March 21: last course
Homework for Math students

April 4: projects defense

CFD

8. Boundary conditions

CFD

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Heat Equation in 1D

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \qquad \text{with} \quad x \in [0, L]$$

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Dirichlet:

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \qquad \text{with} \quad x \in [0, L]$$

$$u = g$$

CFD

Dirichlet:

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \qquad \text{with} \quad x \in [0, L]$$

$$u = g$$

$$\frac{u_1^{n+1} - u_1^n}{\Delta t} = \kappa \frac{u_2^n - 2u_1^n + g}{\Delta x^2}$$

CFD

Neumann:

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \qquad \text{with} \quad x \in [0, L]$$

$$\left.\frac{\partial u}{\partial x}\right|_0 = 0$$

$$\frac{u_1 - u_0}{\Delta x} = 0$$

CFD

Neumann:

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \qquad \text{with} \quad x \in [0, L]$$

$$\left.\frac{\partial u}{\partial x}\right|_0 = 0$$

$$\frac{-3u_0 + 4u_1 - u_2}{2\Delta x} = 0$$

CFD

Neumann:

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \qquad \text{with} \quad x \in [0, L]$$

$$\left.\frac{\partial u}{\partial x}\right|_0 = 0$$

$$\left.\frac{\partial u}{\partial x}\right|_0 = \frac{u_1 - u_{-1}}{2\Delta x} + \mathcal{O}(\Delta x^2) = 0$$

CFD

Neumann:

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \qquad \text{with} \quad x \in [0, L]$$

$$\left.\frac{\partial u}{\partial x}\right|_0 = 0$$

$$\begin{aligned} u_0^{n+1} &= u_0^n + \frac{\kappa \Delta t}{\Delta x^2}\left[u_{-1}^n - 2\,u_0^n + u_1^n\right] \\ &= u_0^n + \frac{\kappa \Delta t}{\Delta x^2}\left[-2u_0^n + 2\,u_1^n\right]. \end{aligned}$$

CFD

Neumann:

$$\frac{\partial u}{\partial t} = \kappa \frac{\partial^2 u}{\partial x^2} \qquad \text{with} \quad x \in [0, L]$$

$$\left.\frac{\partial u}{\partial x}\right|_0 = 0$$

$$\left.\frac{\partial u}{\partial x}\right|_{1/2} = \frac{u_1 - u_0}{\Delta x} + \mathcal{O}(\Delta x^2) = 0$$

$$u_0 = u_1$$

Neumann (non-homogeneous):

$$\frac{\partial u}{\partial t} = \kappa \, \frac{\partial^2 u}{\partial x^2} \qquad \text{with} \quad x \in [0, L]$$

$$\left.\frac{\partial u}{\partial x}\right|_0 = \gamma \qquad\qquad u_0 = u_1 - \gamma \, \Delta x$$

Robin / Fourier / Mixed :

$$\partial u / \partial x = \alpha u$$

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Advection:

$$\frac{\partial u}{\partial t} + c\,\frac{\partial u}{\partial x} = 0$$

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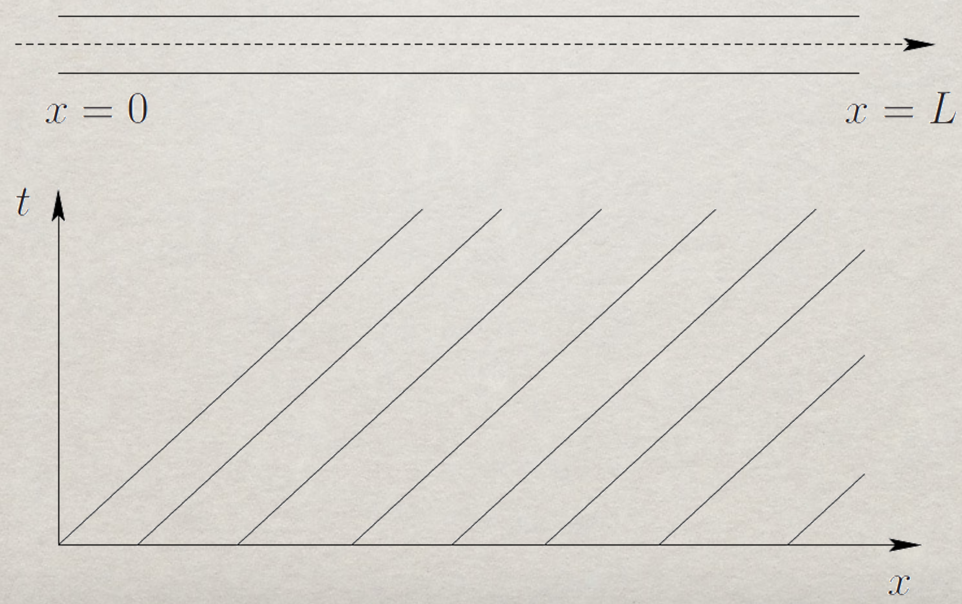
Advection:



CFD

Sound waves

$$
\begin{aligned}
\frac{1}{\rho_0}\frac{\partial \rho'}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{u} &= 0 \\
\rho_0 \frac{\partial \mathbf{u}}{\partial t} + c_0^2\,\boldsymbol{\nabla}\rho' &= 0
\end{aligned}
$$

CFD

Sound waves

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\begin{aligned}
\frac{1}{\rho_0}\frac{\partial \rho'}{\partial t} + \boldsymbol{\nabla}\cdot\mathbf{u} &= 0 \\
\rho_0 \frac{\partial \mathbf{u}}{\partial t} + c_0^2\,\boldsymbol{\nabla}\rho' &= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \tilde{\rho}}{\partial t} + \frac{\partial u}{\partial x} &= 0 \\
\frac{\partial \mathbf{u}}{\partial t} + c_0^2 \frac{\partial \tilde{\rho}}{\partial x} &= 0\,.
\end{aligned}
$$

CFD

$$
\begin{aligned}
\frac{\partial \tilde{\rho}}{\partial t} + \frac{\partial u}{\partial x} &= 0 \\
\frac{\partial \mathbf{u}}{\partial t} + c_0^2 \frac{\partial \tilde{\rho}}{\partial x} &= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial}{\partial t}(c_0\,\tilde{\rho} + u) + c_0\,\frac{\partial}{\partial x}(c_0\,\tilde{\rho} + u) &= 0 \\
\frac{\partial}{\partial t}(c_0\,\tilde{\rho} - u) - c_0\,\frac{\partial}{\partial x}(c_0\,\tilde{\rho} - u) &= 0
\end{aligned}
$$

CFD

$$
\begin{aligned}
\frac{\partial}{\partial t}(c_0\,\tilde{\rho} + u) + c_0\,\frac{\partial}{\partial x}(c_0\,\tilde{\rho} + u) &= 0 \\
\frac{\partial}{\partial t}(c_0\,\tilde{\rho} - u) - c_0\,\frac{\partial}{\partial x}(c_0\,\tilde{\rho} - u) &= 0
\end{aligned}
$$

$$X^{\pm} = c_0\,\tilde{\rho} \pm u$$

$$\frac{\partial}{\partial t}X^{\pm} \pm c_0\,\frac{\partial}{\partial x}X^{\pm} = 0$$

CFD

Shallow water:

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ \eta \end{pmatrix} + \begin{pmatrix} U & 1 \\ c^2 & U \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} u \\ \eta \end{pmatrix} = 0$$

$$c = \sqrt{gH}$$

$$V_{\pm} = u \pm \eta / c$$

CFD

Shallow water:

$$\frac{\partial}{\partial t}\begin{pmatrix} u \\ \eta \end{pmatrix} + \begin{pmatrix} U & 1 \\ c^2 & U \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} u \\ \eta \end{pmatrix} = 0$$

CFD

Shallow water:

$$c = \sqrt{gH} \qquad V_{\pm} = u \pm \eta/c$$

$$\frac{\partial}{\partial t}\begin{pmatrix} V_+ \\ V_- \end{pmatrix} + \begin{pmatrix} U+c & 0 \\ 0 & U-c \end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix} V_+ \\ V_- \end{pmatrix} = 0$$

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Boundary layers

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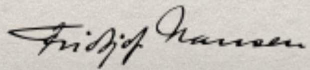
Fridtjof Nansen (1861-1930)



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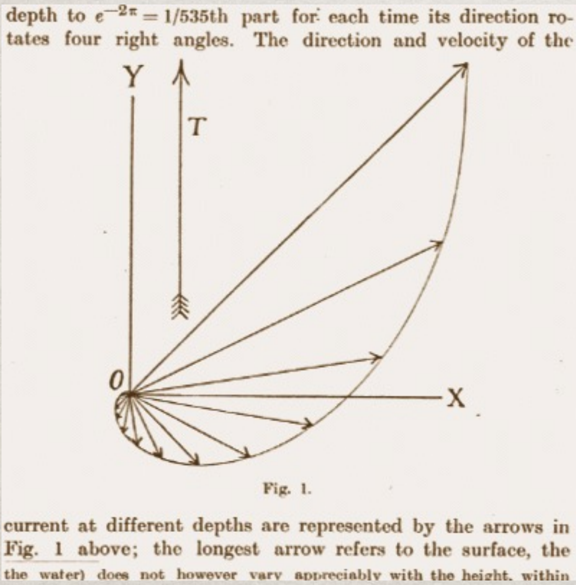
Vagn Ekman (1874-1954)



depth to $e^{-2\pi} = 1/535$th part for each time its direction rotates four right angles. The direction and velocity of the

Fig. 1.

current at different depths are represented by the arrows in Fig. 1 above; the longest arrow refers to the surface, the

the water) does not however vary appreciably with the height, within

CFD

Ekman spiral

$$
\begin{aligned}
-2\Omega\, u_y &= -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \frac{\partial^2 u_x}{\partial z^2}\,,\\
2\Omega\, u_x &= -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu \frac{\partial^2 u_y}{\partial z^2}\,.
\end{aligned}
$$

CFD

Ekman spiral

$$
\begin{aligned}
2\Omega\, u_x^o &= -\frac{1}{\rho}\frac{\partial p^0}{\partial y}, \\
0 &= -\frac{1}{\rho}\frac{\partial p^0}{\partial X},
\end{aligned}
$$

$$
\begin{aligned}
-2\Omega\, u_y &= \nu \frac{\partial^2 u_x}{\partial z^2}, \\
2\Omega\,(u_x - u_x^0) &= \nu \frac{\partial^2 u_y}{\partial z^2}.
\end{aligned}
$$

CFD

Ekman spiral

$$
\begin{aligned}
-2\Omega\, u_y &= \nu \frac{\partial^2 u_x}{\partial z^2}, \\
2\Omega\,(u_x - u_x^0) &= \nu \frac{\partial^2 u_y}{\partial z^2}.
\end{aligned}
$$

$$
\begin{aligned}
u_x &= u_x^0 \left[1 - e^{-z/\delta}\cos(z/\delta)\right], \\
u_y &= u_x^0\, e^{-z/\delta}\sin(z/\delta),
\end{aligned}
$$

CFD

Ekman spiral

$$E\,\partial_{zz}\mathcal{V} = \mathrm{i}\,\mathcal{V}$$

$$\zeta = \sqrt{2\,E}\,z$$

$$\frac{\partial^2 \mathcal{V}}{\partial \zeta^2} = 2\,\mathrm{i}\,\mathcal{V}$$

$$\mathcal{V} \propto e^{-\zeta\,(1+\mathrm{i})}$$

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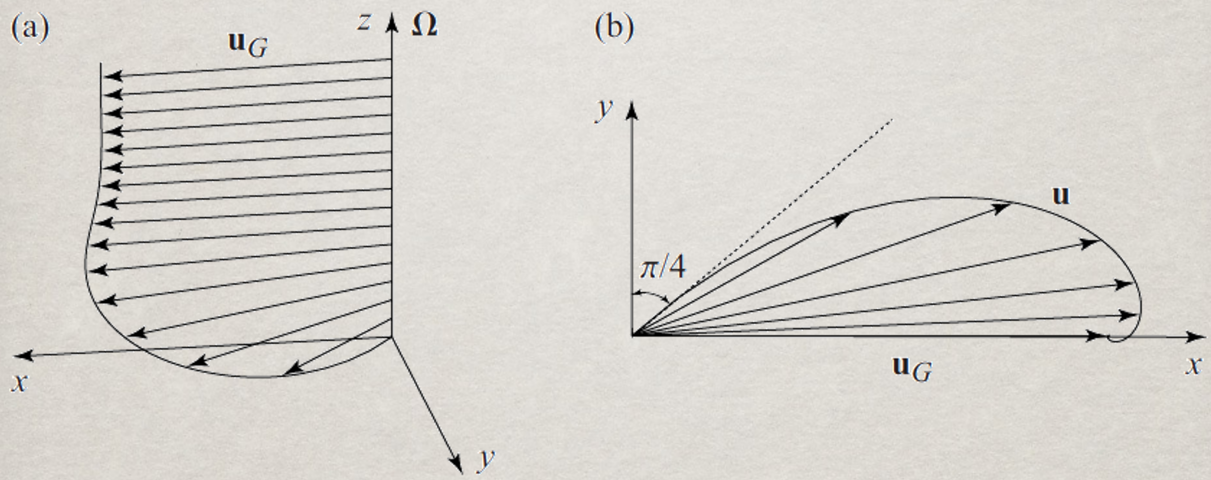

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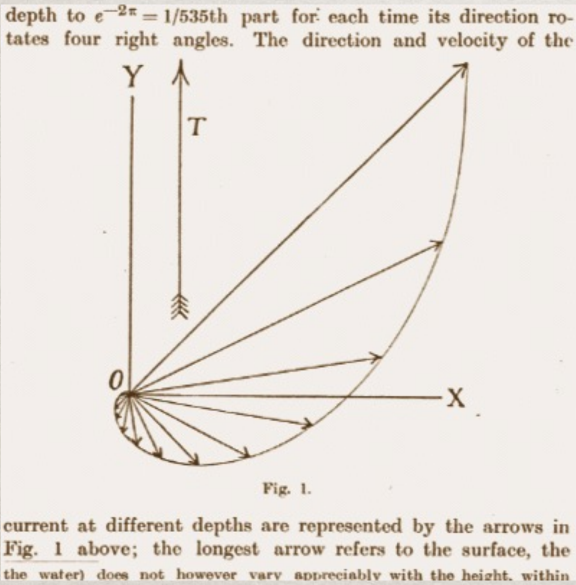
Vagn Ekman (1874-1954)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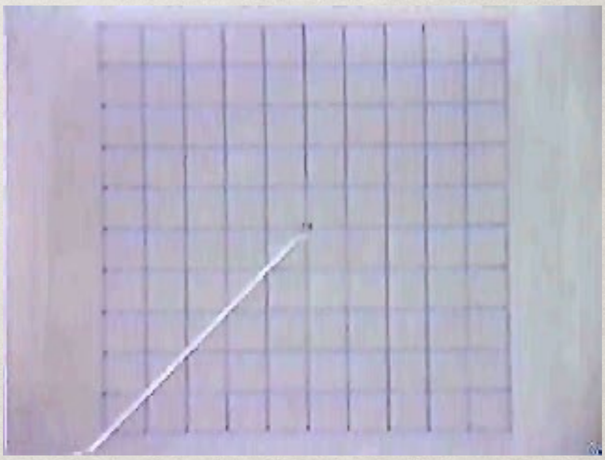

(GFD Dennou Japon)

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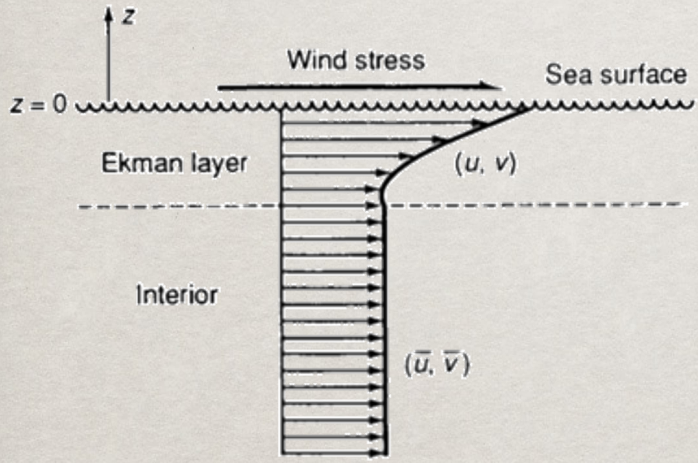
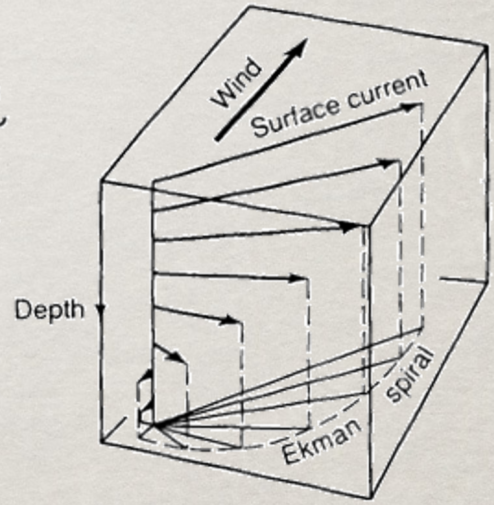
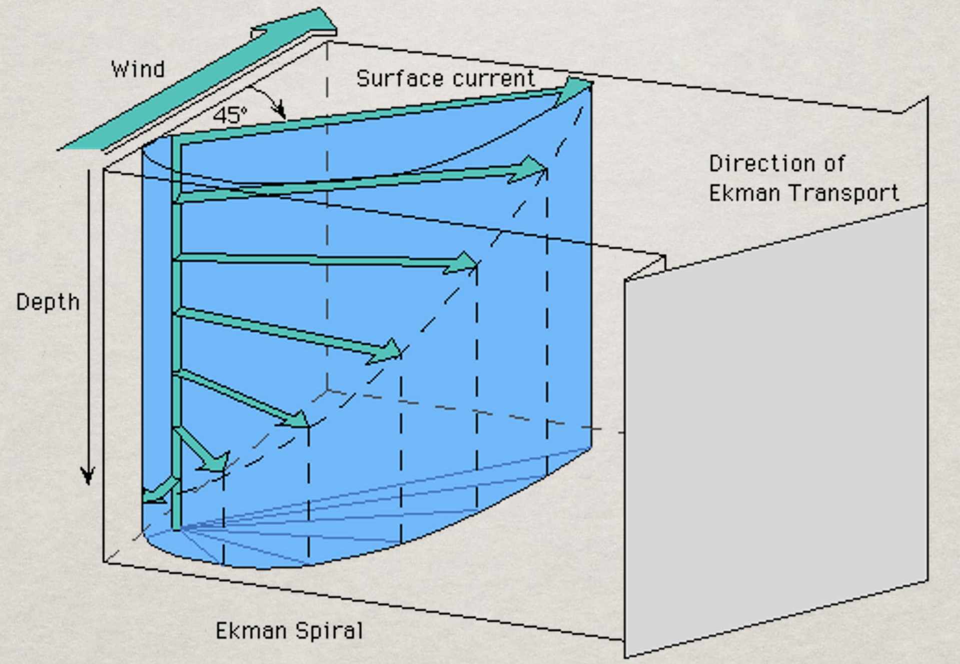




$$\nu \propto e^{-\zeta\,(1+i)}$$

Cushman-Roisin



Adapted from Thurman, Harold V. **Essentials of Oceanography, 5th ed.**
Prentice-Hall, Inc., 1996.

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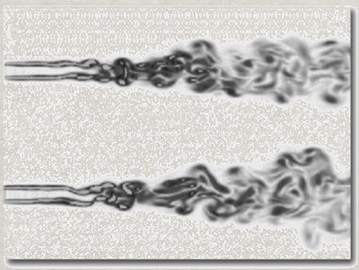
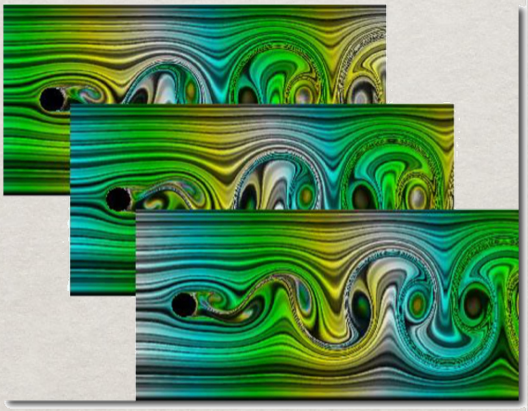
Opened domains:

- **wakes, jets and plumes**
- aerodynamics
- acoustics

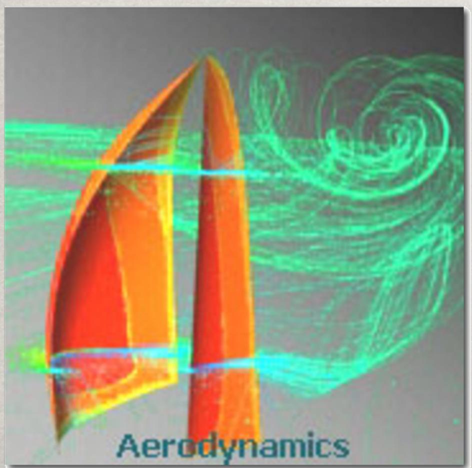


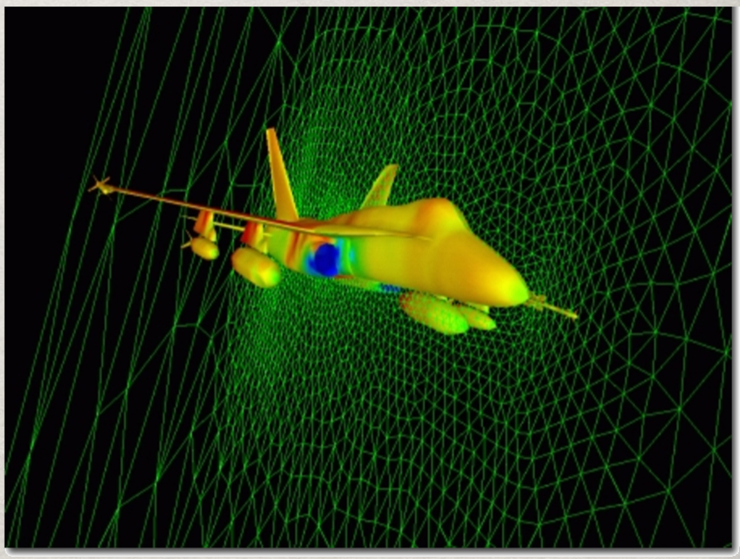


CFD

Opened domains:

- wakes, jets and plumes
- **aerodynamics**
- acoustics

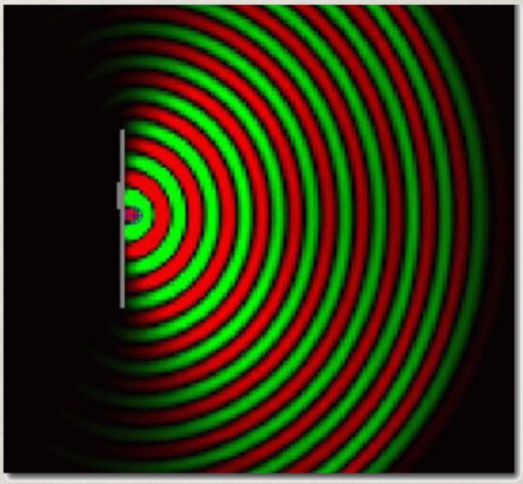


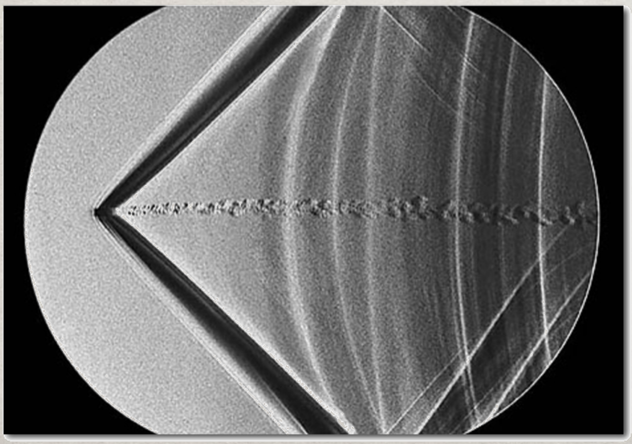


CFD

Opened domains:

- wakes, jets and plumes
- aerodynamics
- **acoustics**





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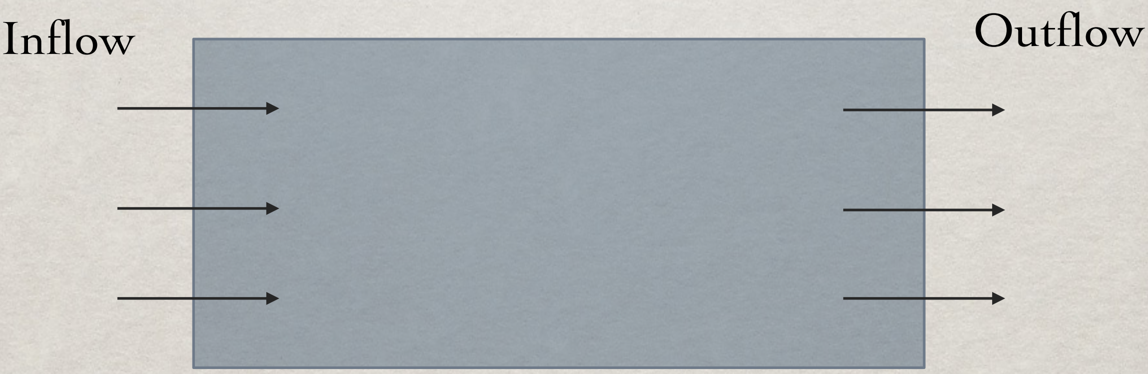
Opened domains:



Opened domains for waves:

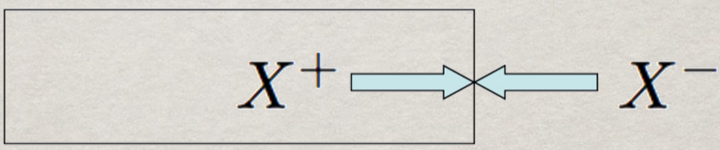
General idea:

If the perturbation can be decomposed into wave components, one can manipulate the dispersion relation at the boundary to suppress the incoming waves.

Engquist-Majda (1977)

Characteristics:

In one dimension, we want to construct an outflow boundary condition at the right boundary



outflow boundary condition $X^- = 0$

Engquist-Majda (1977)

Final Projects

Kelvin wake

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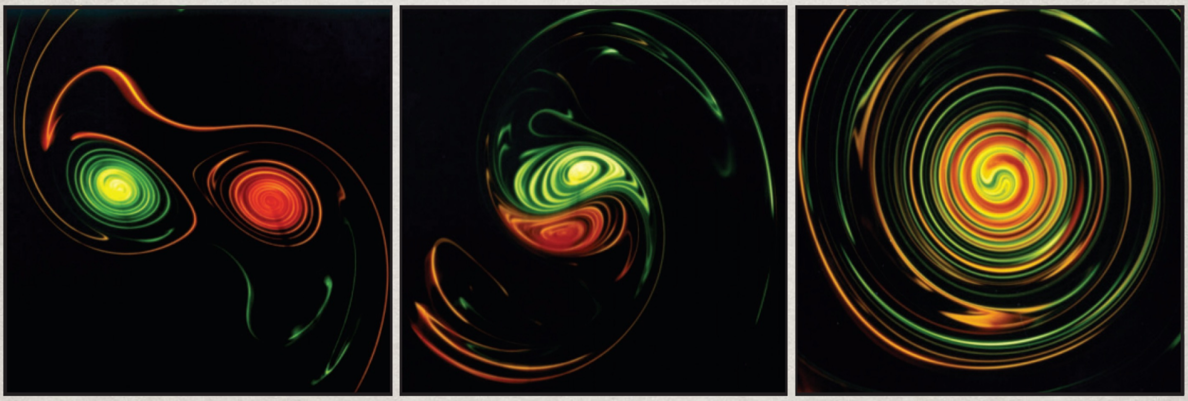
Final Projects



Vortices interactions

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Thermal convection

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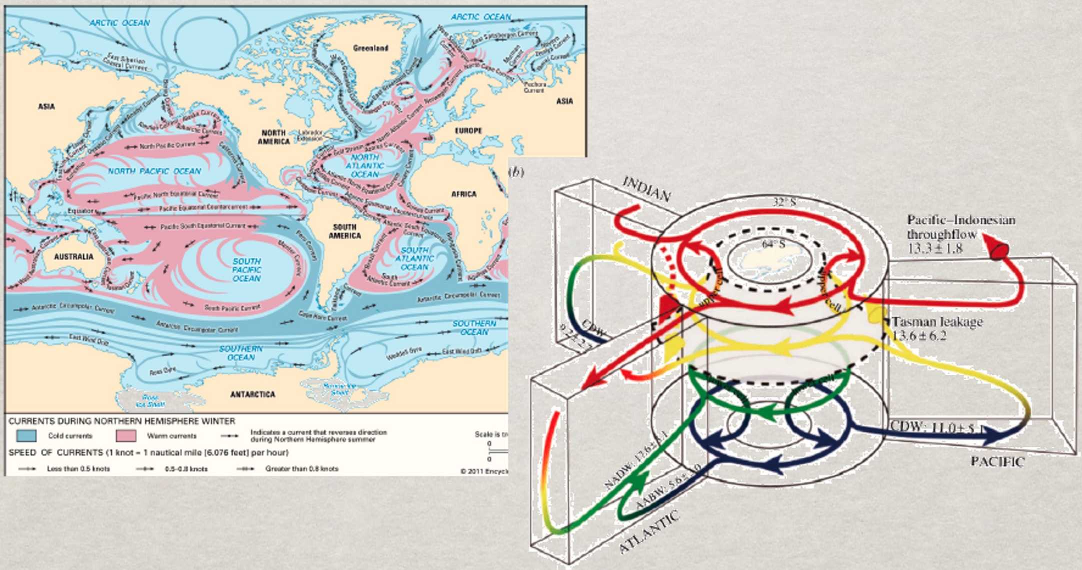

Thermo-haline convection

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Avalanche model

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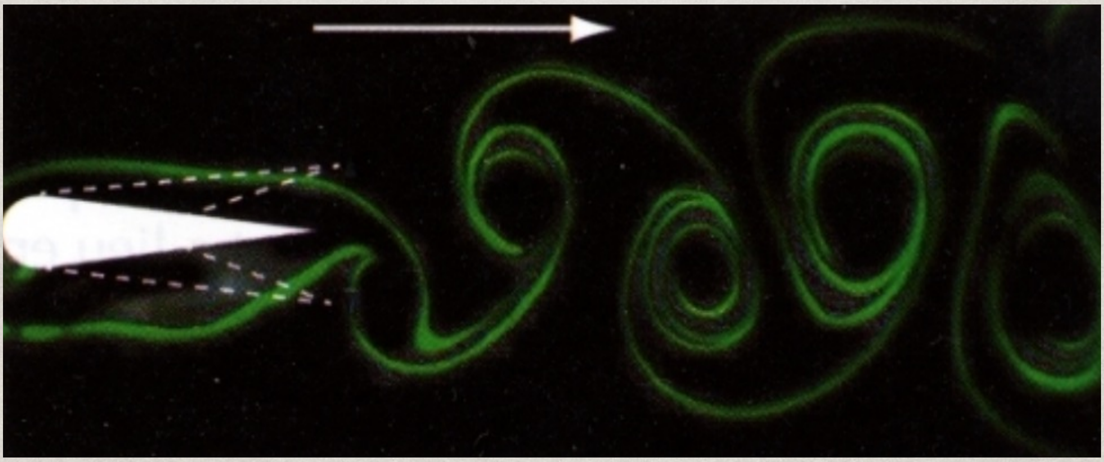

Propulsion

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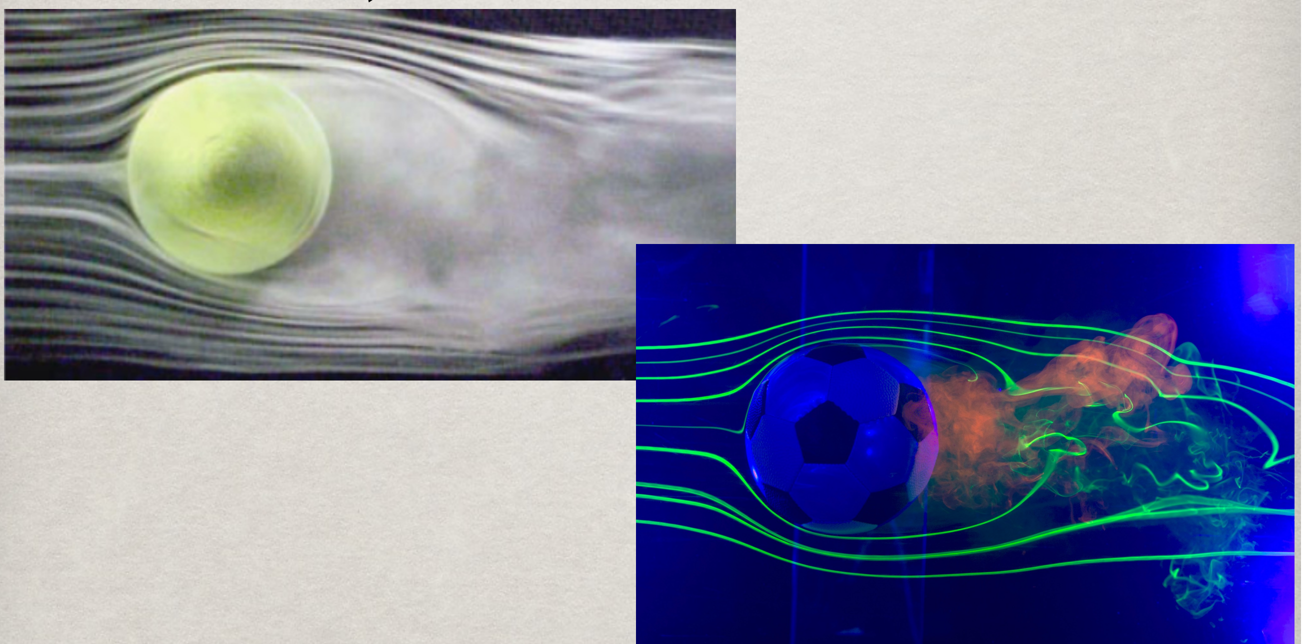

Tennis/Football lift, Magnus effect

Final Projects

Your own idea…