Технические навыки взятия частных производных

$\exists\, f(x,y)$ — „очень хорошая ф-ция"

$$f(x,y) = \cos(x^2y+2x)\cdot\ln(x^2+y^2)$$

$f'_x = \dfrac{\partial f}{\partial x}$ — думаем про $f(x,y)$ как про ф-цию от x с пар-м y, дифф-м по обычным пр-м

$$f'_x(x,y) = -\sin(x^2y+2x)(2xy+2)\ln(x^2+y^2) + \cos(x^2y+2x)\frac{2x}{x^2+y^2}$$

$$f'_y(x,y) = -\sin(x^2)\ln(x^2+y^2) + \cos(x^2y+2x)\frac{2y}{x^2+y^2}$$

$$\begin{tikzcd}
 & & f(x,y) \arrow[dl, "\frac{\partial}{\partial x}"'] \arrow[dr, "\frac{\partial}{\partial y}"] & & \\
 & f'_x(x,y) \arrow[dl, "\frac{\partial}{\partial x}"'] \arrow[d, "\frac{\partial}{\partial y}"] & & f'_y(x,y) \arrow[d, "\frac{\partial}{\partial x}"'] \arrow[dr, "\frac{\partial}{\partial y}"] & \\
\frac{\partial^2 f}{\partial x^2} = f''_{xx} & \frac{\partial^2 f}{\partial x\,\partial y} = f''_{xy} & & \frac{\partial^2 f}{\partial y\,\partial x} = f''_{yx} & \frac{\partial^2 f}{\partial y^2} = f''_{yy}
\end{tikzcd}$$

$$f(x,y) = x^y$$

$f'_x = y\cdot x^{y-1}$; $\quad f'_y = x^y\ln x$

$f''_{xx} = y(y-1)\cdot x^{y-2}$ $\qquad f''_{yx} = yx^{y-1}\ln x + x^y\cdot\frac{1}{x}$

$f''_{xy} = x^{y-1} + y\cdot x^{y-1}\ln x$ $\qquad f''_{yy} = x^y(\ln x)^2$

$=$

не всегда

$$f(x,y,z) = x^{y^z}$$

$f'_x = y^z x^{y^z-1}$ $\qquad f'_y = x^{y^z}\cdot\ln x\cdot z y^{z-1}$ $\qquad f'_z = x^{y^z}\ln x^y$

Дифф-л (техническая сторона)

$f(x,y)$ — „хорошая"

x, y — незав-е пер-е

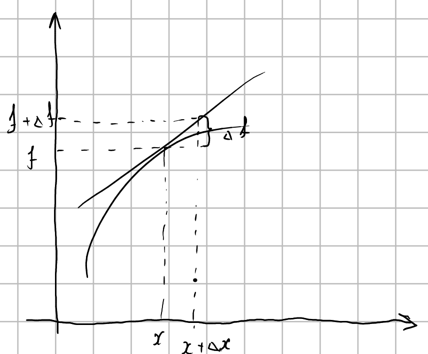
$$df = \frac{\partial f}{\partial x}\underbrace{dx}_{\text{прир-е по } x} + \frac{\partial f}{\partial y}\cdot\underbrace{dy}_{\text{прир-е по } y}$$

$$d^2 f = \frac{\partial^2 f}{\partial x^2}dx^2 + 2\frac{\partial^2 f}{\partial x \partial y}dx dy + \frac{\partial^2 f}{\partial y^2}dy^2$$

$\frac{\partial^3 f}{\partial x \partial^2 y}$

$x\,y\,y$
$y\,x\,y$
$y\,y\,x$

$$d^n f = \left(\frac{\partial \_}{\partial x}dx + \frac{\partial \_}{\partial y}dy\right)^n f = \sum C_n^k \frac{\partial^n f}{\partial_x^k \partial y^{n-k}} dx^k dy^{n-k}$$



Исп-е d для прибл. вычисл-й

$$\Delta f \approx df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$

(x_0, y) — „хорошая" т.

$(x_0+\Delta x,\ y_0+\Delta y)$ — „нужная" т. (Δx — дост-но мал)

$$f(x_0+\Delta x, y_0+\Delta y) \approx f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\Delta x + \frac{\partial f}{\partial y}(x_0,y_0)\Delta y$$

$$\sqrt[3]{0{,}95^2 + 7{,}01} \approx 2\,?$$

$x_0 = 1 \qquad \Delta x = -0{,}05$

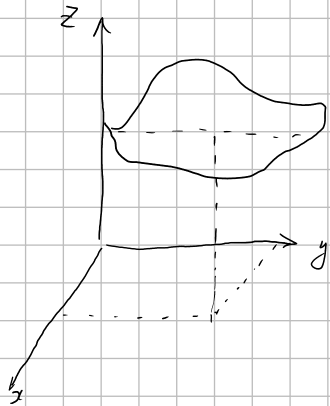
$y_0 = 7 \qquad \Delta y = 0{,}01$

$(x_0+\Delta x, y+\Delta y) = (0{,}95;\ 7{,}01)$

$$\frac{\partial f}{\partial x} = \frac{1}{3}\frac{2x}{\sqrt[3]{x^2+y}^{\,2}}\Bigg|_{(1,7)} = \frac{2}{3\cdot 4} = \frac{1}{6}, \qquad \frac{\partial f}{\partial y} = \frac{1}{3}\frac{1}{\sqrt[3]{x^2+y}^{\,2}} = \frac{1}{12}$$

$$\sqrt[3]{0{,}95^2+7{,}01} \approx 2 + \frac{1}{6}(-0{,}05) + \frac{1}{12}\cdot 0{,}01 = 1{,}985$$

Градиент



$z = f(x,y)$ $\{x, y, f(x,y)\}$

градиент в т. (x_0, y_0)

$$\overline{grad}\, f(x_0,y_0) = \vec{\nabla} f(x_0,y_0) = \left(\frac{\partial f}{\partial x}; \frac{\partial f}{\partial y}\right)_{(x_0,y_0)}$$

„набла"

Смысл: направл-е наискорейш. возр-я

модуль — „скорость"

$\vec{a} = (a_1, a_2, \dots, a_n)$ $f: \mathbb{R}^n \to \mathbb{R}$ $f(x_1, \dots, x_n)$

$$\frac{\partial f}{\partial \vec{a}} = \frac{(grad\, f; a)}{|\vec{a}|} = \left(\frac{\partial f}{\partial x_1} a_1 + \frac{\partial f}{\partial x_2} a_2 + \dots + \frac{\partial f}{\partial x_n} a_n\right) \frac{1}{\sqrt{a_1^2 + \dots + a_n^2}}$$

$$\frac{\partial f}{\partial (grad\, f)} = \frac{(grad\, f; grad\, f)}{|grad\, f|} = |grad\, f|$$

$f(x,y,z) = \operatorname{arctg} \frac{xy}{z^2}$; $M(0;1;2)$

$f'_x = \frac{z^2 y}{z^4 + x^2 y^2}$ $f'_y = \frac{z^2 x}{z^4 + x^2 y^2}$ $f'_z = -2 \frac{xyz}{z^4 + x^2 y^2}$

$grad(0,1,2) = \left(\frac{1}{4}; 0; 0\right)$

$\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ $\vec{\nabla}^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \Delta$ оператор Лапласа

если в $\mathbb{R}^n$

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i}$$

Функция f: $\Delta f = 0$ в некот области $\Rightarrow$ гармонич-я

$f(x,y) = e^x \sin y$ $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$