Homework 4 for Math 173B - Winter 2025

1. Consider the constrained optimization problem

$$\begin{aligned} \min_{x\in\mathbb{R}^n} \quad & f(x) \\ \text{subject to} \quad & g_i(x) \le 0, \quad i = 1, ..., m. \end{aligned}$$

   Recall that the Lagrangian function associated with this problem is

$$L(x,u) = f(x) + \sum_{i=1}^{m} u_i g_i(x) = u^T g(x)$$

   where $g(x) = (g_1(x), g_2(x), ..., g_m(x)) \in \mathbb{R}^m$.
   Recall also that the Lagrangian dual function is given by

$$F(u) = \min_x L(x,u).$$

   Show that F is a concave function.
   Hint: You want to show that for $u_1, u_2 \in \mathbb{R}^m$ and $\alpha \in [0,1]$ with $u = \alpha u_1 + (1-\alpha)u_2$, we have

$$F(u) \ge \alpha F(u_1) + (1-\alpha)F(u_2).$$

   Comment: We never assumed anything about the convexity f. Still, by showing that the dual F is concave anyway, you now know that all local maxima of F are global maxima, and you can use convex optimization techniques to maximize F! Weak duality then tells you that the global minimum of f is bounded below by the global maximum of F.

2. Consider the optimization problem

$$\begin{aligned} \min_{(x_1,x_2)} \quad & x_1 + \frac{1}{2}(x_1^2 + 4x_2^2) \\ \text{subject to} \quad & x_1 + 2x_2 = 1 \\ & x_1 \ge 0 \\ & x_2 \ge 0 \end{aligned}$$

   (a) Write down the Lagrangian function associated with this problem.
   (b) Derive the associated Lagrangian dual function.
   (c) Write down the associated dual optimization problem.

3. For two vectors v and w in $\mathbb{R}^m$, we write $v \leq w$ to indicate that $v_i \leq w_i, \forall i \in \{1, \ldots, m\}$. With that in mind, consider the optimization problem

$$\min_{x \in \mathbb{R}^n} \|x\|^2 \text{ subject to } Ax \leq b$$

where $A \in \mathbb{R}^{m \times n}$, with $m \leq n$, is full rank.

(a) Write down the Lagrangian function associated with this problem.

(b) Derive the associated dual function.

(c) Write down the associated dual optimization problem.

Remark: If you don't like the inequality $Ax \leq b$, recall that it is equivalent to the set of inequalities $a_i^T x \leq b_i, i = 1, ..., m$, where the a_i's are the vectors in the rows of A.

4. Consider the optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n, y \in \mathbb{R}^m} \quad & \frac{1}{2}\|y\|^2 \\ \text{subject to} \quad & Ax - b = y. \end{aligned}$$

(a) Write down the Lagrangian associated with this problem.

(b) Find the associated dual function.

(c) Write down the dual optimization problem.

5. Let $Q \in \mathbb{R}^{n \times n}$ be a symmetric positive definite matrix and consider the optimization problem

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \quad & \frac{1}{2}x^T Q x + c^T x \\ \text{subject to} \quad & Ax \geq b. \end{aligned}$$

(a) Write down the Lagrangian associated with this problem.

(b) Find the associated Lagrangian dual function.

(c) Write down the dual optimization problem.