Numerical Methods for Fluid Dynamics

Emmanuel DORMY (DMA-ENS)
emmanuel.dormy@ens.fr

CFD

6. Spectral Methods and Hydrodynamic Instabilities and Turbulence

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Fourier Transform of a [0,2π) periodic function

$$\phi_k(x) = \mathrm{e}^{ikx}$$

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Fourier Transform of a [0,2π) periodic function

$$\phi_k(x) = \mathrm{e}^{ikx}$$

$$\frac{1}{2\pi}\int_0^{2\pi} \phi_k(x)\phi_l^*(x)\,\mathrm{d}x = \delta_{kl} = \begin{cases} 0 & \text{if } k \neq l\,, \\ 1 & \text{if } k = l\,. \end{cases}$$

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Fourier Transform of a [0,2π) periodic function

$$\hat{u}_k = \frac{1}{2\pi} \int_0^{2\pi} u(x)\, \mathrm{e}^{-ikx}\, \mathrm{d}x$$

$$u(x) = \sum_{k=-\infty}^{+\infty} \hat{u}_k\, \phi_k(x)$$

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Discrete Fourier Transform

$$\hat{u}_k = \frac{1}{N} \sum_{j=0}^{N} u(x_j)\, \mathrm{e}^{-ikx_j}\ .$$

$$\frac{1}{N} \sum_{j=0}^{N-1} \mathrm{e}^{i(k'-k)x_j} = \begin{cases} 1 & \text{if } (k'-k) \in N\mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

$$u(x_j) = \sum_{k=-N/2}^{N/2-1} \hat{u}_k\, \mathrm{e}^{ikx_j}\,.$$

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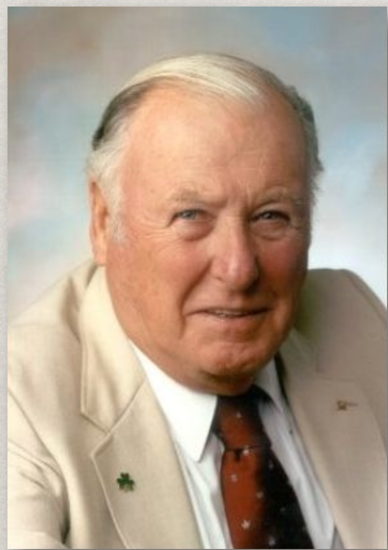
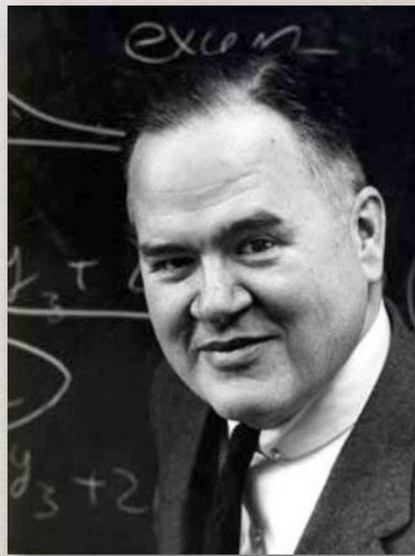
Fast Fourier Transform (FFT)
Cooley & Tukey (1965)



James William Cooley
(1926-2016)



John Wilder Tukey
(1915-2000)

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Fast Fourier Transform (FFT)
Cooley & Tukey (1965)

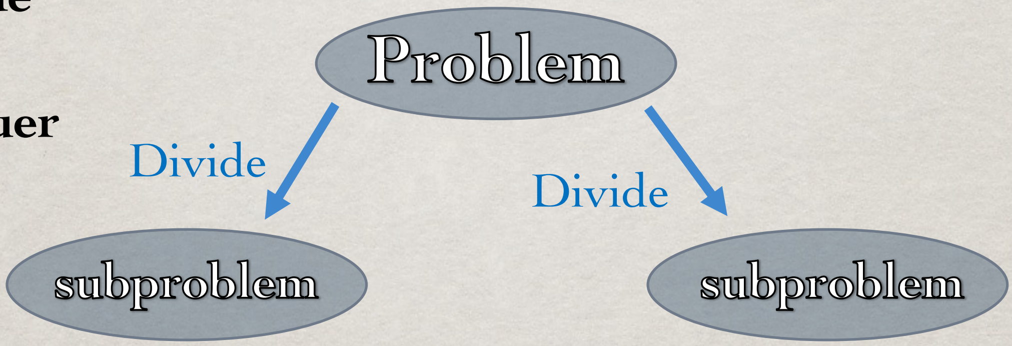
Divide
&
Conquer

Problem

CFD

Fast Fourier Transform (FFT)
Cooley & Tukey (1965)

Divide & Conquer



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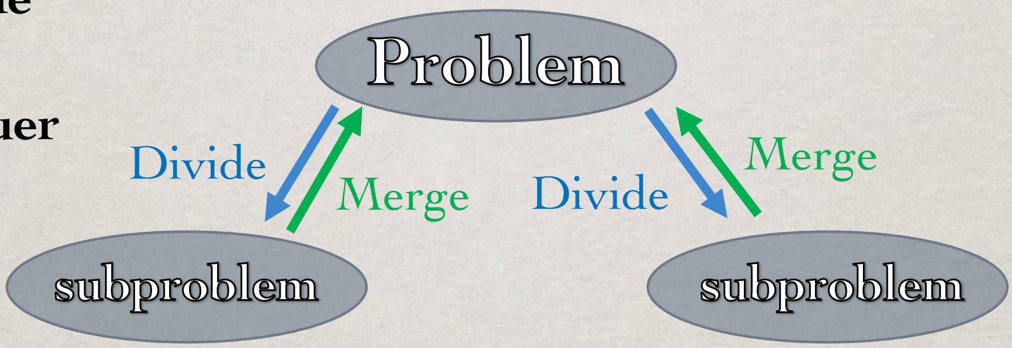
Fast Fourier Transform (FFT)
Cooley & Tukey (1965)

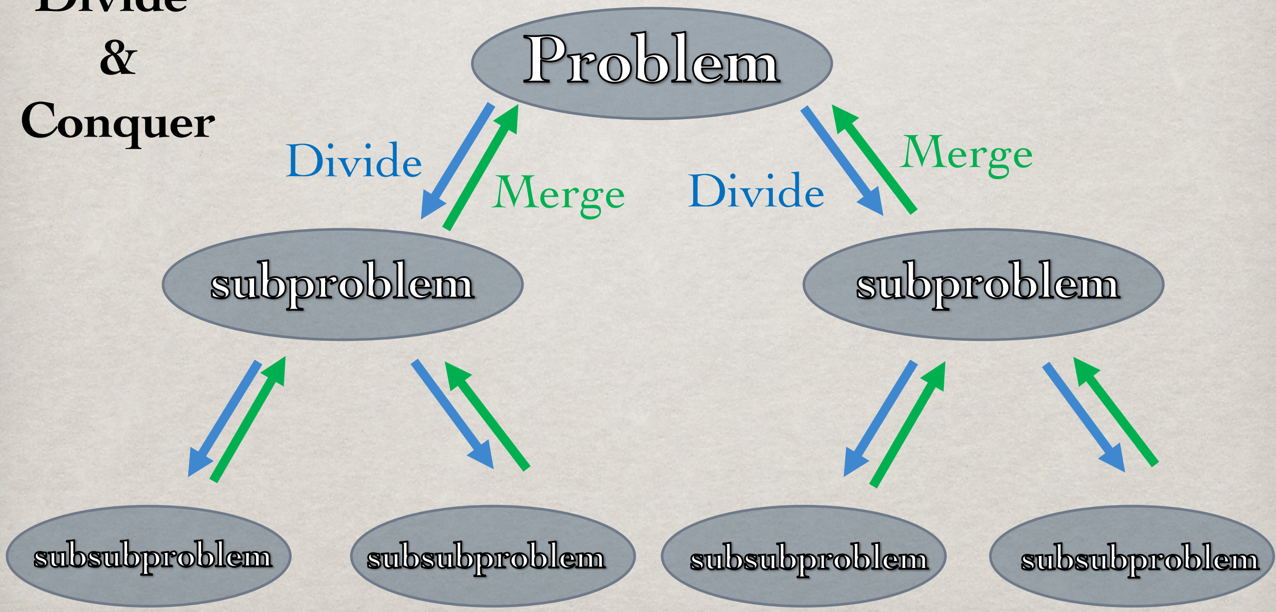
Divide & Conquer



CFD

Fast Fourier Transform (FFT)
Cooley & Tukey (1965)

Divide
&
Conquer



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Convergence study

$$u = \exp(\cos(x))$$

$$u' = -\sin(x)\,u$$

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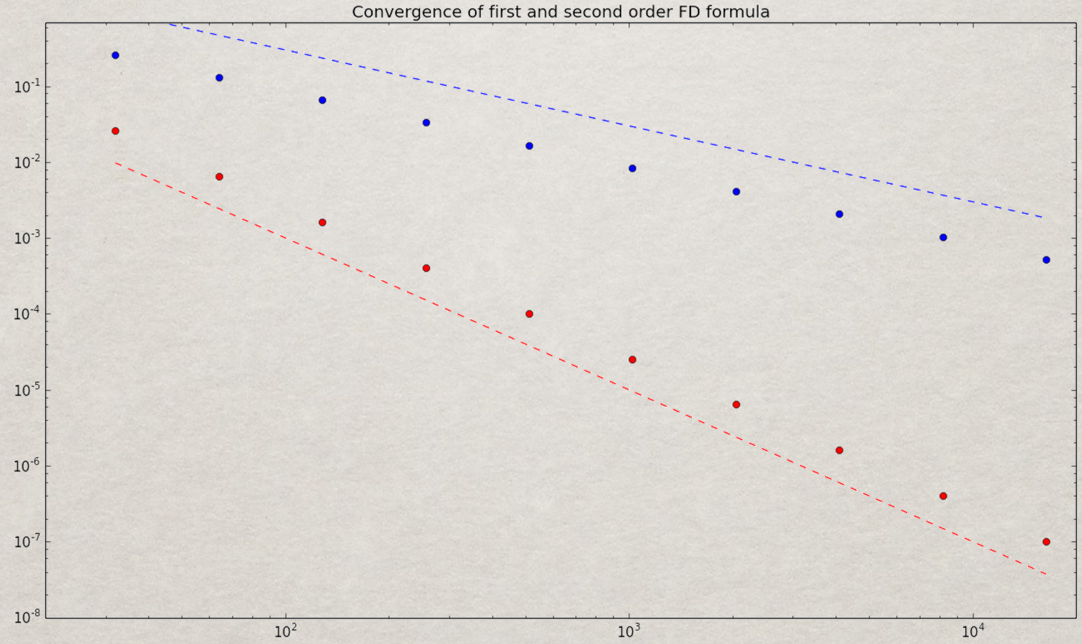
Finite Difference convergence



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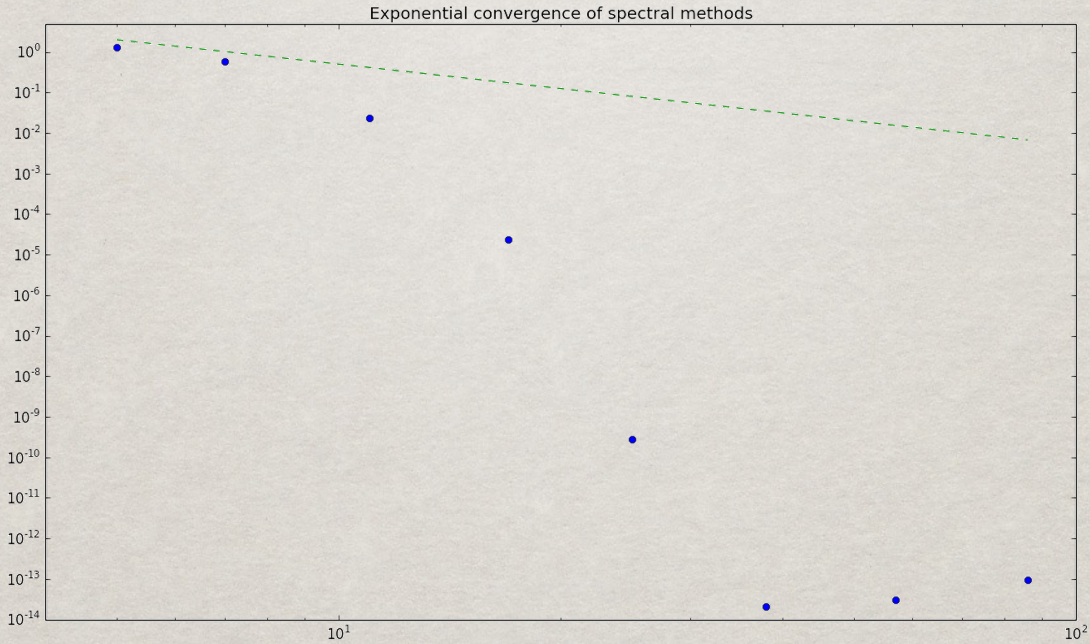
Exponential convergence



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Exponential convergence

$$\hat{u}_k = \frac{1}{2\pi}\int_0^{2\pi} u(x)\mathrm{e}^{-ikx}\,\mathrm{d}x = -\frac{1}{2\pi ik}\int_0^{2\pi}\frac{\mathrm{d}u}{\mathrm{d}x}\mathrm{e}^{-ikx}\,\mathrm{d}x + \underbrace{\left[\frac{u(x)\mathrm{e}^{-ikx}}{-2\pi ik}\right]_0^{2\pi}}_{=0}$$

$$= \cdots = \frac{1}{2\pi(-ik)^n}\int_0^{2\pi}\frac{\mathrm{d}^n u}{\mathrm{d}x^n}\mathrm{e}^{-ikx}\,\mathrm{d}x$$

$$= \mathcal{O}(k^{-n})\,.$$

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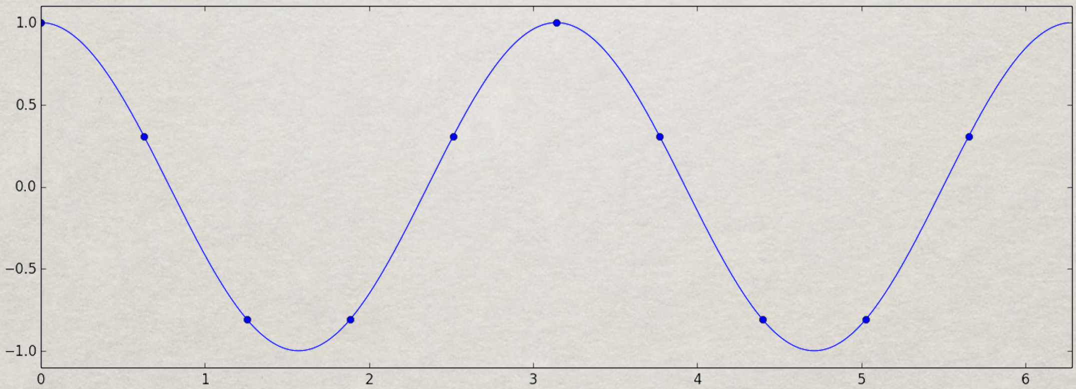
Shannon / Nyquist



$\cos(2\,x)$

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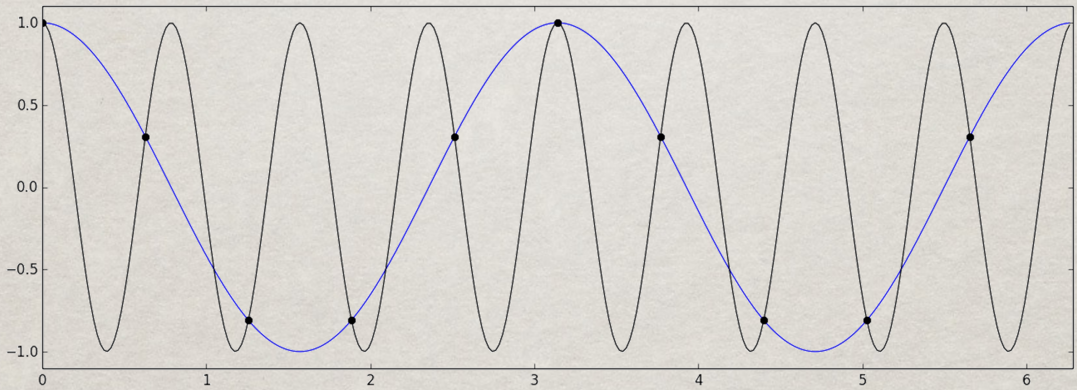
Shannon / Nyquist



$$\cos(2\,x) \qquad \cos((N-2)\,x) = \cos(8\,x)$$

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Shannon / Nyquist

$$F(x) = \cos(k\,x)\,, \qquad G(x) = \cos((N-k)x)\,,$$

$$F_j = \cos(kj\Delta x)\,, \qquad \begin{aligned} G_j &= \cos((N-k)j\Delta x) \\ &= \cos(2\pi j - kj\Delta x) \\ &= \cos(kj\Delta x) = F_j \end{aligned}$$

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Advection-diffusion

$$\frac{\partial u}{\partial t} = -c\frac{\partial u}{\partial x} + \kappa\frac{\partial^2 u}{\partial^2 x}$$

$$\frac{\partial \hat{u}_k}{\partial t} = -cik\hat{u}_k - \kappa k^2 \hat{u}_k$$

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Advection-diffusion

$$\hat{u}_k^{n+1} = \hat{u}_k^n e^{\Delta t(-cik-\kappa k^2)} \,,$$

$$\frac{\hat{u}_k^{n+1} - \hat{u}_k^n}{\Delta t} = (-cik - \kappa k^2)\hat{u}_k^n \,.$$

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The Burgers equation

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x}$$

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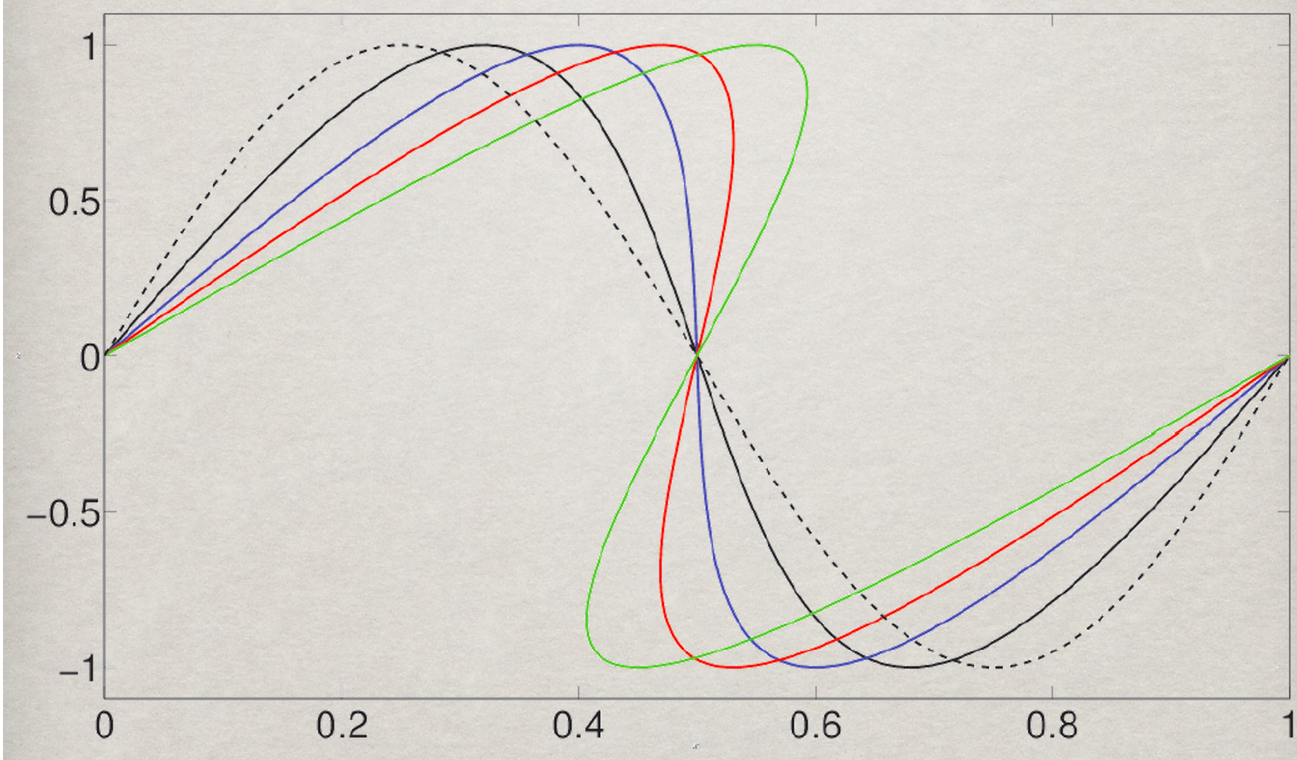

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The Burgers equation

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right)$$

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) + \varepsilon\frac{\partial^2 u}{\partial x^2}$$

KdV: $$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) - \alpha\frac{\partial^3 u}{\partial x^3}$$

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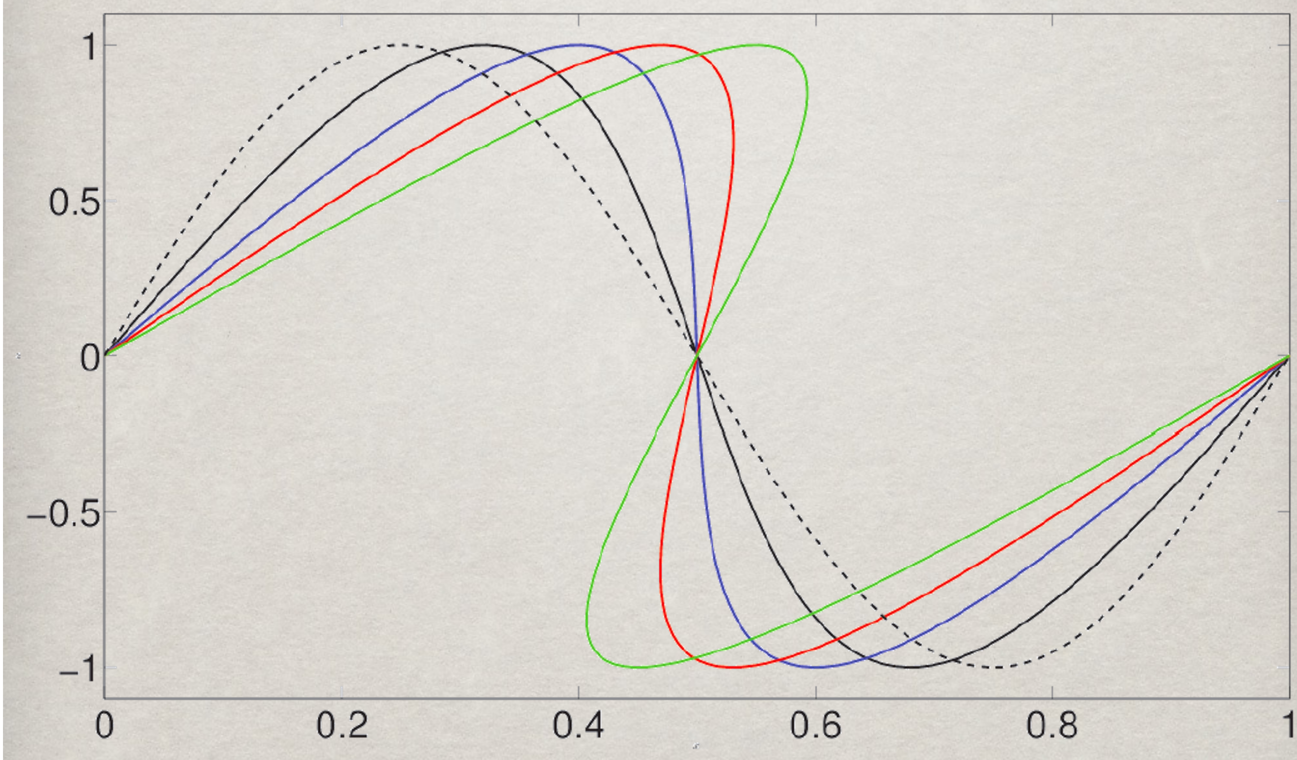

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The Burgers equation

$$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x} + \varepsilon\frac{\partial^2 u}{\partial^2 x}$$

$$\frac{\partial \hat{u}_k}{\partial t} = -\hat{u}_k * ik\hat{u}_k - \varepsilon k^2 \hat{u}_k$$

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The Burgers equation

$$\frac{\partial u}{\partial t} = -\frac{1}{2}\frac{\partial u^2}{\partial x} + \varepsilon\frac{\partial^2 u}{\partial^2 x}$$

$$\frac{\partial \hat{u}_k}{\partial t} = -\frac{ik}{2}\widehat{(u^2)}_k - \varepsilon k^2 \hat{u}_k$$

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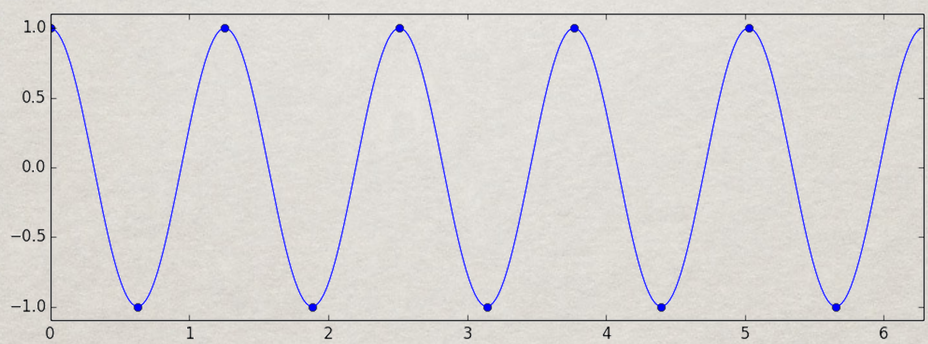
The Burgers equation

$$\hat{u}_k^{n+1} = \hat{u}_k^n e^{-\Delta t \varepsilon k^2} - \frac{ik}{2}\Delta t \widehat{(u^2)}_k^n ,$$

$$\frac{\hat{u}_k^{n+1} - \hat{u}_k^n}{\Delta t} = -\frac{ik}{2}\widehat{(u^2)}_k^n - \varepsilon k^2 \hat{u}_k^n .$$

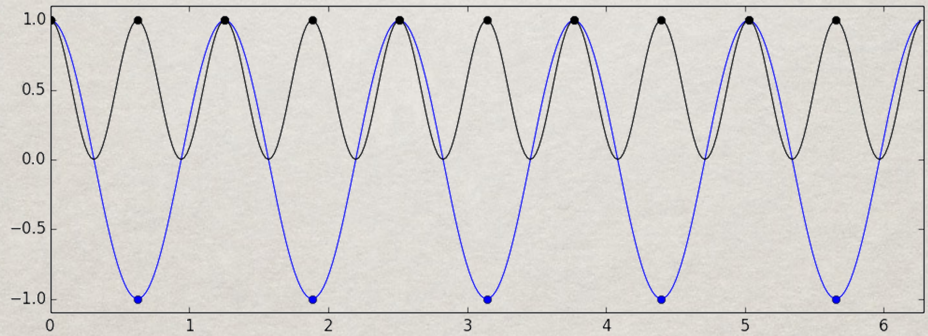
CFD

Aliasing:



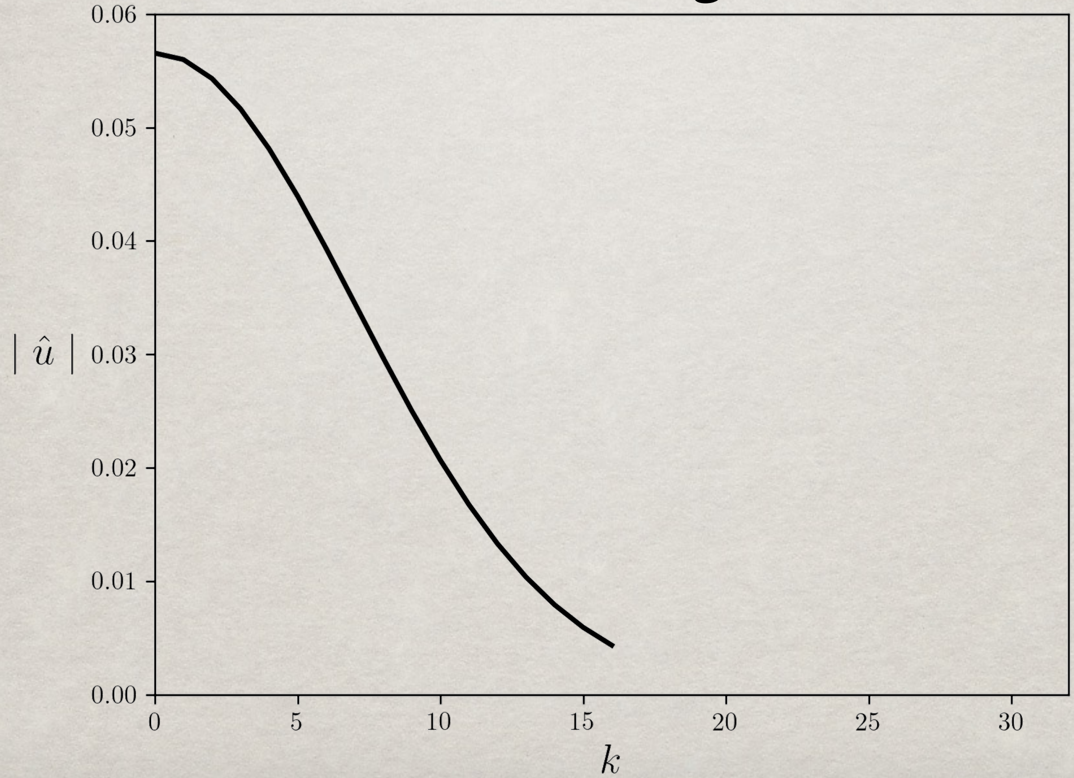
CFD

Aliasing:



CFD

Aliasing:



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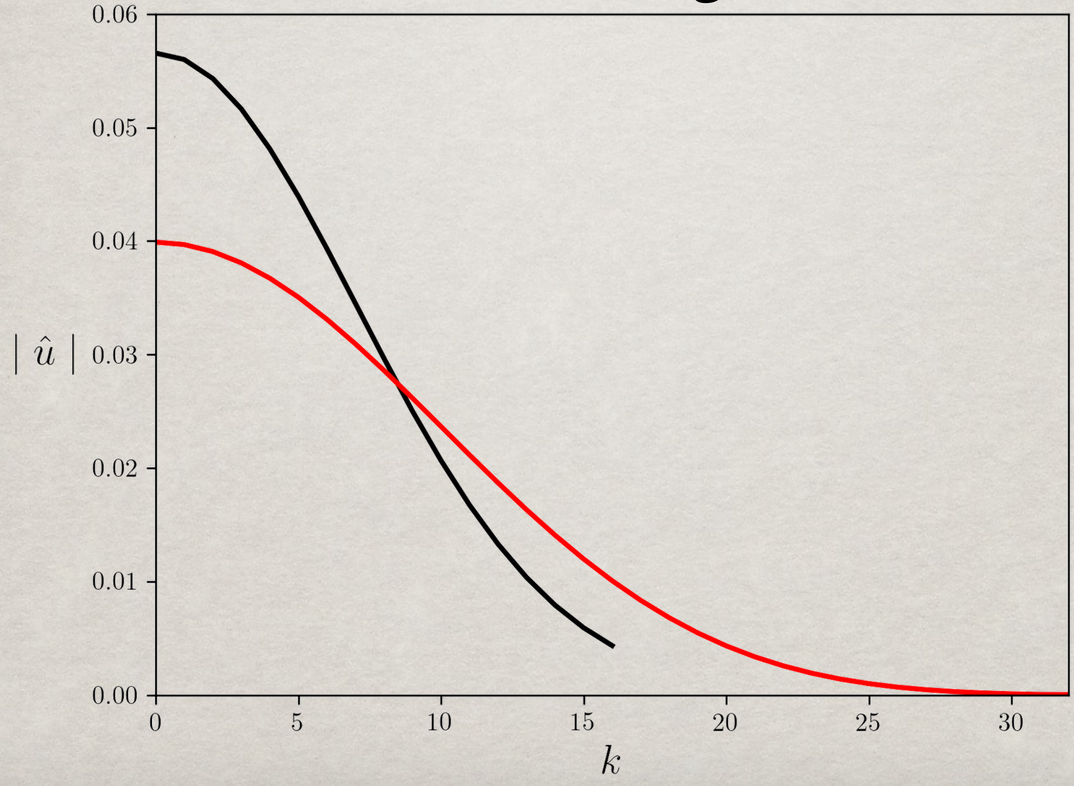
Aliasing:



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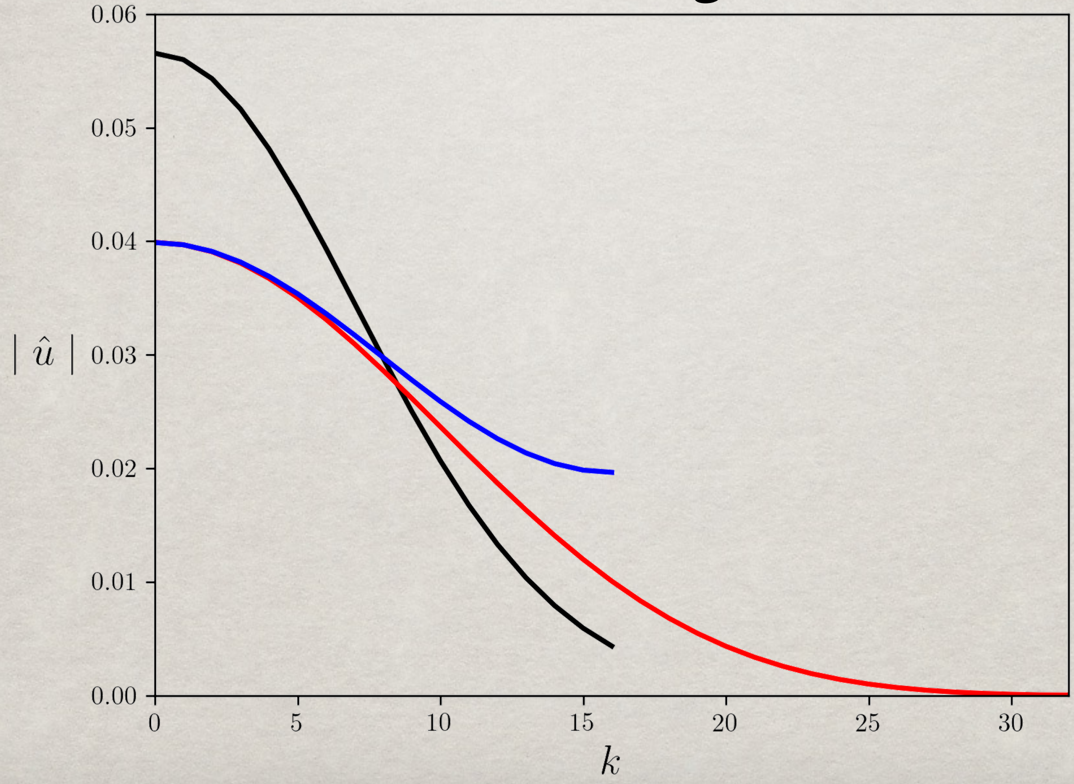

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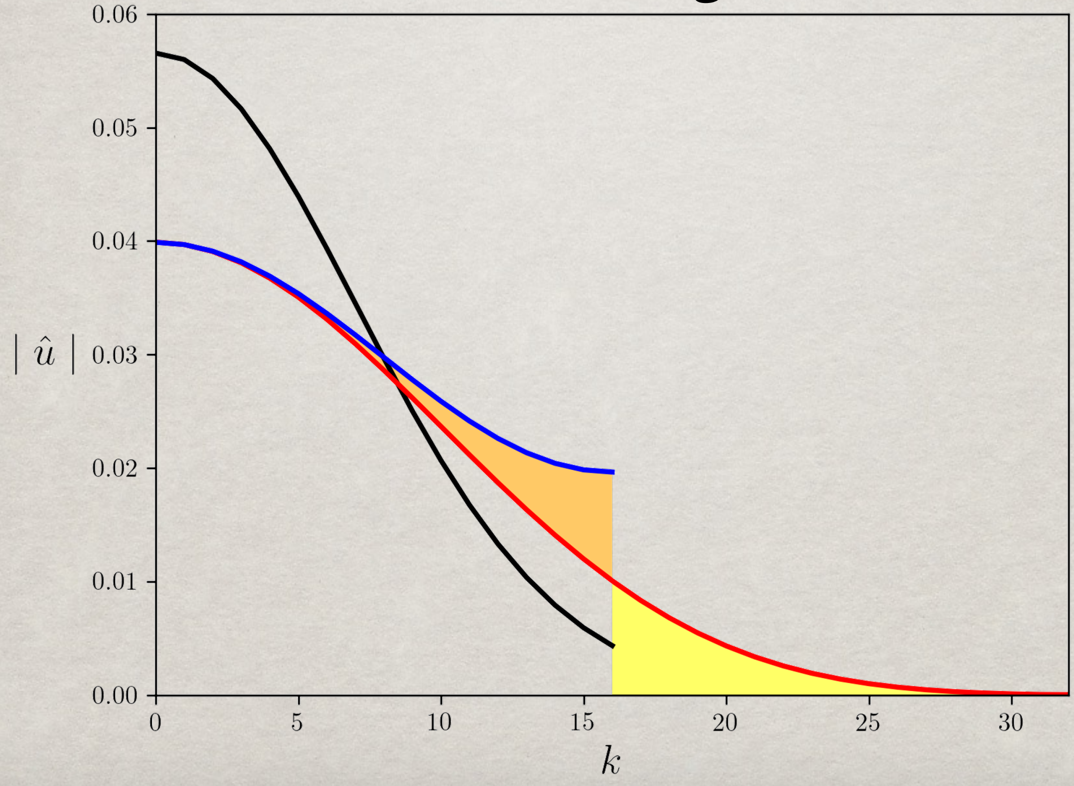

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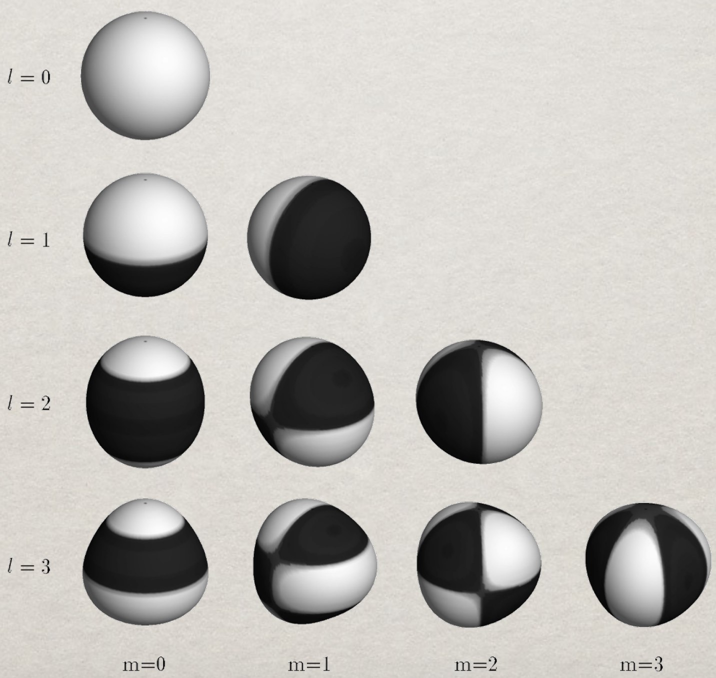
Spherical harmonics



Finite Domain & Chebyshev

$$u(x) = \sum_{k=0}^{N} a_k \, \phi_k(x) \qquad T_k(x) = \cos(k \, \cos^{-1}(x))$$

$$-1 \leq x \leq 1$$

$$T_0(x) = 1,$$

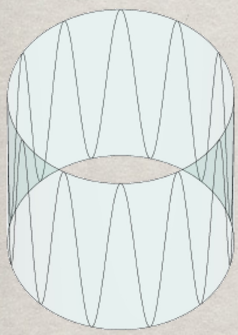
$$T_1(x) = x,$$

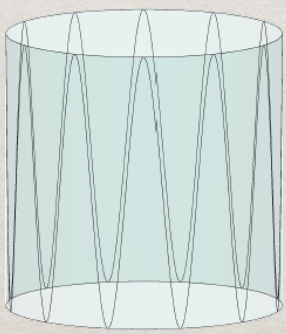
$$T_2(x) = 2\,x^2 - 1,$$

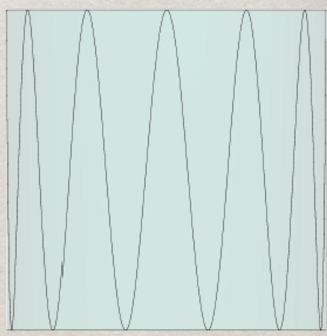
$$T_3(x) = 4\,x^3 - 3\,x$$

Finite Domain & Chebyshev

$$T_k(x) = \cos(k \, \cos^{-1}(x))$$







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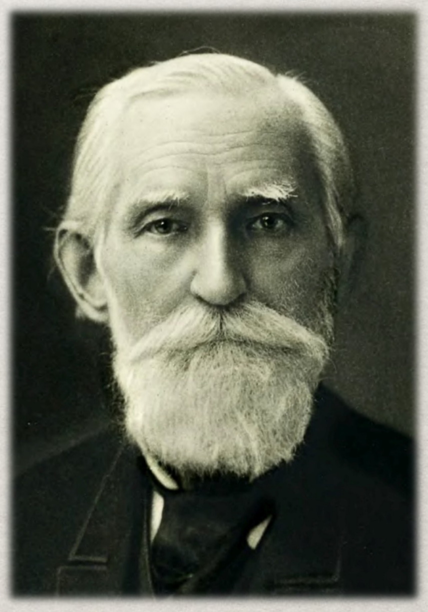
Finite Domain & Chebyshev

$$T_k(x) = \cos(k \cos^{-1}(x))$$

$$T_0(x) = 1, \quad T_1(x) = x,$$

$$T_{k+1}(x) = 2x\,T_k(x) - T_{k-1}(x)$$



Pafnuty Chebyshev
(1821–1894)

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Finite Domain & Chebyshev

Chebychev polynomials are orthogonal:

$$\int_{-1}^{1} \frac{T_n(x)\,T_m(x)}{\sqrt{1-x^2}}\mathrm{d}x = C_n \delta_{nm},$$

$$C_0 = \pi, \quad C_n = \frac{\pi}{2} \quad (n \neq 0)$$

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Finite Domain & Chebyshev

$$f'(x) \simeq \sum_{k=0}^{N} a_k\, T_k'(x) = \sum_{k=0}^{N} b_k\, T_k(x)$$

$$c_n\, b_n = 2 \sum_{\substack{p=n+1 \\ p+n \text{ odd}}}^{\infty} p\, a_p$$

with $c_0 = 2$ and $c_i = 1$ for i>0.

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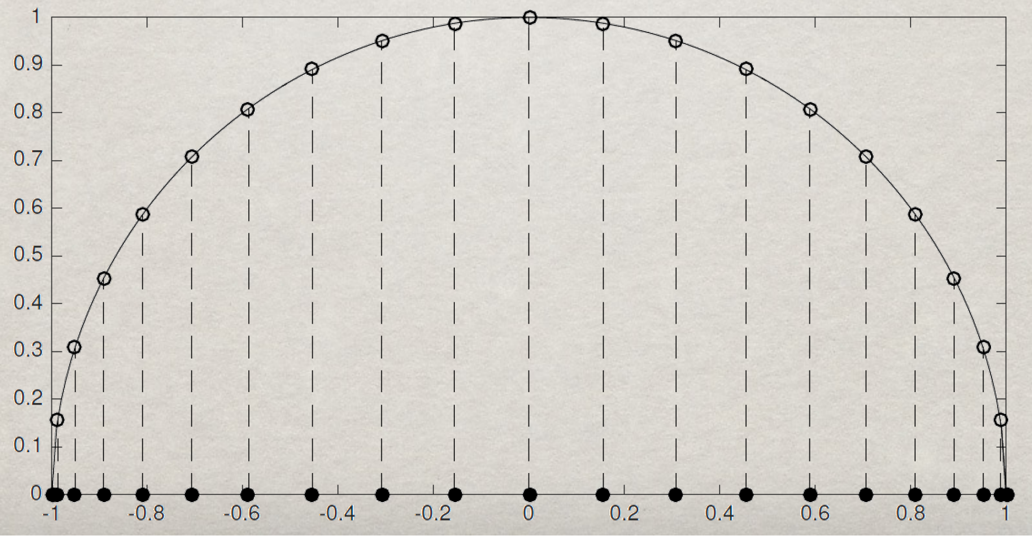
Finite Domain & Chebyshev

The Gauss-Lobato collocation points

$$x_j = \cos\left(\frac{j\pi}{N}\right) \qquad j = 0, \dots, N$$



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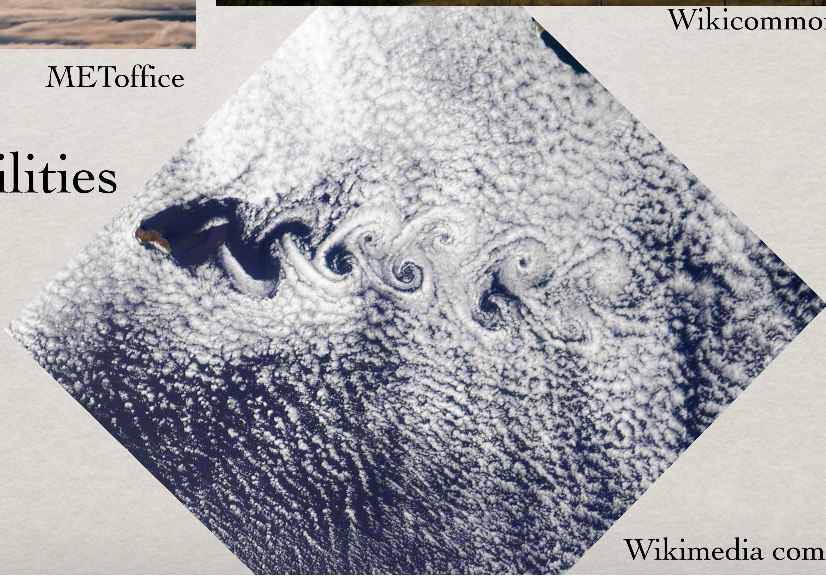
Shear instabilities



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Thermal instabilities

Lord Rayleigh
(John Strutt)
(1842-1919)

Henri Bénard
(1874-1939)

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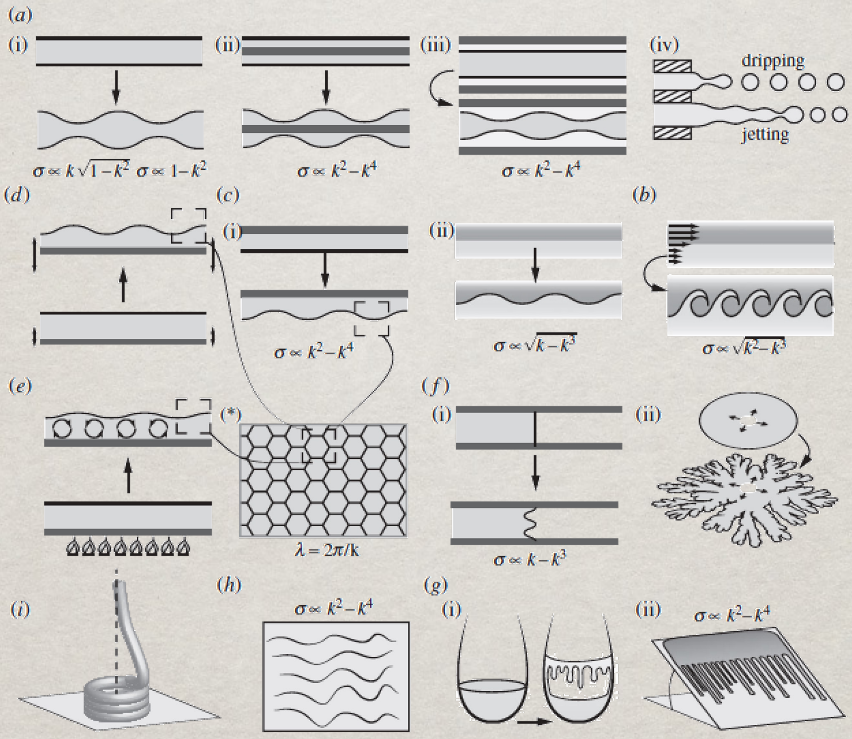
Interfacial instabilities

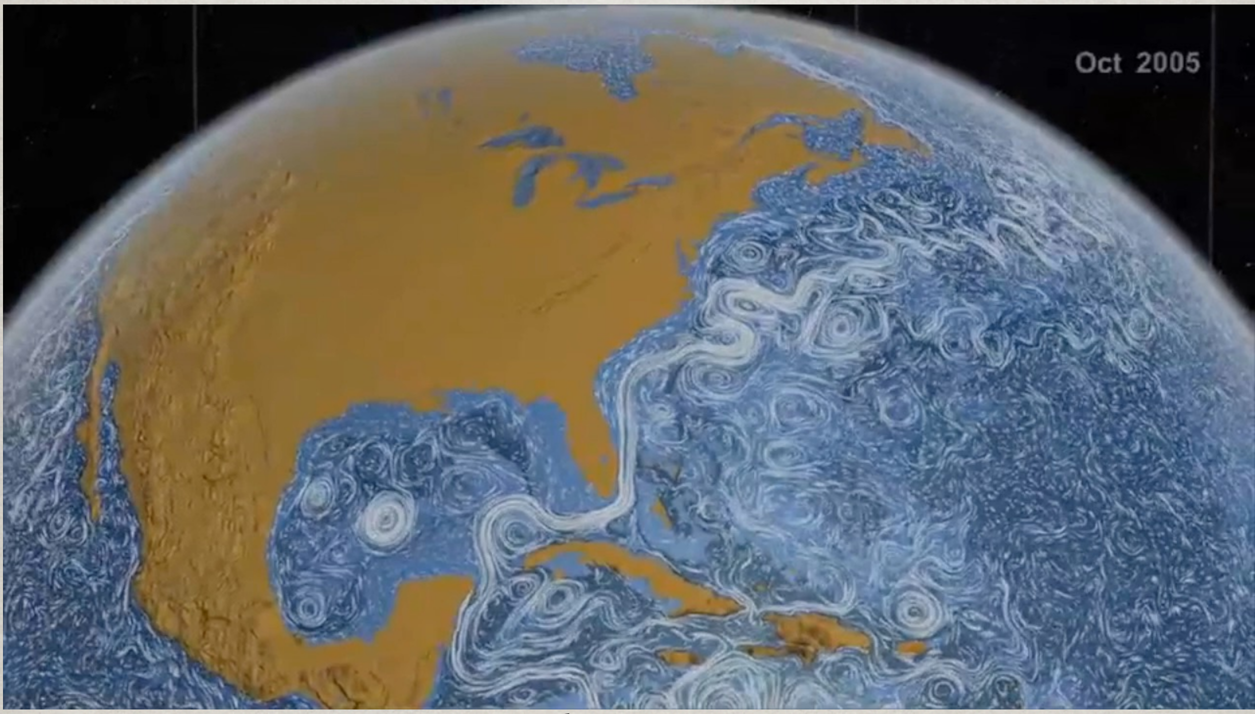


Gallaire & Brun, 2017

Hydrodynamic turbulence

Leonardo da Vinci
c1510

Hydrodynamic turbulence



Perpetual Ocean (NASA)