Numerical study of the 2D Kuramoto - Sivashinsky equation

Víctor Ballester

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Université Paris-Dauphine, PSL
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Abstract

The aim of this report is to give both qualitative and quantitative insight into the chaotic behavior of the 2D Kuramoto-Sivashinsky equation. This equation is more commonly known in its 1D version and this report wants to complement the numerical study carried out in [KKP15] in order to extent the bibliography on the 2D version of the equation. Kuramoto-Sivashinsky types of equations are seen in various physical phenomena such as flame propagation or reaction-diffusion systems [Kur78; Siv77]. We will see that the 2D KS equation exhibits chaotic behavior as we increase the spatial domain size.

1 Introduction

The well known 1D Kuramoto-Sivashinsky (KS) equation can be written as

$$u_t + \frac{1}{2}{u_x}^2 + u_{xx} + u_{xxxx} = 0 \tag{1}$$

It is usually equipped with periodic boundary conditions $u(t, x + L) = u(t, x)$ for some $L > 0$, which defines the domain of definition of the PDE, and an initial condition $u(0, x) = u_0(x)$. The natural extension in the 2D case is the following Dirichlet problem with periodic boundary conditions:

$$\begin{cases} u_t + \frac{1}{2}|\nabla u|^2 + \Delta u + \Delta^2 u = 0 & \text{in } (0,\infty)\times[0,L_x)\times[0,L_y) \\ u(t,x,y) = u(t,x+L_x,y) & \text{in } [0,\infty)\times\mathbb{R}\times[0,L_y) \\ u(t,x,y) = u(t,x,y+L_y) & \text{in } [0,\infty)\times[0,L_x)\times\mathbb{R} \\ u(0,x,y) = u_0(x,y) & \text{for all } x\in[0,L_x), y\in[0,L_y) \end{cases} \tag{2}$$

with $L_x, L_y > 0$. For the sake of simplicity, we will make a rescale of the variables in order to obtain a square domain of definition, namely:

$$x_{\text{new}} = \frac{2\pi}{L_x}x \qquad y_{\text{new}} = \frac{2\pi}{L_y}y \qquad t_{\text{new}} = \left(\frac{L_x}{2\pi}\right)^2 t \tag{3}$$

Using this new variables (and dropping the subindex *new* for simplicity), the equation becomes:

$$\begin{cases} u_t + \frac{1}{2}|\nabla_\nu u|^2 + \Delta_\nu u + {\Delta_\nu}^2 u = 0 & \text{in } (0,\infty)\times[0,2\pi)\times[0,2\pi) \\ u(t,x,y) = u(t,x+2\pi,y) & \text{in } [0,\infty)\times\mathbb{R}\times[0,2\pi) \\ u(t,x,y) = u(t,x,y+2\pi) & \text{in } [0,\infty)\times[0,2\pi)\times\mathbb{R} \\ u(0,x,y) = u_0(x,y) & \text{for all } x\in[0,2\pi), y\in[0,2\pi) \end{cases} \tag{4}$$

where we used the notation from [KKP15]:

$$\nabla_\nu = \left(\partial_x, \sqrt{\frac{\nu_2}{\nu_1}}\partial_y\right) \qquad \mathbf{div}_\nu = \partial_x + \sqrt{\frac{\nu_1}{\nu_2}}\partial_y \tag{5}$$

$$\Delta_\nu = \mathbf{div}_\nu(\nabla_\nu) = \partial_{xx} + \frac{\nu_2}{\nu_1}\partial_{yy} \qquad {\Delta_\nu}^2 = \Delta_\nu(\Delta_\nu) = \partial_{xxxx} + 2\frac{\nu_2}{\nu_1}\partial_{xxyy} + \frac{{\nu_2}^2}{{\nu_1}^2}\partial_{yyyy} \tag{6}$$

and $\nu_1 := \left(\frac{L_x}{2\pi}\right)^2$, $\nu_2 := \left(\frac{L_y}{2\pi}\right)^2$. Note that the new equation is invariant under the transformation $(t, x, y, \nu_1, \nu_2) \mapsto \left(\frac{\nu_2}{\nu_1}t, y, x, \nu_2, \nu_1\right)$. That is, if $u(t, x, y)$ is a solution of the equation with parameters (ν_1, ν_2), then $u\left(\frac{\nu_2}{\nu_1}t, y, x\right)$ is the solution of the equation for the parameters (ν_2, ν_1).

Let's study now the linear stability of the different modes (k_x, k_y) of the equation for $k_x, k_y \in \mathbb{N} \cup \{0\}$. Setting $v = \delta(e^{\lambda t + i(k_x x + k_y y)} + \text{c.c.})$, with $\delta \ll 1$, as a perturbation of the trivial state $u = 0$, we obtain the following equality once we impose that v is a solution of Eq. (4):

$$\lambda = \left({k_x}^2 + \frac{\nu_2}{\nu_1}{k_y}^2\right)\left(1 - \nu_1 {k_x}^2 - \nu_2 {k_y}^2\right) \tag{7}$$

where as usual c.c. denotes the complex conjugate. We see that, for example, if $\nu_1, \nu_2 \geq 1$, then there is no pair (k_x, k_y) that makes $\lambda > 0$ and therefore all the nodes are stable. But as soon as we decrease ν_1 or ν_2 below 1, unstable nodes start to appear in an *increasing*[1] order. For example, for $\nu_1 = \nu_2 = 1/8$ the nodes $(0, 1)$, $(1, 0)$, $(1, 1)$, $(2, 0)$, $(0, 2)$, $(2, 1)$, $(1, 2)$, $(2, 2)$ are unstable. This is illustrated in **??**.

References

[KKP15] A. Kalogirou, E. E. Keaveny, and D. T. Papageorgiou. "An in-depth numerical study of the two-dimensional Kuramoto-Sivashinsky equation." In: *Proc. R. Soc. A* 471.20140932 (2015). DOI: .

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[1] Increasing in the sense the node $(k_x + 1, k_y)$ will become unstable once the node (k_x, k_y) had become unstable and not before.