1 Proofs

1.1 Conics in a nutshell

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Proof. Consider the ellipse E centered at the origin and oriented as in **??**.

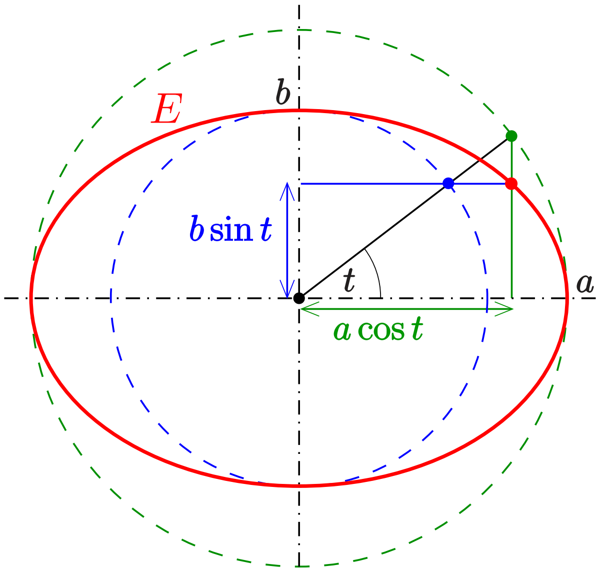

Figure 1: Reference frame centered at the center of the ellipse. Source: [Ag217].

From **??** one can check that it can be parametrized by $(x, y) = (a \cos t, b \sin t)$ with $t \in [0, 2\pi)$. This parametrization satisfies:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

Hence, the area enclosed in the ellipse can be parametrized by $(x, y) = (ar \cos t, br \sin t)$, with $r \in [0, 1]$ and $t \in [0, 2\pi)$. The Jacobian of the transformation $(r, t) \to (x, y)$ transformation is abr. Therefore, from the change of variable theorem we have that:

$$\mathrm{Area}(E) = \iint_E \mathrm{d}x \, \mathrm{d}y = \int_0^{2\pi} \int_0^1 abr \, \mathrm{d}r \, \mathrm{d}t = \pi ab \tag{2}$$

□

1.2 Introduction to astrodynamics and satellite tracking

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Proof. Cross-multiplying **??** by $\mathbf{h}$ we obtain

$$\frac{\mathrm{d}(\dot{\mathbf{r}} \times \mathbf{h})}{\mathrm{d}t} = \ddot{\mathbf{r}} \times \mathbf{h} = -\frac{\mu}{r^3} \mathbf{r} \times \mathbf{h} = -\frac{\mu}{r^3} \mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}}) = \frac{\mu}{r^3} [(\mathbf{r} \cdot \mathbf{r})\dot{\mathbf{r}} - (\mathbf{r} \cdot \dot{\mathbf{r}})\mathbf{r}] \tag{3}$$

where in the last equality we have used the vector equality $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Now note that:

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\mathbf{r}}{r} \right) = \frac{\dot{\mathbf{r}}}{r} - \frac{\dot{r}}{r^2} \mathbf{r} = \frac{1}{r^3} [(\mathbf{r} \cdot \mathbf{r})\dot{\mathbf{r}} - (\mathbf{r} \cdot \dot{\mathbf{r}})\mathbf{r}] \tag{4}$$

because[1] $2r\dot{r} = \dfrac{\mathrm{d}(r^2)}{\mathrm{d}t} = \dfrac{\mathrm{d}(\mathbf{r}\cdot\mathbf{r})}{\mathrm{d}t} = 2\mathbf{r}\cdot\dot{\mathbf{r}}$. Thus:

$$\frac{\mathrm{d}(\dot{\mathbf{r}}\times\mathbf{h})}{\mathrm{d}t} = \mu\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right) \tag{5}$$

Integrating with respect to the time yields

$$\dot{\mathbf{r}}\times\mathbf{h} = \frac{\mu}{r}\mathbf{r} + \mathbf{B} \tag{6}$$

where $\mathbf{B}\in\mathbb{R}^3$ is the constant of integration. Observe that since $\dot{\mathbf{r}}\times\mathbf{h}$ is perpendicular to $\mathbf{h}$, $\dot{\mathbf{r}}\times\mathbf{h}$ lies on the orbital plane and so does $\mathbf{r}$. Hence, $\mathbf{B}$ lies on the orbital plane too. Now, dot-multiplying this last equation by $\mathbf{r}$ and using that $\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = (\mathbf{u}\times\mathbf{v})\cdot\mathbf{w}\ \forall\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^3$ we obtain

$$h^2 = \mathbf{h}\cdot\mathbf{h} = (\mathbf{r}\times\dot{\mathbf{r}})\cdot\mathbf{h} = \mathbf{r}\cdot(\dot{\mathbf{r}}\times\mathbf{h}) = \frac{\mu}{r}\mathbf{r}\cdot\mathbf{r} + \mathbf{r}\cdot\mathbf{B} = \mu r + rB\cos\nu \tag{7}$$

where $h := \|\mathbf{h}\|$, $B := \|\mathbf{B}\|$ and ν denotes the angle between $\mathbf{r}$ and $\mathbf{B}$, called *true anomaly.* Rearranging the terms we finally obtain the equation of a conic section

$$r = \frac{h^2/\mu}{1+(B/\mu)\cos(\nu)} \tag{8}$$

with $p := h^2/\mu$ and $e := B/\mu$. □

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Proof. Recall that the area of a parallelogram generated by two vectors $\mathbf{u},\mathbf{v}\in\mathbb{R}^3$ is given by $\|\mathbf{u}\times\mathbf{v}\|$. Thus, approximating the difference $A(t+k)-A(t)$ by half of the area of the parallelogram generated by $\mathbf{r}(t)$ and $\mathbf{r}(t+k)$ (see **??**) we obtain:

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = \lim_{k\to 0}\frac{A(t+k)-A(t)}{k} = \lim_{k\to 0}\frac{\|\mathbf{r}(t)\times\mathbf{r}(t+k)\|}{2k} = \lim_{k\to 0}\frac{\|\mathbf{r}(t)\times(\mathbf{r}(t+k)-\mathbf{r}(t))\|}{2k} = \\ = \frac{\|\mathbf{r}(t)\times\dot{\mathbf{r}}(t)\|}{2} = \frac{h}{2} \tag{9}$$

where the penultimate equality is due to the continuity and linearity of the cross product. □

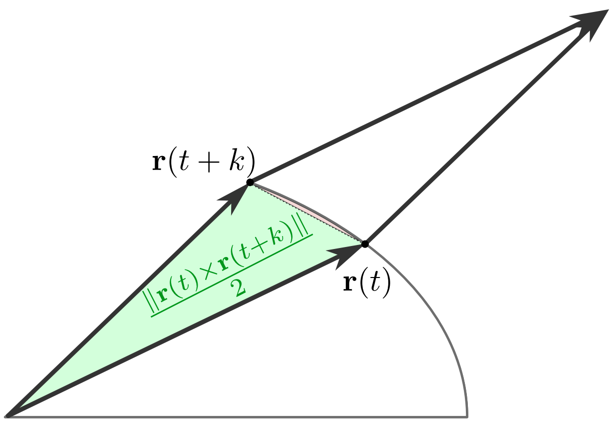

Figure 2: Graphical representation of the error made (red region) when approximating the area swept by the radius vector by half the area of the parallelogram generated by $\mathbf{r}(t)$ and $\mathbf{r}(t+k)$ (green region).

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[1] Bear in mind that in general $\dot{r}\neq\|\dot{\mathbf{r}}\|$. Indeed, if β denotes the angle between $\mathbf{r}$ and $\dot{\mathbf{r}}$ we have that $\dot{r} = \|\dot{\mathbf{r}}\|\cos\beta$. In particular $\dot{r}$ may be negative.

Proof. Integrating **??** with respect to time between 0 and T (the period) yields:

$$\pi ab = A(T) = \int_0^T A'(t)\,\mathrm{d}t = \int_0^T \frac{h}{2}\,\mathrm{d}t = \frac{hT}{2} \implies n = \frac{2\pi}{T} = \frac{h}{ab} = \frac{h}{a^2\sqrt{1-e^2}} = \sqrt{\frac{\mu}{a^3}} \tag{10}$$

where we have used **????**. $\square$

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Proof. We first prove the uniqueness. Clearly $f \in \mathcal{C}^1(\mathbb{R})$ and $f'(E) = 1 - e\cos E > 0$ for all $E \in [0, 2\pi)$ because $e < 1$. Thus, f is strictly increasing and so it has at most one zero. Now, if $0 \leq \bar{M} < \pi$, then:

$$f(M) = -e\sin M \leq 0 \qquad \text{and} \qquad f(M+e) = e(1 - \sin(M+e)) \geq 0 \tag{11}$$

So by Bolzano's theorem, f has a solution in $[M, M+e]$. If $\pi \leq \bar{M} < 2\pi$, then:

$$f(M) = -e\sin M \geq 0 \qquad \text{and} \qquad f(M-e) = -e(1 + \sin(M-e)) \leq 0 \tag{12}$$

So again by Bolzano's theorem, f has a solution in $[M-e, M]$. $\square$

1.3 Earth's gravitational field and other perturbations

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Proof. An easy computation shows that fixed $\mathbf{s} \in \mathbb{R}^3$ we have:

$$\boldsymbol{\nabla}\left(\frac{1}{\|\mathbf{r}-\mathbf{s}\|}\right) = -\frac{1}{\|\mathbf{r}-\mathbf{s}\|^3}(\mathbf{r}-\mathbf{s}) \tag{13}$$

So we need to justify whether the following exchange between the gradient and the integral is correct:

$$\mathbf{g} = -\int_\Omega \frac{G\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|^3}(\mathbf{r}-\mathbf{s})\mathrm{d}^3\mathbf{s} = \int_\Omega G\rho(\mathbf{s})\boldsymbol{\nabla}\left(\frac{1}{\|\mathbf{r}-\mathbf{s}\|}\right)\mathrm{d}^3\mathbf{s} = \boldsymbol{\nabla}\int_\Omega \frac{G\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|}\mathrm{d}^3\mathbf{s} \tag{14}$$

Without loss of generality it suffices to justify that

$$\frac{\partial}{\partial x}\int_\Omega \frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|}\mathrm{d}^3\mathbf{s} = \int_\Omega \frac{\partial}{\partial x}\left(\frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|}\right)\mathrm{d}^3\mathbf{s} \tag{15}$$

assuming $\mathbf{r} = (x, y, z)$ and $\mathbf{s} = (x', y', z')$. In order to apply the theorem of derivation under the integral sign we need to control $\frac{\partial}{\partial x}\left(\frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|}\right) = -\rho(\mathbf{s})\frac{x-x'}{\|\mathbf{r}-\mathbf{s}\|^3}$ by an integrable function $h(\mathbf{s})$. Using spherical coordinates centered at $\mathbf{r}$ and writing $(\mathbf{r}-\mathbf{s})_{\mathrm{sph}} = (\rho_{\mathbf{r}}, \theta, \phi)$, the integrand to bound becomes (in spherical coordinates):

$$\left|-\rho(\mathbf{s})\frac{x-x'}{\|\mathbf{r}-\mathbf{s}\|^3}{\rho_{\mathbf{r}}}^2\sin\phi\right| = |\rho(\mathbf{s})|\left|\frac{\rho_{\mathbf{r}}\cos\theta\sin\phi}{{\rho_{\mathbf{r}}}^3}{\rho_{\mathbf{r}}}^2\sin\phi\right| \leq |\rho(\mathbf{s})| \leq K \tag{16}$$

where the last inequality follows for certain $K \in \mathbb{R}$ by Weierstrass theorem (ρ is continuous and Ω is compact). Thus, since $h(\mathbf{s}) = K$ is integrable, because Ω is bounded, the equality of **??** is correct. $\square$

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Proof. Recall that $\Delta V = \mathbf{div}(V)$. So since $\mathbf{g} = \boldsymbol{\nabla}V$ it suffices to prove that $\mathbf{div}(\mathbf{g}) = 0$. Note that if $\mathbf{r} \in \Omega^c$, then $\exists\delta > 0$ such that $\|\mathbf{r}-\mathbf{s}\| \geq \delta > 0$ $\forall\mathbf{s} \in \Omega$ because Ω is closed. As a result, $\frac{\mathbf{r}-\mathbf{s}}{\|\mathbf{r}-\mathbf{s}\|^3}$ is differentiable and:

$$\begin{aligned}\mathbf{div}\left(\frac{\mathbf{r}-\mathbf{s}}{\|\mathbf{r}-\mathbf{s}\|^3}\right) &= \frac{\partial}{\partial x}\left(\frac{x-x'}{\|\mathbf{r}-\mathbf{s}\|^3}\right) + \frac{\partial}{\partial y}\left(\frac{y-y'}{\|\mathbf{r}-\mathbf{s}\|^3}\right) + \frac{\partial}{\partial z}\left(\frac{z-z'}{\|\mathbf{r}-\mathbf{s}\|^3}\right) = \\ &= \frac{\|\mathbf{r}-\mathbf{s}\|^2 - 3(x-x')^2}{\|\mathbf{r}-\mathbf{s}\|^5} + \frac{\|\mathbf{r}-\mathbf{s}\|^2 - 3(y-y')^2}{\|\mathbf{r}-\mathbf{s}\|^5} + \frac{\|\mathbf{r}-\mathbf{s}\|^2 - 3(z-z')^2}{\|\mathbf{r}-\mathbf{s}\|^5} = 0\end{aligned}$$

Hence, as in **??**, we have that:

$$\left|\rho(\mathbf{s})\frac{\|\mathbf{r}-\mathbf{s}\|^2-3(x-x')^2}{\|\mathbf{r}-\mathbf{s}\|^5}\right| \leq \frac{4|\rho(\mathbf{s})|}{\|\mathbf{r}-\mathbf{s}\|^3} \leq \frac{4|\rho(\mathbf{s})|}{\delta^3} \tag{17}$$

which is integrable by Weierstrass theorem. Thus, by the theorem of derivation under the integral sign:

$$\mathbf{div}(\mathbf{g}) = -\,\mathbf{div}\int_\Omega \frac{G\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|^3}(\mathbf{r}-\mathbf{s})\mathrm{d}^3\mathbf{s} = -\int_\Omega G\rho(\mathbf{s})\,\mathbf{div}\left(\frac{\mathbf{r}-\mathbf{s}}{\|\mathbf{r}-\mathbf{s}\|^3}\right)\mathrm{d}^3\mathbf{s} = 0 \tag{18}$$

□

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Proof. Suppose we have two solutions V_1, V_2 of **??**. Then, $W := V_1 - V_2$ is harmonic in Ω^c, $W = 0$ on $\overline{\partial\,\Omega}$ and $\lim\limits_{\|\mathbf{r}\|\to\infty} W = 0$. So $\forall\varepsilon > 0$, $\exists n \in \mathbb{N}$ large enough such that $\Omega \subseteq B(0,n)$ and $|W| \leq \varepsilon$ on $\mathbb{R}^3 \setminus \overline{B(0,n)}$. Thus, by the maximum principle, $|W| \leq \varepsilon$ on $\overline{B(0,n)} \cap \Omega^c$. Since the ε is arbitrary, we must have $W = 0$ on Ω^c, that is, $V_1 = V_2$. □

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Proof. Let N_{n_1,m_1}, N_{n_2,m_2} be the normalization factors of the spherical harmonics Y_{n_1,m_1}, Y_{n_2,m_2} respectively. Note that we can separate the variables in the integral of **??**. So if $i \neq j$, the integral over θ becomes $\int_0^{2\pi}\cos(m_1\theta)\sin(m_2\theta)\,\mathrm{d}\theta$ which is equal to 0 regardless of the values of m_1 and m_2. So from now on assume that $i = j$. Due to the symmetry between the cosine and the sine we can suppose that $i = \mathrm{c}$. Thus:

$$\int_0^{2\pi}\int_0^{\pi} Y^i_{n_1,m_1}(\theta,\phi)Y^j_{n_2,m_2}(\theta,\phi)\mathrm{d}\Omega =$$

$$= N_{n_1,m_1}N_{n_2,m_2}\int_0^{\pi} P_{n_1,m_1}(\cos\phi)P_{n_2,m_2}(\cos\phi)\sin\phi\mathrm{d}\phi\int_0^{2\pi}\cos(m_1\theta)\cos(m_2\theta)\,\mathrm{d}\theta \tag{19}$$

An easy check shows that if $m_1 \neq m_2$ then the integral over θ is zero (and the same applies with sines). So suppose $m_1 = m_2 = m$. In that case, if $m \neq 0$ we have $\int_0^{2\pi}(\cos m\theta)^2\,\mathrm{d}\theta = \int_0^{2\pi}(\sin m\theta)^2\,\mathrm{d}\theta = \pi$ and if $m = 0$, the cosine integral evaluates to 2π whereas the sine integral is 0. We can omit this latter case because $Y^{\mathrm{s}}_{n,0}$ is identically zero. Thus:

$$\frac{2\pi}{2-\delta_{0,m}}N_{n_1,m}N_{n_2,m}\int_0^{\pi} P_{n_1,m}(\cos\phi)P_{n_2,m}(\cos\phi)\sin\phi\mathrm{d}\phi = \frac{2\pi}{2-\delta_{0,m}}N_{n_1,m}N_{n_2,m}\int_{-1}^{1} P_{n_1,m}(x)P_{n_2,m}(x)\,\mathrm{d}x \tag{20}$$

By **??** this latter integral is $\frac{2}{2n_1+1}\frac{(n_1+m)!}{(n_1-m)!}\delta_{n_1,n_2}$. Finally, if $n_1 = n_2 = n$, putting all normalization factors together we get:

$$\frac{2\pi}{2-\delta_{0,m}}N^m_n N^m_n\frac{2}{2n+1}\frac{(n+m)!}{(n-m)!} = 4\pi \tag{21}$$

□

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Proof. Let $f(r,\theta,\phi)$ be a solution of the Laplace equation:

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial f}{\partial\phi}\right) + \frac{1}{r^2(\sin\phi)^2}\frac{\partial^2 f}{\partial\theta^2} \tag{22}$$

Using separation variables $f(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$ we can write:

$$\frac{\Theta\Phi}{r^2}(r^2R')' + \frac{R\Theta}{r^2\sin\phi}(\sin\phi\Phi')' + \frac{R\Phi}{r^2(\sin\phi)^2}\Theta'' = 0 \tag{23}$$

Here, we are making and abuse of notation denoting all the derivatives with a prime, but the reader should have no confusion with it. Isolating R from Θ and Φ yields:

$$\frac{(r^2R')'}{R} = -\frac{1}{\sin\phi\Phi}(\sin\phi\Phi')' - \frac{1}{(\sin\phi)^2\Theta}\Theta'' \tag{24}$$

Since the left-hand side depends entirely on r and the right-hand side does not, it follows that both sides must be constant. Therefore:

$$\frac{(r^2R')'}{R} = \lambda \tag{25}$$

$$\frac{1}{\sin\phi\Phi}(\sin\phi\Phi')' + \frac{1}{(\sin\phi)^2\Theta}\Theta'' = -\lambda \tag{26}$$

with $\lambda \in \mathbb{R}$. Similarly, separating variables from **??** we obtain that the equations

$$\frac{1}{\Theta}\Theta'' = -m^2 \tag{27}$$

$$\frac{\sin\phi}{\Phi}(\sin\phi\Phi')' + \lambda(\sin\phi)^2 = m^2 \tag{28}$$

must be constant with $m \in \mathbb{C}$ (a priori). The solution to the well-known **??** is a linear combination of the $\cos(m\theta)$ and $\sin(m\theta)$. Note, though, that since Θ must be a 2π-periodic function, that is satisfying $\Theta(\theta + 2\pi) = \Theta(\theta)$ $\forall\theta \in \mathbb{R}$, m must be an integer. On the other hand making the change of variables $x = \cos\phi$ and $y = \Phi(\phi)$ in **??** and using the chain rule, that equation becomes:

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\lambda - \frac{m^2}{1-x^2}\right)y = 0 \tag{29}$$

which is the associate Legendre equation. We have argued in **??** that we need $\lambda = n(n+1)$ and $m \leq n$ in order to obtain regular solutions at $x = \cos\phi = \pm 1$. Moreover, these solutions are $P_{n,m}(\cos\phi)$.

Finally, note that equation **??** is a Cauchy-Euler equation (check [Wik]) and so the general solution of it is given by

$$R(r) = c_1r^n + c_2r^{-n-1} \tag{30}$$

because $\lambda = n(n+1)$ (the reader may check that r^n and r^{-n-1} are indeed two independent solutions of **??**). So the general solution becomes a linear combination of each solution found varying $n \in \mathbb{N}\cup\{0\}$ and $m \in \{0,1,\ldots,n\}$:

$$f(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}(a_nr^n + b_nr^{-n-1})P_{n,m}(\cos\phi)(c_{n,m}\cos(m\theta) + s_{n,m}\sin(m\theta)) \tag{31}$$

□