Numerical study of the 2D Kuramoto-Sivashinsky equation

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Abstract

The aim of this report is to give both qualitative and quantitative insight into the chaotic behavior of the 2D Kuramoto-Sivashinsky equation. This equation is more commonly known in its 1D version and this report wants to complement the numerical study carried out in [KKP15] in order to extend the bibliography on the 2D version of the equation. Kuramoto-Sivashinsky types of equations are seen in various physical phenomena such as flame propagation or reaction-diffusion systems [Kur78; Siv77]. We will see that the 2D KS equation exhibits chaotic behavior as we increase the spatial domain size.

1 Introduction

The well-known 1D Kuramoto-Sivashinsky (KS) equation can be written as

$$u_t + \frac{1}{2}{u_x}^2 + u_{xx} + u_{xxxx} = 0 \tag{1}$$

It is usually equipped with periodic boundary conditions $u(t, x + L) = u(t, x)$ for some $L > 0$, which defines the domain of definition of the PDE, and an initial condition $u(0, x) = u_0(x)$. The natural extension in the 2D case is the following Dirichlet problem with periodic boundary conditions:

$$\begin{cases} u_t + \frac{1}{2}|\boldsymbol{\nabla} u|^2 + \Delta u + \Delta^2 u = 0 & \text{in } (0,\infty) \times [0, L_x) \times [0, L_y) \\ u(t, x, y) = u(t, x + L_x, y) & \text{in } [0,\infty) \times \mathbb{R} \times [0, L_y) \\ u(t, x, y) = u(t, x, y + L_y) & \text{in } [0,\infty) \times [0, L_x) \times \mathbb{R} \\ u(0, x, y) = u_0(x, y) & \text{for all } x \in [0, L_x), y \in [0, L_y) \end{cases} \tag{2}$$

with $L_x, L_y > 0$. For the sake of simplicity, we will rescale the variables in order to obtain a square domain of definition, namely:

$$x_{\text{new}} = \frac{2\pi}{L_x} x \qquad y_{\text{new}} = \frac{2\pi}{L_y} y \qquad t_{\text{new}} = \left(\frac{L_x}{2\pi}\right)^2 t \tag{3}$$

Using this new variables (and dropping the subscript *new* for simplicity), the equation becomes:

$$\begin{cases} u_t + \frac{1}{2}|\boldsymbol{\nabla}_\nu u|^2 + \Delta_\nu u + {\Delta_\nu}^2 u = 0 & \text{in } (0,\infty) \times [0, 2\pi) \times [0, 2\pi) \\ u(t, x, y) = u(t, x + 2\pi, y) & \text{in } [0,\infty) \times \mathbb{R} \times [0, 2\pi) \\ u(t, x, y) = u(t, x, y + 2\pi) & \text{in } [0,\infty) \times [0, 2\pi) \times \mathbb{R} \\ u(0, x, y) = u_0(x, y) & \text{for all } x \in [0, 2\pi), y \in [0, 2\pi) \end{cases} \tag{4}$$

where we used the notation from [KKP15]:

$$\boldsymbol{\nabla}_\nu = \left(\partial_x, \sqrt{\frac{\nu_2}{\nu_1}}\partial_y\right) \qquad \mathbf{div}_\nu = \partial_x + \sqrt{\frac{\nu_1}{\nu_2}}\partial_y \tag{5}$$

$$\Delta_\nu = \mathbf{div}_\nu(\boldsymbol{\nabla}_\nu) = \partial_{xx} + \frac{\nu_2}{\nu_1}\partial_{yy} \qquad {\Delta_\nu}^2 = \Delta_\nu(\Delta_\nu) = {\partial_x}^4 + 2\frac{\nu_2}{\nu_1}{\partial_x}^2{\partial_y}^2 + \frac{{\nu_2}^2}{{\nu_1}^2}{\partial_y}^4 \tag{6}$$

and $\nu_1 := \left(\frac{L_x}{2\pi}\right)^2$, $\nu_2 := \left(\frac{L_y}{2\pi}\right)^2$. Note that the new equation is invariant under the transformation $(t, x, y, \nu_1, \nu_2) \mapsto \left(\frac{\nu_2}{\nu_1}t, y, x, \nu_2, \nu_1\right)$ if and only if the initial condition is symmetric in x and y. In that case, if $u(t, x, y)$ is a solution of the equation with parameters (ν_1, ν_2), then $u\left(\frac{\nu_2}{\nu_1}t, y, x\right)$ is the solution of the equation for the parameters (ν_2, ν_1).

Let's study now the linear stability of the different modes (k_x, k_y) of the equation for $k_x, k_y \in \mathbb{N} \cup \{0\}$. Setting $v = \delta(e^{\lambda t + i(k_x x + k_y y)} + \text{c.c.})$, with $\delta \ll 1$, as a perturbation of the trivial state $u = 0$, we obtain the following equality once we impose that v is a solution of Eq. (4):

$$\lambda = \left({k_x}^2 + \frac{\nu_2}{\nu_1}{k_y}^2\right)\left(1 - \nu_1{k_x}^2 - \nu_2{k_y}^2\right) \tag{7}$$

where, as usual, c.c. denotes the complex conjugate. We see that, for example, if $\nu_1, \nu_2 \geq 1$, then there is no pair (k_x, k_y) that makes $\lambda > 0$ and therefore all the nodes are stable. But as soon as we decrease ν_1 or ν_2 below 1, unstable nodes start to appear in an *increasing*[1] order. For example, for $\nu_1 = \nu_2 = 1/6$ the nodes $(0, 1)$, $(1, 0)$, $(1, 1)$, $(2, 0)$, $(0, 2)$, $(2, 1)$, $(1, 2)$ are unstable and all the others are stable.

In order to contribute to the bibliography on the 2D KS equation, we will study the equation with an initial condition different from the one used in [KKP15], which was $u_0(x, y) = \sin(x) + \sin(y) + \sin(x + y)$. Instead, we will use the following initial condition:

$$u_0(x, y) = \sin(x) + \sin(y) + \cos(x + y) + \sin(4x + 4y) + \cos(7x) + \cos(7y) \tag{8}$$

which is still symmetric in x and y. Note that we are adding the modes $(2, 0)$, $(0, 2)$, $(2, 2)$ to the initial condition used in [KKP15] and so a richer behavior is expected.

In order to distinguish and classify the different kinds of behavior that the equation exhibits, we will monitor the L^2-norm of the solution:

$$E_u(t) := \|u(t)\|_{L^2}^2 = \int_0^{2\pi}\int_0^{2\pi} u(t, x, y)^2 \,\mathrm{d}x\,\mathrm{d}y \tag{9}$$

It will be of interest to study also its time derivative $\dot{E}_u(t)$ and the phase space $(E_u(t), \dot{E}_u(t))$.

Finally, we can easily note that the mean of the solution is decreasing in time. Indeed:

$$\begin{aligned} 4\pi^2\frac{\mathrm{d}\overline{u}}{\mathrm{d}t} = \int_0^{2\pi}\int_0^{2\pi} u_t \,\mathrm{d}x\,\mathrm{d}y = -\int_0^{2\pi}\int_0^{2\pi}\left(\frac{1}{2}|\boldsymbol{\nabla}_\nu u|^2 + \Delta_\nu u + {\Delta_\nu}^2 u\right)\mathrm{d}x\,\mathrm{d}y = \\ = -\frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}\left({u_x}^2 + \frac{\nu_2}{\nu_1}{u_y}^2\right)\mathrm{d}x\,\mathrm{d}y \leq 0 \end{aligned} \tag{10}$$

where we used the fact that the solution is periodic in x and y. If we forget about the trivial state $u = 0$, this later inequality is strict and so the mean of the solution is strictly decreasing in time. In order to avoid this, we will subtract the mean of the solution at each step of integration, or equivalently, we will solve the equation

$$u_t + \frac{1}{2}\left[|\boldsymbol{\nabla}_\nu u|^2 - \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}|\boldsymbol{\nabla}_\nu u|^2 \,\mathrm{d}x\,\mathrm{d}y\right] + \Delta_\nu u + {\Delta_\nu}^2 u = 0 \tag{11}$$

with the same initial condition as before, because we have chosen an initial condition with zero mean. The reader may notice that here we have used the same notation to denote the initial solution and $u - \overline{u}$.

[1] Increasing in the sense the node $(k_x + 1, k_y)$ will become unstable once the node (k_x, k_y) had become unstable and not before.

2 Numerical methods

There are several numerical methods to integrate this kind of nonlinear equations. In this report we will use a pseudo-spectral method. The idea is to divide the spatial grid $[0, 2\pi] \times [0, 2\pi]$ in $N_x \times N_y$ cells and to approximate the solution $u(t, x, y)$ by a truncated Fourier series in each cell:

$$\tilde{u}(t, x_i, y_j) = \sum_{k_x=-N_x/2}^{N_x/2-1} \sum_{k_y=-N_y/2}^{N_y/2-1} \hat{u}(t, k_x, k_y) e^{i(k_x x_i + k_y y_j)} \tag{12}$$

for $i = 0, \ldots, N_x - 1$ and $j = 0, \ldots, N_y - 1$. The coefficients $\hat{u}(t, k_x, k_y)$ are the discrete Fourier coefficients of the solution and they are given by:

$$\hat{u}(t, k_x, k_y) = \frac{1}{N_x N_y} \sum_{i=0}^{N_x-1} \sum_{j=0}^{N_y-1} u(t, x_i, y_j) e^{-i(k_x x_i + k_y y_j)} \tag{13}$$

To efficiently compute the discrete Fourier transform, we will use the Fast Fourier Transform (FFT) algorithm, which reduces the complexity of the computation from $\mathcal{O}({N_x}^2 {N_y}^2)$ to $\mathcal{O}(N_x N_y \log(N_x N_y))$ and attains its maximum performance when N_x and N_y are powers of 2.

To integrate the PDE in time, we discretize the time domain $[0, T]$ in N_t points separated by a time step h. We will use a family of schemes that treat the linear part implicitly and the nonlinear part explicitly, the so called IMEX schemes [Akr+15]. The general theory of IMEX schemes was introduced in [AC04], in which they consider schemes of the form:

$$u_t + \mathcal{L}u = \mathcal{N}(u) \tag{14}$$

where $\mathcal{L}$ is a linear operator and $\mathcal{N}$ is a nonlinear operator. In order for the scheme to be stable and convergent, we need to check the conditions given in [AC04]. These require $\mathcal{L}$ being a self-adjoint and positive definite operator and $\mathcal{N}$ being a locally Lipschitz continuous operator. In our case, $\mathcal{L} = \Delta_\nu + {\Delta_\nu}^2$ but in this case, the operator is not positive definite (see [Kal14] for explicit details). However, adding a large enough constant c to both sides of the equation solves the problem. We will not reproduce the proof of $\mathcal{N}u = -\frac{1}{2}|\boldsymbol{\nabla}_\nu u|^2 + cu$ being locally Lipschitz continuous, as it is not the main interest of this report, and it can be found in [Akr+15]. The constant c that makes the scheme stable and convergent is $c = 1 + \frac{1}{\nu_1}$ [Kal14].

That being set, we will use the 2nd order backward differentiation formula (BDF2) as the scheme to integrate our equation in the Fourier space. We have chosen BDF2 schemes because backward differentiation of order 1 (implicit Euler) produced inconsistencies for small values of ν_1 and ν_2, and higher orders of backward differentiation require much more time to compute. In the Fourier space, **??** becomes, using matrix notation:

$$\tilde{\mathbf{u}}_t + \mathbf{L}\tilde{\mathbf{u}} = \mathbf{N}(\tilde{\mathbf{u}}) \tag{15}$$

The IMEX-BDF2 scheme for Eq. (14) is given by:

$$\frac{3}{2}\tilde{\mathbf{u}}^{n+2} + h\mathbf{L}\tilde{\mathbf{u}}^{n+2} = 2\tilde{\mathbf{u}}^{n+1} - \frac{1}{2}\tilde{\mathbf{u}}^{n} + 2h\mathbf{N}(\tilde{\mathbf{u}}^{n+1}) - h\mathbf{N}(\tilde{\mathbf{u}}^{n}) \tag{16}$$

from which we can iteratively solve using the initial condition as a first step and a BDF1 scheme as a second step:

$$\tilde{\mathbf{u}}^{n+1} + h\mathbf{L}\tilde{\mathbf{u}}^{n+1} = \tilde{\mathbf{u}}^{n} + h\mathbf{N}(\tilde{\mathbf{u}}^{n}) \tag{17}$$

Recall that in practice Eq. (13) has to be approximated with $\tilde{u}(t_k, x_i, y_j)$ instead of using the unknown quantity $u(t_k, x_i, y_j)$, where t_k is the k-th time step.

All the code used in this report, as well as some animations of the presented results, can be found in this GitHub repository.

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3 Results

4 Conclusions

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