Numerical Methods for Fluid Dynamics

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Numerical Methods for Fluid Dynamics

Lectures&Tutorials:

Thursday 1pm – 4pm

Outline:

Lectures: Theory and algorithmic

Tutorials & Homework(not marked): Practical implementation

Webpage for the course:

http://www.math.ens.fr/~dormy/CFD

Login: Fluid
Passwd: Flow

CFD

Weeks 1 & 2 : Basic concepts

Finite differences, consistancy, convergence, compact schemes, iterative solvers, time stepping…

The course will ony « go fluid » on week 3!

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1. Introduction

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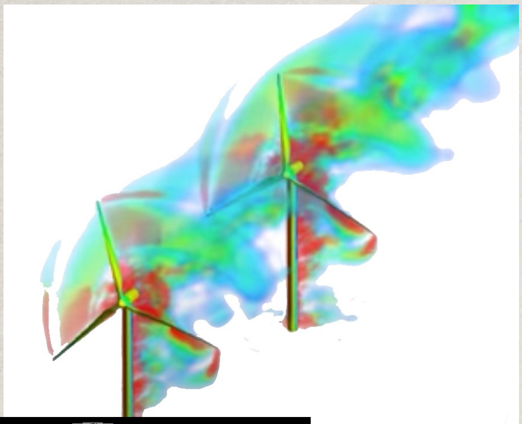

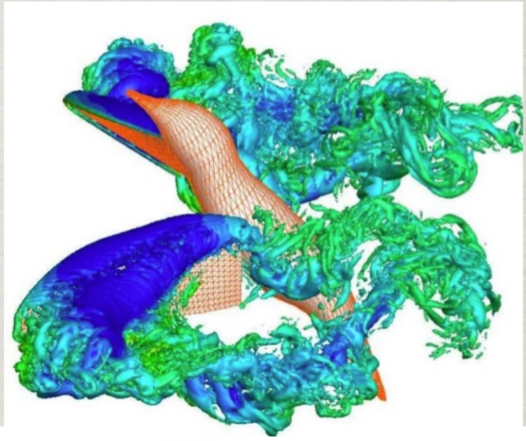

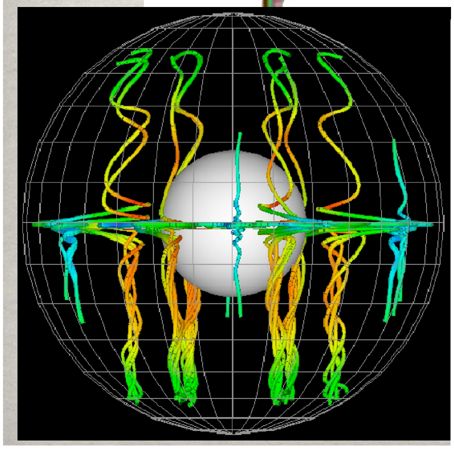

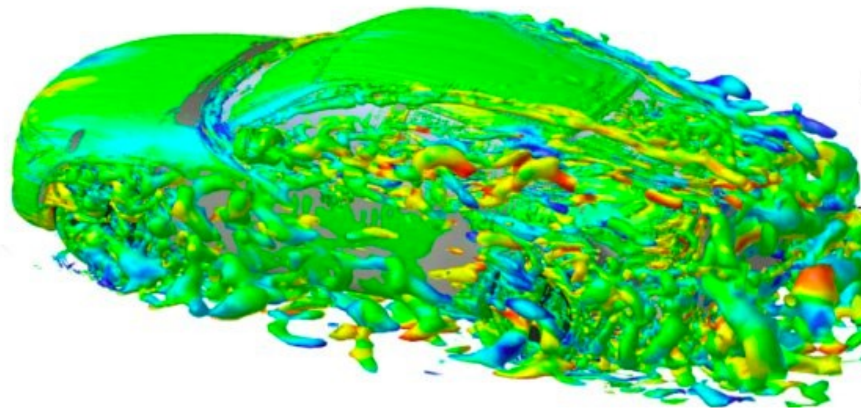

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- Modelling
- Discrete problem
- Resolution (algorithm)
- Coding
- Validation

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Modelling

IncompressibleNavier-Stokes equations

2 dimensions of space

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \frac{1}{Re} \frac{\partial^2 u_i}{\partial x_j \partial x_j}$$

$$\frac{\partial u_i}{\partial x_i} = 0$$

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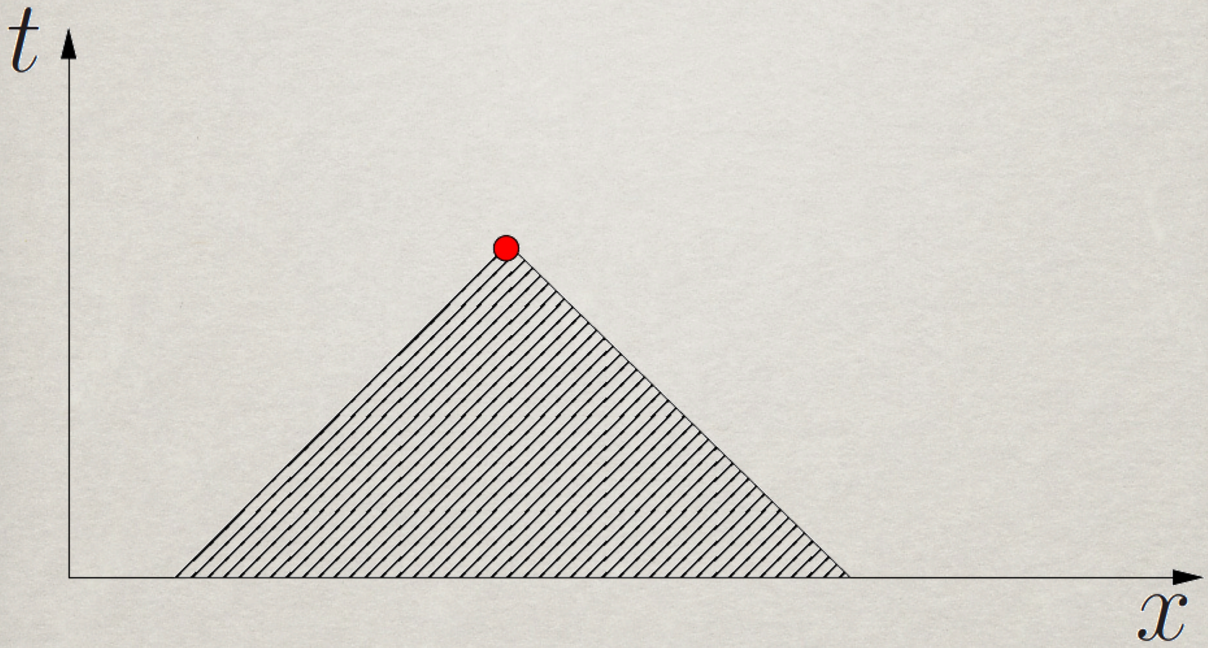

Hyperbolic

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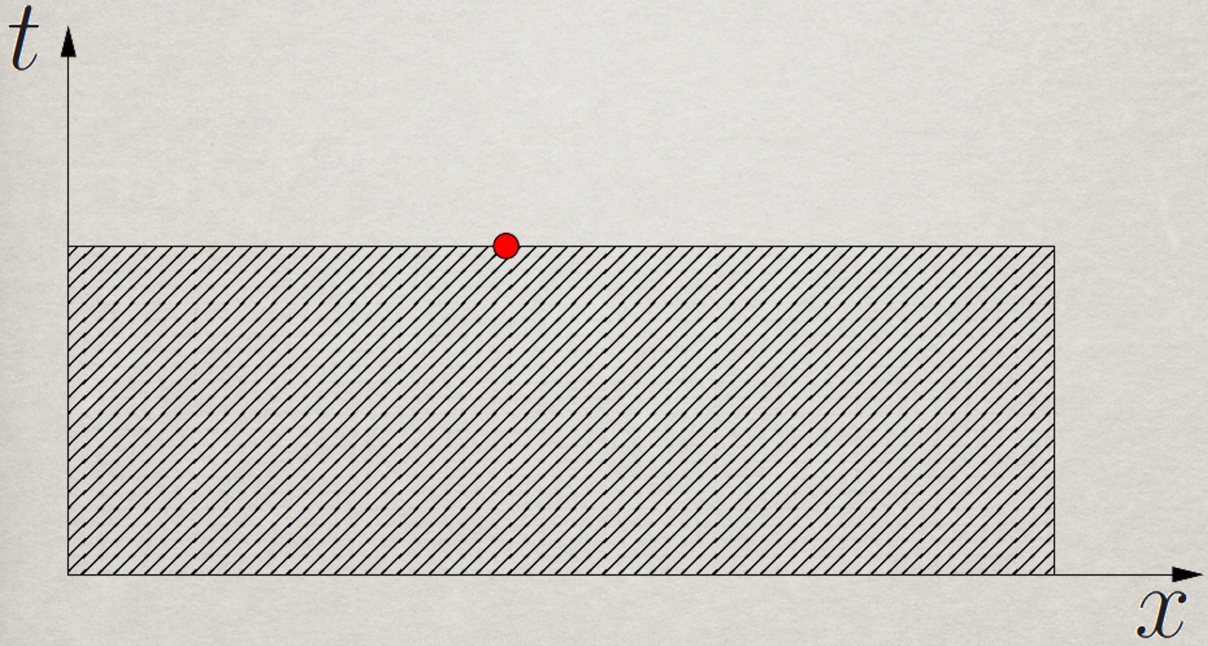

Parabolic

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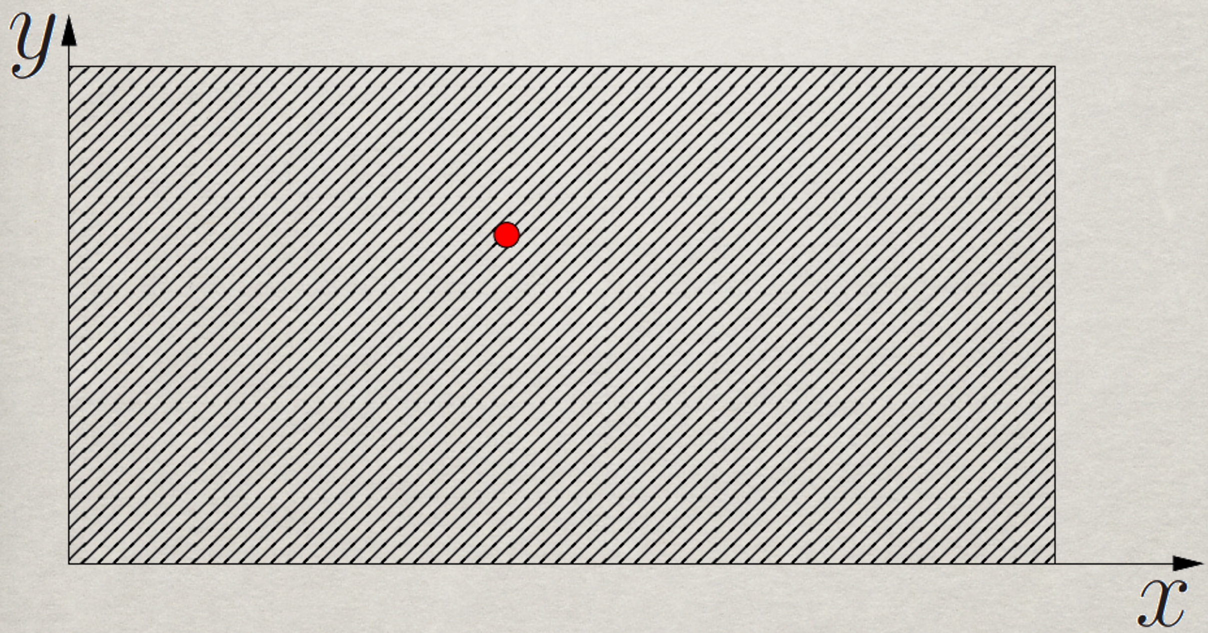

Elliptic

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$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + d\frac{\partial u}{\partial x} + e\frac{\partial u}{\partial y} + f\,u = g$$

$$ax^2 + 2bxy + cy^2 + dx + ey + f = 0\,.$$

$$\Delta = b^2 - ac\,.$$

$\Delta < 0$	$\Delta = 0$	$\Delta > 0$
Elliptic	Parabolic	Hyperbolic

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Discrete problem

For example using Finite Differences

$$x_j = j\Delta x$$

Δx Grid step

$$f_j = f(x_j)$$

CFD

$$\begin{cases} L(\boldsymbol{u}) = \boldsymbol{g} & \text{in} \quad \Omega\,, \\ B(\boldsymbol{u}) = \boldsymbol{\gamma} & \text{on} \quad \partial\Omega\,, \end{cases}$$

CFD

$$\begin{cases} L(\boldsymbol{u}) = \boldsymbol{g} & \text{in} \quad \Omega\,, \\ B(\boldsymbol{u}) = \boldsymbol{\gamma} & \text{on} \quad \partial\Omega\,, \end{cases}$$

$$\begin{cases} L_h(\boldsymbol{u}_h) = \boldsymbol{g} & \text{in} \quad \Omega\,, \\ B_h(\boldsymbol{u}_h) = \boldsymbol{\gamma} & \text{on} \quad \partial\Omega\,, \end{cases}$$

CFD

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$$L_h \underset{h \to 0}{\longrightarrow} L \qquad \qquad \boldsymbol{u}_h \underset{h \to 0}{\longrightarrow} \boldsymbol{u}$$

CFD

$$\begin{cases} L(\boldsymbol{u}) = \boldsymbol{g} & \text{in} \quad \Omega\,, \\ B(\boldsymbol{u}) = \boldsymbol{\gamma} & \text{on} \quad \partial\Omega\,, \end{cases}$$

$$\begin{cases} L_h(\boldsymbol{u}_h) = \boldsymbol{g} & \text{in} \quad \Omega\,, \\ B_h(\boldsymbol{u}_h) = \boldsymbol{\gamma} & \text{on} \quad \partial\Omega\,, \end{cases}$$

$$L_h \underset{h \to 0}{\longrightarrow} L \qquad \qquad \boldsymbol{u}_h \underset{h \to 0}{\longrightarrow} \boldsymbol{u}$$

Consistancy | Convergence

$$R_h(\boldsymbol{u}) = L_h(\boldsymbol{u}) - L(\boldsymbol{u}) = L_h(\boldsymbol{u}) - \boldsymbol{g}$$

Truncation error

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- Modelling
- Discrete problem
- Resolution (algorithm)
- Coding
- Validation

2. Finite Differences

CFD

$$
\begin{aligned}
u_{j+\alpha} &= \sum_{k=0}^{\infty} \frac{1}{k!} (\alpha \Delta x)^k \left(\frac{\partial^k u}{\partial x^k} \right)_j \\
&= u_j + \alpha\, \Delta x \left(\frac{\partial u}{\partial x} \right)_j + \alpha^2 \frac{\Delta x^2}{2} \left(\frac{\partial^2 u}{\partial x^2} \right)_j + \alpha^3 \frac{\Delta x^3}{3!} \left(\frac{\partial^3 u}{\partial x^3} \right)_j \\
&+ \alpha^4 \frac{\Delta x^4}{4!} \left(\frac{\partial^4 u}{\partial x^4} \right)_j + \alpha^5 \frac{\Delta x^5}{5!} \left(\frac{\partial^5 u}{\partial x^5} \right)_j + \mathcal{O}(\Delta x^6),
\end{aligned}
$$

(with $\alpha \in \mathbb{Z}$).

$$
\partial_x u\big|_j = \frac{u_{j+1} - u_{j-1}}{2\Delta x} - \frac{\Delta x^2}{6} \left(\frac{\partial^3 u}{\partial x^3} \right)_j + \mathcal{O}(\Delta x^4)\,.
$$

CFD

$$
\begin{aligned}
u_{j+\alpha} &= \sum_{k=0}^{\infty} \frac{1}{k!} (\alpha \Delta x)^k \left(\frac{\partial^k u}{\partial x^k} \right)_j \\
&= u_j + \alpha\, \Delta x \left(\frac{\partial u}{\partial x} \right)_j + \alpha^2 \frac{\Delta x^2}{2} \left(\frac{\partial^2 u}{\partial x^2} \right)_j + \alpha^3 \frac{\Delta x^3}{3!} \left(\frac{\partial^3 u}{\partial x^3} \right)_j \\
&+ \alpha^4 \frac{\Delta x^4}{4!} \left(\frac{\partial^4 u}{\partial x^4} \right)_j + \alpha^5 \frac{\Delta x^5}{5!} \left(\frac{\partial^5 u}{\partial x^5} \right)_j + \mathcal{O}(\Delta x^6),
\end{aligned}
$$

(with $\alpha \in \mathbb{Z}$).

$$
u_{j-1} + u_{j+1} = 2u_j + \Delta x^2 \left. \frac{\partial^2 u}{\partial x^2} \right|_j + \frac{\Delta x^4}{12} \left. \frac{\partial^4 u}{\partial x^4} \right|_j + \mathcal{O}(\Delta x^6)
$$

CFD

$$u_{j+\alpha} = \sum_{k=0}^{\infty} \frac{1}{k!} (\alpha \Delta x)^k \left(\frac{\partial^k u}{\partial x^k}\right)_j$$

$$= u_j + \alpha\, \Delta x \left(\frac{\partial u}{\partial x}\right)_j + \alpha^2 \frac{\Delta x^2}{2} \left(\frac{\partial^2 u}{\partial x^2}\right)_j + \alpha^3 \frac{\Delta x^3}{3!} \left(\frac{\partial^3 u}{\partial x^3}\right)_j$$

$$+ \alpha^4 \frac{\Delta x^4}{4!} \left(\frac{\partial^4 u}{\partial x^4}\right)_j + \alpha^5 \frac{\Delta x^5}{5!} \left(\frac{\partial^5 u}{\partial x^5}\right)_j + \mathcal{O}(\Delta x^6),$$

(with $\alpha \in \mathbb{Z}$).

$$u_{j-1} + u_{j+1} = 2u_j + \Delta x^2 \left.\frac{\partial^2 u}{\partial x^2}\right|_j + \frac{\Delta x^4}{12} \left.\frac{\partial^4 u}{\partial x^4}\right|_j + \mathcal{O}(\Delta x^6)$$

$$\left.\frac{\partial^2 u}{\partial x^2}\right|_j = \frac{u_{j-1} - 2u_j + u_{j+1}}{\Delta x^2} - \frac{\Delta x^2}{12} \left.\frac{\partial^4 u}{\partial x^4}\right|_j + \mathcal{O}(\Delta x^4)\,.$$

CFD

$$\partial_x u|_j = \frac{u_{j+1} - u_{j-1}}{2\Delta x} - \frac{\Delta x^2}{6} \left(\frac{\partial^3 u}{\partial x^3}\right)_j + \mathcal{O}(\Delta x^4),$$

$$\partial_x u|_j = \frac{u_j - u_{j-1}}{\Delta x} + \frac{\Delta x}{2} \left(\frac{\partial^2 u}{\partial x^2}\right)_j - \frac{\Delta x^2}{6} \left(\frac{\partial^3 u}{\partial x^3}\right)_j + \mathcal{O}(\Delta x^3),$$

$$\partial_x u|_j = \frac{u_{j+1} - u_j}{\Delta x} - \frac{\Delta x}{2} \left(\frac{\partial^2 u}{\partial x^2}\right)_j - \frac{\Delta x^2}{6} \left(\frac{\partial^3 u}{\partial x^3}\right)_j + \mathcal{O}(\Delta x^3)\,.$$

CFD

$$\delta_0 u\big|_j = \frac{u_{j+1} - u_{j-1}}{2\Delta x}, \qquad \delta_- u\big|_j = \frac{u_j - u_{j-1}}{\Delta x}$$

$$\text{and} \qquad \delta_+ u\big|_j = \frac{u_{j+1} - u_j}{\Delta x}\,.$$

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$$\partial_x u\big|_{j+1/2} = \frac{u_{j+1} - u_j}{\Delta x} - \frac{1}{24}\Delta x^2 \left(\frac{\partial^3 u}{\partial x^3}\right)_{j+1/2} + \mathcal{O}(\Delta x^4)\,.$$

CFD

$$\partial_x u|_j = \frac{-3u_j + 4u_{j+1} - u_{j+2}}{2\Delta x} + \frac{\Delta x^2}{3}\left(\frac{\partial^3 u}{\partial x^3}\right)_j + \dots,$$

$$\partial_x u|_j = \frac{3u_j - 4u_{j-1} + u_{j-2}}{2\Delta x} + \frac{\Delta x^2}{3}\left(\frac{\partial^3 u}{\partial x^3}\right)_j + \dots,$$

CFD

$$\partial_x u|_j = \frac{-u_{j+2} + 8u_{j+1} - 8u_{j-1} + u_{j-2}}{12\Delta x} + \frac{\Delta x^4}{3}\left(\frac{\partial^5 u}{\partial x^5}\right)_j + \dots .$$

CFD

$$\partial_{xx}u|_j = \frac{u_{j-1} - 2u_j + u_{j+1}}{\Delta x^2} - \frac{\Delta x^2}{12}\left(\frac{\partial^4 u}{\partial x^4}\right)_j + \ldots,$$

$$\partial_{xx}u|_j = \frac{-u_{j-2} + 16u_{j-1} - 30u_j + 16u_{j+1} - u_{j+2}}{12\Delta x^2} + \frac{\Delta x^4}{90}\left(\frac{\partial^6 u}{\partial x^6}\right)_j + \ldots .$$

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$$u(0) = u_0,\ u(\Delta x) = u_1,\ u(2\Delta x) = u_2\,,$$

$$u(x) = a + b\,x + c\,x^2\,,$$

$$b = \frac{-3u_0 + 4u_1 - u_2}{2\Delta x}, \qquad \text{et} \qquad 2\,c = \frac{u_0 - 2u_1 + u_2}{\Delta x^2}\,.$$

CFD

Lagrange polynomials

$$(x_{j-1}, u_{j-1}), \qquad (x_j, u_j), \qquad (x_{j+1}, u_{j+1}),$$

$$U(x) = \frac{(x - x_j)(x - x_{j+1})}{(x_{j-1} - x_j)(x_{j-1} - x_{j+1})} u_{j-1}$$

$$+ \frac{(x - x_{j-1})(x - x_{j+1})}{(x_j - x_{j-1})(x_j - x_{j+1})} u_j$$

$$+ \frac{(x - x_{j-1})(x - x_j)}{(x_{j+1} - x_{j-1})(x_{j+1} - x_j)} u_{j+1} .$$

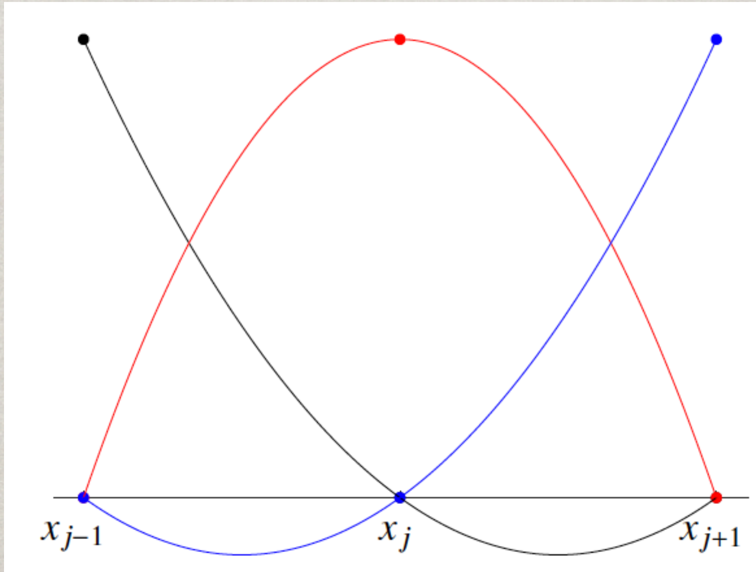
CFD

$$U'(x) = \frac{2x - (x_j + x_{j+1})}{(x_{j-1} - x_j)(x_{j-1} - x_{j+1})} u_{j-1}$$

$$+ \frac{2x - (x_{j-1} + x_{j+1})}{(x_j - x_{j-1})(x_j - x_{j+1})} u_j$$

$$+ \frac{2x - (x_{j-1} + x_j)}{(x_{j+1} - x_{j-1})(x_{j+1} - x_j)} u_{j+1} .$$

CFD



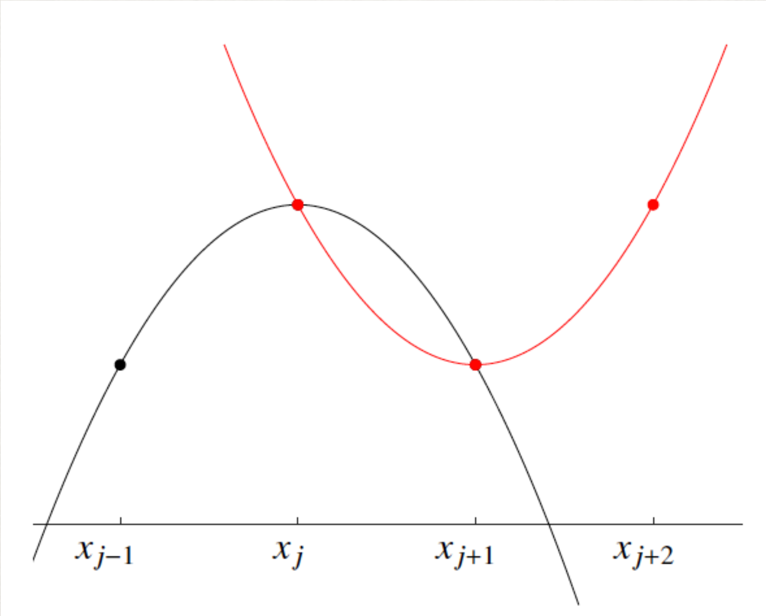
CFD

$$U(x) = \sum_{k=1}^{n} u_k \mathcal{L}_k(x), \quad \text{with} \quad \mathcal{L}_k(x) = \prod_{l=1,\, l\neq k}^{n} \frac{x - x_l}{x_k - x_l}.$$

The exact derivative of $\mathcal{L}_k$ with respect to x takes the form

$$\mathcal{L}'_k(x) = \sum_{m=1,\, m\neq k}^{n} \left(\frac{1}{x_k - x_m} \prod_{l=1,\, l\neq(k,m)}^{n} \frac{x - x_l}{x_k - x_l} \right).$$

CFD



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Compact Finite Differences

$$\frac{\partial^2 u}{\partial x^2} = -f$$

$$\alpha u_{j+1} + \beta u_j + \gamma u_{j-1} \simeq f_j \, .$$

CFD

Compact Finite Differences

$$\frac{\partial^2 u}{\partial x^2} = -f$$

$$\alpha u_{j+1} + \beta u_j + \gamma u_{j-1} \simeq f_j \, .$$

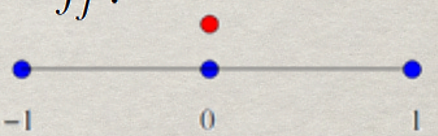
$$\alpha u_{j+1} + \beta u_j + \gamma u_{j-1} \simeq a f_{j-1} + b f_j + c f_{j+1} \, .$$

CFD

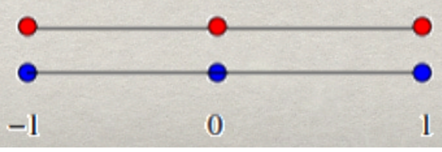
Compact Finite Differences

$$\frac{\partial^2 u}{\partial x^2} = -f$$

$$\alpha u_{j+1} + \beta u_j + \gamma u_{j-1} \simeq f_j \, .$$



$$\alpha u_{j+1} + \beta u_j + \gamma u_{j-1} \simeq a f_{j-1} + b f_j + c f_{j+1} \, .$$



CFD

Compact Finite Differences

$$\frac{\partial^2 u}{\partial x^2} = -f$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} - \frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4} + \mathcal{O}(\Delta x^4)\,.$$

CFD

Compact Finite Differences

$$\frac{\partial^2 u}{\partial x^2} = -f$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} - \frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4} + \mathcal{O}(\Delta x^4)\,.$$

$$-\frac{\partial^4 u}{\partial x^4} = \frac{\partial^2 f}{\partial x^2} = \frac{f_{i+1} - 2f_i + f_{i-1}}{\Delta x^2} + \mathcal{O}(\Delta x^2)\,.$$

CFD

Compact Finite Differences

$$\frac{\partial^2 u}{\partial x^2} = -f$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} - \frac{\Delta x^2}{12}\frac{\partial^4 u}{\partial x^4} + \mathcal{O}(\Delta x^4)\,.$$

$$-\frac{\partial^4 u}{\partial x^4} = \frac{\partial^2 f}{\partial x^2} = \frac{f_{i+1} - 2f_i + f_{i-1}}{\Delta x^2} + \mathcal{O}(\Delta x^2)\,.$$

$$\begin{aligned}\frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2} &= -f_j - \frac{f_{j+1} - 2f_j + f_{j-1}}{12} + \mathcal{O}(\Delta x^4) \\ &= -\frac{f_{i+1} + 10f_i + f_{i-1}}{12} + \mathcal{O}(\Delta x^4)\,.\end{aligned}$$

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$$\partial u/\partial x = f$$

$$\partial_x u|_j = \frac{u_{j+1} - u_{j-1}}{2\Delta x} - \frac{\Delta x^2}{6}\left(\frac{\partial^3 u}{\partial x^3}\right)_j + \mathcal{O}(\Delta x^4),$$

$$\frac{\partial^3 u}{\partial x^3} = \frac{\partial^2 f}{\partial x^2} = \frac{f_{i+1} - 2f_i + f_{i-1}}{\Delta x^2} + \mathcal{O}(\Delta x^2),$$

$$\frac{u_{j+1} - u_{j-1}}{2\Delta x} = \frac{1}{6}f_{j-1} + \frac{2}{3}f_j + \frac{1}{6}f_{j+1} + \mathcal{O}(\Delta x^4)\,.$$

CFD

$$\frac{\partial^2 u}{\partial x^2} = -f$$

$$f_{j-1} = \frac{35}{12} u_{j-1} - \frac{26}{3} u_j + \frac{19}{2} u_{j+1} - \frac{14}{3} u_{j+2} + \frac{11}{12} u_{j+3} + \mathcal{O}(\Delta x^4)$$

$$f_j = \frac{11}{12} u_{j-1} - \frac{5}{3} u_j + \frac{1}{2} u_{j+1} + \frac{1}{3} u_{j+2} - \frac{1}{12} u_{j+3} + \mathcal{O}(\Delta x^4)$$

$$f_{j+1} = -\frac{1}{12} u_{j-1} + \frac{4}{3} u_j - \frac{5}{2} u_{j+1} + \frac{4}{3} u_{j+2} - \frac{1}{12} u_{j+3} + \mathcal{O}(\Delta x^4)$$

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$$f_{j-1} = \frac{35}{12} u_{j-1} - \frac{26}{3} u_j + \frac{19}{2} u_{j+1} - \frac{14}{3} u_{j+2} + \frac{11}{12} u_{j+3} + \mathcal{O}(\Delta x^4)$$

$$f_j = \frac{11}{12} u_{j-1} - \frac{5}{3} u_j + \frac{1}{2} u_{j+1} + \frac{1}{3} u_{j+2} - \frac{1}{12} u_{j+3} + \mathcal{O}(\Delta x^4)$$

$$f_{j+1} = -\frac{1}{12} u_{j-1} + \frac{4}{3} u_j - \frac{5}{2} u_{j+1} + \frac{4}{3} u_{j+2} - \frac{1}{12} u_{j+3} + \mathcal{O}(\Delta x^4)$$

$$f_{j-1} + 10\, f_j + f_{j+1} = \left(12\, u_{j-1} - 24\, u_j + 12\, u_{j+1}\right) / \Delta x^2 + \mathcal{O}(\Delta x^4)\,.$$

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Analogy with Padé expansion

(Henri Padé, 1863-1955)

$$f(x) = \sum_{j=0}^{+\infty} a_j x^j$$

$$f(x) - [L/M](x) = \mathcal{O}(x^{L+M+1})$$

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Analogy with Padé expansion

[L,0] $\quad \exp(x) = \dfrac{\sum_0^L \frac{1}{L!}\, x^L}{1} + \mathcal{O}(x^{L+1})$

[1,1] $\quad \exp(x) = \dfrac{1 + \frac{1}{2}x}{1 - \frac{1}{2}x} + \mathcal{O}(x^3)$

[2,2] $\quad \exp(x) = \dfrac{1 + \frac{1}{2}x + \frac{1}{12}x^2}{1 - \frac{1}{2}x + \frac{1}{12}x^2} + \mathcal{O}(x^5)$

[3,3] $\quad \exp(x) = \dfrac{1 + \frac{1}{2}x + \frac{1}{10}x^2 + \frac{1}{120}x^3}{1 - \frac{1}{2}x + \frac{1}{10}x^2 - \frac{1}{120}x^3} + \mathcal{O}(x^7)$

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Modified wave number $f = e^{i(kx)}$

$$\frac{f_{j+1} - f_{j-1}}{2\Delta x} = i\,k'\,f_j$$

$$i\,k' = \frac{\mathrm{e}^{ik\Delta x} - \mathrm{e}^{-ik\Delta x}}{2\Delta x}, \qquad \frac{k'}{k} = \frac{\sin(k\Delta x)}{k\Delta x}.$$

$$ik' = i\sin(k\Delta x)/\Delta x.$$

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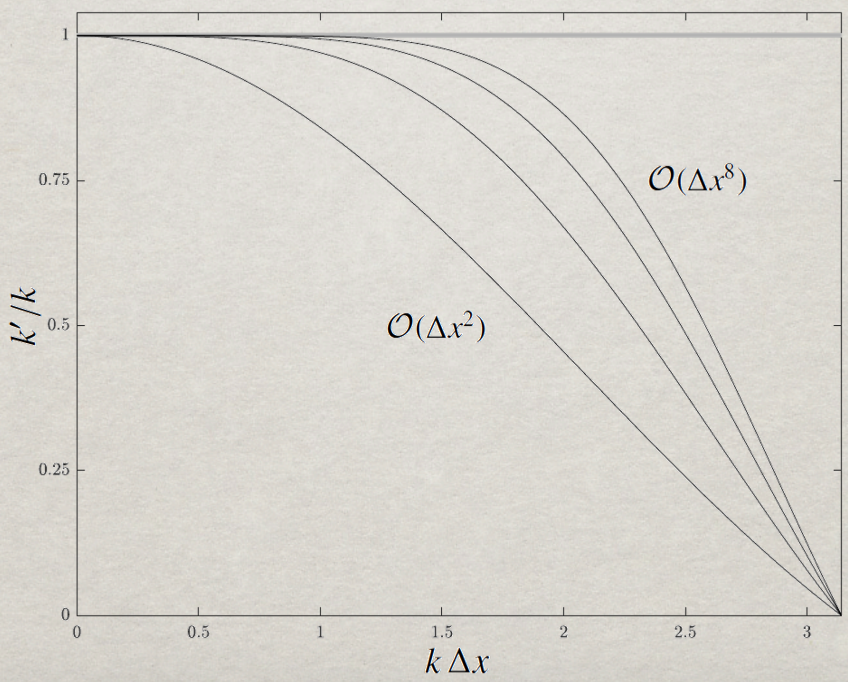
Transfer function for the centred schemes 2,4,6,8 order



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$$\frac{1}{2\Delta x}\left(u_{j+1} - u_{j-1}\right) = \frac{1}{6}\left(f_{j+1} + f_{j-1}\right) + \frac{2}{3} f_j$$

$$i\,k' = \frac{\frac{1}{2}\left(\mathrm{e}^{ik\Delta x} - \mathrm{e}^{-ik\Delta x}\right)}{\frac{1}{6}\left(\mathrm{e}^{ik\Delta x} + \mathrm{e}^{-ik\Delta x}\right) + \frac{2}{3}}\,\frac{1}{\Delta x},$$

$$\frac{k'}{k} = \frac{3\sin(k\delta x)}{\cos(k\Delta x) + 2}\,\frac{1}{k\Delta x}.$$

CFD

Transfer function for the compact schemes 4,6,8,12 order