Curvature of the Probability Weighting Function

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When individuals choose among risky alternatives, the psychological weight attached to an outcome may not correspond to the probability of that outcome. In rank-dependent utility theories, including prospect theory, the probability weighting function permits probabilities to be weighted nonlinearly. Previous empirical studies of the weighting function have suggested an inverse S-shaped function, first concave and then convex. However, these studies suffer from a methodological shortcoming: estimation procedures have required assumptions about the functional form of the value and/or weighting functions. We propose two preference conditions that are necessary and sufficient for concavity and convexity of the weighting function. Empirical tests of these conditions are independent of the form of the value function. We test these conditions using preference "ladders" (a series of questions that differ only by a common consequence). The concavity-convexity ladders validate previous findings of an S-shaped weighting function, concave up to $p < 0.40$, and convex beyond that probability. The tests also show significant nonlinearity away from the boundaries, 0 and 1. Finally, we fit the ladder data with weighting functions proposed by Tversky and Kahneman (1992) and Prelec (1995).
(*Decision Making*; *Expected Utility*; *Nonexpected Utility Theory*; *Prospect Theory*; *Risk*; *Risk Aversion*)

1. Introduction

Numerous examples, of which the Allais Paradox is the most celebrated, demonstrate the descriptive shortcomings of expected utility theory and its critical axiom, the independence axiom. Most notably, Kahneman and Tversky (1979) presented a series of systematic violations of the independence axiom over both gains and losses and proposed a model, prospect theory, that organized these choice patterns. Kahneman and Tversky's "paradoxes" have stood as a minimum hurdle for the slew of nonexpected utility theories that has followed (for reviews, see Machina 1987, Weber and Camerer 1987, Fishburn 1988). At the same time, empiricists have moved beyond these paradoxes, raising the bar for choice theories with a number of new empirical phenomena that theories must explain (summarized in Camerer 1992, 1995).

Camerer (1989) has pronounced the assemblage of descriptive findings "long and depressing," noting that no theory can explain all the empirical evidence. Although the latter is certainly true, the former pronouncement is somewhat exaggerated. Indeed studies have identified two critical empirical regularities that any good descriptive model should accommodate: gain-loss asymmetry and "nonlinearity in probability." Together, these two regularities capture the "fourfold pattern of risk attitudes": risk aversion for most gains and low probability losses, and risk seeking for most losses and low probability gains. The asymmetry between gains and losses is highlighted by the reflection effect and numerous other demonstrations that individuals treat losses and gains differently (Tversky and Kahneman 1991). This paper focuses on the second empirical regularity, nonlinearity in probability. Zeckhauser's Russian Roulette example (Kahneman and Tversky

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1979) provides an intuitive demonstration that the psychological weight assigned to an event may not correspond to the stated probability of that event:

> Suppose you are compelled to play Russian roulette, but are given the opportunity to purchase the removal of one bullet from the loaded gun. Would you pay as much to reduce the number of bullets from four to three as you would to reduce the number of bullets from one to zero?

Most people would pay considerably more to reduce the number of bullets from one to zero than from four to three. Psychologically, a reduction in the probability from 4/6 to 3/6 seems much less significant than a reduction from 1/6 to 0, and likewise from six to five bullets. The Russian Roulette example suggests that individuals do not treat probabilities linearly.[1]

This article focuses on the role of nonlinear probability weights in prospect theory (Kahneman and Tversky 1979, Tversky and Kahneman 1992). We focus on prospect theory because it accounts for most of the current empirical evidence. The two essential features of the prospect theory family are a value function, defined in terms of gains and losses, and a probability weighting function that permits probabilities to be weighted nonlinearly. These two features map directly to the empirical regularities of gain-loss asymmetry and nonlinearity in probability. Recently, Tversky and Kahneman (1992) offered a new version of prospect theory, cumulative prospect theory (CPT). The modified representation addresses two criticisms of original prospect theory: the original representation permitted violations of stochastic dominance and was limited to a maximum of two nonzero outcomes.

The purpose of this article is to provide greater clarity about the properties of the weighting function, both conceptually and empirically. In particular, we are motivated by the following four questions:

(i) What empirical properties does the probability weighting function satisfy?

(ii) For each empirical property of the weighting function, is there a corresponding intuitive (and empirically testable) preference condition?

(iii) Are probability weights nonlinear away from the boundaries (0 and 1)?

(iv) Do there exist parsimonious functional forms of the weighting function that are consistent with these empirical properties?

Empirical studies have consistently suggested an inverse-S-shaped weighting function, which is concave below and convex above some fixed $p^* < 0.40$. Such an S-shaped function is consistent with risk seeking for low probability gains and risk aversion for medium and high probability gains. Returning to the Russian Roulette example, concavity for small probabilities suggests that people would pay more to reduce the number of bullets from one to zero than from two to one, while convexity for large probabilities suggests a higher premium for a reduction from six to five than from five to four.

Previous tests, however, have some empirical shortcomings. First, most experiments were not designed as direct tests of the weighting function, but rather as tests of an axiom critical to some utility representation (e.g., independence, betweenness, comonotonic independence, or ordinal independence),[2] a hypothesis (e.g., fanning-in or fanning-out),[3] or a family of theories (e.g., expected utility, rank-dependent utility, weighted utility, etc.).[4] Furthermore, these tests have generally assumed functional forms for the value and/or weighting functions. Motivated by these shortcomings, we present two common-consequence conditions that are necessary and sufficient for concavity and convexity of the weighting function in the framework of any rank-dependent utility model, including CPT. Methodologically, these conditions permit a systematic test of the empirical properties of the weighting function that does not require assumptions about the shape of the value

[1] Direct empirical evidence of nonlinearity in probability is provided by systematic violations of betweenness, in the directions of both quasi-concavity and quasi-convexity, liking and disliking randomization (Camerer and Ho 1994).

[2] Kahneman and Tversky (1979), Camerer and Ho (1994), Wakker et al. (1994), Wu (1994).

[3] Chew and Waller (1986), Conlisk (1989), Prelec (1990).

[4] Starmer and Sugden (1989), Camerer (1989, 1992), Battalio et al. (1990), Harless (1992), Gigliotti and Sopher (1993).

function. Our tests validate the basic S-shape (concave and then convex) and show that the nonlinearity of the probability weighting function is not due merely to boundary effects.

The paper proceeds as follows. In §2, we review prospect theory and discuss some weaknesses of previous empirical tests of the weighting function. In §3, we establish two conditions that are consistent with the familiar S-shape of the weighting function. Section 4 describes an experiment involving a series of concavity-convexity "ladders." In §5, we fit two recently proposed functional forms of the weighting function to the data. Finally, in §6, we summarize our findings and propose some next steps in this research agenda. All proofs are left for the Appendix.

2. Review of Prospect Theory

In this section, we review Kahneman and Tversky's (1979) original prospect theory (hereafter called OPT) and Tversky and Kahneman's (1992) generalization, cumulative prospect theory (CPT; see also Starmer and Sugden 1988, Luce and Fishburn 1991, Wakker and Tversky 1993). For simplicity, in this paper we restrict ourselves to gains only. Although all the conditions and tests described also apply to losses, they do not apply directly to mixed gambles (losses and gains).

Let $(p, x; q, y)$ denote a prospect that gives a p chance at x, q chance at y, and a $1 - p - q$ chance at 0, where $x > y > 0$. In addition, $p + q < 1$. Under OPT,

$$(p, x; q, y) > (p', x'; q', y') \Leftrightarrow$$
$$\pi(p)v(x) + \pi(q)v(y) > \pi(p')v(x') + \pi(q')v(y'),$$

where π: $[0, 1] \rightarrow [0, 1]$ is a probability weighting function and $v(\cdot)$ is a value function defined with respect to a reference point, typically concave for gains and convex for losses.

For gains, CPT coincides with rank-dependent expected utility (RDEU) (Quiggin 1982, 1993; Quiggin and Wakker 1994; Yaari 1987; Green and Jullien 1988; Wakker 1989; Segal 1989). Whereas CPT and OPT are identical for simple prospects (one nonzero outcome), CPT assigns a different weight to the second highest nonzero outcome:

$$(p, x; q, y) > (p', x'; q', y') \Leftrightarrow$$
$$\pi(p)v(x) + [\pi(p + q) - \pi(p)]v(y)$$
$$> \pi(p')v(x') + [\pi(p' + q') - \pi(p')]v(y').$$

CPT and OPT are different representations unless $\pi(p + q) - \pi(p) = \pi(q)$ for all p and q, which holds only if $\pi(\cdot)$ is linear. Whether CPT assigns more or less weight to the middle outcome than OPT depends on the parameters of the prospect and shape of the weighting function.

Empiricists have employed a variety of methodologies to estimate the shape of the weighting function. For example, Tversky and Kahneman (1992) obtained "cash equivalents" by asking a series of refined choice questions. The weighting function was then estimated using nonlinear regression, assuming a power value function and a single-parameter functional form for the weighting function, $\pi(p) = p^\gamma/(p^\gamma + (1 - p)^\gamma)^{1/\gamma}$. In a second study, Camerer and Ho (1994) assumed the same functional forms for the value and weighting functions and estimated the weighting function parameter, γ, using a maximum likelihood technique and a single-agent stochastic choice model on a variety of betweenness data.

These two studies showed a consistent pattern: the weighting function was concave below some p^* (typically 0.30 to 0.40) and convex above it.[5] However, the estimation techniques used suffer from at least one of two shortcomings. First, previous studies have either assumed parametric forms of the weighting function and/or value function or estimated the value function using nonchoice procedures (e.g., Currim and Sarin 1989). Inferences about the weighting function could be incorrect due to misspecification of the value function. The second problem is the indeterminacy in using only simple prospects, prospects with at most one nonzero outcome in which case the value and weighting functions are only determined up to a power (Prelec 1995). Some studies (e.g., Kryzysztofowicz and Koch 1989) have used only simple prospects.

[5] Camerer and Ho (1994) estimated $\gamma = 0.56$, only slightly different from Tversky and Kahneman's (1992) estimate of $\gamma = 0.62$. See also Preston and Baratta (1948) and Lattimore et al. (1992) for qualitatively similar findings.

These criticisms raise a natural question: Is it possible to test for properties of the weighting function in a cleaner manner? In the next section, we outline straightforward conditions that can be used to test for concavity and convexity of the weighting function. These conditions deal with both objections: they are free of assumptions about the shape of the value or weighting function (thus are nonparametric) and involve nonsimple prospects (thus solving the indeterminacy problem).

3. Conditions on the Shape of the Weighting Function

In this section, we discuss two conditions that establish restrictions on the weighting function. For simplicity, we take $v(\cdot)$ and $\pi(\cdot)$ to be continuous and $\pi(\cdot)$ to be twice-differentiable.[6]

3.1. Concavity

We start with concavity and recall one definition of concavity of $\pi(\cdot)$: the slope of $\pi(\cdot)$ decreases as p increases: $\pi'(p) > \pi'(q)$ for $q > p$ or $\pi(p + \epsilon) - \pi(p) > \pi(q + \epsilon) - \pi(q)$ for $p < q$, $\epsilon > 0$. Put differently, a small extra unit of probability has greater impact for small probabilities than for larger probabilities. The following simple problem captures this very intuitive idea:

> Suppose that you own lottery tickets for lotteries in two different states (A and B). The lotteries will be conducted separately, and each lottery offers a prize of $10,000 to the winner. For **State Lottery A**, you own 1 out of 100 tickets (thus your chance of winning is 1%). For **State Lottery B**, you own 10 out of 100 tickets (thus your chance of winning is 10%). You are now offered one free lottery ticket, either for State Lottery A or State Lottery B. Would you rather improve your chances in State Lottery A or B?

We expect that most people would prefer an extra ticket for Lottery A. A ticket for Lottery A doubles the chance of winning, from 1% to 2%, whereas an extra ticket for Lottery B increases the chance of winning that lottery from 10% to 11%. The intuitive idea captured here is consistent with Zeckhauser's Russian Roulette example: going from 0.01 to 0.02 has a greater psychological impact than going from 0.10 to 0.11. In weighting function terms, $\pi(0.02) - \pi(0.01) > \pi(0.11) - \pi(0.10)$. Note, however, that this particular illustration of concavity is problematic because it depends on assumptions about joint receipts of lottery tickets (Luce 1988, Luce and Fishburn 1991). We correct for this problem by formulating this intuition in the more conventional terms of choice between gambles:

Concavity condition: Let $p < q$, $q' < q''$. Then

$$\text{if } R = (p, x; q', y) \sim (q + q', y) = S,$$

$$\text{then } R' = (p, x; q'', y) \succeq (q + q'', y) = S'.$$

R and S are mnemonic for Risky (higher chance at 0) and Safe gambles. Note that the second pair of gambles is constructed from the first pair merely by adding a $q'' - q'$ chance at y to both R and S. Thus, we term the more general family of these conditions *common-consequence* conditions since R' and S' are constructed by adding common consequences to both R and S. This condition was originally proposed by Segal (1987) who called his condition the Generalized Allais Paradox.

The concavity condition is a necessary and sufficient condition for concavity of the weighting function within the CPT framework.

PROPOSITION 1. *In the context of CPT or RDEU, (i) and (ii) are equivalent*:

(*i*) *$\pi(\cdot)$ is concave in the range $(\underline{r}, \bar{r})$*;

(*ii*) *the concavity condition holds for all p, q, q', q'', $\underline{r} < p$ and $q < \bar{r}$.*

This condition is exploited in the following example. Prelec (1990) found that 73% of subjects preferred S = (0.02, \$20,000) over R = (0.01, \$30,000), while 82% preferred R' = (0.01, \$30,000; .32, \$20,000) over S' = (0.34, \$20,000). Note that R' and S' are created by adding a 0.32 chance at \$20,000 to R and S respectively. The modal preferences, S and R', made by 55% of the subjects, are captured by concavity of $\pi(\cdot)$ in the standard range. Assuming CPT, the modal preferences reduce to $\pi(0.02) - \pi(0.01) > \pi(0.34) - \pi(0.33)$, which follows from concavity of $\pi(\cdot)$ below $p^* = 0.34$. It is important to note that although our condition is necessary and sufficient for concavity of $\pi(\cdot)$, the condition

[6] For rank-dependent representations, continuity of $v(\cdot)$ and $\pi(\cdot)$ typically follows from some version of the continuity axiom (e.g., Quiggin and Wakker 1994 or Wakker and Tversky 1993).

must hold for all parameters within a certain range. Thus, concavity of $\pi(\cdot)$ below $p = 0.34$ is a sufficient but not necessary explanation for the Prelec choice pattern.[7] We comment more on this issue in §4.

The concavity condition is also necessary and sufficient for concavity of $\pi(\cdot)$ under OPT.

PROPOSITION 2. *In the context of OPT, (i) and (ii) are equivalent:*
(i) $\pi(\cdot)$ is concave in the range $(\underline{r}, \bar{r})$;
(ii) the concavity condition holds for all $p, q, q', q'', \underline{r} < q'$ and $q'' < \bar{r}$.

The results for OPT and CPT are nearly identical, except for the range in which concavity applies. Returning to Prelec's example, the model preferences under OPT are captured by $\pi(0.02) > \pi(0.34) - \pi(0.32)$, which follows from concavity of $\pi(\cdot)$ below $p^* = 0.34$. Once again, strict concavity is sufficient, but not necessary—the above inequality follows from $\pi(\cdot)$ *weakly* concave, i.e., $\pi(p) = a + bp$ $(a > 0)$, but $\pi(0) = 0$ and $\pi(1) = 1$.

3.2. Convexity

Next, we turn to convexity of $\pi(\cdot)$. If $\pi(\cdot)$ is convex, then $\pi(p + \epsilon) - \pi(p) < \pi(q + \epsilon) - \pi(q)$ for $p < q, \epsilon > 0$. Note that the definition of convexity is identical to that of concavity, except for the sign of the inequality. Thus our treatment of convexity parallels our previous treatment of concavity. Reconsider the lottery ticket example, supposing that for **State Lottery A**, you own 75 out of 100 tickets (thus your chance of winning is 75%) and for **State Lottery B**, you own 99 out of 100 tickets (thus your chance of winning is 99%). Now most people would prefer an extra ticket for Lottery B. Upgrading Lottery B creates a certainty of winning \$1,000, whereas an extra Lottery A ticket only increases the chance of winning from 75% to 76%. Again, this intuition is captured neatly by the weighting function, $\pi(1) - \pi(0.99) > \pi(0.76) - \pi(0.75)$. As before, we must correct for the problem of joint receipts. The convexity condition is the concavity condition with the preferences reversed in the second pair.

[7] For example, the stated inequality follows from $\pi(\cdot)$ "concave near 0 (e.g., $\pi(0.01) = 0.03$, $\pi(0.02) = 0.05$) and convex" for $p > 0.25$ (e.g., $\pi(0.25) = 0.25$, $\pi(0.26) = 0.254$, $\pi(0.33) = 0.30$, $\pi(0.34) = 0.307$).

Convexity condition: Let $p < q, q' < q''$. Then

$$\text{if } R = (p, x; q', y) - (q + q', y) = S,$$
$$\text{then } R' = (p, x; q'', y) \preceq (q + q'', y) = S'.$$

Under CPT, the convexity condition is necessary and sufficient for convexity of $\pi(\cdot)$.

PROPOSITION 3. *In the context of CPT or RDEU, (i) and (ii) are equivalent:*
(i) $\pi(\cdot)$ is convex in the range $(\underline{r}, \bar{r})$;
(ii) the convexity condition holds for all $p, q, q', q'', \underline{r} < p$ and $q < \bar{r}$.

Kahneman and Tversky (1979) provide evidence for convexity of $\pi(\cdot)$ over the standard range. They found that 83% of subjects preferred $R = (0.33$, \$2,500) to $S = (0.34$, \$2,400), while 82% of subjects chose $S' =$ \$2,400 over $R' = (0.33$, \$2,500, 0.66, \$2,400). If $\pi(\cdot)$ is convex above $p^* = 0.33$, then adding a p chance at \$2,400 to both R and S, a common consequence, should shift preferences in the direction of S. Indeed, a strong reversal is obtained by adding 0.66 chance at \$2,400 to both gambles. Under CPT, preferences for R and S' imply that $\pi(1) - \pi(0.99) > \pi(0.34) - \pi(0.33)$, which is true if $\pi(\cdot)$ is convex above $p^* = 0.33$.

As before, the convexity condition is necessary and sufficient for convexity of $\pi(\cdot)$ within the OPT framework as well.

PROPOSITION 4. *In the context of OPT, the following are equivalent:*
(i) $\pi(\cdot)$ is convex in the range $(\underline{r}, \bar{r})$;
(ii) the convexity condition holds for all $p, q, q', q'', \underline{r} < q'$ and $q'' < \bar{r}$.

In this case, convexity of $\pi(\cdot)$ does not provide an easy explanation for the dominant pattern in Example B, $\pi(1) - \pi(0.66) > \pi(0.34)$, which follows from convexity of $\pi(\cdot)$ in the range (0, 1) but also from more innocuous restrictions on $\pi(\cdot)$, such as subcertainty, $\pi(p) + \pi(1 - p) < 1$.

3.3. Related Conditions

The concavity/convexity conditions are closely related to two conditions proposed by Tversky and Wakker (1995): lower subadditivity and upper subadditivity.

Lower subadditivity condition (Tversky and Wakker 1995): Let $p < q$. Then

$$\text{if } R = (p, x; q, y) \sim y = S,$$

$$\text{then } R' = (p + q, x) \preceq (q, x; 1 - q, y) = S'.$$

R' and S' are obtained by shifting q chance at y in R and S to q chance at x in R' and S'. The lower subadditivity condition is necessary and sufficient for the lower subadditivity of the weighting function, $\pi(p) + \pi(q) \geq \pi(p + q)$. It is important to note that our concavity condition is stronger than lower subadditivity, as lower subadditivity is primarily concerned with nonlinearities (and possible discontinuities) near $p = 0$. For example, a piecewise linear weighting function with a discontinuity at 0 ($\pi(p) = a + bp$, $a > 0$, $\pi(0) = 0$, $\pi(1) = 1$ (Bell 1985)) satisfies lower subadditivity strictly, but concavity weakly.

Tversky and Wakker's upper subadditivity condition is strictly weaker than convexity:

Upper subadditivity condition (Tversky and Wakker, 1995): Let $x > y$. Then

$$\text{if } R = (p, x) \sim (1 - r, y) = S,$$

$$\text{then } R' = (p, x; r, y) \preceq y = S'.$$

This condition is a special case of the convexity condition when $q' = 0$, $q + q' = r$, and $q + q'' = 1$. In this condition, q chance at y is added to both R and S. As with lower subadditivity, upper subadditivity, $1 - \pi(1 - q) \geq \pi(p + q) - \pi(p)$, primarily addresses boundary effects at $p = 1$, i.e., the certainty effect. A linear weighting function with discontinuities at 0 and 1 ($\pi(p) = a + bp$, $a > 0$, $a + b < 1$, $\pi(0) = 0$, $\pi(1) = 1$) satisfies both lower and upper subadditivity strictly, but concavity and convexity only weakly.

The comparison of our conditions to Tversky and Wakker's raises an important issue. The Prelec and Kahneman and Tversky examples use probabilities near the boundaries 0 and 1. These results could be explained by merely postulating discontinuities at the boundaries. Are we imposing unnecessarily strong conditions on $\pi(\cdot)$? Our justification for formulating our tests in terms of the stronger restrictions of concavity and convexity is that these conditions permit a test of the extent of nonlinearities throughout the 0-1 interval, not only at the boundaries. These conditions also test for concavity and convexity without making prior assumptions about the functional form of $\pi(\cdot)$. Curvature can thus be assessed ordinally: concavity requires that the addition of a chance at y improves the riskier gamble, whereas the opposite holds for convexity.

4. Study: Concavity and Convexity Ladders

4.1. Procedure

In this section, we describe a test of the concavity and convexity of the weighting function using the concavity and convexity conditions described in §2. We recruited 420 Harvard University and University of Washington undergraduates and paid them \$3 to \$5 to complete a choice questionnaire. Each choice questionnaire consisted of approximately 25 binary choice questions, some of which were tests of concavity and convexity. Copies of the questionnaires are available from the authors.

Tests of the concavity and convexity conditions were operationalized as a series of eight choices between two gambles. The eight pairs of gambles were arranged in the form of a "ladder" in which each step in the ladder added a common consequence to both gambles. For example, the first *rung* or question of Ladder 1 was a choice between R_1 = (0.05, \$240) and S_1 = (0.07, \$200). We then added a common consequence of 0.10 chance at \$200 to both R_1 and S_1 to get R_2 = (0.05, \$240; 0.10, \$200) and S_2 = (0.17, \$200). We repeated this process to create a total of eight questions. Thus, concavity/convexity ladders consist of eight rungs, each rung differing from another only by the addition or deletion of a common consequence. The complete set of five ladders used in this study appears in Table 1.

For each ladder, the first four rungs were clustered in roughly the first third of the probability range (probability of winning $\leq$ 0.40), while the last four rungs were clustered in the top two-thirds of the probability range (probability of winning $\geq$ 0.40). This unequal spacing reflects previous empirical findings that the inflection point is between 0.30 and 0.40. We designed the questions so that approximately half of the rungs tested

Table 1 Concavity/Convexity Ladders

	Rung 1	Rung 2	Rung 3	Rung 4	Rung 5	Rung 6	Rung 7	Rung 8
				Ladder 1				
R	0.05, $240	0.05, $240 0.10, $200	0.05, $240 0.20, $200	0.05, $240 0.30, $200	0.05, $240 0.45, $200	0.05, $240 0.60, $200	0.05, $240 0.75, $200	0.05, $240 0.90, $200
S	0.07, $200	0.17, $200	0.27, $200	0.37, $200	0.52, $200	0.67, $200	0.82, $200	0.97, $200
				Ladder 2				
R	0.05, $100	0.05, $100 0.10, $50	0.05, $100 0.20, $50	0.05, $100 0.30, $50	0.05, $100 0.45, $50	0.05, $100 0.60, $50	0.05, $100 0.75, $50	0.05, $100 0.90, $50
S	0.10, $50	0.20, $50	0.30, $50	0.40, $50	0.55, $50	0.70, $50	0.85, $50	$50
				Ladder 3				
R	0.01, $300	0.01, $300 0.10, $150	0.01, $300 0.20, $150	0.01, $300 0.30, $150	0.01, $300 0.45, $150	0.01, $300 0.60, $150	0.01, $300 0.80, $150	0.01, $300 0.98, $150
S	0.02, $150	0.12, $150	0.22, $150	0.32, $150	0.47, $150	0.62, $150	0.82, $150	$150
				Ladder 4				
R	0.03, $320	0.03, $320 0.10, $200	0.03, $320 0.20, $200	0.03, $320 0.30, $200	0.03, $320 0.45, $200	0.03, $320 0.65, $200	0.03, $320 0.85, $200	0.03, $320 0.95, $200
S	0.05, $200	0.15, $200	0.25, $200	0.35, $200	0.50, $200	0.70, $200	0.90, $200	$200
				Ladder 5				
R	0.01, $500	0.01, $500 0.10, $100	0.01, $500 0.20, $100	0.01, $500 0.30, $100	0.01, $500 0.45, $100	0.01, $500 0.65, $100	0.01, $500 0.80, $100	0.01, $500 0.95, $100
S	0.05, $100	0.15, $100	0.25, $100	0.35, $100	0.50, $100	0.70, $100	0.85, $100	$100

concavity and half tested convexity. Also, note that $q - p$ is small (at most 0.04) for all ladders. Although our conditions are necessary and sufficient for concavity and convexity of $\pi(\cdot)$, the conditions must hold for *all* parameters within a certain range. Thus, strictly speaking, establishing the curvature of $\pi(\cdot)$ definitively is clearly impractical. However, since $q - p$ is small, the inferred restrictions on $\pi(\cdot)$ reasonably approximate restrictions on the slope of $\pi(\cdot)$, hence implications about $\pi''(\cdot)$.

Each question was answered by a total of 105 subjects. For each of the five ladders, each subject received four of the eight rungs. For Ladders 1, 2, and 4, subjects responded to either the first four or the last four rungs. For Ladders 3 and 5, subjects responded to every other rung (either rungs 1, 3, 5, and 7 or rungs 2, 4, 6, and 8). This design permitted us to perform both between- and within-subject analyses. The order of the alternatives was counter-balanced within each question (for each question, roughly half of the subjects received *R* first and half of the subjects received *S* first). In addition, the order of the questions was randomized across questionnaires.

4.2. Results

A convex weighting function, as proposed by Segal (1987), predicts a decrease in the proportion of *R* choices as one moves along the ladder from Rung 1 to Rung 8, while a concave weighting function predicts an increase in the proportion of *R* choices. The hypothesized S-shaped weighting function, first concave for small probabilities and then convex for large probabilities, should produce the following pattern in

Figure 1 **Concave/Convex Ladder 1**

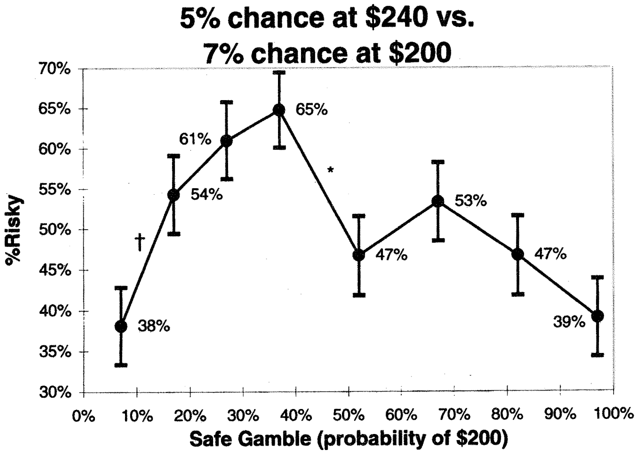
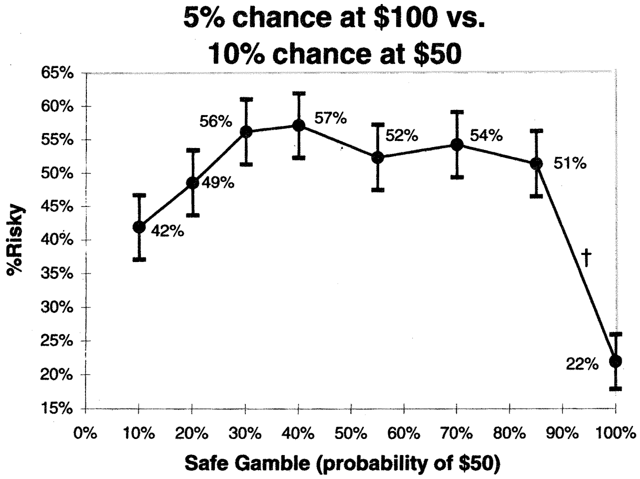


Figure 2 **Concave/Convex Ladder 2**

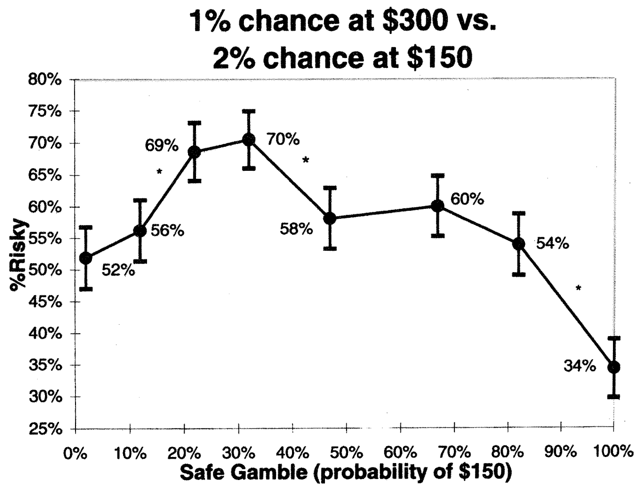


the choice ladders: an increasing percentage of R choices (corresponding to concavity of $\pi(\cdot)$) and then a decreasing percentage of R choices (corresponding to convexity of $\pi(\cdot)$). Thus, we predict that the percentage of R choices will trace an inverted-U curve against the probability associated with the safe outcome at each rung in the ladder.

Figures 1 through 5 show the percentage of subjects choosing the risky prospect in each rung for Ladders 1 through 5, respectively. Consider the first rung of Ladder 1 shown in Figure 1. In Rung 1, 38% of subjects preferred $R_1 = (0.05, \$240)$ to $S_1 = (0.07, \$200)$, denoted $\%R_1 = 38\%$. The vertical bar reflects $\%R_i \pm$ one standard error. With the exception of Rung 5, the data are consistent with the hypothesized S-shape pattern: $\%R_i$ increases when $\pi(\cdot)$ is concave and $\%R_i$ decreases when $\pi(\cdot)$ is convex. Moreover, the data suggest concavity up to p^* between 0.30 and 0.45, and convexity beyond that probability. The inferred inflection point is consistent with previous studies. In Figure 1, statistically significant differences between $\%R_i$ and $\%R_{i+1}$ are denoted by daggers for pairs of rungs involving within-subject comparisons (a McNemar test) and by asterisks for pairs of rungs involving between-subject comparisons (a two-sample binomial test). For Ladder 1, the difference between Rungs 1 and 2, and Rungs 4 and 5 are significant at the 0.05 level (McNemar and binomial tests, respectively).

Figures 2–5 depict the percentages of risky choices in concavity/convexity Ladders 2–5. Although the data are sometimes a bit noisy, the ladders in aggregate show a fairly consistent pattern, concavity up to $p^* = 0.30$ to 0.40 ($\%R_i$ increasing) and then convexity beyond that probability ($\%R_i$ decreasing). Variation among ladders could be explained by a number of causes, including sampling error and editing operations subjects may employ for certain pairs of gambles (Kahneman and Tversky 1979, Wu 1994).

Figure 3 **Concave/Convex Ladder 3**



Unlike Ladder 1, Ladders 2 through 5 test directly how much of the nonlinearity is due to boundary effects. For example, in Ladder 2, Rung 8, subjects chose between $R_8 = (0.05, \$100; 0.90, \$50)$ and $S_8 = \$50$, a sure thing. In this case, $\%R_8$ is substantially lower than $\%R_i$ for any other rung i. The same pattern holds for Ladders 3 and 4. In all but one case, the difference between $\%R_7$ and $\%R_8$ (the sure thing) is greater than between any other two adjacent rungs. Thus, most of the curvature of $\pi(\cdot)$ occurs at the boundaries as previous research has suggested. Even so, $\%R_i$ does not remain constant away from the boundaries, as a piecewise linear weighting function would predict.

We obtain a more complete picture of the patterns in Figures 1 through 5 by performing all possible pairwise comparisons between Rung i and Rung j, $i \neq j$. This involves 28 tests per ladder for a total of 140 tests. Some of the tests are within-subjects (McNemar), and the remaining tests were between-subjects (two-sample binomial), and all tests are one-sided with $\alpha = 0.05$. The results are summarized in Table 2, which presents the counts of statistically significant results across all five ladders for all possible pairwise comparisons. The upper panel presents the count of statistically significant one-sided results for $\%R_i > \%R_j$ while the lower panel presents the same count for $\%R_i < \%R_j$. For example, the upper panel shows that in 3 of the 5 ladders, the difference between $\%R_1$ and $\%R_2$ was statistically significant. Overall, there is a good deal of evidence for concavity in the first third of the probability range (the first four rungs denoted by the shaded region in the upper panel) and for convexity in the top two-thirds of the probability range (the last four rungs denoted by the shaded region in the lower panel). Furthermore, there is no statistically significant evidence of concavity or convexity outside these ranges.[8]

Figure 4 **Concave/Convex Ladder 4**

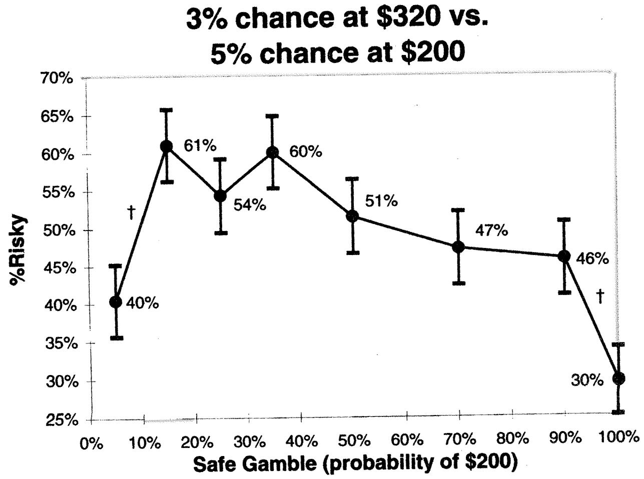
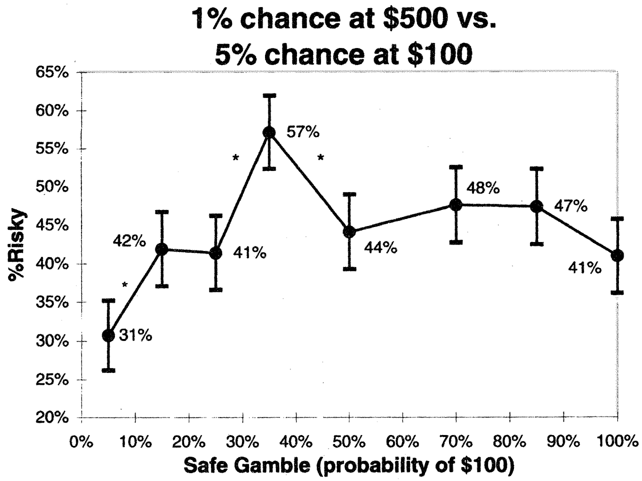


Figure 5 **Concave/Convex Ladder 5**



5. Estimation of Weighting Functions

In this section, we describe a procedure for estimating the parameters of $\pi(\cdot)$. We require additional assumptions to estimate the weighting function. In particular, we fit the data with weighting functions proposed by Tversky and Kahneman (1992) and Prelec (1995). We use a "single-agent stochastic choice model," similar in spirit to Camerer and Ho (1994). All subjects (agents) are assumed to have the same underlying preferences, but the utility of an alternative is stochastic. Subjects choose the alternative with the highest utility, but the utility of each alternative is computed using CPT, the

[8] Preferences for event-splitting (Starmer and Sugden 1993) cannot explain preferences for S then R (explained by concave $\pi(\cdot)$) and R then S (explained by convex $\pi(\cdot)$).

Table 2 Counts of Statistically Significant Differences

counts of $\%R$ increases across all five ladders

	Rungs						
	2	3	4	5	6	7	8
1	3	4	5	1	3	1	0
2		1	2	0	0	0	0
3			1	0	0	0	0
4				0	0	0	0
5					0	0	0
6						0	0
7							0

counts of $\%R$ decreases across all five ladders

	Rungs						
	2	3	4	5	6	7	8
1	0	0	0	0	0	0	3
2		0	0	0	1	1	4
3			0	2	0	2	4
4				3	1	3	5
5					0	0	3
6						0	4
7							3

same value and weighting function, plus a noise term.[9] Naturally, all agents do not have the same preferences. Nevertheless, we justify such a model as an attempt to describe the "typical" subject, or "representative agent" in economic parlance (see Camerer and Ho 1994, p. 186 for a more complete justification; see also McFadden 1981).

For each ladder and each rung i, $\%R_i$ measures the percentage of subjects who preferred the risky gamble. If each subject has the same underlying preferences, variation in choice could only be due to randomness in subject response. Thus, we use a stochastic choice model (Luce and Suppes 1965) and estimate the probability that a subject will choose R over S, conditional on some preference parameters. Furthermore, we use a logistic function to describe the probability of choosing R over S,

$$\Pr(R > S) = 1/(1 + \exp(U(S) - U(R))), \quad \text{where}$$

$$U(p, x; q, y) = \pi(p)v(x) + [\pi(p + q) - \pi(p)]v(y),$$

the CPT form. The fraction of the time an individual chooses R over S depends on the utility difference, $U(R) - U(S)$, which in turn depends on the functional forms (including parameters) of $\pi(\cdot)$ and $v(\cdot)$. If $U(R) = U(S)$, then the choice between R and S is decided by a coin flip, i.e., $\Pr(R > S) = 0.5$.

We assume functional forms for the weighting function and value function. For the value function, we assume a power function, $v(x) = x^{\alpha}$. Whereas, it is straightforward to use other functional forms, other studies (Tversky and Kahneman 1992, Camerer and Ho 1994) have assumed power functions. Assuming this form allows us to compare our results with these studies. For a weighting function, we first employ Tversky and Kahneman's (1992) single-parameter form:

$$\pi(p) = p^{\gamma}/(p^{\gamma} + (1 - p)^{\gamma})^{1/\gamma}.$$

If $\gamma = 1$, then the weighting function is the identity, $\pi(p = p$. Otherwise, $\pi(\cdot)$ becomes more curved as γ gets smaller (up to $\gamma = 0.27$).

We use a least-squares method to estimate γ and α. For each rung i, the "error" is the difference between the actual choice probability and $\Pr(R_i > S_i; \gamma, \alpha)$, the estimated probability that R_i is chosen over S_i. For each ladder, we calculate the sum of square errors (SSE),

$$\mathrm{SSE}(\gamma, \alpha) = \sum_{i=1}^{8} (\%R_i - \Pr(R_i > S_i; \gamma, \alpha))^2$$

for particular γ and α. The best estimates are the γ and α that minimize $\mathrm{SSE}(\gamma, \alpha)$.[10]

We implement the procedure in Splus using a Gauss-Newton algorithm (Bates and Chambers 1990). Table 3 shows the best estimates for each ladder taken separately. For example, in Ladder 1, $\gamma = 0.57$, which is close

[9] More specifically, we assume that $U(p, x; q, y) = \pi(p)v(x) + [\pi(p + q) - \pi(p)]v(y) + \epsilon$, where ϵ is independent and identically distributed for all prospects. This random utility formulation can be translated into the strong utility model (Theorem 33, Luce and Suppes 1965). The strong utility model assumes deterministic utility, but a probabilistic choice function of the utilities of the alternatives.

[10] Under the standard assumptions of normally distributed errors, minimizing SSE is identical to maximum likelihood (Seber and Wild 1989).

Table 3 Sum of Squared Errors and Parameter Estimates Assuming CPT

	Tversky & Kahneman (1992)				Prelec (1995)		
	α	γ	Fixed Point	SSE	α	γ	SSE
1	0.58 (0.12)	0.57 (0.14)	0.32	0.036	0.68 (0.11)	0.63 (0.10)	0.033
2	0.45 (0.12)	0.57 (0.11)	0.32	0.011	0.21 (0.10)	0.03 (0.22)	0.007
3	0.75 (0.10)	0.77 (0.09)	0.41	0.020	0.47 (0.07)	0.48 (0.12)	0.024
4	0.69 (0.32)	0.81 (0.20)	0.43	0.021	0.84 (0.73)	0.92 (0.29)	0.021
5	0.81 (0.27)	0.94 (0.09)	0.48	0.024	0.78 (2.19)	0.95 (0.49)	0.028
Aggregate (1–5)	0.50 (0.12)	0.71 (0.10)	0.39	0.242	0.48 (0.16)	0.74 (0.14)	0.304

Note: Standard Errors in parentheses.

to the estimates by Tversky and Kahneman (1992) of $\gamma = 0.61$ and by Camerer and Ho (1994) of $\gamma = 0.56$. The other ladders typically show less curvature, with three of the estimates above $\gamma = 0.70$. Although our estimates vary from ladder to ladder, they are more uniform than the estimates obtained by Camerer and Ho using a maximum-likelihood estimation technique.[11] We also pool the data from Ladders 1 to 5 to get an aggregate least-squares estimate of $\gamma = 0.71$, slightly more linear than the estimates of Camerer and Ho or Tversky and Kahneman. Table 3 also presents the standard errors of the estimates in parentheses. Taking $\gamma = 1$ ($\pi(p) = p$) to be the null hypothesis, for Ladders 1, 2, and 3 and in aggregate, we can reject the null hypothesis of expected utility. In Figure 6, we plot $\pi(\cdot)$ for $\gamma = 0.56$ (Camerer and Ho), $\gamma = 0.61$ (Tversky and Kahneman), and our estimate $\gamma = 0.71$.[12] Estimates of γ using two other functional forms of the value function, exponential ($\gamma = 0.50$) and log-quadratic ($\gamma = 0.70$), are qualitatively similar.[13]

We repeat the same analysis with Prelec's (1995) function, $\pi(p) = \exp(-(-\ln p)^\gamma)$. Like Tversky and Kahneman's function, Prelec's function is the identity line when $\gamma = 1$ and becomes more regressive as γ decreases. Prelec's function forces the fixed point to be $1/e = 0.36$ for all γ. Although Prelec's function fits worse than Tversky and Kahneman, the parameter estimates are nearly identical. The standard errors for Ladders 4 and 5 are excessively high because the two parameters were highly correlated (greater than 0.999), thus making the parameters unstable.[14]

To provide two benchmarks, we repeated our procedure assuming a power value function, and (i) ex-

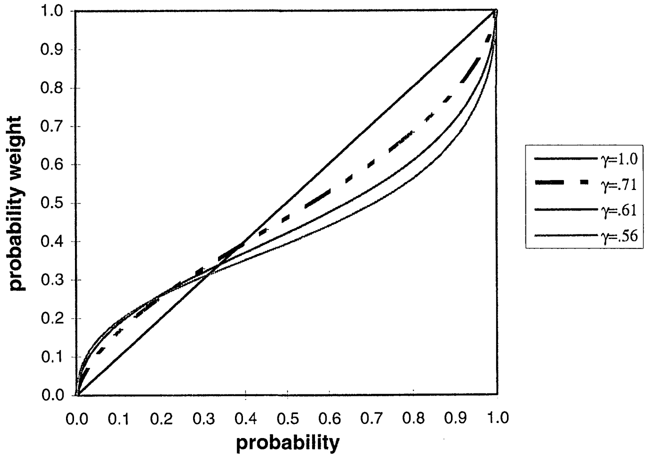
[11] Their estimates vary from $\gamma = 0.28$ to 1.87, an atypically S-shaped function (*convex, then concave*).

[12] Our estimates for α are considerably lower than Tversky and Kahneman's (1992) $\alpha = 0.88$. Camerer and Ho (1994) do not report α, but we repeated their estimation procedure using data generously provided by Teck-Hua Ho and found $\alpha = 0.37$, which is closer to our $\alpha = 0.52$. This suggests that the differences might be due to the estimation procedure and/or elicitation task. We repeated our estimation procedures assuming $\gamma = 0.56$ and $\gamma = 0.61$, and only got slightly worse fits: SSE = 0.249 ($\gamma = 0.61$) and SSE = 0.255 ($\gamma = 0.56$).

[13] The logistic function used is a special case of the more general family of logistic functions, $1/(1 + \exp(\alpha(U(S) - U(R))))$, and thus is arbitrary. To compare results for different value functions, the value functions must all be normalized in some standard way. We choose $v(0) = 0$, and $v(1) = 1$.

[14] We also estimated $\pi(p) = p^\gamma/(p^\gamma + (1 - p)^\gamma)^\delta$, a two-parameter function of which Karmarkar (1978), $\delta = 1$, and Tversky and Kahneman, $\delta = 1/\gamma$, are special cases. For the aggregate data, SSE = 0.241, $\gamma = 0.721$, $\delta = 1.565$, marginally better than Tversky and Kahneman's single-parameter function, and worse if we adjust for degrees-of-freedom. The fit for another two-parameter function, $\pi(p) = \delta p^\gamma/(\delta p^\gamma + (1 - p)^\gamma)$, Lattimore, Baker and Witte (1992), is similar: SSE = 0.247, $\delta = 0.84$, and $\gamma = 0.68$. These estimates are very close to those obtained by Gonzalez (1993) and Tversky and Fox (1995).

Figure 6 Tversky and Kahneman (1992) Weighting Function for Various γ



pected utility ($\pi(p) = p$); or (ii) a piecewise linear weighting function ($\pi(p) = a + bp$, but $\pi(0) = 0$, $\pi(1) = 1$). We then compare the SSE obtained from using the Tversky and Kahneman function, SSE_{TK}, and the linear weighting function, SSE_L, with the SSE obtained by using expected utility, SSE_{EU}. The reductions in variation, a sort of "pseudo R^2," using the Tversky-Kahneman and linear functions, $(SSE_{EU} - SSE_{TK})/SSE_{EU}$ and $(SSE_{EU} - SSE_L)/SSE_{EU}$ respectively, are shown in Table 4. The Tversky-Kahneman function outperforms the linear function in all cases.[15]

Finally, we repeat the procedure using OPT (Table 5). The scale tilts slightly in favor of OPT: OPT outperforms CPT on 3 of 5 ladders and in the aggregate. These mixed results are consistent with previous research. Camerer and Ho (1994) found that OPT fits the betweenness data slightly better than CPT. The few studies on whether separable (OPT) or rank-dependent (CPT) weights are empirically sounder (Camerer and Ho 1994, footnote 27) present problems for each theory. For example, Wu (1994) found that some preference patterns could be explained by OPT but not CPT. More tests are clearly needed.

[15] Although the linear function has two parameters, only one parameter is used. If $\pi(p) = a + bp$, then the addition of a common-consequence, say q chance at y, affects R and S equally, i.e., $U(R) - U(S) = U(R') - U(S')$. Thus, $\Pr(R > S) = \Pr(R' > S')$, unless S' is on the boundary. Since Ladder 1 does not involve any boundary questions, $\pi(p) = a + bp$ cannot improve over $\pi(p) = p$.

Table 4 Percent of Variation Explained ("Pseudo-R^2")

Ladder	$\pi(p) = a + bp$	$\pi(p) = \frac{p^\gamma}{(p^\gamma + (1-p)^\gamma)^{1/\gamma}}$
1	0%	46%
2	82%	87%
3	76%	84%
4	55%	73%
5	3%	40%
Aggregate (1–5)	25%	39%

6. Summary

6.1. Answers

First, we return to the four questions posed in the introduction.

(*i*) *What empirical properties does the probability weighting function satisfy?*

Kahneman and Tversky (1979) originally proposed a weighting function that overweights small probabilities and underweights larger probabilities. Recent studies have offered a more refined set of properties, namely concavity for small probabilities and convexity for large probabilities. Our empirical tests are consistent with these studies: we consistently find that $\pi(\cdot)$ is concave

Table 5 Sum of Squared Errors and Parameter Estimates Assuming OPT

	Tversky & Kahneman (1992)			
	α	γ	Fixed Point	SSE
1	0.68 (0.11)	0.92 (0.02)	0.47	0.031
2	0.67 (1.87)	0.90 (0.60)	0.47	0.048
3	0.52 (0.11)	0.48 (0.13)	0.39	0.022
4	0.85 (0.64)	0.96 (0.10)	0.49	0.033
5	0.79 (0.95)	0.95 (0.23)	0.48	0.021
Aggregate (1–5)	0.72 (0.07)	0.92 (0.02)	0.47	0.217

Note: Standard errors in parentheses.

up to $p \approx 0.40$ and convex beyond that probability. The concavity and convexity of the weighting function permits us to pinpoint where indifference curves will fan in and fan out in the Marschak-Machina triangle (Machina 1982). (Concavity (convexity) is equivalent to horizontal fanning in (fanning out) of indifference curves along the x axis of the MM triangle.)[16]

(*ii*) *For each empirical property of the weighting function, is there a corresponding intuitive (and empirically testable) preference condition?*

In this paper, we described two conditions that are necessary and sufficient for concavity and convexity of $\pi(\cdot)$. These common-consequence conditions permit nonparametric tests of the curvature properties of the weighting function.

(*iii*) *Are probability weights nonlinear away from the boundaries* (0 *and* 1)?

Tversky and Wakker's (1995) goal was to "characterize a weighting function according to which an event has greater impact when it turns impossibility into possibility, or possibility into certainty, than when it merely makes a possibility more or less likely." Camerer (1992) reflects this empirical regularity in his Stylized Fact #1: "EU is not Violated Inside the Triangle." However, Camerer tempers this, noting: "EU violations inside the triangle are still too high to be considered random error, but they are not very systematic. Much as Newtonian mechanics is an adequate working theory at low velocities, EU seems to be an adequate working theory for gambles inside the triangle." Whereas the curvature of the weighting function is more pronounced at the boundaries, our studies show that there is significant curvature strictly within the boundaries. The data-fitting exercise indicates that a weighting function that is strictly nonlinear within the boundaries generally outperforms (at times substantially) a linear weighting function with discontinuities at 0 and 1.

[16] Wu and Gonzalez (1996a) show that an S-shaped weighting function defines three regions in the probability simplex: one region in which fanning out occurs for all shifts (horizontal or vertical) and two regions in which there is mixed fanning.

(*iv*) *Do there exist parsimonious functional forms of the weighting function that are consistent with these empirical properties?*

We fit one-parameter weighting functions proposed by Tversky and Kahneman (1992) and Prelec (1995) to our data. We find that both functions offer a substantial improvement over expected utility as measured by percentage of variation explained (pseudo R^2). In addition, the parameters fit to the Tversky-Kahneman function are remarkably similar to those found by others using very different estimation procedures and experimental methods.

6.2. Future Research

This study and a slew of other work indicate that nonlinearity in probability is essential for a good descriptive theory of decision making under risk. One possible objection to our work is that we have not established the shape of the weighting function at the individual level. Although our preference conditions are all about preferences of individual decision makers, we use choice data and thus lack the statistical power for individual subject estimation. Establishing the shape of $\pi(\cdot)$ on the individual level is crucially important—the only large sample individual estimation of the weighting function we know of is Gonzalez (1993). We plan on using similar procedures to test for curvature properties at the individual level. One nonparametric procedure that exploits the concavity/convexity condition involves asking subjects to *match* probabilities in common-consequence pairs. For example, subjects can be asked to provide indifference probabilities q and q' for choices between $R_1 = (0.05, \$240)$ and $S_1 = (q, \$200)$, and $R_2 = (0.05, \$240; 0.10, \$200)$ and $S_2 = (q', \$200)$. Concavity predicts that $q' - q > 0.10$.

A second objection is that precise probabilities seldom accompany real world prospects. Most meaningful decisions involve outcomes tied to events, about which individuals may or may not attempt to estimate the likelihood. As in Tversky and Wakker's (1995) treatment of lower and upper subadditivity, our conditions readily generalize from risk (probabilities given) to uncertainty (events given). In a recent study (Wu and Gonzalez 1996b), we extended the concavity and convexity conditions from risk to uncertainty, demonstrating how nonadditivity of probability judgments can be disentangled from curvature of the probability weighting function. "Nonlinearity in probability," in particular an S-shaped probability weighting function, seems to play a major role in decision making under uncertainty as it did in decision making under risk (see also, Tversky and Fox 1995, Tversky and Koehler 1994).[17]

[17] We thank Colin Camerer, Jeff Casey, Robin Keller, John Miyamoto, Dražen Prelec, Amos Tversky, Peter Wakker, and an anonymous referee for especially useful and detailed comments. We also thank Susan Coates, Rachel Croson, Tim Erdman, and Cynthia Wachtell for research assistance. This work was supported by the Research Division of the Harvard Business School and Grant SES 91-10572 from the National Science Foundation.

Appendix

PROOF OF PROPOSITION 1. Suppose that CPT holds. First, we prove that (i) implies (ii). If $(p, x; q', y) \sim (q + q', y)$, then

$$\pi(p)v(x) + [\pi(p + q') - \pi(p)]v(y) = \pi(q + q')v(y). \quad \text{(A.1)}$$

Concavity of $\pi(\cdot)$ (and $q' < q''$, $p < q$) implies that $\pi(p + q'') - \pi(p + q') \geq \pi(q + q'') - \pi(q + q')$. Then

$$\pi(q + q')v(y) + [\pi(p + q'') - \pi(p + q')]v(y) \geq \pi(q + q'')v(y). \quad \text{(A.2)}$$

Substituting (A.1) into (A.2) and rearranging, we get

$$\pi(p)v(x) + [\pi(p + q'') - \pi(p)]v(y) \geq \pi(q + q'')v(y).$$

Thus, $(p, x; q'', y) \succeq (q + q'', y)$.

Next, we prove that (ii) implies (i). Let $(p, x) \sim (q, y)$. Note that given p, q, and x, continuity of $v(\cdot)$ ensures solvability. Then, by the concavity condition, $(p, x; \epsilon, y) \succeq (q + \epsilon, y)$. Simplifying, we get

$$\pi(p + \epsilon) - \pi(p) \geq \pi(q + \epsilon) - \pi(q) \quad \text{or}$$

$$\frac{\pi(p + \epsilon) - \pi(p)}{\epsilon} \geq \frac{\pi(q + \epsilon) - \pi(q)}{\epsilon}$$

for all p, q and ϵ. Thus,

$$\lim_{\epsilon \to 0} \frac{\pi(p + \epsilon) - \pi(p)}{\epsilon} = \pi'(p) \geq \lim_{\epsilon \to 0} \frac{\pi(q + \epsilon) - \pi(q)}{\epsilon} = \pi'(q).$$

Since $\pi'(p) \geq \pi'(q)$ for all $p < q$, $\pi(\cdot)$ is concave on (p, q). □

REMARK: Proofs that do not assume differentiability are found in Wu and Gonzalez (1996b).

PROOF OF PROPOSITION 2. Suppose that OPT holds. First, we show that (i) implies (ii). If $(p, x; q', y) \sim (q + q', y)$, then $\pi(p)v(x) + \pi(q')v(y) = \pi(q + q')v(y)$ or

$$\pi(p)v(x) = [\pi(q + q') - \pi(q')]v(y). \quad \text{(A.3)}$$

By concavity of $\pi(\cdot)$, $\pi(q + q') - \pi(q') \geq \pi(q + q'') - \pi(q'')$. Then,

$$[\pi(q + q') - \pi(q')]v(y) \geq [\pi(q + q'') - \pi(q'')]v(y). \quad \text{(A.4)}$$

Substituting (A.3) into (A.4) and rearranging, we get $\pi(p)v(x) + \pi(q'')v(y) \geq \pi(q + q'')v(y)$, which holds if and only if $(p, x; q'', y) \succeq (q + q'', y)$.

Next, we show that (ii) implies (i). If $(p, x; q', y) \sim (q' + \epsilon, y)$ and $(p, x; q'', y) \succeq (q'' + \epsilon, y)$, then $\pi(q' + \epsilon) - \pi(q') \geq \pi(q'' + \epsilon) - \pi(q'')$ for all ϵ, $q' < q''$. Thus,

$$\lim_{\epsilon \to 0} \frac{\pi(q' + \epsilon) - \pi(q')}{\epsilon} = \pi'(q') \geq \lim_{\epsilon \to 0} \frac{\pi(q'' + \epsilon) - \pi(q'')}{\epsilon} = \pi'(q'')$$

for all $q' > q''$, which implies that $\pi(\cdot)$ is concave on (q', q''). □

PROOF OF PROPOSITION 3. Same as Proposition 1, with the inequalities reversed. □

PROOF OF PROPOSITION 4. Same as Proposition 2, with the inequalities reversed. □

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