Homework 1 for Math 173B - Winter 2024

1. Which of these functions is strongly convex? Show your work.
   (a) $f : \mathbb{R} \to \mathbb{R}$, with $f(x) = x^6$.
   (b) $f : \mathbb{R} \to \mathbb{R}$, with $f(x) = \log(1 + e^x) + x^2$.
   (c) $f : \mathbb{R} \to \mathbb{R}$, with $f(x) = e^{x^2}$.
   (d) $f : \mathbb{R}^2 \to \mathbb{R}$, with $f(x, y) = x^2 + 2xy + y^2 + y + 1$.

2. Give an examples of
   (a) a function $f : \mathbb{R}^4 \to \mathbb{R}$ that is convex but not strongly convex.
   (b) a function $f : \mathbb{R}^4 \to \mathbb{R}$ that is strongly convex.

3. Write a computer program that runs gradient descent, with fixed step size, to minimize the 2 functions from Problem 2, and comment on the convergence speed of the algorithm on each of the functions. Repeat this twice, once with $x^{(0)} = (0.1, 0.1, 0.1, 0.1)$ and once with $x^{(0)} = (15, -20, 10, 10)$. Include graphs to illustrate your results. On the x-axis, you should have the iteration number (say t), and on the y-axis, plot $f(x^{(t)})$. You may plot the results for each of the different initializations in the same figure (but clearly labeled), in which case you would provide 2 figures total.

4. Consider the setup where we flip three fair coins.
   (a) What is the sample space Ω associated with this example.
   (b) What is the event space $\mathcal{F}$ associated with this example (you don't have to list all of its elements, it is enough to describe it accurately).
   (c) Describe the associated probability measure (recall that the coins are fair).
   (d) Suppose, now, that we care about the number of coins that come up heads. Define the random variable X corresponding to this, recalling that X is a function. That is, define the domain and range of X, as well as how it assigns values.
   (e) What is the probability mass function associated with X? Remember to show your work.

5. Consider the uniform random variable $X \sim U(a, b)$, where $a < b$. The "$X \sim \mathcal{D}$" notation here just means that the random variable X is "drawn from" the distribution $\mathcal{D}$. Moreover, we define the uniform distribution $U(a, b)$ via its probability density function:

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{if } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$$

   (a) Find, and sketch, the cumulative distribution function associated with $U(a, b)$.
   (b) What is the probability that $x \in [a, a + (b - a)/2]$?
   (c) Let $a = 0, b = 1$. What is the probability that $x \in [1/3, 1/2] \cup [4/5, 5/6]$?