Tips and Tricks in Physics

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Contents

1 Gaussian integral

$$\int dx\ \exp{(-ax^2)} = \sqrt{\frac{\pi}{a}} \tag{1}$$

$$\int dx\ \exp{(-ax^2 + bx + c)} = \sqrt{\frac{\pi}{a}} \exp{(\frac{b^2}{4a} + c)} \tag{2}$$

Let

$$I_n(a) = \int\limits_0^\infty e^{-ax^2} x^n dx \tag{3}$$

Then we have

$$\int\limits_0^\infty e^{-ax^2} x^n dx = \begin{cases} \dfrac{(n-1)!!}{2^{n/2+1} a^{n/2}} \sqrt{\dfrac{\pi}{a}} & \text{for } n \text{ odd} \\ \dfrac{((n-1)/2)!}{2a^{(n+1)/2}} & \text{for } n \text{ even} \end{cases} \tag{4}$$

with important cases

$$I_0(a) = \frac{1}{2}\sqrt{\frac{\pi}{a}} \tag{5}$$

$$I_1(a) = \frac{1}{2a} \tag{6}$$

$$I_0(a) = \frac{1}{4a}\sqrt{\frac{\pi}{a}} \tag{7}$$

$$I_0(a) = \frac{1}{2a^2} \tag{8}$$

2 Bogoliubov transformation

The Bogoliubov transformation is very useful to diagonalize bilinear hamiltonians in systems such as sperconductivity, superfluids and antiferromagnets (spin wave).

Fermions Consider the following hamiltonian with λ real:

$$H = \epsilon({c_1}^\dagger c_1 + {c_2}^\dagger c_2) + \lambda(({c_1}^\dagger {c_2}^\dagger + c_2 c_1) \tag{9}$$

We introduce the Bogoliubov transformations as:

$$ {c_1}^\dagger = u{d_1}^\dagger + v d_2 \tag{10}$$

$$ {c_2}^\dagger = u{d_2}^\dagger - v d_1 \tag{11}$$

The requirement that the canonical anti-commutation relations still be satisfied implies that $u = \cos\theta$ and $v = \sin\theta$, having then $u^2 + v^2 = 1$. Using this transformation gives us the following diagonalized hamiltonian:

$$H = \tilde{\epsilon}({d_1}^\dagger d_1 + {d_2}^\dagger d_2) + \epsilon - \tilde{\epsilon} \tag{12}$$

with $\tilde{\epsilon} = \sqrt{\epsilon^2 + \lambda^2}$.

Bosons Consider the same hamiltonian with λ real:

$$H = \epsilon({c_1}^\dagger c_1 + {c_2}^\dagger c_2) + \lambda(({c_1}^\dagger {c_2}^\dagger + c_2 c_1) \tag{13}$$

We introduce the Bogoliubov transformations as:

$$ {c_1}^\dagger = u{d_1}^\dagger + v d_2 \tag{14}$$

$$ {c_2}^\dagger = u{d_2}^\dagger + v d_1 \tag{15}$$

The requirement that the canonical anti-commutation relations still be satisfied implies that $u = \cosh\theta$ and $v = \sinh\theta$, having then $u^2 - v^2 = 1$. Using this transformation gives us the following diagonalized hamiltonian:

$$H = \tilde{\epsilon}({d_1}^\dagger d_1 + {d_2}^\dagger d_2) - \epsilon + \tilde{\epsilon} \tag{16}$$

with $\tilde{\epsilon} = \sqrt{\epsilon^2 - \lambda^2}$.

Note: The bosonic transformation requires $\epsilon > \lambda$ for a stable equilibrium.

3 Gamma function

The Gamma function is an extension of the factorial. We have the recursive relation:

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} \tag{17}$$

Its formal definition is

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x} dx \tag{18}$$

4 Laplace transform

5 Legendre transforms

General definition For a one dimensional function of $f(x)$, the Legendre transform is defined as the relationship $\{F, x\} \leftrightarrow \{G, s\}$, where $s(x) \equiv \frac{dF(x)}{dx}$. The Legendre transform is then defined as

$$G(s) = s \cdot x(s) - F(x(s)) \tag{19}$$

such that $x(s) = \frac{dG(s)}{ds}$.The Legendre transform of the function $F(x)$ is a more useful encoding of the information when the conditions of:

- Strict convexity (second derivative always positive) and smoothness;
- Easier to measure, control or think about the derivative of F than to measure x itself.

The condition of convexity allows for a one-to-one mapping between the $\{F, x\}$ and $\{G, s\}$. **[For more information and a more explicit geometric derivation, see arXiv:0806.1147v2.]**

Example in 1D clasical spin system In a classical spin system (the spin at position $\mathbf{r}$ is $\mathbf{S}(\mathbf{r})$), the partition function of the system can be writen in term of the Helmholtz free energy $F(\mathbf{B})$

$$Z(B) = e^{-\beta F(\mathbf{B})} \tag{20}$$

with an external magnetic field $\mathbf{B}$, in this case the probe of our system. The magnetization density of the system is therefore

$$\mathbf{m} = \frac{1}{V} \int d^d r \langle \mathbf{S}(\mathbf{r}) \rangle = -\frac{1}{V} \frac{\partial F(\mathbf{B})}{\partial \mathbf{B}} \tag{21}$$

In such a case, the Gibbs free energy is the Legendre transform of the Helmholtz free energy:

$$G(\mathbf{m}) = F(\mathbf{B}(\mathbf{m})) + V\mathbf{m} \cdot \mathbf{B}(\mathbf{m}) \tag{22}$$

where we have inverted the relation $\mathbf{m} = \mathbf{m}(\mathbf{B})$ in order to get $\mathbf{B} = \mathbf{B}(\mathbf{m})$. Furthermore, the Gibbs free energy satisfies

$$\frac{\partial G(\mathbf{m})}{\partial \mathbf{m}} = V\mathbf{B} \tag{23}$$

Both functions contain the same physical information, albeit in a different point of view. The Gibbs free energy is however more interesting in the case of thermodynamic equilibrium, with the magnitization of the system being obtained when the Gibbs free energy is stationary (in the absence of an external field, $\mathbf{B} = 0$).

If the magnetic probe $\mathbf{B}$ is varying in space, it gives rise to a spatially non-uniform magnetization, and $F[\mathbf{B}]$ and $G[\mathbf{m}]$ are now functionals of those fields. The equations are the same (with an integral over all space for the coupling of $\mathbf{m}$ and $\mathbf{B}(\mathbf{m})$), but with functional derivatives instead of classical partial derivatives.

6 Fourier transforms

We can write integrals in x and q spaces (in d-dimension) as:

$$\int_x = \int d^d x \qquad , \qquad \int_q = \frac{d^d q}{(2\pi)^d} \tag{24}$$

The Fourier transform is then defined as:

$$f(x) = \int_q \tilde{f}(q) e^{\imath q x} \qquad , \qquad \tilde{f}(x) = \int_x f(x) e^{-\imath q x} \tag{25}$$

Important Fourier Transforms

- Let $g(x) = \nabla f(x)$, then $\mathcal{F}_g(k) = \imath k \tilde{f}(k)$ (*by integration by parts*).
- Let $h(x) = \nabla^2 f(x)$, then $\mathcal{F}_h(k) = -k^2 \tilde{f}(k)$