Homework 5 for Math 173B - Winter 2025

1. Let $a \in \mathbb{R}^d, b \in \mathbb{R}$ and define the sets $A = \{x \in \mathbb{R}^d : a^T x + b \geq \beta\}$ and $B = \{x \in \mathbb{R}^d : a^T x + b \leq \alpha\}$ where α and β are such that $\alpha < \beta$. The goal of this problem is to show that the "margin", i.e., width of the region separating A and B is $w = \frac{\beta - \alpha}{\|a\|}$

   (a) Sketch the regions A and B.

   (b) Let z be a point in A satisfying $a^T z + b = \alpha$ and notice that the margin is given by the length of the vector λa such that $a^T(z + \lambda a) + b = \beta$. In other words your are moving orthogonally to the (affine) hyperplane $\{x : a^T x + b = \alpha\}$ until you reach the hyperplane $\{x : a^T x + b = \beta\}$. Using this insight, show that the margin is $w = \frac{\beta - \alpha}{\|a\|}$.

2. Let $A = \{(1, 0), (0, 1)\}$ and $B = \{(-2, 0), (0, -2)\}$. Draw A, B on a graph, and find a line which separates A from B. In this two-dimensional example, SVMs can help us find the "best" line that separates points in A and B.

   (a) Write down the primal SVM optimization problem and its dual optimization problem.

   (b) Using the primal problem, find the equation of the best line that separates the two sets.

   (c) Use the dual problem to find a pair of points in the convex hulls of A and B, respectively, that are closest to each other. Is such a pair unique in this example?

3. Consider the optimization problem

$$\begin{aligned} \min_{(x_1, x_2)} \; & 2x_2 + (x_1^2 + x_2^2) \\ \text{subject to} \quad & x_1 + x_2 = 1 \\ & x_1 \geq 0 \\ & x_2 \geq 0 \end{aligned}$$

   (a) Write down the KKT conditions associated with this problem.

   (b) Find a primal point and a dual point that satisfy the KKT conditions. (Hint: it might be helpful to start with the "complementary slackness" conditions, and do the casework.)

   (c) Does the primal point you found solve the optimization problem? Justify your answer.

4. Let A be a $p \times n$ matrix. Consider the linear optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & c^T x \\ \text{subject to} \quad & Ax = b \\ & x_i \geq 0, i = 1, ..., n \end{aligned}$$

(a) Write down the KKT conditions for the problem.

(b) Suppose that a primal feasible x^* and dual feasible (λ^*, ν^*), together satisfy the above KKT conditions. Will these points be optimal for the problem?

5. Using the KKT conditions, find the optimal solution to the problem

$$\min_{x \in \mathbb{R}^n} \quad \|x\|^2 \quad \text{subject to} \quad a^T x = b.$$

Why is this sufficient to find an optimal solution in this case? (Notice that you just found the closest point on the hyperplane $\{x \in \mathbb{R}^n : a^T x = b\}$ to the origin.)