MATH 173B, HW4

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Problem 1

By definition of the dual function:

$$F(u_i) \leq f(x) + u_i^T g(x) \quad \forall x \in \mathbb{R}^n \tag{*}$$

Let $u = \alpha u_1 + (1-\alpha)u_2$ where $\alpha \in [0,1], \quad u_1, u_2 \in \mathbb{R}^m, m, n \in \mathbb{N}$,
and x^* be the minimizer of $L(x,u)$

$$F(u) = L(x^*, u) = f(x^*) + u^T g(x^*) = f(x^*) + \alpha \cdot u_1^T g(x^*) + (1-\alpha) \cdot u_2^T g(x^*)$$

$$F(u) = \left[\alpha \cdot f(x^*) + u_1^T g(x^*)\right] + \left[(1-\alpha) \cdot f(x^*) + u_2^T g(x^*)\right]$$

$$(*) \implies F(u) \geq [\alpha F(u_1)] + [(1-\alpha)F(u_2)]$$

Problem 2

(a)

$\mathcal{L}[x, \lambda, v] = x_1 + \frac{1}{2}(x_1^2 + 4x_2^2) - \lambda_1 \cdot x_1 - \lambda_2 \cdot x_2 + v_1 \cdot (x_1 + 2x_2 - 1)$

$\mathcal{L}[x, \lambda, v] = (1 + v_1 - \lambda_1)x_1 + (2v_1 - \lambda_2)x_2 + \frac{1}{2}(x_1^2 + 4x_2^2) - v_1$

(b)

Since the optimizer is convex in x (linear + second degree is convex) we can set the gradient in x equal to zero.

$$\nabla \mathcal{L}(x, \lambda, v) = \begin{bmatrix} (1 + v_1 - \lambda_1) + x_1 \\ (2v_1 - \lambda_2) + 4x_2 \end{bmatrix}$$

minimizer x^* is:

$$x_1 = \lambda_1 - 1 - v_1$$
$$x_2 = \frac{1}{4}(\lambda_2 - 2v_1)$$

Putting back into $\mathcal{L}(x, \lambda, v)$ yields:

$$F(\lambda, v) = (1+v_1-\lambda_1)\cdot(\lambda_1-v_1-1)+(2v_1-\lambda_2)\cdot\frac{1}{4}(\lambda_2-2v_1)-v_1+\frac{1}{2}(\lambda_1^2-2\lambda_1-2\lambda_1 v_1+1+2v_1+v_1^2)+\frac{1}{8}(\lambda_2^2-4\lambda_2 v_1+v_1^2$$

(c)

Using the function from part (b)

$$\max_{\lambda, v} F(\lambda, v) \quad \text{s.t.} \quad \lambda \geq 0.$$

Problem 3

(a)

Introducing the dual variables $\lambda \geq 0$ for the inequality constraints, the Lagrangian function is:

$$\mathcal{L}(x, \lambda) = \|x\|^2 + \lambda^T(Ax - b).$$

(b)

Taking the gradient with respect to x and setting it to zero:

$$\nabla_x \mathcal{L}(x, \lambda) = 2x + A^T\lambda = 0 \quad \Rightarrow \quad x^* = -\frac{1}{2}A^T\lambda.$$

Substituting x^* into $\mathcal{L}(x, \lambda)$:

$$F(\lambda) = -\frac{1}{4}\lambda^T AA^T\lambda - \lambda^T b.$$

(c)

The dual problem is given by:

$$\max_{\lambda \geq 0} \left(-\frac{1}{4}\lambda^T AA^T\lambda - \lambda^T b\right) \quad s.t. \lambda_i \geq 0 \quad \forall i \in \{1, 2, .., m\}$$

Problem 4

$$\mathcal{L}(x, y, \lambda, v) = \frac{1}{2}||y||^2 + v^T(Ax - b - y)$$

(b)

To find the dual function, we minimize the Lagrangian with respect to x and y:
Minimize with respect to y:

$$\frac{\partial L}{\partial y} = y - v = 0 \quad \Rightarrow \quad y = v.$$

Substitute $y = v$ into the Lagrangian:

$$L(x, v, \lambda, v) = \frac{1}{2}\|v\|^2 + v^T(Ax - b - v).$$

We end up with:

$$L(x, v, \lambda, v) = -\frac{1}{2}\|v\|^2 + v^T(Ax - b).$$

Minimize with respect to x:

$$\inf_x v^T Ax = \begin{cases} 0, & \text{if } A^T v = 0, \\ -\infty, & \text{otherwise.} \end{cases}$$

Answer:

$$F(\lambda, v) = \begin{cases} -\frac{1}{2}\|v\|^2 - v^T b, & \text{if } A^T v = 0, \\ -\infty, & \text{otherwise.} \end{cases}$$

(c)

The dual optimization problem is:

$$\max_{v \in \mathbb{R}^m} \quad -\frac{1}{2}\|v\|^2 - v^T b \quad \text{subject to} \quad A^T v = 0.$$

Problem 5

(a)

The Lagrangian associated with this problem is given by:

$$L(x, \lambda, v) = \frac{1}{2}x^T Qx + c^T x + \lambda^T(b - Ax), \quad \lambda \geq 0$$

(b)

$$\frac{\partial L}{\partial x} = Qx + c - A^T\lambda = 0 \quad \Rightarrow \quad x = -Q^{-1}(c - A^T\lambda).$$

$$L(-Q^{-1}(c - A^T\lambda), \lambda, v) = \frac{1}{2}(c - A^T\lambda)^T Q^{-1}(c - A^T\lambda) - \lambda^T b.$$

Answer: Lagrangian dual function is:

$$F(v, \lambda) = -\frac{1}{2}(c - A^T\lambda)^T Q^{-1}(c - A^T\lambda) - \lambda^T b.$$

(c)

$$\max_{\lambda \geq 0} \quad -\frac{1}{2}(c - A^T\lambda)^T Q^{-1}(c - A^T\lambda) - \lambda^T b.$$