x_i}, \chi_i \varphi) = \sum_i (u, \chi_i \varphi) = (u, \varphi).$$

$\square$

Given a closed cone $\Gamma \subset T^*_0 X$, it will be convenient to introduce the space $\mathcal{D}'_\Gamma(X)$ of distributions u such that $\mathrm{WF}(u) \subset \Gamma$ (hence $\mathcal{D}'(X) = \mathcal{D}'_{T^*_0 X}(X)$). This space is equipped with a slightly more restrictive topology than the usual topology on $\mathcal{D}'(X)$ described in §A.3.

Definition 2.1.8 (Weak convergence for distributions)**.** Let $u \in \mathcal{D}'_\Gamma(X)$ and $(u_j)_{j\in\mathbb{N}}$ be a sequence of distributions in $\mathcal{D}'_\Gamma(X)$. We will say that $u_j \to u$ in $\mathcal{D}'_\Gamma(X)$ if:

(i) $u_j \to u$ in $\mathcal{D}'(X)$,

(ii) for all $x_0 \in X$, there exists an neighborhood U_{x_0} of x_0 such that for all $\varphi \in C^\infty_{\mathrm{comp}}(U_{x_0})$ with $\varphi(x_0) > 0$, for all closed cone $V \subset \mathbb{R}^n$ such that $\Gamma \cap (\mathrm{supp}(\varphi) \times V) = \emptyset$, for all $N > 0$, one has:

$$\sup_{\xi\in V} \langle\xi\rangle^N |\widehat{\varphi u_j}(\xi) - \widehat{\varphi u}(\xi)| \to_{j\to\infty} 0. \tag{2.1.6}$$

In other words, outside of Γ, the convergence of the Fourier transform is very fast. It will also be convenient to have the following remark in mind:

Remark 2.1.9. Let $u \in \mathcal{E}'(X)$. Assume that $u_j \in \mathcal{E}'(X)$ is a family such that $u_j \to u$ in $\mathcal{D}'(X)$. Then, there exists $C, M > 0$ such that:

$$\forall j \in \mathbb{N}, \forall \xi \in \mathbb{R}^n, \qquad |\widehat{u_j}(\xi)| \le C\langle\xi\rangle^M.$$

Indeed, it suffices to apply Lemma A.3.4 with the function $\varphi := e^{-i\xi\bullet}$.

Remark 2.1.10. Note that we could replace (2.1.6) by the following uniform boundedness condition: for all $N > 0$, there exists $C_N > 0$ such that

$$\sup_{j\ge 0} \sup_{\xi\in V} \langle\xi\rangle^N |\widehat{\varphi u_j}(\xi)| < C_N. \tag{2.1.7}$$

Indeed, let us show that (2.1.7) implies (2.1.6).

Fix $N > 0$ and $\varepsilon > 0$. By (2.1.7) applied with $2N$, we obtain for all $\xi \in V$

$$\langle \xi \rangle^N |\widehat{\varphi u_j}(\xi) - \widehat{\varphi u}(\xi)| \leq C_{2N} \langle \xi \rangle^{-N}.$$

Hence, there exists $R > 0$ large enough such that for all $|\xi| \geq R$, $\langle \xi \rangle^N |\widehat{\varphi u_j}(\xi) - \widehat{\varphi u}(\xi)| < \varepsilon$. We now study the case $|\xi| \leq R$. We claim that

$$\sup_{\xi \in V, |\xi| \leq R} |\widehat{\varphi u_j}(\xi) - \widehat{\varphi u}(\xi)| \to_{j \to \infty} 0.$$

Indeed, if not, then (up to a subsequence) there exists $\delta > 0$ and a sequence $(\xi_j)_{j \in \mathbb{N}}$ of points in $V \cap B_R$ (where $B_R := \{\xi \in \mathbb{R}^n \mid |\xi| \leq R\}$) such that

$$|\widehat{\varphi u_j}(\xi_j) - \widehat{\varphi u}(\xi_j)| > \delta. \tag{2.1.8}$$

By compactness of $V \cap B_R$, up to a subsequence, we have $\xi_j \to \xi_\infty$. Then:

$$\begin{aligned} |\widehat{\varphi u_j}(\xi_j) - \widehat{\varphi u}(\xi_j)| \leq |\widehat{\varphi u_j}(\xi_j) - \widehat{\varphi u_j}(\xi_\infty)| + |\widehat{\varphi u_j}(\xi_\infty) - \widehat{\varphi u}(\xi_\infty)| \\ + |\widehat{\varphi u}(\xi_\infty) - \widehat{\varphi u}(\xi_j)|. \end{aligned} \tag{2.1.9}$$

The last term in (2.1.9) is bounded by using that u has finite order N and that $e^{-i\xi_j \bullet} \to e^{-i\xi \bullet}$ converges in $C^\infty(X)$:

$$\begin{aligned} |\widehat{\varphi u}(\xi_\infty) - \widehat{\varphi u}(\xi_j)| &= \left|\left(u, \varphi(e^{-i\xi_\infty \bullet} - e^{-i\xi_j \bullet})\right)\right| \\ &\leq C \|\varphi(e^{-i\xi_\infty \bullet} - e^{-i\xi_j \bullet})\|_{C^N(X)} \to_{j \to \infty} 0. \end{aligned}$$

The second term in (2.1.9) converges to 0 by using the convergence in $\mathcal{D}'(X)$:

$$|\widehat{\varphi u_j}(\xi_\infty) - \widehat{\varphi u}(\xi_\infty)| = \left|\left((u - u_j), \varphi e^{-i\xi_\infty \bullet}\right)\right| \to_{j \to \infty} 0.$$

As to the first term in (2.1.9), we use Lemma A.3.4 which guarantees the existence of a uniform order $M > 0$ bounding the sequence $(u_j)_{j \geq 0}$, that is:

$$\begin{aligned} |\widehat{\varphi u_j}(\xi_j) - \widehat{\varphi u_j}(\xi_\infty)| &= \left|\left(u_j, \varphi(e^{-i\xi_j \bullet} - e^{-\xi_\infty \bullet})\right)\right| \\ &\leq C \|\varphi(e^{-i\xi_j \bullet} - e^{-\xi_\infty \bullet})\|_{C^M(X)} \to_{j \to \infty} 0. \end{aligned}$$

As a consequence, we get by (2.1.9) that $|\widehat{\varphi u_j}(\xi_j) - \widehat{\varphi u}(\xi_j)| \to 0$ which contradicts (2.1.8).

We also have:

Lemma 2.1.11. *$C^\infty_{\mathrm{comp}}(X)$ (resp. $C^\infty(X)$) is dense in $\mathcal{E}'_\Gamma(X)$ (resp. $\mathcal{D}'_\Gamma(X)$).*

Proof. We show that $C^\infty_{\mathrm{comp}}(X)$ is dense in $\mathcal{E}'_\Gamma(X)$. The case of $\mathcal{D}'_\Gamma(X)$ is handled similarly by taking an exhaustion of X by compact sets. Let $u \in \mathcal{E}'_\Gamma(X)$ and define $u_\varepsilon := \chi_\varepsilon \star u$, where $\chi_\varepsilon := \varepsilon^{-n} \chi(\bullet/\varepsilon)$ and χ is a small nonnegative even cutoff function equal to 1 at $x = 0$ and such that

$$\int_{\mathbb{R}^n} \chi = 1.$$

Observe that for $\varepsilon > 0$ small enough, $\mathrm{supp}(u_\varepsilon) \subset X$. We claim that $u_\varepsilon \to u$ in $\mathcal{D}'_\Gamma(X)$. In $\mathcal{D}'(X)$, this is obvious since given $\varphi \in C^\infty_{\mathrm{comp}}(X)$, one has:

$$(u_\varepsilon, \varphi) = (\chi_\varepsilon \star u, \varphi) = (u, \chi_\varepsilon \star \varphi) \to (u, \varphi). \tag{2.1.10}$$

Take $x_0 \in X, \varphi, \varphi' \in C^\infty_{\mathrm{comp}}(X)$ with support near x_0 such that $\varphi \prec \varphi'$ and a cone $V \subset \mathbb{R}^n$ such that $\Gamma \cap (\mathrm{supp}(\varphi) \times V) = \emptyset$. By assumption, $\xi \mapsto \widehat{\varphi' u}(\xi)$ decays super-polynomially in a conic neighborhood of all directions $\xi_0 \in V$. Then, writing $u = \varphi' u + (1 - \varphi')u$, we get for $\varepsilon > 0$ small enough:

$$\begin{aligned}
\widehat{\varphi u_\varepsilon}(\xi) - \widehat{\varphi u}(\xi) &= \mathcal{F}\left(\varphi\left(\chi_\varepsilon \star (\varphi' u)\right)\right)(\xi) + \mathcal{F}\left(\varphi\left(\chi_\varepsilon \star ((1-\varphi')u)\right)\right)(\xi) - \mathcal{F}(\varphi\varphi' u)(\xi) \\
&= \mathcal{F}\left(\varphi\left(\chi_\varepsilon \star (\varphi' u) - \varphi' u\right)\right)(\xi) \\
&= (2\pi)^n \mathcal{F}(\varphi) \star \left(\mathcal{F}\left(\chi_\varepsilon \star (\varphi' u)\right) - \mathcal{F}(\varphi' u)\right)(\xi).
\end{aligned}$$

Arguing as in (2.1.3), we decompose the convolution product as:

$$\widehat{\varphi u_\varepsilon}(\xi) - \widehat{\varphi u}(\xi) = \int_{B(0,\delta|\xi|)} \left(\widehat{\chi}_\varepsilon(\xi - \eta) - 1\right) \widehat{\varphi' u}(\xi - \eta) \widehat{\varphi}(\eta) \mathrm{d}\eta + \int_{\mathbb{R}^n \setminus B(0,\delta|\xi|)} \cdot \tag{2.1.11}$$

Observe that $\widehat{\chi}_\varepsilon(\xi) = \widehat{\chi}(\varepsilon\xi)$. As a consequence, using that $\widehat{\chi}(0) = \int \chi = 1$, we obtain:

$$|\widehat{\chi}_\varepsilon(\xi) - 1| \leq \varepsilon |\xi| \, \|\widehat{\chi}\|_{C^1} \lesssim \varepsilon \langle \xi \rangle.$$

We then argue as in (2.1.3): the first integral in (2.1.11) is bounded using the decay of $\widehat{\varphi' u}$ in a conic neighborhood of the ξ-direction while the second integral is bounded using the Paley-Wiener Theorem A.3.11. This gives for $N \gg 0$ large enough:

$$\begin{aligned}
|\widehat{\varphi u_\varepsilon}(\xi) - \widehat{\varphi u}(\xi)| &\lesssim \varepsilon \int_{B(0,\delta|\xi|)} \langle \xi - \eta \rangle \langle \xi - \eta \rangle^{-2N} \langle \eta \rangle^{-2N} \mathrm{d}\eta \\
&\quad + \varepsilon \int_{\mathbb{R}^n \setminus B(0,\delta|\xi|)} \langle \xi - \eta \rangle \langle \xi - \eta \rangle^{M} \langle \eta \rangle^{-2N} \, \mathrm{d}\eta \\
&\lesssim \varepsilon \langle \xi \rangle^{-N}.
\end{aligned}$$

This concludes the proof. □

Exercise 2.1.12. Make sure you know how to prove (2.1.10). For that, you can use Lemma A.3.7.

2.1.2 Wavefront set of an oscillatory integral

The goal of this paragraph is to relate the wavefront of an oscillatory integral to the symbol and the phase function. We start with the following:

Lemma 2.1.13. *Let $a \in S^m_{\rho,\delta}(X \times \mathbb{R}^N)$. Assume that for all $(x, \theta) \in \mathrm{supp}(a)$, $d_\theta \phi(x, \theta) \neq 0$. Then the oscillatory integral I_a defines a smooth function $I_a \in C^\infty(X)$.*

Proof. This is based on the strategy of proof of Lemma 1.1.17. Indeed, introduce $\Phi(x,\theta) := |\theta|^2|d_\theta\phi(x,\theta)|^2$. By assumption $\Phi \neq 0$ for $\theta \neq 0$. Then define, for some cutoff function χ located near $\theta = 0$:

$$Q := (1-\chi(\theta))\Phi^{-1}(x,\theta)\sum_j |\theta|^2\partial_{\theta_j}\phi(x,\theta)D_{\theta_j} + \chi(\theta)$$

By construction, we have $Qe^{i\phi} = e^{i\phi}$ and we set $L := {}^tQ$. As before, it can be checked that L has the form:

$$L = \sum_{j=1}^N a_j\partial_{\theta_j} + b,$$

where $a_j \in S^0_{1,0}, b \in S^{-1}_{1,0}$. Moreover, we have by construction that $I_a = I_{L^k a}$. Now it suffices to observe that $L^k a \in S^{m-k\rho}_{\rho,\delta}(X \times \mathbb{R}^N)$. Taking k large enough, we can apply Lemma 1.1.13 and we see that I_a is smooth. □

By Example 1.1.21, $\mathrm{WF}(\delta_0) = \{0\} \times (\mathbb{R}^n \setminus \{0\})$ and the phase $\phi(x,\xi) := x\cdot\xi$ allows to define δ_0. Observe that $\partial_\xi\phi(x,\xi) = x$ vanishes only at $x = 0$ and $\partial_x\phi(x = 0,\xi) = \xi$. So it seems that there is a relation between the phase and the wavefront set... We now consider the general case. Given a phase function $\phi \in C^\infty(X \times \mathbb{R}^N)$, we define the closed set:

$$\Lambda_\phi := \{(x, d_x\phi(x,\theta)) \mid d_\theta\phi(x,\theta) = 0, \theta \neq 0\} \subset T^*X \setminus \{0\}.$$

Observe that by 1-homogeneity of ϕ in the θ-variable, this is a *closed conic subset* of T^*X. We have:

Theorem 2.1.14. *The following holds:*

$$\mathrm{WF}(I_a) \subset \Lambda_\phi.$$

Proof. First of all, we write

$$\begin{aligned} I_a(x) &= \int_{\mathbb{R}^N_\theta} e^{i\phi(x,\theta)}a(x,\theta)\mathrm{d}\theta \\ &= \int_{\mathbb{R}^N_\theta} e^{i\phi(x,\theta)}\chi(x,\theta)a(x,\theta)\mathrm{d}\theta + \int_{\mathbb{R}^N_\theta} e^{i\phi(x,\theta)}(1-\chi(x,\theta))a(x,\theta)\mathrm{d}\theta \\ &= I_a^{(1)}(x) + I_a^{(2)}(x), \end{aligned}$$

where $\chi \in C^\infty(X \times \mathbb{R}^N)$ is a smooth cutoff function ≤ 1 such that $\chi = 1$ in a small conic neighborhood of the set

$$C_\phi := \left\{(x,\theta) \in X \times \mathbb{R}^N \mid d_\theta\phi(x,\theta) = 0\right\},$$

and $\chi = 0$ outside (near $\theta = 0$, we take $\chi = 1$). By Lemma 2.1.13, we know that $I_a^{(2)}$ is smooth so it suffices to study $I_a^{(1)}$. We write $a'(x,\theta) := a(x,\theta)\chi(x,\theta)$ which has support near C_ϕ by construction.

Consider $(x_0, \xi_0) \in T_0^*X \setminus \Lambda_\phi$ and $\psi \in C^\infty_{\mathrm{comp}}(X)$ with support near x_0 and such that $\psi(x_0) \neq 0$. We have:

$$\widehat{\psi I_{a'}}(\xi) = \langle I_{a'}, \psi e^{-i\bullet\cdot\xi}\rangle = \int_{\mathbb{R}^n_x}\int_{\mathbb{R}^N_\theta} e^{i(\phi(x,\theta)-x\cdot\xi)}\psi(x)a'(x,\theta)\mathrm{d}\theta\mathrm{d}x.$$

The strategy is the same as before and we make repeated integration by parts to show rapid decay in ξ. We consider a conic neighborhood V of $\xi_0/|\xi_0|$ that is small enough so that (x,ξ) does not belong to Λ_ϕ for all $x \in \mathrm{supp}(\psi), \xi \in V$ (this is always possible since $(x_0,\xi_0) \notin \Lambda_\phi$, up to taking ψ with small enough support and V small enough).

Observe that for all $(x,\theta) \in (\mathrm{supp}(\psi) \times \mathbb{R}^N) \cap C_\phi$, we have $d_x\phi(x,\theta) - \xi_0 \neq 0$ and thus by continuity the same holds for (x,θ) in a conic neighborhood of C_ϕ and ξ in a conic neighborhood of ξ_0.

We then have for all $(x,\theta) \in \mathrm{supp}(a') \cap (\mathrm{supp}(\psi) \times \mathbb{R}^N), \xi \in V$:

$$\begin{aligned}|d_x(\phi(x,\theta) - x\cdot\xi)| &= |d_x\phi(x,\theta) - \xi| \\ &= (|\theta| + |\xi|)\left|d_x\phi\left(x, \frac{\theta}{|\theta| + |\xi|}\right) - \frac{\xi}{|\theta|+|\xi|}\right| \geq C(|\theta| + |\xi|),\end{aligned} \tag{2.1.12}$$

where the last bound follows from the fact that $d_x\phi(x,\theta) - \xi \neq 0$ on the compact subset:

$$(\mathrm{supp}(\psi) \times \mathbb{R}^N_\theta) \cap \mathrm{supp}(a') \cap V \cap (\mathrm{supp}(\psi) \cap K) \subset \mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_\xi$$

where $K := \left\{(\theta,\xi) \in \mathbb{R}^N \times \mathbb{R}^n \mid |\theta| + |\xi| = 1\right\}$.

We then set

$$Q := \frac{1}{|d_x\phi(x,\theta) - \xi|^2}\sum_{j=1}^n (\partial_{x_j}\phi(x,\theta) - \xi_j)D_{x_j}$$

which satisfies $Q(e^{i(\phi - x\cdot\xi)}) = e^{i(\phi-x\cdot\xi)}$ and define $L := {}^tQ$. We observe that

$$(x,\theta,\xi) \mapsto \frac{(\partial_{x_j}\phi(x,\theta) - \xi_j)}{|d_x\phi(x,\theta) - \xi|^2} \in S^{-1}_{1,0}(X \times \mathbb{R}^N \times \mathbb{R}^n),$$

and this is a consequence of the bound (2.1.12). This implies that L has the form

$$L = \sum_{j=1}^n a_j D_{x_j} + b,$$

where $a_j, b \in S^{-1}_{1,0}(X \times \mathbb{R}^N \times \mathbb{R}^n)$. By repeated integration by parts, we get:

$$\widehat{\psi I_{a'}}(\xi) = \int_{\mathbb{R}^n_x}\int_{\mathbb{R}^n_\theta} e^{i(\phi(x,\theta)-x\cdot\xi)}L^k(\psi(x)a'(x,\theta))\mathrm{d}\theta\mathrm{d}x,$$

and this is controlled by:

$$|\widehat{\psi I_{a'}}(\xi)| \lesssim \int_{x\in\mathrm{supp}(\psi)} \int_{\mathbb{R}^n_\theta} (|\theta| + |\xi|)^{-k} \mathrm{d}\theta \mathrm{d}x \lesssim |\xi|^{n-k}.$$

Taking k large enough, we obtain the desired result. □

2.1.3 Exercises

Exercise 1

Let F_1, F_2 be two Fréchet spaces. What does it mean for a linear map $u : F_1 \to F_2$ to be continuous?

Exercise 2

The principal value of $1/x$ is defined as the distribution $\mathrm{vp}(1/x) : C^\infty_{\mathrm{comp}}(\mathbb{R}) \to \mathbb{C}$ such that:

$$\mathrm{vp}(1/x) : \varphi \mapsto \lim_{\varepsilon\to 0^+} \int_{\mathbb{R}\setminus[-\varepsilon,\varepsilon]} \frac{\varphi(x)}{x} \mathrm{d}x.$$

1. What is $\mathrm{supp}\,(\mathrm{vp}(1/x))$?
2. What is the order of $\mathrm{vp}(1/x)$?
3. Compute $\mathrm{WF}(\mathrm{vp}(1/x))$.

Exercise 3

Let $\delta_{\mathbb{R}^k}$ be the Dirac mass on the k-plane $\mathbb{R}^k \times \{0\} \subset \mathbb{R}^n$, that is

$$\delta_{\mathbb{R}^k}(\varphi) := \int_{\mathbb{R}^k} \varphi(x, 0)\mathrm{d}x.$$

Show that $\mathrm{WF}(\delta_{\mathbb{R}^k}) = N^*\mathbb{R}^k \setminus \{0\}$, the *conormal to* $\mathbb{R}^k$, that is

$$\left\{((x, 0), \xi) \mid \forall v \in \mathbb{R}^k \times \{0\}, \langle \xi, v\rangle = 0\right\}$$

Exercise 4

Define for $z \in \mathbb{C}$ the function x^z_+ by:

$$x^z_+ = \begin{cases} 0 & \text{on } (-\infty, 0], \\ \exp(z \log(\bullet)) & \text{on } (0, \infty). \end{cases}$$

1. Show that x^z_+ defines a distribution on $\mathbb{R}$ for $\Re(z) > -1$. Compute its support and its wavefront set.
2. Show that $\{\Re(z) > -1\} \ni z \mapsto x^z_+ \in \mathcal{D}'(\mathbb{R})$ is holomorphic in the sense that for all $\varphi \in C^\infty_{\mathrm{comp}}(\mathbb{R})$, the function $\{\Re(z) > -1\} \ni z \mapsto (x^z_+, \varphi) \in \mathbb{C}$ is holomorphic.

Our goal is to show that $\mathbb{C} \ni z \mapsto x_+^z \in \mathcal{D}'(\mathbb{R})$ extends to a meromorphic family of distributions. This means that there exists a maximal countable and isolated subset $\mathcal{P} \subset \mathbb{C}$ and a map $n : \mathcal{P} \to \mathbb{Z}_+^*$ such that for all $\varphi \in C^\infty_{\text{comp}}(\mathbb{R})$, the function $\mathbb{C} \ni z \mapsto (x_+^z, \varphi) \in \mathbb{C}$ is meromorphic, with poles contained in $\mathcal{P}$, and of order at most given by n.

For $z \in \mathbb{C} \setminus \mathbb{Z}_-^*$, define for $k > -\Re(z) - 1$:

$$(\mathrm{pf}(x_+^z), \varphi) := (-1)^k \int_0^{+\infty} \frac{x^{z+k}}{(z+1)...(z+k)} \partial_x^k \varphi(x) \mathrm{d}x.$$

3. Show that the definition of $\mathrm{pf}(x_+^z)$ is independent of k as long as $k > -\Re(z) - 1$. Show that it coincides with x_+^z when $\Re(z) > -1$.

Let Γ be the Euler function. Recall that $\Gamma(n+1) = n!$ and that Γ admits a meromorphic extension to $\mathbb{C}$. We define for $z \in \mathbb{C} \setminus \mathbb{Z}_-^*$:

$$\chi_+^z := \frac{\mathrm{pf}(x_+^z)}{\Gamma(z+1)}.$$

4. Show that $\partial_x \chi_+^z = \chi_+^{z-1}$ in $\mathcal{D}'(\mathbb{R})$ for all $z \in \mathbb{C}$ such that $\{\Re(z) > 0\}$.
5. Deduce that $\mathbb{C} \ni z \mapsto \chi_+^z \in \mathcal{D}'(\mathbb{R})$ is holomorphic.
6. Conclude that $\mathbb{C} \ni z \mapsto x_+^z \in \mathcal{D}'(\mathbb{R})$ admits a meromorphic extension from $\{\Re(z) > -1\}$ to $\mathbb{C}$.

Exercise 5

Let $f \in C^\infty(X), \Im(f) \geq 0$, where $X \subset \mathbb{R}^n$ an open subset. Fix $\varepsilon > 0$.

1. Show that
$$\frac{1}{f(x) + i\varepsilon} = \frac{1}{i} \int_0^{+\infty} e^{i(f(x)+i\varepsilon)\tau} \mathrm{d}\tau.$$
2. We assume that $df(x) \neq 0$ when $f(x) = 0$. Show that the limit
$$\frac{1}{f(x) + i0} := \lim_{\varepsilon \to 0} \frac{1}{f(x) + i\varepsilon}$$
exists in $\mathcal{D}'(X)$.
3. What is $\mathrm{singsupp}\left((f(x) + i0)^{-1}\right)$?
4. Compute $\mathrm{WF}\left((f(x) + i0)^{-1}\right)$.
5. For $n = 1$, show that
$$\frac{1}{x \pm i0} = \mp i\pi\delta_0 + \mathrm{vp}(1/x), \qquad \delta_0 = \frac{1}{2i\pi}\left(\frac{1}{x - i0} - \frac{1}{x + i0}\right).$$

Exercise 6:

Let X be an open subset of $\mathbb{R}^n$ and let $u \in \mathcal{D}'(X)$.

1. What does it mean for a distribution to be real?
2. Show that if u is real, then $\mathrm{WF}(u)$ is invariant by the action of the fiberwise antipodal map $(x, \xi) \mapsto (x, -\xi)$.
3. Assume X is such that $R(X) = X$, where $R(x) = -x$. What does it mean for u to be even or odd?
4. Show that if u is even or odd, then: $(x, \xi) \in \mathrm{WF}(u)$ iff $(-x, -\xi) \in \mathrm{WF}(u)$.

2.2 Extension of linear operators to distributions

2.2.1 Bilinear integration of distributions

We start with the following technical lemma:

Lemma 2.2.1 (Bilinear integration)**.** *Let $\Gamma_1, \Gamma_2 \subset T^*X \setminus \{0\}$ be two closed cones such that $\Gamma_1 \cap (-\Gamma_2) = \emptyset$. The bilinear map $B : C^\infty(X) \times C^\infty_{\mathrm{comp}}(X) \to \mathbb{C}$, defined by*

$$B(f_1, f_2) := \int_X f_1(x) f_2(x) \mathrm{d}x \tag{2.2.1}$$

has a unique sequentially continuous extension to $\mathcal{D}'_{\Gamma_1}(X) \times \mathcal{E}'_{\Gamma_2}(X) \to \mathbb{C}$.

Recall that the convergence in $\mathcal{D}'_\Gamma(X)$ is slightly more restrictive than that in $\mathcal{D}'(X)$, see Definition 2.1.8. In particular, this lemma has the following remarkable consequence:

Corollary 2.2.2 (Multiplication of distributions)**.** *Let $\Gamma_1, \Gamma_2 \subset T^*X \setminus \{0\}$ be two closed cones such that $\Gamma_1 \cap (-\Gamma_2) = \emptyset$. Then, the multiplication*

$$\times : C^\infty(X) \times C^\infty(X) \to C^\infty(X) \hookrightarrow \mathcal{D}'(X), \qquad (f_1, f_2) \mapsto f_1 \times f_2,$$

has a unique sequentially continuous extension to $\mathcal{D}'_{\Gamma_1}(X) \times \mathcal{D}'_{\Gamma_2}(X) \to \mathcal{D}'(X)$.

One can actually refine the previous corollary by computing the wavefront set of the product. This is carried out in Lemma A.5.7. We start by proving the Corollary and then go on with proving Lemma 2.2.1.

Proof of Corollary 2.2.2. For $\varphi \in C^\infty_{\mathrm{comp}}(X)$, $f_1 \in \mathcal{D}'_{\Gamma_1}(X)$, $f_2 \in \mathcal{D}'_{\Gamma_2}(X)$, define $(f_1 \times f_2, \varphi) := B(f_1, \varphi f_2)$. This is well-defined as $\varphi f_2 \in \mathcal{E}'_{\Gamma_2}(X)$ and this is a distribution (i.e. a linear continuous functional on $C^\infty_{\mathrm{comp}}(X)$) because if $\varphi_j \to \varphi$ in $C^\infty_{\mathrm{comp}}(X)$, then $\varphi_j f_2 \to \varphi f_2$ in $\mathcal{E}'_{\Gamma_2}(X)$ and B is continuous by Lemma 2.2.1 so $(f_1 \times f_2, \varphi_j) \to (f_1 \times f_2, \varphi)$. $\square$

Proof of Lemma 2.2.1. Uniqueness is a mere consequence of the density of $C^\infty(X)$ (resp. $C^\infty_{\mathrm{comp}}(X)$) in $\mathcal{D}'_\Gamma(X)$ (resp. $\mathcal{E}'_\Gamma(X)$) proved in Lemma 2.1.11. Indeed, if B_1 and B_2 are two smooth continuous extensions of B from $C^\infty(X)\times C^\infty_{\mathrm{comp}}(X)$ to $\mathcal{D}'_{\Gamma_1}(X)\times\mathcal{E}'_{\Gamma_2}(X)$, then given $f_1\in\mathcal{D}'_{\Gamma_1}(X), f_2\in\mathcal{E}'_{\Gamma_2}(X)$, $u_j\in C^\infty(X)$ such that $u_j\to f_1$ in $\mathcal{D}'_{\Gamma_1}(X)$ and $v_j\in C^\infty_{\mathrm{comp}}(X)$ such that $v_j\to f_2$, we must have $B(u_j,v_j)\to B_1(f_1,f_2)=B_2(f_1,f_2)$.

We now show existence and continuity. For that, let $x_0\in X,\varphi\in C^\infty_{\mathrm{comp}}(X)$ with support near x_0 such that $\varphi\geq 0,\varphi(x_0)=1$ and consider open conic subsets $V_1,V_2\subset\mathbb{R}^n$ such that $V_1\cap(-V_2)=\emptyset$, and

$$V_i\supset\{\xi\in\mathbb{R}^n\setminus\{0\}\mid\exists x\in\mathrm{supp}(\varphi),(x,\xi)\in\Gamma_i\},$$

is a small conic neighborhood. The existence of such a φ near x_0 is guaranteed by the fact that $\Gamma_1\cap(-\Gamma_2)=\emptyset$. We call such a function φ a *good* function.

We define (2.2.1) locally (near x_0) by means of the Parseval identity. For $f_1\in C^\infty(X), f_2\in C^\infty_{\mathrm{comp}}(X)$, we have:

$$(\varphi f_1,\varphi f_2)=\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{\varphi f_1}(\xi)\widehat{\varphi f_2}(-\xi)\mathrm{d}\xi=:B_\varphi(f_1,f_2). \tag{2.2.2}$$

We claim that this is extends continuously on $\mathcal{D}'_{\Gamma_1}(X)\times\mathcal{E}'_{\Gamma_2}(X)$. If $f_1\in\mathcal{D}'_{\Gamma_1}(X)$, $f_2\in\mathcal{E}'_{\Gamma_2}(X)$, then f_2 has compact support and we can take a finite sum[3] such that $\sum_{i\in\mathcal{I}}\varphi^2_{x_i}>0$ on a relatively open subset of X containing the support of f_2, and each φ_{x_i} is good. We can define $\varphi\in C^\infty(X)$ such that $\varphi^2=\sum_{i\in\mathcal{I}}\varphi^2_{x_i}$ on X and it suffices to set:

$$(f_1,f_2):=\sum_{i\in\mathcal{I}}(\varphi^{-1}\varphi_i f_1,\varphi^{-1}\varphi_i f_2).$$

So the proof boils down to a local statement near a point x_0. We need to show that B_φ extends continuously from $C^\infty(X)\times C^\infty_{\mathrm{comp}}(X)$ to $\mathcal{D}'_{\Gamma_1}(X)\times\mathcal{E}'_{\Gamma_2}(X)$. Let us first check that if f_1 and f_2 are two distributions in $\mathcal{D}'_{\Gamma_1}(X)\times\mathcal{E}'_{\Gamma_2}(X)$ such that $\Gamma_1\cap(-\Gamma_2)=\emptyset$, then B_φ is well-defined. We decompose the integral as:

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{\varphi f_1}(\xi)\widehat{\varphi f_2}(-\xi)\mathrm{d}\xi=\frac{1}{(2\pi)^n}\left(\int_{V_1}+\int_{V_2}+\int_{\mathbb{R}^n\setminus(V_1\cup V_2)}\right) \tag{2.2.3}$$

For any direction ξ, either $\widehat{\varphi f_1}(\xi)$ or $\widehat{\varphi f_2}(-\xi)$ decreases faster than any polynomial by the condition $\Gamma_1\cap(-\Gamma_2)=\emptyset$, while the other one is polynomially bounded in ξ since the distribution has finite order. As a consequence, we get for some $M>0$ (provided by the order of the distribution φf_1) and for any positive $N\gg 0$ (that we choose very large compared to M):

$$\int_{V_1}|\widehat{\varphi f_1}(\xi)\widehat{\varphi f_2}(-\xi)|\mathrm{d}\xi\lesssim\int_{V_1}\langle\xi\rangle^M\langle\xi\rangle^{-N}\mathrm{d}\xi<\infty.$$

The same holds for the V_2 conic neighborhood. As to $\mathbb{R}^n\setminus(V_1\cup V_2)$, the bound is

[3] That is, for every open subset $U\Subset X$, the the number of functions φ_{x_i} with support intersecting U is finite.

even easier since both Fourier transforms decay super-polynomially. This shows that the right-hand side of (2.2.2) is well-defined for distributions in $\mathcal{D}'_{\Gamma_1}(X) \times \mathcal{E}'_{\Gamma_2}(X)$.

So we take the (2.2.2) for a definition. Of course, when f_1 and f_2 are smooth, this coincides with the expected formula, namely (2.2.1) (modulo multiplication by φ^2).

Let us now prove continuity of B_φ. This is more or less the same as existence. If $(u_j)_{j\in\mathbb{N}}, (v_j)_{j\in\mathbb{N}}$ are families of distributions in $\mathcal{D}'_{\Gamma_1}(X)$ (resp. $\mathcal{E}'_{\Gamma_2}(X)$) such that $u_j \to f_1$ in $\mathcal{D}'_{\Gamma_1}(X)$ (resp. $v_j \to f_2$ in $\mathcal{D}'_{\Gamma_2}(X)$), we want to show that $B_\varphi(u_j, v_j) \to B_\varphi(f_1, f_2)$. Note that all v_j's can be chosen with support in some fixed compact subset of X. Observe that by Remark 2.1.9 (which follows itself from Lemma A.3.4) convergence in $\mathcal{D}'(X)$ guarantees the existence of uniform constants $C, M > 0$ such that:

$$\forall j \in \mathbb{N}, \forall \xi \in \mathbb{R}^n, \qquad |\widehat{\varphi u_j}(\xi)|, |\widehat{\varphi v_j}(\xi)| \quad \leq C\langle\xi\rangle^M. \tag{2.2.4}$$

We use the decomposition (2.2.3) once again for $B_\varphi(u_j, v_j) - B_\varphi(f_1, f_2)$. We are thus left with showing that

$$\int_Y |\widehat{\varphi f_1}(\xi)\widehat{\varphi f_2}(-\xi) - \widehat{\varphi u_j}(\xi)\widehat{\varphi v_j}(-\xi)|\mathrm{d}\xi \to_{j\to\infty} 0, \tag{2.2.5}$$

for $Y = V_1, V_2$ and $\mathbb{R}^n \setminus (V_1 \cup V_2)$. Let us deal with V_2 for instance.

First of all, observe that we have the pointwise convergence for all $\xi \in \mathbb{R}^n$:

$$\widehat{\varphi u_j}(\xi)\widehat{\varphi v_j}(-\xi) = (u_j, \varphi e^{-i\xi\bullet})(v_j, \varphi e^{-i\xi\bullet}) \to \widehat{\varphi f_1}(\xi)\widehat{\varphi f_2}(-\xi),$$

by convergence in $\mathcal{D}'(X)$. In order to apply the dominated convergence Theorem A.3.13 in (2.2.5), it thus suffices to bound the integrand by an integrable function that is independent of j. By definition of convergence in $\mathcal{D}'_{\Gamma_1}(X)$ (see Definition 2.1.8), we have that for all $N \geq 0$, there exists $\varepsilon_j > 0$ such that $\varepsilon_j \to 0$ and:

$$\forall \xi \in V_2, \qquad |\widehat{\varphi f_1}(\xi) - \widehat{\varphi u_j}(\xi)| \leq \varepsilon_j \langle\xi\rangle^{-N}. \tag{2.2.6}$$

As a consequence, combining (2.2.4) and (2.2.6), we obtain that for all $\xi \in V_2$:

$$\begin{aligned}
|\widehat{\varphi u_j}(\xi)\widehat{\varphi v_j}(-\xi)| &\leq |(\widehat{\varphi u_j}(\xi) - \widehat{\varphi f_1}(\xi))\widehat{\varphi v_j}(-\xi)| + |\widehat{\varphi f_1}(\xi))\widehat{\varphi v_j}(-\xi)| \\
&\leq \varepsilon_j\langle\xi\rangle^{-N} C\langle\xi\rangle^M + C\langle\xi\rangle^{-N} C\langle\xi\rangle^M \leq C\langle\xi\rangle^{-N'},
\end{aligned}$$

for some other $N' > 0$ which can be taken large enough to make this function integrable. This eventually proves continuity.

□

Remark 2.2.3. Observe that in the previous proof, it seems *a priori* quite surprising that we do not use the full strength of the convergence in $\mathcal{D}'_\Gamma(X)$, that is, it would have sufficed to use that

$$\sup_{j\geq 0}\sup_{\xi\in V}\langle\xi\rangle^N |\widehat{\varphi u_j}(\xi)| < +\infty.$$

But actually, this condition is equivalent to convergence in $\mathcal{D}'_\Gamma(X)$ as proved in

Remark 2.1.10.

2.2.2 Extension theorem

Let $A : C^\infty_{\text{comp}}(Y) \to \mathcal{D}'(X)$ be an operator with Schwartz kernel $K_A \in \mathcal{D}'(X \times Y)$. We want to understand what are the restrictions to the existence of an extension of A (by continuity) to an operator defined on $\mathcal{E}'(Y)$. Let us start with a heuristic discussion. Formally, we want to define the distribution

$$Au(x) = \int_Y K_A(x, y)u(y)\mathrm{d}y,$$

for some well-chosen distribution $u \in \mathcal{E}'(Y)$ (namely, with some good wavefront set conditions), that is for all $\varphi \in C^\infty_{\text{comp}}(X)$, we want to define

$$(Au, \varphi) = (K, \varphi \otimes u)_{X\times Y} = \int_{X\times Y} K(x, y)(\varphi \otimes u)(x, y)\mathrm{d}x\mathrm{d}y = B(K, \varphi \otimes u). \quad (2.2.7)$$

Now, we see that this actually boils down to Lemma 2.2.1 with the distributions $K_A \in \mathcal{D}'(X \times Y)$ and $\varphi \otimes u \in \mathcal{D}'(X \times Y)$. Since A — or equivalently K_A is given —, we need to find some conditions on the wavefront set of u so that the wavefront set of $\varphi \otimes u$ does not intersect the (opposite of the) wavefront set of K_A in order to be able to apply Lemma 2.2.1. This will be the content of the Theorem 2.2.4 below.

Let us introduce some notation first. Generic points in $T^*(X \times Y)$ are denoted by (x, ξ, y, η). We define:

$$\mathrm{WF}'(K) := \{(x, \xi, y, -\eta) \mid (x, \xi, y, \eta) \in \mathrm{WF}(K)\}, \quad (2.2.8)$$

$$\mathrm{WF}'_X(K) := \{(x, \xi) \in T^*_0X \mid \exists y \in Y, (x, \xi, y, 0) \in \mathrm{WF}'(K)\}, \quad (2.2.9)$$

$$\mathrm{WF}'_Y(K) := \{(y, \eta) \in T^*Y_0 \mid \ \exists x \in X, (x, 0, y, \eta) \in \mathrm{WF}'(K)\}. \quad (2.2.10)$$

We also introduce the *composition* operator $\mathrm{WF}'(K)\circ : T^*Y \to T^*X$ defined in the following way: given $\Omega \subset T^*Y$, we set:

$$\mathrm{WF}'(K) \circ \Omega := \{(x, \xi) \in T^*_0X \mid \exists(y, \eta) \in \Omega, (x, \xi, y, \eta) \in \mathrm{WF}'(K)\}.$$

The most common use of this will be

$$\mathrm{WF}'(K) \circ \mathrm{WF}(u) = \{(x, \xi) \in T^*_0X \mid \exists(x, \xi, y, \eta) \in \mathrm{WF}'(K), (y, \eta) \in \mathrm{WF}(u)\}.$$

Writing $O_X := X \times \{0\}$ and $O_Y := Y \times \{0\}$, we see that

$$\mathrm{WF}'_X(K) = \mathrm{WF}'(K) \circ O_Y, O_X \supset \mathrm{WF}'(K) \circ WF'_Y(K).$$

We then have the following important result:

Theorem 2.2.4 (Extension of linear operators to distributions)**.** *Let $A : C^\infty_{\text{comp}}(Y) \to \mathcal{D}'(X)$ be a continuous linear operator with Schwartz kernel $K \in \mathcal{D}'(X\times Y)$. Let $\Gamma \subset T^*Y_0$ be a closed cone such that $\mathrm{WF}'_Y(K)\cap\Gamma = \emptyset$. Then $\widetilde{\Gamma} := \mathrm{WF}'(K)\circ\Gamma\cup\mathrm{WF}'_X(K)$ is closed in T^*_0X and $A : \mathcal{E}'_\Gamma(Y) \to \mathcal{D}'_{\widetilde{\Gamma}}(X)$ extends uniquely as a continuous operator.*

In particular, if $u \in \mathcal{D}'(Y)$ is such that $\mathrm{WF}(u) \cap \mathrm{WF}'_Y(K) = \emptyset$, *then*

$$\mathrm{WF}(Au) \subset \mathrm{WF}'(K) \circ \mathrm{WF}(u) \cup \mathrm{WF}'_X(K).$$

Let us formulate an important "practical" remark:

Remark 2.2.5. As we shall see in the proof, the operator A acting on $\mathcal{E}'_\Gamma(Y)$ will be therefore defined by (2.2.7) which is itself defined by means of the multiplication Lemma 2.2.1. Of course, in practice, one never uses these definitions! Since all these operators are continuous extensions from smooth functions to distributions (satisfying some good wavefront set conditions), in order to compute Au for $u \in \mathcal{E}'_\Gamma(Y)$, one considers a sequence of smooth functions $u_\varepsilon \in C^\infty_{\mathrm{comp}}(Y)$ such that $u_\varepsilon \to u$ in $\mathcal{E}'_\Gamma(Y)$ and then compute $\lim_{\varepsilon\to 0} Au_\varepsilon$ in $\mathcal{D}'(X)$.

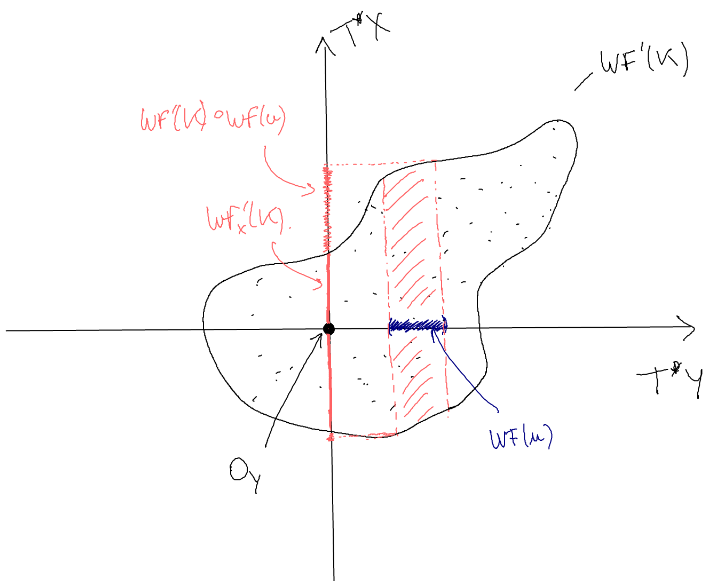

Figure 2.3: A schematic representation of the composition of wavefront sets.

In order to prove Theorem 2.2.4, we need a preliminary lemma on the wavefront set of the tensor product of distributions.

Lemma 2.2.6. *Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ be open subsets and $\Gamma_1 \subset T_0^*X, \Gamma_2 \subset T_0^*Y$ be two closed cones. Then the map*

$$\mathcal{D}'_{\Gamma_1}(X) \times \mathcal{D}'_{\Gamma_2}(Y) \ni (u,v) \mapsto u \otimes v \in \mathcal{D}'_{\Gamma_3}(X \times Y)$$

is (sequentially) continuous, where

$$\Gamma_3 = (\Gamma_1 \times \Gamma_2) \cup (\Gamma_1 \times O_Y) \cup (O_X \times \Gamma_2).$$

Proof. As in Lemma 2.2.1, continuity is straightforward once existence is proved. So it suffices to check that $\mathrm{WF}(u \otimes v) \subset (\Gamma_1 \times \Gamma_2) \cup (\Gamma_1 \times O_Y) \cup (O_X \times \Gamma_2)$ if $\mathrm{WF}(u) \subset \Gamma_1, \mathrm{WF}(v) \subset \Gamma_2$. Let $(x_0, \xi_0, y_0, \eta_0) \notin (\Gamma_1 \times \Gamma_2) \cup (\Gamma_1 \times O_Y) \cup (O_X \times \Gamma_2)$ and let $\varphi_{x_0} \in C^\infty_{\mathrm{comp}}(X), \varphi_{y_0} \in C^\infty_{\mathrm{comp}}(X)$ be two functions localized near x_0 and y_0 such that $\varphi_{x_0}(x_0) > 0, \varphi_{y_0}(y_0) > 0$. Define $\varphi := \varphi_{x_0}\varphi_{y_0}$. Then, writing $J := (\xi, \eta)$

for a generic point:

$$\widehat{\varphi(u \otimes v)}(J) = \widehat{\varphi_{x_0} u}(\xi)\widehat{\varphi_{y_0} u}(\eta).$$

We then argue as follows. Assume first that $(x_0, \xi_0) \in \Gamma_1$. Then by assumption $\eta_0 \neq 0$ (i.e. $(y_0, \eta_0) \notin O_Y$) and $(y_0, \eta_0) \notin \Gamma_2$. This implies that for all $N \gg 1$, there exists $C_N > 0$ such that for all $|\eta| \gg 1$ and η in a conic neighborhood of η_0, one has:

$$|\widehat{\varphi_{y_0} u}(\eta)| \leq C_N \langle \eta \rangle^{-N}.$$

Since we also have $|\widehat{\varphi_{x_0} u}(\xi)| \leq C\langle \xi \rangle^M$ for some $C, M > 0$, this implies that:

$$|\widehat{\varphi(u \otimes v)}(J)| \lesssim \langle \eta \rangle^{-N} \langle \xi \rangle^M.$$

Now, observe that, since $\eta_0 \neq 0$, for all $J = (\xi, \eta)$ in the conic neighborhood of (ξ_0, η_0), one has $|\xi| \leq c|\eta|$ for some constant $c > 0$. This implies that:

$$\langle \eta \rangle^2 + \langle \xi \rangle^2 \lesssim \langle J \rangle^2 = 1 + |\xi|^2 + |\eta|^2 \lesssim \langle \eta \rangle^2.$$

Hence:

$$|\widehat{\varphi(u \otimes v)}(J)| \lesssim \langle J \rangle^{-N'},$$

for some (other) constant $N' > 0$. This proves super-polynomial decay in a conic neighborhood of $J_0 = (\xi_0, \eta_0)$.

We now assume that $(x_0, \xi_0) \notin \Gamma_1$. If $\xi_0 = 0$, that is $(x_0, \xi_0) \in O_X$, then $(y_0, \eta_0) \notin \Gamma_2$ and we can apply the same argument as before, using that $|\xi| \leq c|\eta|$. If $\xi_0 \neq 0$, then the same argument also works but using $|\eta| \leq c|\xi|$ this time. $\square$

Proof of Theorem 2.2.4. As in the proof of Lemma 2.2.1, uniqueness follows from the density of smooth functions and continuity is a routine computation once the existence has been established. So we are left with proving the existence of such a natural extension. Let us quickly repeat the discussion leading to (2.2.7). We want to define the distribution

$$Au = \int_Y K(x, y)u(y)\mathrm{d}y,$$

that is for all $\varphi \in C^\infty_{\mathrm{comp}}(X)$, we want to define

$$(Au, \varphi) = (K, \varphi \otimes u)_{X \times Y} = \int_{X \times Y} K(x, y)(\varphi \otimes u)(x, y)\mathrm{d}x\mathrm{d}y. \tag{2.2.11}$$

First of all, observe that since φ is smooth:

$$\mathrm{WF}(\varphi \otimes u) \subset O_X \times \mathrm{WF}(u), \tag{2.2.12}$$

by Lemma 2.2.6. As a consequence, in order to apply Lemma 2.2.1, it suffices to have

$$\mathrm{WF}(K) \cap (-\mathrm{WF}(\varphi \otimes u)) = \emptyset,$$

and this is satisfied if $\mathrm{WF}(K) \cap (-(O_X \times \mathrm{WF}(u))) = \emptyset$. Note that

$$\mathrm{WF}(K) \cap (-(O_X \times \mathrm{WF}(u))) = \{(x, 0, y, -\eta) \in \mathrm{WF}(K) \mid (y, \eta) \in \mathrm{WF}(u)\},$$

and thus this condition is equivalent to

$$\mathrm{WF}(u) \cap \mathrm{WF}'_Y(K) = \emptyset.$$

As a consequence, (2.2.11) is well-defined on $\mathcal{D}'_\Gamma(Y)$, for any cone $\Gamma \subset T^*_0 Y$ such that $\Gamma \cap \mathrm{WF}'_Y(K) = \emptyset$. This is the first part of the statement in Theorem 2.2.4. It is also not difficult to check that $A : \mathcal{E}'_\Gamma(Y) \to \mathcal{D}'(X)$ is sequentially continuous.

We now need to bound $\mathrm{WF}(Au)$, that is we need to show that $\mathrm{WF}(Au) \subset \mathrm{WF}'(K) \circ \mathrm{WF}(u) \cup \mathrm{WF}'_X(K)$. Let $\varphi \in C^\infty_{\mathrm{comp}}(X)$ be a smooth cutoff function and $V \subset \mathbb{R}^n$ be a cone such that

$$(\operatorname{supp} \varphi \times V) \cap (\mathrm{WF}'(K) \circ \mathrm{WF}(u) \cup \mathrm{WF}'_X(K)) = \emptyset.$$

Define for $N > 0, \xi \in V$, $\psi_{N,\xi}(x) := \varphi(x) \langle \xi \rangle^N e^{-ix\xi}$. Now, given $u \in \mathcal{D}'(Y)$ such that $\mathrm{WF}(u) \cap \mathrm{WF}'_Y(K) = \emptyset$, we may compute:

$$(K, \psi_{N,\xi} \otimes u) = \langle \xi \rangle^N \widehat{\varphi K u}(\xi)$$

and we need to show that this is bounded by a constant $C > 0$ (uniformly in $\xi \in V$). More precisely, we want to show that this converges to 0 and for that it suffices to show that there exists some cone $\Gamma_2 \subset T^*_0(X \times Y)$ such that $\Gamma_2 \cap (-\mathrm{WF}(K)) = \emptyset$ and $\psi_{N,\xi} \otimes u \to 0$ in $\mathcal{D}'_{\Gamma_2}(X \times Y)$ since we can then apply the continuity property of Lemma 2.2.1 to conclude.

Now, setting $\Gamma_1 := \mathrm{supp}(\varphi) \times V$, observe that as $\xi \to \infty$ (and $\xi \in V$), we have $\psi_{N,\xi} \to 0$ in $\mathcal{D}'_{-\Gamma_1}(X)$ (indeed, as ξ increases, this tends to be more and more irregular in the $-\xi$ direction due to the $e^{-ix\xi}$ factor; but the Fourier transform still decays fast outside V) and this implies that $\psi_{N,\xi} \otimes u \to 0$ in $\mathcal{D}'_{\Gamma_2}(X \times Y)$ for some cone Γ_2 given by

$$\Gamma_2 = (-\Gamma_1 \times \mathrm{WF}(u)) \cup (-\Gamma_1 \times O_Y) \cup (O_X \times \mathrm{WF}(u)), \tag{2.2.13}$$

as follows from Lemma 2.2.6. It now suffices to check that $\mathrm{WF}(K) \cap -(\Gamma_2)$ is empty but this is just some simple verifications. This concludes the proof. □

Exercise 2.2.7. Check (2.2.13).

2.2.3 Operations on distributions

We review elementary operations on distributions. These can be expressed as oscillatory integrals and the previous results allow a precise description of their action on the wavefront set of distributions.

2.2.3.1 Multiplication of distributions

We can now discuss the multiplication of distributions. The following lemma refines Corollary 2.2.2.

Lemma 2.2.8 (Multiplication of distributions). *Let $\Gamma_1,\ \Gamma_2 \subset T_0^*X$ be two closed conic subsets such that $\Gamma_1 \cap (-\Gamma_2) = \emptyset$. Then the product*

$$\times : C^\infty(X) \times C^\infty(X) \to C^\infty(X), \qquad (f,g) \mapsto f \times g,$$

admits a unique continuous extension as a map

$$\times : \mathcal{D}'_{\Gamma_1}(X) \times \mathcal{D}'_{\Gamma_2}(X) \to \mathcal{D}'_{\Gamma_3}(X),$$

where

$$\Gamma_3 := \Gamma_1 \cup \Gamma_2 \cup (\Gamma_1 \oplus \Gamma_2), \tag{2.2.14}$$

and the direct sum is defined as:

$$\Gamma_1 \oplus \Gamma_2 := \{(x, \xi + \eta) \in T^*X_0 \mid (x,\xi) \in \Gamma_1, (x,\eta) \in \Gamma_2\}.$$

We refer to Exercise 7 in §2.1.3 below for some examples of products of distributions. The proof can be done by a direct adaptation of the proof of Lemma 2.2.1 (it suffices to add a smooth function φ in front of f_1 in Lemma 2.2.1 basically). However, even if it might seem complicated, we find it more instructive to use the full power of Theorem 2.2.4 to define the multiplication.

Proof. Let $\iota : X \to X \times X$ be the diagonal embedding $\iota(x) := (x,x)$. This map induces a pullback map on smooth functions

$$\iota^* : C^\infty(X \times X) \to C^\infty(X), \qquad \iota^* f(x) := f(x,x).$$

The multiplication of two smooth functions $f, g \in C^\infty(X)$ can be written as

$$f \times g = \iota^*(f \otimes g). \tag{2.2.15}$$

As a consequence, the question boils down to understand to what extent (2.2.15) can be extended to distributions.

We take $\Gamma_1, \Gamma_2 \subset T^*X_0$ two closed conic subsets such that $\Gamma_1 \cap (-\Gamma_2) = \emptyset$. Let $f \in \mathcal{D}'_{\Gamma_1}(X), g \in \mathcal{D}'_{\Gamma_2}(X)$. By Lemma 2.2.6, we know that

$$\mathrm{WF}(f \otimes g) \subset (\Gamma_1 \times \Gamma_2) \cup (\Gamma_1 \times O_X) \cup (O_X \times \Gamma_2). \tag{2.2.16}$$

In order to apply Theorem 2.2.4, we thus need to compute the Schwartz kernel K_{ι^*} of ι^*. Observe that this Schwartz kernel lives in $\mathcal{D}'(X \times X \times X)$, that is, $\mathcal{D}'(X^3)$ (this is called the *triple space*). Moreover, it satisfies (in the formal sense):

$$\iota^* f(x) = f(x,x) = \int_{X \times X} K_{\iota^*}(x,(y,z)) f(y,z) \mathrm{d}y\mathrm{d}z.$$

It is thus clear that $K_{\iota^*}(x,(y,z)) = \delta_0(x-y, x-z)$, in the sense that, given $\varphi \in$

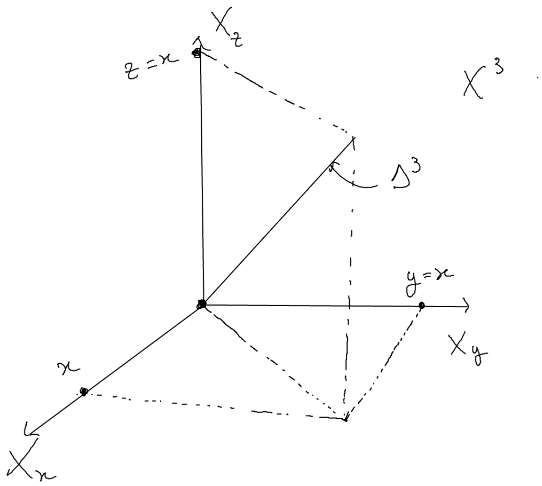

Figure 2.4: A picture of the triple space X^3 and the triple diagonal Δ_3.

$C^\infty_{\mathrm{comp}}(X)$,

$$(\iota^* f, \varphi) = \int_X f(x,x)\varphi(x)\mathrm{d}x = \int_{X^3} \delta_0(x-y, x-z)(f\otimes\varphi)\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Let us call *triple diagonal* the set $\Delta_3 := \{(x,x,x) \mid x \in X\} \subset X^3$. This is a linear subspace of $X^3 \subset (\mathbb{R}^n)^3$. Since K_A consists in integrating with respect to the smooth measure $\mathrm{d}x$ on Δ_3 (it is the Dirac mass on Δ_3), we deduce by Exercise 2.1.5 that its wavefront set is contained in the conormal $N^*\Delta_3$ of Δ_3 (minus the 0 section), namely,

$$N_0^*\Delta_3 := \left\{(z,\xi) \in T_0^*(X^3) \mid z \in \Delta_3, \forall v \in \Delta_3, (\xi, v) = 0\right\}.$$

Let us compute this conormal. Given $\xi \in (\mathbb{R}^n)^3$, we write $\xi = (\xi_1, \xi_2, \xi_3)$ corresponding to each factor. Then, if $v = (w,w,w) \in \Delta_3$, we have

$$(\xi, v) = ((\xi_1,\xi_2,\xi_3),(w,w,w)) = (\xi_1, w) + (\xi_2, w) + (\xi_3, w).$$

As a consequence, this is 0 for all $w \in \mathbb{R}^n$ if and only if $\xi_1 + \xi_2 + \xi_3 = 0$, that is, $\xi_1 = -(\xi_2 + \xi_3)$. Hence:

$$\mathrm{WF}(K_{\iota^*}) = \{((x,x,x), -(\xi_2+\xi_3), \xi_2, \xi_3) \mid x \in X, \xi_2, \xi_3 \in \mathbb{R}^n_0\}. \tag{2.2.17}$$

By Theorem 2.2.4, we deduce that, given $u \in \mathcal{D}'(X,X)$, $\iota^* u$ is well-defined if

$$\mathrm{WF}(u) \cap \mathrm{WF}'_{X\times X} K_{\iota^*} = \emptyset.$$

Now,

$$\begin{aligned}
&\mathrm{WF}'_{X\times X} K_{\iota^*} \\
&= \left\{((y,z),(\xi_2,\xi_3)) \in T_0^*(X^2) \mid \exists x \in X, (x, 0, ((y,z),(\xi_2,\xi_3))) \in \mathrm{WF}'(K_{\iota^*})\right\} \\
&= \left\{((y,z),(\xi_2,\xi_3)) \in T_0^*(X^2) \mid \exists x \in X, (x, 0, ((y,z),-(\xi_2,\xi_3))) \in \mathrm{WF}(K_{\iota^*})\right\}.
\end{aligned}$$

But if $(x, 0, ((y, z), -(\xi_2, \xi_3))) \in \mathrm{WF}(K_{\iota^*})$, then by (2.2.17), we get that $x = y = z$ and $\xi_2 + \xi_3 = 0$, that is,

$$\mathrm{WF}'_{X\times X} K_{\iota^*} = \left\{(x, x, \xi, -\xi) \in T_0^* X^2 \mid x \in X, \xi \in \mathbb{R}_0^n\right\}.$$

Observe that this is nothing but the conormal to the double diagonal

$$\Delta_2 := \left\{(x, x) \in X^2 \mid x \in X\right\} \subset X^2,$$

that is

$$\mathrm{WF}'_{X\times X} K_{\iota^*} = N_0^* \Delta_2.$$

We deduce from this computation that under the condition $\Gamma_1 \cap (-\Gamma_2) = \emptyset$, we get by (2.2.16) that the condition $\mathrm{WF}(f \otimes g) \cap \mathrm{WF}'_{X\times X} K_{\iota^*} = \emptyset$ is satisfied. Hence the the product $\iota^*(f \times g)$ is well-defined as a distribution in $\mathcal{D}'(X)$.

It then remains to bound its wavefront set. Following Theorem 2.2.4, we need to compute

$$\mathrm{WF}'_X(K_{\iota^*}) := \left\{(x, \xi) \in T^* X_0 \mid \exists (y, z) \in X^2, (x, \xi, (y, z), 0) \in \mathrm{WF}'(K_{\iota^*})\right\},$$

and this is clearly equal to the empty set $\emptyset$ by (2.2.17). Hence

$$\begin{aligned}
&\mathrm{WF}(f \times g)\\
&\subset \mathrm{WF}'(K_{\iota^*}) \circ \mathrm{WF}(f \otimes g)\\
&= \{(x, \xi) \in T^* X_0 \mid \exists (y, z, \xi_2, \xi_3) \in \mathrm{WF}(f \otimes g), (x, \xi, y, z, \xi_2, \xi_3) \in \mathrm{WF}'(K_{\iota^*})\}\\
&= \{(x, \xi) \in T^* X_0 \mid \exists (y, z, \xi_2, \xi_3) \in \mathrm{WF}(f \otimes g), (x, \xi, y, z, -\xi_2, -\xi_3) \in \mathrm{WF}(K_{\iota^*})\}\\
&\subset \mathrm{WF}(f) \cup \mathrm{WF}(g) \cup (\mathrm{WF}(f) \oplus \mathrm{WF}(g)),
\end{aligned}$$

where the last inclusion follows from (2.2.16) and (2.2.17). This concludes the proof. □

2.2.3.2 Pullback

Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ be two open subsets and $\kappa : X \to Y$ be a smooth map. If $u \in C^\infty_{\mathrm{comp}}(Y)$, then the pullback of u by κ is naturally defined as $\kappa^* u(x) := u(\kappa(x))$. We would like to extend the action of κ to compactly supported distributions $\mathcal{E}'(Y)$. For that, we use the following formula, showing that κ^* has a Schwartz kernel which can be expressed as an oscillatory integral:

$$\kappa^* u(x) = u(\kappa(x)) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_y} \int_{\mathbb{R}^n_\eta} e^{i(\kappa(x)-y)\cdot\eta} u(y) \mathrm{d}y \mathrm{d}\eta. \tag{2.2.18}$$

Eventually, we introduce the *conormal to the range of* κ:

$$N_\kappa := N^* \kappa(X) = \left\{(\kappa(x), \eta) \mid x \in M, \eta \in T^*_{\kappa(x)} Y, d\kappa(x)^\top \eta = 0\right\}.$$

The following holds:

Lemma 2.2.9. *Let $\mathcal{D}'_\kappa(Y)$ be the set of distributions $u \in \mathcal{D}'(Y)$ such that $\mathrm{WF}(u) \cap N_\kappa = \emptyset$. Then the operator $\kappa^* : C^\infty(Y) \to C^\infty(X)$ extends uniquely to a continuous map $\kappa^* : \mathcal{D}'_\kappa(Y) \to \mathcal{D}'(X)$. Moreover, if $u \in \mathcal{D}'_\kappa(Y)$, then*

$$\mathrm{WF}(\kappa^* u) \subset \left\{(x, \xi) \in T^*_0 X \mid \exists (y, \eta) \in \mathrm{WF}(u), y = \kappa(x), \xi = d\kappa(x)^\top \eta\right\}.$$

Remark 2.2.10. As we saw earlier, it is convenient to think of distributions on manifolds as a generalization of densities. As a consequence, the extension of the operator κ^* should rather be achieved from $C^\infty(Y, \Omega^1 Y) \to C^\infty(X, \Omega^1 X)$ to $\mathcal{D}'_\kappa(Y) \to \mathcal{D}'(X)$. For that, one can simply pick arbitrary densities $|\mu_Y|$ and $|\mu_X|$ and declare that

$$\kappa^*(f|\mu_Y|) := \kappa^* f |\mu_X|. \tag{2.2.19}$$

But sometimes the induced action on densities is already defined and differs from (2.2.19). For instance, if $\kappa : X \to Y$ is a diffeomorphism, then the pullback of densities involve a determinant, see (A.4.4). We also refer to Remark 2.2.12 below for further details in the diffeomorphism case.

Proof. This is a straightforward consequence of (2.2.18) together with Theorem 2.2.4. Indeed, by (2.2.18), the operator κ^* has kernel

$$K(x, y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\eta} e^{i(\kappa(x) - y)\cdot\eta} \mathrm{d}\eta.$$

This has to be understood as an oscillatory integral and $K \in \mathcal{D}'(X \times Y)$. The phase is $\phi(x, y, \eta) := (\kappa(x) - y) \cdot \eta$. By Theorem 2.1.14, we have

$$\begin{aligned}\mathrm{WF}(K) \subset \Lambda_\phi &= \{(x, y, d_x\phi(x, y, \eta), d_y\phi(x, y, \eta)) \mid d_\eta\phi(x, y, \eta) = 0, \eta \neq 0\} \\ &= \left\{(x, y, d\kappa(x)^\top \eta, -\eta) \mid \kappa(x) = y, \eta \neq 0\right\}.\end{aligned}$$

To be consistent with the notations of the previous paragraphs, we rather denote this as:

$$\mathrm{WF}'(K) \subset \left\{(x, d\kappa(x)^\top \eta, y, \eta) \mid \kappa(x) = y, \eta \neq 0\right\}.$$

We then have

$$\begin{aligned}\mathrm{WF}'_X(K) &= \mathrm{WF}'(K) \circ O_X = \{(x, \xi) \in T^*_0 X \mid \exists y \in Y, (x, \xi, y, 0) \in \mathrm{WF}'(K)\} \\ &= \emptyset. \\ \mathrm{WF}'_Y(K) &:= \{(y, \eta) \in T^* Y_0 \mid \ \exists x \in X, (x, 0, y, \eta) \in \mathrm{WF}'(K)\} \\ &= \left\{(y, \eta) \in T^* Y_0 \mid \exists x \in X, \kappa(x) = y, d\kappa(x)^\top \eta = 0, \eta \neq 0\right\} \\ &= N_\kappa.\end{aligned}$$

By Theorem 2.2.4, we deduce that for $u \in \mathcal{D}'(Y)$ such that $\mathrm{WF}(u) \cap N_\kappa = \emptyset$, $\kappa^* u \in \mathcal{D}'(X)$ is well-defined and

$$\mathrm{WF}(\kappa^* u) \subset \mathrm{WF}'(K) \circ \mathrm{WF}(u) = \left\{(x, d\kappa(x)^\top \eta) \in T^*_0 X \mid (\kappa(x), \eta) \in \mathrm{WF}(u)\right\}.$$

$\square$

A particular but important example is when $X, Y \subset \mathbb{R}^n$ and $\kappa : X \to Y$ is a diffeomorphism. Observe that in this case $\mathrm{WF}'_Y(K) = \emptyset$ and thus the pullback is always well-defined. We have as a particular case of Lemma 2.2.9:

Lemma 2.2.11 (Wavefront set under a change of coordinates)**.** *Let $\kappa : X \to Y$ be a diffeomorphism. The induced action $\kappa^* : C^\infty(Y, \Omega^1 Y) \to C^\infty(X, \Omega^1 X)$ admits a unique continuous extension to $\kappa^* : \mathcal{D}'(Y) \to \mathcal{D}'(X)$. Given $u \in \mathcal{D}'(Y)$, $\kappa^* u \in \mathcal{D}'(X)$ is well-defined,*

$$\mathrm{WF}(\kappa^* u) = \mathrm{d}\kappa^\top (\mathrm{WF}(u)) := \left\{(x, \xi) \in X \times \mathbb{R}^n \mid (\kappa(x), \mathrm{d}\kappa^{-\top}(\xi)) \in \mathrm{WF}(u)\right\}.$$

and we have the formula:

$$(\kappa^* u, \varphi) = (u, \kappa_* \varphi), \tag{2.2.20}$$

where $\kappa_* \varphi(y) := \varphi(\kappa^{-1}(y))$.

Remark 2.2.12. The pairing (A.4.4) is the same as (A.4.2). As we already saw in Remark 2.2.10, there are two natural definitions of the pullback of distributions: one can either consider the continuous extension of the canonical pullback map on densities $P_1 := \kappa^* : C^\infty(Y, \Omega^1 Y) \to C^\infty(X, \Omega^1 X)$ given by $(P_1|\mu|)(x) := |\mu|_{\kappa(x)}(d\kappa(x)\bullet, ..., d\kappa(x)\bullet)$ (in which case (A.4.4) holds) *or* one takes the continuous extension of the less natural mapping $P_2 : C^\infty(Y, \Omega^1 Y) \to C^\infty(X, \Omega^1 X)$ by $P_2(f|dy|) = (\kappa^* f)|dx|$. But in this case, the pairing (A.4.4) has to be changed to

$$(\kappa^* u, \varphi) = (u, |\det d\kappa(\kappa^{-1}(y))|^{-1} \kappa_* \varphi).$$

As an exercise, compute $\kappa^* \delta_p$ with these two definitions.

Proof. Straightforward consequence of Lemma 2.2.9. The equality follows from the double inclusion applying Lemma 2.2.9 for the inverse. The last formula is simply the change of coordinates in $\mathbb{R}^n$ which extends naturally to distributions by continuity. □

2.2.3.3 Discussion on manifolds

The previous lemma has an important consequence: it shows that notion of wavefront set is well-defined on a manifold.

Definition 2.2.13. A point $(x_0, \xi_0) \in T_0^* M$ belongs to $\mathrm{WF}(u)$ if and only if there exists a small open neighborhood U of x_0, $\chi \in C^\infty_{\mathrm{comp}}(U)$ such that $\chi(x_0) > 0$ and a local diffeomorphism $\kappa : U \to \mathbb{R}^n$ such that:

$$d\kappa^{-\top}(x_0, \xi_0) \in \mathrm{WF}(\kappa_*(\chi u)).$$

Equivalently, the existence of such a triple (U, χ, κ) could be replaced by "for all such triples".

We need to prove that this is well-defined, that is, it does not depend on the triple chosen.

Proof. We need to prove that if $(x_0, \xi_0) \in d\kappa_0^\top(\mathrm{WF}(\kappa_{0*}(\chi u)))$, that is to say, $d\kappa_0^{-\top}(x_0, \xi_0) \in \mathrm{WF}(\kappa_{0*}(\chi u))$ for some κ_0, then $(x_0, \xi_0) \in d\kappa^\top(\mathrm{WF}(\kappa_*(\chi u)))$, that is $d\kappa^{-\top}(x_0, \xi_0) \in \mathrm{WF}(\kappa_*(\chi u)$, for any other local diffeomorphism κ. But using Lemma 2.2.11, we get:

$$\begin{aligned} d\kappa^{-\top}(x_0, \xi_0) &= d(\kappa \circ \kappa_0^{-1})^{-\top} \underbrace{\kappa_0^{-\top}(x_0, \xi_0)}_{\in \mathrm{WF}(\kappa_{0*}(\chi u))} \\ &\in \mathrm{WF}\left((\kappa \circ \kappa_0^{-1})_* \kappa_{0*}(\chi u)\right) = \mathrm{WF}(\kappa_*(\chi u)). \end{aligned}$$

(Technically, we would need to change the function χ as well, but this is a harmless detail.) ☐

Remark 2.2.14. The wavefront set is well-defined on manifolds because of the relation $\kappa^* \mathrm{WF}(u) = \mathrm{WF}(\kappa^* u)$ on $\mathbb{R}^n$. This is a more general principle: in order to define an invariant object on smooth manifolds, one has to be able to define this object on $\mathbb{R}^n$ and to show that this object transforms invariantly by the action of smooth diffeomorphism. Another example is the usual exterior derivative d, which also commutes with the action of diffeomorphisms $\kappa^* d = d\kappa^*$ and thus induces a well-defined operator on M.

Example 2.2.15. Let $Y \subset M$ be a smooth submanifold, let $|\mu|$ be a smooth density on Y, that is a section of the density bundle $\Omega^1 Y \to Y$, as in Definition A.4.1. Then:

$$(|\mu|, f) := \int_Y f|\mu|$$

is distribution of order 0. Its wavefront set is contained in N^*Y.

The previous example is a particular case of a *conormal distribution*, i.e. whose singularities are contained in a conormal bundle.

As an illustration, let us consider a diffeomorphism $\kappa : M \to M$ on a smooth closed manifold M and study its induced action on distributions. Recall the notion of density bundle $\Omega^1 M \to M$ of Definition A.4.1. The diffeomorphism κ induces an operator $\kappa^* : C^\infty(M) \to C^\infty(M)$ given by $\kappa^*\varphi(x) := \varphi(\kappa(x))$ and also on densities $\kappa^* : C^\infty(M, \Omega^1 M) \to C^\infty(M, \Omega^1 M)$. How to compute the induced action $\kappa^* : \mathcal{D}'(M) \to \mathcal{D}'(M)$? Since smooth densities embed naturally into distributions, we want the action on distributions to extend continuously that of densities in order to be consistent. In other words, if $u \in \mathcal{D}'(M)$, and $|\mu_\varepsilon| \to u$ is a sequence of smooth densities converging to u in $\mathcal{D}'(M)$, we must have by (A.4.4):

$$(\kappa^* u, \psi) = \lim_{\varepsilon \to 0} (\kappa^* |\mu_\varepsilon|, \psi) = \lim_{\varepsilon \to 0} (|\mu_\varepsilon|, \kappa_* \psi) = (u, \kappa_* \psi).$$

Exercise 2.2.16. Given a diffeomorphism $\kappa : M \to M$, show that $\kappa^* \delta_p = \delta_{\kappa^{-1}(p)}$ for $p \in M$.

2.2.3.4 Flat trace

This is an important consequence of the previous discussion. We let M be a smooth closed manifold with fixed density $|\mu|$ and $A : C^\infty(M) \to C^\infty(M)$ be an operator with smooth Schwartz kernel $K_A \in \mathcal{D}'(M \times M)$, that is $K_A = R_A|\mu|(x)|\mu|(y)$ with $R_A \in C^\infty(M \times M)$ as in §A.4 (it is computed with respect to the density $|\mu|$). (The discussion can also be naturally carried out on an open set $X \subset \mathbb{R}^n$; then K has to be taken with compact support in $X \times X$.) The *trace* of the operator A is defined as

$$\mathrm{Tr}(A) := \int_M R_A(x,x)|\mu|(x). \tag{2.2.21}$$

It is *independent of the choice of density* $|\mu|$. Indeed, choosing another density $|\mu'| = a|\mu|$ for some smooth positive function $a \in C^\infty(M)$, we saw in §A.4 that R_A is changed to $\widetilde{R_A}(x,y) = R_A(x,y)a^{-1}(y)$ and thus

$$\int_M \widetilde{R_A}(x,x)|\mu'|(x) = \int_M R_A(x,x)a^{-1}(x)a(x)|\mu|(x) = \int_M R_A(x,x)|\mu|(x).$$

We would like to extend this definition of the trace to operators with non-smooth Schwartz kernel (typically, oscillatory integrals such as the pullback operator). For that, define $\iota : M \to M \times M, x \mapsto (x,x)$ the embedding of the diagonal into $M \times M$ and the induced action[4] on densities $\iota^* : C^\infty(M \times M, \Omega^1(M \times M)) \to C^\infty(M, \Omega^1 M)$ by

$$\iota^*(a(x,y)|\mu|(x)|\mu|(y)) := a(x,x)|\mu|(x).$$

We then rewrite (2.2.21) as:

$$\mathrm{Tr}(A) = (\iota^* K, \mathbf{1}). \tag{2.2.22}$$

For $v \in T_xM, \xi, \xi' \in T_x^*M$, we have

$$\begin{aligned}(\xi,\xi')\,(d\iota(x)(v)) &= (\xi,\xi')\,(v,v)\\ &= \xi(v) + \xi'(v) = d\iota(x)^\top(\xi,\xi')(v) = (\xi+\xi')(v).\end{aligned}$$

The conormal to the range of the map ι is

$$\begin{aligned}N_\iota &= \left\{(x,\xi,x',\xi') \in T_0^*(M \times M) \mid x = x', d\iota(x)^\top(\xi,\xi') = 0\right\}\\ &= \{(x,\xi,x,-\xi) \mid (x,\xi) \in T_0^*M\} = N^*\Delta,\end{aligned}$$

that is the *conormal to the diagonal* $\Delta := \{(x,x) \in M \times M \mid x \in M\} \subset M \times M$.

Lemma 2.2.17. *Let $A : C^\infty(M) \to \mathcal{D}'(M)$ be a continuous linear operator with*

[4] This choice ensures that the following diagram is commutative:

$$\begin{array}{ccc} C^\infty(M \times M) & \xrightarrow{\iota^*} & C^\infty(M) \\ \big\downarrow{\scriptstyle |\mu|\otimes|\mu|} & & \big\downarrow{\scriptstyle |\mu|} \\ C^\infty(M \times M, \Omega^1(M \times M)) & \xrightarrow[\iota^*]{} & C^\infty(M, \Omega^1 M). \end{array}$$

Schwartz kernel $K \in \mathcal{D}'(M \times M)$*. If* $\mathrm{WF}(K) \cap N^*\Delta = \emptyset$*, then*

$$\mathrm{Tr}^\flat(A) := (\iota^* K,\ \mathbf{1})$$

is well-defined and called the flat trace *of* A*. If* A *has smooth Schwartz kernel, the flat trace coincides with the usual trace.*

For distributions, this is still independent of the choice of density $|\mu|$ since smooth functions are dense in distributions and we saw that the flat trace of an operator with smooth Schwartz kernel is independent of the choice of density. The proof is almost straightforward: it consists in observing that, under the assumption that $\mathrm{WF}(K) \cap N^*\Delta = \emptyset$, the extension Theorem 2.2.4 together with Lemma 2.2.9, guarantee that the map ι^* (initially defined on densities) induce a continuous map $\iota^* : \mathcal{D}'_\Gamma(M \times M) \to \mathcal{D}'(M)$, where Γ is any closed cone not intersecting $N^*\Delta$. In practice, we compute the flat trace by approximating the kernel by smooth Schwartz kernels. Indeed, we can always find $R_\varepsilon \in C^\infty(M \times M)$ such that $K_\varepsilon := R_\varepsilon(x, y)|\mu|(x)|\mu|(y) \to K$ in $\mathcal{D}'_\Gamma(M \times M)$. Writing A_ε for the operator with kernel K_ε, we then have that $\iota^* K_\varepsilon \to_{\varepsilon \to 0} \iota^* K$ in $\mathcal{D}'(M)$ and thus:

$$\mathrm{Tr}^\flat(A_\varepsilon) = \int_M R_\varepsilon(x, x)|\mu|(x) \to_{\varepsilon \to 0} \mathrm{Tr}^\flat(A).$$

The following example is very important:

Example 2.2.18. We let M be a smooth closed manifold and $\kappa : M \to M$ be a smooth diffeomorphism with isolated non-degenerate fixed points. By this, we mean that the set of fixed points of κ (i.e. such that $\kappa(x) = x$) is isolated (hence finite) and that for each fixed point $x_\star$, $d\kappa(x_\star) - \mathbb{1}$ is invertible. The operator κ induces a map $L : C^\infty(M, \Lambda^* T^* M) \to C^\infty(M, \Lambda^* T^* M)$ defined as

$$L\left(\sum_{k=0}^{n} \varphi_k\right) = \sum_{j=0}^{n} (-1)^k \kappa^* \varphi_k,$$

where $\varphi_k \in C^\infty(M, \Lambda^k T^* M)$ and $\kappa^* : C^\infty(M, \Lambda^k T^* M) \to C^\infty(M, \Lambda^k T^* M)$ is the pullback map on forms. It can be shown that the wavefront set of the Schwartz kernel of L is disjoint from the conormal to diagonal $N^*\Delta$. As a consequence of Lemma 2.2.17, its flat trace is well-defined and:

$$\mathrm{Tr}^\flat(L) = \sum_{k=1}^{n} (-1)^k \sum_{j=1}^{N} \frac{\mathrm{Tr}(\Lambda^k d\kappa(x_j))}{|\det(\mathbb{1} - d\kappa(x_j))|} = \sum_{j=1}^{N} \operatorname{sgn} \det(\mathbb{1} - d\kappa(x_j)), \qquad (2.2.23)$$

We refer to Exercises 7 and 8 below.

2.2.4 Exercises

Exercise 1: The Cauchy problem for the wave equation

We consider the following Cauchy problem for the wave equation:

$$\begin{cases} \partial_t^2 f - \Delta f = 0 \\ f(t=0) = 0, \partial_t f(t=0) = u, \end{cases} \tag{2.2.24}$$

where $u \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$.

1. Show existence and uniqueness of a solution $f \in C^\infty([0,\infty), \mathcal{S}(\mathbb{R}^n))$ for (2.2.24) if $u \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$.
2. Show that for all $t \geq 0, x \in \mathbb{R}^n$:

$$f(t,x) = \int_{\mathbb{R}^n_\xi} \int_{\mathbb{R}^n_y} e^{i(x-y)\cdot\xi}(2i|\xi|)^{-1}(e^{it|\xi|} - e^{-it|\xi|})u(y)\mathrm{d}y\mathrm{d}\xi$$

We let $\chi \in C^\infty(\mathbb{R}^n)$ be a smooth cutoff function such that $\chi = 0$ near $\xi = 0$ and $\chi = 1$ for $|\xi| \geq 1$. We decompose the solution as

$$f(t,x) = f_+(t,x) + f_-(t,x) + k(t,x),$$

where

$$f_\pm(t,x) = \pm \int_{\mathbb{R}^n_\xi} \int_{\mathbb{R}^n_y} e^{i(x-y)\cdot\xi}(2i|\xi|)^{-1}e^{\pm it|\xi|}\chi(\xi)u(y)\mathrm{d}y\mathrm{d}\xi =: F_\pm(t)u$$

$$k(t,x) = \int_{\mathbb{R}^n_\xi} \int_{\mathbb{R}^n_y} e^{i(x-y)\cdot\xi}(2i|\xi|)^{-1}(e^{it|\xi|} - e^{-it|\xi|})(1-\chi(\xi))u(y)\mathrm{d}y\mathrm{d}\xi =: K(t)u$$

3. Show that the operator $K(t)$ is smoothing.
4. Show that $F_\pm(t)$ is an operator whose Schwartz kernel $K_\pm(t)$ is given by an oscillatory integral. Compute $\mathrm{WF}(K_\pm(t))$.
5. Show that $F_\pm(t) : \mathcal{E}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n)$ is continuous. Given $u \in \mathcal{E}'(\mathbb{R}^n)$, compute $\mathrm{WF}(F_\pm(t)u)$ in terms of $\mathrm{WF}(u)$.
6. Show existence and uniqueness of a solution $f \in C^\infty([0,\infty), \mathcal{S}'(\mathbb{R}^n))$ if $u \in \mathcal{E}'(\mathbb{R}^n)$.
7. Take $u := \delta_0$. Compute $\mathrm{WF}(f(t,\bullet))$. Can you explain this from a physical perspective?

Exercise 2: multiplication of distributions

1. For $x_1, x_2 \in \mathbb{R}^n$, $x_1 \neq x_2$, compute $\delta_{x_1} \times \delta_{x_2}$. Why is $\delta_{x_1}^2$ not (a priori) well-defined?

2. In $\mathbb{R}^2$, define

$$(\delta_x, \varphi) := \int_{\mathbb{R}^2} \varphi(x,0)\mathrm{d}x, \quad (\delta_y, \varphi) := \int_{\mathbb{R}^2} \varphi(0,y)\mathrm{d}y.$$

Show that $\delta_x \times \delta_y$ is well-defined and compute it.

3. Find an example where the inclusion (2.2.14) is an equality and an example where it is not.

Exercise 3

Let $X \subset \mathbb{R}^n$ be an open subset and $S \subset T_0^*X$ be a closed conic subset. Show that there exists $u \in \mathcal{D}'(X)$ such that $\mathrm{WF}(u) = S$.

Exercise 4: pullback, pushforward of distributions

Let $X \subset \mathbb{R}^n, F \subset \mathbb{R}^k$ be two open subsets. Let $\pi : X \times F \to X$ be given by $\pi(x,y) = x$. For $u \in C^\infty_{\mathrm{comp}}(X \times F), f \in C^\infty_{\mathrm{comp}}(X)$, define:

$$\pi^* f(x,y) := f(x), \quad \pi_* u(x) := \int_F u(x,y)\mathrm{d}y.$$

1. Show that $\pi^* : \mathcal{E}'(X) \to \mathcal{D}'(X \times F)$ extends continuously and bound $\mathrm{WF}(\pi^* f)$ in terms of $\mathrm{WF}(f)$.
2. Deduce that $\pi^* : \mathcal{E}'(X) \to \mathcal{D}'_\Gamma(X \times F)$ extends continuously for some well-chosen conic subset $\Gamma \subset T_0^*(X \times F)$.
3. Show that $\pi_* : \mathcal{E}'(X \times F) \to \mathcal{E}'(X)$ extends continuously and bound $\mathrm{WF}(\pi_* u)$ in terms of $\mathrm{WF}(u)$.

Exercise 5

Let M^n be a smooth closed oriented n-dimensional manifold and $\pi : E \to M$ be an oriented fiber bundle, with fiber diffeomorphic to F^k, a closed oriented k-dimensional manifold. Let ω_E be a smooth volume form on E and ω_M be a smooth volume form on M.

1. Recall the definition of a fiber bundle.
2. Show the existence of $\nu \in C^\infty(E, \Lambda^k T^*E)$ such that $\omega_E = \nu \wedge \pi^*\omega_M$. Show that the restriction of ν to each fiber $E_x \hookrightarrow E$ is a (positive) volume form.
3. Consider the pullback operator $\pi^* : C^\infty(M) \to C^\infty(E)$, defined by $\pi^* f(x,v) := f(x)$. Show that it extends uniquely to a continuous map $\pi^* : L^2(M, \omega_M) \to L^2(E, \omega_E)$.
4. We let $\pi_* : L^2(E, \omega_E) \to L^2(M, \omega_M)$ be the adjoint of π^*. Compute π_*.

5. Show that $\pi^* : \mathcal{D}'(M) \to \mathcal{D}'(E)$ extends continuously. Bound $\mathrm{WF}(\pi^* f)$ in terms of $\mathrm{WF}(f)$.

6. Show that $\pi_* : \mathcal{D}'(E) \to \mathcal{D}'(M)$ extends continuously. Bound $\mathrm{WF}(\pi_* u)$ in terms of $\mathrm{WF}(u)$.

Exercise 6

Let M^n be a smooth closed manifold and let $X \in C^\infty(M, TM)$ be a smooth vector field. Let $(\varphi_t)_{t\in\mathbb{R}}$ be the flow generated by X. It acts by pullback on smooth functions as $\varphi_t^* : C^\infty(M) \to C^\infty(M)$, $\varphi_t^* f(x) := f(\varphi_t(x))$.

1. Show that $\varphi_t^* : \mathcal{D}'(M) \to \mathcal{D}'(M)$ extends continuously.

2. Compute $\mathrm{WF}(\varphi_t^* f)$ in terms of $\mathrm{WF}(f)$. Explain this heuristically.

Let $\chi \in C^\infty(\mathbb{R})_{\mathrm{comp}}$ be a smooth cutoff function. Define the operator

$$E := \int_{-\infty}^{+\infty} \chi(t)\varphi_t^* \mathrm{d}t.$$

3. Compute $\mathrm{WF}(Eu)$ in terms of $\mathrm{WF}(u)$. *Hint: Consider the projection* $\pi : M \times \mathbb{R} \to M, \pi(x,t) = x$.

Exercise 7: Flat trace of diffeomorphisms with non-degenerate fixed points

Let M be a smooth closed manifold and $\Psi : M \to M$ be a smooth diffeomorphism with non-degenerate fixed points. By this, we mean that for every fixed point $x_\star$ of Ψ (i.e. such that $\Psi(x_\star) = x_\star$), $d\Psi(x_\star) - \mathbb{1}$ is invertible. The operator Ψ induces a map $\Psi^* : C^\infty(M) \to C^\infty(M)$ whose Schwartz kernel is denoted by K.

1. Show that, under these assumptions, the number of fixed points is finite.

2. What is $\mathrm{supp}(K)$? What is $\mathrm{WF}(K)$?

3. Show that $\mathrm{WF}(K) \cap N^*\Delta = \emptyset$. Deduce that $\mathrm{Tr}^\flat(\Psi^*)$ is well-defined.

We now want to compute $\mathrm{Tr}^\flat(\Psi^*)$. For that, let $X, Y \subset \mathbb{R}^n$ be open sets containing 0 and let $\psi : X \to Y$ be a smooth diffeomorphism such that $\psi(0) = 0, d\psi(0) - \mathbb{1}$ is invertible and 0 is the only fixed point in X. Let $\chi : \mathbb{R}^n \to [0,1]$ be a smooth nonnegative cutoff function with support near 0, such that

$$\int_{\mathbb{R}^n} \chi = 1.$$

We define $\widetilde{\chi}_\varepsilon := \varepsilon^{-n}\chi(\bullet/\varepsilon)$ and $\chi_\varepsilon \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ by

$$\chi_\varepsilon(x,y) := \widetilde{\chi}_\varepsilon(x)\widetilde{\chi}_\varepsilon(y).$$

We let $K_\psi \in \mathcal{D}'(Y \times X)$ be the Schwartz kernel of ψ^* and let $\imath : \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ be the diagonal embedding $x \mapsto (x, x)$. We define

$$K_\varepsilon := \chi_\varepsilon \star K \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n),$$

where $\star$ is the usual convolution product. Eventually, we consider $\varphi \in C^\infty_{\text{comp}}(\mathbb{R}^n, [0,1])$, a smooth cutoff function with support near 0.

5. Why is $\operatorname{Tr}^\flat(\varphi\psi^*\varphi)$ well-defined if the support of φ is close enough to 0?
6. Show that $\imath^*(\varphi K_\varepsilon \varphi) \to \imath^*(\varphi K_\psi \varphi)$ in $\mathcal{D}'_\Gamma(X)$ as $\varepsilon \to 0$, where Γ is some well-chosen closed conic subset of T^*X that you will introduce.
7. Compute $\imath^*(\varphi K_\varepsilon \varphi)$.
8. Show that
$$\operatorname{Tr}^\flat(\varphi\psi^*\varphi) = |\det(d\psi(0) - \mathbb{1})|^{-1}.$$
9. Deduce the value of $\operatorname{Tr}^\flat(\Psi^*)$.

Exercise 8: Flat trace and fixed points

Let $A \in \mathcal{M}_n(\mathbb{C})$. The action of A on $\mathbb{C}^n$ extends naturally to $\Lambda^k\mathbb{C}^n$ (for $k = 0, ..., n$) by setting on pure elements:

$$A(\eta_1 \wedge ... \wedge \eta_k) := (A\eta_1) \wedge ... \wedge (A\eta_k),$$

where $\eta_1, ..., \eta_k \in \mathbb{C}^n$.

1. Show that:
$$\det(\mathbb{1} - A) = \sum_{k=1}^{n} (-1)^k \operatorname{Tr}(\Lambda^k A).$$

The action of Ψ on functions/distributions by pullback can be naturally extended to k-forms. More precisely, if $f \in C^\infty(M, \Lambda^k T^*M)$, $x \in M, v_1, ..., v_k \in T_xM$, then we can define:

$$\left[\Psi^*_{(k)} f\right]_x (v_1, ..., v_k) := f_{\Psi(x)} \left(d\Psi(x)(v_1), ..., d\Psi(x)(v_k)\right).$$

We let $K_{(k)}$ be the Schwartz kernel of the operator acting on the bundle of k-forms.

1. Given $k = 0, ..., n$, what is $\operatorname{supp}(K_{(k)})$? $\operatorname{WF}(K_{(k)})$?
2. Show that the flat trace $\operatorname{Tr}^\flat(\Psi^*_{(k)})$ is well-defined and that
$$\operatorname{Tr}^\flat(\Psi^*_{(k)}) = \sum_{j=1}^{N} \frac{\operatorname{Tr}(\Lambda^k d\Psi(x_j)^\top)}{|\det(\mathbb{1} - d\Psi(x_j))|},$$
where $x_1, ..., x_N$ are the fixed points and $\Lambda^k d\Psi(x_j)^\top$ denotes the linear operator induced by $d\Psi(x_j) : T_{x_j}M \to T_{x_j}M$ on $\Lambda^k T^*_{x_j} M$.

3. Let
$$\Lambda^\star T^*M := \oplus_{k=0}^n \Lambda^k T^*M.$$
Let $L_\Psi : C^\infty(M, \Lambda^\star TM) \to C^\infty(M, \Lambda^\star TM)$ be the operator defined as
$$L_\Psi\left(\sum_{k=0}^n f_k\right) = \sum_{k=0}^n (-1)^k \Psi_{(k)}^* f_k,$$
where $f_k \in C^\infty(M, \Lambda^k T^*M)$. We denote by the same letter L its continuous extension as a map
$$L_\Psi : \mathcal{D}'(M, \Lambda^\star T^*M) \to \mathcal{D}'(M, \Lambda^\star T^*M),$$
Show that $\mathrm{Tr}^\flat(L_\Psi)$ is well-defined and that
$$\mathrm{Tr}^\flat(L_\Psi) = \sum_{j=1}^N \operatorname{sgn}\det(\mathbb{1} - d\Psi(x_j)),$$
that is the number of fixed points of Ψ counted with signs.

Exercise 9: Examples of flat traces

Consider the sphere
$$\mathbb{S}^2 := \left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\right\} \subset \mathbb{R}^3.$$

1. Let $R : \mathbb{S}^2 \to \mathbb{S}^2$ be the antipodal map, given by $R(v) := -v$. Show that $\mathrm{Tr}^\flat(L_R)$ is well-defined and compute it.
2. Let $N := (0, 0, 1)$ be the North pole. Consider the exponential map
$$\mathbb{S}^1 \times [0, \pi] \in (u, r) \mapsto \exp_N(ru) \in \mathbb{S}^2,$$
where $\mathbb{S}^1$ is identified here with $\{v \in T_N\mathbb{S}^2 \mid |v| = 1\}$. Show that the vector field defined in coordinates by
$$X(u, r) := -r(\pi - r)\partial_r$$
is well-defined on $\mathbb{S}^2$ and smooth.
3. Let $(\varphi_t)_{t\in\mathbb{R}}$ be the flow generated by X. Describe (and draw on a picture) its flowlines.
4. Define $\Psi := \varphi_1$. Show that $\mathrm{Tr}^\flat(L_\Psi) = 2$. What does 2 represent for the sphere?

2.3 First properties of pseudodifferential operators

2.3.1 Rough properties

We can start giving a few of the properties of ΨDOs:

Lemma 2.3.1. *We have:*

(i) $A : C^\infty_{\mathrm{comp}}(X) \to C^\infty(X)$ *is continuous and it admits a unique continuous extension* $A : \mathcal{E}'(X) \to \mathcal{D}'(X)$.

(ii) $\mathrm{WF}(K_A) \subset N^*\Delta$, *where* $\Delta := \{(x,x) \mid x \in X\} \subset X \times X$ *is the diagonal embedding of* X *into* $X \times X$. *In particular,* $K_A \in C^\infty(X \times X \setminus \Delta)$.

(iii) $\mathrm{WF}(Au) \subset \mathrm{WF}(u)$ *(*pseudolocality*).*

Proof. We just to apply our previous results.

(1) The first item follows from Theorem 1.2.8 after insertion of a small cutoff function χ supported near $\{\xi = 0\}$:

$$K_A(x,y) := \int_{\mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi} a(x,y,\xi)\chi(\xi)\mathrm{d}\xi + \int_{\mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi} a(x,y,\xi)(1-\chi(\xi))\mathrm{d}\xi.$$

The first term is obviously smooth, i.e. the kernel of the induced operator is smooth and thus the mapping property is immediate. As to the second term, it suffices to observe that for all $x \in X$, $d_{(y,\xi)}\phi(x,y,\xi) = (-\xi, x-y) \neq 0$ on the support of the symbol (due to the cutoff function). Also for all $y \in X$, $d_{(x,\xi)}\phi(x,y,\xi) \neq 0$ so we can apply Theorem 1.2.8.

(2) As far as the second item is concerned, we have to compute Λ_ϕ for this specific choice of phase function and then the conclusion follows from Theorem 2.1.14. We have $d_\xi\phi(x,y,\xi) = x - y$ so this vanishes on the diagonal $x = y$. Moreover, we have: $d_x\phi(x,x,\xi) = \xi$ and $d_y\phi(x,x,\xi) = -\xi$, that is

$$\mathrm{WF}(K_A) \subset \{(x,x,\xi,-\xi) \mid (x,\xi) \in T^*_0 X\} = N^*\Delta. \tag{2.3.1}$$

(3) The last point is a straightforward consequence of Theorem 2.2.4. Indeed, it is clear that $\mathrm{WF}'_X(K_A) = \mathrm{WF}'_Y(K_A) = \emptyset$ (in the notations of Theorem 2.2.4 and $\mathrm{WF}'(K_A) = \{(x,\xi,x,\xi) \in T^*_0(X \times X) \mid (x,\xi) \in T^*_0 X\}$. Hence:

$$\mathrm{WF}(Au) \subset \mathrm{WF}'(K_A) \circ \mathrm{WF}(u) = \{(x,\xi) \in T^*_0 X \mid (x,\xi) \in \mathrm{WF}(u)\} = \mathrm{WF}(u).$$

□

We also have the following:

Lemma 2.3.2. $\Psi^{-\infty}(X)$ *is the set of continuous linear operators* $C^\infty_{\mathrm{comp}}(X) \to \mathcal{D}'(X)$ *with smooth Schwartz kernel.*

Proof. Let $A \in \Psi^{-\infty}(X)$, that is $A = \mathrm{Op}(a)$ for some $a \in S^{-\infty}(X \times X \times \mathbb{R}^n)$, that is for all $\alpha, \beta, \beta' \in \mathbb{N}^n$, for all $N \geq 0$, for all $K \subset X \times X$, there exists $C_N > 0$ such that:

$$|\partial_x^\beta \partial_y^{\beta'} \partial_\xi^\alpha a(x,y,\xi)| \leq C_N \langle \xi \rangle^{-N}, \qquad \forall (x,y,\xi) \in K \times \mathbb{R}^n.$$

Then

$$K_A(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi} a(x,y,\xi) \mathrm{d}\xi,$$

and this is clearly a smooth function of (x,y) as the integral (as well as all the derivatives) is absolutely convergent due to the decaying property of a.

Conversely, if $A : \mathcal{E}'(Y) \to C^\infty(X)$, that is A has smooth Schwartz kernel (by Lemma 1.2.3), then $A = \mathrm{Op}(a)$ with $a(x,y,\xi) := K_A(x,y) e^{-i(x-y)\xi} \chi(\xi) \in S^{-\infty}(X \times X \times \mathbb{R}^n)$ where $\chi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$ is such that $\int_{\mathbb{R}^n_\xi} \chi(\xi) \mathrm{d}\xi = (2\pi)^n$. $\square$

2.3.2 Properly supported ΨDOs

We now investigate the support of distributions under the action of a pseudodifferential operator. We start with the following elementary result:

Lemma 2.3.3. *Let $A : C^\infty_{\mathrm{comp}}(Y) \to \mathcal{D}'(X)$ be a continuous linear operator with Schwartz kernel $K_A \in \mathcal{D}'(X \times Y)$. Then:*

$$\mathrm{supp}(Au) \subset \mathrm{supp}(K_A) \circ \mathrm{supp}(u) := \{x \in X \mid \exists y \in \mathrm{supp}(u), (x,y) \in \mathrm{supp}(K_A)\}.$$

Sketch of proof. The proof is left as an exercise to the reader. The idea is simply that if $u = \delta_{y_0}$ for some $y_0 \in Y$, then

$$Au(x) = \int_Y K_A(x,y) u(y) \mathrm{d}y = K_A(x, y_0),$$

so the support of Au is contained in $\{x \in X \mid (x, y_0) \in \mathrm{supp}(K_A)\}$. $\square$

Now, let M, N be two topological spaces. Recall that a continuous map $f : M \to N$ is said to be *proper* if for any compact subset $K \subset N$, $f^{-1}(K)$ is also a compact subset of M. Given a pseudodifferential operator $A \in \Psi^m_{\rho,\delta}(X)$, the support of its Schwartz kernel $K_A \in \mathcal{D}'(X \times X)$ is contained in $X \times X$. We let $\Pi_{1,2} : X \times X \to X$ be the projection onto the first (resp. second) factor.

Definition 2.3.4. The pseudodifferential operator $A \in \Psi^m_{\rho,\delta}(X)$ is said to be *properly supported* if $\Pi_{1,2} : \mathrm{supp}(K_A) \to X$ are both proper maps.

From Lemma 2.3.3, we directly see that the following holds:

Lemma 2.3.5. *Let $A \in \Psi^m_{\rho,\delta}(X)$ be properly supported. Then:*

$$A : C^\infty_{\mathrm{comp}}(X) \to C^\infty_{\mathrm{comp}}(X), \quad \mathcal{E}'(X) \to \mathcal{E}'(X),$$

are continuous.

We also have:

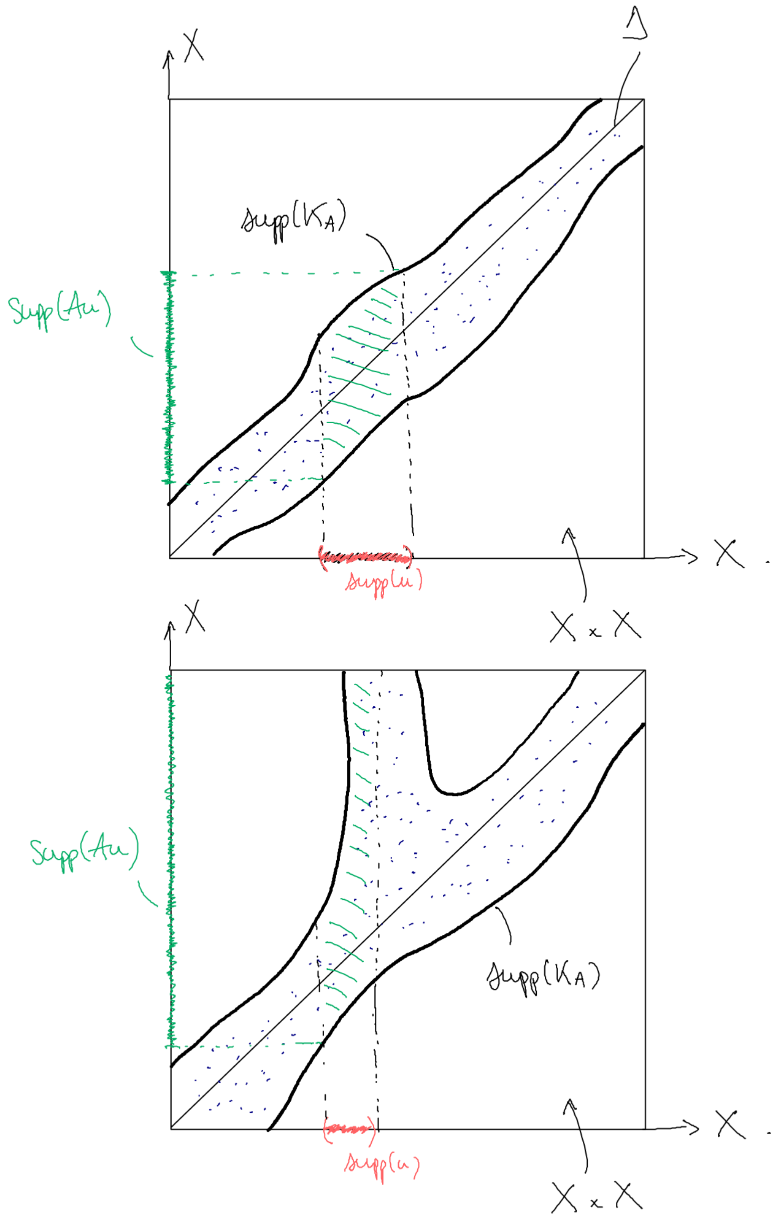

Figure 2.5: On top: a properly supported operator; it maps a distribution with compact support to a distribution with compact support. Bottom: a non-properly supported operator; it maps a distribution with compact support to a distribution with non-compact support.

Lemma 2.3.6. *Let $A \in \Psi^m_{\rho,\delta}(X)$. Then, there exists $A_1 \in \Psi^m_{\rho,\delta}(X)$, properly supported, and $A_2 \in \Psi^{-\infty}(X)$ such that $A = A_1 + A_2$.*

Proof. Let $\chi \in C^\infty(X \times X)$ be a function such that $\chi = 1$ in a neighborhood of the diagonal $\Delta \subset X \times X$ and properly supported. Then:

$$\begin{aligned} Au(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_y} \int_{\mathbb{R}^n_\xi} e^{i(x-y)\xi} a(x,y,\xi) u(y) \mathrm{d}\xi \mathrm{d}y \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_y} \int_{\mathbb{R}^n_\xi} e^{i(x-y)\xi} \chi(x,y) a(x,y,\xi) u(y) \mathrm{d}\xi \mathrm{d}y \\ &\qquad + \int_{\mathbb{R}^n_y} (1-\chi(x,y)) K_A(x,y) \mathrm{d}y. \end{aligned}$$

The first operator is obviously in $\Psi^m_{\rho,\delta}(X)$ and properly supported. As to the second operator, it is smoothing, that is it has smooth Schwartz kernel since $\mathrm{WF}(K_A) \subset \Delta$ and $1 - \chi$ vanishes in a neighborhood of Δ. $\square$

2.3.3 Bijection between symbols and quantization

Theorem 2.3.7. *We consider $A := \mathrm{Op}(a)$ for some symbol $a \in S^m_{\rho,\delta}(X \times X \times \mathbb{R}^n)$. We also assume that $\rho > \delta$ and that A is properly supported. Define $\sigma^{\mathrm{full}}_A(x,\xi) := e^{-ix\xi} A(e^{i\bullet\xi})$. Then $\sigma^{\mathrm{full}}_A \in S^m_{\rho,\delta}(X \times \mathbb{R}^n)$ and it has the asymptotic development*

$$\sigma^{\mathrm{full}}_A(x,\xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{1}{i^{|\alpha|}\alpha!} \left(\partial^\alpha_\xi \partial^\alpha_y a\right)(x,x,\xi).$$

Moreover, for all $u \in C^\infty_{\mathrm{comp}}(X)$, we have:

$$\begin{aligned} Au(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\xi} e^{ix\cdot\xi} \sigma^{\mathrm{full}}_A(x,\xi) \widehat{u}(\xi) \mathrm{d}\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_y} \int_{\mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi} \sigma^{\mathrm{full}}_A(x,\xi) u(y) \mathrm{d}\xi \mathrm{d}y. \end{aligned} \tag{2.3.2}$$

The symbol σ^{full}_A is called the full symbol *of A. It is seen as a section of the cotangent bundle, namely $\sigma^{\mathrm{full}}_A \in C^\infty(T^*X)$ (and it satisfies the usual symbolic estimates (1.1.1)).*

Remark 2.3.8. The fact that the symbol in (2.3.2) depends only on (x,ξ) (and not on the y-variable) is called a *left-quantization*. A right-quantization consists in writing the operator in terms of a symbol depending only on (y,ξ).

Note that the last integral in (2.3.2) is understood in the following sense. Writing $K_A(x,y) := \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi} b(x,\xi) \mathrm{d}\xi$, we know that $K_A \in \mathcal{D}'(X \times X)$ is an oscillatory integral. Then

$$\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_y} \int_{\mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi} b(x,\xi) u(y) \mathrm{d}\xi \mathrm{d}y = \langle K_A(x,\bullet), u(\bullet)\rangle_{\mathbb{R}^n_y}.$$

The proof is a bit technical and relies on the stationary phase lemma. Nevertheless, let us state an important consequence:

Corollary 2.3.9. *Take $\rho > \delta$. The quantization map defined in Definition (3.1.4) is bijective as a map*

$$\mathrm{Op} : S^m_{\rho,\delta}(T^*X)/S^{-\infty} \to \Psi^m_{\rho,\delta}(X)/\Psi^{-\infty}(X)$$

with inverse given by

$$\sigma^{\mathrm{full}} : \Psi^m_{\rho,\delta}(X)/\Psi^{-\infty}(X) \to S^m_{\rho,\delta}(T^*X)/S^{-\infty}(T^*X).$$

(This is only an algebraic identification, regardless of topology.)

Proof. We argue on the full symbol rather than on Op directly. Given $A \in \Psi^m_{\rho,\delta}(X)$ (not necessarily properly supported), the full symbol $\sigma^{\mathrm{full}}(A)$ is defined as follows: write $A = A_1 + A_2$ with $A_1 \in \Psi^m_{\rho,\delta}(X)$ properly supported, $A_2 \in \Psi^{-\infty}(X)$ smoothing, and set $\sigma^{\mathrm{full}}(A) := \sigma^{\mathrm{full}}(A_1)$. It is almost straightforward to check that this is indeed well-defined (independently of the choice of A_1) as an element in $S^m_{\rho,\delta}(T^*X)/S^{-\infty}(T^*X)$.

For the surjectivity of the full symbol, we take $a \in S^m_{\rho,\delta}(T^*X)$ and a cutoff function $\chi \in C^\infty(X \times X)$ such that $\chi(x, y) = 1$ in a neighborhood of the diagonal $\Delta \subset X \times X$. Then, by Theorem 2.3.7

$$\sigma^{\mathrm{full}}(\mathrm{Op}(\chi a))(x, \xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial_y^\alpha \partial_\xi^\alpha (a\chi)(x, x, \xi) = a(x, \xi),$$

that is, $\sigma^{\mathrm{full}}(\mathrm{Op}(\chi a)) - a$ is an element of $S^{-\infty}(T^*X)$.

As to the injectivity, assume that $A = A_1 + A_2$ and $\sigma^{\mathrm{full}}(A) = \sigma^{\mathrm{full}}(A_1) = 0$. Modulo smoothing operators, we can write $A_1 = \mathrm{Op}(\sigma^{\mathrm{full}}(A_1)) \pmod{\Psi^{-\infty}}$ and thus $A_1 \in \Psi^{-\infty}(X)$, that is, $A \in \Psi^{-\infty}(X)$. This proves the claim. $\square$

We now start the proof of Theorem 2.3.7. For the sake of simplicity, take $\rho = 1, \delta = 0$.

Proof of Theorem 2.3.7. We fix a compact $K \subset X$ and take $x \in K$. Note that by assumption, the support of $a(x, \bullet, \bullet)$ is then compact in $y \in X$. So all the integrals below are performed over a compact set in the y-variable. We have:

$$\begin{aligned} b(x, \xi) = e^{-ix\xi} A(e^{i\bullet\xi})(x) &= e^{-ix\cdot\xi} \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_y} \int_{\mathbb{R}^n_\eta} e^{i(x-y)\eta} a(x, y, \eta) e^{iy\xi} \mathrm{d}\eta \mathrm{d}y \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_y} \int_{\mathbb{R}^n_\eta} e^{i(x-y)(\eta-\xi)} a(x, y, \eta) \mathrm{d}\eta \mathrm{d}y, \end{aligned}$$

where this is as usual understood as an oscillatory integral. Note that $b(x, \xi)$ is a smooth function of (x, ξ): for the x variable, this follows from the fact that A maps smooth function to smooth functions by Lemma 2.3.1; for the ξ variable, this is a parametric version of the previous statement. The critical points of the phase

$\phi(x, \xi, y, \eta) := (x - y)(\eta - \xi)$ are obtained for $d_{(y,\eta)}\phi = 0 = (-(\eta - \xi), (x - y))$, that is $x = y, \eta = \xi$.

We are going to cut our integral in two pieces according to the closeness with the critical set (eventually, we want to apply the stationary phase lemma, so we need to integrate over a compact set). Take a cutoff function $\chi \in C^\infty_{\text{comp}}(\mathbb{R}_+)$ such that $\chi = 1$ on $[0, 1/4]$ and $\chi = 0$ outside $[0, 1/2]$. Then write (for $|\xi| \geq 1$):

$$\begin{aligned} b(x,\xi) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n_y}\int_{\mathbb{R}^n_\eta} e^{i(x-y)(\eta-\xi)}\chi\left(\frac{|\eta-\xi|}{|\xi|}\right)a(x,y,\eta)\mathrm{d}\eta\mathrm{d}y \\ &+ \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n_y}\int_{\mathbb{R}^n_\eta} e^{i(x-y)(\eta-\xi)}\left(1-\chi\left(\frac{|\eta-\xi|}{|\xi|}\right)\right)a(x,y,\eta)\mathrm{d}\eta\mathrm{d}y \\ &= b_1(x,\xi) + b_2(x,\xi). \end{aligned}$$

Let us show that $b_2 \in S^{-\infty}(T^*X)$. For that, we observe that the function under the integral sign is non-zero when $|\eta - \xi| \geq |\xi|$. We can thus, as usual, integrate by parts. We set:

$${}^tL = \frac{1}{|\eta-\xi|^2}\sum_{j=1}^n(\xi_j - \eta_j)D_{y_j},$$

so that ${}^tLe^{i\phi} = e^{i\phi}$ and $L = {}^tL$. Then:

$$b_2(x,\xi) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n_y}\int_{\mathbb{R}^n_\eta} e^{i(x-y)(\eta-\xi)}\left(1-\chi\left(\frac{|\eta-\xi|}{|\xi|}\right)\right)L^k(a)(x,y,\eta)\mathrm{d}\eta\mathrm{d}y$$

Note that, on the region of integration, we have $|\xi| \leq \frac{1}{4}|\eta - \xi|$ and

$$|\eta| = |\eta - \xi + \xi| \leq |\eta - \xi| + |\xi| \leq \frac{5}{4}|\eta - \xi|,$$

that is:

$$\frac{2}{3}(|\eta| + |\xi|) \leq |\eta - \xi|.$$

From this and the expression of L, we get that

$$L^k(a)(x, y, \eta) \lesssim (|\eta| + |\xi|)^{m-k}.$$

Hence, after repeated integration by parts, b_2 eventually becomes a converging integral, and we get as in the proof of Theorem 2.1.14:

$$b_2(x, \xi) \lesssim |\xi|^{n-k}$$

Since one can take $k \in \mathbb{N}$ to be arbitrarily large, this proves that b_2 decays faster than any polynomial. The derivatives of b_2 are treated similarly, which eventually proves that $b_2 \in S^{-\infty}$.

Let us now deal with b_1. Note that the integration for b_1 is performed over a compact set. It will be convenient for that to use the characterization of Lemma 1.1.10. We introduce the change of variable $s := y - x, \theta := \eta - \xi$ and then we set

$\xi := \omega/h$, where $h = 1/|\xi|, |\omega| = 1$, $\theta = \sigma/h$. We get:

$$\begin{aligned} b_1(x,\xi) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n_s}\int_{\mathbb{R}^n_\theta} e^{-is\cdot\theta}\chi\left(\frac{|\theta|}{|\xi|}\right)a(x,x+s,\xi+\theta)\mathrm{d}\theta\mathrm{d}s \\ &= \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n_s}\int_{\mathbb{R}^n_\sigma} e^{-\frac{i}{h}s\cdot\sigma}\chi\left(|\sigma|\right)a\left(x,x+s,\frac{\sigma+\omega}{h}\right)\mathrm{d}\sigma\mathrm{d}s \end{aligned}$$

We then apply the stationary phase lemma (the support of the integrand is compact) and more precisely Example 1.3.6. We get:

$$b_1(x,\xi) = \sum_{|\alpha|\ \leq N-1}\frac{h^{|\alpha|}}{\alpha! i^{|\alpha|}}\partial_s^\alpha\partial_\sigma^\alpha\left(\chi\left(|\sigma|\right)a\left(x,x+s,\frac{\sigma+\omega}{h}\right)\right)\Big|_{s=0,\sigma=0} + S_N(h,x,\xi),$$

Note that the cutoff function χ is constant equal to 1 near $|\sigma| = 0$, so we can remove it. Every ∂_σ^α derivative yields a $h^{-|\alpha|}$ so these will cancel out and we get:

$$b_1(x,\xi) = \sum_{|\alpha|\ \leq N-1}\frac{1}{\alpha! i^{|\alpha|}}\partial_s^\alpha\partial_\sigma^\alpha\left(a\left(x,x+s,\xi+\sigma\right)\right|_{s=0,\sigma=0} + S_N(h,x,\xi). \tag{2.3.3}$$

The remainder is bounded by:

$$\begin{aligned} &S_N(h,x,\xi) \\ &\lesssim h^N\sum_{|\alpha+\beta|\ \leq 2n+1}\sup_{s,|\sigma|\ \leq 2}\left|\partial_s^\beta\partial_\sigma^\alpha(\partial_s\cdot\partial_\sigma)^N\left(\chi\left(|\sigma|\right)a\left(x,x+s,\frac{\sigma+\omega}{h}\right)\right)\right|. \end{aligned}$$

If we forget about the χ, this is a sum of terms of the form

$$h^{-|\alpha|}(\partial_s^\beta\partial_\sigma^\alpha)a\left(x,x+s,\frac{\sigma+\omega}{h}\right) \tag{2.3.4}$$

and using the symbolic estimates (1.1.1) for a, we get that this is controlled by:

$$\lesssim h^{-|\alpha|}\langle(\sigma+\omega)h^{-1}\rangle^{m-|\alpha|} \lesssim |\xi|^{|\alpha|}\left(1+|\xi||\sigma+\omega|\right)^{m-|\alpha|} \lesssim \langle\xi\rangle^m.$$

Note that the last bound follows from the fact that $|\sigma| \leq 1/2$ on the support of χ, which implies that $|\sigma+\omega|\ \geq 1/2$. As a consequence, the overall remainder is bounded by:

$$|S_N(x,\xi)|\ \lesssim \langle\xi\rangle^{m-N}.$$

Let us point out that in the general case where on takes symbols in $S^m_{\rho,\delta}$, the assumption $\rho > \delta$ comes into play in the bound of the terms (2.3.4). It can also be checked, following the same proof, that the higher order derivatives of b_1 are temperate in the sense of Lemma 1.1.10, namely $|\partial_x^\beta\partial_\xi^\alpha b_1|\ \lesssim \langle\xi\rangle^M$. So here is what we have proved so far:

- Gathering the terms together in (2.3.3) according to the value of $|\alpha| = k \in \mathbb{N}$, we have the existence of $a_j \in S^{m-j}_{1,0}(T^*X)$ such that $b = \sum_{j=0}^{N-1} a_j + S_N$, where

for all $(x, \xi) \in T^*K$:
$$|S_N(x, \xi)| \lesssim \langle\xi\rangle^{m-N}.$$

- For all $\alpha, \beta \in \mathbb{N}^n$, there exists $C := C(K, \alpha, \beta), M = M(\alpha, \beta)$ such that for all $(x, \xi) \in T^*K$:
$$|\partial_\xi^\alpha \partial_x^\beta b_1(x, \xi)| \leq C\langle\xi\rangle^M.$$

We are then in the position of applying Lemma 1.1.10 to conclude that $b_1 \in S^m_{1,0}(T^*X)$ and has the asymptotic development $b_1 \sim \sum_{j\geq 0} a_j$. Hence, $b = b_1 + b_2$ has the desired asymptotic expansion.

It remains to prove (2.3.2). We write

$$u := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n_\xi} e^{i\bullet\cdot\xi}\widehat{u}(\xi)\mathrm{d}\xi.$$

Define for $h > 0$ small and χ a cutoff function near 0:

$$u_h := \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n_\xi} e^{i\bullet\cdot\xi}\widehat{u}(\xi)\chi(h|\xi|)\mathrm{d}\xi.$$

and observe that $u_h \to u$ in $C^\infty(X)$. Then:

$$\begin{aligned} Au_h &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n_\xi} A(e^{i\bullet\cdot\xi})\widehat{u}(\xi)\chi(h|\xi|)\mathrm{d}\xi \\ &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n_\xi} e^{i\bullet\cdot\xi}b(\bullet, \xi)\widehat{u}(\xi)\chi(h|\xi|)\mathrm{d}\xi \end{aligned}$$

Taking the limit as $h \to 0$, using the continuity of A on $C^\infty(X)$:

$$Au = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n_\xi} e^{ix\cdot\xi}b(x, \xi)\widehat{u}(\xi)\mathrm{d}\xi = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi}b(x, \xi)u(y)\mathrm{d}y\mathrm{d}\xi$$

□

Exercise 2.3.10. Verify that $u_h \to u$ in $C^\infty(X)$.

Remark 2.3.11. Let us also point out that in Theorem 2.3.7 if $a \in S^m_{\rho,\delta}(X \times \mathbb{R}^n)$ depends only on (x, ξ) (i.e. $a(x, y, \xi) = a(x, \xi)$ is a *left-quantization*) and $b \in S^{m'}_{\rho,\delta}$, one has
$$e^{-ix\xi}A(e^{i\bullet\xi}b(\bullet, \xi)) \sim \sum_{\alpha\in\mathbb{N}^n}\frac{1}{\alpha!}\partial_\xi^\alpha a(x, \xi)D_x^\alpha b(x, \xi) \in S^{m+m'}_{\rho,\delta}.$$

The proof is similar to that of Theorem 2.3.7 since, up to a smoothing remainder, we would have had to consider

$$\begin{aligned}
\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n_y}\int_{\mathbb{R}^n_\eta} & e^{i(x-y)(\eta-\xi)}\chi\left(\frac{|\eta-\xi|}{|\xi|}\right)a(x,\eta)b(y,\xi)\mathrm{d}\eta\mathrm{d}y \\
&= \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n_s}\int_{\mathbb{R}^n_\sigma} e^{-\frac{i}{h}s\cdot\sigma}\chi\left(|\sigma|\right)a(x,\frac{\sigma+\omega}{h})b(x+s,\xi)\mathrm{d}\eta\mathrm{d}y \\
&= \sum_{|\alpha|\ \leq N-1}\frac{h^{|\alpha|}}{\alpha! i^{|\alpha|}}\partial_s^\alpha\partial_\sigma^\alpha\left(\chi\left(|\sigma|\right)a(x,\frac{\sigma+\omega}{h})b(x+s,\xi)\right)\Big|_{s=0,\sigma=0} \\
&\qquad\qquad + S_N(h,x,\xi) \\
&= \sum_{|\alpha|\ \leq N-1}\frac{1}{\alpha! i^{|\alpha|}}\partial_s^\alpha\partial_\sigma^\alpha\left(a(x,\xi+\sigma)b(x+s,\xi)\right)\big|_{s=0,\sigma=0} + S_N(h,x,\xi) \\
&\sim \sum_\alpha \frac{1}{\alpha!}\partial_\xi^\alpha a(x,\xi)D_x^\alpha b(x,\xi).
\end{aligned}$$

Chapter 3

Pseudodifferential operators

The aim of this chapter is to study the basic properties of pseudodifferential operators, namely, composition rules, continuity on L^2 and Sobolev spaces, and ellipticity.

Contents

3.1 Pseudodifferential calculus

We now study pseudodifferential operators as they were defined previously and their main properties. First of all, there are a few canonical operations on ΨDOs one needs to understand.

3.1.1 Basic operations

3.1.1.1 Adjoint

Recall that we use the convention $\langle \bullet, \bullet \rangle_{L^2}$ to denote the L^2-scalar product on X (induced from $\mathbb{R}^n$), namely

$$\langle \varphi, \psi \rangle_{L^2} = \int_X \varphi(x)\overline{\psi}(x)\mathrm{d}x.$$

(This should not be confused with $\langle \bullet, \bullet \rangle$ which is $\mathbb{C}$-linear in both variables.) Given $A : C^\infty_{\mathrm{comp}}(X) \to \mathcal{D}'(X)$, its *formal adjoint* $A^* : C^\infty_{\mathrm{comp}}(X) \to \mathcal{D}'(X)$ is defined as the complex transpose of A, namely such that for all $\varphi, \psi \in C^\infty_{\mathrm{comp}}(X)$:

$$\langle A\varphi, \psi \rangle_{L^2} = \langle \varphi, A^*\psi \rangle_{L^2}.$$

The kernel of A^* satisfies the relation

$$K_{A^*}(x, y) = \overline{K_A(y, x)}.$$

Lemma 3.1.1. *Let $A \in \Psi^m_{\rho,\delta}(X)$ with $\rho > \delta$. Then $A^* \in \Psi^m_{\rho,\delta}(X)$ and*

$$\sigma^{\mathrm{full}}_{A^*}(x, \xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial^\alpha_\xi D^\alpha_x \overline{\sigma^{\mathrm{full}}_A}(x, \xi).$$

Proof. Up to changing $A \in \Psi^m_{\rho,\delta}(X)$ by smoothing remainders, we can assume directly that A is properly supported and given by:

$$A\varphi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_y} \int_{\mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi} \sigma^{\mathrm{full}}_A(x, \xi)\varphi(y)\mathrm{d}\xi\mathrm{d}y.$$

Then

$$A^*\varphi(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_y} \int_{\mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi} \overline{\sigma^{\mathrm{full}}_A(y, \xi)}\varphi(y)\mathrm{d}\xi\mathrm{d}y$$

and by Theorem 2.3.7, we get

$$\sigma^{\mathrm{full}}_{A^*}(x, \xi) = \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \left(\partial^\alpha_\xi D^\alpha_y \overline{\sigma^{\mathrm{full}}_A(y, \xi)} \right)\Big|_{y=x}.$$

□

Remark 3.1.2. In particular, we observe that the leading term in this expansion is $\overline{\sigma^{\mathrm{full}}_A}(x, \xi)$.

In the case where one has a vector bundle $\mathbb{C}^r \to X$ equipped with a Hermitian metric h, the L^2-scalar product is then given by: for all $\varphi, \psi \in C^\infty(X, \mathbb{C}^r)$,

$$\langle \varphi, \psi \rangle_{L^2} = \int_X h_x(\varphi(x), \psi(x)) \mathrm{d}x.$$

The case of L^2-functions is obtained for the usual Hermitian metric on $\mathbb{C}$, $h(z, w) := z\overline{w}$. Then the formal adjoint of an operator $A : C^\infty_{\text{comp}}(X, \mathbb{C}^r) \to \mathcal{D}'(X, \mathbb{C}^r)$ is defined in the same way as an operator $A^* : C^\infty_{\text{comp}}(X, \mathbb{C}^r) \to \mathcal{D}'(X, \mathbb{C}^r)$ such that $\langle A\varphi, \psi \rangle_{L^2} = \langle \varphi, A^*\psi \rangle_{L^2}$.

3.1.1.2 Composition

We now discuss the composition of pseudodifferential operators.

Lemma 3.1.3. *Let $A \in \Psi^{m_1}_{\rho,\delta}(X), B \in \Psi^{m_2}_{\rho,\delta}(X)$ such that one of them is properly supported and $\rho > \delta$. Then $AB \in \Psi^{m_1+m_2}_{\rho,\delta}(X)$ and*

$$\sigma^{\text{full}}_{AB}(x, \xi) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial^\alpha_\xi \sigma^{\text{full}}_A(x, \xi) D^\alpha_x \sigma^{\text{full}}_B(x, \xi). \tag{3.1.1}$$

Proof. Up to smoothing remainders, we can always assume that A, B are properly supported and written in left-quantization with full symbols σ_A, σ_B. Hence for $u \in C^\infty_{\text{comp}}(X)$, Bu is compactly supported and

$$Bu(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\xi} e^{ix\cdot\xi} \sigma^{\text{full}}_B(x, \xi) \widehat{u}(\xi) \mathrm{d}\xi.$$

This is an absolutely converging integral (in $C^\infty(X)$[1]) and we can apply the operator A which gives

$$\begin{aligned} ABu(x) &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\xi} A\left(e^{i\bullet\cdot\xi} \sigma^{\text{full}}_B(\bullet, \xi)\right)(x) \widehat{u}(\xi) \mathrm{d}\xi \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\xi} e^{ix\xi} \underbrace{e^{-ix\xi} A\left(e^{i\bullet\cdot\xi} \sigma^{\text{full}}_B(\bullet, \xi)\right)}_{:=c(x,\xi)} \widehat{u}(\xi) \mathrm{d}\xi = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_\xi} e^{ix\xi} c(x, \xi) \widehat{u}(\xi) \mathrm{d}\xi, \end{aligned}$$

with c given by Remark 2.3.11 (following Theorem 2.3.7):

$$c \sim \sum_\alpha \frac{1}{\alpha!} \partial^\alpha_\xi a(x, \xi) D^\alpha_x b(x, \xi) \in S^{m_1+m_2}_{\rho,\delta}(T^*X).$$

This proves the claim. $\square$

[1] In the sense that

$$\int_{\mathbb{R}^n_\xi} \| e^{i\bullet\xi} \sigma^{\text{full}}_B(\bullet, \xi) \widehat{u}(\xi) \|_{C^k(K)} \mathrm{d}\xi < \infty,$$

for all compact $K \subset X, k \in \mathbb{N}$.

Remark 3.1.4. Note that the previous Lemma and (3.1.1) shows that there is a natural operation of "multiplication" of symbols corresponding to the multiplication of pseudodifferential operators. More precisely, for $\rho > \delta$, $a \in S^{m_1}_{\rho,\delta}(T^*X), b \in S^{m_2}_{\rho,\delta}(T^*X)$, one can uniquely define (modulo $S^{-\infty}$):

$$a \diamond b \sim \sum_{\alpha} \frac{1}{\alpha!} \partial_\xi^\alpha a(x,\xi) D_x^\alpha b(x,\xi).$$

It can be checked that this bilinear map is associative, namely $a \diamond (b \diamond c) = (a \diamond b) \diamond c$ but it is clearly not commutative (otherwise, it would imply that the product of pseudodifferential operators is commutative modulo smoothing operators!).

3.1.1.3 Changes of coordinates

We let $\kappa : X \to Y$ be a diffeomorphism between two open subsets $X, Y \subset \mathbb{R}^n$. If $u \in C^\infty(Y)$, then $\kappa^* u \in C^\infty(X)$ and κ also acts on distributions $\kappa^* : \mathcal{D}'(Y) \to \mathcal{D}'(X)$, as seen before. We let $B \in \Psi^m(Y)$ be a pseudodifferential operator on Y obtained as the quantization of $b \in S^m(Y \times Y \times \mathbb{R}^n)$ and define:

$$A := \kappa^* B \kappa_* : C^\infty(X) \to C^\infty(X).$$

Theorem 3.1.5. *A is a pseudodifferential operator in $\Psi^m(X)$ with full symbol given by*

$$\sigma_A^{\text{full}}(x,\eta) \sim \sum_{\alpha \in \mathbb{N}^n} \frac{1}{\alpha!} \partial_\eta^\alpha D_z^\alpha \left(b(\kappa(x), \kappa(z), G(x,z)^{-\top}\eta) \frac{|\det d\kappa(z)|}{|\det G(x,z)|} \right) \Big|_{x=z}, \tag{3.1.2}$$

where $^{-\top}$ stands for the inverse transpose.

In particular, the equality reads:

$$\sigma_A^{\text{full}}(x,\eta) = \sigma_B^{\text{full}}(\kappa(x), d\kappa(x)^{-\top}(\eta)) + \mathcal{O}(S^{m-1}) = \widetilde{\kappa}^* \sigma_B^{\text{full}}(x,\eta) + \mathcal{O}(S^{m-1}),$$

where $\widetilde{\kappa} : T^*X \to T^*Y$ is the diffeomorphism induced by $\kappa : X \to Y$ on the cotangent bundle.

Proof. We write:

$$\begin{aligned} A\varphi(x) = B(\varphi \circ \kappa^{-1})(\kappa(x)) &= \frac{1}{(2\pi)^n} \int e^{i(\kappa(x)-y)\cdot\xi} b(\kappa(x), y, \xi) \varphi(\kappa^{-1}(y)) \mathrm{d}y \mathrm{d}\xi \\ &= \frac{1}{(2\pi)^n} \int e^{i(\kappa(x)-\kappa(z))\cdot\xi} b(\kappa(x), \kappa(z), \xi) \varphi(z) |\det d\kappa(z)| \mathrm{d}z \mathrm{d}\xi. \end{aligned}$$

Observe that $\kappa(x) - \kappa(z) = G(x,z)(x-z)$, where

$$G(x,z) = \int_0^1 d\kappa_{z+t(x-z)} \mathrm{d}t,$$

and thus $(\kappa(x) - \kappa(z)) \cdot \xi = (x-z) \cdot G(x,z)^\top \xi$. Note that $G(x,x) = \mathrm{d}\kappa(x)$ and this is invertible. As a consequence, for z close enough to x, $G(x,z)$ is also invertible. By introducing an adequate cutoff function near $x = z$, we can always assume that that $G(x,z)$ is invertible (the remaining terms corresponding to integration away from the diagonal are obviously smoothing). We then make the change of variable $\eta := G(x,z)^\top \xi$ so that:

$$\begin{aligned} A\varphi(x) &= \frac{1}{(2\pi)^n}\int e^{i(x-z)\cdot\eta} b(\kappa(x),\kappa(z),G(x,z)^{-\top}\eta)\varphi(z)\frac{|\det d\kappa(z)|}{|\det {}^tG(x,z)|}\mathrm{d}z\mathrm{d}\eta. \\ &= \frac{1}{(2\pi)^n}\int e^{i(x-z)\cdot\eta} a(x,z,\eta)\varphi(z)\mathrm{d}z\mathrm{d}\eta. \end{aligned}$$

where

$$a(x,z,\eta) = b(\kappa(x),\kappa(z),G(x,z)^{-\top}\eta)\frac{|\det d\kappa(z)|}{|\det G(x,z)|}.$$

One then checks that this symbol a is indeed in $S^m(X \times X \times \mathbb{R}^n)$. Moreover, the full symbols is then given by

$$\sigma_A^{\mathrm{full}}(x,\eta) \sim \sum_{\alpha\in\mathbb{N}^n} \frac{1}{\alpha!}\partial_\eta^\alpha D_z^\alpha\left(b(\kappa(x),\kappa(z),G(x,z)^{-\top}\eta)\frac{|\det d\kappa(z)|}{|\det G(x,z)^\top|}\right)\Big|_{x=z}$$

□

We observe that the leading term in (3.1.2), obtained for $\alpha = 0$ is:

$$b(\kappa(x),\kappa(x),G(x,x)^{-\top}\eta)\frac{|\det d\kappa(x)|}{|\det G(x,x)^\top|} = b(x,x,d\kappa^{-\top}(x)\eta),$$

that is

$$\sigma_A^{\mathrm{full}}(x,\eta) = b(x,x,d\kappa^{-\top}(x)\eta) + \mathcal{O}_{S^{m-1}}(1) = \kappa^*\sigma_B^{\mathrm{full}} + \mathcal{O}_{S^{m-1}}(1),$$

where the pullback $\kappa^* : C^\infty(T^*Y) \to C^\infty(T^*X)$ for functions on the cotangent bundle is given by:

$$\kappa^* f(x,\eta) = f(\kappa(x), d\kappa^{-\top}(x)\eta).$$

3.1.2 The principal symbol

The principal symbol is defined as the *leading part* of the full symbol. It is the only part of the full symbol that behaves nicely under changes of coordinates. This can be observed on a very simple example:

Example 3.1.6. In spherical coordinates, the Laplacian is given by:

$$\Delta = \partial_r^2 + \frac{n-1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2,$$

and thus its full symbol in these coordinates, using the (λ,η) variables for the dual variables to (r, ω) (where $\omega \in \mathbb{S}^{n-1}$), is

$$q(r, \omega, \lambda, \eta) = -\lambda^2 - \frac{1}{r^2}\eta^2 + \frac{n-1}{r} i\lambda.$$

For the sake of simplicity, we now assume $n = 2$. The polar coordinates are then given by the diffeomorphism:

$$\kappa : (0, \infty) \times \mathbb{S}^1 \ni (r, \theta) \mapsto (r\cos(\theta), r\sin(\theta)),$$

The symbol q seems unrelated to the previous symbol we had in the standard coordinates of $\mathbb{R}^2$, namely $p_\Delta(x, y, \xi_x, \xi_y) = -(\xi_x^2 + \xi_y^2)$, since p_Δ is a homogeneous polynomial of order 2, while q contains a degree 1 monomial. However, we observe that if we only keep the top part of these polynomials and neglect the lower order terms, we find that these expressions agree. Indeed, a vector $v = v_r\partial_r + v_\theta\partial_\theta$ is identified to the vector $d\kappa(v)$ in the (x, y)-coordinates with

$$d\kappa_{(r,\theta)} = \begin{pmatrix} \cos(\theta) & -r\sin(\theta) \\ \sin(\theta) & r\cos(\theta) \end{pmatrix}.$$

Similarly, a *covector* $\xi = \xi_x dx + \xi_y dy$ is identified with the covector $d\kappa^\top(\xi)$ in the (r, θ)-coordinates, that is ξ is identified with

$$\begin{pmatrix} \lambda \\ \eta \end{pmatrix} = d\kappa^\top \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix} = \begin{pmatrix} \cos(\theta)\xi_x + \sin(\theta)\xi_y \\ -r\sin(\theta)\xi_x + r\cos(\theta)\xi_y \end{pmatrix}.$$

We then observe that, considering the top part of the polynomial q and neglecting the term of order 1, we get:

$$\begin{aligned} -\lambda^2 - \frac{1}{r^2}\,\eta^2 &= -\left(\cos(\theta)\xi_x + \sin(\theta)\xi_y\right)^2 - \frac{1}{r^2}\left(-r\sin(\theta)\xi_x + r\cos(\theta)\xi_y\right)^2 \\ &= -\ (\xi_x^2 + \xi_y^2), \end{aligned}$$

that is the two symbols, at their *principal level* coincide.

This motivates the following definition:

Definition 3.1.7. Given $A \in \Psi^m(X)$, we define its *principal symbol* σ_A as the equivalence class $[\sigma_A^{\text{full}}] \in S^m(T^*X)/S^{m-1}(T^*X)$.

The principal symbol is well-defined and behaves like a function on the cotangent bundle. Formally, we should add an index m to the principal symbol, namely for every $m \in \mathbb{R}$, there is a principal symbol map $\sigma^{(m)}$ but we omit this notation for the sake of simplicity. Note that $\sigma_A = 0$ is equivalent to $\sigma_A^{\text{full}} \in S^{m-1}(T^*X)$ which means that $A \in \Psi^{m-1}(X)$. In other words:

$$\ker\sigma = \ \Psi^{m-1}(X). \tag{3.1.3}$$

We have:

Proposition 3.1.8. *The principal symbol is a bijective homomorphism*

$$\sigma : \Psi^m(X)/\Psi^{m-1}(X) \to S^m(T^*X)/S^{m-1}(T^*X),$$

and behaves as a function on the cotangent bundle by a change of coordinates. Moreover, $\sigma_{A^*} = \overline{\sigma_A}$. In particular, if A is formally self-adjoint, that is $A = A^*$, then σ_A is real-valued.

Proof. The homomorphism property, namely

$$\sigma_{A\circ B} = \sigma_A \times \sigma_B$$

is a straightforward consequence of the composition formula (3.1.1) for the full symbol. The formula for the adjoint is a consequence of the remark below Lemma 3.1.1. □

In the (ρ, δ)-classes of symbols, for $\rho > 1/2$ and $\delta = 1 - \rho$, the principal symbol is well-defined as a map

$$\sigma : \Psi^m_{\rho,\delta}(X)/\Psi^{m-(2\rho-1)}_{\rho,\delta}(X) \to S^m_{\rho,\delta}(T^*X)/S^{m-(2\rho-1)}_{\rho,\delta}(T^*X)$$

More generally, we can extend this to vector bundles i.e. for $\Psi^m(X, E \to F)$ and we get a bijection

$$\begin{aligned}\sigma : \Psi^m(X, E \to F)/\Psi^{m-1}(X, E \to F)& \\ \to S^m(T^*X, \mathrm{Hom}(E, F))/S^{m-1}(T^*X, \mathrm{Hom}(E, F)),&\end{aligned}$$

is a bijection. Given $B \in \Psi^m(X, E \to F), A \in \Psi^m(X, F \to G)$, it still satisfies the relation (as homeomorphisms):

$$\sigma_{A\circ B} = \sigma_A \circ \sigma_B.$$

Moreover, $\sigma_{A^*} = \sigma_A^*$, where the adjoint on the right-hand side is understood with respect to the Hermitian structure of the vector bundle.

3.1.3 Pseudodifferential operators on closed manifolds

3.1.3.1 Quantization on closed manifolds

The goal of this paragraph is to define pseudodifferential operators on smooth closed (compact, boundaryless) manifolds. Given a smooth manifold M, we say that (κ, U) is a *chart* if $U \subset M$ is an open subset and $\kappa : U \to X \subset \mathbb{R}^n$ is a diffeomorphism. We say that a family $(\kappa_i, U_i)_{i=1}^N$ is a family of *cutoff charts* if these are all charts and $\cup_{i=1}^N U_i = M$ covers M. We will use the notation $\chi \prec \chi'$ for two smooth cutoff functions with compact support in the same open subset $U \subset M$ such that $\chi' = 1$ on the support of χ. When given a family of cutoff charts, we will always consider a partition of unity $\sum_{i=1}^N \chi_i = \mathbf{1}$ subordinated to that cover, as well as other cutoff functions $\chi'_i \in C^\infty_{\mathrm{comp}}(U_i)$ such that $\chi_i \prec \chi'_i$.

Definition 3.1.9. Let M be a smooth closed manifold. We define the class $\Psi^m(M)$ of pseudodifferential operators as the set of continuous linear operators $A : C^\infty(M) \to C^\infty(M)$ such that for every chart (κ, U), for every $\chi, \chi' \in C^\infty_{\mathrm{comp}}(U)$ such that $\chi \prec \chi'$, the operator

$$C^\infty_{\mathrm{comp}}(X) \ni \varphi \mapsto A_{\kappa,\chi',\chi}\varphi := \kappa_* \chi' A \chi \kappa^* \varphi \in C^\infty_{\mathrm{comp}}(X)$$

belongs to $\Psi^m(X)$.

It is straightforward to extend this definition to the case of vector bundles: in this case, one chooses the charts (κ, U) small enough so that the bundles $E, F \to M$ are trivial over the charts. Definition (3.1.9) is quite painful to use. In particular, one usually has symbols and wants to quantize them. We define a *quantization procedure* as a map:

$$\mathrm{Op} : S^m(T^*M) \to \Psi^m(T^*M),$$

defined by:

$$\mathrm{Op}(a)\varphi := \sum_{i=1}^{N} \kappa_i^* \left(\Psi_i' \,\mathrm{Op}_{\mathbb{R}^n}((\kappa_i)_*(\chi_i' a))(\kappa_i)_* \chi_i \varphi\right), \tag{3.1.4}$$

where $\Psi_i' := (\kappa_i)_* \chi_i'$ and $\mathrm{Op}_{\mathbb{R}^n}$ is the standard left-quantization on $\mathbb{R}^n$. This quantization is *highly non-canonical* as it depends on a choice of cutoff charts. Note that we always have $\mathrm{Op}(1) = \mathbb{1}$ and we have:

Proposition 3.1.10. *Let* Op *be an arbitrary quantization procedure. Then:*

$$\Psi^m(M) = \left\{\mathrm{Op}(a) + K \mid a \in S^m(T^*M), K \ \in \Psi^{-\infty}(M)\right\}$$

Proof. We omit the details, see [Abe12, Section 3.10] for instance. □

In other words, if $A \in \Psi^m(M)$ in the sense of Definition 3.1.9, then there exists $a \in S^m(T^*M), K \in \Psi^{-\infty}(M)$ such that $A = \mathrm{Op}(a) + K$. However, observe that this decomposition is not unique as we can always change a by $a + r$ for some $r \in S^{-\infty}(T^*M)$. We now define the *principal symbol* of pseudodifferentials operators in $\Psi^m(M)$. Recall that if $U \subset M$ is an open set and $\kappa : U \to \kappa(U) := X \subset \mathbb{R}^n$ is a diffeomorphism, it induces a map $\kappa^* : C^\infty(T^*X) \to C^\infty(T^*U)$ on functions on the cotangent bundle by $\kappa^* f(x,\xi) = f(\kappa(x), d\kappa(x)^{-\top}\xi)$.

Definition 3.1.11. Given $A \in \Psi^m(M)$, we define its principal symbol as $\sigma_A|_{\mathrm{supp}(\chi)} := \kappa^* \sigma_{A_{\kappa,\chi',\chi}}|_{\mathrm{supp}(\chi)}$.

Proof. One needs to check that this is well-defined, that is, independent of the choice of coordinates. But, taking another cutoff chart, we end up with the change of coordinates formula (3.1.2) for the principal symbol in $\mathbb{R}^n$. □

Proposition 3.1.12. *If $A \in \Psi^{m_1}(M), B \in \Psi^{m_2}(M)$, then $AB \in \Psi^{m_1+m_2}(M)$ and $\sigma_{A\circ B} = \sigma_A \ \sigma_B = \sigma_{B\circ A}$. In particular, $\Psi^{m_1}(M) \circ \Psi^{m_2}(M) \subset \Psi^{m_1+m_2}(M)$ and $\cup_{m\in\mathbb{R}} \Psi^m(M)$ is a* graded algebra*. We call it the* pseudodifferential algebra.

Proof. Immediate. □

This quantization procedure can also be defined for Hermitian vector bundles. We let $E \to M$ be a smooth Hermitian bundle over M. Recall that this means that E is the data of an underlying topological bundle over M (which we also denote by E in order not to burden the notation) and a Hermitian structure h, that is for every $x \in M$, h_x is a Hermitian metric on E_x. Given two Hermitian vector bundles $E, F \to M$, we can also consider a quantization procedure:

$$\mathrm{Op} : S^m(T^*M, \mathrm{Hom}(E,F)) \to \Psi^m(M, E \to F).$$

There is a slight abuse of notations here: indeed, $\mathrm{Hom}(E,F) \to M$ is technically a vector bundle *over* M whereas we want to consider symbols *defined on* T^*M and taking values in $\mathrm{Hom}(E,F)$. For this to make sense, one has to consider the projection $\pi : T^*M \to M$ and then pullback the bundles to get $\pi^*E, \pi^*F \to T^*M$. The symbols are therefore elements of $S^m(T^*M, \pi^*\mathrm{Hom}(E,F))$. In order not to burden the notations, we will always drop the π^*.

3.1.3.2 More on the principal symbol

A pseudodifferential operator $A \in \Psi^m(M)$ is said to be *classical* if its full symbol, expressed in any coordinate chart, admits a polyhomogeneous expansion i.e. for all $N > 0$, one can write:

$$\sigma_A^{\mathrm{full}} = a_m + a_{m-1} + ... + a_{m-N+1} + r_N,$$

where $a_k \in S^k(T^*\mathbb{R}^n)$ is k-homogeneous for ξ large enough and $r_N \in S^{-N}(T^*\mathbb{R}^n)$. A useful characterization of the principal symbol is the following:

Lemma 3.1.13. *Let $A \in \Psi^m(M, E \to F)$ be* classical. *Let $f \in C^\infty(M,E)$ and $S \in C^\infty(M)$ such that $dS \neq 0$ on* supp(f). *Then for all $x \in$ supp(f):*

$$\lim_{h \to 0} e^{-\frac{i}{h}S(x)} h^m A(e^{\frac{i}{h}S} f)(x) = \sigma_A(x, dS(x)) f(x)$$

Proof. Let us consider the case of the trivial bundle $E = \mathbb{C}$ for the sake of simplicity. First of all, up to a smoothing remainder, we can assume $A = \mathrm{Op}(a)$ for some $a \in S^m(T^*M)$ and an arbitrary quantization. Up taking a partition of unity, we can always chop $f = \sum_{i=1}^N f\chi_i = \sum_{i=1}^N f_i$ into small pieces so that each of them is supported in a cutoff chart of the quantization. As a consequence, the statement boils down to proving the result for the usual quantization in $\mathbb{R}^n$. For that, we consider in $\mathbb{R}^n$ for some cutoff function χ:

$$A(e^{\frac{i}{h}S}\chi)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n_y} \int_{\mathbb{R}^n_\xi} e^{i(x-y)\cdot\xi} a(x,\xi) e^{\frac{i}{h}S(y)} \chi(y) \mathrm{d}y \mathrm{d}\xi,$$

where, as usual, this is understood as an oscillatory integral. We make the change of variable $\xi' := \xi/h$ and we get:

$$A(e^{\frac{i}{h}S}\chi)(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n_y} \int_{\mathbb{R}^n_\xi} e^{\frac{i}{h}((x-y)\cdot\xi + S(y))} a(x,\xi/h) \chi(y) \mathrm{d}y \mathrm{d}\xi.$$

We let $\phi(x,y,\xi) := (x-y)\cdot\xi + S(y)$ be the phase. Note that $d_{(y,\xi)}\phi(x,y,\xi) = (-\xi + dS(y), x-y)$ so this vanishes for $x=y, \xi = dS(y)$. We introduce a cutoff function $\psi(y,\xi)$ such that $\psi = 1$ near $\{\xi = dS(y)\}$. Then:

$$\begin{aligned} A(e^{\frac{i}{h}S}\chi)(x) &= \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n_y}\int_{\mathbb{R}^n_\xi} e^{\frac{i}{h}((x-y)\cdot\xi+S(y))}\psi(y,\xi)a(x,\xi/h)\chi(y)\mathrm{d}y\mathrm{d}\xi \\ &+ \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n_y}\int_{\mathbb{R}^n_\xi} e^{\frac{i}{h}((x-y)\cdot\xi+S(y))}(1-\psi(y,\xi))a(x,\xi/h)\chi(y)\mathrm{d}y\mathrm{d}\xi. \end{aligned}$$

By construction, in the second integral, the differential of the phase does not vanish on the support of the amplitude and thus by repeated integration by parts in the y-variable, we can make this integral absolutely convergent and get that this second term is $\mathcal{O}(h^\infty)$. As far as the first term is concerned, it is now well-defined and we integrate over a compact set. The fact that the operator is a classical pseudodifferential operator implies that $a(x,\xi/h) = a_m(x,\xi/h) + r(x,\xi/h) = h^{-m}a_m(x,\xi) + r(x,\xi/h)$, with $r \in S^{m-1}(T^*\mathbb{R}^n)$. Multiplying everything by h^m, we thus get:

$$\begin{aligned} h^m A(e^{\frac{i}{h}S}\chi)(x) &= \mathcal{O}(h^\infty) + \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n_y}\int_{\mathbb{R}^n_\xi} e^{\frac{i}{h}((x-y)\cdot\xi+S(y))}\psi(y,\xi)a_m(x,\xi)\chi(y)\mathrm{d}y\mathrm{d}\xi \\ &+ \frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n_y}\int_{\mathbb{R}^n_\xi} e^{\frac{i}{h}((x-y)\cdot\xi+S(y))}\psi(y,\xi)h^m r(x,\xi/h)\chi(y)\mathrm{d}y\mathrm{d}\xi. \end{aligned}$$

The first integral is performed over $\{|\xi| \leq h\}$, for $h > 0$ small enough. We are thus in position to apply the stationary phase Lemma 1.3.2 to the first integral and it is immediate that, after multiplication by $e^{-\frac{i}{h}S(x)}$ it converges as $h \to 0$ to $a_m(x, dS(x))\chi(x)$. As far as the second integral is concerned, we need to show that it converges to 0. For this, we can write $r = a_{m-1} + ... + a_{m-N} + r_N$ where $a_k \in S^k(T^*\mathbb{R}^n)$ is k-homogeneous and $r_N \in S^{m-N-1}(T^*\mathbb{R}^n)$. Each of the a_k contributions will give zero by the stationary phase lemma. As to the r_N contribution, due to the cutoff ψ, the ξ under the integral varies in an annulus in the cotangent bundle, that is $1/C \leq |\xi| \leq C$. Moreover, for all $|\xi| \geq 1$, we have $|r_N(x,\xi)| \leq C/|\xi|^{N+1-m}$. As a consequence, we get for $h > 0$ small enough: $|r_N(x,\xi/h)| \leq Ch^{N+1-m}/|\xi|^{N+1-m}$ and:

$$\begin{aligned} &\left|\frac{1}{(2\pi h)^n}\int_{\mathbb{R}^n_y}\int_{\mathbb{R}^n_\xi} e^{\frac{i}{h}((x-y)\cdot\xi+S(y))}\psi(y,\xi)h^m r(x,\xi/h)\chi(y)\mathrm{d}y\mathrm{d}\xi\right| \\ &\qquad \lesssim h^{-n}\int_{\mathbb{R}^n_y}\int_{\mathbb{R}^n_\xi}\psi(y,\xi)\chi(y)h^m h^{N+1-m}|\xi|^{m-N-1}\mathrm{d}y\mathrm{d}\xi \lesssim h^{N+1-n}, \end{aligned}$$

and this clearly converges to 0 by taking N large enough. □

Remark 3.1.14. If the principal symbol is not homogeneous, then there is no hope to be able to get a limit for $h^m a_m(x,\xi/h)$ as $h \to 0$.

Example 3.1.15. Let M be a smooth closed manifold. Let $X \in C^\infty(M, TM)$ be a smooth vector field. It can be seen in particular as a differential operator of order

1, that is $X \in \Psi^1(M)$. Then

$$X(e^{\frac{i}{h}S}) = \frac{i}{h} dS(X) e^{\frac{i}{h}S},$$

that is: $\sigma_X(x, \xi) = i\langle \xi, X(x)\rangle$.

3.1.3.3 Adjoint of a ΨDO

Let $E \to M$ be a smooth Hermitian vector bundle and let $\mathrm{d}\mu$ be a smooth measure on M. We have a natural L^2-scalar product given for $\varphi, \psi \in C^\infty(M, E)$ by:

$$\langle \varphi, \psi \rangle_{L^2(M,E)} := \int_M h_x(\varphi(x), \psi(x)) \mathrm{d}\mu(x).$$

Given $A : C^\infty(M) \to \mathcal{D}'(M)$, this allows one to define as before the formal adjoint $A^* : C^\infty(M) \to \mathcal{D}'(M)$ of A so that for all

$$\langle A\varphi, \psi \rangle_{L^2(M,E)} = \langle \varphi, A^*\psi \rangle_{L^2(M,E)}.$$

We sum up the main properties:

Proposition 3.1.16. *If $A \in \Psi^m(M)$, then $A^* \in \Psi^m(M)$ and $\sigma_{A^*} = \overline{\sigma_A}$. More generally, if $E, F \to M$ are Hermitian vector bundles and $A \in \Psi^m(M, E \to F)$, then $A^* \in \Psi^m(M, F \to E)$ and $\sigma_{A^*} = \sigma_A^*$, where the adjoint is understood in the sense of complex linear algebra.*

Proof. Immediate. □

3.1.4 Exercises

Exercise 1

Recall that the gradient $\nabla : C^\infty(M) \to C^\infty(M, T_{\mathbb{C}}M)$ is defined by $\nabla\varphi := (d\varphi)^\sharp$, where $\sharp : T^*M \to TM$ denotes the musical isomorphism[2]. It is a differential operator of order 1, that is $\nabla \in \Psi^1(M, \mathbb{C} \to T_{\mathbb{C}}M)$.

1. Show that its principal symbol is given by $\sigma_\nabla(x, \xi) = \xi^\sharp$.

2. We define the (non-negative) Laplacian $\Delta := \nabla^*\nabla$. What is the order of this operator? Show that $\sigma_\Delta(x, \xi) = |\xi|^2_{g^{-1}}$.

Exercise 2

The exterior derivative acts as $d : C^\infty(M, \Lambda^k T^*M) \to C^\infty(M, \Lambda^{k+1} T^*M)$ by the usual formula in coordinates:

$$d(f_I dx_I) = \sum_{i=1}^n \partial_{x_i} f_I dx_i \wedge dx_I,$$

[2] $\xi^\sharp = Z$ is defined such that $\langle \xi, \bullet \rangle = g(Z, \bullet)$.

where $dx_I = dx_{i_1} \wedge ... \wedge dx_{i_k}$ for some indices $i_1, ..., i_k \in \{1, ..., n\}$.

1. Show that $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^p \alpha \wedge d\beta$, if α is a p-form.
2. Compute σ_d.
3. Is it injective/surjective? Describe the kernel and the range.
4. We let $d^* : C^\infty(M, \Lambda^{k+1}T^*M) \to C^\infty(M, \Lambda^k T^*M)$ be the adjoint (divergence operator). Compute σ_{d^*}.
5. We define $\Delta := (d + d^*)^2$, the Hodge Laplacian acting on k-forms. Compute σ_Δ.

Exercise 3

Let M be a smooth closed manifold and μ be a smooth density on M. Let $P \in \Psi^m(X)$. Recall that the *formal adjoint* P^* of P is the (unique) operator satisfying the equality: for all $\varphi, \psi \in C^\infty(M)$:

$$\langle P\varphi, \psi\rangle_{L^2(M,\mathrm{d}\mu)} := \int_M (P\varphi)\psi \ \ \mathrm{d}\mu = \int_M \varphi(P^*\psi) \ \mathrm{d}\mu = \langle \varphi, P^*\psi\rangle_{L^2(M,\mathrm{d}\mu)}.$$

We showed in class that $P^* \in \Psi^m(X)$. Nevertheless, the formal adjoint P^* *depends* on a choice of density μ. As a consequence, we will rather write P^*_μ for the formal adjoint of P with respect to the density μ.

1. Let $\mu' := a \cdot \mu$ be another smooth density on M, where $a \in C^\infty(M)$ is a positive function. Compute $P^*_{\mu'}$ in terms of a and P^*_μ.
2. Deduce that the principal symbol σ_{P^*} is well-defined, independently of the choice of density μ.

Let $X \in C^\infty(M, TM)$ be a smooth vector field, seen as a differential operator of order 1 acting on $C^\infty(M)$. If μ is a smooth density on M, we define the *divergence* of X with respect to μ as the (unique) function satisfying the equality $\mathcal{L}_X\mu = \mathrm{div}_\mu(X)\mu$.

3. Recall the value of $\sigma_X \in S^1(T^*M)$.
4. Show that $\int_M \mathcal{L}_X\mu = 0$.
5. Deduce the formal adjoint X^*_μ of X, computed with respect to μ. *Hint: consider the quantity* $\int_M \mathcal{L}_X(\varphi\psi\mu)$.
6. What is σ_{X^*}?

3.2 Ellipticity, parametrices and continuity

For the sake of simplicity, we now restrict to $\rho = 1, \delta = 0$ and consider pseudodifferential operators on a closed manifold M. We will write $\Psi^m(M) := \Psi^m_{\rho,\delta}(M)$.

3.2.1 Elliptic operators

3.2.1.1 Definition

We now define the notion of ellipticity.

Definition 3.2.1. Let $A \in \Psi^m$ and $(x_0, \xi_0) \in T_0^*M$. Then A is said to be *elliptic* at (x_0, ξ_0) if there exists $C > 0$ and a conic neighborhood $V \subset T^*M$ of (x_0, ξ_0) such that for all $(x, \xi) \in V$ and $|\xi| \geq C$:

$$|\sigma_A(x, \xi)| \geq \frac{1}{C} \langle \xi \rangle^m. \tag{3.2.1}$$

We say that A is elliptic if it is elliptic on T^*M. We will also say that a symbol $a \in S^m(T^*M)$ is elliptic at $(x_0, \xi_0) \in T_0^*M$ if it satisfies (3.2.1).

The principal symbol is only well-defined as an element of S^m/S^{m-1}. As a consequence, any property related to the principal symbol has to be S^{m-1}-invariant.

Exercise 3.2.2. Check that the property (3.2.1) is indeed well-defined on the quotient space $S^m(T^*M)/S^{m-1}(T^*M)$.

Remark 3.2.3. If A is elliptic at (x_0, ξ_0), then it is also elliptic at $(x_0, \lambda\xi_0)$ for all $\lambda > 0$. The elliptic set $\mathrm{ell}(A) \subset T^*M$ (the set of points that are elliptic) is an open conic subset of T_0^*M.

A similar definition can be given for vector bundles. If $E, F \to M$ are two Hermitian vector bundles *of same rank*, then we say that $A \in \Psi^m(M, E \to F)$ is elliptic at (x_0, ξ_0) if there exists $C > 0$ and a conic neighborhood $V \subset T^*M$ of (x_0, ξ_0) such that for all $(x, \xi) \in V$, $|\xi| \geq C$ and for all $f \in E_x$:

$$\|\sigma_A(x, \xi) f\|_{F_x} \geq \frac{1}{C} \langle \xi \rangle^m \|f\|_{E_x}$$

The norms here are the ones carried by the Hermitian bundles. Note that this definition is actually *independent* of the choice of metrics on the vector bundles.

Exercise 3.2.4. Check this.

Example 3.2.5. Let (M, g) be a smooth closed Riemannian manifold. Let $\nabla : C^\infty(M) \to C^\infty(M, TM)$ be the gradient. Define $\Delta := \nabla^*\nabla$. The principal symbol of the Laplacian is then given by $\sigma_\Delta(x, \xi) = |\xi|^2$ as we saw previously. It is obviously elliptic.

Another important example is the following:

Example 3.2.6. Let $a \in S^m(T^*M)$ be homogeneous of order $m \in \mathbb{R}$ for $\{|\xi| \geq 1\}$. Let $(x_0, \xi_0) \in T^*M$ such that $|\xi_0| \geq 1$. Then a is elliptic at (x_0, ξ_0) if and only if $a(x_0, \xi_0) \neq 0$. There is a interesting geometric way of describing these homogeneous functions. Indeed, they induce a function $a_\infty \in C^\infty(\partial_\infty T^*M)$ on the boundary at infinity of the cotangent space by:

$$a_\infty(x, \omega) := \lim_{\lambda \to \infty} \lambda^{-m} a(x, \lambda\xi),$$

where $[\xi] = \omega$. This is a smooth function defined on the compact set $\partial_\infty T^*M$. Then the ell$(A) = \mathrm{int}(\mathrm{supp}(a_\infty))$

We have:

Lemma 3.2.7. *Let $a \in S^m(T^*M)$ be an elliptic symbol. Then, there exists $\chi \in C^\infty(T^*M)$, a cutoff function such that $\chi = 1$ for $|\xi| \gg 1$ large enough, such that the symbol b defined by $b(x,\xi) = a^{-1}(x,\xi)\chi(x,\xi)$ (for $|\xi| \gg 1$ large enough) lies in $S^{-m}(T^*M)$. Moreover, $ab - 1 \in S^{-\infty}(T^*M)$.*

Proof. It suffices to show that $b(x,\xi) := a(x,\xi)^{-1}$, defined for $|\xi| \gg 1$ large enough is indeed a symbol in $S^{-m}(T^*M)$. For that, we can take local coordinates. We show by induction on $k := |\alpha|+|\beta| \in \mathbb{N}$ that the derivatives $\partial_\xi^\alpha \partial_\xi^\beta b$ of b satisfy the required bounds (1.1.1). For $k = 0$, this is immediate. Then for $|\xi| \gg 1$ large enough and $|\alpha| + |\beta| = k+1$, we have

$$\begin{aligned}
\partial_\xi^\alpha \partial_x^\beta (ab) &= \partial_\xi^\alpha \partial_x^\beta (1) = 0 \\
&= \sum_{\alpha_1+\alpha_2=\alpha, \beta_1+\beta_2=\beta} C(\alpha_1,\alpha_2,\beta_1,\beta_2) \partial_\xi^{\alpha_1}\partial_x^{\beta_1}(a) \partial_\xi^{\alpha_2}\partial_x^{\beta_2}(b) \\
&= a\partial_\xi^\alpha \partial_x^\beta (b) + \sum_{\alpha_1+\alpha_2=\alpha, \beta_1+\beta_2=\beta, |\alpha_1|+|\beta_2| \geq 1} C(\alpha_1,\alpha_2,\beta_1,\beta_2) \partial_\xi^{\alpha_1}\partial_x^{\beta_1}(a) \partial_\xi^{\alpha_2}\partial_x^{\beta_2}(b),
\end{aligned}$$

that is

$$\partial_\xi^\alpha \partial_x^\beta (b) = -a^{-1} \sum_{\alpha_1+\alpha_2=\alpha, \beta_1+\beta_2=\beta, |\alpha_1|+|\beta_2| \geq 1} C(\alpha_1,\alpha_2,\beta_1,\beta_2) \partial_\xi^{\alpha_1}\partial_x^{\beta_1}(a) \partial_\xi^{\alpha_2}\partial_x^{\beta_2}(b).$$

On the right-hand side, we have $|\alpha_2| + |\beta_2| \leq k$, so we can apply the induction hypothesis and we get the result. ☐

3.2.1.2 Inverting elliptic operators

Now, the crucial point is that elliptic operators can be inverted within the class of pseudodifferential operators.

Theorem 3.2.8 (Global parametrix)**.** *Let $A \in \Psi^m(M)$ be elliptic. Then, there exists $B \in \Psi^{-m}(M)$ such that*

$$AB - \mathbb{1} = K_1, \quad BA - \mathbb{1} = K_2$$

where $K_1, K_2 \in \Psi^{-\infty}(M)$.

Such a B is called a *parametrix* or an approximate inverse or a quasi-inverse. In particular, we will see below that, given an invertible elliptic differential operator, this *genuine inverse* will indeed be *pseudodifferential*. A straightforward corollary is:

Corollary 3.2.9 (Rough elliptic regularity)**.** *Let $A \in \Psi^m(M)$ be elliptic. Then:*

$$A : \mathcal{D}'(M)/C^\infty(M) \to \mathcal{D}'(M)/C^\infty(M)$$

is a bijection.

Proof. If $[Au] = 0$ for some $u \in \mathcal{D}'(M)$, that is $Au = f \in C^\infty(M)$ then $BAu = Bf = u + K_2 u \in C^\infty(M)$. Since K_2 is smoothing, $K_2 u \in C^\infty(M)$ and thus $u \in C^\infty(M)$. This shows injectivity. Conversely, if $v \in \mathcal{D}'(M)$, then setting $u := Bv \in \mathcal{D}'(M)$, we have $A(Bv) = v + K_1 v = v \bmod C^\infty$. This shows surjectivity. □

Before proving Theorem 3.2.8, we state a version of Borel's lemma for ΨDOs. The Borel summation lemma for symbols implies a summation lemma for ΨDOs:

Lemma 3.2.10. *Let $A_j \in \Psi^{m_j}(M)$, for $j \in \mathbb{N}$ where $(m_j)_{j\in\mathbb{N}}$ is a decreasing sequence of real numbers diverging to $-\infty$. Then, there exists $A \in \Psi^{m_0}(M)$, unique modulo $\Psi^{-\infty}(X)$ such that $A \sim \sum_{j\geq 0} A_j$, that is for all $k \in \mathbb{N}$:*

$$A - \sum_{j=0}^{k-1} A_j \in \Psi^{m_k}(M).$$

Proof. We can write $A_j = \mathrm{Op}(a_j) + K_j$ for some $a_j \in S^{m_j}(T^*M)$ and $K_j \in \Psi^{-\infty}(M)$. By Borel's Lemma 1.1.8, there exists a symbol $a \in S^{m_0}(T^*M)$ such that $a \sim \sum_j a_j$. We then set $A := \mathrm{Op}(a)$. Observe that

$$A - \sum_{j=0}^{N-1} A_j = \mathrm{Op}(a - \sum_{j=0}^{N-1} a_j) + \sum_{j=0}^{N-1} K_j \in \Psi^{m_N}(M).$$

□

Proof of Theorem 3.2.8. Define $a_0 := \sigma_A \in S^m(T^*X)$, the principal symbol of A. Set $B_0 := \mathrm{Op}(b_0)$, where $b_0 \in S^{-m}(T^*X)$ is provided by Lemma 3.2.7. Then: $-R := AB_0 - \mathbb{1} \in \Psi^0(M)$ has principal symbol $\sigma_R = a_0 b_0 - 1 = 0$, so $R \in \Psi^{-1}$, and $AB_0 = \mathbb{1} - R$. We then apply Lemma 3.2.10 and choose $Q \sim \sum_{k\geq 0} R^k \in \Psi^0(M)$ (Neumann series). Observe that $(\mathbb{1} - R)Q - \mathbb{1} \in \Psi^{-\infty}(M)$. As a consequence,

$$AB_0 Q = (\mathbb{1} - R)Q = \mathbb{1} + \mathcal{O}_{\Psi^{-\infty}}(1),$$

and $B_{\mathrm{right}} := B_0 Q$ is a right-parametrix. The same can be done on the left, and we obtain B_{left}. Eventually, it suffices to observe that, modulo $\Psi^{-\infty}$, we have:

$$B_{\mathrm{left}} = B_{\mathrm{left}}(AB_{\mathrm{right}}) = (B_{\mathrm{left}}A)B_{\mathrm{right}} = B_{\mathrm{right}}.$$

□

Remark 3.2.11. We also mention the existence of local parametrices. Let $A \in \Psi^m(X)$ be elliptic at $(x_0, \xi_0) \in T_0^*M$. Then, there exists $B \in \Psi^{-m}(X)$ elliptic at $(x_0, \xi_0) \in T^*M$ such that

$$AB = \mathrm{Op}(\chi) + K_1, \quad BA = \mathrm{Op}(\chi) + K_2,$$

where $\chi \in S^0(T^*M)$ is equal to 1 in a conic neighborhood of (x_0, ξ_0) and $K_1, K_2 \in \Psi^{-\infty}(M)$.

3.2.2 Boundedness on $L^2(M)$

Let M be a smooth closed oriented manifold. Functional spaces are defined in the following way. First of all, we consider a smooth volume form (identified with a measure) $\mathrm{d}\mu$ on M. Then:

$$\|\varphi\|^2_{L^2(M)} := \int_M |\varphi(x)|^2 \mathrm{d}\mu(x), \tag{3.2.2}$$

and $L^2(M)$ is defined as the space of measurable functions on M such that the norm in (3.2.2) is finite. (Equivalently, it is the completion of $C^\infty(M)$ with respect to the norm (3.2.2).) Note that the space $L^2(M)$ *does not depend* on the choice of smooth measure $\mathrm{d}\mu$. Moreover, for any other choice of measure $\mathrm{d}\mu'$, one obtains a new norm that is equivalent to the previous one.

Exercise 3.2.12. Prove this claim.

We start with the following important result:

Theorem 3.2.13. *Let $A \in \Psi^0(M)$ and define:*

$$M := \limsup_{x\in M, |\xi|\to\infty} |\sigma_A(x,\xi)|$$

Then, for all $\varepsilon > 0$, there exists $A_\varepsilon \in \Psi^0(M), K_\varepsilon \in \Psi^{-\infty}(M)$ such that $A = A_\varepsilon + K_\varepsilon$ and $\|A_\varepsilon\|_{L^2\to L^2} \leq M + \varepsilon$.

Remark 3.2.14. We will actually prove a slightly weaker version of the theorem than what is stated (but this will be harmless for the main consequence of Theorem 3.2.13, namely Corollary 3.2.16), namely we will prove that there exists a constant $C > 0$ (depending only on some choice of cutoff functions) such that for all $A \in \Psi^0(M)$, for all $\varepsilon > 0$, there exists $A_\varepsilon \in \Psi^0(M), K_\varepsilon \in \Psi^{-\infty}(M)$ such that $A = A_\varepsilon + K_\varepsilon$ and $\|A_\varepsilon\|_{L^2\to L^2} \leq C(M+\varepsilon)$. In order to prove the exact bound of Theorem 3.2.13, we will need a specific *mollifier* (see Remark 3.2.15 below for a definition) whose definition is postponed to §4.3.

Proof. Fix $\varepsilon > 0$ and define $B := (M+\varepsilon)^2\mathbb{1} - A^*A \in \Psi^0(M)$ which satisfies $\sigma_B(x,\xi) \geq C_\varepsilon > 0$ for $|\xi| \gg 1$ large enough and some constant $C_\varepsilon > 0$.

Step 1: a preliminary bound. First of all, we show that we can write $B = R^*R - K$, for some $R \in \Psi^0(M)$ and $K \in \Psi^{-\infty}(M)$. We start by defining

$$r_0 := \sqrt{(M+\varepsilon)^2 - |\sigma_A|^2}, R_0 := \mathrm{Op}(r_0) \in \Psi^0(M),$$

which satisfies $B = R_0^*R_0 + K_0$ with $K_0 \in \Psi^{-1}(M)$ (formally selfadjoint) and $R_0 = R_0^* + \mathcal{O}_{\Psi^{-1}}(1)$. We then look for $R_1 \in \Psi^{-1}(M)$ such that $B = (R_0+R_1)^*(R_0+R_1)+K_1$ with $K_1 \in \Psi^{-2}(M)$. We develop and obtain:

$$B = R_0^*R_0 + R_1^*R_0 + R_0^*R_1 + R_1^*R_1 + K_1 = B - K_0 + 2R_1R_0 + \mathcal{O}_{\Psi^{-2}(M)}(1),$$

that is $2R_1R_0 = K_0 + \mathcal{O}_{\Psi^{-2}(M)}(1)$. We set: $R_1 = \frac{1}{2}K_0R_0^{-1}$, where R_0^{-1} is a parametrix for the elliptic operator R_0. We iterate this process and assume that we have constructed $R_0, ..., R_{k-1}$, where $R_i \in \Psi^{-i}(M)$ and

$$B = (R_0 + ... + R_{k-1})^*(R_0 + ... + R_{k-1}) + K_{k-1},$$

where $K_{k-1} \in \Psi^{-k}(M)$. Then setting:

$$R_k = \frac{1}{2}K_{k-1}(R_0 + ... + R_{k-1})^{-1},$$

we get the operators for the k-th step. Eventually, setting $R \sim \sum_{k\geq 0} R_k$, we see that $B - R^*R \in \Psi^{-\infty}(M)$.

This implies that for $\varphi \in C^\infty(M)$:

$$\begin{aligned}\|A\varphi\|^2_{L^2(M)} = \langle A^*A\varphi, \varphi\rangle_{L^2} &= (M+\varepsilon)^2\|\varphi\|^2_{L^2} - \|C\varphi\|^2_{L^2} + \langle K\varphi, \varphi\rangle_{L^2} \\ &\leq (M+\varepsilon)^2\|\varphi\|^2_{L^2} + \langle K\varphi, \varphi\rangle_{L^2} \leq C\|\varphi\|^2_{L^2},\end{aligned} \tag{3.2.3}$$

as K is obviously bounded on $L^2(M)$ since it has smooth Schwartz kernel. Note that at this stage, we have already proved the L^2-boundedness of pseudodifferential operators of order 0. To get a more accurate bound, we need some extra work.

Step 2: proof of Theorem 3.2.13. We now consider a family of cutoff charts $(\kappa_i, U_i)_{i=1}^N$ and we further assume that $(\kappa_i)_*\mathrm{d}\mu = \mathrm{d}x$ is the Lebesgue measure in $\mathbb{R}^n$ (this is always possible, see the Exercises §3.2.4). We let $\sum_{i=1}^n \chi_i^2 = \mathbf{1}$ be a partition of unity subordinated to that cover (i.e. each χ_i has support in U_i). We let $\widetilde{\eta} \in C^\infty(\mathbb{R}^n)$ be a smooth (real-valued) non-negative cutoff function with support near 0 such that $\int_{\mathbb{R}^n} \widetilde{\eta} = 1$. Observe that $0 \leq |\widehat{\widetilde{\eta}}| \leq 1$ but we would like this to be satisfied by the Fourier transform (without absolute values). For that, set $\eta := \widetilde{\eta} \star \mathcal{R}\widetilde{\eta}$ which satisfies

$$\widehat{\eta} = \widehat{\widetilde{\eta}} \cdot \mathcal{F}\mathcal{R}\widetilde{\eta} = \widehat{\widetilde{\eta}} \cdot \mathcal{R}\mathcal{F}\widetilde{\eta} = |\widehat{\widetilde{\eta}}|^2$$

Note that $\int_{\mathbb{R}^n} \eta = \widehat{\eta}(0) = 1$ and $0 \leq \widehat{\eta} \leq 1$. Define $\eta_h := h^{-n}\eta(\bullet/h)$ for $h > 0$ small enough (approximation of unity). Then, set

$$E_h\varphi = \sum_{i=1}^N \chi_i\kappa_i^* \left(\eta_h \star (\kappa_i)_* \underbrace{\chi_i\varphi}_{:=\varphi_i}\right). \tag{3.2.4}$$

Note that

$$\mathrm{supp}(\eta_h \star (\kappa_i)_*\varphi_i) \subset \mathrm{supp}(\eta_h) + \mathrm{supp}((\kappa_i)_*\varphi_i) = \mathrm{supp}((\kappa_i)_*\varphi_i) + \mathcal{O}(h),$$

and thus for h small enough this support is contained in $X_i := \kappa_i(U_i)$ so the pullback by κ_i^* is well-defined. Moreover, the construction ensures that $E_h^* = E_h$. Indeed, this follows from the volume-preserving property of the κ_i and the fact that convolution by η_h is self-adjoint since we have by the Parseval identity (A.3.2) (using $\widehat{\eta_h} \in \mathbb{R}$):

$$\begin{aligned}\langle \eta_h \star f_1, f_2 \rangle_{L^2(\mathbb{R}^n, \mathrm{d}x)} &= \frac{1}{(2\pi)^n} \langle \widehat{\eta_h \star f_1}, \widehat{f_2} \rangle_{L^2(\mathbb{R}^n, \mathrm{d}x)} \\ &= \frac{1}{(2\pi)^n} \langle \widehat{f_1}, \widehat{\eta_h} \widehat{f_2} \rangle_{L^2(\mathbb{R}^n, \mathrm{d}x)} = \langle f_1, \eta_h \star f_2 \rangle_{L^2(\mathbb{R}^n, \mathrm{d}x)}.\end{aligned}$$

We claim that $E_h \varphi \to \varphi$ in $L^2(M)$ as $h \to 0$. For that it suffices to show that each term in (3.2.4) converges to φ_i in $L^2(M)$. By using the charts, it amounts to proving that $\eta_h \star (\kappa_i)_* \varphi \ \to (\kappa_i)_* \varphi$ in $L^2(\mathbb{R}^n, \mathrm{d}x)$. Now this is a very standard result based on Parseval's identity: if $\psi \in L^2_{\mathrm{comp}}(\mathbb{R}^n)$, then:

$$\begin{aligned}\|\eta_h \star \psi - \psi\|^2_{L^2(\mathbb{R}^n)} &= \frac{1}{(2\pi)^n} \|\widehat{\eta_h \star \psi} - \widehat{\psi}\|^2_{L^2(\mathbb{R}^n)} \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left|(\widehat{\eta_h} - \mathbf{1})(\xi) \widehat{\psi}(\xi)\right|^2 \mathrm{d}\xi \leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \left|\widehat{\psi}(\xi)\right|^2 \mathrm{d}\xi.\end{aligned} \tag{3.2.5}$$

Note that $\widehat{\eta_h}(\xi) = \widehat{\eta}(h\xi)$ and thus the term under the integral converges simply to 0 (since $\widehat{\eta}(0) = \int_{\mathbb{R}^n} \eta = 1$). Moreover, it satisfies the dominated convergence theorem and thus the above integral converges to 0 as $h \to 0$. This proves the claim.

We also observe that by Young's inequality, we have

$$\|E_h \varphi\|_{L^2} \leq \sum_{i=1}^N \|\eta_h \star (\kappa_i)_* \varphi_i\|_{L^2(\mathbb{R}^n, \mathrm{d}x)} \leq \sum_{i=1}^N \|\eta_h\|_{L^1} \|\varphi_i\|_{L^2} \leq C \sum_{i=1}^N \|\varphi_i\|_{L^2} \leq CN \|\varphi\|_{L^2},$$

where $C := \|\eta\|_{L^1}$. Hence, by (3.2.3):

$$\begin{aligned}\|A(E_h - \mathbb{1})\varphi\|^2_{L^2(M)} &\leq (M + \varepsilon)^2 \|E_h \varphi - \varphi\|^2_{L^2} + \langle K(E_h - \mathbb{1})\varphi, (E_h - \mathbb{1})\varphi \rangle_{L^2} \\ &\leq (M + \varepsilon)^2 (1 + CN)^2 \|\varphi\|^2_{L^2} + \langle (E_h - \mathbb{1}) K (E_h - \mathbb{1})\varphi, \varphi \rangle_{L^2} \\ &\leq (M + \varepsilon)^2 (1 + CN)^2 \|\varphi\|^2_{L^2} + (1 + CN) \|(E_h - \mathbb{1})K\|_{L^2 \to L^2} \|\varphi\|^2_{L^2}\end{aligned}$$

We now claim that $\|(E_h - \mathbb{1})K\|_{L^2 \to L^2} \to 0$ as $h \to 0$. Indeed, identifying operators and their Schwartz kernel, the kernel of $(E_h - \mathbb{1})K$ is given by:

$$(E_h - \mathbb{1})K(x, y) = \int_M E_h(x, z) K(z, y) \mathrm{d}z - K(x, y) = [E_h K(\bullet, y)](x) - K(x, y)$$

Now, we know by our previous computation that $E_h K(\bullet, y) \to K(\bullet, y)$ in $L^2(M)$ (and uniformly in $y \in M$), that is:

$$\sup_{y \in M} \int_{M_x} |(E_h - \mathbb{1})K(x, y)|^2 \mathrm{d}x \to_{h \to 0} 0.$$

By Schur's lemma[3], we get that

$$\|(E_h - \mathbb{1})K\|^2_{L^2 \to L^2} \leq \int_{M_y} \left(\int_{M_x} |(E_h - \mathbb{1})K(x,y)|^2 \mathrm{d}x \right) \mathrm{d}y \to 0.$$

As a consequence, we have that for h small enough,

$$\begin{aligned} \|A(E_h - \mathbb{1})\varphi\|^2_{L^2(M)} &\leq (M+\varepsilon)^2(1+CN)^2\|\varphi\|^2_{L^2} + (1+CN)\varepsilon\|\varphi\|^2_{L^2} \\ &\leq (M+2\varepsilon)^2(1+CN)^2\|\varphi\|^2_{L^2}, \end{aligned}$$

which proves the result. □

Remark 3.2.15. The operators $E_h := \mathrm{Op}(\chi_h)$ play an important role in the theory and are called *mollifiers*. They are a generalization of the well-known notion of convolution by a smooth function, that is they provide approximations of distributions by smooth functions. Note that, in order to get the sharp bound $(M+\varepsilon)$ instead of $(M+\varepsilon)(1+CN)$, we would need a mollifier E_h such that $\|\mathbb{1} - E_h\|_{L^2 \to L^2} \leq 1$. It is not clear for the moment how to construct such a family of mollifiers but it will become in a few chapters, once we have defined the spectral theory of elliptic pseudodifferential operators. Another route is to use semiclassical analysis which we have avoided so far.

As a consequence of Theorem 3.2.13, we have:

Corollary 3.2.16. *Let $A \in \Psi^0(M)$. Assume that*

$$\limsup_{x \in M, |\xi| \to \infty} |\sigma_A(x,\xi)| = 0.$$

Then $A : L^2(M) \to L^2(M)$ is compact.

Proof. First of all, let us prove that $A \in \Psi^{-\infty}(M)$ is compact on $L^2(M)$. Since A has smooth Schwartz kernel $K_A \in \mathcal{D}'(M \times M)$ which can be written as $K_A(x,y) = R_A(x,y)\mathrm{d}\mu(x)\mathrm{d}\mu(y)$ with $R_A \in C^\infty(M \times M)$. We now show that A can be obtained as a limit (in $\mathcal{L}(L^2(M))$) of finite-rank operators. Let $(\phi_i)_{i \geq 0}$ be an orthonormal basis of $L^2(M)$. Then $(\phi_i \otimes \phi_j)_{i,j \geq 0}$ is an orthonormal basis of $L^2(M \times M)$. We can thus write $R_A(x,y) = \sum_{i,j \geq 0} c_{ij}\phi_i(x)\phi_j(y)$. We set $R_A^{(N)}(x,y) := \sum_{i,j \leq N} c_{ij}\phi_i(x)\phi_j(y)$. Then $R_A^{(N)} \to R_A$ in $L^2(M \times M)$. We define $A^{(N)} \in \mathcal{L}(L^2(M))$, the operator whose Schwartz kernel is $R_A^{(N)}(x,y)\mathrm{d}\mu(x)\mathrm{d}\mu(y)$, that is,

$$A^{(N)}\varphi(x) = \int_M R_A^{(N)}(x,y)\varphi(y)\mathrm{d}\mu(y).$$

[3] Let $K : L^2(M) \to L^2(M)$ be an operator with smooth kernel. Then:

$$\|K\varphi\|^2_{L^2} = \int_{M_x} |K\varphi(x)|^2 \mathrm{d}x = \int_{M_x} \left| \int_{M_y} K(x,y)\varphi(y)\mathrm{d}y \right|^2 \leq \int_{M_x}\int_{M_y} |K(x,y)|^2 \mathrm{d}y\mathrm{d}x \|\varphi\|^2_{L^2},$$

that is $\|K\|_{L^2 \to L^2} \leq \|K\|_{L^2(M_x \times M_y, \mathrm{d}x \otimes \mathrm{d}y)}$.

Note that this is an finite-rank operator and we claim that $A^{(N)} \to A$ in $\mathcal{L}(L^2(M))$. Indeed:

$$\begin{aligned}
\|(A^{(N)} - A)\varphi\|^2_{L^2(M)} &= \int_M |(A^{(N)} - A)\varphi(x)|^2 \mathrm{d}\mu(x) \\
&= \int_M \left| \int_M (R_A^{(N)}(x,y) - R_A(x,y))\varphi(y)\mathrm{d}\mu(y) \right|^2 \mathrm{d}\mu(x) \\
&\leq \int_M \left(\int_M |R_A^{(N)}(x,y) - R_A(x,y)|^2 \mathrm{d}\mu(y) \right) \|f\|^2_{L^2(M)} \mathrm{d}\mu(x) \\
&\leq \|R_A^{(N)} - R_A\|^2_{L^2(M\ \times M)} \|f\|^2_{L^2(M)}.
\end{aligned}$$

Hence $\|A^{(N)} - A\|_{\mathcal{L}(L^2(M))} \leq \|R_A^{(N)} - R_A\|_{L^2(M\ \times M)} \to 0$. This proves the claim.

We then apply Theorem 3.2.13 with $M = 0$. We get for every $\varepsilon > 0$ a decomposition $A = A_\varepsilon + K_\varepsilon$ with $\|A_\varepsilon\|_{L^2 \to L^2} \ \leq \varepsilon$, that is $A_\varepsilon \ \to 0$ in $\mathcal{L}(L^2)$. This implies that $A = \lim_{\varepsilon \to 0} K_\varepsilon$. Since $K_\varepsilon \in \Psi^{-\infty}(M)$, every $K_\varepsilon \in \mathcal{L}(L^2)$ is compact and thus A is compact. $\square$

3.2.3 Boundedness on Sobolev spaces

3.2.3.1 Definition

The Sobolev spaces are defined in the following way. For $s > 0$, we introduce

$$\Lambda_s := \mathbb{1} + \mathrm{Op}(\langle \xi \rangle^{s/2})^* \,\mathrm{Op}(\langle \xi \rangle^{s/2}) \in \Psi^s(M),$$

which has principal symbol $\sigma_{\Lambda_s}(x,\xi) = \langle \xi \rangle^s$. We then set for $s > 0$ and $\varphi \in C^\infty(M)$:

$$\|\varphi\|_{H^s(M)} := \|\Lambda_s \varphi\|_{L^2(M)}, \quad \langle \varphi, \psi \rangle_{H^s(M)} := \langle \Lambda_s \varphi, \Lambda_s \psi \rangle_{L^2(M)}$$

and:

$$H^s(M) := \overline{C^\infty(M)}^{\|\cdot\|_{H^s(M)}}.$$

It is clear from the very definition of the norm that for $\varphi \in C^\infty(M)$:

$$\begin{aligned}
\|\varphi\|_{H^s(M)} &= \langle (\mathbb{1} + \mathrm{Op}(\langle \xi \rangle^{s/2})^* \,\mathrm{Op}(\langle \xi \rangle^{s/2}))\varphi, (\mathbb{1} + \mathrm{Op}(\langle \xi \rangle^{s/2})^* \,\mathrm{Op}(\langle \xi \rangle^{s/2}))\varphi \rangle_{L^2(M)} \\
&\geq \|\varphi\|^2_{L^2(M)},
\end{aligned}$$

and thus $H^s(M) \subset L^2(M)$ is a continuous embedding. Moreover, $C^\infty(M) \subset H^s(M)$ is dense (by construction) and thus $H^s(M) \subset L^2(M)$ is also dense. We have the elementary properties:

Lemma 3.2.17. *The following holds:*

(i) $\big(H^s(M), \langle \bullet, \bullet \rangle_{H^s(M)}\big)$ *is a Hilbert space.*

(ii) $\Lambda_s : H^s(M) \to L^2(M)$ *is an isomorphism and an isometry. Moreover, the inverse operator* $\Lambda_s^{-1} : L^2(M) \to H^s(M)$ *is the restriction of a pseudodifferential operator in* $\Psi^{-s}(M)$ *to the space* $L^2(M)$.

(iii) *The embedding* $H^s(M) \hookrightarrow L^2(M)$ *is compact.*

(iv) *More generally, for all* $t < s$, $H^s(M) \hookrightarrow H^t(M)$ *is compact and* $\Lambda_t : H^s(M) \to H^{s-t}(M)$ *is an isomorphism.*

(v) *The space* $(H^s(M), \| \bullet \|_{H^s})$ *is intrinsic, i.e. independent of any choice made during the construction.*

Proof. (1) It is clear that $\langle \bullet, \bullet \rangle_{H^s(M)}$ is a bilinear conjugate symmetric and non-negative inner product. It remains to show that it is definite. But if $\|\varphi\|_{H^s(M)} = 0 \geq \|\varphi\|_{L^2(M)}$, then $\varphi = 0$.

(2) It is tautological that $\|\Lambda_s \varphi\|_{L^2(M)} = \|\varphi\|_{H^s(M)}$ for all $\varphi \in H^s(M)$. Injectivity of $\Lambda_s : H^s(M) \to L^2(M)$ is then obvious. It remains to show surjectivity. For that, let us consider a parametrix $B \in \Psi^{-s}(M)$ as provided by Theorem 3.2.8 i.e. such that $\Lambda_s B = \mathbb{1} + K_1$ and $B \Lambda_s = \mathbb{1} + K_2$, where $K_1, K_2 \in \Psi^{-\infty}(M)$. Observe that $B : L^2 \to H^s$ is continuous since

$$\|Bu\|_{H^s} = \| \underbrace{\Lambda_s B}_{\in \Psi^0} u\|_{L^2} \leq C \|u\|_{L^2},$$

by the L^2-continuity Theorem 3.2.13. The operator K_1 is compact on $L^2(M)$. Moreover:

$$(\mathbb{1} + K_1)(L^2(M)) = \Lambda_s B(L^2(M)) \subset \Lambda_s(H^s(M)) \subset L^2(M).$$

The space on the left-hand side is closed with finite codimension in $L^2(M)$ by Theorem A.2.10. By Lemma A.2.9, this implies that $\Lambda_s(H^s(M))$ is also closed with finite codimension. Let us now show that its codimension is 0; for that, it suffices to compute its orthogonal. Let $v \in L^2(M)$ such that for all $\varphi \in C^\infty(M)$:

$$\langle \Lambda_s \varphi, v \rangle_{L^2} = 0 = \langle \varphi, \Lambda_s v \rangle_{L^2}.$$

(Note that the last bracket is well-defined i.e. $\Lambda_s v$ is well-defined as a distribution.) This implies that $\Lambda_s v = 0$ and thus $v = 0$ by (1). Hence $\Lambda_s : H^s(M) \to L^2(M)$ is an isomorphism and an isometry.

The operator $\Lambda_s^{-1} : L^2(M) \to H^s(M)$ is therefore well-defined. Moreover, $(B - \Lambda_s^{-1})\Lambda_s = K_2$. Hence:

$$(B - \Lambda_s^{-1})\Lambda_s B = (B - \Lambda_s^{-1})(\mathbb{1} + K_1) = K_2 B,$$

that is

$$\Lambda_s^{-1} = B + BK_1 - K_2 B - \Lambda_s^{-1} K_1. \tag{3.2.6}$$

The equality holds *a priori* only on $L^2(M)$ since the operator Λ_s^{-1} is only defined as a map $L^2(M) \to H^s(M)$ so far. Nevertheless, we observe that Λ_s^{-1} maps $C^\infty(M) \to C^\infty(M)$ boundedly. Indeed, similarly to (3.2.6), we can use the parametrix to obtain

$$\Lambda_s^{-1} = B - BK_1 + K_2(B - \Lambda_s^{-1})$$

and all these terms are bounded as $C^\infty(M) \to C^\infty(M)$. (For $K_2\, \Lambda_s^{-1}$, one uses the factorization:

$$C^\infty(M) \xrightarrow{\iota} L^2(M) \xrightarrow{\Lambda_s^{-1}} H^s(M) \xrightarrow{K_2} C^\infty(M),$$

where ι is the embedding.) We then go back to (3.2.6). We claim that the operator $\Lambda_s^{-1}K_1$ is smoothing: indeed, since K_1 is smoothing we have the following composition of bounded operators:

$$\mathcal{D}'(M) \xrightarrow{K_1} C^\infty(M) \xrightarrow{\Lambda_s^{-1}} C^\infty(M).$$

As a consequence, (3.2.6) shows that $\Lambda_s^{-1} = B + \mathcal{O}_{\Psi^{-\infty}}(1)$, that is $\Lambda_s^{-1} \in \Psi^{-s}(M)$.

(3) To show compactness, we look at the bounded operator $\Lambda_s^{-1} : L^2(M) \to L^2(M)$. Note that $\Lambda_s^{-1} \in \Psi^{-s}(M)$ so it is compact on $L^2(M)$ by Corollary 3.2.16. This operator can be factorized as

$$L^2(M) \xrightarrow{\Lambda_s^{-1}} H^s(M) \xrightarrow{\iota} L^2(M),$$

where the first operator is an isometry, that is $K = \iota \circ U$ where $K := \Lambda_s^{-1}|_{L^2 \to L^2}$ (compact), $U := \Lambda_s^{-1}|_{L^2 \to H^s}$ (unitary) and $\iota : H^s(M) \to L^2(M)$ is the embedding. Hence $\iota = KU^{-1}$ so ι is compact.

(4) We have the following commutative diagram:

$$\begin{tikzcd} H^s(M) \arrow[r, "\iota"] \arrow[d, "\Lambda_s^{-1}"] & H^t(M) \arrow[d, "\Lambda_t"] \\ L^2(M) \arrow[r, "\Lambda_t\Lambda_s^{-1}"'] & L^2(M) \end{tikzcd}$$

that is

$$\Lambda_t\Lambda_s^{-1}|_{L^2 \to L^2} = \Lambda_t|_{H^t \to L^2} \circ \iota \circ \Lambda_s^{-1}|_{L^2 \to H^s}.$$

Note that $\Lambda_t\Lambda_s^{-1} \in \Psi^{t-s}(M)$ and since $t < s$, this operator is compact on $L^2(M)$. Since $\Lambda_t|_{H^t \to L^2}, \Lambda_s^{-1}|_{L^2 \to H^s}$ are both isometries, we conclude that ι is compact.

(5) It remains to prove that $H^s(M) = \Lambda_s^{-1}L^2(M)$ is intrinsic. Assume that we make other choices during the construction and define $H'^s(M) := \Lambda'^{-1}_s \Psi^s(M)$ for some other pseudodifferential operator $\Lambda'_s \in L^2(M)$. Then $\Lambda_s = A\Lambda'_s$ for some $A \in \Psi^0(M)$ and thus:

$$\|\varphi\|_{H^s(M)} = \|\Lambda_s\varphi\|_{L^2} = \|A\Lambda'_s\varphi\|_{L^2} \leq C\|\Lambda'_s\varphi\|_{L^2} = C\|\varphi\|_{H'^s(M)}$$

by Theorem 3.2.13. The reverse inequality is also straightforward and this shows that $H^s(M) = H'^s(M)$ with equivalent norms. $\square$

An important remark that follows directly from the proof is:

Remark 3.2.18. If $A \in \Psi^m(M)$ is elliptic, then any continuous linear operator

$B : C^\infty(M) \to \mathcal{D}'(M)$ satisfying $AB - \mathbb{1}, BA - \mathbb{1} \in \Psi^{-\infty}(M)$ is bounded as a map

$$\mathcal{D}'(M) \to \mathcal{D}'(M), C^\infty(M) \to C^\infty(M),$$

and pseudodifferential. We did the proof for Λ_s but it works more generally for any other operator. In particular, the proof shows that the inverse of a pseudodifferential operator is also pseudodifferential.

3.2.3.2 Duality

We now set for $s > 0$, $\Lambda_{-s} := \Lambda_s^{-1} \in \Psi^{-s}(M)$ and this is a pseudodifferential operator of order $-s$ by (3) of the previous Lemma. We let $H^{-s}(M)$ be the completion of $C^\infty(M)$ for the norm

$$\|\varphi\|_{H^{-s}} := \|\Lambda_{-s}\varphi\|_{L^2(M)}$$

Note that by Theorem 3.2.13, we have:

$$\|\varphi\|_{H^{-s}} \leq C\|\varphi\|_{L^2(M)},$$

and thus $L^2(M) \hookrightarrow H^{-s}(M)$ is continuous. The space H^{-s} can be identified with the dual $(H^s(M))'$ by means of the L^2-scalar product.

Indeed, given $T \in (H^s(M))'$, thanks to the Riesz representation Theorem, it can be represented by an element $T^\sharp \in H^s(M)$, that is for all $\varphi \in C^\infty(M)$:

$$T(\varphi) = \langle T^\sharp, \varphi\rangle_{H^s(M)} = \langle \Lambda_s T^\sharp, \Lambda_s\varphi\rangle_{L^2} = \langle \Lambda_s^2 T^\sharp, \varphi\rangle_{L^2}.$$

Also note that

$$\|T\|_{(H^s)'} = \sup_{\|\varphi\|_{H^s}=1} |T(\varphi)| = \|T^\sharp\|_{H^s} = \|\Lambda_s^2 T^\sharp\|_{H^{-s}},$$

that is

$$\Phi : (H^s(M))' \to H^{-s}(M), \quad T \mapsto \Lambda_s^2 T^\sharp,$$

is an isometry.

Lemma 3.2.19. *Φ is an isomorphism and an isometry.*

Proof. By the previous discussion, Φ is injective and isometric. It suffices to check that it is surjective. If $u \in H^{-s}(M)$, then $\Lambda_{-s}u \in L^2(M)$ and $\Lambda_{-s}^2 u \in H^s(M)$ can then be identified with a covector $T := (\Lambda_{-s}^2)^\sharp \in (H^s(M))'$. □

In other words, we will keep identifying elements of $H^{-s}(M)$ with linear functionals on $H^s(M)$, i.e. distributions on $H^s(M)$. Also note that there is a natural extension of the L^2-pairing to $H^{-s} \times H^s$, namely:

Lemma 3.2.20. *The L^2-pairing:*

$$C^\infty(M) \times C^\infty(M) \ni (f, g) \mapsto \langle f, g\rangle_{L^2(M)}$$

extends by continuity to

$$H^{-s}(M) \times H^s(M) \ni (f, g) \mapsto \langle f, g\rangle_{L^2(M)}.$$

Proof. It suffices to prove that for all $f, g \in C^\infty(M)$, $|\langle f, g\rangle_{L^2}| \leq C\|f\|_{H^{-s}}\|g\|_{H^s}$. But by Cauchy-Schwarz:

$$|\langle f, g\rangle_{L^2}| = |\langle \Lambda_{-s}f, \Lambda_s g\rangle_{L^2}| \leq \|\Lambda_{-s}f\|_{L^2}\ \|\Lambda_s g\|_{L^2} = \|f\|_{H^{-s}}\|g\|_{H^s}.$$

□

Exercise 3.2.21. Let $x_0 \in M$ and $\delta_{x_0} \in \mathcal{D}'(M)$ be the Dirac mass at x_0. Show that $\delta_{x_0} \in H^{-n/2-}(M)$ (that is $\delta_{x_0} \in H^{-n/2-\varepsilon}(M)$ for all $\varepsilon > 0$) but $\delta_{x_0} \notin H^{-n/2}$.

Remark 3.2.22. There is also another way of defining Sobolev spaces by chopping distributions in small pieces, pulling them back to $\mathbb{R}^n$ and then using the standard Sobolev norm on the Euclidean space. But it is not difficult to check that this also gives another equivalent way of defining the Sobolev spaces $H^s(M)$ (with equivalent norms). For integer values of $k \in \mathbb{N}$, it can be shown that $H^k(M)$ has a more geometric description. We let g be an arbitrary metric on M, ∇ be the Levi-Civita connection and $\nabla^k : C^\infty(M) \to C^\infty(M, \mathrm{Sym}^k T^*M)$ be the k-th symmetric covariant derivative. It is defined in the following way: consider $x \in M, v \in T_xM$ and $t \mapsto \gamma(t)$ the local geodesic segment such that $\gamma(0) = x, \dot\gamma(0) = v$. Then

$$\nabla^k\varphi(v, ..., v) := \frac{d^k}{dt^k}\varphi(\gamma(t))\Big|_{t=0}.$$

Note that this is a symmetric k-homogeneous polynomial in the v variable so it is completely determined by its evaluation on the elements $v^{\otimes k}$.

In particular, for $k = 1$, we retrieve the usual definition of the gradient $\nabla\varphi$. We then set

$$\|\varphi\|^2_{H^k_{\mathrm{geo}}} := \sum_{j \leq k} \|\nabla^j\varphi\|^2_{L^2(M,\mathrm{Sym}^j T^*M)} \tag{3.2.7}$$

and we define $H^k_{\mathrm{geo}}(M)$ to be the completion of $C^\infty(M)$ with respect to the norm (3.2.7). It can be checked that for all $k \geq 0$, the space $H^k_{\mathrm{geo}}(M)$ does not depend on the choice of metric g and the norms obtained for two different choices of metrics are equivalent. Moreover, $H^k_{\mathrm{geo}}(M) = H^k(M)$ with equivalence of norms.

Exercise 3.2.23. Prove the remark.

3.2.3.3 Boundedness of ΨDOs

We now prove the important boundedness result. They will be almost tautological consequences of the previous results.

Theorem 3.2.24. *Let $A \in \Psi^m(M)$. Then, for all $s \in \mathbb{R}$, $A : H^{s+m}(M) \to H^s(M)$ is bounded.*

Now that we have all the tools, the proof is almost tautological.

Proof. Boundedness of $A : H^{s+m}(M) \to H^s(M)$ is equivalent to the boundedness of $\Lambda_s A \Lambda^{-1}_{s+m} : L^2(M) \to L^2(M)$. But $\Lambda_s A \Lambda^{-1}_{s+m} \in \Psi^0(M)$ so it is bounded on $L^2(M)$ by Theorem 3.2.13. □

A useful corollary is the following:

Corollary 3.2.25. *Let $K \in \Psi^{-\infty}(M)$. Then for all $s, t \in \mathbb{R}$, $K : H^s(M) \to H^t(M)$ is bounded. In particular, $K : H^s(M) \to H^s(M)$ is compact.*

Proof. The first part of the statement follows directly from Theorem 3.2.24. The second part is a consequence of the fact that $K : H^s(M) \to H^s(M)$ can be factored as $H^s(M) \xrightarrow{K} H^{s+2022}(M) \overset{\iota}{\hookrightarrow} H^s(M)$, where $\iota : H^{s+2022}(M) \to H^s(M)$ is the compact embedding. □

Actually, one can check that the converse is also true:

Exercise 3.2.26. Show that if $K : \mathcal{D}'(M) \to \mathcal{D}'(M)$ is a linear operator with the property that $K : H^s(M) \to H^t(M)$ is bounded for every $s, t \in \mathbb{R}$, then it is smoothing. (Hint: use Lemma 1.2.3 and the embedding estimates of Theorem **??**.)

3.2.4 Exercises

Exercise 1

Let $U \subset M$ be an open subset of M diffeomorphic to $B(0,1) \subset \mathbb{R}^n$ and ω be a volume form on U. Show that there exists $r > 0$ and $\kappa : U \to B(0,r)$ such that $\kappa_*\omega = \mathrm{d}x$, the standard volume form in $\mathbb{R}^n$.

3.3 Properties of elliptic operators

3.3.1 Index of an elliptic operator

We now study the Fredholm index of elliptic operators. This is an important achievement of the theory and a (small) step towards the Atiyah-Singer index theorem, see [BGV92].

Theorem 3.3.1. *Let $A \in \Psi^m(M)$ be an elliptic pseudodifferential operator of order $m \in \mathbb{R}$. Then:*

(i) *For all $s \in \mathbb{R}$, $A \in \mathcal{F}(H^{s+m}(M), H^s(M))$ and $A \in \mathcal{F}(C^\infty(M), C^\infty(M))$.*

(ii) *One has: $\ker A, \ker A^* \subset C^\infty(M)$. Thus the Fredholm index $\mathrm{ind}(A)$ of A is independent of s ans given by the formula*

$$\mathrm{ind}(A) = \dim\ker A - \dim\ker A^*.$$

In particular, if $\dim\ \ker A = \dim\ker A^* = 0$, then $A : H^{s+m}(M) \to H^s(M)$ is an isomorphism.

(iii) *For any $R \in \Psi^{m'}(M)$, where $m' < m$, $\mathrm{ind}(A + R) = \mathrm{ind}(A)$.*

Proof. Fix $s \in \mathbb{R}$, $A \in \Psi^m(M)$ and consider a parametrix $B \in \Psi^{-m}(M)$ such that $AB = \mathbb{1} + K_1$ and $BA = \mathbb{1} + K_2$ with $K_1, K_2 \in \Psi^{-\infty}(M)$.

(1, 2) In order to show that $A : H^{s+m}(M) \to H^s(M)$ is Fredholm, we need to show that $\ker A$ is finite-dimensional and $\operatorname{ran} A$ is closed with finite codimension. First of all, if $\varphi \in H^{s+m}(M)$ satisfies $A\varphi = 0$, then $BA\varphi = (\mathbb{1} + K_2)\varphi = 0$ which implies that $\varphi \in C^\infty(M)$ (since K_2 is smoothing) and $\varphi \in \ker(\mathbb{1}+K_2)$. In particular, we obtain $\ker A|_{H^{s+m}} \subset \ker(\mathbb{1} + K_2)|_{L^2(M)}$ for all $s \in \mathbb{R}$. So we get two things: 1) that the kernel is finite-dimensional and 2) that it is independent of s.

Let us now deal with the range. Using the boundedness of $B : H^s(M) \to H^{s+m}(M)$ by Theorem 3.2.24, we get:

$$AB(H^s(M)) = (\mathbb{1} + K_1)(H^s(M)) \subset A(H^{s+m}(M)) \subset H^s(M).$$

Since $K_1 : H^s(M) \to H^s(M)$ is compact by Corollary 3.2.25, we get that the space on the left-hand side is closed with finite codimension and thus by Lemma A.2.9, the intermediate space is also closed with finite codimension in $H^s(M)$. By Lemma A.5.4, the dimension of the cokernel is also that of $\ker A^*$, where

$$A^* : (H^s(M))' = H^{-s}(M) \to (H^{s+m}(M))' = H^{-s-m}(M).$$

Now, the operator $A^* \in \Psi^m(M)$ is also elliptic so $\ker A^*|_{H^{-s}} \subset C^\infty(M)$ is finite-dimensional and does not depend on s.

(3) If $R \in \Psi^{m'}(M)$ with $m' < m$, then $R : H^{s+m}(M) \to H^s(M)$ is compact. We then use the stability of the index by compact perturbations, see Lemma A.5.10. □

Corollary 3.3.2. *If $A \in \Psi^m(M)$ is elliptic and formally self-adjoint, namely $A = A^*$, then* $\operatorname{ind}(A) = 0$. *As a consequence $A : H^{s+m}(M) \to H^s(M)$ is an isomorphism iff its is injective iff it is surjective.*

Proof. Immediate. □

A direct consequence is the following is that for all $t, s \in \mathbb{R}$, $\Lambda_t : H^s(M) \to H^{s-t}(M)$ is an isomorphism.

3.3.2 Sharp parametrix

We now refine the parametrix construction. We let $A \in \Psi^m(M)$ be an elliptic pseudodifferential operator. The manifold M is endowed with a density; we thus have a $L^2(M)$ space and we can build from this a canonical “inverse" $B \in \Psi^{-m}(M)$ for the elliptic operator A.

Theorem 3.3.3. *Let $A \in \Psi^m(M)$ be elliptic. Then there exists $B \in \Psi^{-m}(M)$ such that*

$$BA = \mathbb{1} - \Pi_{\ker A}, \quad AB = \Pi_{\operatorname{ran} A} = \mathbb{1} - \Pi_{(\operatorname{ran} A)^\perp},$$

where $\Pi_{\ker A}$ is the L^2-orthogonal projection onto $\ker A$ *and $\Pi_{\operatorname{ran} A}$ is the H^{-m}-orthogonal projection onto* $\operatorname{ran} A$.

Proof. We know that $\ker A \subset C^\infty(M)$ and $\ker A$ is finite-dimensonal. As a consequence, we can write

$$L^2(M) = \ker A \oplus^\perp F, \tag{3.3.1}$$

where F is the L^2-orthogonal to $\ker A$. The space $AF = \mathrm{ran}(A|_{L^2(M)}) \subset H^{-m}(M)$ is closed with finite codimension. We let $L \subset H^{-m}(M)$ be the orthogonal complement of AF (with respect to the $H^{-m}(M)$ inner product). We define the operator $B : H^{-m}(M) \to L^2(M)$ in the following way: $B(Au) = u$ for all $u \in F$ and $B|_L = 0$. It is immediate to see that $BA = \mathbb{1} - \Pi_{\ker A} = \Pi_F$, where $\Pi_{\ker A}$ is the L^2-orthogonal projection onto the first factor in (3.3.1) (and Π_F the orthogonal projection onto the second factor) and that $AB = \Pi_{\mathrm{ran}\, A} = \mathbb{1} - \Pi_L$, where $\Pi_{\mathrm{ran}\, A}$ is the $H^{-m}(M)$-orthogonal projection onto $\mathrm{ran}\, A$.

Let us now show that $\Pi_{\ker A}$ and Π_L are smoothing operators. We let $(e_1, ..., e_k)$ be an L^2-orthonormal basis of $\ker A$. Then

$$\Pi_{\ker A} = \sum_{j=1}^{k} \langle \bullet, e_j \rangle_{L^2} e_j,$$

and this operator has Schwartz kernel

$$K_{\Pi_{\ker A}}(x, y) = \sum_{j=1}^{k} e_j(x) \overline{e_j(y)}.$$

Note that by ellipticity of A, the e_j's are smooth functions on M, so it is clear that $K_{\Pi_{\ker A}} \in C^\infty(M \times M)$. Similarly, we may first identify L with the kernel of $A^* : H^m(M) \to L^2(M)$. Indeed, we have: for all $u \in L^2(M), f \in L$,

$$\langle Au, f \rangle_{H^{-m}(M)} = \langle Au, \Lambda_{-2m} f \rangle_{L^2} = \langle u, A^* \Lambda_{-2m} f \rangle_{L^2},$$

that is $A^* \Lambda_{-2m} f = 0$ and $\Lambda_{-2m} f \in H^m(M)$. By ellipticity of A^*, we get that $f \in C^\infty(M)$ (i.e. $L \subset C^\infty(M)$) and $\Lambda_{2m} : \ker A^* \to L$ is an isometry. Taking $(f_1, ..., f_p)$, an $H^{-m}(M)$-orthonormal basis of L, we have

$$\Pi_L = \sum_{j=1}^{n} \langle \bullet, f_j \rangle_{H^{-m}} f_j = \langle \bullet, \Lambda_{-2m} f_j \rangle_{L^2} f_j,$$

and this operator has smooth Schwartz kernel given by

$$K_{\Pi_L}(x, y) = f_j(x) \overline{(\Lambda_{-2m} f_j)(y)}.$$

It now remains to show that $B \in \Psi^{-m}(M)$, that is that the operator B previously defined is the restriction to $H^{-m}(M)$ of a pseudodifferential operator acting on $\mathcal{D}'(M)$. But this follows exactly the arguments in the proof as Lemma 3.2.17, see also Remark 3.2.18. ☐

3.3.3 Elliptic estimates

First of all, we start with a basic bootstrap result:

Lemma 3.3.4 (Elliptic regularity)**.** *Let $u \in \mathcal{D}'(M)$ and $A \in \Psi^m(M)$. Assume that $Au \in H^s(M)$. Then $u \in H^{s+m}(M)$.*

Proof. This is a direct consequence of the existence of a parametrix: take $B \in \Psi^{-m}(M)$ such that $BA - \mathbb{1} \in \Psi^{-\infty}(M)$. Then $BAu = u + \mathcal{O}_{C^\infty}(1)$. Since $B : H^s \to H^{s+m}$ is bounded, $BAu \in H^{s+m}$ and thus $u \in H^{s+m}$. □

Example 3.3.5. For instance, we retrieve the well-known statement concerning harmonic functions: if $u \in \mathcal{D}'(M)$ satisfies $\Delta u = 0$, then $u \in C^\infty(M)$.

We now refine quantitatively the previous statement:

Proposition 3.3.6 (Global elliptic estimate)**.** *Let $A \in \Psi^m(M)$ and assume that A is elliptic. Then, for every $N \in \mathbb{N}, s \in \mathbb{R}$, there exists $C > 0$ such that for every $\varphi \in C^\infty(M)$:*

$$\|\varphi\|_{H^{s+m}(M)} \leq C \left(\|A\varphi\|_{H^s(M)} + \|\varphi\|_{H^{-N}(M)}\right) \tag{3.3.2}$$

Moreover, if A is injective, then for all $s \in \mathbb{R}$, there exists $C > 0$ such that for all $\varphi \in C^\infty(M)$:

$$\|\varphi\|_{H^{s+m}(M)} \leq C\|A\varphi\|_{H^s(M)}.$$

By density of $C^\infty(M)$ in $H^{s+m}(M)$, these estimates also hold for $\varphi \in H^{s+m}(M)$.

More generally, as long as the right-hand side of the inequality makes sense, we get that $\varphi \in H^{s+m}(M)$ by Lemma 3.3.4 and the inequality can be applied.

Proof. Take a parametrix $B \in \Psi^{-m}(M)$ such that $BA = \mathbb{1} + K, K \in \Psi^{-\infty}(M)$. Then using the boundedness of $B : H^m \to H^{s+m}, K : H^{-N} \to H^{s+m}$, we get:

$$\|\varphi\|_{H^{s+m}} \leq \|BA\varphi\|_{H^{s+m}} + \|K\varphi\|_{H^{s+m}} \leq C\left(\|A\varphi\|_{H^s} + \|\varphi\|_{H^{-N}}\right).$$

When A is injective, it suffices to take the sharp parametrix provided by Theorem 3.3.3. □

Remark 3.3.7. We also mention the local elliptic estimate. Given an open set $U \subset M$, we say that A is elliptic on U if it is elliptic at all points in T^*U. Let $A \in \Psi^m(M)$ and $U \subset M$. Assume that A is elliptic on U. Then, for every open neighborhood V of U, for every $N \in \mathbb{N}, s \in \mathbb{R}$, there exists $C > 0$ such that for all $\varphi \in C^\infty_{\mathrm{comp}}(U)$:

$$\|\varphi\|_{H^{s+m}(U)} \leq C\left(\|A\varphi\|_{H^s(V)} + \|\varphi\|_{H^{-N}(V)}\right)$$

Chapter 4

Advanced topics

The aim of this chapter is to discuss more advanced topics: elliptic complexes and the Hodge Theorem, Gårding's inequalities, spectral theory of elliptic pseudodifferential operators and propagation of singularities.

Contents

4.1 Elliptic complexes

4.1.1 General discussion

We now introduce the notion of *elliptic complex.* We consider a finite sequence of Hermitian vector bundles $E_0, ..., E_N \to M$ and (pseudo)differential operators of order $m > 0$:

$$... \to C^\infty(M, E_j) \stackrel{d_j}{\to} C^\infty(M, E_{j+1}) \stackrel{d_{j+1}}{\to} C^\infty(M, E_{j+2}) \to ... \tag{4.1.1}$$

For every $(x, \xi) \in T_0^*M$, it induces a long sequence of finite-dimensional vector spaces:

$$0 \to E_0(x) \to ... \to E_j(x) \stackrel{\sigma_{d_j}(x,\xi)}{\to} E_{j+1}(x) \stackrel{\sigma_{d_{j+1}}(x,\xi)}{\to} E_{j+2}(x) \to ... \to E_N(x) \to 0 \tag{4.1.2}$$

Definition 4.1.1. We say that (4.1.1) defines a *differential complex* if the operators d_j are differential of same order and satisfy $d_{j+1} \circ d_j = 0$ for all $0 \leq j \leq N-1$. We say that the complex is *elliptic* if the long sequence (4.1.2) is exact.

The relation $d_{j+1} \circ d_j = 0$ implies at the level of principal symbols that $\sigma_{d_{j+1}} \circ \sigma_{d_j} = 0$, that is $\operatorname{ran} \sigma_{d_j} \subset \ker \sigma_{d_{j+1}}$ and the elliptic property is equivalent to requiring that this inclusion is an equality. We let

$$\Delta_j := d_j^* d_j + d_{j-1} d_{j-1}^*. \tag{4.1.3}$$

When $m = 1$, we will call this operator the *Hodge Laplacian.* More generally, if the complex is differential of order $m > 0$, it is a differential operator of order $2m$ acting on $C^\infty(M, E_j) \to C^\infty(M, E_j)$. We let $\Delta := \oplus_{j \geq 0} \Delta_j$ be the sum operator acting on the sections of $E := \oplus_{j \geq 0} E_j \to M$.

We also define the operators

$$D_j : C^\infty(M, E_j) \to C^\infty(M, E_{j-1}) \oplus C^\infty(M, E_{j+1}), \qquad D_j := d_{j-1}^* + d_j.$$

We set:

$$E_{\text{even}} := \oplus_{j \geq 0} E_{2j}, E_{\text{odd}} := \oplus_{j \geq 0} E_{2j+1}, E := \oplus_{j \geq 0} E_j,$$

and $D_+ := \oplus_{j \geq 0} D_{2j}, D_- := \oplus_{j \geq 0} D_{2j+1}$. Note that:

$$D_\pm : C^\infty(M, E_{\text{even/odd}}) \to C^\infty(M, E_{\text{odd/even}}),$$

and $D_+^* = D_-$. Eventually, we set:

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix},$$

where this is understood as an operator acting on the decomposition $E = E_{\text{even}} \oplus E_{\text{odd}}$ as $D = d + d^*$. This is a differential operator of order m acting on $C^\infty(M, E)$ which satisfies the equality $D^2 = \Delta$ since:

$$\begin{aligned} D^2(f_j) = D(d_j f_j + d_{j-1}^* f_j) &= d_{j+1} d_j f_j + d_j^* d_j f_j + d_{j-2}^* d_{j-1}^* f_j + d_{j-1} d_{j-1}^* f_j \\ &= d_j^* d_j f_j + d_{j-1} d_{j-1}^* f_j = \Delta_j f_j. \end{aligned}$$

It is also formally self-adjoint. Note that

$$\ker D_+ = \oplus_j \ker \Delta_{2j}, \ \ker D_- = \oplus_j \ker \Delta_{2j+1}.$$

Indeed, if $D_+ u = 0$, then $D\widetilde{u} = 0$ where $\widetilde{u} = (u, 0)^\top$ and thus $D^2 \widetilde{u} = \Delta \widetilde{u} = 0$, that is $\Delta_{2j} u_{2j} = 0$. Conversely, if $\Delta u = 0$ and $u = \sum_j u_{2j}$, then using $D^* D = \Delta$, we get $Du = 0 = D_+ u$.

Remark 4.1.2. The operator D is called a *Dirac operator*. It is of fundamental importance in the Atiyah-Singer index theorem. This theorem relates the index $\operatorname{ind}(D)$ of the operator (which we have defined already) to some geometric quantity.

Lemma 4.1.3. *Let $\mathcal{C}$ be a differential complex. The following items are equivalent:*

(i) *$\mathcal{C}$ is elliptic;*

(ii) *$\Delta : C^\infty(M, E) \to C^\infty(M, E)$ is elliptic;*

(iii) *$D : C^\infty(M, E) \to C^\infty(M, E)$ is elliptic;*

(iv) *$D_+ : C^\infty(M, E_{\mathrm{even}}) \to C^\infty(M, E_{\mathrm{odd}})$ is elliptic.*

In particular, a necessary condition for the complex to be elliptic is that $E_{\mathrm{even}} \simeq E_{\mathrm{odd}}$.

Proof. Items (2) and (3) are obviously equivalent since $\sigma_D(x, \xi) \in \operatorname{End}(E_x)$ is invertible if and only if $\sigma_\Delta(x, \xi) = \sigma_D(x, \xi)^2$ is invertible. As to the equivalence between (3) and (4), we have

$$\sigma_D = \begin{pmatrix} 0 & \sigma_{D_-} \\ \sigma_{D_+} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sigma_{D_+}^* \\ \sigma_{D_+} & 0 \end{pmatrix},$$

so σ_D is invertible if and only if $\dim(E_{\mathrm{even}}) = \dim(E_{\mathrm{odd}})$ and σ_{D_+} is invertible.

We now assume $\mathcal{C}$ is elliptic. Since the operators are differential, they have homogeneous principal symbols and it thus suffices to check that $\sigma_{\Delta_j}(x, \xi) \in \operatorname{End}(E_j)$ is everywhere invertible on $T_0^* M$. Assume that $f_j \in E_j(x)$ satisfies $\sigma_{\Delta_j}(x, \xi) f_j = 0$. Then:

$$\begin{aligned} 0 = \langle \sigma_{\Delta_j}(x, \xi) f_j, f_j \rangle &= \langle \sigma_{d_j^* d_j + d_{j-1} d_{j-1}^*} f_j, f_j \rangle \\ &= \langle \sigma_{d_j}^* \sigma_{d_j} f_j, f_j \rangle + \langle \sigma_{d_{j-1}} \sigma_{d_{j-1}}^* f_j, f_j \rangle \\ &= \|\sigma_{d_j} f_j\|^2 + \|\sigma_{d_{j-1}}^* f_j\|^2, \end{aligned}$$

that is $\sigma_{d_j} f_j = 0 = \sigma_{d_{j-1}}^* f_j$. By assumption, $\operatorname{ran} \sigma_{d_{j-1}} = \ker \sigma_{d_j}$ so this implies that $f_j = \sigma_{d_{j-1}} f_{j-1}$ and thus

$$\sigma_{d_{j-1}}^* f_j = \sigma_{d_{j-1}}^* \sigma_{d_{j-1}} f_{j-1} = 0,$$

that is $\sigma_{d_{j-1}} f_{j-1} = 0 = f_j$. Hence σ_{Δ_j} is invertible.

Now conversely, if σ_Δ is invertible and $f_j \in \ker \sigma_{d_j}$, we need to show that $f_j \in \operatorname{ran} \sigma_{d_{j-1}}$. Let v_j be such that

$$f_j = \sigma_{\Delta_j} v_j = \sigma_{d_j}^* \sigma_{d_j} v_j + \sigma_{d_{j-1}} \sigma_{d_{j-1}}^* v_j. \tag{4.1.4}$$

We claim that $\sigma_{d_j} v_j = 0$. Indeed,

$$0 = \sigma_{d_j} f_j = \sigma_{d_j} \sigma_{d_j}^* (\sigma_{d_j} v_j) + 0$$

and thus

$$\Delta_{j+1}(\sigma_{d_j} v_j) = \sigma_{d_j} \sigma_{d_j}^* (\sigma_{d_j} v_j) + \sigma_{d_{j+1}}^* \sigma_{d_{j+1}} (\sigma_{d_j} v_j) = 0.$$

By assumption $\sigma_{\Delta_{j+1}}$ is invertible so $\sigma_{d_j} v_j = 0$ and thus by (4.1.4), we get that $f_j \in \operatorname{ran} \sigma_{d_{j-1}}$. $\square$

Example 4.1.4. One of the most important example is provided by the complex of differential forms, namely:

$$\ldots \to C^\infty(M, \Lambda^j T^* M) \xrightarrow{d} C^\infty(M, \Lambda^{j+1} T^* M) \xrightarrow{d} C^\infty(M, \Lambda^{j+2} T^* M) \to \ldots,$$

which satisfies indeed $d \circ d = 0$. This complex is elliptic since the Hodge Laplacian Δ is elliptic by §3.1.4, Exercise 2.

4.1.2 Hodge's Theorem

We now state the general Hodge Theorem.

Theorem 4.1.5 (Hodge)**.** *Let $\mathcal{C}$ be an elliptic differential complex. Then for all $0 \leq j \leq N$, we have:*

$$L^2(M, E_j) = \ker \Delta_j \oplus^\perp \operatorname{ran} d_{j-1}|_{H^1(M, E_{j-1})} \oplus^\perp \operatorname{ran} d_j^*|_{H^1(M, E_{j+1})},$$

with the convention that the summands are empty when $j-1 = -1$ and $j+1 = N+1$. Each summand is a closed subspace of $L^2(M, E_j)$ and the sum is orthogonal for the L^2-scalar product. It also holds in any other Sobolev space $H^s(M, E_j), s \in \mathbb{R}$ instead of $L^2(M, E_j)$[1] *and also in the smooth case:*

$$C^\infty(M, E_j) = \ker \Delta_j \oplus \operatorname{ran} d_{j-1}|_{C^\infty(M, E_{j-1})} \oplus \operatorname{ran} d_j^*|_{C^\infty(M, E_{j+1})}. \tag{4.1.5}$$

Moreover:

$$\begin{aligned} \ker d_j|_{C^\infty(M, E_j)} &= \ker \Delta_j \oplus \operatorname{ran} d_{j-1}|_{C^\infty(M, E_{j-1})}, \\ \ker d_j^*|_{C^\infty(M, E_j)} &= \ker \Delta_j \oplus \operatorname{ran} d_j^*|_{C^\infty(M, E_{j+1})}. \end{aligned}$$

[1]Namely:

$$H^s(M, E_j) = \ker \Delta_j \oplus \operatorname{ran} d_{j-1}|_{H^{s+1}(M, E_{j-1})} \oplus \operatorname{ran} d_j^*|_{H^{s+1}(M, E_{j+1})},$$

We set:

$$H^j(\mathcal{C}) := \ker d_j|_{C^\infty(M,E_j)}/d_{j-1}(C^\infty(M,E_{j-1}))$$

and call it the j-th cohomology group *of the complex. We then have:*

$$\ker \Delta_j \xrightarrow{\sim} H^j(\mathcal{C}).$$

In particular, the cohomology groups are finite-dimensional.

Technically, the Hodge theorem only consists of the very last part of the statement on the isomorphism $\ker \Delta_j \xrightarrow{\sim} H^j(\mathcal{C})$. The arrow is simply the projection composed with the embedding:

$$\ker \Delta_j \xrightarrow{\iota} \ker d_j|_{C^\infty(M,E_j)} \xrightarrow{\pi} \ker d_j|_{C^\infty(M,E_j)}/d_{j-1}(C^\infty(M,E_{j-1})).$$

The most important case is given by the complex of differential forms. In this case,

$$\begin{aligned} H^j(M) &:= \ker d|_{C^\infty(M,\Lambda^j T^*M)}/d(C^\infty(M,\Lambda^{j-1}T^*M)) \\ &= \text{closed forms/exact forms} \end{aligned}$$

is called the j-th cohomology group of the manifold.

For the proof, we will need the following lemma:

Lemma 4.1.6. *Let $E_1, E_2 \to M$ be two Hermitian vector bundles and $P : C^\infty(M,E_1) \to C^\infty(M,E_2)$ be an elliptic pseudodifferential operator of order $m \in \mathbb{R}$ in $\Psi^m(M, E_1 \to E_2)$. Then for all $s \in \mathbb{R}$:*

$$H^s(M,E_1) = \ker P \oplus \operatorname{ran} P^*_{|H^{s+m}(M,E_2)}.$$

This decomposition also holds in the smooth case, namely

$$C^\infty(M,E_1) = \ker P \oplus \operatorname{ran} P^*_{|C^\infty(M,E_2)}.$$

For $s = 0$, the above sum is orthogonal with respect to the L^2-scalar product. In particular, if $E_1 = E_2$ and $P = P^*$ is formally self-adjoint, then

$$H^s(M,E) = \ker P \oplus \operatorname{ran} P_{|H^{s+m}(M,E)}.$$

Proof. Let us prove the smooth case for instance. First of all, we show that $\ker P \cap \operatorname{ran} P^*_{|C^\infty(M,E_2)} = \{0\}$. If $Pf_1 = 0$ and $f_1 = P^*f_2$, then $PP^*f_2 = 0$, that is $\langle PP^*f_2, f_2\rangle_{L^2} = \|P^*f_2\|^2 = 0$ and thus $f_1 = P^*f_2 = 0$. We now take $f \in C^\infty(M,E_1)$. First of all, let us assume that we can decompose $f = h + v$, with $h \in \ker P, v = P^*w \in \operatorname{ran} P^*_{|C^\infty(M,E_2)}$. Then $Pf = PP^*w$. Let $Q \in \Psi^{-2m}(M,E_2)$ be a sharp parametrix for PP^* as in Theorem 3.3.3, namely such that $QPP^* = \mathbb{1} - \Pi_{\ker PP^*} = \mathbb{1} - \Pi_{\ker P^*}$ (here $\Pi_{\ker P^*}$ is the L^2-orthogonal projection onto $\ker P^*$) and $PP^*Q = \Pi_{\operatorname{ran} PP^*}$ (here $\Pi_{\operatorname{ran} PP^*}$ is the H^{-2m}-orthogonal projection onto $PP^*(L^2(M,E_2))$). Observe that

$$PP^*(L^2(M,E_2)) = P(H^{-m}(M,E_1)).$$

Indeed, there is an obvious inclusion, these spaces have both finite codimension and

$$\operatorname{codim} PP^*(L^2(M,E_2)) = \dim\ker PP^* = \dim\ker P^* = \operatorname{codim} P(H^{-m}(M,E_1)).$$

Thus $\Pi_{\mathrm{ran}\,PP^*} = \Pi_{\mathrm{ran}\,P}$, where the latter is the H^{-2m}-orthogonal projection onto $P(H^{-m}(M,E_1))$. We then have $QPf = QPP^*w = w - \Pi_{\ker P^*}w = w$, if we further assume that $\Pi_{\ker P^*}w = 0$, that is $v = P^*QPf$. Now, we set $v := P^*QPf \in \mathrm{ran}\, P^*_{|C^\infty(M,E_2)}$ and $h := f - v$. Observe that

$$Ph = Pf - Pv = Pf - PP^*QPf = Pf - \Pi_{\mathrm{ran}\,P}Pf = 0.$$

This proves the lemma. □

The proof of Theorem 4.1.5 is now straightforward.

Proof of Theorem 4.1.5. We apply the previous lemma with $P = D_j = d_j + d_{j-1}^* : C^\infty(M,E_j) \to C^\infty(M,E_{j-1}\oplus E_{j+1})$ which satisfies $\ker D_j = \ker \Delta_j$. Any $f \in C^\infty(M,E)$ can be uniquely decomposed as $f = h + D_j^* w$, where $h \in \ker\Delta_j, w \in C^\infty(M,E_{j-1}\oplus E_{j+1})$, that is

$$f = h + d_{j-1}w_{j-1} + d_{j-1}^* w_{j+1},$$

with $w_{j-1} \in C^\infty(M,E_{j-1}), w_{j+1} \in C^\infty(M,E_{j+1})$. The rest of the theorem is immediate. □

4.1.3 Index of the complex

We define the Euler characteristic of a differential complex as:

$$\chi(\mathcal{C}) := \mathrm{ind}(D_+). \tag{4.1.6}$$

Lemma 4.1.7. *We have:*

$$\chi(\mathcal{C}) = \sum_{j=0}^{N} (-1)^j \dim H^j$$

Proof. Direct computation:

$$\begin{aligned}\mathrm{ind}(D_+) &= \dim\ker D_+ - \dim\ker(D_+)^* \\ &= \dim\ker D_+ - \dim\ker D_- = \sum_j H^{2j} - \sum_j H^{2j+1}.\end{aligned}$$

□

When $\mathcal{C}$ is the complex of differential forms, $\chi(\mathcal{C}) = \chi(M)$ is called *the Euler characteristic of* M:

$$\chi(M) := \sum_{j=0}^{n} (-1)^j \dim H^j(M).$$

The number $b_j := \dim H^j(M)$ are called *the Betti numbers* of M. When M is orientable, the Poincaré duality asserts that $b_j = b_{n-j}$. In particular, odd-dimensional orientable manifolds have vanishing Euler characteristic. The Euler characteristic has important topological interpretations. Computing Betti numbers can be achieved in many ways. In particular, the Mayer-Vietoris sequence is a great tool for that but this is below the scope of this course, see [BT82] for instance.

Example 4.1.8 (Cohomology of $\mathbb{S}^2$)**.** The sphere

$$\mathbb{S}^2 := \left\{x^2 + y^2 + z^2 = 1 \mid (x, y, z) \in \mathbb{R}^3\right\} \subset \mathbb{R}^3$$

is an orientable smooth surface. It is immediate that $H^0(\mathbb{S}^2) \simeq \ker \Delta_0 = \mathbb{R} \cdot \mathbf{1}$ (the constant function) since it is connected. Thus $b_0 = 1$ and admitting the Poincaré duality, we also have $b_2 = 1$. Moreover, $\mathbb{S}^2$ is simply connected which implies (try to prove it!) that $H^1(\mathbb{S}^2) = 0$ and $b_1 = 0$. Hence:

$$\chi(\mathbb{S}^2) = 1 - 0 + 1 = 2.$$

Exercise 4.1.9. Show that $H^1(\mathbb{S}^2) = 0$. *Hint: Consider $\omega \in C^\infty(M, T^*M)$ such that $d\omega = 0$ and look at the function $f(x) := \int_{\gamma_x} \omega$, where γ_x is any path joining the North Pole of $\mathbb{S}^2$ to $x \in \mathbb{S}^2$.*

4.2 Gårding's inequalities

We state without proof a fundamental and non-trivial result, relating the non-negativity of the principal symbol of a pseudodifferential operator to a lower bound on the operator.

Theorem 4.2.1 (Sharp Gårding inequality)**.** *Let $A \in \Psi^m_{\mathrm{cl}}(M)$ with $m \geq 0$ and assume that $\Re(\sigma_A) \geq 0$. Then, there exists a constant $C > 0$ such that*

$$\Re\langle A\varphi, \varphi\rangle_{L^2} \geq -C\|\varphi\|^2_{H^{\frac{m-1}{2}}}.$$

Proof. Omitted, see [GS94, Exercise 4.8]. □

There is a weaker statement which is easier to prove and will be sufficient for our purposes:

Lemma 4.2.2 (Weak Gårding inequality)**.** *Let $A \in \Psi^m_{\mathrm{cl}}(M)$ with $m \geq 0$ and assume that*

$$\Re(\sigma_A) \geq C_0\langle\xi\rangle^m,$$

for some constant $C_0 > 0$. Then, there exists a constant $C > 0$ such that for all $\varphi \in H^{m/2}(M)$,

$$\Re\langle A\varphi, \varphi\rangle_{L^2} \geq 1/C \times \|\varphi\|^2_{H^{m/2}} - C\|\varphi\|^2_{L^2}.$$

Proof. It suffices to prove the statement for $\varphi \in C^\infty(M)$ since it then holds for $\varphi \in H^{m/2}(M)$ by density. Taking $B := (A + A^*)/2$, we see that $\sigma_B = \Re(\sigma_A)$ and $\Re\langle A\varphi, \varphi\rangle_{L^2} = \langle B\varphi, \varphi\rangle_{L^2}$. Following the parametrix construction (or the square root construction), one shows that $B = C^*C + K$, for some elliptic $C \in \Psi^m_{\mathrm{cl}}(M), K \in \Psi^{-\infty}(M)$. This implies that

$$\begin{aligned}\Re\langle A\varphi, \varphi\rangle_{L^2} &= \|C\varphi\|^2_{L^2} + \langle K\varphi, \varphi\rangle_{L^2}\\ &\geq \|C\varphi\|^2_{L^2} - C_1\|\varphi\|^2_{L^2},\end{aligned}$$

where $C_1 > 0$ is some constant, using the boundedness of K on $L^2(M)$. Then, using the global elliptic estimate (3.3.2) with the operator C and $s = 0, N = 0$, we get the announced result. □

4.3 Spectral theory

4.3.1 Basic results

We now consider an elliptic pseudodifferential operator $A \in \Psi^m(M)$ with $m > 0$. The operator $A : H^m(M) \to L^2(M)$ is bounded. In order to do spectral theory, we need to stick to the space $L^2(M)$, that is we consider the *unbounded operator* A on $L^2(M)$ with dense domain $H^m(M) \subset L^2(M)$.

Lemma 4.3.1. (i) *$(A, H^m(M))$ is closed.*

(ii) *It is the closure of $(A, C^\infty(M))$.*

Recall that an operator $A : \mathcal{H} \to \mathcal{H}$ with dense domain $\mathcal{D}(A)$ is closed if, given $(u_n)_{n\geq 0}$ such that $u_n \in \mathcal{D}(A)$, $u_n \to u$ and $Au_n \to f$ in $\mathcal{H}$, one has $u \in \mathcal{D}(A)$ and $Au = f$.

Proof. (1) Take $(u_n)_{n\in\mathbb{N}}$, $u_n \in H^m(M)$ such that $u_n \to u$ and $Au_n \to f$ in $L^2(M)$. Since $A : \mathcal{D}'(M) \to \mathcal{D}'(M)$ is continuous, the equality $Au = f$ holds in $\mathcal{D}'(M)$ and we know that $u, f \in L^2(M)$. By Lemma 3.3.4, we deduce that $u \in H^m(M)$ and $Au = f$.

(2) Follows from the fact that $C^\infty(M)$ is dense in $H^m(M)$ (for the $H^m(M)$-topology). □

The *resolvent set* $\rho(A) \subset \mathbb{C}$ of A is defined as the set of points $z \in \mathbb{C}$ such that $(A-z)^{-1} : L^2(M) \to L^2(M)$ is defined, that is $A-z : H^m(M) \to L^2(M)$ is invertible (note that $(A-z)^{-1}$ then takes values in $\mathcal{D} = H^m(M)$). The *spectrum of* A, denoted by $\mathrm{spec}(A) \subset \mathbb{C}$, is the complement of the resolvent set. It can be checked that $\rho(A)$ is open and thus $\mathrm{spec}(A)$ is closed. The function $R(z) := (A - z)^{-1}$ is called the *resolvent* of A.

Lemma 4.3.2. *$z \in \rho(A)$ if and only if* $\ker(A - z) = \ker(A^* - \overline{z}) = \{0\}$.

We do not indicate the functional space on which we consider the kernel or the cokernel anymore since they are always included in $C^\infty(M)$ as the operators are elliptic.

Proof. Immediate consequence of Theorem 3.3.1. □

The following result is important. It asserts that the inverse (when defined) of an elliptic pseudodifferential operator is also pseudodifferential.

Theorem 4.3.3 (Inverse operator theorem)**.** *Assume $A \in \Psi^m(M)$ with $m > 0$ is elliptic. Let $z \in \rho(A)$. Then $(A-z)^{-1} : L^2(M) \to H^m(M)$ is the restriction of a pseudodifferential operator in $\Psi^{-m}(M)$ from $\mathcal{D}'(M)$ to $L^2(M)$ (or the extension from $C^\infty(M)$ to $L^2(M)$). Moreover, $[(A-z)^{-1}]^* = (A^* - \overline{z})^{-1}$.*

Proof. Similar to Lemma 3.2.17, item (2), see also Remark 3.2.18. Note that here, the assumption $m > 0$ guarantees that $\sigma(A-z) = \sigma(A)$ so $A - z$ is also elliptic. The last statement is more or less obvious from the equalities $(A-z)(A-z)^{-1} = \mathbb{1}$ thus $[(A-z)^{-1}]^*(A^* - \overline{z}) = \mathbb{1}$, that is $[(A-z)^{-1}]^* = (A^* - \overline{z})^{-1}$. □

We now further assume that the operator A is *formally self-adjoint*, that is $A = A^*$. We then have the following:

Lemma 4.3.4. *If $A \in \Psi^m(M)$ is elliptic formally self-adjoint, then* $\operatorname{spec}(A) \subset \mathbb{R}$.

Proof. Take $z \notin \mathbb{R}$. Then $A - z$ and $A^* - \overline{z}$ both have trivial kernel. Indeed if $(A-z)u = 0$ then $\langle (A-z)u, u\rangle_{L^2} = \langle Au, u\rangle_{L^2} - z\|u\|^2_{L^2} = 0$. By symmetry of A, $\langle Au, u\rangle_{L^2} \in \mathbb{R}$ and since $\Im(z) \neq 0$, we get $u = 0$. Same argument for A^*. □

Theorem 4.3.5. *Let $A \in \Psi^m(M)$ (for some $m > 0$) be elliptic and formally self-adjoint. Then, $(A, H^m(M))$ is self-adjoint and there exists a complete orthonormal basis $(\varphi_j)_{j=0}^\infty$ of $L^2(M)$ such that $A\varphi_j = \lambda_j \varphi_j$, $\lambda_j \in \mathbb{R}$ and $|\lambda_j| \to_{j\to\infty} \infty$. The set $\{\lambda_j\}$ is precisely equal to the spectrum $\sigma(A)$.*

There is an extension of this result to the non-selfadjoint case. We refer to [Shu01, Theorem 8.4] for further details.

Proof. First of all, combining Lemma 4.3.1 together with the fact that A is formally self-adjoint, we easily get that $(A, H^m(M))$ is a *closed symmetric operator with dense domain*. In order to show that it is self-adjoint, we can thus apply Theorem A.6.12: it suffices to check that $\operatorname{ran}(A \pm i)|_{H^m(M)} = L^2(M)$. But since $\operatorname{spec}(A) \subset \mathbb{R}$ by Lemma 4.3.4, this is immediate.

Let us now show that $\operatorname{spec}(A) \neq \mathbb{R}$. Assume the contrary. Then by Lemma 4.3.2, there is for every $\lambda \in \mathbb{R}$ a function $\varphi_\lambda \in C^\infty(M)$ such that $A\,\varphi_\lambda = \lambda\varphi_\lambda$. Now, the symmetry implies that for $\lambda \neq \mu$, one has $\langle \varphi_\lambda, \varphi_\mu\rangle_{L^2} = 0$. Indeed:

$$\langle A\varphi_\lambda, A\varphi_\mu\rangle_{L^2} = \lambda^2 \langle \varphi_\lambda, \varphi_\mu\rangle_{L^2} = \mu^2 \langle \varphi_\lambda, \varphi_\mu\rangle_{L^2} = \lambda\mu\langle \varphi_\lambda, \varphi_\mu\rangle_{L^2},$$

so if $\langle \varphi_\lambda, \varphi_\mu\rangle_{L^2} \neq 0$, we get a contradiction. But then, in turn, this contradicts the separability of $L^2(M)$.

We now pick a point $\lambda_0 \in \mathbb{R} \cap \rho(A)$. By Theorem 4.3.3, $R(\lambda_0) \in \Psi^{-m}$ is well-defined, compact on $L^2(M)$ and self-adjoint. As a consequence, we can apply the usual spectral theorem for compact operators: there is a sequence $\theta_j \to 0$ of eigenvalues and a complete orthonormal basis $(\varphi_j)_{j\in\mathbb{N}}$ of $L^2(M)$ such that

$$R(\lambda_0)\varphi_j = \theta_j \varphi_j. \tag{4.3.1}$$

Moreover, $\theta_j \neq 0$ since $\ker R(\lambda_0) = \ker(A-\lambda_0)^{-1} = 0$. Applying $(A-\lambda_0)$ to (4.3.1), we get $\varphi_j = \theta_j(A-\lambda_0)\varphi_j$, that is

$$A\varphi_j = (\lambda_0 + \theta_j^{-1})\varphi_j. \tag{4.3.2}$$

Obviously, $\lambda_j := \lambda_0 + \theta_j^{-1}$ satisfies $|\lambda_j| \to \infty$. Moreover, by ellipticity of $A - \lambda_j$ (note that $\sigma(A-\lambda_j) = \sigma(A)$ since the operator has order $m > 0$), the φ_j are smooth. The fact that $\sigma(A)$ is precisely equal to the set $\{\lambda_j\}$ is a consequence of Lemma 4.3.2: indeed, if $\lambda \in \{\lambda_j\}$, then $\ker A - \lambda \neq 0$; conversely, if $\lambda \in \sigma(A)$, then $R(\lambda_0)u = (\lambda-\lambda_0)^{-1}u$, that is $(\lambda-\lambda_0)^{-1} = \theta_j$ for some j and thus $\lambda = \lambda_0 + \theta_j$. $\square$

Remark 4.3.6. Usually, one deals with operators with elliptic and *positive* principal symbol $\sigma_A > 0$. By the (weak) Gårding inequality (i.e. $\langle Au, u\rangle_{L^2} \geq -C\|u\|^2_{L^2}$), one immediately gets that there is only a *finite number of negative eigenvalues.*

Example 4.3.7. The most standard example is that of the Laplacian. Let (M, g) be a smooth connected closed Riemannian manifold and let Δ_g be the (non-negative) Laplace operator. It is an elliptic differential operator of order 2 and thus satisfies Theorem 4.3.5. As a consequence, there is a complete orthonormal basis of $L^2(M)$ which consists of smooth functions $(\varphi_j)_{j\in\mathbb{N}}$ such that $\Delta_g\varphi_j = \lambda_j\varphi_j$ and $\lambda_j \geq 0$. Without loss of generality, we sort out the eigenvalues by increasing order, that is $\lambda_0 \leq \lambda_1 \leq ...$. Note that $\lambda_0 = 0$ is associated to the constant eigenfunction $\mathbf{1}$ (it is a 1-dimensional eigenspace). The eigenvalues are therefore $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq ...$ (counted without multiplicities).

Using this rough spectral theory for the Laplacian for instance, we are now in position to define a natural mollifier E_h on our manifold which will both satisfy $\|E_h\|_{L^2\to L^2}, \|\mathbb{1} - E_h\|_{L^2\to L^2} \leq 1$. Such a mollifier was needed in the proof of Theorem 3.2.13 in order to obtain the sharp bound for the L^2-norm of ΨDOs. For that, we let Π_j be the L^2-orthogonal projection onto $\ker(\Delta_g - \lambda_j)$. We then define for $h > 0$,

$$E_h := \sum_{\lambda_j \leq h^{-1}} \Pi_j. \tag{4.3.3}$$

We have:

Lemma 4.3.8. *For all $h > 0$, $E_h : \mathcal{D}'(M) \to C^\infty(M)$ has smooth Schwartz kernel. It satisfies the bounds $\|E_h\|_{L^2\to L^2}, \|\mathbb{1} - E_h\|_{L^2\to L^2} \leq 1$. Moreover, for all $\varphi \in L^2(M)$, $E_h\varphi \to \varphi$ in $L^2(M)$ as $h \to 0$.*

Proof. Taking an orthonormal basis $(\varphi_j)_{j\in\mathbb{N}}$ of $L^2(M)$ given by eigenfunctions of the Laplacian, we have

$$\Pi_j = \langle \bullet, e_j\rangle_{L^2} e_j.$$

Its Schwartz kernel is given by $K_j(x, y) = e_j(x)e_j(y)$, which is obviously smooth. The rest of the statement follows from the fact that for all $\varphi \in L^2(M)$,

$$\varphi = \sum_j \Pi_j\varphi,$$

where the convergence takes place in $L^2(M)$. $\square$

We also indicate here as a concluding remark that the previous statement can be directly generalized to the case where the operator $A \in \Psi^m(M, E \to E)$ acts on a Hermitian vector bundle $E \to M$. We leave it as an exercise for the reader to check the details.

4.3.2 Weyl's law

Once one has defined the eigenvalues of an elliptic formally self-adjoint operator, one may wonder how these equidistribute on the real line. This is still an active field of research. A partial (and rough) answer is provided by the famous *Weyl law*, going back to the beginning of the XXth century:

Theorem 4.3.9 (Weyl's law)**.** *Let $A \in \Psi^m(M)$ (with $m > 0$) be elliptic, formally self-adjoint and further assume that $\sigma(A) > 0$. We define for $\lambda \in \mathbb{R}$:*

$$N_A(\lambda) := \sharp \mathrm{spec}(A) \cap]-\infty, \lambda].$$

Then:

$$N_A(\lambda) = \frac{1}{(2\pi)^n} \mathrm{vol}_{T^*M}\left(\{\sigma(A) < \lambda\}\ \right) + \mathcal{O}(\lambda^{(n-1)/m}).$$

For a proof, see [GS94, Chapter 12]. Taking $A = \Delta_g, m = 2$, we see that

$$\begin{aligned}
\mathrm{vol}_{T^*M}\left(\{\sigma(A) < \lambda\}\ \right) &= \int_{|\xi|_g^2 < \lambda} \mathrm{d}x\mathrm{d}\xi \\
&= \int_M \left(\int_{T_x^*M} \mathbb{1}(|\xi|_g < \lambda^{1/2}) \mathrm{d}\xi \right) \mathrm{d}x \\
&= \mathrm{vol}_g(M) \lambda^{n/2} \,\mathrm{vol}(\mathbb{S}^{n-1}),
\end{aligned}$$

and thus

$$N_{\Delta_g}(\lambda) = \frac{1}{(2\pi)^n} \mathrm{vol}_g(M) \lambda^{n/2} \,\mathrm{vol}(\mathbb{S}^{n-1}) + \mathcal{O}(\lambda^{(n-1)/2}). \tag{4.3.4}$$

4.4 Propagation of singularities

TODO

Appendix A

Appendix

Contents

A.1 Notation

- $\alpha \in \mathbb{N}^n$ is a multi-index, namely $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ and $|\alpha| := \alpha_1 + ... + \alpha_n$ its *length*. The factorial is defined as $\alpha! := \alpha_1!...\alpha_n!$;
- $\partial_x^\alpha \varphi := \partial_{x_1}^{\alpha_1}...\partial_{x_n}^{\alpha_n}\varphi$ are usual derivatives for a smooth function $\varphi \in C^\infty(\mathbb{R}^n)$ (this does not depend on the order in which the derivatives are taken);
- $x^\alpha = x_1^{\alpha_1}...x_n^{\alpha_n}$;
- $D_x^\alpha := i^{-|\alpha|}\partial_x^\alpha$ (so that $D_\xi^\alpha(e^{ix\cdot\xi}) = x^\alpha e^{ix\cdot\xi}$);
- $\langle x \rangle := \sqrt{1+|x|^2}$ is the *Japanese bracket* of $x \in \mathbb{R}^n$, where $|\bullet|$ is the Euclidean norm in $\mathbb{R}^n$;
- $\widehat{\cdot}$ or $\mathcal{F}$ is the Fourier transform of a function or a distribution;
- $C^k(X)$ is the space of functions of regularity $k \in \mathbb{N}$ on an open subset $X \subset \mathbb{R}^n$;
- $C^\infty(X)$ is the space of smooth functions on X and $C^\infty(X) = \cap_{k\in\mathbb{N}} C^k(X)$;
- $\mathcal{D}'(X), \mathcal{E}'(X)$ are respectively the spaces of distributions and compactly supported distributions in the open subset $X \subset \mathbb{R}^n$;
- $\mathcal{S}(\mathbb{R}^n)$ is the space of Schwartz functions i.e. $\varphi \in \mathcal{S}(\mathbb{R}^n)$ if and only if $\varphi \in C^\infty(\mathbb{R}^n)$ and for all $\alpha \in \mathbb{N}^n, k \in \mathbb{N}$:
$$\sup_{x\in\mathbb{R}^n} \langle x \rangle^k |\partial_x^\alpha \varphi(x)| < \infty,$$
- $\mathcal{S}'(\mathbb{R}^n)$ is the topological dual of $\mathcal{S}(\mathbb{R}^n)$, called the space of temperate distributions;
- $A \lesssim B$ means that there exists a uniform constant $C > 0$ (independent of A and B) such that $A \leq CB$;
- $\varphi \prec \varphi'$ means that $\varphi' = 1$ on the support of φ.

A.2 Functional analysis

A.2.1 Fréchet spaces

We refer to [Rud91, Chapter 1] for further details on the topology of vector spaces.

Definition A.2.1. A Fréchet space E is a topological vector space satisfying the following three properties:

(i) It is a Hausdorff topological space;

(ii) Its topology is induced by a countable family of seminorms[1] $\{\| \bullet \|_k \mid k \in \mathbb{N}\}$ in the sense that $U \subset E$ is *open* if and only if for all $x \in U$, there exists $k \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$\{y \in E \mid \forall j \leq k, \|y - x\|_j < r\} \subset U.$$

(iii) It is complete with respect to this family of seminorms.

Completeness with respect to the family of seminorms means that if $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence with respect to *every* seminorm[2], then there exists $x_\infty \in E$ such that for all $k \in \mathbb{N}$, $\|x_n - x_\infty\|_k \to_{n\to\infty} 0$. Note that a sequence of vectors $(x_n)_{n\in\mathbb{N}}$ converges to x_∞ in the Fréchet space defined by the family of seminorms $\{\| \bullet \|_k \mid k \in \mathbb{N}\}$ if $\|x_n - x_\infty\|_k \to_{n\to\infty} 0$ for all $k \in \mathbb{N}$.

Lemma A.2.2. *The metric*

$$d(x, y) := \sum_{k=0}^{\infty} \frac{1}{2^k} \frac{\|x - y\|_k}{1 + \|x - y\|_k}$$

is translation-invariant, induces the topology of E and the metric space (E, d) is complete.

Example A.2.3. A typical example is $C^\infty(X)$, where $X \subset \mathbb{R}^n$ is an open subset. The seminorms are defined as follows: take $(K_\ell)_{\ell\in\mathbb{N}}$ a nested sequence of compact subsets such that $\cup_{\ell \geq 0} K_\ell = X$ and define for $\varphi \in C^\infty(X)$,

$$\|f\|_{C^k(K_\ell)} := \sup_{|\alpha| \leq k} \sup_{x\in K_\ell} |\partial_x^\alpha \varphi(x)|.$$

A.2.2 General Theorems

Given E, F, two topological vector spaces, we will denote by $\mathcal{L}(E, F)$ the set of continuous linear operators from E to F. First of all, we recall the Hahn-Banach Theorem (a weaker version of it actually):

Theorem A.2.4 (Hahn-Banach)**.** *Let E be a Banach space, $V \subset E$ be a subspace and $f \in \mathcal{L}(V, \mathbb{C})$ be a continuous linear functional defined over V with norm ≤ 1. Then, there exists a continuous linear extension of f to E with norm ≤ 1.*

For a proof of Theorem A.2.4, see [Rud91, Chapter 3]. The following is known as Banach's theorem or the open mapping Theorem, see [Rud91, Chapter 2, Theorem 2.11]:

Theorem A.2.5 (Banach)**.** *Let $A : E_1 \to E_2$ be a* surjective *linear map between the Banach spaces E_1, E_2. Then A is* open, *that is the image of an open set is an open set.*

[1] A seminorm satisfies the same properties as a norm except that $\|x\| = 0$ does not imply $x = 0$ anymore.

[2] Namely, for all $k \in \mathbb{N}$, for all $\varepsilon > 0$, there exists $n_0(k, \varepsilon) \in \mathbb{N}$ such that for all $m, n \geq n_0(\varepsilon)$, one has $\|x_n - x_m\|_k < \varepsilon$.

A useful corollary is:

Corollary A.2.6. *Let $A \in \mathcal{L}(E_1, E_2)$ and bijective. Then A is a homeomorphism, that is A^{-1} is continuous.*

Proof. Let $U \subset E_1$ be an open set. Then $(A^{-1})^{-1}(U) = A(U)$ is open by Theorem A.2.5, that is A^{-1} is continuous. Also note that if $F \subset E_1$ is closed, then $A(F) = (A^{-1})^{-1}(F) \subset E_2$ is closed. □

Eventually, we recall the Banach-Steinhaus Theorem in Banach spaces asserting that a collection of continuous linear operators is *simply bounded* if and only if it *uniformly bounded*:

Theorem A.2.7 (Banach-Steinhaus)**.** *Let E be a Banach space, Y be a normed vector space, and $\mathcal{F}$ be a collection of continuous linear operators from X to Y. If for all $x \in E$, $\sup_{T\in\mathcal{F}} \|T(x)\|_Y < \infty$, then $\sup_{T\in\mathcal{F}} \|T\|_{\mathcal{L}(E,Y)} < \infty$, or equivalently*

$$\sup_{\|x\|_E=1, T\in\mathcal{F}} \|T(x)\|_Y < \infty.$$

In Fréchet spaces, the statement is almost the same, except that, in the conclusion of the Theorem, the uniform boundedness has to be taken with respect to one of the seminorms defining the Fréchet space. For a proof of Theorem A.2.7, see [Rud91, Chapter 2, Theorem 2.5].

A.2.3 Compact operators

We start with a discussion on **complemented spaces in Banach spaces**, see [Rud91, Definition 4.20] and the discussion below for further details. Let E be a Banach space and $F \subset E$ be a closed subspace. Note that if $\dim F < \infty$, then F is automatically closed. We say that F is *complemented* in E if there exists a closed subspace $L \subset E$ such that $E = F + L$ and $F \cap L = \{0\}$. In this case, we say that E is the *direct sum* of F and L and we write as usual

$$E = F \oplus L.$$

A vector subspace $F \subset E$ is said to have *finite codimension* if the quotient space E/F is finite-dimensional. We define $\mathrm{codim}(F) := \dim(E/F)$. We will need a useful lemma:

Lemma A.2.8. *Let E be a Banach space. Then:*

(i) *If $L \subset E$ is finite-dimensional space, there exists a closed subspace $F \subset E$ such that $L \oplus F = E$.*

(ii) *If $F \subset E$ is closed and $\dim E/F < \infty$, then there exists a finite-dimensional space L such that $E = F \oplus L$. In particular, $\mathrm{codim}(F) = \dim(L)$.*

Proof of Lemma A.2.8. (1) The proof relies on the Hahn-Banach Theorem A.2.4. Consider $(\ell_1, ..., \ell_p)$ a basis of L and $(\xi_1, ..., \xi_p)$ a dual basis such that $\xi_i(\ell_j) = \delta_{ij}$.

By the Hahn-Banach Theorem A.2.4, we can extend continuously the linear forms ξ_i to E. Define $F := \cap_i \ker \xi_i$. This is a closed subspace as all the ξ_i are continuous and thus the $\ker \xi_i$ are closed. We claim that $F \oplus L = E$. Indeed, if $x \in F \cap L$, then $x = \sum_i \lambda_i \ell_i$ but $\xi_j(x) = 0 = \lambda_j$, that is $x = 0$. Moreover, given $x \in E$, setting $\lambda_i := \xi_i(x)$, we then have $x = \sum_i \lambda_i \ell_i + (x - \sum_i \lambda_i \ell_i)$, where $x - \sum_i \lambda_i \ell_i \in F$.

(2) Consider $\pi : E \to E/F$ the quotient map. Pick as basis $e_1, ..., e_n \in E/F$ and $x_1, ..., x_n \in E$ such that $\pi x_i = e_i$. Then $L = \mathrm{Span}(x_1, ..., x_n)$ satisfies $E = F \oplus L$. $\square$

We also have the following useful result:

Lemma A.2.9. *Let E be a Banach space and $F_1 \subset F_2 \subset E$ be nested spaces such that F_1 is closed with finite codimension. Then F_2 is closed with finite codimension and* $\operatorname{codim} F_2 \leq \operatorname{codim} F_1$.

We now discuss the important notion of **compact operators**. Let E_1, E_2 be two Banach spaces. Let $A \in \mathcal{L}(E_1, E_2)$. Recall that its *kernel* $\ker(A)$ is the set of elements $x \in E_1$ such that $Ax = 0$ while its *cokernel* $\mathrm{coker}(A)$ is the space E_2/AE_1. We let $\mathcal{K}(E_1, E_2) \subset \mathcal{L}(E_1, E_2)$ be the set of *compact operators*, namely continuous operators such that for any bounded sequence $(x_n)_{n \in \mathbb{N}}$ in E_1, the sequence $(T(x_n))_{n \in \mathbb{N}}$ admits a converging subsequence. Typical examples of compact operators are provided by *finite rank operators*, namely operators whose image is a finite-dimensional subspace of E_2. We recall the following properties:

Theorem A.2.10. (i) *$\mathcal{K}(E_1, E_2)$ is closed (for the operator norm);*

(ii) *If E_2 is a Hilbert space, then $\mathcal{K}(E_1, E_2)$ is the closure (for the operator norm) of the space of finite rank operators;*

(iii) *$\mathcal{K}(E_1)$ is a two-sided ideal of $\mathcal{L}(E_1)$, namely for $A \in \mathcal{L}(E_1), K \in \mathcal{K}(E_1)$, one has $AK, KA \in \mathcal{K}(E_1)$;*

(iv) *If $K \in \mathcal{K}(E_1)$, $\lambda \in \mathbb{C} \setminus \{0\}$, then $\ker(K+\lambda)$ is finite-dimensional and $\mathrm{ran}(K+\lambda)$ is closed with finite codimension.*

(v) *If $K \in \mathcal{K}(E_1)$, K has discrete spectrum $\sigma(K) \subset \mathbb{C}$. Every nonzero $\lambda \in \sigma(K)$ is an eigenvalue associated to a finite-dimensional eigenvalue. Moreover, $\sigma(K)$ can only accumulate at 0.*

(vi) *An operator $K : E_1 \to E_2$ is compact iff its adjoint $K^* : E_2^* \to E_1^*$ is compact (Schauder's Theorem).*

(vii) *If $K \in \mathcal{K}(E_1)$, $\lambda \in \sigma(K)$ and $\lambda \neq 0$, then $\lambda \in \sigma(K^*)$ and:*

$$\begin{aligned} \dim(K - \lambda \mathbb{1}_{E_1}) &= \dim(K^* - \lambda \mathbb{1}_{E_1^*}) \\ &= \dim E_1/(K-\lambda)(E_1) \\ &= \dim E_1^*/(K^*-\lambda)(E_1^*). \end{aligned}$$

For a detailed proof of these facts, we refer to [Rud91, Chapter 4]. We will use the following lemma due to Riesz:

Lemma A.2.11 (Riesz)**.** *Let $(E, \|\bullet\|_E)$ be a Banach space. Let $F \subset E$ be a closed subspace such that $F \neq E$. Then, for all $\varepsilon > 0$, there exists $x \in E$ such that $\|x\| = 1$ and $d(x, F) > 1 - \varepsilon$.*

Proof. Let $F \subset E, F \neq E$ be a closed subspace of E. Let $x_0 \in E$ such that $x_0 \notin F$ and $\|x_0\| = 1$. Since F is closed, $d(x_0, F) =: \delta_0 > 0$. In particular, for all $\varepsilon > 0$, there exists $f_\varepsilon \in F$ such that $\|x_0 - f_\varepsilon\| = \delta_0 + \mathcal{O}(\varepsilon)$ (that is, $\|x_0 - f_\varepsilon\| \in [\delta_0 - \varepsilon, \delta_0 + \varepsilon]$). For $\varepsilon > 0$ fixed, we set $y_\varepsilon := \frac{x_0 - f_\varepsilon}{\|x_0 - f_\varepsilon\|}$. Observe that for all $f \in F$:

$$\|y_\varepsilon - f\| = \frac{1}{\delta_0 + \mathcal{O}(\varepsilon)} \|x_0 - \underbrace{(f_\varepsilon + (\delta_0 + \mathcal{O}(\varepsilon) f)}_{\in F}\| > \frac{\delta_0}{\delta_0 + \mathcal{O}(\varepsilon)},$$

that is $d(y_\varepsilon, F) > \frac{\delta_0}{\delta_0 + \mathcal{O}(\varepsilon)}$. Taking $\varepsilon > 0$ small enough, the last quantity can be made arbitrarily close to 1. This concludes the proof as $\|y_\varepsilon\| = 1$. $\square$

This leads to the well-known characterization of finite dimension:

Lemma A.2.12. *Let $(E, \|\bullet\|_E)$ be a normed space. Then the closed unit ball $\overline{B(0,1)}$ is compact if and only if E is finite-dimensional.*

Proof. If E is finite-dimensional, then the closed unit ball is compact. Assume now that E is infinite-dimensional. Consider $x_1 \in E$ such that $\|x_1\| = 1$ and write $F_1 := \mathrm{Span}(x_1)$. By Lemma A.2.11, there exists $x_2 \in E$ such that $x_2 \notin F_1, d(x_2, F_1) > 1/2$, and we set $F_2 := \mathrm{Span}(x_1, x_2)$. Iteratively, we construct a sequence of vectors $(x_n)_{n \in \mathbb{N}}$ and closed (finite-dimensional) subspaces $F_n := \mathrm{Span}(x_1, ..., x_n)$ such that $d(x_{n+1}, F_{n+1}) > 1/2$. By construction, for all $n > m$, $d(x_n, x_m) > 1/2$ so there are no converging subsequences. $\square$

We now prove Theorem A.2.10:

Proof of Theorem A.2.10. (1) This is based on the standard diagonal argument. Let $u_n \in \mathcal{K}(E_1, E_2)$ such that $u_n \to u \in \mathcal{L}(E_1, E_2)$ in the operator norm, that is for all $\varepsilon > 0$, for all $n \geq N(\varepsilon)$ large enough:

$$\forall x \in E_1, \quad \|u_n(x) - u(x)\| \leq \varepsilon \|x\|.$$

We want to show that u is compact. Let $(x_k)_{k \in \mathbb{N}}$ be a bounded sequence in E_1. Without loss of generality (up to rescaling), we can assume that $\|x_k\|_{E_1} \leq 1$. By compactness, we can extract a first subsequence $\sigma_1 : \mathbb{N} \to \mathbb{N}$ such that $u_1(x_{\sigma_1(k)}) \to x_\infty^{(1)}$. We set $\sigma(1) := \sigma_1(1)$. We then extract a second subsequence such that $u_2(x_{\sigma_1 \circ \sigma_2(k)}) \to x_\infty^{(2)}$ with $\sigma_2(1) = 1$ and we set $\sigma(2) := \sigma_1 \circ \sigma_2(2)$. We then iterate this process and construct for $\ell > 0$, $\sigma_\ell : \mathbb{N} \to \mathbb{N}$ such that $u_\ell(x_{\sigma_1 \circ ... \circ \sigma_\ell(k)}) \to x_\infty^{(\ell)}$, $\sigma_\ell(j) = j$ for $j \leq \ell - 1$, and we set $\sigma(\ell) := x_{\sigma_1 \circ ... \circ \sigma_\ell(\ell)}$. By construction, we have that for all $\ell > 0$, $u_\ell(x_{\sigma(k)}) \to x_\infty^{(\ell)}$.

As a consequence, taking $\varepsilon > 0$ and $\ell > 0$ such that $\|u_\ell - u\|_{\mathcal{L}(E_1,E_2)} < \varepsilon$, we have:

$$\begin{aligned}\|u(x_{\sigma(k)}) - u(x_{\sigma(k')})\|_{E_2} &\leq \|(u-u_\ell)(x_{\sigma(k)})\|_{E_2} \\ &\quad + \|(u-u_\ell)(x_{\sigma(k')})\|_{E_2} + \|u_\ell(x_{\sigma(k)} - x_{\sigma(k')})\|_{E_2} \\ &\leq 2\varepsilon + \|u_\ell(x_{\sigma(k)} - x_{\sigma(k')})\|_{E_2}.\end{aligned}$$

Since $u_\ell(x_{\sigma(k)}) \to x_\infty^{(\ell)}$ in E_2, it is a Cauchy sequence and there exists $k_0 > 0$ such that for all $k, k' \geq k_0$, $\|u_\ell(x_{\sigma(k)} - x_{\sigma(k')})\|_{E_2} < \varepsilon$. As a consequence, for all $k, k' \geq k_0$, we have $\|u(x_{\sigma(k)}) - u(x_{\sigma(k')})\|_{E_2} < \varepsilon$, that is $(u(x_{\sigma(k)}))_{k\geq 0}$ is Cauchy, so it converges to some element in E_2.

(2) Admitted (or left as an exercise to the reader).

(3) Immediate.

(4) Up to rescaling, we can always assume that $\lambda = 1$. In order to show that $\ker(\mathbb{1} + K)$ is finite-dimensional, it suffices to show that its unit sphere is compact (and we can then conclude by the Riesz Lemma A.2.11). For that, let $x_n \in \ker(\mathbb{1}+K)$ such that $\|x_n\|_{E_1} = 1$. By compactness of K, up to extraction, $Kx_n \to y$ in E_1. But $Kx_n = -x_n$ so $x_n \to -y$ and we conclude that the unit sphere is compact, so $\ker(\mathbb{1} + K)$ is finite-dimensional.

We now show that $\mathbb{1} + K$ has closed range. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points such that $(\mathbb{1} + K)x_n \to y \in E_1$. Note that, since $\ker(\mathbb{1} + K)$ is finite-dimensional, we can consider a complement $F_1 \subset E_1$ such that $F_1 \oplus \ker(\mathbb{1} + K) = E_1$. Then, up to changing x_n, we can always assume that it is $x_n \in F_1$. First of all, we show that (x_n) is bounded. Assume that it is not bounded, that is up to taking a subsequence $\|x_n\| \to \infty$. We set $x'_n := x_n/\|x_n\|$. Then, using the equality

$$B_2 A x'_n = B_2 A x_n/\|x_n\| = x'_n + K_1 x'_n, \tag{A.2.1}$$

we see that the left-hand side converges to 0. As (x'_n) is bounded and K_1 is compact, we can extract so that up to a subsequence $K_1 x'_n \to z$. Using (A.2.1) once again, we get that $x'_n \to z$, $\|z\| = 1$ and $(\mathbb{1} + K_1)z = 0$. Since F_1 is closed, we have $z \in F_1$ but this contradicts the fact that $\|z\| = 1$ and $z \in \ker(\mathbb{1} + K_1)$. So (x_n) is bounded. As a consequence, up to a subsequence, we have that $Kx_n \to w$, and then $x_n = (\mathbb{1} + K)x_n - Kx_n \to y - w =: x \in E_1$. Moreover, $(\mathbb{1} + K)x = y$ which concludes the proof.

(5) Let $\lambda \in \sigma(K) \setminus \{0\}$, that is, such that $K - \lambda : E_1 \to E_1$ is not bijective. Up to rescaling, we can assume $\lambda = -1$. First of all, we want to show that -1 is an eigenvalue of K, that is, $\ker(\mathbb{1} + K) \neq \{0\}$. We argue by contradiction and assume that K is injective but not surjective. We set $E^{(1)} := (\mathbb{1} + K)(E_1) \subset E_1$ which is a strict closed subspace of E_1 and $E^{(n)} := (\mathbb{1} + K)^n(E_1)$ which is a nested sequence of closed subspaces with finite codimension (i.e. such that $E^{(n+1)} \subset E^{(n)} \subset \ldots \subset E_1$). Note that by injectivity of $\mathbb{1} + K$, these inclusions are proper, namely $E^{(n+1)} \neq E^{(n)}$.

By Lemma A.2.11, we can then consider $x_n \in E^{(n)}$ such that $d(x_n, E^{(n+1)}) > 1/2$ and $\|x_n\|_{E_1} = 1$. By compactness of K, we have that, up to a subsequence, $Kx_n \to y$. Hence, taking n large enough, we have:

$$1/10 \geq \|Kx_n - Kx_{n+1}\|_{E_1} = \|(\mathbb{1}+K)x_n - (\mathbb{1}+K)x_{n+1} + x_{n+1} - x_n\|_{E_1}$$

Observe that $(\mathbb{1}+K)x_n - (\mathbb{1}+K)x_{n+1} + x_{n+1} \in E^{(n+1)}$ and $x_n \in E^{(n)}$ so the last distance is greater than $1/2$ and this is a contradiction, that is -1 is an eigenvalue of K.

We now show that for every $\varepsilon > 0$, the number of eigenvalues of K of modulus $> \varepsilon$ is finite. We argue by contradiction and assume that it is not, that is, we can find a sequence of distinct eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ for K such that $|\lambda_n| > \varepsilon$ and unit eigenvectors $(x_n)_{n\in\mathbb{N}}$ associated to the λ_n's. We set $Y_n := \mathrm{Span}(x_1, ..., x_n)$. We can take unit vectors $(y_n)_{n\in\mathbb{N}}, y_n \in Y_n$ such that $d(y_n, Y_{n-1}) > 1/2$. The sequence $(y_n)_{n\in\mathbb{N}}$ being bounded, we have that, up to extraction, $Ky_n \to y_\infty$. In particular, this is a Cauchy sequence and taking n large enough, we can ensure that $\|Ky_{n+1} - Ky_n\| < \varepsilon/2022$. But, writing $y_{n+1} = \mu_{n+1}x_{n+1} + v_n$ for some $\mu_{n+1} \neq 0, v_n \in Y_n$, we have:

$$\begin{aligned}\varepsilon/2022 \geq \|Ky_{n+1} - Ky_n\| &= \|\lambda_{n+1}\mu_{n+1}x_{n+1} + Kv_n - Ky_n\| \\ &= |\lambda_{n+1}|\|y_{n+1} \underbrace{-v_n + \lambda_{n+1}^{-1}(Kv_n - Ky_n)}_{\in Y_n}\| \geq \varepsilon/2,\end{aligned}$$

which is a contradiction. This also shows that the spectrum can only accumulate at 0.

(6,7) See [Rud91, Theorem 4.19].

□

A.3 Distribution and integration theory

A.3.1 Definition

See [Rud91, Chapter 6]. Let X be an open set in $\mathbb{R}^n$. The space of distributions $\mathcal{D}'(X)$ is the topological dual of $C^\infty_{\mathrm{comp}}(X)$, namely the set of continuous linear forms $u : C^\infty_{\mathrm{comp}}(X) \to \mathbb{C}$ such that for all compact $K \subset X$, there exists $n_K \in \mathbb{N}, C_K > 0$ such that for all $\varphi \in C^\infty(X)$ with support in K:

$$|(u,\varphi)| \leq C_K \sup_{|\alpha| \leq n_K} \|\partial^\alpha \varphi\|_{L^\infty(X)}.$$

The *order of the distribution* (which may depend on K) is the lowest integer n_K one can take. In general, distributions that we will manipulate will have finite order. The standard Lebesgue measure gives an embedding of $C^\infty(X) \hookrightarrow \mathcal{D}'(X)$ by simply defining for $f \in C^\infty(X)$ the associated distribution $f|dx|$ such that:

$$(f|dx|, \varphi) := \int_X f(x)\varphi(x)\mathrm{d}x = (f,\varphi) = \langle f, \overline{\varphi}\rangle_{L^2(X)}.$$

(Note that this is $\mathbb{C}$-linear.) For the sake of simplicity, we will simply write f instead of $f|dx|$.

The *support* of a distribution u is the minimal closed set[3] F, in the sense of inclusions, such that for all $\varphi \in C^\infty(X \setminus F)$, one has $(u, \varphi) = 0$. We will write $\mathcal{E}'(X)$ for the space of distributions with compact support. Given $u \in \mathcal{E}'(X)$, we can always see it as a distribution in $\mathbb{R}^n$ since for any $\varphi \in C^\infty(\mathbb{R}^n)$, the pairing (u, φ) makes sense. We will freely use the embedding $\mathcal{E}'(X) \hookrightarrow \mathcal{E}'(\mathbb{R}^n)$.

We now define the notion of convergence of test functions.

Definition A.3.1 (Convergence of test functions)**.** Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of elements in $C^\infty_{\mathrm{comp}}(X)$ and $\varphi \in C^\infty_{\mathrm{comp}}(X)$. We say that $\varphi_n \to \varphi$ in $C^\infty_{\mathrm{comp}}(X)$ if there exists a compact $K \subset X$ (containing $\mathrm{supp}(\varphi) \subset K$) such that $\mathrm{supp}(\varphi_n) \subset K$ for all n large enough and for all multi-index $\alpha \in \mathbb{N}^n$, $\partial^\alpha \varphi_n \to \partial^\alpha \varphi$ uniformly on K.

This is a very strong notion of convergence. We have:

Lemma A.3.2 (Sequential continuity of distributions)**.** *Let $u \in \mathcal{D}'(X)$, $\varphi \in C^\infty(X)$ and $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence of elements in $C^\infty_{\mathrm{comp}}(X)$ such that $\varphi_n \to \varphi$ in $C^\infty(X)$. Then:*

$$(u, \varphi_n) \to (u, \varphi).$$

The proof is immediate and follows from the definitions. There is also a notion of convergence in the sense of distributions:

Definition A.3.3 (Convergence of distributions)**.** Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of distributions in $\mathcal{D}'(X)$ and $u \in \mathcal{D}'(X)$. We say that $u_n \to u$ in $\mathcal{D}'(X)$ (in the sense of distributions) if for all $\varphi \in C^\infty_{\mathrm{comp}}(X)$,

$$(u_n, \varphi) \to (u, \varphi).$$

It will be convenient to have the following result at hand for converging sequences in $\mathcal{D}'(X)$.

Lemma A.3.4. *Let $u \in \mathcal{E}'(X)$. Assume that $u_j \in \mathcal{E}'(X)$ is a family such that $u_j \to u$ in $\mathcal{D}'(X)$. Then, there exists $C, N > 0$ such that:*

$$\forall j \in \mathbb{N}, \forall \varphi \in C^\infty(X), \qquad |(u_j, \varphi)| \ \leq C \|\varphi\|_{C^N(X)}.$$

The proof follows from the Banach-Steinhaus Theorem in Fréchet spaces asserting that a family of continuous linear operators is *simply bounded* if and only if it is *uniformly bounded*, see Theorem A.2.7 for a statement in Banach spaces. But let us do the proof by hand for the sake of completeness. It relies on the Baire Theorem which we recall here for the reader's convenience:

Theorem A.3.5 (Baire Theorem)**.** *Let (X, d) be a complete metric space and $(F_n)_{n\geq 0}$ be a sequence of closed sets with empty interior. Then the interior $\cup_{n\geq 0} F_n$ is also empty.*

[3] X is endowed with the induced topology from $\mathbb{R}^n$ so X is closed in X in particular.

Equivalently, a countable intersection of open dense sets is also dense. The proof of Theorem A.3.5 is fairly classical and left as an exercise for the reader. We now prove Lemma A.3.4

Proof. Define

$$\begin{aligned} A_n &:= \bigcap_{j\geq 0} \left\{\varphi \in C^\infty(X) \mid |(u_j,\varphi)| \ \leq n\|\varphi\|_{C^n(X)}\right\} \\ &= \left\{\varphi \in C^\infty_{\mathrm{comp}}(X) \mid \forall j\geq 0, |(u_j,\varphi)| \ \leq n\|\varphi\|_{C^n(X)}\right\}. \end{aligned}$$

It is immediate to check that A_n is closed in the Fréchet space $C^\infty(X)$ as an intersection of closed subsets. Moreover, we have $C^\infty(X) = \cup_{n\in\mathbb{N}} A_n$. Indeed, given $\varphi \in C^\infty(X)$, one has $(u_j,\varphi) \to (u,\varphi)$ by assumption and $|(u,\varphi)| \ \leq C\|\varphi\|_{C^N(X)}$ for some uniform constants $C, N > 0$ since u is a distribution with compact support. As a consequence, taking $j \geq j_0$ large enough, there exists some integer $n \geq \max(2C, N)$ such that:

$$|(u_j,\varphi)| \ \leq 2C\|\varphi\|_{C^N(X)} \leq n\|\varphi\|_{C^n(X)}.$$

For $j < j_0$, it suffices to use the fact that all distributions u_j have finite order and since there is only a finite number of integers $j < j_0$, this eventually shows that $\varphi \in A_n$ for some $n \geq 0$.

By the Baire Theorem A.3.5, this implies that one of the A_n has non-empty interior (otherwise, if they all had empty interior, their union would have empty interior as well, which contradicts $\cup_n A_n = C^\infty(X)$). Let $n_0 \geq 0$ be such that A_{n_0} has non-empty interior, that is, there exists some $\varphi_0 \in A_{n_0}, k_0, \varepsilon > 0$ such that:

$$\left\{\varphi \in C^\infty(X) \mid \|\varphi - \varphi_0\|_{C^{k_0}(X)} < \varepsilon\right\} \subset A_{n_0}.$$

We set $N := \max(k_0, n_0)$. For $\varphi \in C^N(X)$ such that $\|\varphi\|_{C^N(X)} = 1$, we then have:

$$\begin{aligned} |(u_j,\varphi)| &= 2\varepsilon^{-1}\left|(u_j,\varphi_0 + \tfrac{1}{2}\varepsilon\varphi) - (u_j,\varphi_0)\right| \\ &\leq 2\varepsilon^{-1}\left(n_0\|\varphi_0 + \tfrac{1}{2}\varepsilon\varphi\|_{C^{n_0}(X)} + n_0\|\varphi_0\|_{C^{n_0}(X)}\right) \\ &\leq C\|\varphi\|_{C^N(X)} \leq C. \end{aligned}$$

This eventually prove the claim. □

We also have the important density result:

Lemma A.3.6 (Density of smooth functions in distributions). *$C^\infty_{\mathrm{comp}}(X)$ (resp. $C^\infty(X)$) is dense in $\mathcal{E}'(X)$ (resp. $\mathcal{D}'(X)$).*

This is understood in the sequential sense i.e. for any $u \in \mathcal{D}'(X)$, there is a sequence $(u_n)_{n\in\mathbb{N}}$ of smooth functions in $C^\infty(X)$ such that $u_n \to u$ in $\mathcal{D}'(X)$. The proof of Lemma A.3.6 relies on the following useful result and the convolution product introduced in the following paragraph. It is therefore postponed to the end of §A.3.2.

Lemma A.3.7. *Let $X \subset \mathbb{R}^n, Y \subset \mathbb{R}^m$ and let $\psi \in C^\infty_{\mathrm{comp}}(X \times Y)$. Let $T \in \mathcal{D}'(X)$. Then the function*

$$Y \ni y \mapsto (T, \psi(\bullet, y))$$

belongs to $C^\infty_{\mathrm{comp}}(Y)$ *and*

$$\partial_y^\alpha\left((T, \psi(\bullet, y))\right) = (T, \partial_y^\alpha \psi(\bullet, y)).$$

We also have the Fubini-type formula:

$$\left(T, \int_Y \psi(\bullet, y)\ \mathrm{d}y\right) = \int_Y (T, \psi(\bullet, y))\mathrm{d}y.$$

A.3.2 Convolution

See [Rud91, Chapter 6. Convolutions] for further details. Given $\varphi, \psi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, we define their convolution product as

$$\varphi \star \psi(x) := \int_{\mathbb{R}^n} \varphi(y)\psi(x-y)\mathrm{d}y.$$

The product is obviously bilinear and it satisfies the following usual properties:

Lemma A.3.8. (i) *(Commutativity, associativity) For* $\varphi, \psi, \chi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, $\varphi \star \psi = \psi \star \varphi$ *and* $(\varphi \star \psi) \star \chi = \varphi \star (\psi \star \chi)$.

(ii) *(Derivation) For* $\varphi, \psi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n), \alpha \in \mathbb{N}^n$, $\partial_x^\alpha(\varphi \star \psi) = (\partial_x^\alpha \varphi) \star \psi = \varphi \star (\partial_x^\alpha \psi)$.

(iii) *(Support)* $\mathrm{supp}(\varphi \star \psi) \subset \mathrm{supp}(\varphi) + \mathrm{supp}(\psi)$.

These formulas also extend to distributions with compact support. We can now prove the density lemma fo smooth functions in distributions.

Proof of Lemma A.3.6. todo □

A.3.3 Fourier transform

See [Rud91, Chapter 7] for further details. Given $\varphi \in C^\infty_{\mathrm{comp}}(\mathbb{R}^n)$, the Fourier transform is defined as:

$$\widehat{\varphi}(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)\mathrm{d}x. \tag{A.3.1}$$

We will also use the notation $\mathcal{F}$ sometimes. It is a continuous isomorphism $\mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ with inverse is given by:

$$\mathcal{F}^{-1}\psi(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\cdot\xi}\psi(\xi)\mathrm{d}\xi = \frac{1}{(2\pi)^n}\mathcal{R}\mathcal{F}\psi(x),$$

where $\mathcal{R}\varphi(x) = \varphi(-x)$. It extends by duality to a continuous isomorphism $\mathcal{S}'(\mathbb{R}^n) \to \mathcal{S}'(\mathbb{R}^n)$. It satisfies the following properties:

Lemma A.3.9. (i) *(Riemann-Lebesgue) If* $\varphi \in L^1(\mathbb{R}^n)$, *then* $\widehat{\varphi} \in C^0(\mathbb{R}^n)$ *and* $\widehat{\varphi}(\xi) \to 0$ *as* $|\xi| \to \infty$.

(ii) *(Integral identities) For all* $\varphi, \psi \in L^2(\mathbb{R}^n)$*, we have:*

$$\int_{\mathbb{R}^n} \varphi(x)\widehat{\psi}(x)\mathrm{d}x = \int_{\mathbb{R}^n} \widehat{\varphi}(x)\psi(x)\mathrm{d}x,$$

and the Parseval identity:

$$\langle \varphi, \psi \rangle_{L^2} = \frac{1}{(2\pi)^n} \langle \widehat{\varphi}, \widehat{\psi} \rangle_{L^2} \tag{A.3.2}$$

In particular, $\|\varphi\|_{L^2}^2 = (2\pi)^{-n}\|\widehat{\varphi}\|_{L^2}^2$.

(iii) $\widehat{\varphi \star \psi} = \widehat{\varphi}\widehat{\psi}$ *and* $\widehat{\varphi\psi} = (2\pi)^n \widehat{\varphi} \star \widehat{\psi}$.

(iv) *For all* $\varphi \in \mathcal{S}(\mathbb{R}^n)$ *and* $\alpha \in \mathbb{N}^n$, $\widehat{D_x^\alpha \varphi}(\xi) = \xi^\alpha \widehat{\varphi}(\xi)$ *and* $\widehat{x^\alpha \varphi}(\xi) = (-1)^{|\alpha|} D_\xi^\alpha \widehat{\varphi}(\xi)$.

We also recall some elementary but useful estimates:

Lemma A.3.10. *For* $\varphi \in \mathcal{S}'(\mathbb{R}^n)$*:*

(i) $\|\widehat{\varphi}\|_{L^\infty} \leq \|\varphi\|_{L^1}$,

(ii) $\|\varphi\|_{L^\infty} \leq \dfrac{1}{(2\pi)^n}\|\widehat{\varphi}\|_{L^1}$,

(iii) $\|\widehat{\varphi}\|_{L^1} \leq C \sup_{|\alpha| \leq n+1} \|\partial_x^\alpha \varphi\|_{L^1}$.

The proofs are left as exercise for the reader. We now recall the Paley-Wiener Theorem, see [Rud91, Theorem 7.23]. Let $X \subset \mathbb{R}^n$ be an open set. If $u \in \mathcal{E}'(X)$ is a distribution with compact support in X, then its Fourier transform $\widehat{u}(\xi) := (u, e^{-i\xi \bullet})$ is well-defined, for $\xi \in \mathbb{R}^n$. More generally, one can consider the Fourier transform as a function of the complex variable $z \in \mathbb{R}^n$ and $\widehat{u}(z) := (u, e^{-iz\bullet})$. This is an *entire function* on $\mathbb{C}^n$ (whose restriction to $\mathbb{R}^n$ is the usual Fourier transform of u).

Theorem A.3.11. *Let* $u \in \mathcal{E}'(\mathbb{R}^n)$ *be a distribution of order* N *with compact support inside* $B(0, R)$*. Then* $\mathbb{C}^n \ni z \mapsto \widehat{u}(z)$ *is entire and there exists a constant* $C > 0$ *such that:*

$$|\widehat{u}(z)| \leq C \langle z \rangle^N e^{R|\Im(z)|}. \tag{A.3.3}$$

Conversely, if f *is an entire function in* $\mathbb{C}^n$ *satisfying* (A.3.3)*, for some* C, N *and* R*, then there is a distribution* $u \in \mathcal{E}'(\mathbb{R}^n)$ *with support in* $B(0, R)$ *such that* $f = \widehat{u}$.

A.3.4 Integration theory

See [Rud87, Chapter 3]. For $1 \leq p < \infty$, we let $L^p(\mathbb{R}^n)$ be the space of measurable functions whose p-th power of the absolute value is integrable, that is such that

$$\|\varphi\|_{L(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |\varphi(x)|^p \, \mathrm{d}x \right)^{1/p} < \infty.$$

We let $L^\infty(\mathbb{R}^n)$ be the space of measurable almost-everywhere bounded functions. We have the usual estimates:

Lemma A.3.12 (Standard estimates). (i) *(Hölder's inequality) Let* $1 \leq p, q \leq \infty$ *such that* $1/p + 1/q = 1$*. Given* $f \in L^p(\mathbb{R}^n), g \in L^q(\mathbb{R}^n)$*:*

$$\|fg\|_{L^1(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}. \tag{A.3.4}$$

(ii) *(Young's convolution inequality) Let* $1 \leq p, q \leq r \leq \infty$ *such that* $1 + 1/r = 1/p + 1/q$*. Given* $f \in L^p(\mathbb{R}^n), g \in L^q(\mathbb{R}^n)$*:*

$$\|f \star g\|_{L^r(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^q(\mathbb{R}^n)}. \tag{A.3.5}$$

We now recall the dominated convergence theorem:

Theorem A.3.13. *Let* $(f_n)_{n \in \mathbb{N}}$ *be a sequence of measurable functions on* $\mathbb{R}^n$ *converging pointwise to a measurable function* f*. Assume that there exists* $g \in L^1(\mathbb{R}^n)$ *such that for almost every* $x \in \mathbb{R}^n$*, for all* $n \in \mathbb{N}$*,* $|f_n(x)| \leq g(x)$*. Then* $f \in L^1(\mathbb{R}^n)$ *and*

$$\int_{\mathbb{R}^n} f_n(x) \mathrm{d}x \to \int_{\mathbb{R}^n} f(x) \mathrm{d}x.$$

More generally, these results hold on any measure space.

A.4 Manifolds and distributions

Let M be a smooth manifold. We say that (κ, U) is a *chart* if $U \subset M$ is an open subset and $\kappa : U \to X \subset \mathbb{R}^n$ is a diffeomorphism. We say that a family $(\kappa_i, U_i)_{i=1}^N$ is a family of *cutoff charts* if these are all charts and $\cup_{i=1}^N U_i = M$ covers M. The manifold is *smooth* if the change of coordinates $\psi_{ij} := \kappa_j \circ \kappa_i^{-1}$ are smooth diffeomorphisms whenever they are defined. We now introduce the notion of distributions on manifolds. There are different ways of doing it, some being more elementary than others.

A.4.1 The density bundle

We will use the notion of *densities* on manifolds.

Definition A.4.1 (Density bundle). The density bundle $\Omega^1 M \to M$ is the line bundle over M whose sections at $\Omega^1_x M$ is the set of functions $|\mu| : T_x M \times ... \times T_x M \to \mathbb{R}$ such that, if $A \in \mathrm{GL}(T_x M)$, then

$$|\mu|(Av_1, ..., Av_n) = |\det A| |\mu|(v_1, ..., v_n). \tag{A.4.1}$$

Equivalently, a density is an alternating function[4] $|\mu| : T_x M \times ... \times T_x M \to \mathbb{R}$ such that $\mu(v_1, ..., \lambda v_i, ..., v_n) = |\lambda| \mu(v_1, ..., v_n)$ for all $i \in \{1, ..., n\}, \lambda \in \mathbb{R}$. More general, the bundle of s-densities (for $s \geq 0$) is defined similarly by requiring $|\mu|(Av_1, ..., Av_n) = |\det A|^s \, |\mu|(v_1, ..., v_n)$ instead of (A.4.1).

[4] A function is said to be alternating if $\mu(v_1, ..., v_n) = 0$ whenever $(v_1, ..., v_n)$ is not free.

It can be checked that the line bundle $\Omega^1 M \to M$ is the vector bundle whose transition matrices are given by the family $(|\det d\psi_{ij}|^{-1})_{ij}$ if $\psi_{ij} = \kappa_j \circ \kappa_j^{-1}$ are the changes of coordinates of the atlas (U_i, κ_i)

Exercise A.4.2. Check that this is indeed a cocycle defining a (trivial) line bundle over M.

Note that if $x \in M, \omega \in \Lambda^n T_x^* M$, then $|\omega|$ defined by $|\omega|(v_1, ..., v_n) := |\omega(v_1, ..., v_n)|$ is an element of the density line $\Omega_x^1 M$. The manifold M is *orientable* if and only if $\Lambda^n T^* M \to M$ is trivial while $\Omega^1 \to M$ is *always* trivial. If M is orientable and $\omega \in C^\infty(M, \Lambda^n T^* M)$ is a smooth nowhere vanishing density such that $\int_M \omega > 0$, then ω can be naturally identified with a smooth positive density. Similarly to volume forms, there is a well-defined notion of integration of densities:

$$\int : C^\infty(M, \Omega^1 M) \to \mathbb{R}.$$

(It suffices to pullback the density to $\mathbb{R}^n$ and to integrate it against the Lebesgue measure on $\mathbb{R}^n$, just like volume forms.) In particular, if $|\mu| \in C^\infty(M, \Omega^1 M), \varphi \in C^\infty(M)$, there is a well-defined pairing $(|\mu|, \varphi)$ simply defined by integration of the density $\varphi|\mu|$ on M, namely:

$$(|\mu|, \varphi) := \int_M \varphi|\mu|. \tag{A.4.2}$$

Observe that if $\kappa : M \to M$ is a diffeomorphism, then it induces a natural action on smooth sections of $\Omega^1 M$ by:

$$(\kappa^*|\mu|)_x := |\mu|_{\kappa(x)}(d\kappa(x)v_1, ..., d\kappa(x)v_n). \tag{A.4.3}$$

(Check that this is a well-defined density.) Note that this is totally similar to the induced action on k-forms given by

$$(\kappa^*\alpha)_x := \alpha_{\kappa(x)}(d\kappa(x)v_1, ..., d\kappa(x)v_n),$$

if $\alpha \in C^\infty(M, \Lambda^k T^* M)$. In particular, if $\kappa : X \to Y$ is a diffeomorphism between open subsets of $\mathbb{R}^n$, one can write $|dx| = |dx_1 \wedge ... \wedge dx_n|, |dy| = |dy_1 \wedge ... \wedge dy_n|$ the natural densities correspond to the Lebesgue measure. It is then straightforward to see that $\kappa^*|dy| = |\det d\kappa(x)||dx|$. (Check that as an exercise.)

As in $\mathbb{R}^n$, the pairing (A.4.2) satisfies the following rule:

$$(\kappa^*|\mu|, \varphi) = (|\mu|, \kappa_*\varphi), \tag{A.4.4}$$

where $\kappa_*\varphi(y) := (\kappa^{-1})^*\varphi(y) = \varphi(\kappa^{-1}(y))$ (pushforward of functions).

Exercise A.4.3. Apply (A.4.4) in the case where $\kappa : X \to Y$ is a diffeomorphism between open subsets in $\mathbb{R}^n$. Make sure you retrieve the change of variable formula.

As $\Omega^1 M \to M$ is a line bundle, one can write $\kappa^*|\mu| = a|\mu|$ for some smooth function $a \in C^\infty(M)$. This equality *defines* the (absolute value of) determinant of κ with respect to $|\mu|$.

A.4.2 Definition of distributions

The previous discussion and the pairing formula (A.4.2) show that smooth densities are a particular case of distributions on manifolds. On manifolds, we will therefore think of distributions as a *generalization of densities*. In particular, they will be behave like densities with respect to the action of the diffeomorphism group.

Definition A.4.4. A distribution $u \in \mathcal{D}'(M)$ is a continuous linear functional on the space of smooth functions $C^\infty(M)$ of (locally) finite order on M, that is such that for all compact $K \subset M$, there exists $N \in \mathbb{N}$ such that:

$$\forall \varphi \in C^\infty_{\mathrm{comp}}(K), \qquad |(u,\varphi)| \leq C_K \|\varphi\|_{C^N(K)}.$$

Here is the C^N-norm may be defined by different means: by local charts or via a metric for instance. If M is closed (which we will assume in the following), all the C^N norms (defined either by a metric or by means of local charts) coincide and they only depend on the differentiable structure of the manifold.

We saw on $\mathbb{R}^n$, that there is a quite natural way to embed $C^\infty(X) \hookrightarrow \mathcal{D}'(X)$ as in Example 2.1.3 by means of the map $C^\infty(X) \ni f \mapsto f|dx| \in \mathcal{D}'(X)$. However, this embedding *depends* on a choice of (smooth) density in X and the distribution

$$(fa|dx|, \varphi) := \int_X f(x)\varphi(x) \quad a(x)\mathrm{d}x,$$

could have also been chosen, if $a \in C^\infty(X)$ is some positive function. In $\mathbb{R}^n$, there is natural density given by the Lebesgue measure (the unique translation-invariant smooth measure) but this will no longer be the case on smooth manifolds. If M is a smooth closed manifold, we therefore fix an arbitrary smooth density $|\mu|$ on M and consider the embedding $C^\infty(M) \ni f \mapsto f|\mu| \in \mathcal{D}'(M)$ defined as

$$(f|\mu|, \varphi) := \int_M f(x)\varphi(x)|\mu|(x).$$

Here and below, the terminology *smooth density* refers to a smooth real nowhere vanishing section of the density bundle introduced in Definition A.4.1 whose integral is a strictly positive real number.

We now discuss the Schwartz kernel Theorem 1.2.1. It holds on manifolds as well but there is one subtlety. If M, N are smooth closed manifolds and $A : C^\infty(M) \to \mathcal{D}'(N)$, then there is a unique well-defined distribution $K_A \in \mathcal{D}'(N \times M)$ satisfying the equality

$$(A\varphi, \psi) = (K_A, \psi \otimes \varphi),$$

for all smooth functions $\varphi \in C^\infty(M), \psi \in C^\infty(N)$. In other words, $\mathcal{D}'(N \times M)$ is naturally isomorphic to $\mathrm{Hom}(C^\infty(M), \mathcal{D}'(N))$. For instance, if $y_0 \in M, x_0 \in N$, then $A\varphi := \varphi(y_0)\delta_{x_0}$ is a bounded operator $A : C^\infty(M) \to \mathcal{D}'(N)$, whose Schwartz kernel is given by $K_A(x,y) = \delta_{x_0}(x)\delta_{y_0}(y)$ and this requires no choices. Nevertheless, in practice, one considers bounded operators $A : C^\infty(M) \to C^\infty(N)$ and, in order to produce an operator $A_{\mathrm{induced}} : C^\infty(M) \to \mathcal{D}'(N)$ (and thus a Schwartz kernel $K_{A_{\mathrm{induced}}}$), one needs to *choose* an embedding $C^\infty(N) \hookrightarrow \mathcal{D}'(N)$ which, in turn, requires the choice of a density. Hence, the Schwartz kernel $K_A \in \mathcal{D}'(N \times M)$ is the distribution such that

$$((A\varphi)|\mu|, \psi) = \langle K_A, \psi \otimes \varphi).$$

Changing the density by $a(\bullet)|\mu|$ then changes the Schwartz kernel according to $\widetilde{K}_A(x,y) = a(x)K_A(x,y)$. Note that the density lives on the target space N.

To complete the discussion, let us see what happens if $A : C^\infty(M) \to C^\infty(M)$ has smooth Schwartz kernel (smoothness of the kernel does not require any choice of density). We take $M = N$ for the sake of simplicity. This means that there exists a function $R_A \in C^\infty(M \times M)$ such that

$$A\varphi(x) = \int_M R_A(x,y)\varphi(y)|\mu|(y).$$

Of course, this function R_A depends on the choice of density $|\mu|$. What is the connection between $R_A \in C^\infty(M \times M)$ and the Schwartz kernel $K_A \in \mathcal{D}'(M \times M)$ previously defined? Well, the following equality holds:

$$\begin{aligned}(A\varphi|\mu|, \psi) &= \int_M A\varphi(x)\psi(x)|\mu|(x) \\ &= (K_A, \psi \otimes \varphi) = \int_{M\times M} R_A(x,y)\varphi(y)\psi(x)|\mu|(y)|\mu|(x),\end{aligned}$$

that is $K_A = R_A|\mu|(x)|\mu|(y) \in \mathcal{D}'(M \times M)$. In other words, K_A corresponds to R_A via the embedding $C^\infty(M \times M) \hookrightarrow \mathcal{D}'(M \times M)$ provided by the product density $|\mu|_x \otimes |\mu|_y$. We also observe that changing $|\mu|$ by $a(\bullet)|\mu|$ changes K_A (resp. R_A) by $\widetilde{K}_A = a(x)K_A$ (resp. $\widetilde{R}_A(x,y) = R_A(x,y)a(y)^{-1}$). Usually, in what follows, even if A does not have smooth Schwartz kernel, we will freely identify K_A and R_A.

One way to avoid such choices is to define $\mathcal{D}'(M)$ as the dual of smooth sections of the density bundle, in which case there is an obvious canonical embedding $C^\infty(M) \hookrightarrow \mathcal{D}'(M)$.

A.4.3 Vector bundles

The previous discussion can be extended to vector bundles. We let $E \to M, F \to N$ be two smooth real vector bundles over the manifolds M, N. We let g_E, g_F be real metrics on E, F. The complexification of E and F are given by $E_{\mathbb{C}} := E \otimes_{\mathbb{R}} \otimes\mathbb{C}, F_{\mathbb{C}} := F \otimes_{\mathbb{R}} \mathbb{C}$. Without further notice, we will indifferently speak of E or its complexification $E_{\mathbb{C}}$. For $\varphi, \psi \in C^\infty(M, E)$, as in the case of the trivial line bundle, we have the pairings

$$(\varphi, \psi) = \int_M g_E(\varphi(x), \psi(x))\mathrm{d}\mu(x), \quad \langle \varphi, \psi \rangle = \int_M g_E(\varphi(x), \overline{\psi}(x))\mathrm{d}\mu(x).$$

A distribution $u \in \mathcal{D}'(M, E)$ is the data of a (local) section $u = \sum_i u_i \mathbf{e}_i$, where $(\mathbf{e}_i)_{1\leq i\leq r}$ is a local orthonormal basis of E and $u_i \in \mathcal{D}'(M)$ are locally defined distributions. Given $\varphi \in C^\infty(M, E)$ which can be written locally as $\varphi = \sum_i \varphi_i \mathbf{e}_i$, the pairing is given by

$$(u, \varphi) := \sum_i (u_i, \varphi_i).$$

This is independent of the choice of basis $(\mathbf{e}_i)_{1\leq i\leq r}$. There are other ways of defining distributions in order to avoid having to choose a metric and a basis but we prefer to stick to this down-to-earth description.

We let $\pi_L : M \times N \to M, \pi_R : M \times N \to N$ be the respective left and right projections i.e. $\pi_L(x, y) = x, \pi_R(x, y) = y$. We then define the vector bundle

$$E \boxtimes F := \pi_L^* E \otimes \pi_R^* F,$$

that is given $(x, y) \in M \times N$, the fibers over $(x, y) \in M \times N$ is $E_x \otimes F_y$. An operator with smooth Schwartz kernel $A : \mathcal{D}'(N, F) \to C^\infty(M, E)$ can be given similarly by a smooth section $R_A \in C^\infty(M \times N, E \boxtimes F^*)$ such that

$$A\varphi(x) = \int_M R_A(x, y)\varphi(y)\mathrm{d}\mu(y).$$

Observe that for every $x \in M, y \in N$, using that $E \boxtimes F^*(x, y) = E_x \otimes F_y^* = \mathrm{Hom}(F_y, E_x)$, the quantity $K_A(x, y)\varphi(y)$ belongs to E_x. More generally, continuous operators $A : C^\infty(N, F) \to \mathcal{D}'(M, F)$ have Schwartz kernel in $\mathcal{D}'(M \times N, E \boxtimes F^*)$ and it satisfies the identity:

$$(A\varphi, \psi) = (K_A, \psi \otimes \varphi^\sharp),$$

where, if $\varphi = \sum_i \varphi_i \mathbf{f}_i$ locally, we have $\varphi^\sharp = \sum_i \varphi_i \mathbf{f}_i^*$ (and $(\mathbf{f}_i^*)_{1\leq i\leq r}$ is a local orthonormal basis for F^*).

A.5 Fredholm operators

In order to read the following section, it will be convenient to have in mind results concerning *compact* operators on Banach spaces, which can be found in §A.2.3 (see especially Theorem A.2.10).

A.5.1 Definition of Fredholm operators

Let E_1, E_2 be two Banach spaces and denote by $\mathcal{L}(E_1, E_2)$ the space of continuous linear operators from E_1 to E_2. We start with the following general definition:

Definition A.5.1. Let $A \in \mathcal{L}(E_1, E_2)$. We say that A is *Fredholm* if $\dim \ker(A) < \infty$ and $\dim \operatorname{coker}(A) < \infty$. We let $\mathcal{F}(E_1, E_2) \subset \mathcal{L}(E_1, E_2)$ be the set of all Fredholm operators. We define the *index* of Fredholm operators as the map:

$$\mathrm{ind} : \mathcal{F}(E_1, E_2) \to \mathbb{Z}, \quad \mathrm{ind}(A) := \dim \ker(A) - \dim \operatorname{coker}(A).$$

If E_1, E_2 were finite-dimensional, we would have $\dim E_1 = \dim \ker(A) + \dim \operatorname{ran}(A) =$

$\dim \ker(A) + \dim(E_2) - \dim \operatorname{coker}(A)$, that is:

$$\operatorname{ind}(A) = \dim \ker(A) - \dim \operatorname{coker}(A) = \dim E_1 - \dim E_2.$$

One can think of the index as a sort of generalization of this formula to the infinite-dimensional setting.

Lemma A.5.2. *If* $\dim \operatorname{coker}(A) < \infty$*, then* $\operatorname{ran}(A)$ *is closed in* E_2*.*

The proof is postponed below. Recall that the open mapping Theorem A.2.5 asserts that a continuous linear surjective map between Banach spaces is open. A useful corollary is:

Corollary A.5.3. *Let* $A \in \mathcal{L}(E_1, E_2)$ *and bijective. Then* A *is a homeomorphism, that is* A^{-1} *is continuous.*

Proof. Let $U \subset E_1$ be an open set. Then $(A^{-1})^{-1}(U) = A(U)$ is open by Theorem A.2.5, that is A^{-1} is continuous. Also note that if $F \subset E_1$ is closed, then $A(F) = (A^{-1})^{-1}(F) \subset E_2$ is closed. $\square$

Proof of Lemma A.5.2. Let C be an algebraic complement of $\operatorname{ran}(A)$ i.e. a finite-dimensional subspace $C \subset E_2$ such that $\operatorname{ran}(A) \oplus C = E_2$. Define:

$$\Psi : E_1/\ker(A) \oplus C \to E_2, \quad \Psi(x, c) := A(x) + c.$$

We endow $E_1/\ker(A) \oplus C$ with the norm $\|(x, c)\| := \|x\|_{E_1} + \|c\|_{E_2}$. Then $\|\Psi(x, c)\|_{E_2} \leq \|A\|\|x\|_{E_1} + \|c\|_{E_2} \leq \max(1, \|A\|)\|(x, c)\|$, that is Ψ is continuous. Moreover, it is bijective by construction. By the Banach Theorem A.2.5, we deduce that Ψ^{-1} is continuous. In particular, $\Psi(E_1/\ker A \oplus \{0\}) = (\Psi^{-1})^{-1}(E_1/\ker A \oplus \{0\})$ is closed as $E_1/\ker A \oplus \{0\}$ is closed. $\square$

The following characterization is more tractable for the dimension of the cokernel:

Lemma A.5.4. *Let* $A \in \mathcal{L}(E_1, E_2)$ *and assume that* $\dim \operatorname{coker}(A) < \infty$*. Then* $\dim \operatorname{coker} A = \dim \ker A^*$*, where* $A^* : E_2^* \to E_1^*$ *is the adjoint of* A*.*

Proof. We need to construct an isomorphism $\Psi : \operatorname{coker} A \to \ker A^*$. We have:

$$\ker A^* = \{\xi \in E_2^* \mid \langle A^*\xi, x\rangle = 0, \forall x \in E_1\} = \{\xi \in E_2^* \mid \langle \xi, \operatorname{ran}(A)\rangle = 0\}.$$

Take C, a finite-dimensional space such that $\operatorname{ran}(A) \oplus C = E_2$ so that $\operatorname{coker}(A) \simeq C$. Take a basis $(c_1, ..., c_p)$ of C and define $\xi_i =: \Psi(c_i) \in \ker A^*$ to be the linear form vanishing on $\operatorname{ran}(A) \oplus \operatorname{Span}(c_1, ..., \hat{c}_i, ..., c_p)$ and such that $\xi_i(c_i) = 1$. Note that this indeed a continuous linear form as its kernel is closed. Injectivity is straightforward: if $\Psi(\sum_i \lambda_i c_i) = \sum_i \lambda_i \xi_i = 0$, then evaluating at c_j, we get $\lambda_j = 0$. Surjectivity is also immediate as $\xi \in \ker A^*$ actually defines an element $\xi \in C^*$ (since ξ vanishes on $\operatorname{ran}(A)$) and any element of C^* can be written as a linear combination of the ξ_i's. $\square$

We start by characterizing a first class of Fredholm operators:

Lemma A.5.5. *Let* $K \in \mathcal{K}(E_1)$*. Then* $\mathbb{1} + K$ *is Fredholm and* $\operatorname{ind}(\mathbb{1} + K) = 0$*.*

Proof. Consequence of the last items in Theorem A.2.10 $\square$

A.5.2 Quasi-inverses of Fredholm operators

We now study the existence of inverses for Fredholm operators.

Lemma A.5.6 (Quasi-inverse of Fredholm operators)**.** *Let $A \in \mathcal{F}(E_1, E_2)$. Then, there exists $B \in \mathcal{F}(E_2, E_2)$ such that*

$$BA = \mathbb{1} - \Pi_{\ker A}, \quad AB = \Pi_{\operatorname{ran} A},$$

where $\Pi_{\ker A}$ is a finite-rank projection onto $\ker A$ and $\Pi_{\operatorname{ran} A}$ is a projection onto $\operatorname{ran} A$. If E_1 and E_2 are Hilbert spaces, one can choose these as the orthogonal projections and thus $B, \Pi_{\ker A}, \Pi_{\operatorname{ran} A}$ are unique.

Proof of Lemma A.5.6. By Lemma A.2.8, we can consider a closed subspace $F_1 \subset E_1$ such that $F_1 \oplus \ker A = E_1$. Then $A : F_1 \to \operatorname{ran}(A)$ is an isomorphism. We also consider a finite-dimensional subspace $F_2 \subset E_2$ such that $F_2 \oplus \operatorname{ran}(A) = E_2$. We define $B : E_2 \to F_1$ as $B|_{\operatorname{ran}(A)} := A^{-1}$ and $B|_{F_2} = 0$. It is then immediate to check that $BA = \mathbb{1} - \Pi_{\ker A}$, where $\Pi_{\ker A}$ is the projection onto $\ker A$ with kernel F_1 and $AB = \Pi_{\operatorname{ran} A}$, where $\Pi_{\operatorname{ran} A}$ is the projection onto $\operatorname{ran}(A)$ with kernel F_2. The fact that B is Fredholm is a consequence of the construction. In the case of Hilbert spaces, one can take orthogonal complements F_1 and F_2 which make the projections orthogonal. $\square$

A.5.3 Properties of the index

We now describe the important properties of the index. We start by its multiplicative character:

Lemma A.5.7. *Let $A \in \mathcal{F}(E_1, E_2), B \in \mathcal{F}(E_2, E_3)$. Then $BA \in \mathcal{F}(E_1, E_3)$ with:*

$$\operatorname{ind}(BA) = \operatorname{ind}(A) + \operatorname{ind}(B).$$

Proof. The first step is to construct spaces of finite codimension $L_i \subset E_i$ for $i = 1, 2, 3$ such that $\ker A|_{L_1} = \{0\}$, $AL_1 = L_2$, $\ker B|_{L_2} = \{0\}$ and $BL_2 = L_3$. Indeed, if this is the case, then we may look at the induced operator $\widehat{A} : E_1/L_1 \to E_2/L_2$ (resp. $\widehat{B} : E_2/L_2 \to E_3/L_3$). We claim that $\operatorname{ind} \widehat{A} = \operatorname{ind} A$ (resp. $\operatorname{ind} \widehat{B} = \operatorname{ind} B$). Indeed, $\ker A \oplus L_1$ admits a finite-dimensional complement M_1 such that

$$\ker A \oplus L_1 \oplus M_1 = \ker A \oplus L_1' = E_1.$$

We can decompose $E_2 = AL_1' \oplus M_2 = AL_1 \oplus AM_1 \oplus M_2$, and then $\operatorname{ind}(A) = \dim L_1' - \dim M_2$. Using that E_i/L_i are finite-dimensional, we have:

$$\begin{aligned}
\operatorname{ind} \widehat{A} &= \dim E_1/L_1 - \dim E_2/L_2 \\
&= \dim \ker A + \dim M_1 - (\dim AM_1 + \dim M_2) \\
&= \dim \ker A + \dim M_1 - \dim M_1 - \dim M_2 = \operatorname{ind} A.
\end{aligned}$$

Hence:

$$\begin{aligned}
\operatorname{ind}(BA) &= \operatorname{ind}(\widehat{BA}) \\
&= \dim E_1/L_1 - \dim E_3/L_3 \\
&= (\dim E_1/L_1 - \dim E_2/L_2) + (\dim E_2/L_2 - \dim E_1/L_1) \\
&= \operatorname{ind}(\widehat{A}) + \operatorname{ind}(\widehat{B}) = \operatorname{ind}(A) + \operatorname{ind}(B).
\end{aligned}$$

It now remains to construct the spaces L_1, L_2. We let L_1' and L_2' be closed complements of $\ker A$ and $\ker B$ in E_1 and E_2. Then $L_2 := L_2' \cap \operatorname{ran} A$, $L_1 := A|_{L_1'}^{-1}(L_2)$ and $L_3 := BL_2$ satisfy the requirements. □

There is also a converse statement to the quasi-inverse Lemma A.5.6.

Lemma A.5.8. *Let $A \in \mathcal{L}(E_1, E_2)$ and assume that there exists $B_1, B_2 \in \mathcal{L}(E_2, E_1)$ such that $B_2A - \mathbb{1} \in \mathcal{K}(E_1), AB_1 - \mathbb{1} \in \mathcal{K}(E_2)$. Then A, B_1, B_2 are Fredholm. Moreover,* $\operatorname{ind}(A) = -\operatorname{ind}(B_1) = -\operatorname{ind}(B_2)$.

Proof. Write $K_1 := B_2A - \mathbb{1}, K_2 := AB_1 - \mathbb{1}$. If $x \in E_1$ satisfies $Ax = 0$, then $(K_1 - \mathbb{1})x = 0$, that is $x \in \ker(K_1 - \mathbb{1})$. Since K_1 is compact, this is a finite-dimensional subspace and thus $\dim\ker A < \infty$. The fact that A has closed range follows from the equality $AB_1 = \mathbb{1} + K_2$, as in the proof of Theorem A.2.10. Eventually, $\operatorname{ran}(\mathbb{1} + K_2) = \operatorname{ran}(AB_1) \subset \operatorname{ran}(A)$ and by Theorem A.2.10, $\mathbb{1} + K_2$ has finite codimension, and so does $\operatorname{ran}(A)$. □

Theorem A.5.9 (Stability of the index by small continuous perturbations)**.** *The set of Fredholm operators $\mathcal{F}(E_1, E_2)$ is an open subset of $\mathcal{L}(E_1, E_2)$ (for the operator norm topology) and the index map* $\operatorname{ind} : \mathcal{F}(E_1, E_2) \to \mathbb{Z}$ *is continuous. As a consequence, if $(A_t)_{t\in[0,1]}$ is a continuous family of Fredholm operators $A_t \in \mathcal{F}(E_1, E_2)$, then there index is constant.*

Proof. Let $A \in \mathcal{F}(E_1, E_2)$. We need to show that there exists some $\varepsilon > 0$ such that any perturbation $A+P$, with $P \in \mathcal{L}(E_1, E_2), \|P\| < \varepsilon$, is Fredholm with same index. By Lemma A.5.6, we can consider $B \in \mathcal{F}(E_2, E_1)$ such that $AB = \mathbb{1} + T_2, BA = \mathbb{1} + T_1$, where T_i is of finite rank. We set $\varepsilon := \|B\|^{-1}$ so that $\mathbb{1} + PB$ is invertible by Neumann series and $(\mathbb{1} + PB)^{-1} \in \mathcal{L}(E_2)$. Setting $B' := B(\mathbb{1} + PB)^{-1}$, we then have:

$$\begin{aligned}
(A + P)B' &= AB(\mathbb{1} + PB)^{-1}) + PB(\mathbb{1} + PB)^{-1}) \\
&= (\mathbb{1} + PB)^{-1}) + T_2(\mathbb{1} + PB)^{-1}) + \mathbb{1} - (\mathbb{1} + PB)^{-1}) \\
&= \mathbb{1} + T_2',
\end{aligned}$$

where $T_2' := T_2(\mathbb{1} + PB)^{-1})$ has finite rank. Similarly, we can construct a left-inverse for $A + P$. By Lemma A.5.8, we deduce that $A + P$ is Fredholm. Moreover, by Lemma A.5.7, we have $\operatorname{ind}(B') = \operatorname{ind}(B)$ as $\operatorname{ind}(\mathbb{1} + PB)^{-1}) = 0$ since the operator is invertible. Hence by Lemma A.5.8:

$$\operatorname{ind}(A) = -\operatorname{ind}(B) = -\operatorname{ind}(B') = \operatorname{ind}(A + P).$$

□

Lemma A.5.10 (Stability of the index by compact perturbations)**.** *Let* $A \in \mathcal{F}(E_1, E_2)$, $K \in \mathcal{K}(E_1, E_2)$. *Then* $A + K \in \mathcal{F}(E_1, E_2)$ *and:*

$$\operatorname{ind}(A + K) = \operatorname{ind}(A)$$

Proof. It suffices to show that for any compact perturbation $K \in \mathcal{K}(E_1, E_2)$, one has $A + K \in \mathcal{F}(E_1, E_2)$. Indeed, considering $A + tK$, we have a continuous family of Fredholm operators so they must of constant index, thus proving the last part of the Lemma. □

A.5.4 Exercises

Exercise 1

Let $\mathbb{S}^1 := \mathbb{R}/(2\pi\mathbb{Z})$ be the circle and define $P_s := D_x - s$ on $\mathbb{S}^1$, where $D_x := i^{-1}\partial_x$.

1. Show that $\mathbb{R} \ni s \mapsto P_s \in \mathrm{Diff}^1(\mathbb{S}^1)$ is an analytic family of Fredholm operators of index 0.

2. Compute $\ker P_s$ and $\operatorname{ran} P_s$ for $s \in \mathbb{R}$. Are there special values?

Exercise 2

Let H be a (separable) Hilbert space. Let $(\pi_0(\mathcal{F}(H)), \circ)$ be the monoid[5] of connected components of $\mathcal{F}(H)$ endowed with the composition law $\circ$.

1. Show that $(\pi_0(\mathcal{F}(H)), \circ)$ is a well-defined monoid. What is the identity element?

2. Prove that $(\pi_0(\mathcal{F}(H)), \circ)$ can actually be turned into a group. What is the inverse of an element?

3. Deduce that $\operatorname{ind} : \pi_0(\mathcal{F}(H)) \to \mathbb{Z}$ is a well-defined group homomorphism.

4. Show that $0 \to \pi_0(\mathcal{F}(H)) \to \mathbb{Z} \to 0$ is exact.

A.6 Spectral theory

See [Rud91, Chapter 13]. We now quickly discuss the spectral theorem for unbounded self-adjoint operators. Before this, let us recall the useful Riesz representation theorem.

Theorem A.6.1. *Let* $(\mathcal{H}, \langle \bullet, \bullet \rangle)$ *be a Hilbert space. Let* $T : \mathcal{H} \to \mathbb{C}$ *be a continuous linear form on* $\mathcal{H}$*. Then, there exists a unique* $v \in \mathcal{H}$ *such that for all* $u \in \mathcal{H}$, $T(u) = \langle u, v \rangle$.

[5] A monoid is a set endowed with an associative binary operation together with an identity element. As opposed to groups, there is *a priori* no inverses in monoids.

In particular, we get by Cauchy-Schwarz:

$$\|T\|_{\mathcal{H}'} := \sup_{u\in\mathcal{H},\|u\|=1} |T(u)| = \|v\|_{\mathcal{H}}.$$

Let $(\mathcal{H}, \langle \bullet, \bullet \rangle)$ be a separable Hilbert space. Let $A : \mathcal{H} \to \mathcal{H}$ be an unbounded operator with dense domain $\mathcal{D}(A) \subset \mathcal{H}$. The *adjoint operator* A^* is defined in the following way. We let $\mathcal{D}(A^*) \subset \mathcal{H}$ be the space of all $v \in \mathcal{H}$ such that the linear map

$$\mathcal{H} \ni u \mapsto \langle Au, v\rangle$$

is continuous.

Remark A.6.2. Note that this map is initially only well-defined for $u \in \mathcal{D}(A)$. Continuity is the requirement that there exists a constant $C > 0$ such that for all $u \in \mathcal{D}(A)$, $|\langle Au, v\rangle| \leq C\|u\|_{\mathcal{H}}$. By density of $\mathcal{D}(A) \subset \mathcal{H}$, this admits a continuous extension to $u \in \mathcal{H}$.

For $v \in \mathcal{D}(A^*)$, we can then apply the Riesz representation theorem in order to obtain an element $w(v) \in \mathcal{H}$ such that for all $u \in \mathcal{H}$, $\langle Au, v\rangle = \langle u, w(v)\rangle$. We set $A^*v := w(v)$. This is a linear unbounded operator with domain (not necessarily dense $\mathcal{D}(A^*)$). In particular, the following equality holds for all $u \in \mathcal{D}(A), v \in \mathcal{H}$ *or* for all $u \in \mathcal{H}, v \in \mathcal{D}(A^*)$:

$$\langle Au, v\rangle = \langle u, A^*v\rangle. \tag{A.6.1}$$

Definition A.6.3. An unbounded operator $A : \mathcal{H} \to \mathcal{H}$ with dense domain $\mathcal{D}(A)$ is said to be self-adjoint if $(A, \mathcal{D}(A)) = (A^*, \mathcal{D}(A^*))$, that is $\mathcal{D}(A) = \mathcal{D}(A^*)$ and $A = A^*$ on $\mathcal{D}(A)$.

We now describe some standard and useful notions in operator theory (on Hilbert spaces). If A, B are two unbounded operators defined on the Hilbert space $\mathcal{H}$, we will write $A \subset B$ if the graph of A is contained in that of B and we will say that B *is an extension of* A.

Definition A.6.4 (Closed operators)**.** An unbounded operator $A : \mathcal{H} \to \mathcal{H}$ with domain $\mathcal{D}(A)$ is closed if its graph $\mathcal{G}(A) := \{(u, Au) \mid u \in \mathcal{D}(A)\}$ is closed. Equivalently, if $(u_n)_{n\geq 0}$ is a sequence of elements $u_n \in \mathcal{D}(A)$ such that $u_n \to u$ and $Au_n \to v$ in $\mathcal{H}$, then $u \in \mathcal{D}(A)$ and $Au = v$.

We have the following, know as the closed graph theorem, see [Rud91, Theorem 2.15]:

Theorem A.6.5 (Closed gr aph theorem)**.** *Let $A : \mathcal{H} \to \mathcal{H}$ be a linear operator with domain $\mathcal{D}(A)$. Then A is continuous if and only if $\mathcal{D}(A) = \mathcal{H}$ and A is closed.*

Consider the operator $A := D_x$ acting as an unbounded operator on $L^2([0, 1], \mathbb{C})$. Consider the domain $\mathcal{D}(A) := H^1([0, 1], \mathbb{C})$, the space of L^2-functions on $[0, 1]$ such that the derivative is also in L^2. Then (D_x, H^1) is closed. Indeed, if $u_n \in H^1$ satisfies $u_n \to u, D_xu_n \to f$ in L^2, then it is clear that $f = D_xu$ in the sense of distributions and thus $u \in L^2, u' \in L^2$, that is $u \in H^1$ and $D_xu = f$ in L^2.

Definition A.6.6 (Symmetric operators)**.** An (unbounded) operator $A : \mathcal{H} \to \mathcal{H}$ is said to be symmetric if

$$\langle Au, v \rangle = \langle u, Av \rangle,$$

for all $u, v \in \mathcal{D}(A)$.

As a consequence, we see that

$$|\langle Au, v \rangle = |\langle u, Av \rangle \leq C\|u\|, \quad C = \|Av\|,$$

that is $u \mapsto \langle Au, v \rangle$ is continuous for all $v \in \mathcal{D}(A)$ which implies that $\mathcal{D}(A) \subset \mathcal{D}(A^*)$, that is

$$A \subset A^*. \tag{A.6.2}$$

Conversely, an operator such that $A \subset A^*$ is obviously symmetric. As a consequence:

Lemma A.6.7. *Symmetric densely defined operators are precisely those for which the inclusion* (A.6.2) *is an equality.*

The operator $A = D_x$ with dense domain $\mathcal{D}(A) := C_0^\infty([0,1], \mathbb{C})$ (smooth functions vanishing at $x = 0$ and 1) is symmetric since it satisfies integration by parts without boundary terms. However, it is not self-adjoint. See [Rud91, Example 13.4] for further discussion.

Definition A.6.8 (Closable operators, closure of an operator)**.** We say that $A : \mathcal{H} \to \mathcal{H}$ with domain $\mathcal{D}(A)$ is closable if it admits a closed extension $\overline{A}$. The smallest closed extension of A is called the *closure* of A and denoted by $\overline{A}$.

Of course, if A is closable, then A and its closure $\overline{A}$ have the same closed extensions. We have:

Lemma A.6.9. *Symmetric operators are closable.*

We also have the following:

Lemma A.6.10. *If A is a densely defined operator in $\mathcal{H}$, then A^* is closed. In particular, a selfadjoint operator is closed.*

For the sake of completeness, let us also mention the following notion

Definition A.6.11 (Essential self-adjointness)**.** Let $A : \mathcal{H} \to \mathcal{H}$ be a symmetric operator with dense domain $\mathcal{D}(A)$. We say that A is *essentially self-adjoint* if the closure of A is self-adjoint. (Equivalently, it is essentially self-adjoint if it has a unique self-adjoint extension.)

The following criterium is useful to characterize self-adjoint operators, see [Rud91, Section 13.20]:

Theorem A.6.12. *Let $A : \mathcal{H} \to \mathcal{H}$ be a closed symmetric operator. Then A is self-adjoint if and only if* $\operatorname{ran}(A \pm i) = \mathcal{H}$.

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