Numerical Methods for Fluid Dynamics

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CFD

```
lv=np.size(vec) # lv = n+1;

D1=np.zeros([lv,lv])
D2=np.zeros([lv,lv])
D3=np.zeros([lv,lv])
D4=np.zeros([lv,lv])
D1[:,1]=D0[:,0]
D1[:,2]=4*D0[:,1]
D2[:,2]=4*D0[:,0]
for i in range(n-2): # We compute Tj, which is in column j+1
  j=i+2
  jj=j+1
  D1[:,j+1]=2*jj*D0[:,j]+jj*D1[:,j-1]/(jj-2);
  D2[:,j+1]=2*jj*D1[:,j]+jj*D2[:,j-1]/(jj-2);
  D3[:,j+1]=2*jj*D2[:,j]+jj*D3[:,j-1]/(jj-2);
  D4[:,j+1]=2*jj*D3[:,j]+jj*D4[:,j-1]/(jj-2);
return D0,D1,D2,D4
```

```
au=np.linalg.solve(D0,u)
du_Cheb=D1.dot(au)
err_Cheb[k]=np.max(abs(du-du_Cheb))
```

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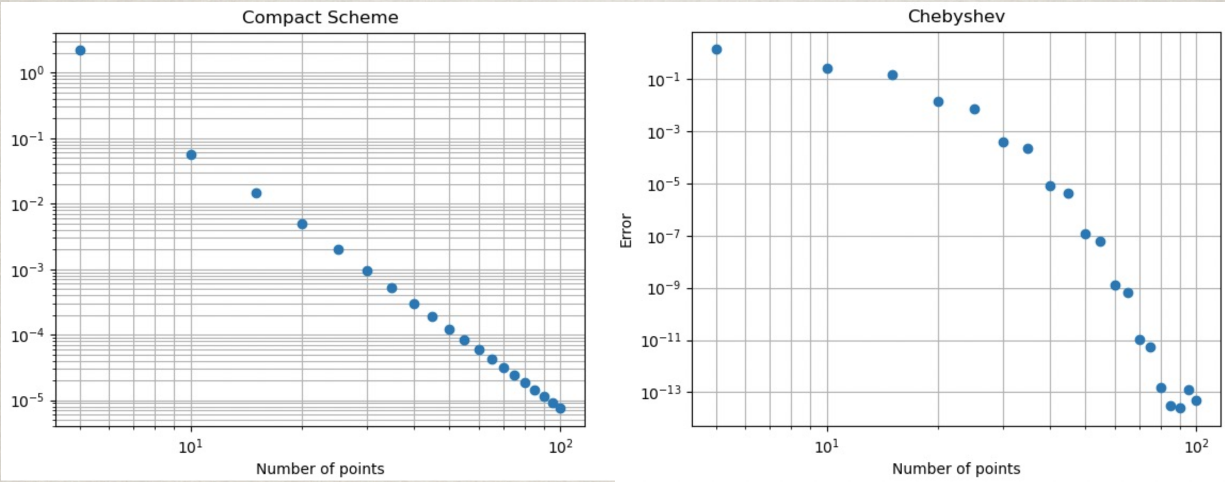

CFD

```
for i in range(NT):
    # Laplacian(psi) = -w
    psi_hat = -w_hat/k2_fixed

    # Velocity from streamfunction
    u = spec_phys(ky*psi_hat,NX,NY,NXSp,NYSp)
    v = spec_phys(-kx*psi_hat,NX,NY,NXSp,NYSp)

    # Gradients of vorticity
    w_x = spec_phys(kx*w_hat,NX,NY,NXSp,NYSp)
    w_y = spec_phys(ky*w_hat,NX,NY,NXSp,NYSp)

    # Non-linear term
    NL     = u*w_x + v*w_y
    NL_hat = phys_spec(NL,NX,NY,NXSp,NYSp)

    # Time stepping (Crank-Nicholson)
    w_hat = (1/dt+k2/(2*Re))*w_hat
    w_hat = w_hat-NL_hat

    w_hat = w_hat/(1/dt-k2/(2*Re))
```

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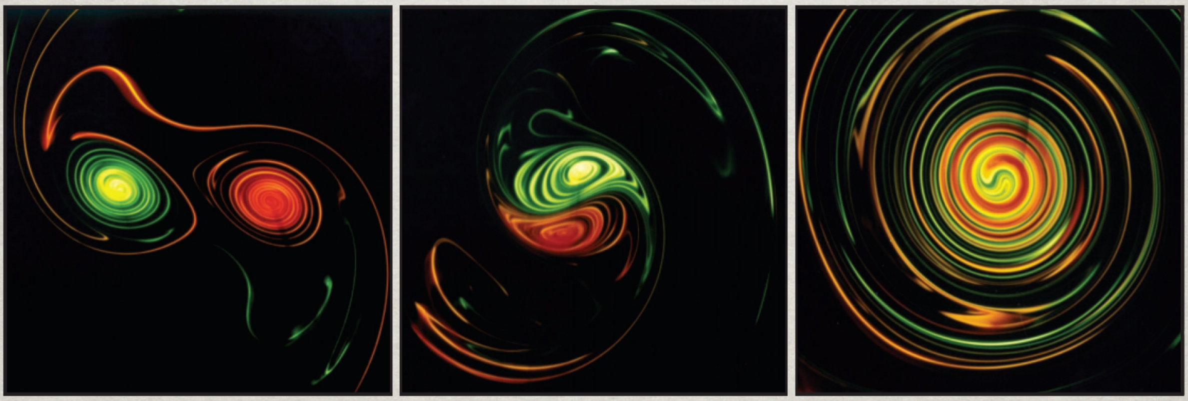
Vortices interactions



Leweke et al (2016)

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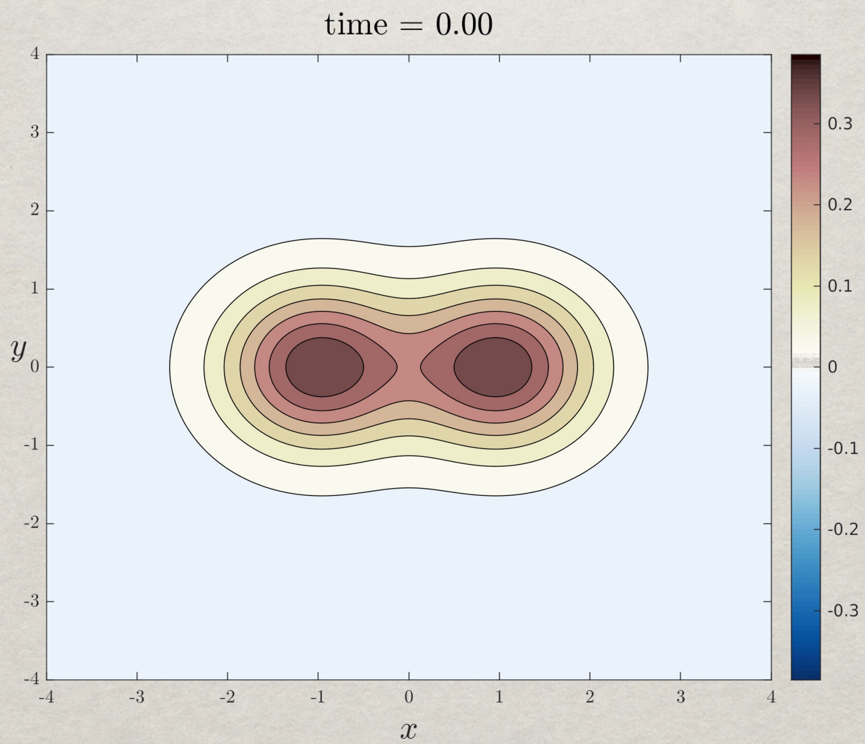

Hydrodynamic turbulence

Leonardo da Vinci
c1510

Hydrodynamic turbulence

Kolmogorov K41

$$\left[\hat{E}(k)\right] = L^3\, T^{-2}\,.$$

$$[k] = L^{-1} \quad \text{et} \quad [\varepsilon] = L^2 T^{-3}\,.$$

$$\hat{E}(k) \propto \varepsilon^{2/3}\, k^{-5/3}\,.$$

Hydrodynamic turbulence

Kolmogorov K41

$$\ell_{\text{diss}} \sim \left(\nu^3/\varepsilon\right)^{1/4},$$

$$\ell_{\text{diss}} \sim \mathrm{Re}^{-3/4}$$

$$N \sim \mathrm{Re}^{9/4}$$

Hydrodynamic turbulence

The Lorenz system

$$\dot{X} = \sigma(Y - X)$$

$$\dot{Y} = X\left(\rho - Z\right) - Y$$

$$\dot{Z} = \beta Z + XY$$

CFD

There are two different stability concepts:

• **Zero stability**: If $t > 0$ is held constant, do the values $u(t)$ remain bounded as $\Delta t \rightarrow 0$?
(i.e. vanishing time-step for fixed time)

• **Absolute stability**: If Δt is kept fixed, do the computed values $u(t)$ remain bounded as $t \rightarrow \infty$?
(i.e. finite time-step, infinite time)

What's the relevance of numerical simulations?

- The attractor is robust
- Shadowing lemma in dynamical systems

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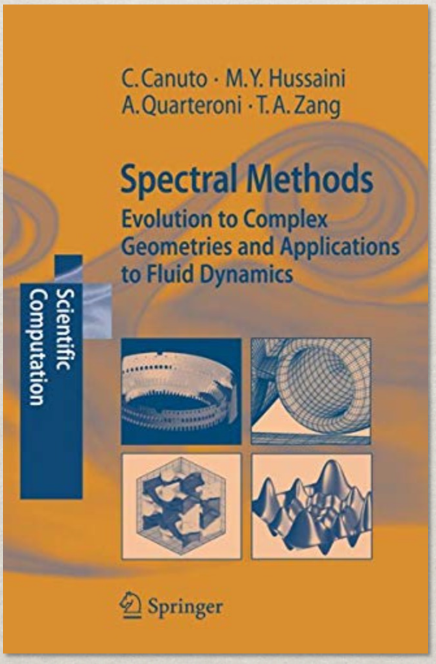
To explore further:



CFD

7. Compressibility and waves

CFD

Waves are a common feature in many fluid dynamic applications:

- **oceanography/geology**
- acoustics
- instabilities

CFD

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CFD

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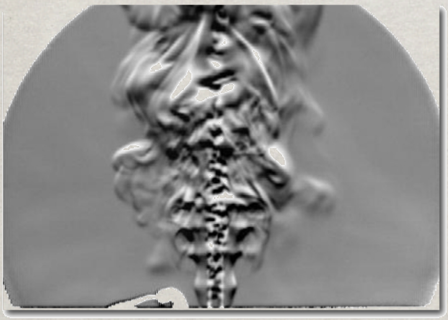

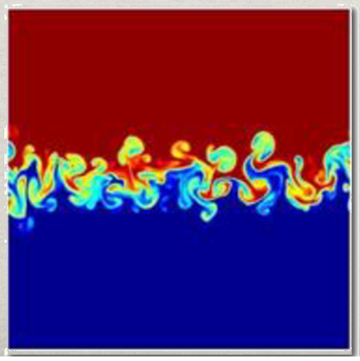

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7. Compressibility and waves

7.1 Staggering

CFD

Linear acoustics

Start with compressible Euler equations

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

$$\rho \frac{D\mathbf{u}}{Dt} = -\nabla p + \eta \left(\nabla^2 \mathbf{u} + \frac{1}{3} \nabla (\nabla \cdot \mathbf{u}) \right) + \mathbf{F}$$

Homentropic fluid $\quad p = p(\rho)$

CFD

Linear acoustics

Inertial effects are negligible; leads to linearized Euler equations

$$\frac{1}{\rho_0} \frac{\partial \rho'}{\partial t} + \nabla . \mathbf{u} = q$$

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} + \nabla p' = \mathbf{F}$$

Eliminate velocity field by cross-differentiation

CFD

Linear acoustics

$$\frac{\partial^2 \rho'}{\partial t^2} - \nabla^2 p' = \rho_0 \frac{\partial q}{\partial t} - \nabla \cdot \mathbf{F}$$

Density and pressure are related via the EOS

$$p_0 + p' = p(\rho_0 + \rho') = p(\rho_0) + \left(\frac{\partial p}{\partial \rho}\right)_0 \rho' + \cdots$$

$$c_0^2$$

CFD

Linear acoustics

Finally, we get an inhomogeneous wave equation for the pressure

$$\left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) p = \boxed{\rho_0 \frac{\partial q}{\partial t} - \nabla \cdot \mathbf{F}}$$

Source term

CFD

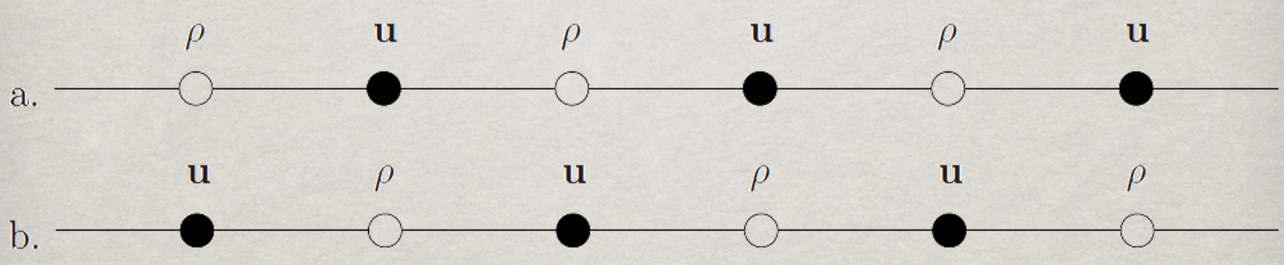
$$\frac{1}{\rho_0}\frac{\partial \rho'}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{u} = q$$

$$\rho_0\,\frac{\partial \mathbf{u}}{\partial t} + c_0^2\,\boldsymbol{\nabla}\rho' = \mathbf{F}$$

CFD

$$\frac{1}{\rho_0}\frac{\partial \rho'}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{u} = q$$

$$\rho_0\,\frac{\partial \mathbf{u}}{\partial t} + c_0^2\,\boldsymbol{\nabla}\rho' = \mathbf{F}$$



CFD

$$\frac{1}{\rho_0}\frac{\partial \rho'}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{u} = q$$

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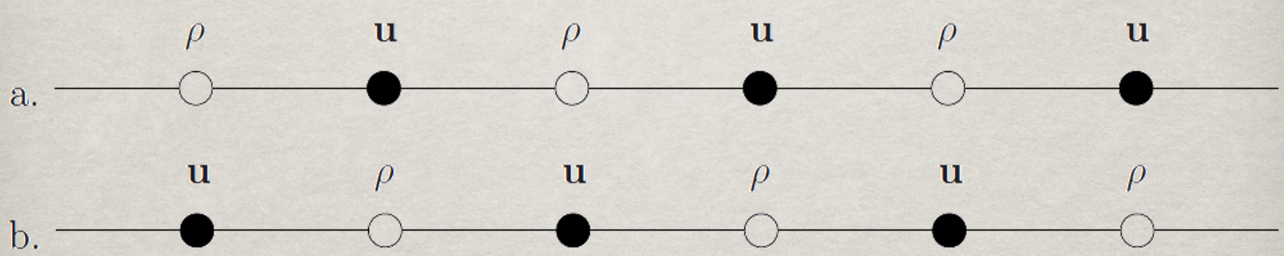

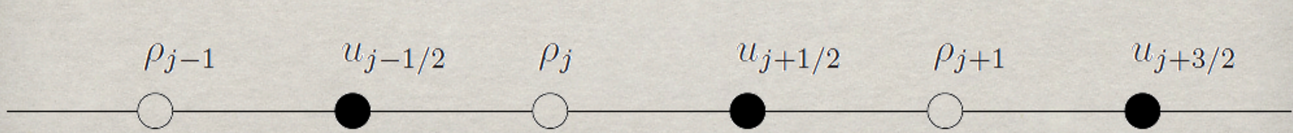

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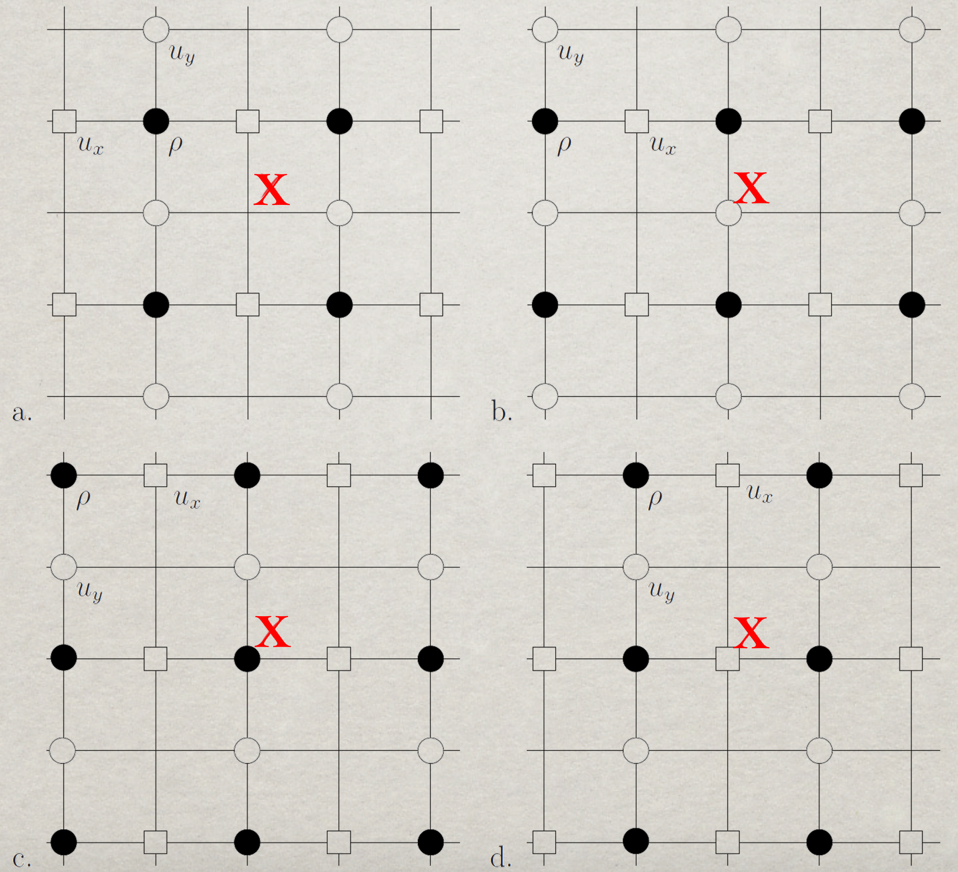

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$$\frac{1}{\rho_0}\frac{\partial \rho'}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{u} = q$$

$$\rho_0\frac{\partial \mathbf{u}}{\partial t} + c_0^2\,\boldsymbol{\nabla}\rho' = \mathbf{F}$$

$$\left(\frac{1}{c_0^2} - \Delta\right)p = \rho_0\frac{\partial q}{\partial t} - \boldsymbol{\nabla}\cdot\mathbf{F}$$

$$\frac{1}{c_0^2}\frac{p_j^{n+1} - 2p_j^n + p_j^{n-1}}{\Delta t^2} = \frac{p_{j+1}^n - 2p_j^n + p_{j-1}^n}{\Delta x^2}$$

CFD

$$\frac{1}{c_0^2}\frac{p_j^{n+1} - 2p_j^n + p_j^{n-1}}{\Delta t^2} = \frac{p_{j+1}^n - 2p_j^n + p_{j-1}^n}{\Delta x^2}$$

This scheme is second-order in time and second-order in space.

CFD

Von Neuman stability analysis

$$\frac{1}{c_0^2}\,\frac{p_j^{n+1}-2p_j^n+p_j^{n-1}}{\Delta t^2}=\frac{p_{j+1}^n-2p_j^n+p_{j-1}^n}{\Delta x^2},$$

$$p_j^{n+1}=2p_j^n-p_j^{n-1}-\frac{c_0^2\,\Delta t^2}{\Delta x^2}\left(p_{j+1}^n-2p_j^n+p_{j-1}^n\right)$$

CFD

Von Neuman stability analysis

$$\frac{\xi_j^{n+1}-2\xi_j^n+\xi_j^{n-1}}{\Delta t^2}=c^2\,\xi^n\,\frac{\mathrm{e}^{ik\Delta x}-2+\mathrm{e}^{-ik\Delta x}}{\Delta x^2}$$

$$\xi^2-2\left[2+\lambda^2(2\cos k\Delta x-2)\right]\xi+1=0$$

$$\lambda=c\Delta t/\Delta x$$

CFD

$$\xi^2 - 2\left[2 + \lambda^2(2\cos k\Delta x - 2)\right]\xi + 1 = 0$$

Both roots have to be inside the unit disk for stability.

CFL restriction $\lambda \leq 1$

$\lambda > 1 : \quad \pi/\Delta x$ waves become unstable first

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von Neumann stability analysis

In two dimensions, the CFL restriction extends to

$$\lambda \leq \frac{1}{\sqrt{2}}$$

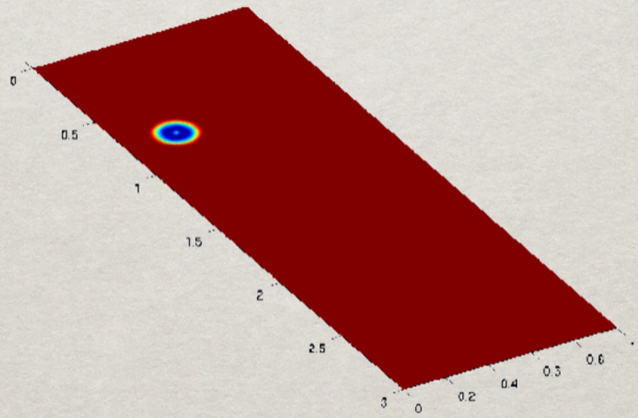
In three dimensions, the CFL restriction extends to

$$\lambda \leq \frac{1}{\sqrt{3}}$$

CFD

Application

Propagation of a pressure pulse with reflecting boundaries



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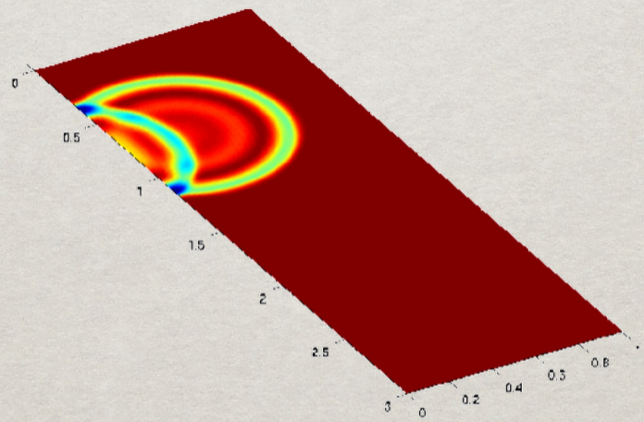
Application

Propagation of a pressure pulse with reflecting boundaries



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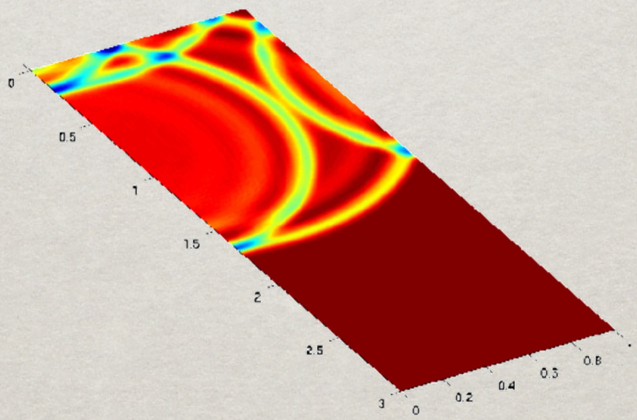
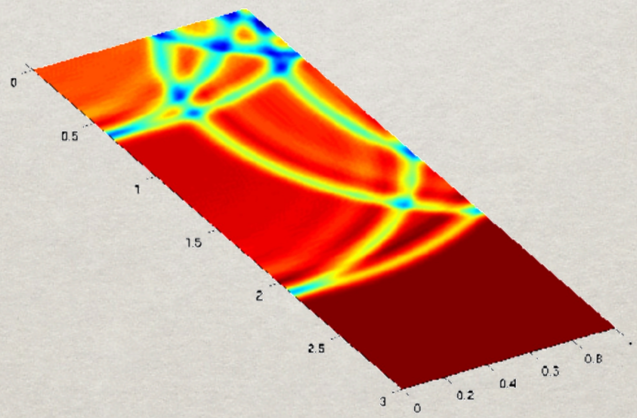
Propagation of a pressure pulse with reflecting boundaries



CFD

Application

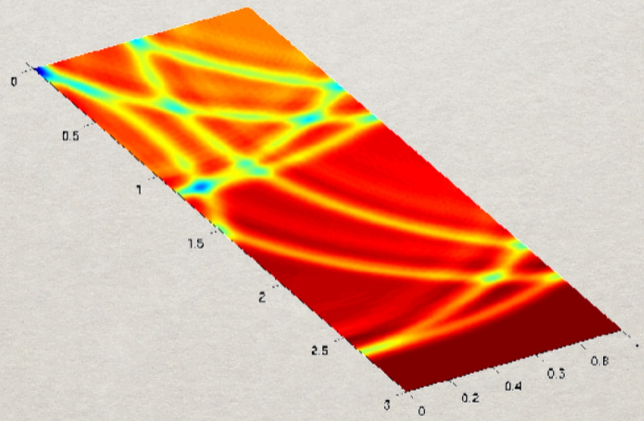
Propagation of a pressure pulse with reflecting boundaries



CFD

Application

Propagation of a pressure pulse with reflecting boundaries



CFD

Grid anisotropy

Different waves travel at different speeds on the grid, although continuous wave equation is nondispersive

$$c_p = \omega / k \qquad\qquad \xi = \mathrm{e}^{i\omega\Delta t}$$

$$c_p(k_x, k_y) = \left| \frac{\ln\left(\xi(k_x, k_y)/|\xi(k_x, k_y)|\right)}{i\sqrt{k_x^2 + k_y^2}\,\Delta t} \right|$$

CFD

With:

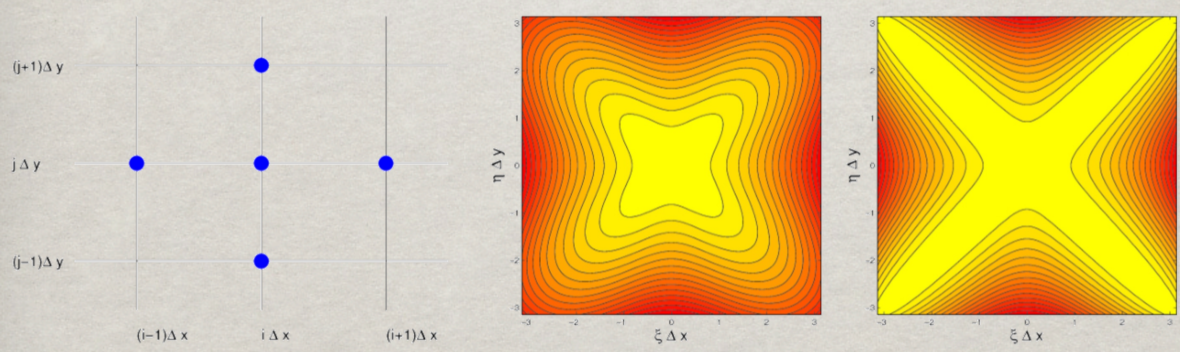
$$\xi^2(k_x, k_y) + B(k_x, k_y)\,\xi(k_x, k_y) + 1 = 0$$

$$B(k_x, k_y) = -2\left(1 + \lambda^2(\cos(k_x\Delta) + \cos(k_y\Delta) - 2)\right)$$

CFD

Grid anisotropy

$$B(k_x, k_y) = -2\left(1 + \lambda^2(\cos(k_x\Delta) + \cos(k_y\Delta) - 2)\right)$$



strongly anisotropic wave propagation

CFD

Grid anisotropy

$$
\begin{aligned}
p_{i,j}^{n+1} = -p_{i,j}^{n-1} &+ \lambda^2\, a\, \left(p_{i,j+1}^{n} + p_{i,j-1}^{n} + p_{i+1,j}^{n} + p_{i-1,j}^{n}\right) \\
&+ \lambda^2\, b\, \left(p_{i+1,j+1}^{n} + p_{i+1,j-1}^{n} + p_{i-1,j+1}^{n} + p_{i-1,j-1}^{n}\right) \\
&+ \lambda^2\, c\, p_{i,j}^{n}
\end{aligned}
$$

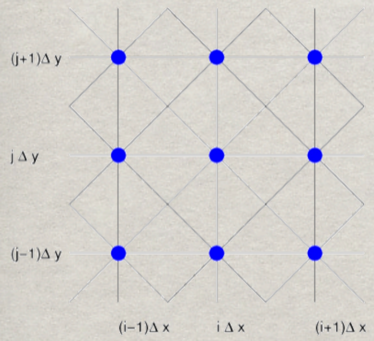
$$
\begin{aligned}
B(k_x, k_y) = -2\lambda^2\, (a(\cos(k_x\Delta) + \cos(k_y\Delta)) \\
+ (1-a)\cos(k_x\Delta)\cos(k_y\Delta) - 1 - a) - 2
\end{aligned}
$$

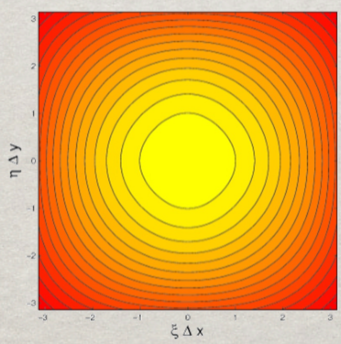
CFD

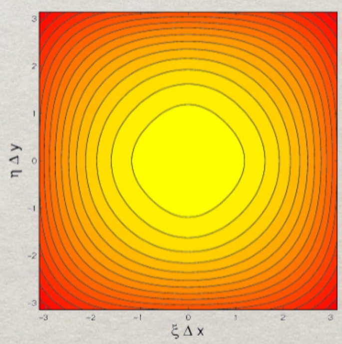
Grid anisotropy

$$
\begin{aligned}
B(k_x, k_y) = -2\lambda^2\, (a(\cos(k_x\Delta) + \cos(k_y\Delta)) \\
+ (1-a)\cos(k_x\Delta)\cos(k_y\Delta) - 1 - a) - 2
\end{aligned}
$$

$a = 0.62$







more isotropic wave propagation

7. Compressibility and waves

7.2 Water waves

CFD

$$\omega^2 = gk \tanh kH \,.$$

Shallow water

$$\omega^2 = gHk^2 \,, \qquad c = \omega/k = \pm\sqrt{gH} \,.$$

Deep water

$$\omega = \pm\sqrt{gk} \,, \qquad c = \omega/k = \pm\sqrt{g/k} \,.$$

CFD

$$\rho \frac{\mathrm{D}u}{\mathrm{D}t} = \rho \frac{\partial u}{\partial t} + \rho\, u \frac{\partial u}{\partial x} + \rho\, w \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x},$$

$$\rho \frac{\mathrm{D}w}{\mathrm{D}t} = \rho \frac{\partial w}{\partial t} + \rho\, u \frac{\partial w}{\partial x} + \rho\, w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} - \rho\, g\,.$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0\,.$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = w\,|_{z=h} \quad \text{and} \quad p\,|_{z=h} = p_0\,.$$

CFD

Hydrostatic basic state

$$u = w = 0\,, \quad h = H\,, \qquad p = p_0 + \rho\, g\,(H - z)\,,$$

$$\rho \frac{\mathrm{D}u}{\mathrm{D}t} = \rho \frac{\partial u}{\partial t} + \rho\, u \frac{\partial u}{\partial x} + \rho\, w \frac{\partial u}{\partial z} = -\frac{\partial p}{\partial x},$$

$$\rho \frac{\mathrm{D}w}{\mathrm{D}t} = \rho \frac{\partial w}{\partial t} + \rho\, u \frac{\partial w}{\partial x} + \rho\, w \frac{\partial w}{\partial z} = -\frac{\partial p}{\partial z} - \rho\, g\,.$$

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0\,.$$

CFD

Hydrostatic basic state

$$u = w = 0\,, \quad h = H\,, \qquad p = p_0 + \rho\, g\,(H - z)\,,$$

Perturbation

$$\rho\,\frac{\partial u'}{\partial t} = -\frac{\partial p'}{\partial x}\,, \qquad \rho\,\frac{\partial w'}{\partial t} = -\frac{\partial p'}{\partial z}\,,$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0\,.$$

CFD

Hydrostatic basic state

$$u = w = 0\,, \quad h = H\,, \qquad p = p_0 + \rho\, g\,(H - z)\,,$$

$$\rho\,\frac{\partial u'}{\partial t} = -\frac{\partial p'}{\partial x}\,, \qquad \rho\,\frac{\partial w'}{\partial t} = -\frac{\partial p'}{\partial z}\,,$$

$$\frac{\partial u'}{\partial x} + \frac{\partial w'}{\partial z} = 0\,.$$

$$\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial z^2} = 0\,.$$

CFD

$$p'(x,z,t) = \hat{p}(z,t)\,\mathrm{e}^{ikx} + c.c.\,,$$

$$\frac{\partial^2 \hat{p}}{\partial z^2} - k^2\,\hat{p} = 0\,,$$

$$\hat{p}(z,t) = p_1(t)\,\mathrm{e}^{kz} + p_2(t)\,\mathrm{e}^{-kz}\,.$$

CFD

$$\hat{p}(z,t) = p_1(t)\,\mathrm{e}^{kz} + p_2(t)\,\mathrm{e}^{-kz}\,.$$

$$\frac{\partial \hat{p}}{\partial z}(0,t) = 0\,.$$

$$P = p_1/2 = p_2/2$$

CFD

$$p'(x,z,t) = P(t)\ \cosh kz\, \mathrm{e}^{ikx} + c.c.\,.$$

$$u'(x,z,t) = -i(k/\rho)\, Q(t)\ \cosh kz\, \mathrm{e}^{ikx} + c.c.\,,$$

$$w'(x,z,t) = -(k/\rho)\, Q(t)\ \sinh kz\, \mathrm{e}^{ikx} + c.c.\,,$$

$$\mathrm{d}Q/\mathrm{dt} = P \qquad\qquad p' = \rho g h'$$

$$h'(x,t) = \frac{1}{g\rho} P(t) \cosh kH\, \mathrm{e}^{ikx} + c.c.\,,$$

$$\frac{\partial p'}{\partial t}(x,H,t) = g\rho w'(x,H,t)\,.$$

CFD

$$\frac{\mathrm{d}P}{\mathrm{dt}}\ \cosh kH = -gk\, Q(t)\ \sinh kH\,,$$

$$\frac{\mathrm{d}^2 Q}{\mathrm{d}t^2} + gk\, Q(t)\ \tanh kH = 0\,.$$

$$\omega^2 = gk \tanh kH\,.$$

CFD

$$\omega^2 = gk \tanh kH \,.$$

Shallow water

$$\omega^2 = gHk^2 \,, \qquad c = \omega/k = \pm\sqrt{gH}\,.$$

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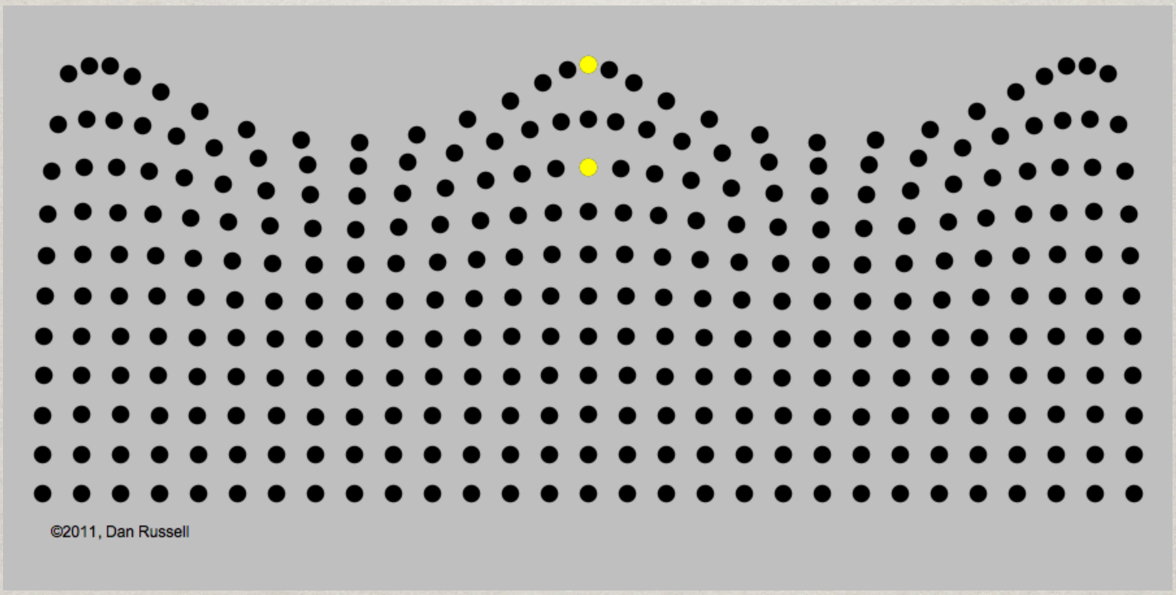
CFD



CFD

CFD

7. Compressibility and waves

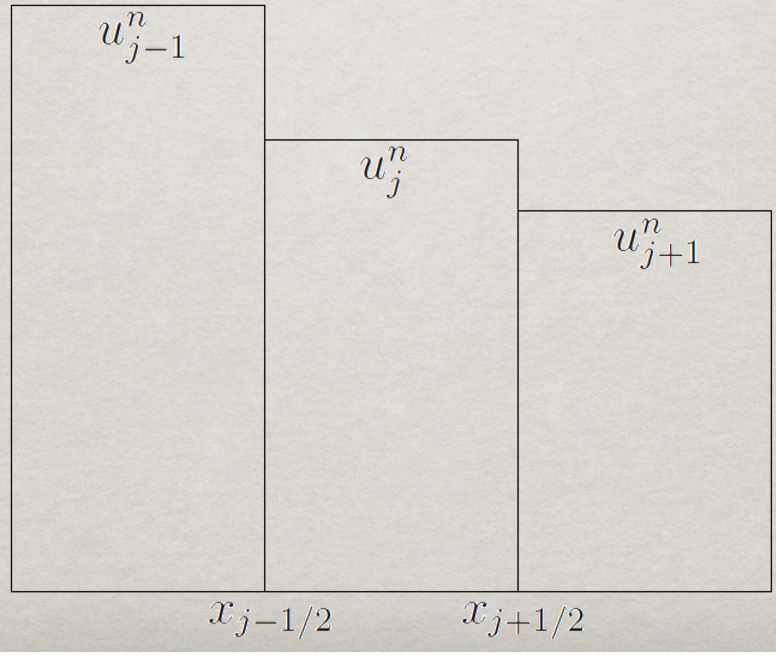
7.3 Conservation laws and Finite Volumes

Finite Volumes:

$$u_j^n = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} u(x,t)\mathrm{d}x$$

CFD

Finite Volumes:



CFD

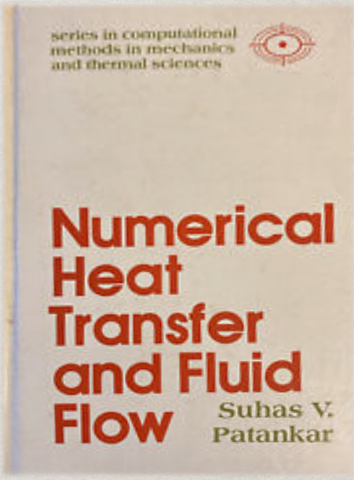
Finite Volumes:

$$u_j^{n+1} - u_j^n = -\frac{\Delta t}{\Delta x}\left(f_{j+1/2}^{n+1/2} - f_{j-1/2}^{n+1/2}\right)$$

$$f_{j+1/2}^{n+1/2} = \frac{1}{\Delta t}\int_{t^n}^{t^{n+1}} f(u(x_{j+1/2}, t)\mathrm{d}t$$

CFD

Suhas V. Patankar
(1941)



CFD

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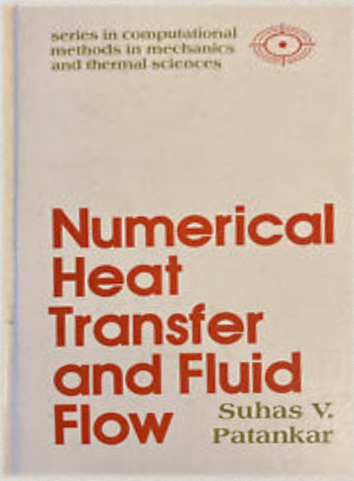


"much like preparing mashed potatoes from dehydrated potato powder. After all, textbook derivations of differential equations always start from the conservation principle applied to a small control volume."

CFD

Upwind:

$$f_{j+1/2}^{up} = \begin{cases} c\,u_j & if \quad c > 0 \\ c\,u_{j+1} & if \quad c < 0 \end{cases}$$

Lax-Wendroff:

$$f_{j+1/2}^{LW} = \frac{1}{2}c\left(u_j + u_{j+1}\right) - \frac{\Delta t}{2\Delta x}c^2\left(u_{j+1} - u_j\right)$$

CFD

Finite Volumes in 2D:

$$\psi_{i,j}^{n+1} = \psi_{i,j}^{n} - \frac{\Delta t}{\Delta x}\left(F_{i+1/2,j} - F_{i-1/2,j}\right) - \frac{\Delta t}{\Delta y}\left(G_{i,j+1/2} - G_{i,j-1/2}\right)$$

CFD

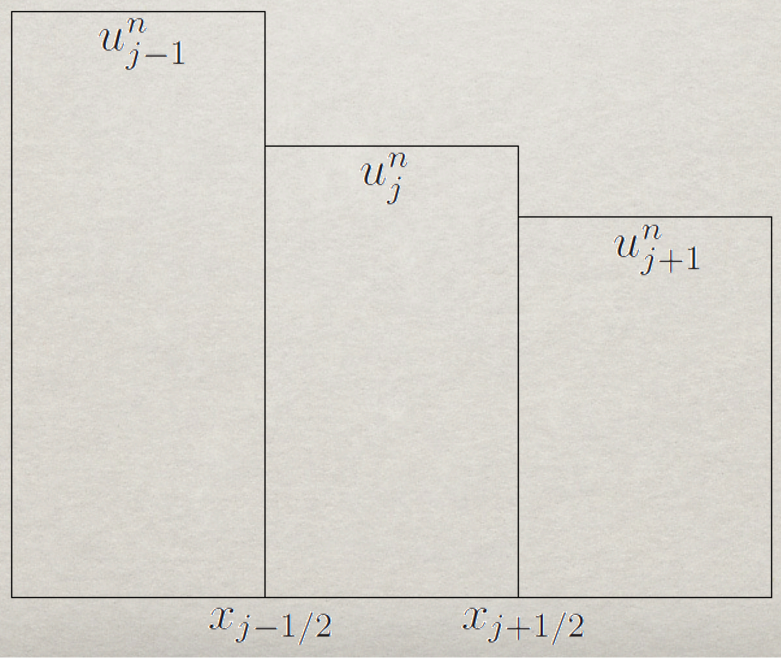
Non-linear conservation laws

$$\frac{\partial u}{\partial t} = -\boldsymbol{\nabla} \cdot \mathbf{F}(u)$$

$$\frac{\partial u}{\partial t} = -\frac{\partial f(u)}{\partial x}$$

CFD

$$f_{j+1/2}^{n+1/2} = \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} f(u(x_{j+1/2}, t)\mathrm{d}t$$

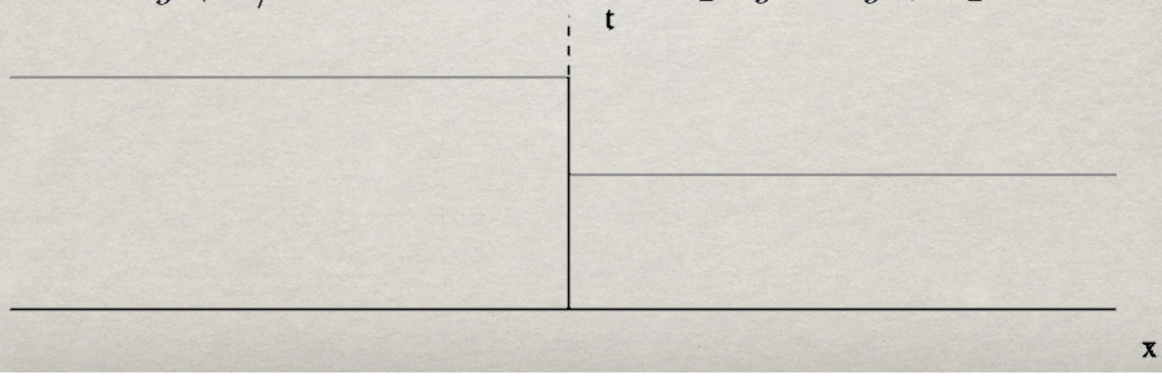


CFD

Riemann problem:

$$u(x,0) = \begin{cases} u_L & \text{if } x < 0 \\ u_R & \text{if } x > 0 \end{cases}$$

$$u^\star_{j+1/2}(x/t) = RP\left[u^n_j, u^n_{j+1}\right]$$



CFD

Riemann problem:

$$u(x,0) = \begin{cases} u_L & \text{if } x < 0 \\ u_R & \text{if } x > 0 \end{cases}$$

$$u^\star_{j+1/2}(x/t) = RP\left[u^n_j, u^n_{j+1}\right]$$

$$f^{n+1/2}_{j+1/2} = f(u^\star_{j+1/2}(0))$$

CFD

The Burgers equation

$$\frac{\partial u}{\partial t} = -\frac{\partial f(u)}{\partial x}$$

$$f(u) = \frac{u^2}{2}$$

CFD

The Burgers equation

$$f(u) = \frac{u^2}{2}$$

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)$$

CFD

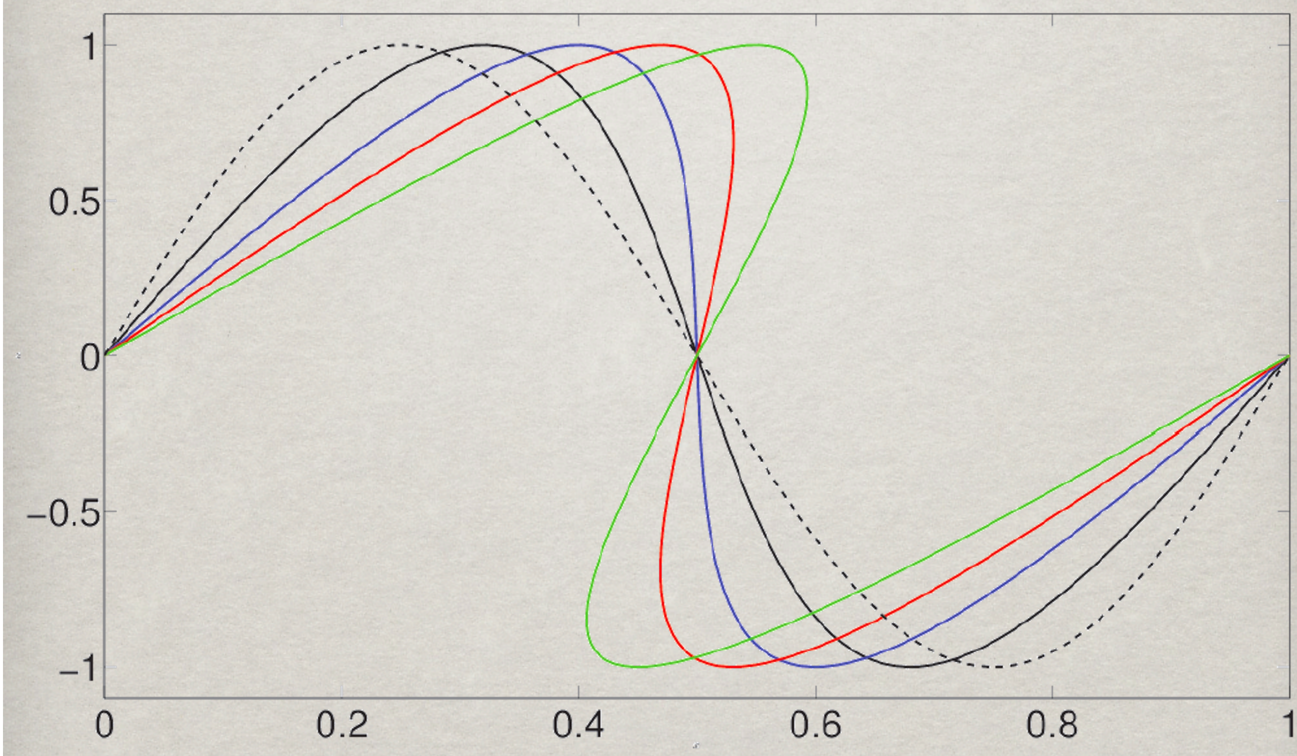
The Burgers equation

$$f(u) = \frac{u^2}{2}$$

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)$$

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x}$$

CFD



CFD

The Burgers equation

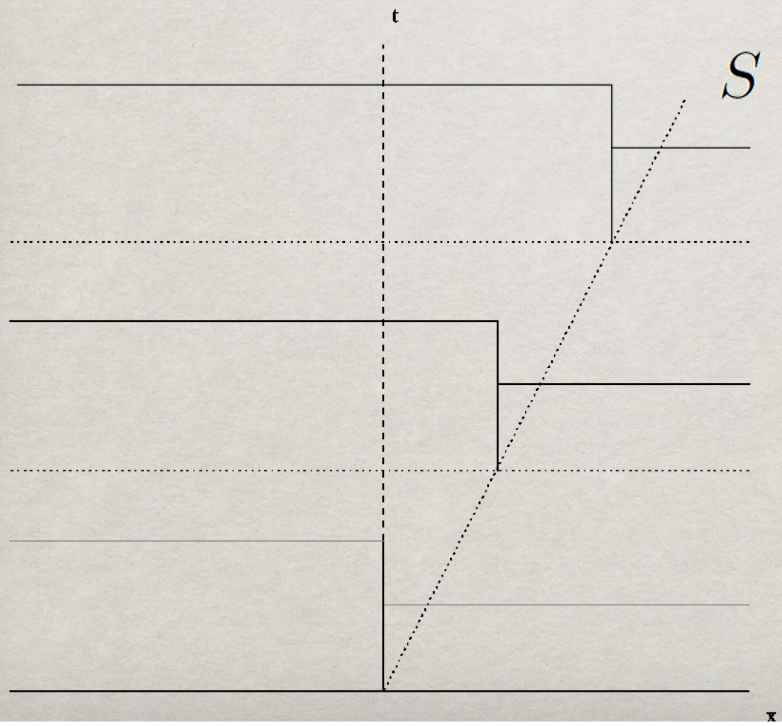
$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)$$

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) + \varepsilon \frac{\partial^2 u}{\partial x^2}$$

KdV: $$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) - \alpha \frac{\partial^3 u}{\partial x^3}$$

CFD

$u_L > u_R$ **Shock**



$$S = \frac{u_L + u_R}{2}$$

CFD

$u_L > u_R$ **Shock**

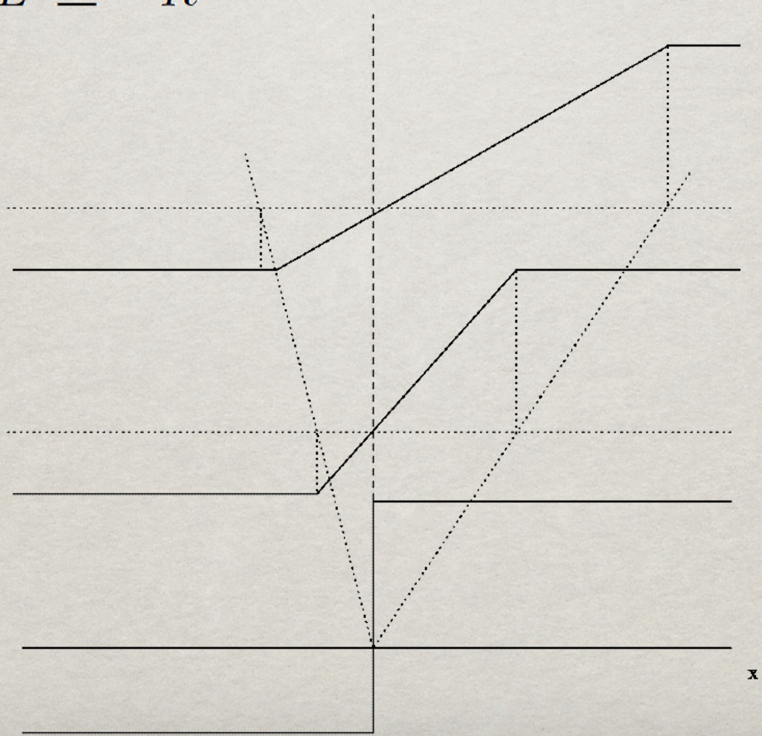
$$S = \frac{u_L + u_R}{2}$$

$$u(\frac{x}{t}) = \begin{cases} u_L & \text{if} \quad S \geq x/t \\ u_R & \text{if} \quad S < x/t \end{cases}$$

$$\begin{cases} u_L & \text{if} \quad S > 0 \\ u_R & \text{if} \quad S < 0 \end{cases}$$

CFD

$u_L \leq u_R$ **Rarefaction**



CFD

$u_L \leq u_R$ **Rarefaction**

$$u(\frac{x}{t}) = \begin{cases} u_L & \text{if} & & & x/t & \leq & u_L \\ x/t & \text{if} & u_L & < & x/t & < & u_R \\ u_R & \text{if} & u_R & \leq & x/t & & \end{cases}$$

$$\begin{cases} u_L & \text{if} & & & 0 & \leq & u_L \\ 0 & \text{if} & u_L & < & 0 & < & u_R \\ u_R & \text{if} & u_R & \leq & 0 & & \end{cases}$$

CFD

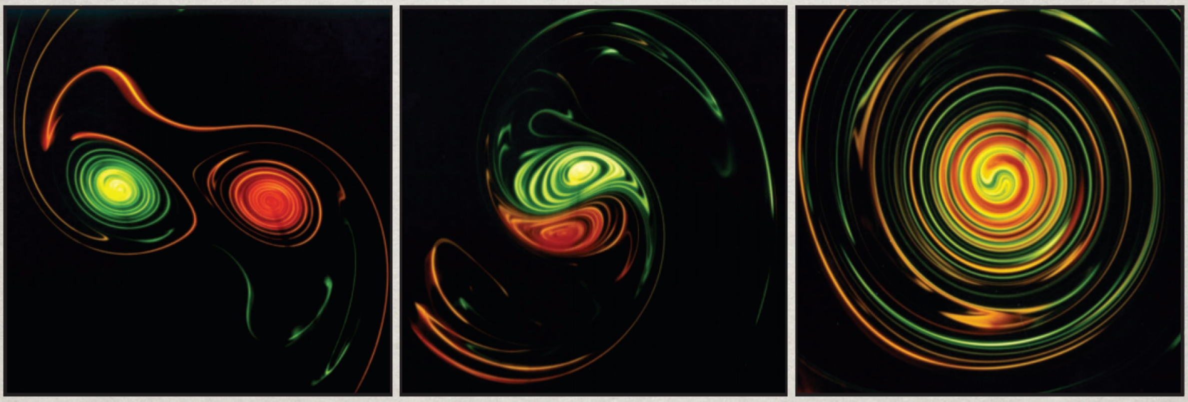
Final Projects

CFD

Kelvin wake

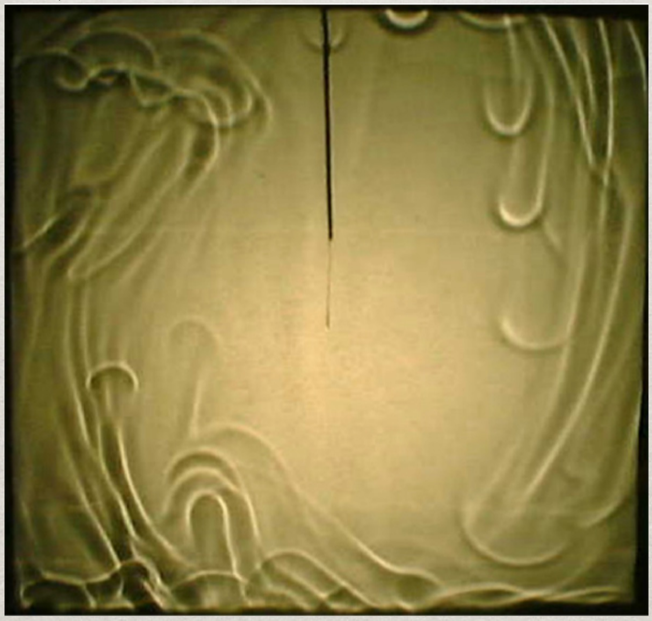
CFD

Final Projects



Vortices interactions

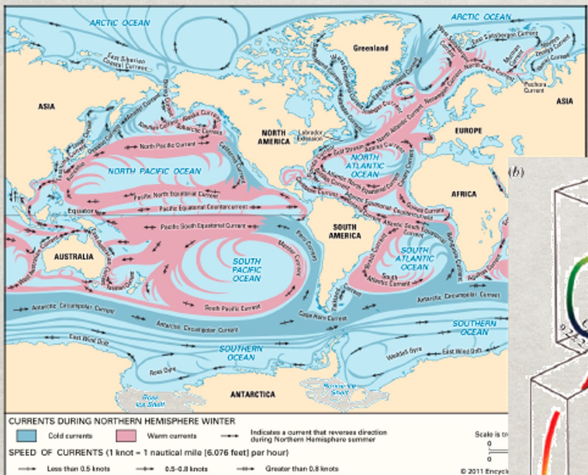
Final Projects

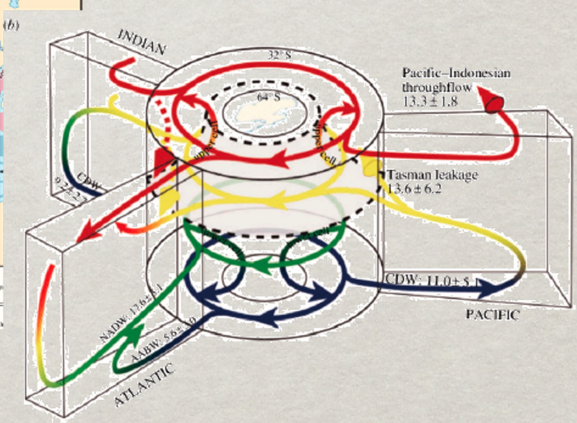


Thermal convection

CFD

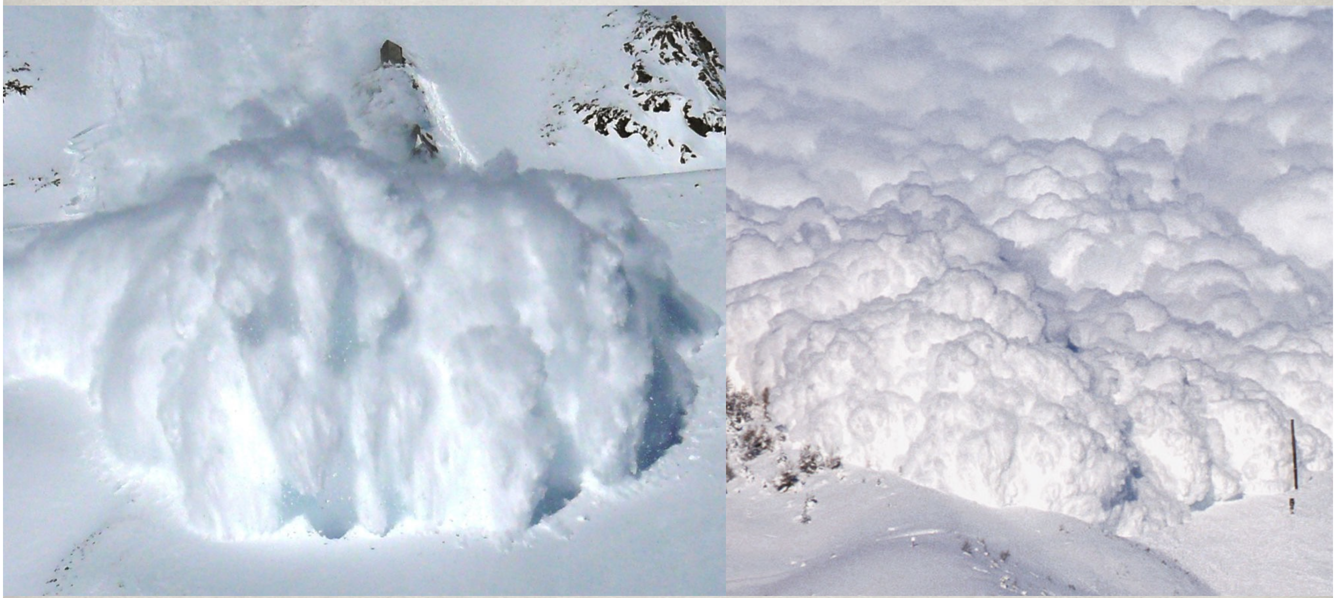
Final Projects





Thermo-haline convection

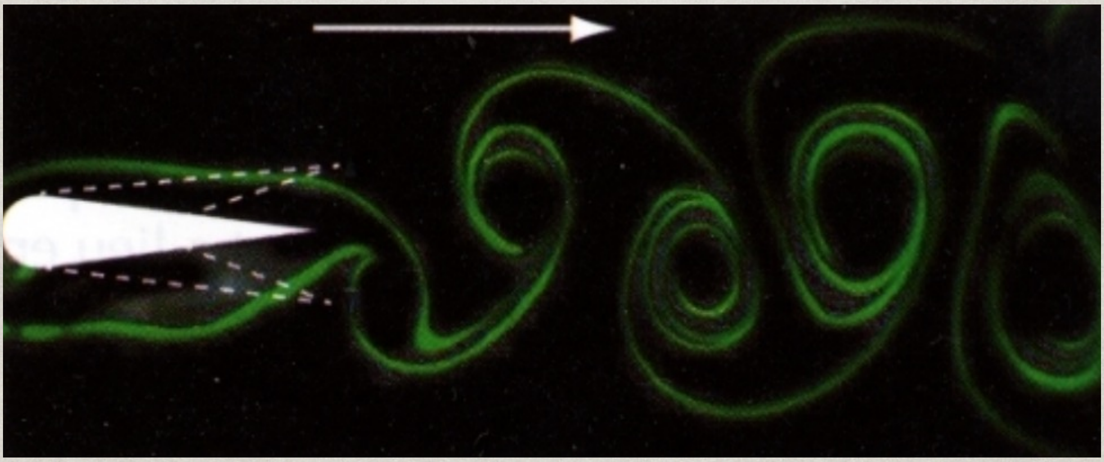
Final Projects



Avalanche model

CFD

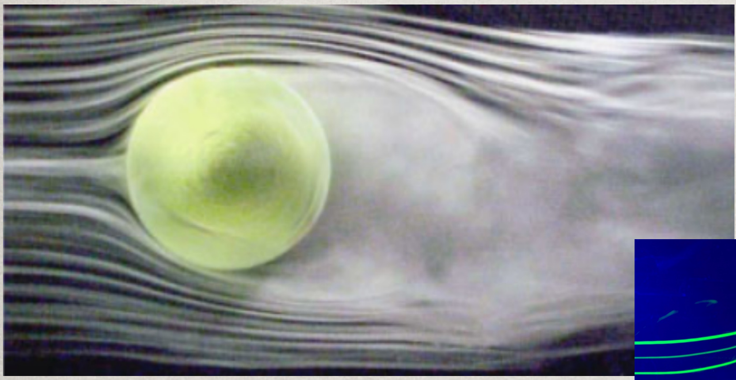
Final Projects



Propulsion

CFD

Final Projects



Tennis/Football lift, Magnus effect