Numerical Methods for Fluid Dynamics TD2

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Foreword

Please download the TD2.tar archive from the course website, move it to a directory dedicated to the course, and then type

```
$ tar xvf TD2.tar
$ cd TD2
$ ipython --pylab
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1 Advection-diffusion problem

We want to consider the following advection-diffusion equation $\dfrac{\partial u}{\partial t} = -U\,\dfrac{\partial u}{\partial x} + \nu\,\dfrac{\partial^2 u}{\partial x^2}\,,$ with $u(t=0,x) = u_0(x)\,,$ and where ν and U are given.

1. We want to write a dispersion relation for this continuous problem. To that end we consider elementary solutions of the form $u = A\,\mathrm{e}^{\sigma t}\,\mathrm{e}^{ikx}\,.$ Relate σ and k through a dispersion relation.

2. Let us now consider a centered finite difference scheme in space

$$\frac{\partial u}{\partial t} = -U\,\frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x} + \nu\,\frac{u(x-\Delta x) - 2u(x) + u(x+\Delta x)}{\Delta x^2}\,.$$

   Introducing $u(x,t) = \hat{u}(t)\,\mathrm{e}^{ikx}$, and plugging this function into the above numerical scheme, what is the equation governing $\mathrm{d}\hat{u}/\mathrm{d}t$? Compare to the equation obtained in question 1. How is the above expression modified if we now consider the case of a periodic domain of size L?
   (Hint: then $k = 2\pi n/L$ with n an integer.)

3. We will now consider a matrix formulation. We introduce a uniform grid with N points in the periodic interval $[0,L)\,,$ i.e.

$$[x_1 = 0,\; x_2 = 0 + \Delta x,\; x_3 = 0 + 2\Delta x,\; \dots,\; x_N = L - \Delta x]\,.$$

We still consider the same numerical scheme in space as previously

$$\frac{\mathrm{d}}{\mathrm{d}t}u_j = -U\,\frac{u_{j+1}-u_{j-1}}{2\Delta x} + \nu\,\frac{u_{j+1}-2u_j+u_{j-1}}{\Delta x^2}\,.$$

(a) What is the matrix A such that $\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u_1 \\ u_2 \\ \dots \\ u_{N-1} \\ u_N \end{pmatrix} = A \begin{pmatrix} u1 \\ u_2 \\ \dots \\ u_{N-1} \\ u_N \end{pmatrix}$?

(b) Now implement this matrix into the file *AdvDiff.py*.
(Look for "TO BE COMPLETED" in the file.)

4. Read the program and understand in particular what the variables "p", "pp" and "ppp" correspond to. Compare the 3 different curves. (The black dashed curve bounds the absolute stability region for the explicit Euler scheme.)

5. In the case of pure diffusion ($U = 0$), do we recover analytical the criterion for the explicit Euler scheme to be stable?

6. What happens in the case $U \neq 0$ and $\nu = 0$ (i.e. pure advection)?

Suggested homework (analytical):

We want to study the diffusion equation in one dimension of space

$$\frac{\partial u}{\partial t} = \nu\,\frac{\partial^2 u}{\partial x^2} \quad \text{with the scheme} \quad \frac{u_j^{n+1}-u_j^{n-1}}{2\,\Delta t} = \nu\,\frac{u_{j+1}^n - 2\,u_j^n + u_{j-1}^n}{\Delta x^2}\,.$$

Can you show that this scheme is second order in space and in time?
Under which condition is this scheme stable?

Suggested homework (numerical):

Write a program in *Python* to compute $u(t=1,x)$ in a periodic domain of size L using the following numerical scheme

$$\frac{u_j^{n+1}-u_j^n}{\Delta t} = -U\,\frac{u_{j+1}^n-u_{j-1}^n}{2\Delta x} + \nu\,\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{\Delta x^2}\,,$$

with initial conditions

$$\begin{cases} u(t=0,x) = 1 \quad \text{for} \quad x \leq L/2\,, \\ u(t=0,x) = 0 \quad \text{for} \quad x > L/2\,. \end{cases}$$

It is interesting to study the behaviour of $u(t=1,x)$ for $U = 1$ and small values of ν.