Numerical Methods for Fluid Dynamics

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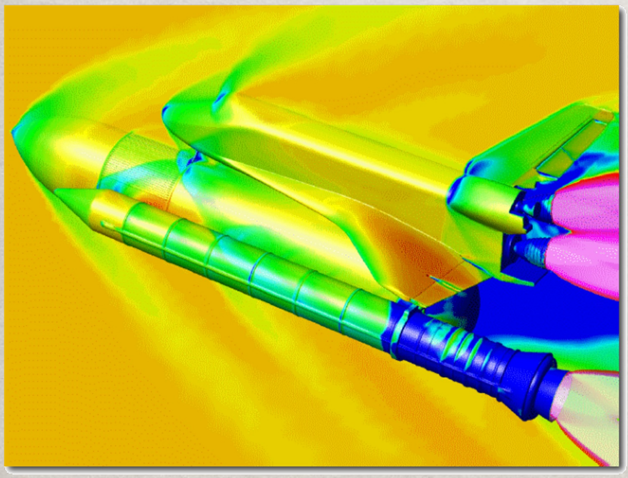
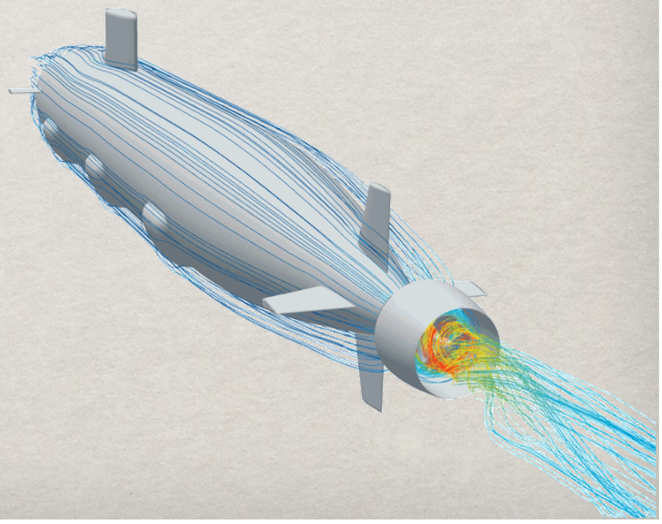
9. Flows in Complex Domains

CFD

Most industrial applications of computational fluid dynamics take place in complex geometries.





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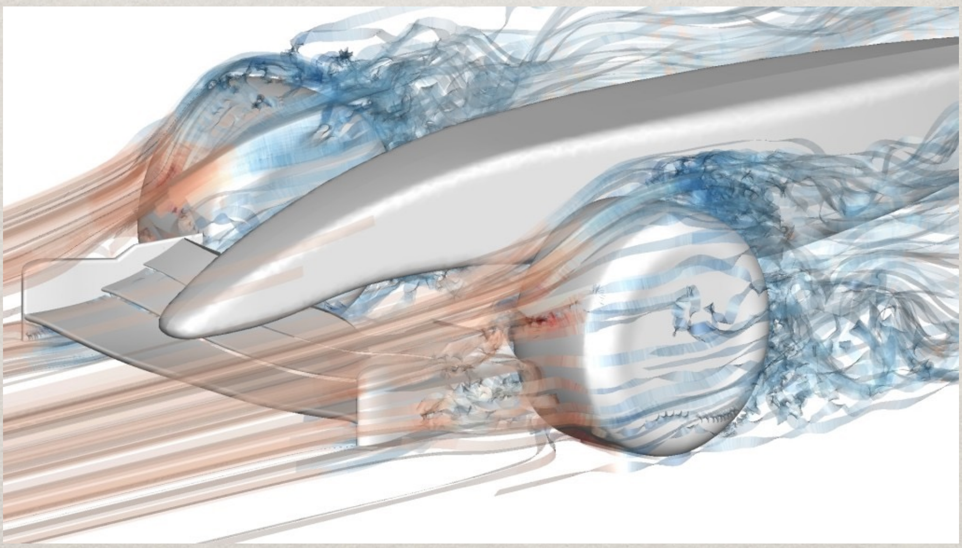

CFD

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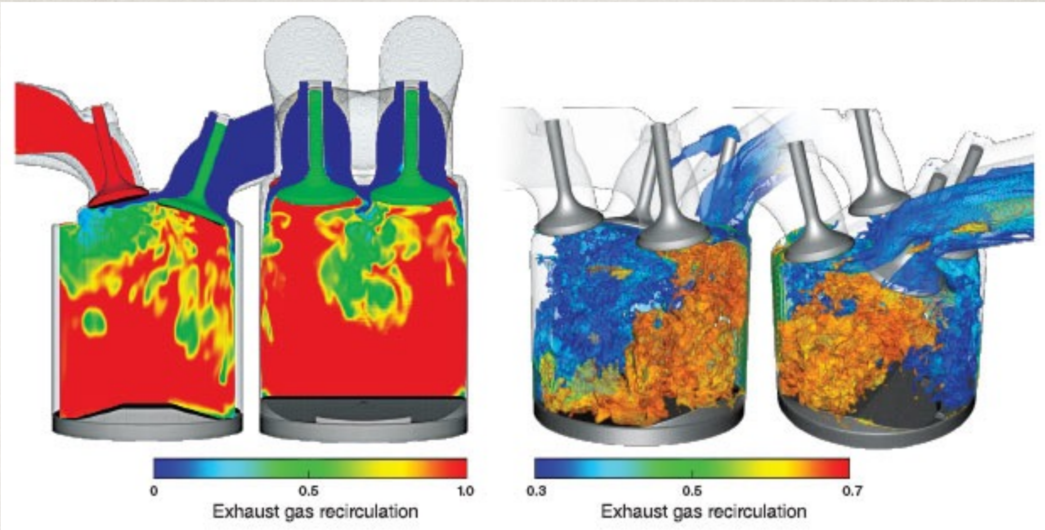

CFD

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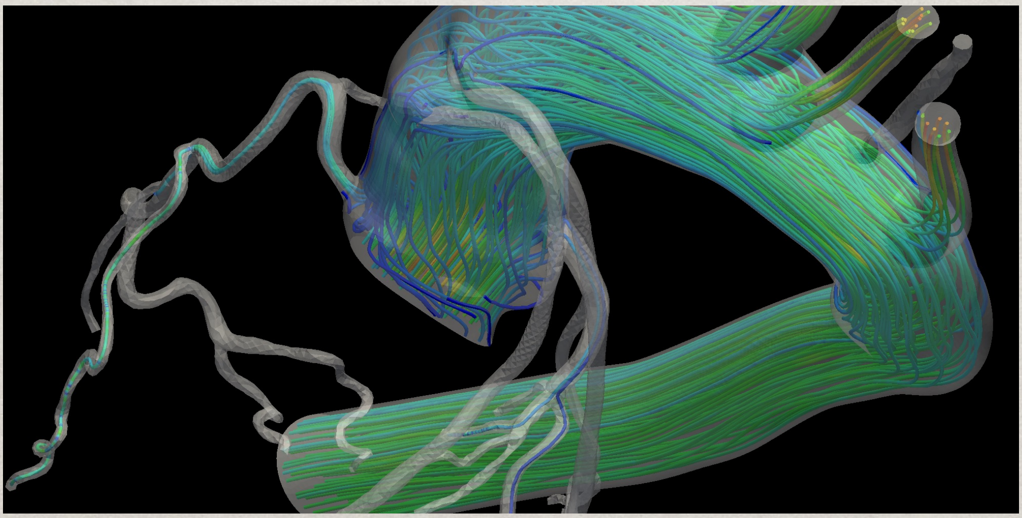

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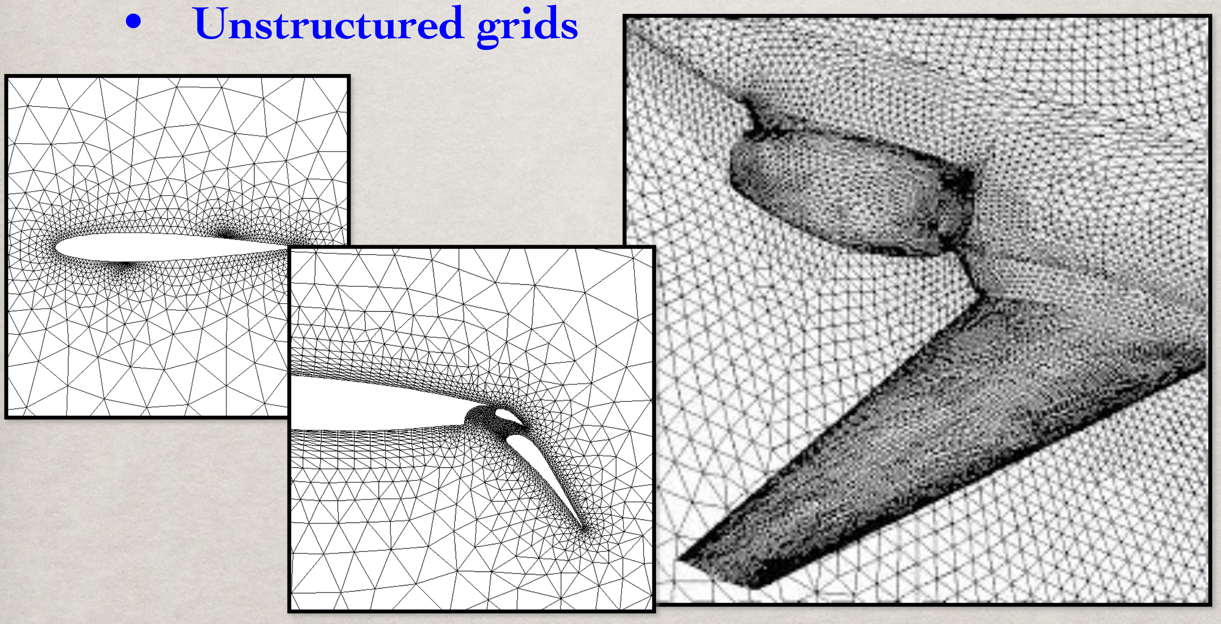
Two different approaches:

- Unstructured grids



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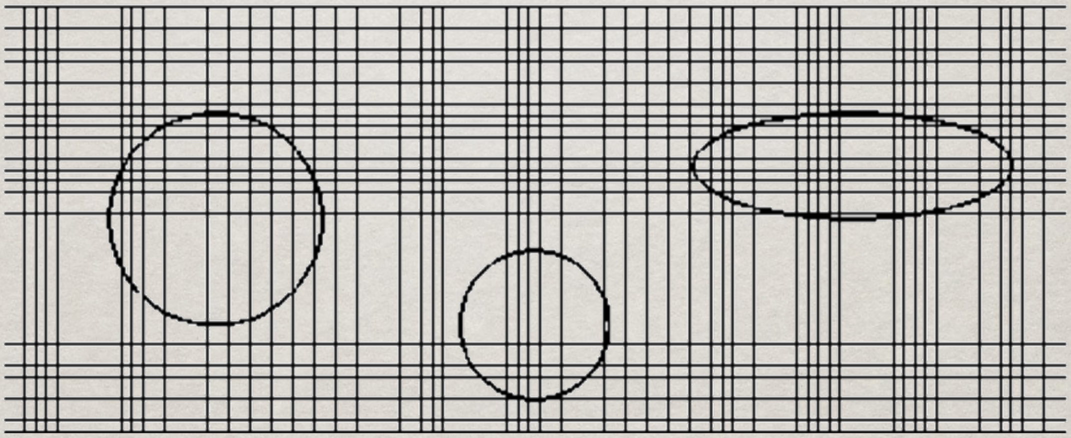
Two different approaches:

- Domain cutting

cut region from a Cartesian grid



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Variational form

$$\begin{cases} \Delta u = -f & \text{in} \quad \Omega\,, \\ u = 0 & \text{on} \quad \partial\Omega\,. \end{cases}$$

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Variational form

$$\begin{cases} \Delta u = -f & \text{in} \quad \Omega\,, \\ u = 0 & \text{on} \quad \partial\Omega\,. \end{cases}$$

$$H^1(\Omega) = \{w \in L^2(\Omega),\ \frac{\partial w}{\partial x},\ \frac{\partial w}{\partial y} \in L^2(\Omega)\}\,.$$

Variational form

$$\text{Find} \quad u \in H_0^1(\Omega)$$

$$\text{such that} \quad \int_\Omega \Delta u \, w \, \mathrm{d}\mathbf{x} = -\int_\Omega f \, w \, \mathrm{d}\mathbf{x}\,,$$

$$\forall w \in H_0^1(\Omega)$$

Variational form

$$\text{Find} \quad u \in H_0^1(\Omega)$$

$$\text{such that} \quad \int_\Omega \boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} w \, \mathrm{d}\mathbf{x} = \int_\Omega f w \, \mathrm{d}\mathbf{x}\,,$$

$$\forall w \in H_0^1(\Omega)$$

A simple 1D periodic example

$$\frac{\partial^2 u}{\partial x^2} = -f$$

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A simple 1D periodic example

$$\frac{\partial^2 u}{\partial x^2} = -f$$

$$\int_\Omega \frac{\partial^2 u}{\partial x^2} w \, \mathrm{d}x = -\int_\Omega f w \, \mathrm{d}x \qquad \forall w \in H^1(\Omega)$$

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A simple 1D periodic example

$$\frac{\partial^2 u}{\partial x^2} = -f$$

$$\int_\Omega \frac{\partial^2 u}{\partial x^2} w \,\mathrm{d}x = -\int_\Omega f w \,\mathrm{d}x \qquad \forall w \in H^1(\Omega)$$

$$\int_\Omega \frac{\partial u}{\partial x}\frac{\partial w}{\partial x} \,\mathrm{d}x = \int_\Omega f w \,\mathrm{d}x \qquad \forall w \in H^1(\Omega)$$

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Finite Element « shape functions »

$$\phi_i(x) = \begin{cases} 0 & \text{if} \quad & x \le x_{i-1}\,, \\ \dfrac{x - x_{i-1}}{x_i - x_{i-1}} & \text{if} \quad x_{i-1} < & x \le x_i\,, \\ \dfrac{x_{i+1} - x}{x_{i+1} - x_i} & \text{if} \quad x_i < & x \le x_{i+1}\,, \\ 0 & \text{if} \quad x_{i+1} < & x\,. \end{cases}$$

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Finite Element « shape functions »

$$u(x) = \sum_{i=1}^{N} u_i \phi_i(x)\,,$$

$$f(x) = \sum_{i=1}^{N} f_i \phi_i(x)\,,$$

$$w(x) = \sum_{i=1}^{N} w_i \phi_i(x)\,.$$

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Discrete problem

$$\int_\Omega \frac{\partial u}{\partial x}\frac{\partial w}{\partial x}\,\mathrm{d}x = \int_\Omega f w\,\mathrm{d}x \qquad \forall w \in H^1(\Omega)$$

$$\int_\Omega \sum_i u_i \frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x}\,\mathrm{d}x = \int_\Omega \sum_i f_i \phi_i \phi_j\,\mathrm{d}x\,, \qquad \forall j \in [1,\, N]\,.$$

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$$\int_\Omega \sum_i u_i \frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x}\,\mathrm{d}x = \int_\Omega \sum_i f_i \phi_i \phi_j \,\mathrm{d}x\,, \qquad \forall j \in [1,\, N]\,.$$

CFD

$$\int_\Omega \sum_i u_i \frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x}\,\mathrm{d}x = \int_\Omega \sum_i f_i \phi_i \phi_j \,\mathrm{d}x\,, \qquad \forall j \in [1,\, N]\,.$$

$$\int_\Omega \frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x}\,\mathrm{d}x = \begin{cases} 2/h & \text{if } i = j\,, \\ -1/h & \text{if } |i-j| = 1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

CFD

$$\int_\Omega \sum_i u_i \frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x}\,\mathrm{d}x = \int_\Omega \sum_i f_i \phi_i \phi_j\,\mathrm{d}x\,, \qquad \forall j \in [1,\, N]\,.$$

$$\int_\Omega \frac{\partial \phi_i}{\partial x}\frac{\partial \phi_j}{\partial x}\,\mathrm{d}x = \begin{cases} 2/h & \text{if } i=j\,, \\ -1/h & \text{if } |i-j|=1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

$$\int_\Omega \phi_i\,\phi_j\,\mathrm{d}x = \begin{cases} 2h/3 & \text{if } i=j\,, \\ h/6 & \text{if } |i-j|=1\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

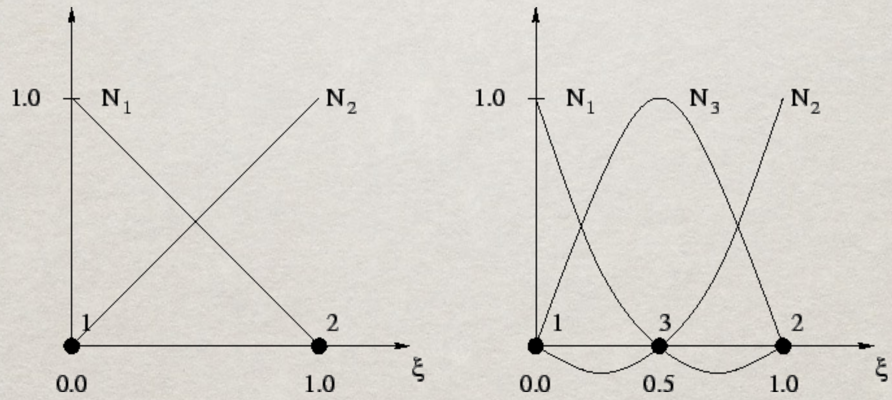
CFD

$$\frac{1}{h}\begin{pmatrix} 2 & -1 & & & & -1 \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & -1 \\ -1 & & & & -1 & 2 \end{pmatrix}\begin{pmatrix} u_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ u_N \end{pmatrix}$$

$$= h\begin{pmatrix} 2/3 & 1/6 & & & & 1/6 \\ 1/6 & 2/3 & 1/6 & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & 1/6 \\ 1/6 & & & & 1/6 & 2/3 \end{pmatrix}\begin{pmatrix} f_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ f_N \end{pmatrix}$$

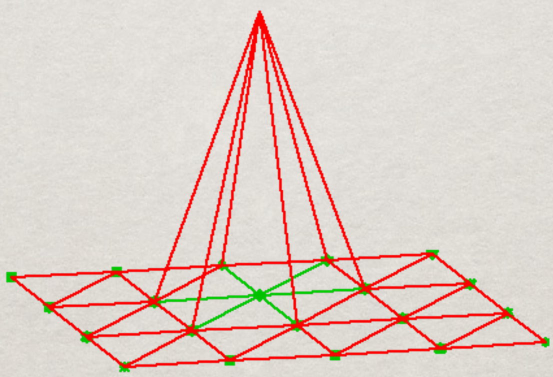
CFD

Shape functions



CFD

Shape functions in 2D, P1



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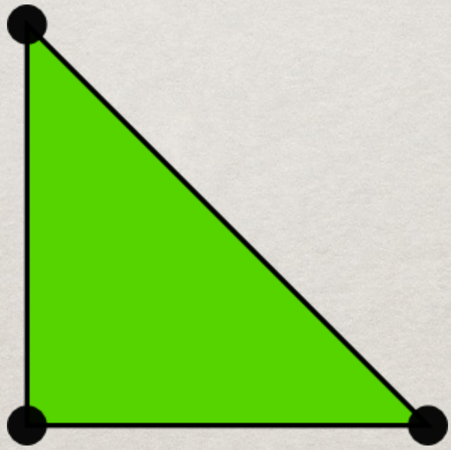
Shape functions in 2D, P1



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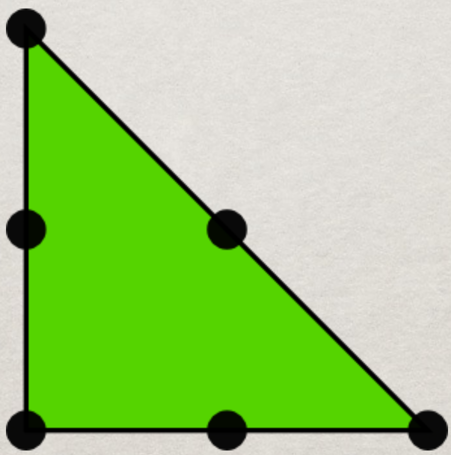
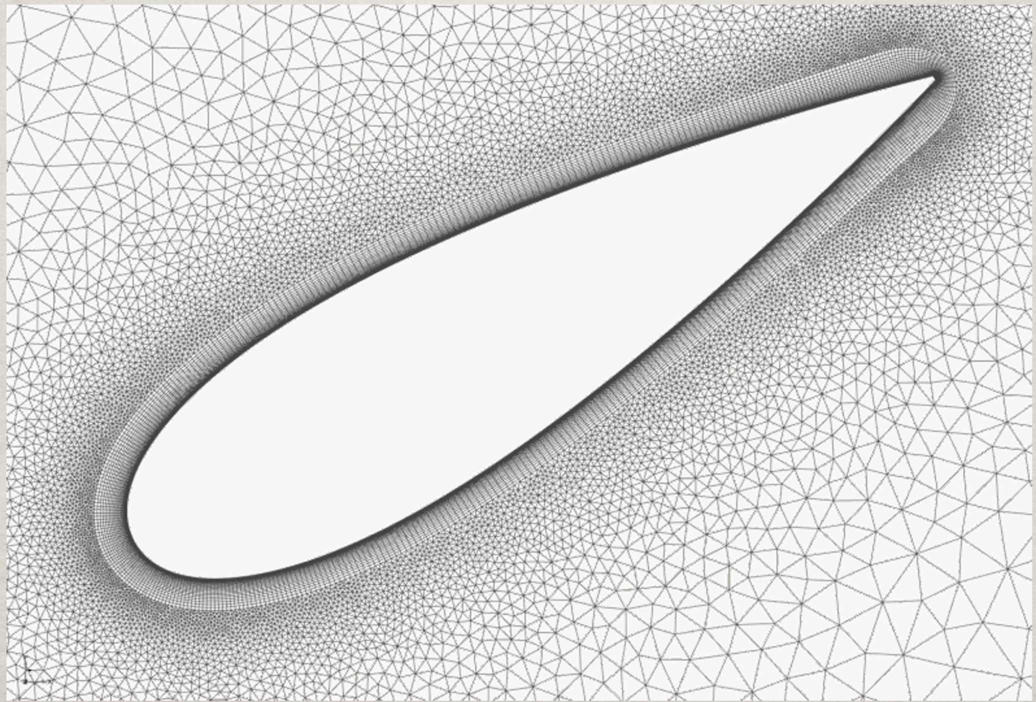
Shape functions in 2D, P2



CFD

Mesh generation



CFD

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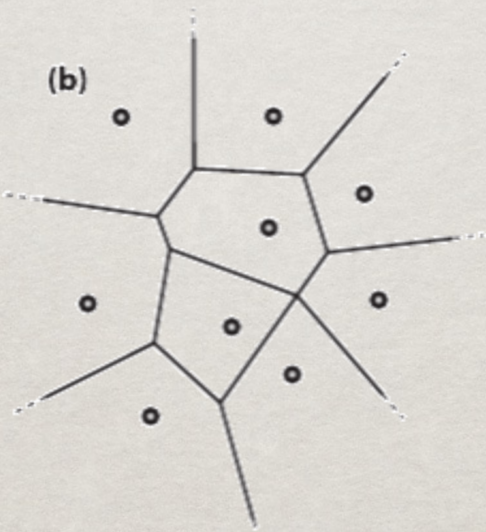
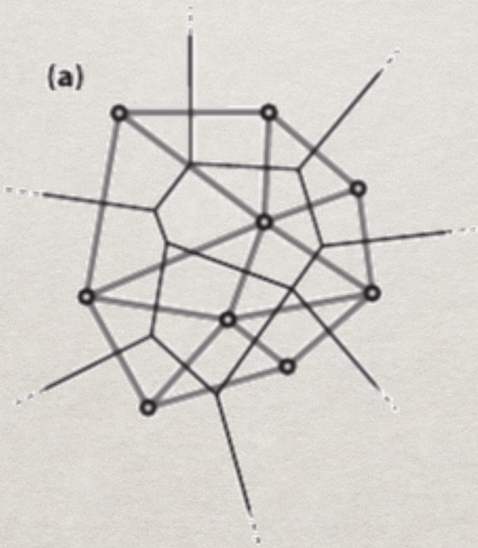


Voronoï

Georgi Fedoseevich Voronoï (1868-1908)

CFD

Mesh generation



Boris Delaunay
(1890-1980)

Delaunay - Voronoï

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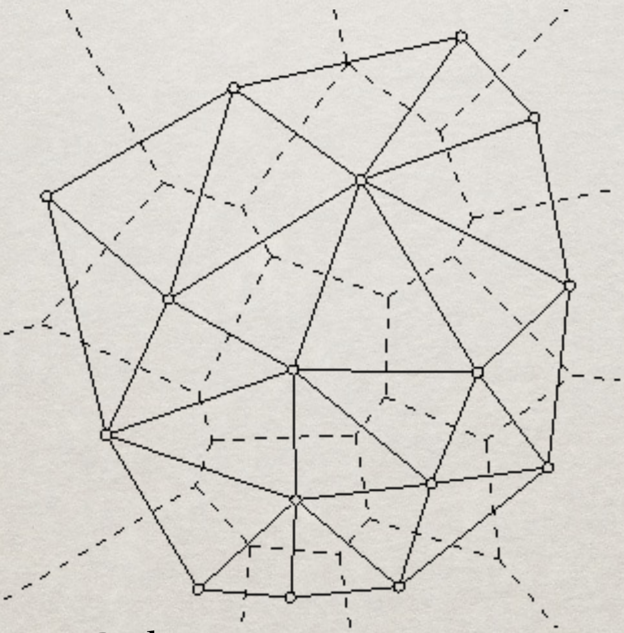
Mesh generation



Delaunay - Voronoï

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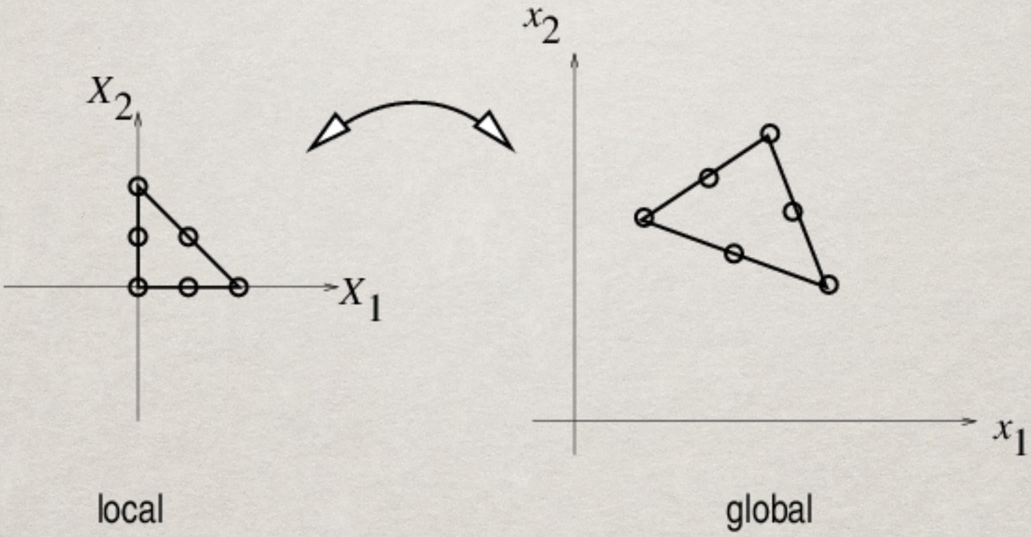
Integration



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Velocity Pressure

$$\int_\Omega [\boldsymbol{\nabla}\mathbf{u} : \boldsymbol{\nabla}\mathbf{w}](\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_\Omega [\mathbf{w} \cdot \boldsymbol{\nabla} p](\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \quad \forall \mathbf{w} \in [H_0^1(\Omega)]^2\,,$$

$$\int_\Omega [q\,\boldsymbol{\nabla} \cdot \mathbf{u}](\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \qquad \forall q \in L^2(\Omega)\,.$$

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Velocity Pressure

$$\int_\Omega [\boldsymbol{\nabla}\mathbf{u} : \boldsymbol{\nabla}\mathbf{w}](\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_\Omega [\mathbf{w}\cdot\boldsymbol{\nabla}p](\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \qquad \forall \mathbf{w} \in [H_0^1(\Omega)]^2\,,$$
$$\int_\Omega [q\,\boldsymbol{\nabla}\cdot\mathbf{u}](\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \qquad \forall q \in L^2(\Omega)\,.$$

$$\int_{\Omega_h} [\boldsymbol{\nabla}\mathbf{u}_h : \boldsymbol{\nabla}\mathbf{w}_h](\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{\Omega_h} [\mathbf{w}_h\cdot\boldsymbol{\nabla}p_h](\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \qquad \forall \mathbf{w}_h \in V_{0h}\,,$$
$$\int_{\Omega_h} [q_h\,\boldsymbol{\nabla}\cdot\mathbf{u}_h](\mathbf{x})\,\mathrm{d}\mathbf{x} = 0 \qquad \forall q_h \in Q_h\,.$$

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Ladyzenskaya-Brezzi-Babuska

$$\inf_{q_h\in Q_h}\ \sup_{\mathbf{u}_h\in V_{0h}} \frac{1}{\|\mathbf{u}_h\|_{V_{0h}}\,\|q_h\|_{Q_h}} \int_{\Omega_h} [q_h\,\boldsymbol{\nabla}\cdot\mathbf{u}_h](\mathbf{x})\,\mathrm{d}\mathbf{x} \geq C > 0\,.$$

CFD

Ladyzenskaya-Brezzi-Babuska

(LBB or Inf-Sup condition)

$$\inf_{q_h \in Q_h} \sup_{\mathbf{u}_h \in V_{0h}} \frac{1}{\|\mathbf{u}_h\|_{V_{0h}} \, \|q_h\|_{Q_h}} \int_{\Omega_h} [q_h \, \boldsymbol{\nabla} \cdot \mathbf{u}_h](\mathbf{x}) \, \mathrm{d}\mathbf{x} \geq C > 0 \, .$$

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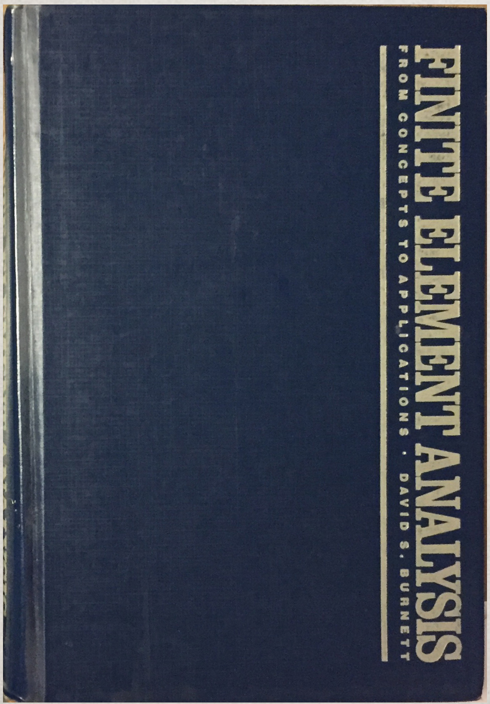
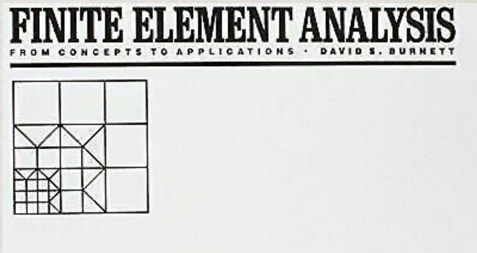
Freefem++

https://freefem.org

CFD

To explore further:





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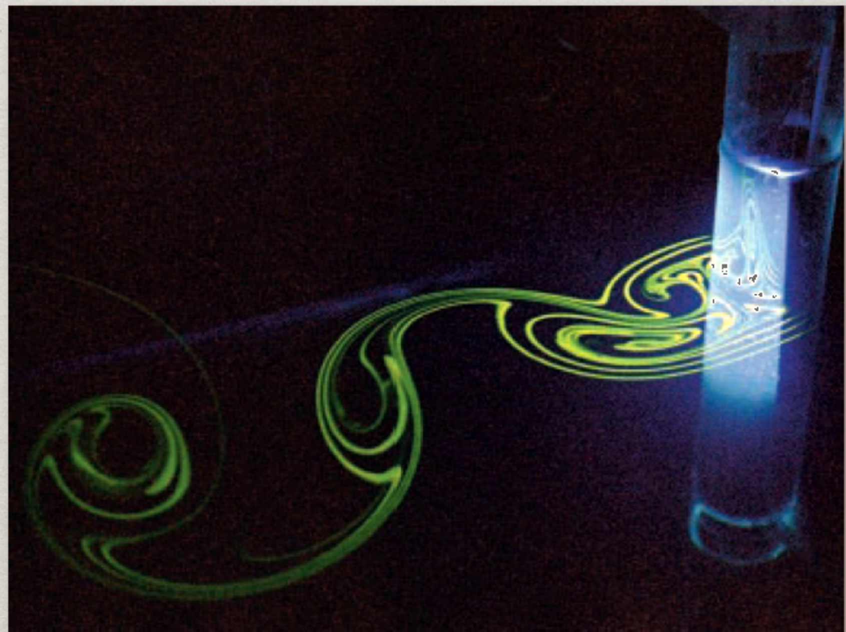
Adaptative stencils



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Adaptive stencils

Very intuitive idea: adjust the finite-difference discretization near the boundary and include the boundary values directly.

- Shortley-Weller technique
- ghost-point reflection

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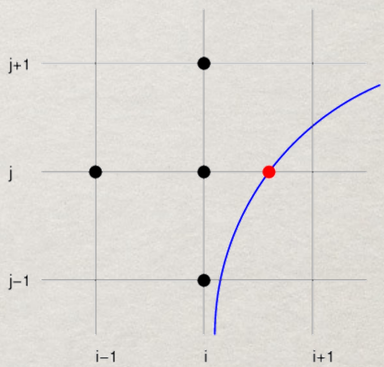
Adaptive stencils

Shortley-Weller technique



$$u_W \frac{2}{\Delta x(\Delta_E + \Delta x)} + u_N \frac{1}{\Delta y^2} + u_S \frac{1}{\Delta y^2} -$$

$$u_C \left(\frac{2}{\Delta_E \Delta x} + \frac{2}{\Delta y^2} \right) = f_C - U \frac{2}{\Delta_E(\Delta_E + \Delta x)}$$

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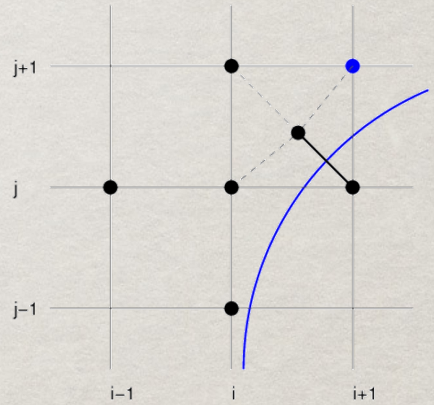
Adaptive stencils

ghost-point reflection

reflection along the boundary normal



bilinear interpolation from neighboring points to mapped point

ghost value = +/- interpolated mapped value

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Adaptive stencils

Advantage: ease of implementation

Disadvantage: uniform stencil structure is compromised; convergence issues; algorithmic issues;

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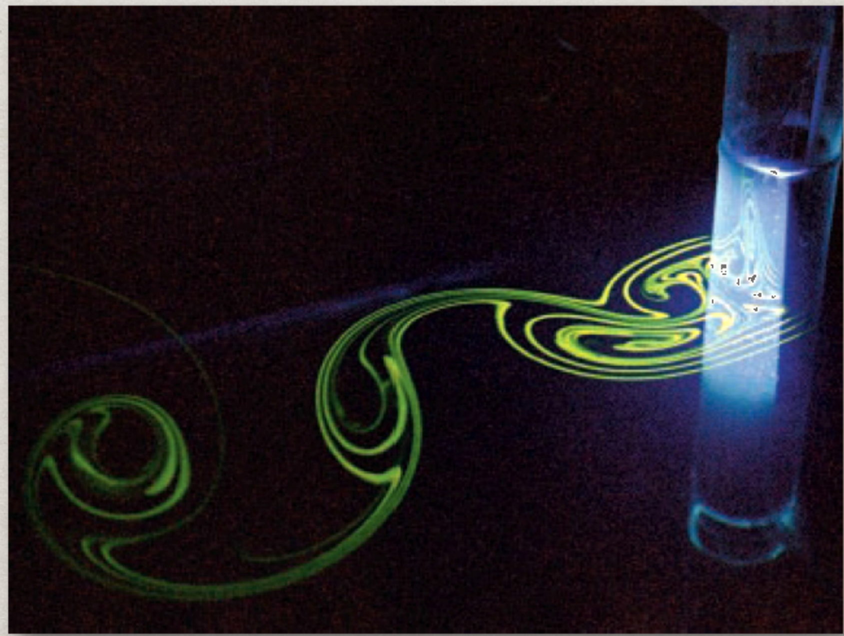
Penalisation



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Penalisation

$$\nabla \cdot \mathbf{u} = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} = -\nabla p + \frac{1}{Re}\Delta \mathbf{u} + \mathbf{f}$$

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Penalisation