

# Numerical study of the 2D Kuramoto-Sivashinsky equation

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# Introduction

2D Kuramoto-Sivashinsky equation:

$$u_t + \frac{1}{2} |\nabla u|^2 + \Delta u + \Delta^2 u = 0$$



$$u_t + \frac{1}{2} \left( u_x^2 + \frac{\nu_2}{\nu_1} u_y^2 \right) + u_{xx} + \frac{\nu_2}{\nu_1} u_{yy} + \nu_1 \left[ u_{xxxx} + \frac{\nu_2}{\nu_1} u_{yyyy} + \frac{\nu_2^2}{\nu_1^2} u_{xxyy} \right] = 0$$

In order to distinguish between different types of solutions, we have monitored the evolution of the  $L^2$  energy:

$$E(t) = \int_0^{2\pi} \int_0^{2\pi} u(x, y, t)^2 \, dx \, dy$$

# Numerical methods

We have used a pseudo-spectral method.

- Fourier transform in space.
- Implicit-Explicit BDF2 in time [Akr+15].

$$\tilde{\mathbf{u}}_t + \mathbf{L}\tilde{\mathbf{u}} = \mathbf{N}(\tilde{\mathbf{u}})$$



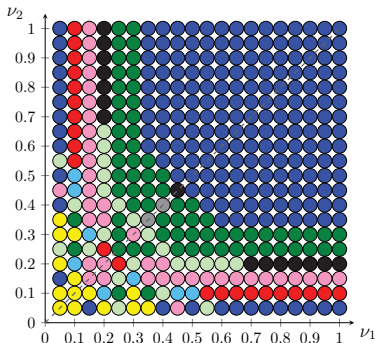
$$\frac{3}{2}\tilde{\mathbf{u}}^{n+2} + h\mathbf{L}\tilde{\mathbf{u}}^{n+2} = 2\tilde{\mathbf{u}}^{n+1} - \frac{1}{2}\tilde{\mathbf{u}}^n + 2h\mathbf{N}(\tilde{\mathbf{u}}^{n+1}) - h\mathbf{N}(\tilde{\mathbf{u}}^n)$$

First step with Implicit-Explicit Euler:

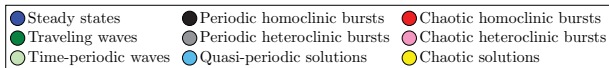
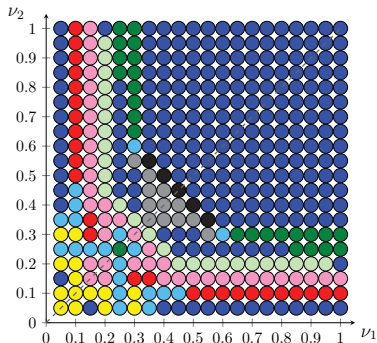
$$\tilde{\mathbf{u}}^{n+1} = \tilde{\mathbf{u}}^n + h\mathbf{N}(\tilde{\mathbf{u}}^{n+1})$$

# Results

$$u_0(x, y) = \sin(x) + \sin(y) + \sin(x + y)$$



$$u_0(x, y) = \sin(x) + \sin(y) + \cos(x + y) \\ + \sin(4x + 4y) + \cos(7x) + \cos(7y)$$

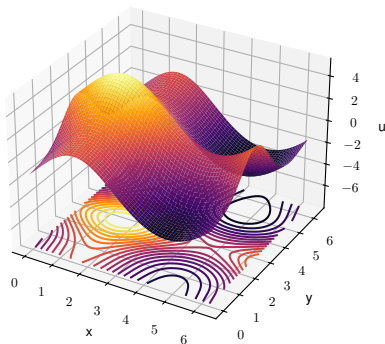


The top left figure is taken from [KKP15].

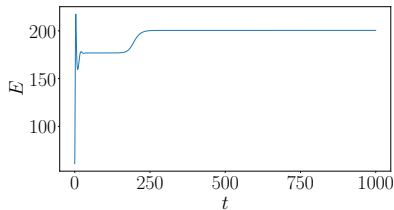
# Results - Travelling waves

$$\nu_1 = 0.35, \nu_2 = 0.3$$

- Periodic in both space and time.
- Constant energy.



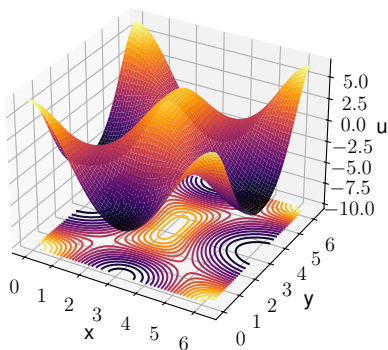
Solution at  $t = 900$



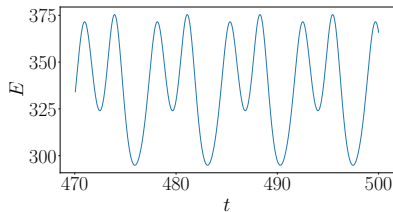
Energy evolution

# Results - Time-periodic waves

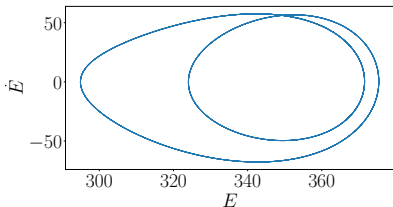
$$\nu_1 = 0.35, \nu_2 = 0.3$$



Solution at  $t = 493.61$



Energy evolution

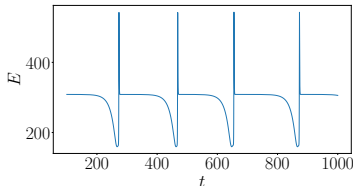


Phase space  $(E, \dot{E})$

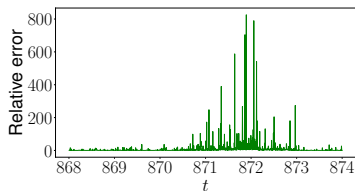
# Results - Periodic bursts

## Homoclinic burst

$$\nu_1 = \nu_2 = 0.45$$



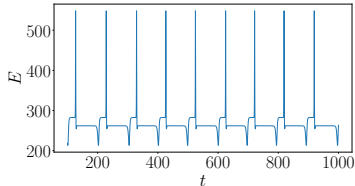
Energy evolution



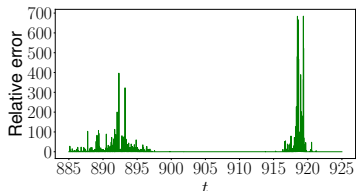
$$\text{Relative error} \max_{x,y \in [0,2\pi]} \frac{|u(t,x,y) - u(t-h,x,y)|}{|u(t,x,y)|}$$

## Heteroclinic burst

$$\nu_1 = \nu_2 = 0.4$$



Energy evolution

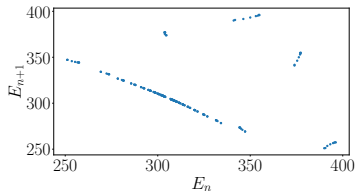
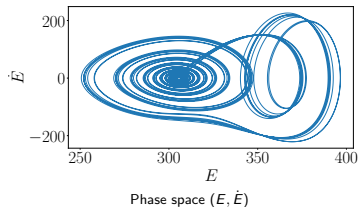


$$\text{Relative error} \max_{x,y \in [0,2\pi]} \frac{|u(t,x,y) - u(t-h,x,y)|}{|u(t,x,y)|}$$

# Results - Chaotic solutions

## Quasi-periodic

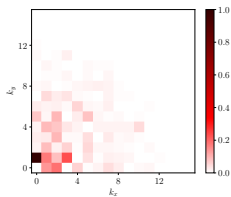
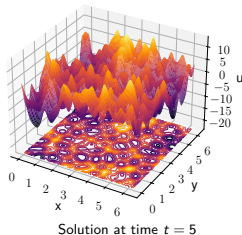
$$\nu_1 = 0.25, \nu_2 = 0.2$$



Return map with section  $\dot{E} = 0$

## Chaotic solution

$$\nu_1 = \nu_2 = 0.008$$



Fourier spectrum at time  $t = 5$



# Conclusions

- Periods of time-periodic waves are generally much smaller than periods of travelling waves, which, in turn, are smaller than periods of periodic bursts.
- A finer mesh grid is needed in a vicinity of bursts, specially for chaotic homoclinic and heteroclinic bursts.
- Quasi-periodic seem to be a transition between travelling waves or time-periodic waves and periodic bursts.
- For our purposes, the number of Fourier modes is relatively limited, but we cannot decrease the mesh size for small values of  $\nu_1$  and  $\nu_2$ .

$$\nu_1 = \left(\frac{2\pi}{L_x}\right)^2 \quad \nu_2 = \left(\frac{2\pi}{L_y}\right)^2$$

- [Akr+15] G. Akrivis et al. “Linearly implicit schemes for multi-dimensional Kuramoto-Sivashinsky type equations arising in falling film flows.” In: *IMA Journal of Numerical Analysis* 36.1 (Apr. 2015), pp. 317–336. ISSN: 0272-4979. DOI: [10.1093/imanum/drv011](https://doi.org/10.1093/imanum/drv011).
- [KKP15] A. Kalogirou, E. E. Keaveny, and D. T. Papageorgiou. “An in-depth numerical study of the two-dimensional Kuramoto-Sivashinsky equation.” In: *Proc. R. Soc. A* 471.20140932 (2015). DOI: [10.1098/rspa.2014.0932](https://doi.org/10.1098/rspa.2014.0932).