2019 ES 156 Problem Set 9

Announcements

- This is another "pick your own adventure" problem set. Submission of your solution to this problem set are made electronically via Canvas no later than **10 am on Friday, April 19th, 2018** Late problem sets will be penalized (see syllabus).

  You have three option in this problem set:

  1. Solve the problems below. The purpose of these problems is for you to familiarize yourself with properties of the sampling theorem and time-bandwith trade-offs.
  2. Do the "Image processing" Python programming exercise, where you get to implement a simplified version of Shazam. You can find the link to the Colab notebook on Canvas. You can implement your solution directly on colab.
  3. Do both the programming exercise and the problems below! In this case, you get to skip a future problem set (or, if you do all PSets, the lowest grade will be replaced). Recall that we only consider the 9 highest scores out of the 10 problem sets. If you do both components, we would consider the 9 highest scores out of 11.

  We only kindly ask that you pick one of the options above and stick to it. For example, please don't do half of the problems below, and then half of the lab.

Problems

1. [**The Heisenberg Uncertainty Principle, 20 points**] We will prove next that if we increase the time duration of a signal, its bandwidth must be reduced. For a signal $x(t)$, let

$$\Delta_t \triangleq \left(\frac{\int_{-\infty}^{\infty} t^2 |x(t)|^2 dt}{E_x} \right)^{1/2},$$

$$\Delta_\omega \triangleq \left(\frac{\int_{-\infty}^{\infty} \omega^2 |\hat{X}(j\omega)|^2 d\omega}{2\pi E_x} \right)^{1/2},$$

$$E_x \triangleq \int_{-\infty}^{\infty} |x(t)|^2 dt.$$

Our goal is to prove that

$$2\Delta_t \Delta_\omega \geq 1.$$

It's ok if you are a little sloppy with your proof, and you can assume that $x(t)$ is real and has finite energy (i.e. $E_x < \infty$).

(a) Use properties of the F.T. to show that

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\omega^2\left|\hat{X}(j\omega)\right|^2 d\omega = \int_{-\infty}^{\infty}\left|\frac{dx(t)}{dt}\right|^2 dt$$

(b) Using integration by parts and the fact that $x(t)$ has finite energy to show that

$$E_x = \int_{-\infty}^{\infty}|x(t)|^2 dt = -2\int_{-\infty}^{\infty}\left[\frac{dx(t)}{dt}\right][tx(t)]\,dt.$$

(c) Now use parts (a), (b), and the Cauchy-Shwarz inequality

$$\left|\int_{-\infty}^{\infty}u(t)v(t)dt\right|^2 \leq \left(\int_{-\infty}^{\infty}|u(t)|^2 dt\right)\left(\int_{-\infty}^{\infty}|v(t)|^2 dt\right)$$

to prove the uncertainty principle.

(d) Argue how the uncertainty principle implies that increasing the time duration of a signal must decrease its bandwidth (and vice-versa).

2. [**Time limited vs. band limited, 15 points**].

This question is somewhat open-ended – do your best.

(a) We saw in class that a bandlimited signal $x(t)$ can be written as a weighted sum of since functions (you can also find this in equation Eq. 7.9–7.11 in the book):

$$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\mathsf{sinc}\left(\frac{t-nT}{T}\right),$$

where T is larger than the Nyquist sampling rate. Use this fact to argue as best as you can that time-limited functions cannot be bandlimited (you don't have to be too formal). Reverse the argument to have a similar conclusion when the function is time limited.

(b) How do you reconcile this fact? Real world signals are bandlimited, but very high frequencies (say 10^20 GHz) make no physical sense. You can base your answer on the first few pages of this paper:

- D. Slepian, "*On bandwidth*," in Proceedings of the IEEE, vol. 64, no. 3, pp. 292-300, March 1976.

3. [**More about Sampling, 25 points**] Consider the Fourier transform of the signal $x_c(t)$, given in Fig. 1.

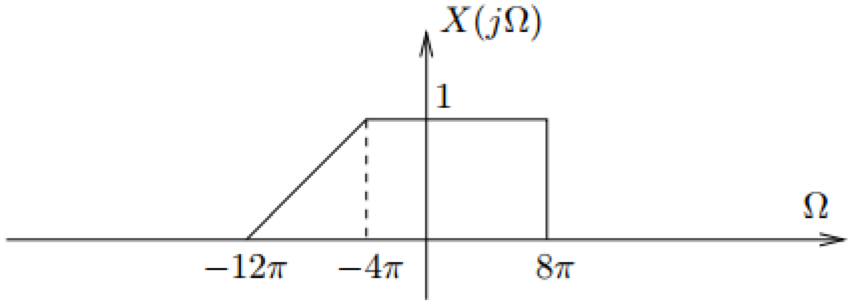

Figure 1: $X(j\omega)$

(a) What is the bandwith of the signal, i.e., the minimum Ω_N such that $X(j\Omega) = 0$ for $|\Omega| > \Omega_N$? What is the Nyquist sampling frequency for this signal?

(b) We sample $x_c(t)$ with sampling period $T_s = \frac{1}{12}$ sec. Draw the sampled spectrum of the signal, $X_s(j\Omega)$. Specify the value of the important points on both the axes.

(c) Draw the spectrum of $X(e^{j\omega})$, the DTFT of the discrete signal $x[n] = x_c(\frac{n}{12})$.

(d) Assume we want to recover the signal from its sampled version $X_s(j\Omega)$. Is it possible to do the exact reconstruction? If yes, propose a filter that recovers the original signal.

(e) Repeat parts (b) and (d) for $T_s = \frac{1}{8}$ sec.

(f) Define a new function in terms of $x_c(t)$ as $y_c(t) = e^{j2\pi t}x_c(t)$. Find $Y(j\Omega)$, the Fourier transform of $y_c(t)$, and draw it.

(g) Consider that the new signal is sampled with sampling period $T_s = \frac{1}{10}$ sec. Draw the corresponding sampled spectrum, $Y_s(j\Omega)$.

(h) Is there any aliasing effect in $Y_s(j\Omega)$? Is it possible to recover the original signal $x_c(t)$ from $Y_s(j\Omega)$? If yes, explain the required steps and write down the explicit formula, otherwise, prove your answer.

(i) Recall the Nyquist sampling frequency found in (a). Does it contradict the result in part (h)? Why?

4. [**Order of Up- and Downsampling, 20 points**] We consider the two system given in Figure 1 (note that all the signals here are discrete-time, thus we consider DTFT for spectrum).

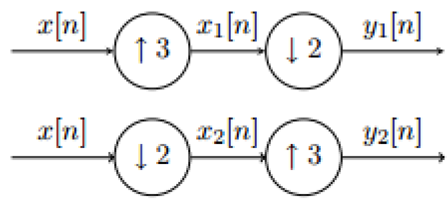

Figure 1: Systems for Problems 1 (a)–(e).

(a) For $X(e^{j\omega})$ as given in Figure 2, sketch $X_1(e^{j\omega})$ and $Y_1(e^{j\omega})$.

(b) Again for $X(e^{j\omega})$ as given in Figure 2, sketch $X_2(e^{j\omega})$ and $Y_2(e^{j\omega})$.

(c) What is the relationship between $Y_1(e^{j\omega})$ and $Y_2(e^{j\omega})$.

For parts (d)(e), we consider a general spectrum $X(e^{j\omega})$ and **not** specific to Figure 2. Therefore, the answers would be in terms of a general spectrum $X(e^{j\omega})$.

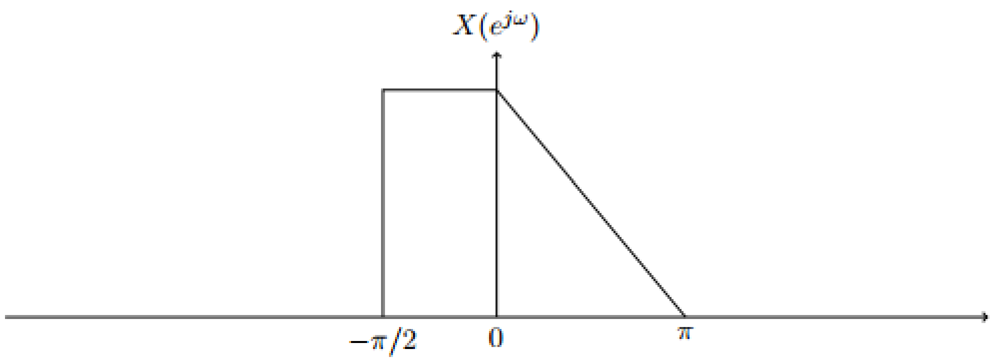

Figure 2: $X(e^{j\omega})$ for parts (a)-(c)

(d) For general $X(e^{j\omega})$, write both $Y_1(e^{j\omega})$ and $Y_2(e^{j\omega})$ in terms of $X(e^{j\omega})$.

(e) Using the expressions you have obtained in part (d) prove the relationship between $Y_1(e^{j\omega})$ and $Y_2(e^{j\omega})$ (Generalize the relationship discovered in part (c) to an arbitrary $X(e^{j\omega})$).

5. [**Amplitude Modulation with a Pulse-Train Carrier, 20 points**] Amplitude modulation with a pulse-train carrier may be modeled as in Figure 3(a). The output of the system is $q(t)$.

   (a) Let $x(t)$ be a band-limited signal [i.e., $X(j\omega) = 0$, $|\omega| \geq \pi/T$], as shown in Figure 3(b). Determine and sketch $R(j\omega)$ and $Q(j\omega)$.

(b) Find the maximum value of Δ such that $w(t) = x(t)$ with an appropriate filter $M(j\omega)$.

(c) Determine and sketch the compensating filter $M(j\omega)$ such that $w(t) = x(t)$.

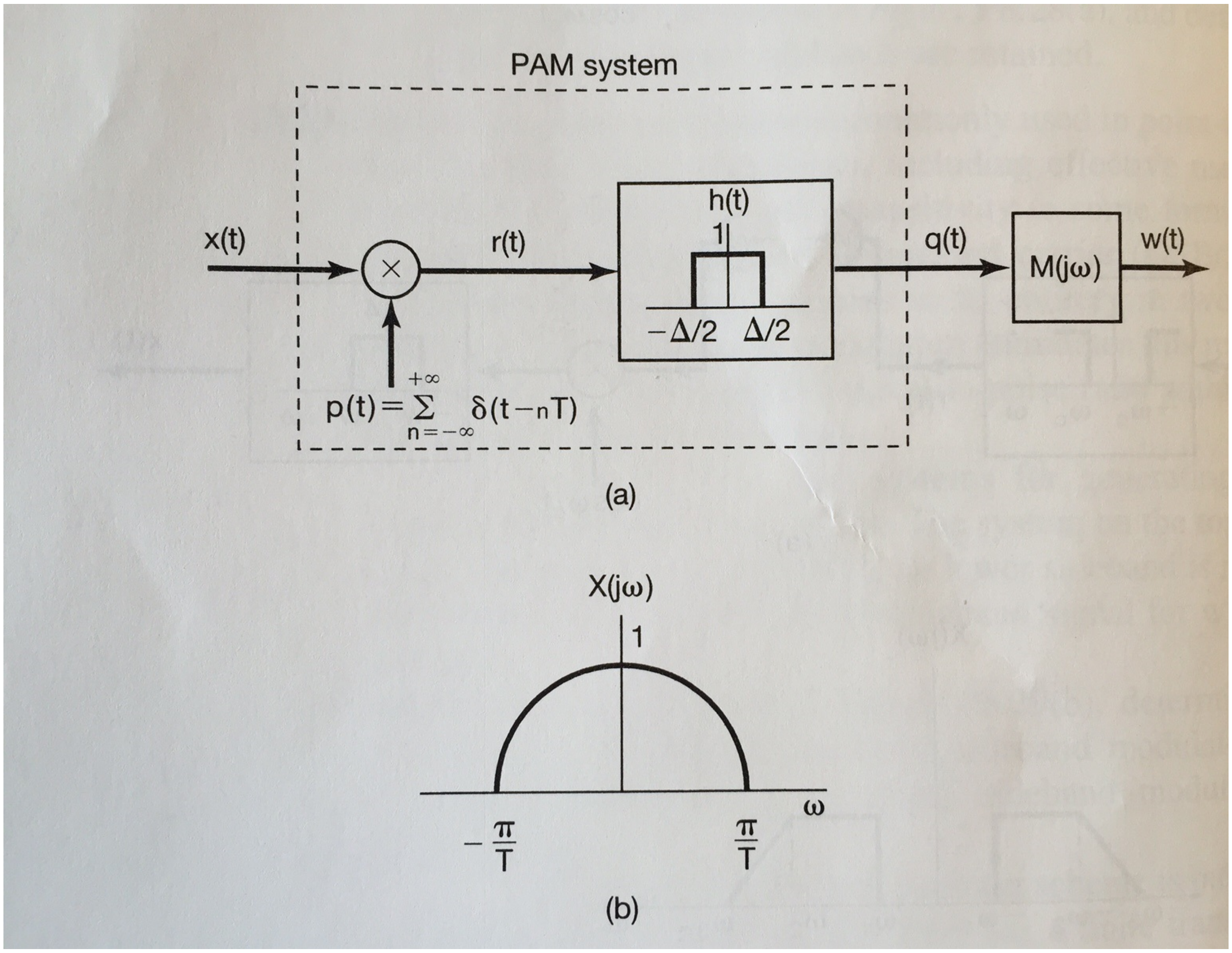

Figure 3. Amplitude Modulation with a Pulse-Train Carrier