Numerical study of the 2D Kuramoto-Sivashinsky equation

Víctor Ballester Ribó

Instabilities and Nonlinear Phenomena
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Université Paris-Dauphine, PSL

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Introduction

2D Kuramoto-Sivashinsky equation:

$$u_t + \frac{1}{2}|\nabla u|^2 + \Delta u + \Delta^2 u = 0$$

$$\rightsquigarrow$$

$$u_t + \frac{1}{2}\left({u_x}^2 + \frac{\nu_2}{\nu_1}{u_y}^2\right) + u_{xx} + \frac{\nu_2}{\nu_1}u_{yy} + \nu_1\left[u_{xxxx} + \frac{\nu_2}{\nu_1}u_{yyyy} + \frac{{\nu_2}^2}{{\nu_1}^2}u_{xxyy}\right] = 0$$

In order to distinguish between different types of solutions, we have monitored the evolution of the L^2 energy:

$$E(t) = \int_0^{2\pi}\int_0^{2\pi} u(x,y,t)^2 \,\mathrm{d}x\,\mathrm{d}y$$

Numerical methods

We have used a pseudo-spectral method.

- Fourier transform in space.
- Implicit-Explicit BDF2 in time [Akr+15].

$$\tilde{\mathbf{u}}_t + \mathbf{L}\tilde{\mathbf{u}} = \mathbf{N}(\tilde{\mathbf{u}})$$

$$\frac{3}{2}\tilde{\mathbf{u}}^{n+2} + h\mathbf{L}\tilde{\mathbf{u}}^{n+2} = 2\tilde{\mathbf{u}}^{n+1} - \frac{1}{2}\tilde{\mathbf{u}}^{n} + 2h\mathbf{N}(\tilde{\mathbf{u}}^{n+1}) - h\mathbf{N}(\tilde{\mathbf{u}}^{n})$$

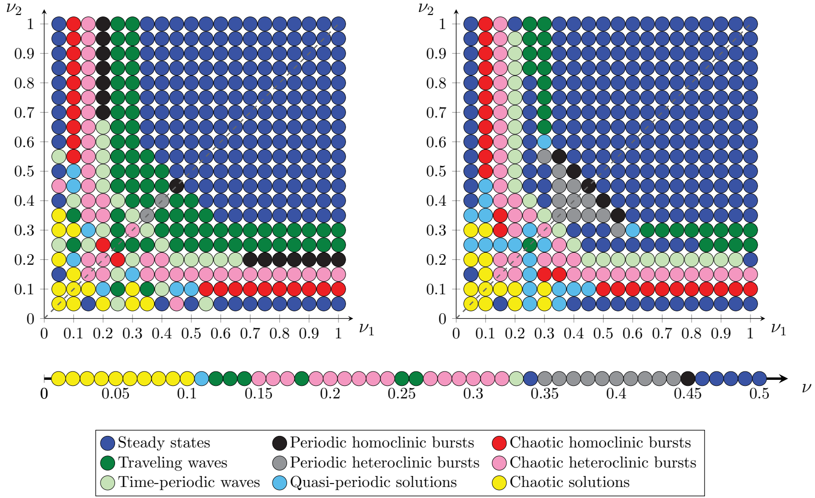
First step with Implicit-Explicit Euler:

$$\tilde{\mathbf{u}}^{n+1} = \tilde{\mathbf{u}}^{n} + h\mathbf{N}(\tilde{\mathbf{u}}^{n+1})$$

Results

$$u_0(x,y) = \sin(x) + \sin(y) + \sin(x+y)$$

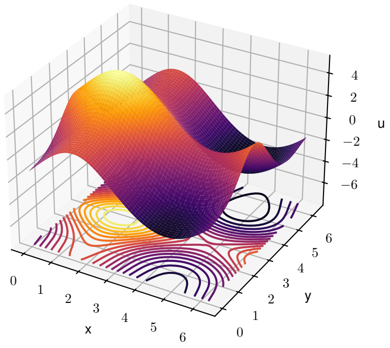
$$u_0(x,y) = \sin(x) + \sin(y) + \cos(x+y) + \sin(4x+4y) + \cos(7x) + \cos(7y)$$



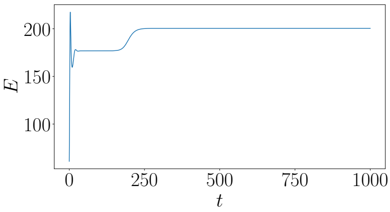
Classification of solutions varying the parameters ν_1, ν_2 with a step of 0.05. The top left figure is taken from [KKP15].

$\nu_1 = 0.85$, $\nu_2 = 0.3$

- Periodic in both space and time.
- Constant energy.



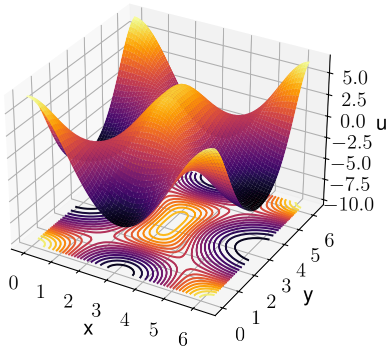
Solution at $t = 900$



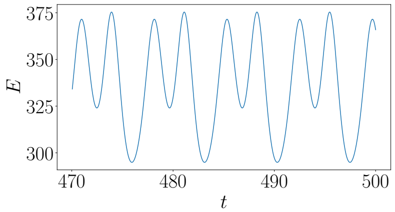
Energy evolution

Results - Time-periodic waves

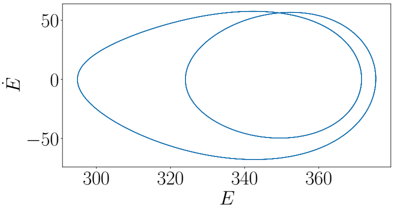
$\nu_1 = 0.35,\ \nu_2 = 0.3$



Solution at $t = 493.61$



Energy evolution

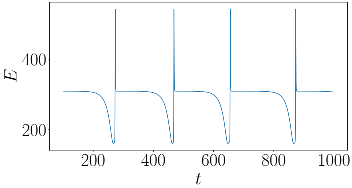


Phase space $(E, \dot{E})$

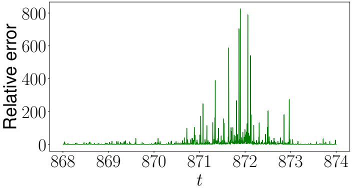
Results - Periodic bursts

Homoclinic burst

$\nu_1 = \nu_2 = 0.45$



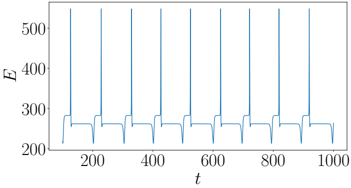
Energy evolution



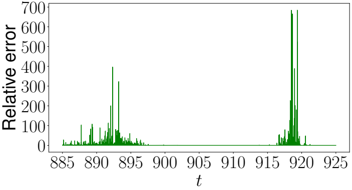
Relative error $\max_{x,y\in[0,2\pi]} \frac{|u(t,x,y) - u(t-h,x,y)|}{|u(t,x,y)|}$

Heteroclinic burst

$\nu_1 = \nu_2 = 0.4$



Energy evolution

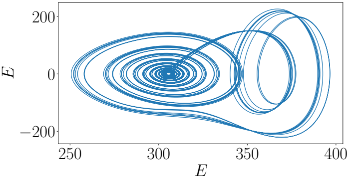


Relative error $\max_{x,y\in[0,2\pi]} \frac{|u(t,x,y) - u(t-h,x,y)|}{|u(t,x,y)|}$

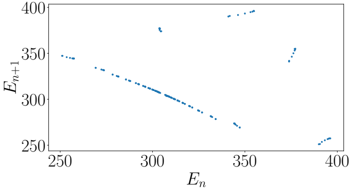
Results - Chaotic solutions

Quasi-periodic

$\nu_1 = 0.25,\ \nu_2 = 0.2$



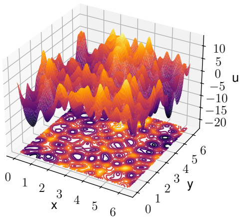
Phase space $(E, \dot{E})$



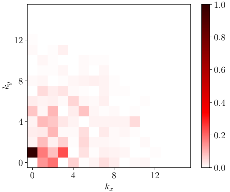
Return map with section $\dot{E} = 0$

Chaotic solution

$\nu_1 = \nu_2 = 0.008$



Solution at time $t = 5$



Fourier spectrum at time $t = 5$

Conclusions

- Periods of time-periodic waves are generally much smaller than periods of travelling waves, which, in turn, are smaller than periods of periodic bursts.
- A finer mesh grid is needed in a vicinity of bursts, specially for chaotic homoclinic and heteroclinic bursts.
- Quasi-periodic seem to be a transition between travelling waves or time-periodic waves and periodic bursts.
- For our purposes, the number of Fourier modes is relatively limited, but we cannot decrease the mesh size for small values of ν_1 and ν_2.

$$\nu_1 = \left(\frac{2\pi}{L_x}\right)^2 \qquad \nu_2 = \left(\frac{2\pi}{L_y}\right)^2$$

References

[Akr+15] G. Akrivis et al. “Linearly implicit schemes for multi-dimensional Kuramoto-Sivashinsky type equations arising in falling film flows.” In: *IMA Journal of Numerical Analysis* 36.1 (Apr. 2015), pp. 317–336. ISSN: 0272-4979. DOI: 10.1093/imanum/drv011.

[KKP15] A. Kalogirou, E. E. Keaveny, and D. T. Papageorgiou. “An in-depth numerical study of the two-dimensional Kuramoto-Sivashinsky equation.” In: *Proc. R. Soc. A* 471.20140932 (2015). DOI: 10.1098/rspa.2014.0932.