Tutorial 4

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Exercise 1: Let us consider the heat equation in an open bounded domain $\Omega \subset \mathbb{R}^n$, over a time interval $[0, T]$, with a homogeneous Dirichlet boundary condition

$$\begin{cases} L(u) = \frac{\partial u}{\partial t} - \kappa \frac{\partial^2 u}{\partial x^2} & \text{if } (x,t) \in \Omega \times (0,T] \\ u = 0 & \text{if } \partial\Omega \times (0,T] \\ u(x,0) = u_0(x) & \text{if } x \in \Omega \end{cases} \tag{1}$$

We remind the *maximum principle* for this continuous problem, i.e.

$$\min\left(0, \inf_{x\in\Omega} u_0(x)\right) \leq u(x,t) \leq \max\left(0, \sup_{x\in\Omega} u_0(x)\right), \quad \forall (x,t) \in \Omega \times (0,T]$$

1. We consider Eq. (1), and want to show that the explicit Euler scheme

$$L_h(u) = \frac{u_j^{n+1} - u_j^n}{\Delta t} - \kappa \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} = 0$$

satisfies, under some conditions (to be determined) the discrete maximum principle, i.e.

$$\min\left(0, \min_{j\in[1,N]} u_j^0\right) \leq u_j^n \leq \max\left(0, \max_{j\in[1,N]} u_j^0\right), \quad \forall j \in [1,N] \times [0,M]$$

for all initial condition u_0. In other words, if the initial data is bounded by two constants $m \leq 0 \leq M$,

$$m \leq u_j^0 \leq M, \quad \forall j \in [1,N]$$

then

$$m \leq u_j^n \leq M, \quad \forall j \in [1,N] \wedge n \geq 0$$

Solution: We proceed by induction. The case $n = 0$, it's already given so we assume that the result holds for n and we prove it for $n+1$. Let $\lambda := \frac{\Delta t}{(\Delta x)^2}\kappa$ and suppose we have $\lambda < \frac{1}{2}$. Then:

$$
\begin{gathered}
u_j^n + \lambda\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) = u_j^{n+1} = u_j^n + \lambda\left(u_{j+1}^n - 2u_j^n + u_{j-1}^n\right) \\
(1-2\lambda)u_j^n + \lambda\left(u_{j+1}^n + u_{j-1}^n\right) = u_j^{n+1} = (1-2\lambda)u_j^n + \lambda\left(u_{j+1}^n + u_{j-1}^n\right) \\
(1-2\lambda)m + \lambda(m+m) \leq u_j^{n+1} \leq (1-2\lambda)M + \lambda(M+M) \\
m \leq u_j^{n+1} \leq M
\end{gathered}
$$

2. Let us define the following discrete norms, for all $1 \leq p \leq \infty$,

$$
\|u^n\|_p = \left(\sum_{j=1}^{N} |u_j^n|^p \Delta x\right)^{\frac{1}{p}}
$$

and $\|u^n\|_\infty = \max_{j\in[1,N]} |u_j^n|$. A numerical scheme is said to be stable in the $\|\cdot\|$ norm if there exists a constant $K > 0$, independent of the Δx and Δt, such that for all initial conditions u_0,

$$
\|u^n\| \leq K\ \|u^0\|, \quad \forall n \geq 0 \tag{2}
$$

Deduce the stability condition in the L^∞ norm for the explicit Euler scheme. Compare to the criterion derived during the lecture for stability in the L^2 norm.

Solution: We have that

$$
\begin{aligned}
\|u^{n+1}\|_\infty &= \max_{j\in[1,N]} |u_j^{n+1}| \\
&\leq |1-2\lambda| \max_{j\in[1,N]} |u_j^n| + \lambda \max_{j\in[0,N-1]} |u_j^n| + \lambda \max_{j\in[2,N+1]} |u_{j+1}^n| \\
&= |1-2\lambda|\ \|u^n\|_\infty + 2\lambda\ \|u^n\|_\infty
\end{aligned}
$$

Now if $\lambda \leq \frac{1}{2}$, we get $\|u^{n+1}\|_\infty \leq \|u^n\|_\infty$, which iteratively implies $\|u^n\|_\infty \leq \|u^0\|_\infty$ (with $K = 1$). Now assume $\lambda > \frac{1}{2}$. Assume we have an initial condition u_0 such that $u_0(i\Delta x) = c > 0$ for some i and $u_0(j\Delta x) = 0\ \forall j \neq i$. Then:

$$
u_i^1 = (1-2\lambda)u_i^0 + \lambda(u_{i+1}^0 + u_{i-1}^0) = (1-2\lambda)c < 0
$$

Now, the scheme is consistent and by the maximum principle for the continuous problem we have that $0 \leq v(x,t) \leq c$, where v is the solution of the continuous problem. Then, by the Lax equivalence theorem, we cannot have convergence of the scheme, and therefore it is not stable. Thus, the stability condition for the explicit Euler scheme in the L^∞ norm is $\lambda \leq \frac{1}{2}$.

Exercise 2: We now want to perform the same analysis of the L^∞ stability but now for the implicit Euler scheme. We want to show that

$$
L_h(u) = \frac{u_j^{n+1} - u_j^n}{\Delta t} - \kappa \frac{u_{j+1}^{n+1} - 2u_j^{n+1} + u_{j-1}^{n+1}}{\Delta x^2} = 0 \tag{3}
$$

satisfies the discrete maximum principle, this time with no condition on Δx and Δt. We consider Dirichlet boundary conditions, i.e. that formula Eq. (3) holds for $j \in [2, N-1]$, and we impose $u_1^n = u_N^n = 0$ for all n. Show, by relating u_j^{n+1} to u_j^n, that if $m \leq 0 \leq M$ are two constants such that

$$
m \leq u_j^0 \leq M, \quad \forall j \in [1, N]
$$

then

$$m \leq u_j^n \leq M, \quad \forall j \in [1, N] \wedge n \geq 0$$

Solution: Doing a similar analysis as in Exercise 1, we can write the implicit scheme as

$$\boldsymbol{A}\boldsymbol{u}^{n+1} = \boldsymbol{u}^n$$

with

$$\boldsymbol{A} = \begin{pmatrix} 1+2\lambda & -\lambda & 0 & \cdots & 0 \\ -\lambda & 1+2\lambda & -\lambda & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -\lambda & 1+2\lambda & -\lambda \\ 0 & \cdots & 0 & -\lambda & 1+2\lambda \end{pmatrix} = \boldsymbol{I} + \lambda \boldsymbol{B}$$

where $\boldsymbol{B} = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$. It suffices that the eigenvalues of $\boldsymbol{A}^{-1}$ are positive and less than 1 to have the discrete maximum principle. It can be seen (check [1]) that the eigenvalues of the tridiagonal matrix $\boldsymbol{B}$ are

$$\lambda_k = 2 - 2\cos\left(\frac{k\pi}{N+1}\right), \quad k = 1, ..., N$$

which implies that the eigenvalues of $\boldsymbol{A}^{-1}$ are

$$\mu_k = \frac{1}{1 + \lambda\lambda_k} = \frac{1}{1 + \lambda\left(2 - 2\cos\left(\frac{k\pi}{N+1}\right)\right)}, \quad k = 1, ..., N$$

which are positive and less or equal to 1. Thus, the implicit scheme satisfies the discrete maximum principle.

Exercise 3: We want to demosntrate Lax equivalence theorem for the heat equation using a linear finite difference scheme, with two levels. We introduce the solution $u(x, t)$ (supposed to be regular) of the heat equation Eq. (1), and u_j^n the numerical approached solution using finite differences with $u_j^0 = u_0(x_j)$. We want to show that if the scheme is consistent and stable for a given norm $\|\cdot\|$, then the scheme is convergent in the sense that the error vector $e_j^n = u_j^n - u(x_j, t_n)$ satisfies

$$\lim_{\Delta x, \Delta t \to 0} \left(\sup_{n\Delta t \leq T} \|e^n\|\right) = 0 \quad \forall T > 0$$

We will suppose the existence and uniqueness of the solution to the heat equation and use the definition of stability Eq. (2).

Solution: From the hypothesis we can write our problem as:

$$\boldsymbol{A}_1 \boldsymbol{u}^{n+1} = \boldsymbol{A}_2 \boldsymbol{u}^n$$

where we assume that the matrix $\boldsymbol{A}_1$ is invertible (otherwise the problem is ill-posed or it has no uniqueness of solutions). Without loss of generality we may assume $\left\|\boldsymbol{A}_1^{-1}\right\| \leq C\Delta t$ (we can assume this either to $\boldsymbol{A}_1^{-1}$ or to $\boldsymbol{A}_2$). Let v be the exact solution of the continuous problem and $\boldsymbol{v}$

the vector composed of the values of v at the grid points. We define the truncation error at time n as $\boldsymbol{T}^n = \boldsymbol{A}_1 \boldsymbol{v}^{n+1} - \boldsymbol{A}_2 \boldsymbol{v}^n$. We have that

$$\begin{aligned} \boldsymbol{u}^{n+1} &= \boldsymbol{A}_1^{-1} \boldsymbol{A}_2 \boldsymbol{u}^n \\ \boldsymbol{v}^{n+1} &= \boldsymbol{A}_1^{-1} (\boldsymbol{A}_2 \boldsymbol{v}^n + \boldsymbol{T}^n) \end{aligned}$$

Let $\boldsymbol{B} := \boldsymbol{A}_1^{-1} \boldsymbol{A}_2$. We then have:

$$\begin{aligned} \boldsymbol{u}^{n+1} - \boldsymbol{v}^{n+1} &= \boldsymbol{B}(\boldsymbol{u}^n - \boldsymbol{v}^n) - \boldsymbol{A}_1^{-1} \boldsymbol{T}^n \\ &\leq \boldsymbol{B}(\boldsymbol{u}^n - \boldsymbol{v}^n) - \boldsymbol{A}_1^{-1} \boldsymbol{T}^n \\ &\leq \boldsymbol{B}^2 (\boldsymbol{u}^{n-1} - \boldsymbol{v}^{n-1}) - \boldsymbol{B} \boldsymbol{A}_1^{-1} \boldsymbol{T}^{n-1} - \boldsymbol{A}_1^{-1} \boldsymbol{T}^n \\ &\leq \boldsymbol{B}^{n+1} (\boldsymbol{u}^0 - \boldsymbol{v}^0) - \sum_{k=0}^{n} \boldsymbol{B}^k \boldsymbol{A}_1^{-1} \boldsymbol{T}^{n-k} \\ &= - \sum_{k=0}^{n} \boldsymbol{B}^k \boldsymbol{A}_1^{-1} \boldsymbol{T}^{n-k} \end{aligned} \tag{4}$$

If we assume that the scheme is consistent of order p in space and q in time we have that $\|\boldsymbol{T}^{\boldsymbol{n}}\| = \mathcal{O}((\Delta t)^q + (\Delta x)^p)$. Now, it can be seen that stability condition is equivalent to having $\|\boldsymbol{B}^k\| \leq K \ \forall k \in \mathbb{N}$ and K given in Eq. (2). Indeed, if we have Eq. (2), then:

$$\boldsymbol{u}^k = \boldsymbol{B} \boldsymbol{u}^{k-1} = ... = \boldsymbol{B}^k \boldsymbol{u}^0 \Longrightarrow \|\boldsymbol{B}^k \boldsymbol{u}^0\| = \|\boldsymbol{u}^k\| \leq K \|\boldsymbol{u}^0\|$$

which from the definition of matrix norm implies that $\|\boldsymbol{B}^k\| \leq C$, and this is valid $\forall k$ such that $k \Delta t \leq T$. Now suppose that we have $\|\boldsymbol{B}^k\| \leq K$, then:

$$\|\boldsymbol{u}^k\| = \|\boldsymbol{B}^k \boldsymbol{u}^0\| \leq K \|\boldsymbol{u}^0\|$$

Thus, the scheme is stable. With that in mind, continuing from Eq. (4), we have that

$$\begin{aligned} \|\boldsymbol{u}^{n+1} - \boldsymbol{v}^{n+1}\| &\leq \sum_{k=0}^{n} \|\boldsymbol{B}^k\| \|\boldsymbol{A}_1^{-1}\| \|\boldsymbol{T}^{n-k}\| \\ &\leq K C \Delta t \sum_{k=0}^{n} \|\boldsymbol{T}^{n-k}\| \\ &= K C (n+1) \Delta t \mathcal{O}((\Delta t)^q + (\Delta x)^p) \\ &\leq K C (T + \Delta t) \mathcal{O}((\Delta t)^q + (\Delta x)^p) \end{aligned}$$

which goes to zero as Δt and Δx go to zero at the same order as the consistency of the scheme.

Problem: We now consider the following one dimensional problem

$$\frac{\partial^2 u}{\partial x^2} = -f, \quad \forall x \in (0,1) \tag{5}$$

where $f \in \mathcal{C}^1([0,1], \mathbb{R})$, with homogeneous Dirichlet boundary conditions $u(0) = u(1) = 0$. We now introduce a Finite Volume discretization of the $[0,1]$ interval:

$$(\Omega_j)_{j=1,\dots,N} \quad \text{defined by} \quad \Omega_j = \left(x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}\right),$$

with

$$x_{\frac{1}{2}} = 0 < x_{\frac{3}{2}} < \ldots < x_{j-\frac{1}{2}} < x_{j+\frac{1}{2}} < \ldots < x_{N-\frac{1}{2}} < x_{N+\frac{1}{2}} = 1$$

Finally, we consider the dual grid composed of N points $\left(x_j\right)_{j=1,\ldots,N}$ such that

$$x_{\frac{1}{2}} = 0 < x_1 < x_{\frac{3}{2}} < \ldots < x_{j-\frac{1}{2}} < x_j < x_{j+\frac{1}{2}} < \ldots < x_N < x_{N+\frac{1}{2}} = 1$$

We introduce $h_{j+\frac{1}{2}} = x_{j+1} - x_j$ and $h_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, where for simplicity we set $x_0 = 0$ and $x_{N+1} = 1$.

1. By integrating Eq. (5) over Ω_j, write the discrete scheme in the finite volume sense for the unknows u_j. We will define f_j for the discrete right-hand side of Eq. (5) and we will introduce $\tilde{F}_{j+\frac{1}{2}}$ to identify the numerical approximate fluxes in $x_{j+\frac{1}{2}}$.

Solution: Recall that we define u_j as the average of u over Ω_j, i.e. $u_j = \frac{1}{h_j} \int_{\Omega_j} u(x)\,\mathrm{d}x$. Integrating Eq. (5) over Ω_j we get:

$$\begin{gathered}\int_{\Omega_j} \frac{\partial^2 u}{\partial x^2}\,\mathrm{d}x = -\int_{\Omega_j} f\,\mathrm{d}x \\ \frac{\partial u}{\partial x}\left(x_{j+\frac{1}{2}}\right) - \frac{\partial u}{\partial x}\left(x_{j-\frac{1}{2}}\right) = -\overline{f}_j h_j\end{gathered} \tag{6}$$

where $\overline{f}_j = \frac{1}{h_j} \int_{\Omega_j} f\,\mathrm{d}x$. Using centered finite differences for the derivative, we get:

$$\frac{u_{j+1} - u_j}{h_{j+\frac{1}{2}}} - \frac{u_j - u_{j-1}}{h_{j-\frac{1}{2}}} = -\overline{f}_j h_j$$

2. Under which hypothesis on the mesh is the scheme consistent? (Important: we will not make this hypothesis below)

Solution: From one side, $\overline{f}_j$ tends to $f\left(x_j\right)$ as h_j tends to zero. Now, let $h_j^+ := x_{j+\frac{1}{2}} - x_j$, $h_j^- := x_j - x_{j-\frac{1}{2}}$, $h_{j+\frac{1}{2}}^+ := x_{j+1} - x_{j+\frac{1}{2}}$ and $h_{j+\frac{1}{2}}^- := x_{j+\frac{1}{2}} - x_j$. Then, $h_j = h_j^+ + h_j^-$ and $h_{j+\frac{1}{2}} = h_{j+\frac{1}{2}}^+ + h_{j+\frac{1}{2}}^-$. Taylor-expanding arround $\frac{\partial u}{\partial x}\left(x_j\right)$ we have:

$$\begin{aligned}\frac{\partial u}{\partial x}\left(x_{j+\frac{1}{2}}\right) &= \frac{\partial u}{\partial x}\left(x_j\right) + \frac{\partial^2 u}{\partial x^2}\left(x_j\right) h_j^+ + \frac{\partial^3 u}{\partial x^3}\left(x_j\right) \frac{\left(h_j^+\right)^2}{2} + \mathcal{O}\left(\left(h_j^+\right)^2\right) \\ \frac{\partial u}{\partial x}\left(x_{j-\frac{1}{2}}\right) &= \frac{\partial u}{\partial x}\left(x_j\right) - \frac{\partial^2 u}{\partial x^2}\left(x_j\right) h_j^- + \frac{\partial^3 u}{\partial x^3}\left(x_j\right) \frac{\left(h_j^-\right)^2}{2} + \mathcal{O}\left(\left(h_j^-\right)^3\right)\end{aligned}$$

This implies:

$$
\begin{aligned}
\frac{\partial^2 u}{\partial x^2}(x_j) &= \frac{\frac{\partial u}{\partial x}\left(x_{j+\frac{1}{2}}\right) - \frac{\partial u}{\partial x}\left(x_{j-\frac{1}{2}}\right)}{h_j} + \frac{1}{h_j}\frac{1}{2}\frac{\partial^3 u}{\partial x^3}(x_j)\left(h_j^2 - 2h_j h_j^-\right) + \mathcal{O}\left(\left(h_j^+\right)^2\right) + \mathcal{O}\left(\left(h_j^-\right)^2\right) \\
&= \frac{\frac{\partial u}{\partial x}\left(x_{j+\frac{1}{2}}\right) - \frac{\partial u}{\partial x}\left(x_{j-\frac{1}{2}}\right)}{h_j} + \frac{1}{2}\frac{\partial^3 u}{\partial x^3}(x_j)\left(h_j - 2h_j^-\right) + \mathcal{O}\left(\left(h_j^+\right)^2\right) + \mathcal{O}\left(\left(h_j^-\right)^2\right) \\
&= \frac{1}{h_j}\left[\frac{u_{j+1} - u_j}{h_{j+\frac{1}{2}}} - \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\left(x_{j+\frac{1}{2}}\right)\left(h_{j+\frac{1}{2}}^+ - h_{j+\frac{1}{2}}^-\right) + \mathcal{O}\left(\left(h_{j+\frac{1}{2}}\right)^2\right) - \right. \\
&\qquad \left. - \frac{u_j - u_{j-1}}{h_{j-\frac{1}{2}}} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\left(x_{j-\frac{1}{2}}\right)\left(h_{j-\frac{1}{2}}^+ - h_{j-\frac{1}{2}}^-\right) + \mathcal{O}\left(\left(h_{j-\frac{1}{2}}\right)^2\right)\right] + \mathcal{O}(h_j)
\end{aligned}
$$

Thus, in order to have a consistent scheme we need to have:

$$
h_{j+\frac{1}{2}}^+ = h_{j+\frac{1}{2}}^-, \quad \forall j = 0, \dots, N
$$

3. We now want to prove uniqueness of the solution for the discrete system obtained in question 1. We note that this is equivalent to showing that the only solution to this problem with no right-hand side is uniformly zero $u_j = 0 \ \forall j$. To this end, we suggest to multiply the expression obtained at question 1 by u_j in order to introduce squared quantities.

Solution: We have the scheme:

$$
\frac{u_{j+1} - u_j}{h_{j+\frac{1}{2}}} - \frac{u_j - u_{j-1}}{h_{j-\frac{1}{2}}} = 0, \quad \forall j = 1, \dots, N
$$

Multiplying by u_j we get:

$$
\frac{u_j u_{j+1}}{h_{j+\frac{1}{2}}} - u_j^2\left(\frac{1}{h_{j+\frac{1}{2}}} + \frac{1}{h_{j-\frac{1}{2}}}\right) + \frac{u_{j-1} u_j}{h_{j-\frac{1}{2}}} = 0, \quad \forall j = 1, \dots, N
$$

Summing over j, and using the fact that $u_0 = u_{N+1} = 0$, we get:

$$
\begin{aligned}
0 &= \sum_{j=1}^{N}\left[\frac{u_j u_{j+1}}{h_{j+\frac{1}{2}}} - u_j^2\left(\frac{1}{h_{j+\frac{1}{2}}} + \frac{1}{h_{j-\frac{1}{2}}}\right) + \frac{u_{j-1} u_j}{h_{j-\frac{1}{2}}}\right] \\
&= \sum_{j=1}^{N-1}\frac{u_j u_{j+1}}{h_{j+\frac{1}{2}}} - \frac{u_N^2}{h_{N+\frac{1}{2}}} - \sum_{j=1}^{N-1}\frac{u_j^2}{h_{j+\frac{1}{2}}} - \frac{u_1^2}{h_{\frac{1}{2}}} - \sum_{j=2}^{N}\frac{u_j^2}{h_{j-\frac{1}{2}}} + \sum_{j=2}^{N}\frac{u_{j-1} u_j}{h_{j-\frac{1}{2}}} \\
&= -\frac{u_N^2}{h_{N+\frac{1}{2}}} - \frac{u_1^2}{h_{\frac{1}{2}}} + \frac{1}{h_{j+\frac{1}{2}}}\sum_{j=1}^{N-1}\left(2u_j u_{j+1} - u_j^2 - u_{j+1}^2\right) \\
&= -\frac{u_N^2}{h_{N+\frac{1}{2}}} - \frac{u_1^2}{h_{\frac{1}{2}}} - \frac{1}{h_{j+\frac{1}{2}}}\sum_{j=1}^{N-1}\left(u_{j+1} - u_j\right)^2
\end{aligned}
$$

which implies that all the terms are zero, and thus $u_j = 0 \ \forall j$.

4. Let $u \in \mathcal{C}^2([0,1], \mathbb{R})$ be a solution of Eq. (5). Introducing

$$
\mathcal{F}_{j+\frac{1}{2}} = -\frac{\partial u}{\partial x}\left(x_{j+\frac{1}{2}}\right)
$$

the exact flux in $x_{j+\frac{1}{2}}$ and $h = \max_{j(h_j)}$ show the "consistancy of fluxes", in the sense that:

$$|\mathcal{F}_{j+\frac{1}{2}} - \tilde{F}_{j+\frac{1}{2}}| \leq C_1 h \quad \text{with } C_1 > 0$$

Under which hypothesis on the mesh is the approximation 2nd order? That is to say, satisfying

$$|\mathcal{F}_{j+\frac{1}{2}} - \tilde{F}_{j+\frac{1}{2}}| \leq C_2 h^2 \quad \text{with } C_2 > 0$$

Solution: Using Eq. (6) the numerical fluxes are

$$\tilde{F}_{j+\frac{1}{2}} = -\frac{u_{j+1} - u_j}{h_{j+\frac{1}{2}}}, \quad \forall j = 1, ..., N$$

and we define $F_{j+\frac{1}{2}} := -\frac{u(x_{j+1}) - u(x_j)}{h_{j+\frac{1}{2}}}$. Recall the definitions of $h^+_{j+\frac{1}{2}}$ and $h^-_{j+\frac{1}{2}}$. Then:

$$u(x_{j+1}) = u\left(x_{j+\frac{1}{2}}\right) + \frac{\partial u}{\partial x}\left(x_{j+\frac{1}{2}}\right) h^+_{j+\frac{1}{2}} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\left(\xi_{j+\frac{1}{2}}\right)\left(h^+_{j+\frac{1}{2}}\right)^2$$
$$u(x_j) = u\left(x_{j+\frac{1}{2}}\right) - \frac{\partial u}{\partial x}\left(x_{j+\frac{1}{2}}\right) h^-_{j+\frac{1}{2}} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\left(\eta_{j+\frac{1}{2}}\right)\left(h^-_{j+\frac{1}{2}}\right)^2$$

for some $\xi_{j+\frac{1}{2}} \in \left(x_{j+\frac{1}{2}}, x_{j+1}\right)$ and $\eta_{j+\frac{1}{2}} \in \left(x_j, x_{j+\frac{1}{2}}\right)$. Substracting the equations and dividing by $h_{j+\frac{1}{2}}$ we get:

$$\begin{aligned}
\left|\mathcal{F}_{j+\frac{1}{2}} - F_{j+\frac{1}{2}}\right| &= \left|\frac{\partial u}{\partial x}\left(x_{j+\frac{1}{2}}\right) - \frac{u(x_{j+1}) - u(x_j)}{h_{j+\frac{1}{2}}}\right| \\
&= \frac{1}{h_{j+\frac{1}{2}}}\left|C_{11}\left(h^+_{j+\frac{1}{2}}\right)^2 - C_{12}\left(h^-_{j+\frac{1}{2}}\right)^2\right| \\
&\leq \frac{1}{h_{j+\frac{1}{2}}}\left(\tilde{C}_{11}\left[\left(h^+_{j+\frac{1}{2}}\right)^2 - \left(h^-_{j+\frac{1}{2}}\right)^2\right] + \tilde{C}_{12}\left(h^-_{j+\frac{1}{2}}\right)^2\right) \\
&\leq \tilde{C}_{11}\left(h^+_{j+\frac{1}{2}} - h^-_{j+\frac{1}{2}}\right) + \tilde{C}_{12} h^-_{j+\frac{1}{2}} \\
&\leq C_1 h
\end{aligned}$$

for some $C_1, \tilde{C}_{11}, \tilde{C}_{12} > 0$. If we want to prove 2nd order consistency, we need $u \in \mathcal{C}^3([0,1], \mathbb{R})$. In that case we have:

$$u(x_{j+1}) = u\left(x_{j+\frac{1}{2}}\right) + \frac{\partial u}{\partial x}\left(x_{j+\frac{1}{2}}\right) h^+_{j+\frac{1}{2}} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\left(x_{j+\frac{1}{2}}\right)\left(h^+_{j+\frac{1}{2}}\right)^2 + \mathcal{O}\left(\left(h^+_{j+\frac{1}{2}}\right)^3\right)$$
$$u(x_j) = u\left(x_{j+\frac{1}{2}}\right) - \frac{\partial u}{\partial x}\left(x_{j+\frac{1}{2}}\right) h^-_{j+\frac{1}{2}} + \frac{1}{2}\frac{\partial^2 u}{\partial x^2}\left(x_{j+\frac{1}{2}}\right)\left(h^-_{j+\frac{1}{2}}\right)^2 + \mathcal{O}\left(\left(h^-_{j+\frac{1}{2}}\right)^3\right)$$

Substracting the equations, we notice that to cancel out the second order term we need to have $h^+_{j+\frac{1}{2}} = h^-_{j+\frac{1}{2}}$. The terms for the 3rd order coefficients follow as in the previous case.

5. We now want to show convergence of the scheme (even when consistancy is not met). We suppose that the solution of Eq. (5) is in $\mathcal{C}^2([0,1], \mathbb{R})$ and introduce the error $e_j = u(x_j) - u_j$ for $j = 1, ..., N$, $e_0 = e_{N+1} = 0$. Using the consistancy of fluxes and Cauchy-Schwarz inequality, show that there exists $C_3 \geq 0$ such that

$$\sum_{j=0}^{N} \frac{\left(e_{j+1} - e_j\right)^2}{h_{j+\frac{1}{2}}} \leq C_3^2 h^2$$

Solution: From the previous work we have the following equalities:

$$\begin{aligned} \mathcal{F}_{j+\frac{1}{2}} - \mathcal{F}_{j-\frac{1}{2}} &= \overline{f}_j h_j \\ \tilde{F}_{j+\frac{1}{2}} - \tilde{F}_{j-\frac{1}{2}} &= \overline{f}_j h_j \\ \mathcal{F}_{j+\frac{1}{2}} - F_{j+\frac{1}{2}} &= T_{j+\frac{1}{2}} \quad \text{with } |T_{j+\frac{1}{2}}| \leq C_1 h \end{aligned} \tag{7}$$

Now we have that:

$$F_{j+\frac{1}{2}} - \tilde{F}_{j+\frac{1}{2}} = -\frac{e_{j+1} - e_j}{h_{j+\frac{1}{2}}} \tag{8}$$

Thus, putting together Eq. (7) and Eq. (8) we get:

$$\frac{e_{j+1} - e_j}{h_{j+\frac{1}{2}}} - \frac{e_j - e_{j-1}}{h_{j-\frac{1}{2}}} = -T_{j+\frac{1}{2}} + T_{j-\frac{1}{2}}, \quad \forall j = 1, ..., N$$

Multiplying this last expression by e_j and summing over all j we get:

$$\sum_{j=1}^{N} \frac{\left(e_{j+1} - e_j\right)e_j}{h_{j+\frac{1}{2}}} - \sum_{j=1}^{N} \frac{\left(e_j - e_{j-1}\right)e_j}{h_{j-\frac{1}{2}}} = -\sum_{j=1}^{N} T_{j+\frac{1}{2}} e_j + \sum_{j=1}^{N} T_{j-\frac{1}{2}} e_j$$

$$\left(e_{N+1} - e_N\right)\frac{e_N}{h_{N+\frac{1}{2}}} - \sum_{j=1}^{N-1} \frac{\left(e_{j+1} - e_j\right)e_j}{h_{j+\frac{1}{2}}} - \left(e_1 - e_0\right)\frac{e_1}{h_{\frac{1}{2}}} - \sum_{j=1}^{N-1} \frac{\left(e_{j+1} - e_j\right)e_j}{h_{j+\frac{1}{2}}} =$$

$$= -T_{N+\frac{1}{2}} e_N - \sum_{j=1}^{N} T_{j+\frac{1}{2}} e_j + T_{\frac{1}{2}} e_1 + \sum_{j=1}^{N-1} T_{j+\frac{1}{2}} e_{j+1}$$

$$-\frac{e_N^2}{h_{N+\frac{1}{2}}} - \frac{e_1^2}{h_{\frac{1}{2}}} - \sum_{j=1}^{N-1} \frac{\left(e_{j+1} - e_j\right)^2}{h_{j+\frac{1}{2}}} = -T_{N+\frac{1}{2}} e_N + T_{\frac{1}{2}} e_1 + \sum_{j=1}^{N-1} T_{j+\frac{1}{2}}\left(e_{j+1} - e_j\right)$$

$$\sum_{j=0}^{N} \frac{\left(e_{j+1} - e_j\right)^2}{h_{j+\frac{1}{2}}} = -\sum_{j=0}^{N} T_{j+\frac{1}{2}}\left(e_{j+1} - e_j\right)$$

Finally taking absolutes values an using Cauchy-Schwarz inequality we get:

$$\begin{aligned} \sum_{j=0}^{N} \frac{\left(e_{j+1} - e_j\right)^2}{h_{j+\frac{1}{2}}} &\leq C_1 h \sum_{j=0}^{N} \frac{|e_{j+1} - e_j|}{\sqrt{h_{j+\frac{1}{2}}}} \sqrt{h_{j+\frac{1}{2}}} \\ &\leq C_1 h \sqrt{\sum_{j=0}^{N} \frac{\left(e_{j+1} - e_j\right)^2}{h_{j+\frac{1}{2}}}} \sqrt{\sum_{j=0}^{N} h_{j+\frac{1}{2}}} \\ &= C_1 h \sqrt{\sum_{j=0}^{N} \frac{\left(e_{j+1} - e_j\right)^2}{h_{j+\frac{1}{2}}}} \end{aligned}$$

where we have used that $\sum_{j=0}^{N} h_{j+\frac{1}{2}} = 1$. Thus:

$$\sum_{j=0}^{N} \frac{\left(e_{j+1} - e_j\right)^2}{h_{j+\frac{1}{2}}} \leq C_1^2 h^2$$

6. Noting that

$$|e_j| = \left| \sum_{k=1}^{j} e_k - e_{k-1} \right|$$

conclude that

$$|e_j| \leq C_3 h \quad \text{for } j = 1, ..., N$$

which implies convergence of the finite volume scheme for Eq. (5).

Solution: Fix $i \in \{1, ..., N\}$. We have that:

$$|e_i| = \left| \sum_{j=1}^{i} e_j - e_{j-1} \right| \leq \sum_{j=1}^{i} \frac{|e_{j+1} - e_j|}{\sqrt{h_{j+\frac{1}{2}}}} \sqrt{h_{j+\frac{1}{2}}} \leq \sqrt{\sum_{j=1}^{i} \frac{\left(e_{j+1} - e_j\right)^2}{h_{j+\frac{1}{2}}}} \sqrt{\sum_{j=1}^{i} h_{j+\frac{1}{2}}} \leq C_1 h$$

again by the Cauchy-Schwarz inequality, $\sum_{j=1}^{i} h_{j+\frac{1}{2}} \leq 1$ and the fact that:

$$\sum_{j=1}^{i} \frac{\left(e_{j+1} - e_j\right)^2}{h_{j+\frac{1}{2}}} \leq \sum_{j=0}^{N} \frac{\left(e_{j+1} - e_j\right)^2}{h_{j+\frac{1}{2}}} \leq C_1^2 h^2$$

Bibliography

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