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2008 Nonlinearity 21 T45

(http://iopscience.iop.org/0951-7715/21/4/T02)

IOP Publishing NONLINEARITY

Nonlinearity **21** (2008) T45–T60 doi:10.1088/0951-7715/21/4/T02

OPEN PROBLEM

Spatially localized structures in dissipative systems: open problems

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Received 19 December 2007, in final form 1 February 2008
Published 28 February 2008
Online at stacks.iop.org/Non/21/T45

Abstract
Stationary spatially localized structures, sometimes called dissipative solitons, arise in many interesting and important applications, including buckling of slender structures under compression, nonlinear optics, fluid flow, surface catalysis, neurobiology and many more. The recent resurgence in interest in these structures has led to significant advances in our understanding of the origin and properties of these states, and these in turn suggest new questions, both general and system-specific. This paper surveys these results focusing on open problems, both mathematical and computational, as well as on new applications.

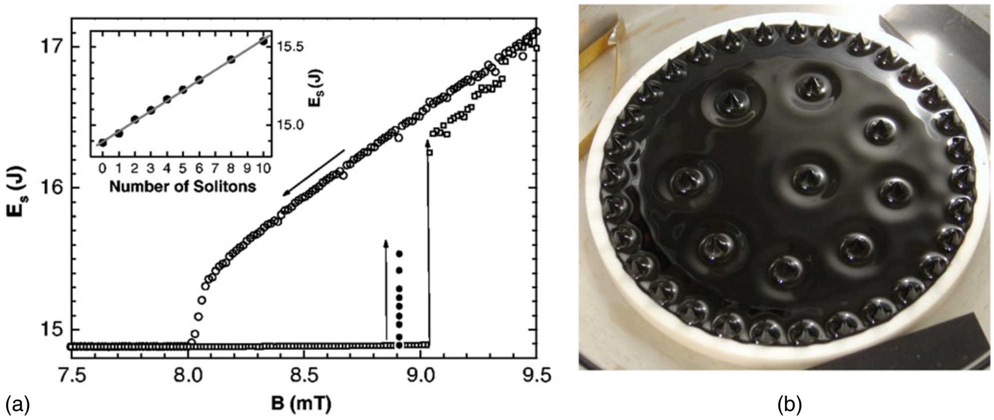
Mathematics Subject Classification: 35B32, 35B60, 35G30

(Some figures in this article are in colour only in the electronic version)

Theory

The study of stationary spatially localized states in driven dissipative systems has a long history. This paper is not intended as a review of this vast subject [13]. Instead it briefly summarizes, in nontechnical language, some of the key recent developments in our understanding of these fascinating and often surprising structures, followed by a discussion of issues that the author believes will be resolved in the near future, as well as some more challenging questions that may be around for a longer time. Propagating solitary waves or pulses are not considered.

The paper focuses on continuum systems while acknowledging the existence of parallel phenomena in discrete dynamical systems, typically lattice systems. Throughout the term ‘spatially localized structure’ will be used to refer to the presence of one state ‘embedded’ in a background consisting of a different state. Examples include a finite amplitude homogeneous state (hereafter, a nontrivial state) embedded in a background of the zero or trivial state, a localized spatial or spatio-temporal oscillation in a background trivial state, etc. Figure 1 shows a recent example from an experiment on a ferrofluid in an applied vertical dc magnetic field [57]. The figure shows a number of stable steady spatially localized ’solitons’ created by suitable perturbation in the bistable region where both the flat surface and a periodic hexagonal array of peaks coexist. These individual localized states may form different types of stable bound states resembling ‘molecules’.



(a) (b)

Figure 1. (*a*) The surface energy E_s of a ferrofluid layer for increasing (open squares, flat surface) and decreasing (open circles, periodic hexagonal array of peaks) magnetic induction B. The full circles indicate the energy of different numbers of spatially localized 'solitons', shown in (*b*), all at nominally identical parameter values ($B = 8.91$ mT) within the region of bistability between the flat surface and the periodic hexagonal pattern. The peaks along the boundary in (*b*) are an edge effect and are excluded from E_s. Reprinted figure with permission from [57]. Copyright (2005) by the American Physical Society.

The theory is simplest for reversible systems on unbounded domains in one spatial dimension, that is, translation-invariant systems that are also equivariant under spatial reflections $\mathcal{R} : x \to -x$. In this case the traditional picture associates time-independent spatially localized structures with homoclinic orbits on the real line. This formulation is usually referred to as *spatial dynamics*. Pomeau [53] pointed out that looking at localized states as approximate *heteroclinic cycles*, i.e. as connections from the background to the included state and back again, is more fruitful. This is particularly so for variational systems, i.e. systems that evolve at all times towards states of lower energy. In this picture one focuses attention on regions in parameter space with bistability between the two competing states, and in particular, on parameter values near the so-called Maxwell point where the two competing states have the same energy. The heteroclinic cycle then becomes a bound state of two 'fronts', the structures that separate the background from the inclusion. Pomeau observed a crucial difference between the case in which the two competing states are both spatially homogeneous and the case in which the localized state is structured. The former case is analogous to a standard first order phase transition, with the location of the phase transition determined by the Maxwell construction. At this parameter value a front connecting the two states will be stationary: the two competing states coexist. Moreover, an infinite multiplicity of spatially localized states of different lengths can be constructed by combining back-to-back fronts. However, this is not possible at other parameter values where one or other state is energetically preferred, and the fronts move to minimize energy by 'expelling' the higher energy state. In the case of bistability with a structured state the situation is quite different, however. Here the fronts at either end can 'lock' or 'pin' to the structure within the localized state, and an infinite number of localized states exists not only at the Maxwell point but in an entire interval of parameter values around the Maxwell point. The resulting interval is called the pinning region, or, for reasons explained below, the *snaking* region [53, 67].

The structure of solutions within the pinning region has been elucidated only recently, and the results are summarized in a recent paper [11] in a special issue of *Chaos* devoted to spatially localized structures. In variational systems on the real line, such as the bistable

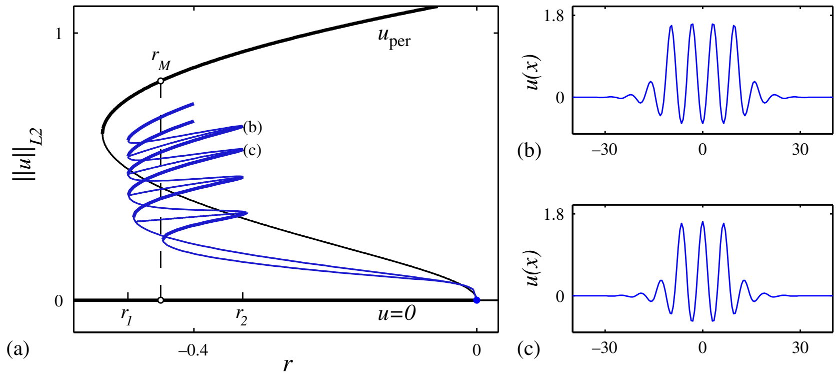


Figure 2. (*a*) Bifurcation diagram showing the snakes-and-ladders structure associated with homoclinic snaking in the Swift–Hohenberg equation (1). The sidepanels show sample solution profiles at the locations (b,c) indicated in the left panel, one from each snaking branch. Heavy lines indicate stable solutions. The pinning (snaking) region $r_1 < r < r_2$ straddles the Maxwell point r_M and contains an infinite number of coexisting stable localized states of different lengths. Courtesy of J Burke.

Swift–Hohenberg equation

$$u_t = ru - \left(\partial_x^2 + k_0^2\right)^2 u + f(u), \tag{1}$$

where $f(u) \equiv b_2u^2 - u^3$, $b_2^2 > 27k_0^4/38$, a branch of spatially periodic (i.e. structured) states bifurcates subcritically from the trivial state $u = 0$ at $r = 0$, so that for $r < 0$ stable large amplitude periodic states u_{per} coexist with the stable trivial state $u = 0$ (figure 2). There is, in addition, a pair of branches of spatially localized steady states that also bifurcates subcritically from $u = 0$ at $r = 0$, and also do so subcritically. Each of these states is even under spatial reflection, in $x = 0$ say, but one has a (global) maximum at $x = 0$ while the other has a (global) minimum at $x = 0$ (figure 2). Since the steady states of this equation satisfy a fourth order ordinary differential equation, we identify the periodic state with a periodic orbit in four-dimensional phase space, while $u = 0$ is identified with a fixed point at the origin. Near $r = 0$ the localized states correspond to homoclinic orbits to the origin, but with decreasing r these orbits approach the period orbit and begin to wrap around it forming an approximate heteroclinic connection between $u = 0$ and the periodic orbit and back again, i.e. an (approximate) heteroclinic cycle. As this occurs the branches of localized states enter the pinning region and begin to snake back and forth; as one proceeds up the snaking branches that result the localized solution adds one wavelength of the periodic state on either side between successive saddle-nodes. In this way the localized state gradually grows in length and approaches the periodic state. In addition, the two snaking branches are connected by short segments of asymmetric steady states forming a DNA-like structure referred to as the *snakes-and-ladders* structure. In variational systems the stability along each snaking branch changes back and forth at every saddle-node, resulting in the coexistence of an infinite number of stable localized states of both types within the pinning region $r_1 < r < r_2$. The asymmetric localized states are always unstable.

It is helpful to provide a geometrical interpretation of the pinning region as well. For this purpose we consider time-independent reversible equations of the form (1), focusing on the subcritical regime in which a reflection-symmetric hyperbolic periodic orbit u_{per} coexists with a hyperbolic fixed point $u = 0$ in the four-dimensional phase space of the equation viewed as a dynamical system in space. Reversibility implies that each stable eigenvalue $-\lambda$ of the fixed point is accompanied by an unstable eigenvalue $+\lambda$ and vice versa, and likewise for the

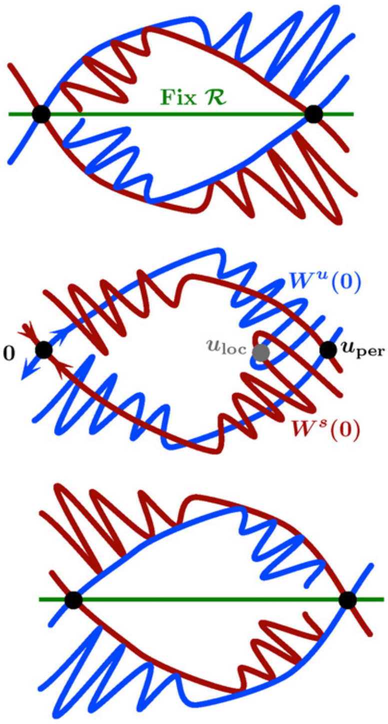


Figure 3. The stable and unstable manifolds of the fixed point $u = 0$ and the periodic state u_{per} showing the first and last tangencies that form the boundary of the pinning region, as well as the secondary intersections of the stable and unstable manifolds of $u = 0$ in the symmetry plane $x = 0$ responsible for the presence of the localized states u_{loc} within it. Courtesy of D Lloyd.

periodic orbit. Moreover, since a periodic orbit has a +1 Floquet multiplier due to phase shifts, there must necessarily be a second +1 multiplier as well, in addition to one stable and one unstable multiplier. It follows that in the subcritical regime $r < 0$ the equilibrium $u = 0$ has two-dimensional stable and unstable manifolds, while the coexisting periodic orbit u_{per} has three-dimensional centre-stable and centre-unstable manifolds. In four dimensions the intersection of the two-dimensional unstable manifold of $u = 0$ with the three-dimensional centre-stable manifold of u_{per} is robust and likewise for the corresponding intersection of the three-dimensional centre-unstable manifold of u_{per} with the two-dimensional stable manifold of $u = 0$ required by reversibility. Secondary intersections between the stable and unstable manifolds of $u = 0$ lie in the symmetry plane $x = 0$ and hence correspond to spatially localized states with reflection symmetry, starting at $u = 0$ as $x \to -\infty$, winding around the periodic orbit, and returning to $u = 0$ as $x \to \infty$ (figure 3). In particular, the boundaries r_1, r_2 of the pinning region correspond to the first and last tangencies between the two-dimensional unstable manifold of $u = 0$ and the three-dimensional centre-stable manifold of the periodic orbit [15, 67]. In fact, these geometrical ideas can be used to understand the whole snakes-and-ladders structure of the pinning region [5]. In contrast, the intersection between the two-dimensional unstable manifold of a fixed point and the two-dimensional stable manifold of a distinct fixed point is of codimension one, implying that such an intersection will occur at a single value of r only. This value corresponds to the usual Maxwell point between two spatially homogeneous states familiar from first order phase transition theory. In this case the pinning region is absent.

Q1. Structure of the snaking region. The pinning or snaking region $r_1 < r < r_2$ is filled with heteroclinic cycles between the periodic and trivial states, forming arbitrarily long localized states. These states are all of single-pulse type. However, it is clear that the pinning region is filled with great many additional states, which we call multipulse states [65]. These states are essentially two or more copies of the localized states already found. These states can be assembled in a variety of ways on the real line and separated by different, regular or irregular distances determined by the pinning or locking between their overlapping tails. Many of these are likely to be stable. Thus the structure of the pinning or snaking region is much more intricate than suggested by figure 2. The full details are unknown although it is clear that the multipulse states generally lie on isolas and not on complete snaking branches since they cannot grow indefinitely in extent without interaction [40, 65].

Q2. Creation of the snaking region. The origin of the pinning or snaking region can be traced to the fact that equation (1) is *reversible* in space, i.e. that it is invariant under $x \to -x$, $u \to u$. This property is responsible for the presence of the so-called spatial reversible Hopf bifurcation with 1 : 1 resonance from $u = 0$ that takes place at $r = 0$ [33]. The unfolding of the normal form for this bifurcation near the codimension two point $r = 0$, $b_2^2 = 27k_0^4/38$ establishes the existence of an exponentially thin heteroclinic region, of width of order $\exp\{-C(b_2 - k_0^2\sqrt{27/38})^2\}$, due to the transversal intersection between the unstable manifold of $u = 0$ and the centre-stable manifold of the periodic orbit. As already mentioned, such intersections are structurally stable. This has been done for equation (1) by Kozyreff and Chapman [43] using beyond-all-orders matched asymptotic expansions but proofs are not available. There appear to be few mathematical results available once $b_2^2 - 27k_0^4/38 = O(1)$, and the snaking region opens out ($r_2 - r_1 = O(1)$).

Q3. Destruction of the snaking region. Away from the codimension two point the width of the snaking region increases and the region may be abruptly terminated, for example, by a collision with another Maxwell point [10]. One possible mechanism leading to the destruction of a snaking region is the formation of a tangency between the unstable manifold of the periodic orbit and the stable manifold of a distinct fixed point [30], but the details of the transitions in the vicinity of such tangencies are unclear. A systematic study of the mechanisms responsible for the closure of snaking regions in reversible systems would be welcome.

Q4. Duality. High up the snaking branches the localized states can be viewed as holes in an otherwise spatially periodic state, instead of as inclusions of a structured state in a background trivial state. This notion has been called duality [16] although the term reciprocity seems more appropriate [69]. Multiple scale perturbation theory on the real line shows that a pair of branches of hole-like states bifurcates near the saddle-node on the branch of periodic states. These branches enter the pinning region and begin to snake as the hole deepens and fills with the trivial state. Thus the pinning region is filled with hole-like states (and multihole states!) *in addition* to the (classical) localized states already discussed.

In fact the notion of duality is imprecise. In the four-dimensional problem defined by equation (1), periodic orbits above the saddle-node have one stable and one unstable Floquet multiplier in space, as well as two Floquet multipliers at +1; at the saddle-node all four multipliers collide at +1, and below it two lie on the unit circle and are complex. This reversible bifurcation differs, therefore, from the bifurcation at $r = 0$. The situation is further complicated by the fact that the periodic orbit does not have a well-defined period or wavenumber: for each wavenumber there is a branch of periodic orbits with that wavenumber, and the stable wavenumbers occupy a region known as the Busse balloon. Heteroclinic connections between periodic orbits with the same or different wavenumbers are likely to be present, and in variational systems cycles composed of such connections may constitute steady solutions. These questions are closely related to recent work on the classification of wave defects by Sandstede and Scheel [59].

Numerically it is known that large amplitude pulse-like states also emerge from the saddle-node of periodic orbits. The corresponding bifurcation is the equivalent of a 'sniper' bifurcation but here occurs in higher dimensions and in a reversible system. In related problems both periodic and nonperiodic states accumulate on the saddle-node [68]. Details are not known. See [56] for related work.

Q5. Wavelength selection. The wavelength within localized states of large spatial extent high up the snaking branches is spatially uniform, unique and depends on r. This wavelength is reduced from its value at the Maxwell point when $r < r_{\mathrm{M}}$ and increased when $r > r_{\mathrm{M}}$. These observations are a consequence of energetics: when $r < r_{\mathrm{M}}$ the fronts 'want' to move inwards to eliminate the higher energy periodic state but are prevented from doing so by the pinning. The result is a compression of the pattern. The opposite occurs when $r > r_{\mathrm{M}}$. Quantitative predictions of the wavelength dependence on r within the pinning region rely on the presence of a conserved spatial Hamiltonian for equation (1). The requirement that the heteroclinic cycle lie in the level set of this function containing the trivial state leads to a prediction of the wavelength that is in complete agreement with the numerical determination [11]. There are no ideas how to compute the selected wavelength in systems which are not Hamiltonian in space. However, the same collapse of the Busse balloon to a unique wavelength is observed in such systems as well. Note that even fronts that are infinitely far apart lead to unique wavenumber selection. The relation of this observation to (the absence of) wavelength selection among spatially *periodic* states [17] is unclear.

Q6. Finite length periodic domain. On an unbounded domain the two pairs of snaking branches (classical and hole-like) remain distinct. This is not so, however, once the domain becomes finite. In this case neither set of branches can snake forever, and the snaking process must terminate when the width of the localized periodic state approaches the domain size, and likewise for the width of the localized hole. Thus both pairs of branches must turn over and exit the snaking region. Typically the two pairs of branches reconnect pairwise, so that the small amplitude branch with maxima at $x = 0$ now connects to the corresponding branch of hole-like states, and likewise for the branch with minima at $x = 0$ (figure 4). This figure, computed for equation (1), shows that the classical localized states enter the pinning region from small amplitude on the right and exit this region at large amplitude towards the left; however, the same figure can also be viewed as showing that the hole-like states enter the snaking region at the top from the left and exit it near the bottom towards the right. One finds, in addition, that in domains of finite size, the branches of small amplitude localized states no longer bifurcate directly from the trivial state, but now do so in secondary bifurcations on the primary branch of periodic states. The finite domain size likewise shifts the bifurcation to the hole-like states. However, overall in domains of finite but large size the behaviour within the snaking region remains essentially identical to that present on the real line, with the effects of the finite size confined to the vicinity of bifurcations creating the localized and hole-like states in the first place. Details of termination of the snaking branches at either end are not fully known, although it is known that the branches can 'unzip' at large amplitude and terminate on *different* periodic branches [6].

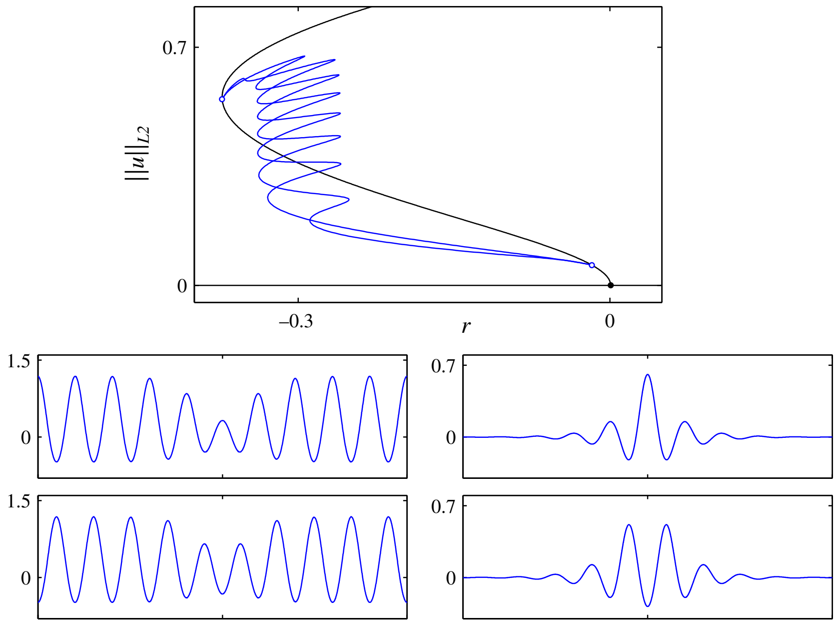


Figure 4. Bifurcation diagram showing the pinning (snaking) region in the Swift–Hohenberg equation (1) on a finite domain. The left panels show a pair of hole-like states near the saddle-node on the branch of periodic states, while the right panels show a pair of spatially localized states near the primary bifurcation. Courtesy of J Burke.

Q7. Nonvariational systems. There are many closely related systems which are nonvariational but still spatially reversible. In such systems energy is no longer defined and the existence of a pinning region must be determined numerically. One finds, generally speaking, that these systems behave similarly to variational systems although the absence of an energy makes the observed behaviour more mysterious. There are, however, some important differences. In particular, the asymmetric states are now produced in parity-breaking bifurcations from the symmetric states and lead to *drifting* localized states. Whether these states continue to form the 'rungs' of a snakes-and-ladders structure is not known. Nor is their stability. However, it is clear that Hopf bifurcations can now occur and hence that the stability assignments along the snakes-and-ladders structure can substantially change.

Q8. Local bifurcations in reversible systems. We have seen that the spatial $1:1$ reversible Hopf bifurcation provides the key to the origin of the snaking region in the Swift–Hohenberg equation. However, other local bifurcations in fourth order systems arise as well, including the codimension-one bifurcation where all eigenvalues are purely imaginary with a pair colliding at the origin forming one positive and one negative eigenvalue [32, 42, 48], and the case with four real eigenvalues, two stable and two unstable, with a pair colliding at the origin and becoming purely imaginary. Of particular interest is the generation of global bifurcations within the unfoldings of these bifurcations since these correspond to localized states or fronts. In optics these include the codimension-one embedded solitons, as well as quasi-solitons, i.e. solitary waves embedded in a (typically exponentially small) background oscillation. The codimension two bifurcation in which all four eigenvalues are at the origin combines these two possibilities [34]. Detailed understanding of these bifurcations is very useful for the study of spatially localized structures but the problems are mathematically difficult largely because of the presence of purely imaginary eigenvalues [48], and the results remain incomplete. Applications lead to higher order problems as well.

Q9. Stability. The stability of the different types of localized states already mentioned has typically been determined numerically by posing the problem on a domain of large but finite spatial period. It is important, however, to fully understand the spectrum of the linearized problem on the real line, and a systematic use of techniques such as the (numerically determined) Evans function would be very helpful.

More theory

Numerical simulations in higher dimensions reveal similar phenomena to those just described in one spatial dimension. However, the theoretical understanding of these results is much more difficult, even for systems that are variational. One of the main reasons is that one can no longer use the approach of spatial dynamics. The equations describing axisymmetric or spherically symmetric patterns continue to depend on a single spatial variable, but are now nonautonomous. Although these complications can be overcome in specific cases the associated analysis becomes much more involved. Even more problematic is the case of patterns localized in two or three spatial dimensions since there appears to be no way of dealing simultaneously with more than one time-like variable. In fact, to the author's knowledge, there have been no studies of fully three-dimensional localized structures, despite the fact that such structures are expected to form in subcritical Turing bifurcations in three-dimensional media. The numerical difficulty appears to lie partly in visualization of such structures since their spatial structure may be quite complex, cf [12].

Q10. Two and three spatial dimensions. Questions 1–9 have natural analogues in two and three spatial dimensions. However, as just mentioned, to understand the behaviour of even the simplest states, the quasi-one-dimensional radially symmetric states, the notion of spatial dynamics must be extended to nonautonomous systems. There are, in addition, localized patches of both stripes and hexagons in the two-dimensional generalization of equation (1); the details of the associated snaking behaviour have been worked by Lloyd *et al* [47]. In addition, there are two-dimensional states that are localized in one dimension and extended in the other [11] and these also snake. In bistable systems with the additional symmetry $u \to -u$ snaking involving other two-dimensional patterns, such as regular triangles or the so-called patchwork quilt state [29] is also anticipated. These types of problems are the focus of research by several groups and detailed results will become available. Extension to three dimensions should be relatively straightforward except for the presence of many more coexisting periodic structures [12] that make the details of the snaking behaviour rather more involved. Likewise snaking involving the various known superpatterns in two dimensions [20, 36] is also expected. Complications involve multiple Maxwell points, overlapping pinning regions and the fact that as the pattern grows by nucleating individual structures it repeatedly breaks and restores the basic symmetry of the state [47].

Q11. Sidewinding. In systems with global coupling snaking may exist but be slanted [19, 26, 27] rather than being 'vertical'—hence the name [26]. The practical significance of this behaviour is that the localized states may exist over a much broader region in parameter space, and indeed even outside the region of bistability. The predictions from this type of theory explain qualitatively several experiments, most notably those on gas discharges by Purwins *et al* [55], but detailed understanding of the origin of the required nonlocal terms is lacking. In many problems such terms arise through an asymptotic reduction of a larger system, cf [19]. Thus a transition from snakes to sidewinders must be a property of local equations as well, although the origin of such a transition remains to be worked out.

Q12. Localized oscillations. Spatially localized oscillations or *oscillons* are found in parametrically driven systems [46, 52, 63]; in some cases these form stable bound states that resemble 'molecules'. Many of these systems possess a natural oscillation frequency because of the proximity to an oscillatory instability, and this oscillation may be either homogeneous or at finite wavenumber. In such systems the application of time-periodic near-resonant parametric forcing of sufficient amplitude may lead to bistability between the (time-independent) background and forced response that is phase-locked to the forcing. Thus in such systems it is the forcing that is responsible for the presence of bistability that is critical for the appearance of localized structures, and this effect is largest within the $1:2$ resonance tongue. In the simplest case, where the natural oscillations are homogeneous in space and weakly damped, the system is described by a single amplitude equation, the forced Ginzburg–Landau equation, whose steady states correspond to driven oscillations at half the forcing frequency. In particular, structured spatially localized steady solutions of this equation in a zero amplitude background correspond to the so-called standard oscillons [69]. However, in this case the competing state is typically the homogeneous phase-locked state, and hence a full snaking region is absent. Nonetheless, the oscillon branch may still oscillate about the 'Maxwell point' although only a finite number of oscillon states is present at any other parameter value.

The hole states are also present and again bifurcate from the saddle-node on the branch of phase-locked states, i.e. from the boundary of the resonance tongue [69]. These states correspond to what have been called superoscillons or reciprocal oscillons, i.e. localized oscillations embedded in a nonzero background oscillation at the same frequency. As before, these states are also tied to the presence of the same 'Maxwell point' as the standard oscillons. Here, however, the equation is nonvariational and the location of the 'Maxwell point' can only be determined numerically by searching for a heteroclinic cycle between the zero and spatially uniform phase-locked states. The bound states, corresponding to the multipulse states already mentioned, should also be present in this region.

One finds, moreover, that there is a number of different types of fronts connecting in-phase and out-of-phase oscillations, with internal structure that reflects the spatial eigenvalues of the embedded zero state. These are also organized by the same 'Maxwell point'. It appears that such fronts are in turn unstable to two-dimensional instabilities that lead to the formation of labyrinthine patterns, much as seen in some experiments [52]. The precise nature of this relation remains unclear.

It should be mentioned that related behaviour also accompanies the presence of other resonance tongues. In particular, the spatial dynamics within the $1:1$ resonance tongue are very rich, with the details still to be worked out. Related phenomena in which an envelope description is invalid, and the full time-dependence must be retained, have hardly been addressed [35]. Of particular interest is the possibility that related states are present in the Faraday problem, i.e. parametrically driven gravity-capillary waves on the surface of a liquid.

Q13. Snaking of localized time-periodic states. The oscillon discussion in the preceding paragraph suggests that temporally periodic spatially localized states should also undergo snaking. In the above example, however, the use of an amplitude equation like the forced

Ginzburg–Landau equation turned the oscillations in the original system into equilibria of the amplitude equation. At present we have no example of snaking of a truly time-dependent state, although numerical simulations of Marangoni convection in a binary fluid suggest that this does indeed occur [2]. However, no theory for these systems appears to exist at the present time, and detailed studies will require software to follow unstable time-periodic oscillations of this type in parameter space.

Q14. Fluctuations. The localized structures discussed here all correspond to equilibria (or periodic orbits) of a dynamical system, but none corresponds to an equilibrium in the thermodynamic sense. In applications nonthermal small amplitude temporal fluctuations, hereafter *noise*, are expected to be present. In variational systems each stable localized state corresponds to a local minimum of the energy function. Thus the addition of noise will lead to eventual escape over the separating saddle points from one minimum to a lower energy minimum, and hence to the gradual erosion of the localized state. Thus noise will result in front motion, or depinning, even inside the pinning region. This motion will either collapse the localized state (if $r < r_M$) or grow it (if $r > r_M$), but existing studies are confined to a few simulations only [14, 58].

Q15. Forced symmetry-breaking. It is of considerable interest to understand the fate of the time-independent and time-dependent localized states present in spatially reversible systems when spatial reversibility is weakly broken, for example, by inclining the system or subjecting it to a horizontal force. The steady states in general turn into drifting or travelling pulses that we may call solitary waves. The multiplicity of such states in physical systems is unknown in general but perturbing away from reversibility could shed much light on this question and on possible snaking behaviour of such travelling pulses and their stability.

Trapping of such pulses in imposed spatial heterogeneity is also of interest, and is expected to lead to complex dynamics, cf [18].

Applications

The questions raised in the previous section have, first and foremost, applications to a great many systems that are described by the bistable Swift–Hohenberg equation and its relatives. These include problems involving the buckling of thin struts [31], the theory of shallow water waves with small dispersion described by a Korteweg–de Vries equation with higher derivative terms [9], neural models [44] as well as a variety of phase-like equations arising in longwave description of patterns [38]. In particular, they are frequently found in equations arising in nonlinear optics [30, 60, 61, 64], as the recent special issue of *Chaos* makes clear. The recent experiments on a ferrofluid in an applied dc magnetic field [57] have already been mentioned.

In the following we describe some applications to more complex systems, and in particular to systems in which the boundary conditions and structure in the transverse direction play an important role. Many other applications, particularly in the theory of localized buckling, surface catalysis and nonlinear optics are at least as compelling, but are omitted here due to space constraints.

Q16. Natural doubly diffusive convection. Stable localized dissipative structures in fluid mechanics were first discovered in natural doubly diffusive convection [28], i.e. convection driven by horizontal gradients of temperature and concentration, in two-dimensional simulations in a closed box with no-slip boundary conditions along all walls. More recent work [7] studied the same system in a vertically extended slot. With periodic boundary conditions in the vertical the conduction solution present when the imposed temperature and concentration gradients balance possesses the $1:1$ reversible Hopf bifurcation in space familiar from equation (1) even though reflections in horizontal planes now act by -1 instead of $+1$. The presence of this bifurcation guarantees the presence of an exponentially thin snaking region in the weakly subcritical regime, and this region rapidly broadens with increasing subcriticality of the primary branch of periodic solutions. Although neither variational nor Hamiltonian in space this system exhibits the same type of homoclinic snaking as seen in equation (1), and qualitatively the same wavelength selection process. However, for the parameter values explored the snaking branches do not end on the branch from which they originate. It is believed that much the same structure will persist even when the horizontal gradients do not balance, and a zero flux mean vertical flow is present as the base state. The complexity of the space of solutions observed in this system may be related to that observed in related systems with a vertical concentration gradient and sideways heating [62].

Q17. Binary fluid convection. In this system a mixture of two miscible fluids is heated from below. When the mixture is characterized by a negative separation ratio the heavier component migrates towards the warm lower boundary and so sets up a stabilizing concentration gradient in response to the applied destabilizing temperature gradient. The resulting system is characterized by bistability between the conduction state and steady overturning convection (SOC), but the behaviour is complicated by the fact that convection sets in at a temporal Hopf bifurcation at $R = R_{\mathrm{H}}$ much before the $1:1$ reversible spatial Hopf bifurcation already discussed, which is present at $R = R_{\mathrm{c}}$ (off-scale in figure 5). Here R is the Rayleigh number and is a measure of the dimensionless temperature difference applied across the system. Despite this stable time-independent localized states connecting steady convection to the *unstable* conduction state are found in the regime $R_{\mathrm{H}} < R < R_{\mathrm{c}}$ [4]. This surprising behaviour (figure 5) can be understood provided the snaking region lies below the threshold R_{abs} for absolute instability of the conduction state, and relies on the spontaneous selection of travelling wave (TW) states beyond the primary Hopf bifurcation. When the snaking region extends beyond R_{abs} the 'voids' between adjacent localized states fill with waves, and the localized states are now embedded in a background of waves, and hence are no longer entirely steady. However, a proper mathematical understanding of the stability of these localized states is still lacking, and the nature of the transition from this state to steady spatially periodic convection remains unclear. In particular, in larger domains the single-pulse states should become unstable (on physical grounds) and the system should select multipulse states instead. Whether this occurs remains unknown.

Convection in a binary fluid mixture in a porous medium, a closely related system, has been studied in great detail by A Bergeon (private communication). The study of the snaking region in this system is the most complete to date, and includes a variety of branches of single and multipulse states. The study reveals just how incomplete our understanding of this region still is.

It should be mentioned that both these systems have the symmetry $O(2) \times Z_2$ owing to identical boundary conditions at the top and bottom. This difference is responsible for the presence of *four* snaking branches instead of two, and the localized states can now be classified by their parity under spatial reflection into odd and even states (figure 6). In the literature these have been called *convectons* [4, 8].

Q18. Dynamics outside of the pinning region. At parameter values just above the pinning region no time-independent states are present except for the two competing spatially extended

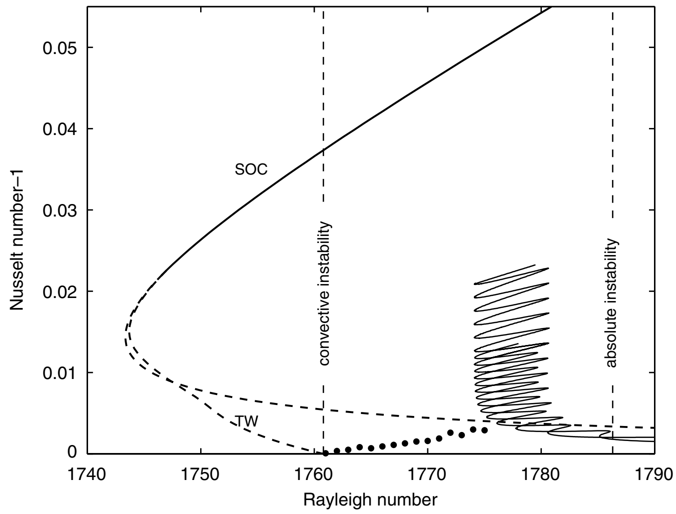


Figure 5. Bifurcation diagram showing the dimensionless time-averaged heat flux $\overline{N}$ as a function of the Rayleigh number R in an aspect ratio $\Gamma = 60$ periodic domain. The conduction state loses instability at $R_{\mathrm{H}} = 1760.8$. Above the threshold small amplitude dispersive chaos is present (solid dots), which leads into the pinning region ($1774 < R < 1781$) containing a multiplicity of stable localized states of both even and odd parity. Branches of SOC and of TWs are also shown. From [4].

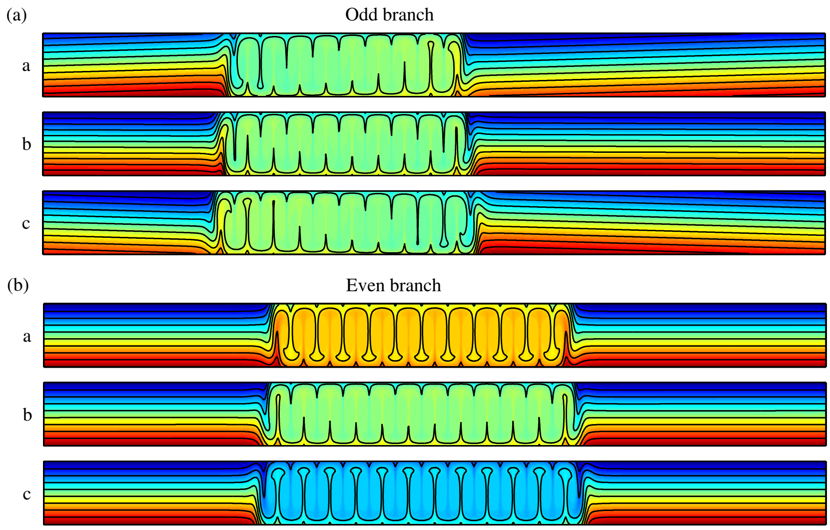


Figure 6. Convecton profiles at successive turning points on the (*a*) odd and (*b*) even parity convecton branches in terms of contours of constant concentration. Each wavelength contains a pair of rolls. From [4].

states. As a result the fronts bounding the localized state unpin in this regime, and since the periodic pattern is energetically preferred the fronts gradually move outwards, at the expense of the higher energy trivial state. The fronts propagate by sequential nucleation of new structures at either end of the localized state, all the time maintaining its symmetry. The structures already present do not move. This motion of the fronts is well understood [10] and persists until the spatially periodic state occupies the available domain. Likewise, at parameter values just below the pinning region it is the trivial state that is preferred and localized states are slowly eroded from both sides, again maintaining the symmetry of the state. In variational systems this process terminates once the localized state is completely gone. However, in some systems (such as binary fluid convection [4]) the trivial state is unstable, and as a result the localized structure regrows abruptly, and the process of erosion repeats. Relaxation oscillations, periodic or chaotic, are the unexpected result. Depending on other parameters the localized state may either form in the same location or in a different, and arbitrary, location. In much the same way the dynamics above the pinning region will strongly depend on whether the threshold $R_{\rm abs}$ for absolute instability lies within the pinning region or above it. Moreover, any instabilities of either of the competing states within the pinning region must have profound consequences for the stability of the broad localized states high up the snaking branches, but the details remain to be clarified.

Q19. Turbulent bands in Couette flow. Spiral turbulence, i.e. spiral bands of turbulent flow embedded in a background laminar flow were first observed in narrow gap Taylor–Couette flow (see [1] and references therein). More recently, experiments on plane Couette flow ([54] and references therein) have revealed the presence of a transition to stationary oblique turbulent *bands* interspersed with laminar flow, when the Reynolds number of the flow is reduced from the fully turbulent regime. Simulations confirm this finding [3]. These observations suggest the presence of spatially localized turbulent patches in a laminar background, or perhaps more appropriately the presence of 'holes' in a turbulent background filled with the laminar state. The ideas summarized above appear to be applicable. The system is reversible in space under the same 180° rotation as natural doubly diffusive convection. Moreover, shear flow instabilities are almost always highly subcritical, and this is all the more so for Couette flow which is in fact stable with respect to all infinitesimal three-dimensional disturbances. Localized states, whether involving an inclusion of the large amplitude turbulent state in a laminar background or vice versa, form naturally under these circumstances, with the bands corresponding to multipulse (or multihole) states. The differences between this scenario and the earlier discussion revolve around the fact that (a) the transverse direction is essential for the presence of the turbulence, and that (b) the state that coexists with the laminar flow is in these circumstances turbulent rather than simply periodic in space. Since the bands are turbulent the presence of noise might well destroy the multiplicity of different 'stable' states present in the vicinity of the 'Maxwell point' for this system.

In principle one can get a handle on this state in two ways. Neither is straightforward. Since the basic Couette flow is stable all such localized states must bifurcate from infinity, together with the branch of turbulent states. To avoid this difficulty one could start with a narrow annulus Taylor–Couette system, or with Couette flow with a superposed Poiseuille flow, both of which have stability thresholds at finite Reynolds number, and compute small amplitude localized states which can then be continued homotopically to the plane Couette problem of interest. In two dimensions these states would not be turbulent and so could be followed even when unstable; the third dimension could then be turned on to test the stability of the solution with respect to three-dimensional perturbations and for possible evolution into localized turbulence. Alternatively one can view the hole-like states as bifurcating from the 'saddle-node on the branch of turbulent states'. In fact there are many saddle-nodes, for different wavelength periodic states, and unstable hole-like states could be continued from these in two dimensions, and once again tested for stability in three. The plethora of unstable TW states present in pipe flow near onset may be related to the same mechanism [22, 25, 37, 66]. The open questions concerning the transition to turbulence in pipe flow are discussed by Eckhardt [23] focusing on spatially extended coherent structures that can be studied by imposing periodic boundary conditions with a relatively short period. It is this author's view that much is to be learned by examining properties of spatially localized structures (i.e. 'puffs') as well. However, these require computations on large domains that have hitherto not been attempted.

Q20. Reaction–diffusion equations. The discovery of spontaneous replication of localized spots in numerical simulations of the two-dimensional Gray–Scott model [50], and its subsequent experimental confirmation [45], has stimulated much interest in such states in reaction–diffusion equations. Many models consist of equations for only one activator and only one inhibitor and hence are of fourth order in space and reversible; many also exhibit bistability between two distinct states. In many cases these states are spatially homogeneous; the resulting localized structures have been called 'mesas' and their properties have been studied in the abstract [39]; stability is only known in particular cases [41]. In other cases the bistability that occurs is between a homogeneous and spatially periodic state, and the phenomenology discussed earlier in this paper carries over. In particular, the basic idea behind the experimentally observed spot replication is that of sequential nucleation that occurs just outside the pinning region, and indeed calculations based on the Gray–Scott model [49] confirm this idea. No doubt similar behaviour occurs in two and three spatial dimensions. Other reaction–diffusion systems, such as the Gierer–Meinhardt system, exhibit similar behaviour [70]; it is reasonable to expect that this behaviour is in fact typical of reaction–diffusion systems in appropriate parameter regimes. It would be very helpful if the work done on this class of systems were integrated into a general picture, as attempted in this paper.

Final remark. In this paper I have listed a number of questions about spatially localized states in dissipative systems. These questions all arose from the presence of a conceptually simple structure in parameter space, the pinning region. This structure unifies a great variety of physically and mathematically disparate systems, and it is for this reason that a detailed understanding of its consequences would be invaluable in applications. None of this would have been possible without the software that has been written for this purpose such as AUTO [21] and XPPAUT [24]. It appears that at present the numerical investigations are far ahead of our mathematical understanding of these interesting phenomena [51]. It is to be hoped that mathematics will catch up over the next 10 years or so, and even suggest new types of structures that have not hitherto been observed.

Acknowledgment

The preparation of this paper was supported by the National Science Foundation under grant DMS-0605238. I am grateful to J Burke for helpful comments, and to A Alonso, P Assemat, O Batiste, A Bergeon, J Burke, I Mercader and A Yochelis for collaboration on questions related to the topic of this paper.

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