Homework 1 for Math 173A - Fall 2024

1. Use the definition of convex functions to answer the following:
   (a) Show that $f : \mathbb{R}^d \to \mathbb{R}$ given by $f(x_1, ..., x_d) = \|x\|_2^2 = \sum_{i=1}^d x_i^2$ is convex.
   (b) Show that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = |x|$ is convex. **Hint:** You may have to break up the argument into several cases around the sign of the inputs. You may also consider using triangle inequality of absolute values: for any $a, b \in \mathbb{R}$, $|a + b| \leq |a| + |b|$.
   (c) For (b), show that f is not *strictly* convex.
   (d) Show that $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \sqrt{|x|}$ is not convex.
2. Use the definition of convex sets to answer the following:
   (a) Show that if the sets S and T are convex, then $S \cap T$ is convex.
   (b) Show that the intersection of any number of convex sets is convex.
   (c) A hyperplane in $\mathbb{R}^d$ is a set of points of the form $\{x : a^\intercal x = b\}$ where $a \in \mathbb{R}^d$ and $b \in \mathbb{R}$. Show that hyperplanes are convex. **Hint:** If you're having trouble seeing why this is true in general, try the problem with a simple concrete example in 2-dimensions.
3. Use the definition of convex functions and sets to answer the following. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function and define the set

$$E_f = \{(x, w) \in \mathbb{R}^{n+1} \quad | \quad x \in \mathbb{R}^n, \quad w \in \mathbb{R}, \quad f(x) \leq w\}.$$

   (a) Show that for all $x \in \mathbb{R}^n$, $(x, f(x)) \in E_f$.
   (b) Show that if f is a convex function, then E_f is a convex set.
   (c) Show conversely that if E_f is a convex set, then f is a convex function.

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4. Find the gradient and Hessian of the following functions, and determine whether the functions are convex.

   (a) $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x_1, x_2) = \frac{1}{2}x_1^4 + x_1 x_2 - e^{x_2}$.

   (b) $f : \mathbb{R}^d \to \mathbb{R}$ given by $f(x) = \langle a, x\rangle^2 + \langle b, x\rangle$.

5. For each problem below, find the gradient and show your work.

   (a) $f : \mathbb{R}^n \to \mathbb{R}$ for $f(x) = \|x\|_2^2$.

   (b) $f : \mathbb{R}^n \to \mathbb{R}$ for $f(x) = \|Ax\|_2^2$ where $A \in \mathbb{R}^{m\times n}$. **Hint:** It may be beneficial to think of the chain rule for the function $f(x) = g(h(x))$ where $g(z) = \|z\|_2^2$ and $h(w) = Aw$.

   (c) $f : \mathbb{R}^n \to \mathbb{R}$ for $f(x) = \|Ax - b\|_2^2$ for $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$.

   (d) $f : \mathbb{R}^n \to \mathbb{R}$ for $f(x) = \|Ax - b\|_2^2 + \gamma\|x\|_2^2$ for $A \in \mathbb{R}^{m\times n}$ and $b \in \mathbb{R}^m$ and $\gamma > 0$.

6. This problem builds on the results from problem 5.

   (a) For part 5(c), use the Hessian of $f(x)$ to show that f is convex. Under what conditions is f strictly convex?

   (b) For 5(d), show that $f(x)$ is always strictly convex.