Numerical Methods for Fluid Dynamics

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Fluid Mechanics...

... at last!!!

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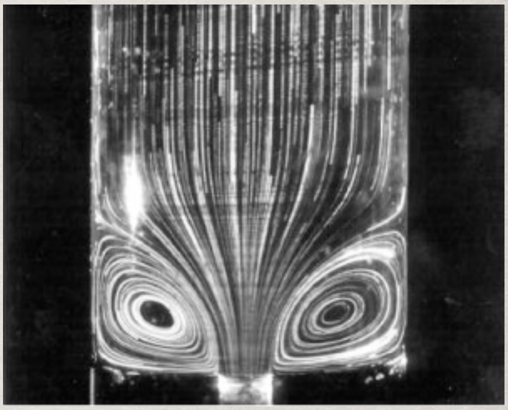
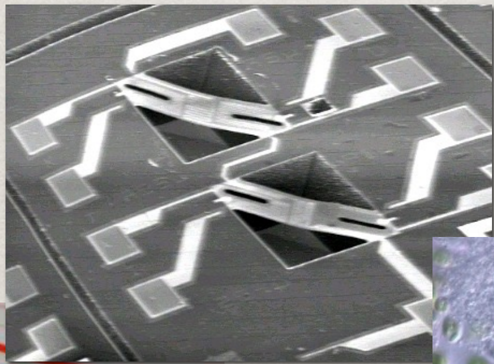
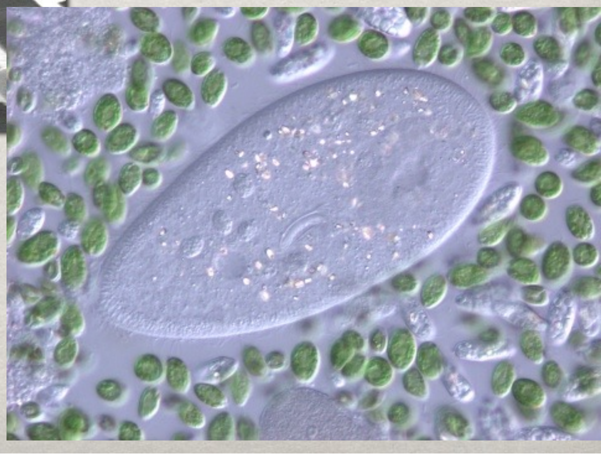
Viscous flows







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Viscous flows

or Stokes flows

George Gabriel Stokes
(1819-1903)

5. Stokes flows

5.1 Introduction

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0\,,$$

$$\rho\left(\mathbf{u} \cdot \boldsymbol{\nabla}\right)\,\mathbf{u} = -\boldsymbol{\nabla} p + \mu \Delta \mathbf{u}\,.$$

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$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0\,,$$

$$\mathrm{Re}\,\left(\mathbf{u} \cdot \boldsymbol{\nabla}\right)\mathbf{u} = -\boldsymbol{\nabla} p + \Delta \mathbf{u}\,.$$

$$\mathrm{Re} \;=\; UL\rho/\mu \;=\; UL/\nu$$

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Stokes Flows:

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0\,,$$

$$\boldsymbol{\nabla} p = \Delta \mathbf{u}\,.$$

5. Stokes flows

5.2 A classical example

Stokes Flows:

$$\nabla \cdot \mathbf{u} = 0\,,$$

$$\nabla p = \Delta \mathbf{u}\,.$$

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$$\boldsymbol{\nabla} \times \boldsymbol{\Psi} = \mathbf{u} \qquad \text{and} \qquad \boldsymbol{\nabla} \cdot \boldsymbol{\Psi} = 0$$

$$\Delta^2 \boldsymbol{\Psi} = 0$$

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$$\boldsymbol{\Psi} = \frac{\Psi}{r \sin\theta} \mathbf{e}_z$$

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{\cot\theta}{r^2} \frac{\partial}{\partial \theta} \right)^2 \Psi = 0$$

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On the obstacle:

$$\frac{\partial \Psi}{\partial r} = \frac{\partial \Psi}{\partial \theta} = 0$$

$$\frac{\partial \Psi}{\partial r} = \Psi = 0$$

And when: $r \longrightarrow \infty$

$$\psi \longrightarrow \frac{1}{2} r^2 \sin^2 \theta$$

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$$\psi = f(r) \sin^2 \theta$$

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{2}{r^2} \right)^2 f = 0$$

$$\psi = \left(\frac{1}{2} r^2 - \frac{3}{2} r + \frac{1}{4} r^{-1} \right) \sin^2 \theta$$

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$$\psi = \left(\frac{1}{2}r^2 - \frac{3}{2}r + \frac{1}{4}r^{-1}\right)\sin^2\theta$$

$$u_r = \left(1 - \frac{3}{2r} + \frac{1}{2r^3}\right)\cos\theta$$

$$u_\theta = \left(-1 + \frac{3}{4r} + \frac{1}{4r^3}\right)\sin\theta$$

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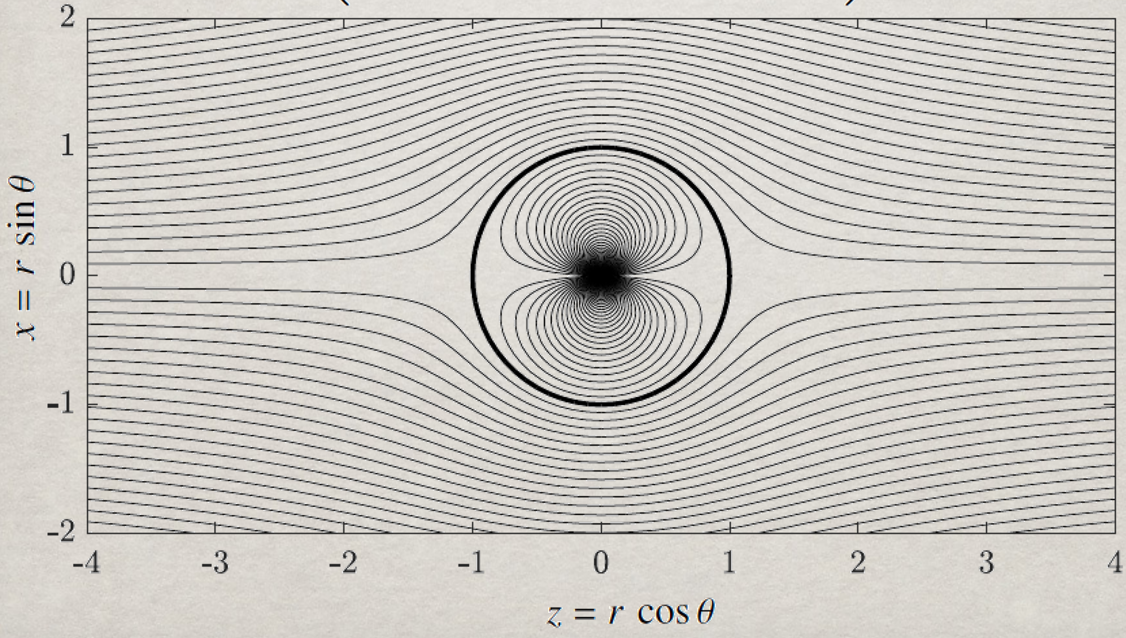
$$\psi = \left(\frac{1}{2}r^2 - \frac{3}{2}r + \frac{1}{4}r^{-1}\right)\sin^2\theta$$



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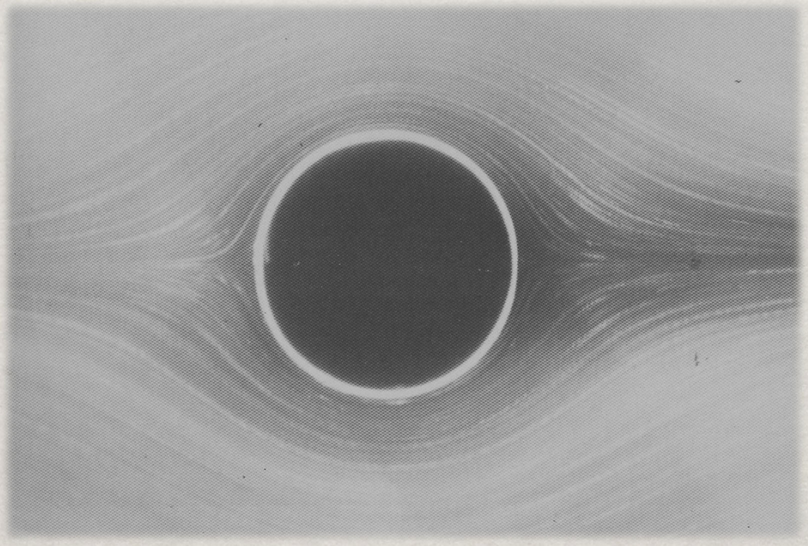
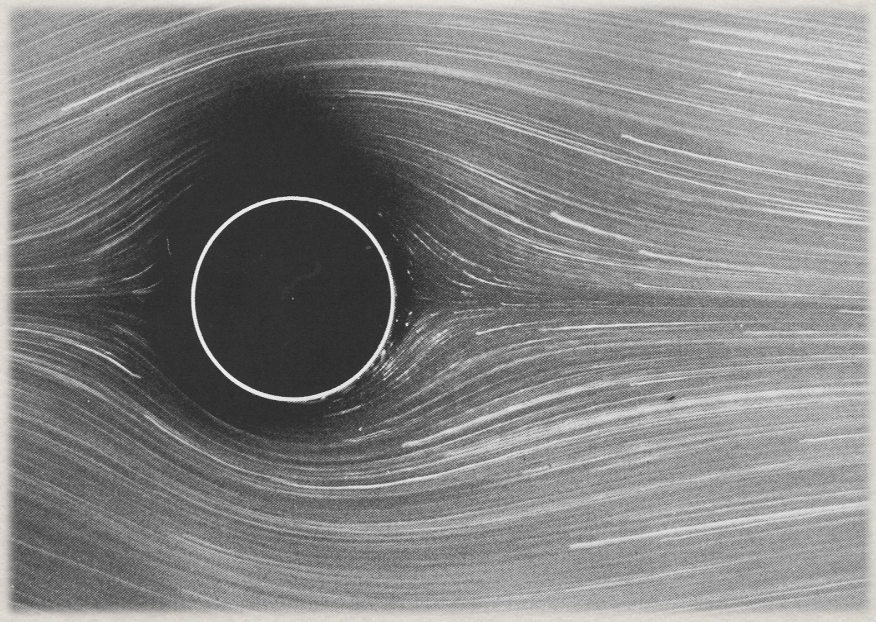
Re=0.16



Van Dycke, An Album of Fluid Motions

CFD

Re=1.54



Van Dycke, An Album of Fluid Motions

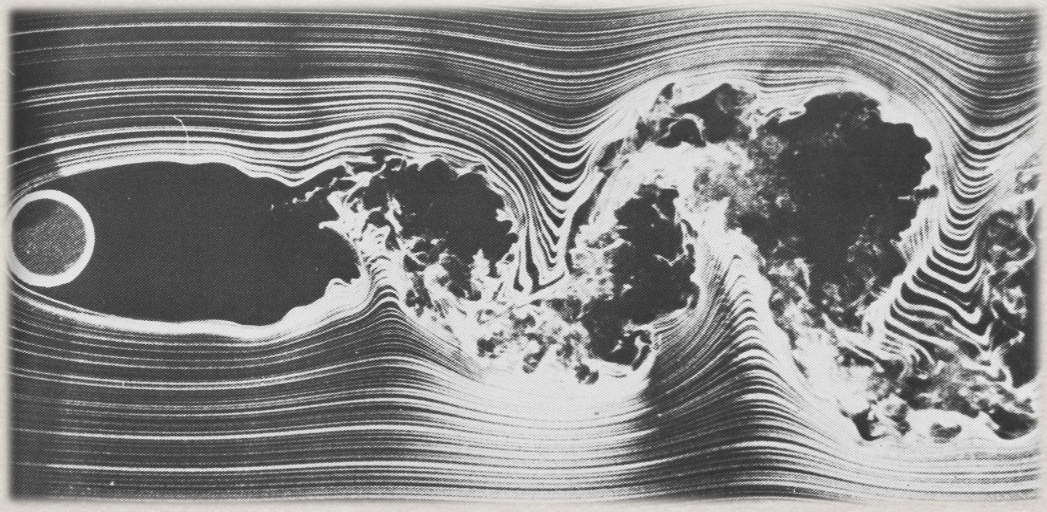
CFD

Re=26

Van Dycke, An Album of Fluid Motions

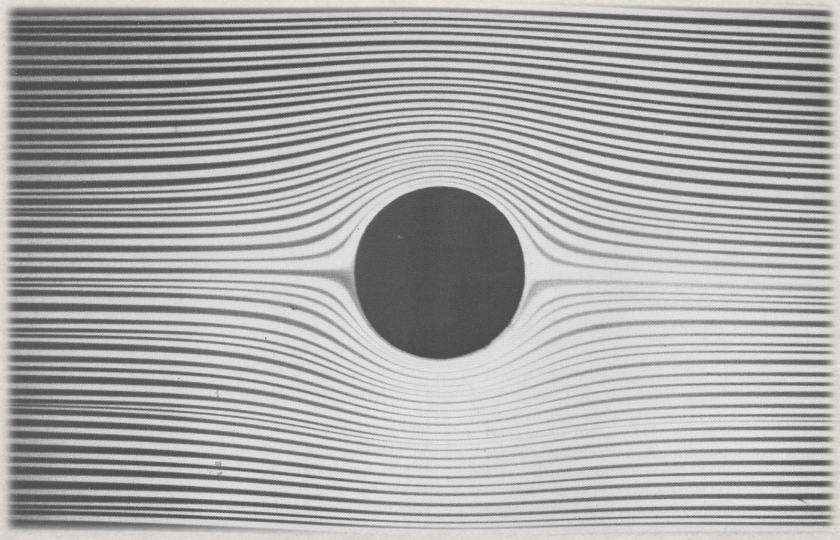
CFD

Re=10^4



Van Dycke, An Album of Fluid Motions

Hele-Shaw



Van Dycke, An Album of Fluid Motions

5. Stokes flows

5.3 Numerical solution

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Stokes Flows:

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0\,,$$

$$\boldsymbol{\nabla} p = \Delta \mathbf{u}\,.$$

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Time-dependent Stokes Flows:

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0\,,$$

$$\frac{\partial \mathbf{u}}{\partial t} = -\boldsymbol{\nabla} p + \Delta \mathbf{u}\,.$$

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$$\nabla \cdot \nabla p = \Delta p = -f$$

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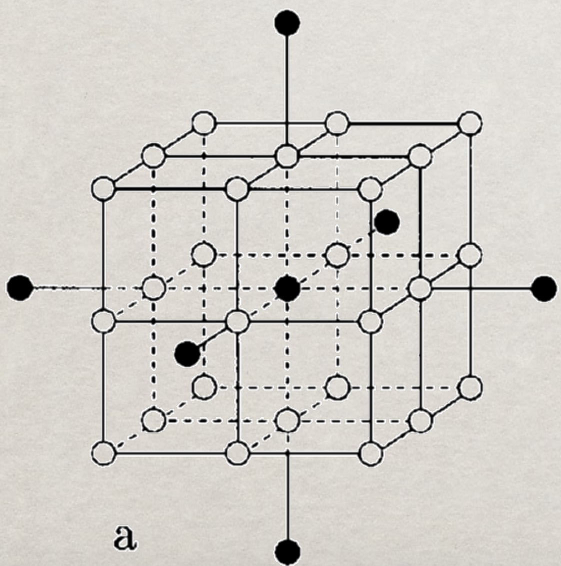
$$\nabla \cdot \nabla p = \Delta p = -f$$



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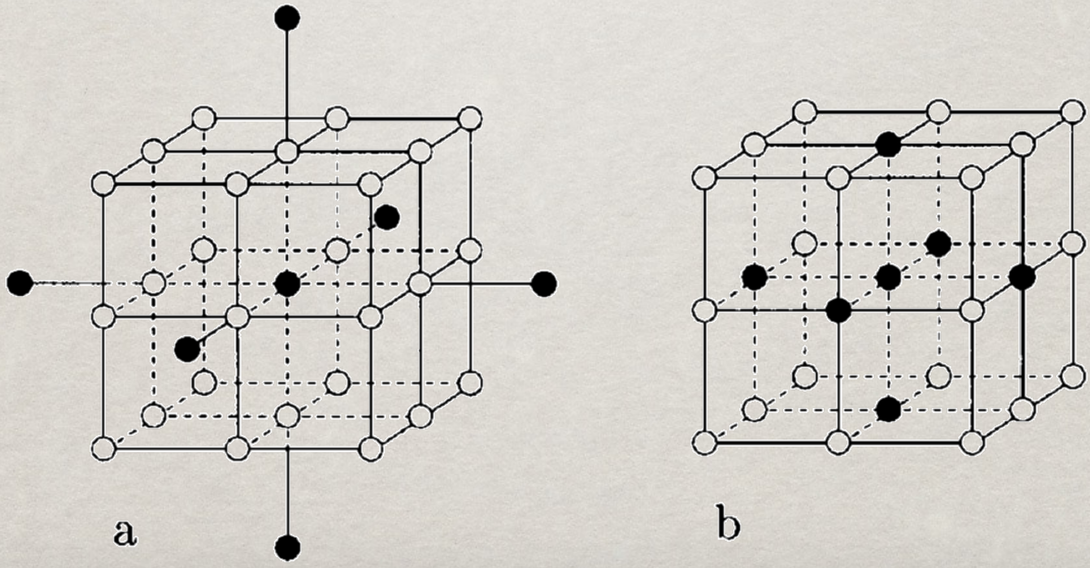
$$\nabla \cdot \nabla p = \Delta p = -f$$



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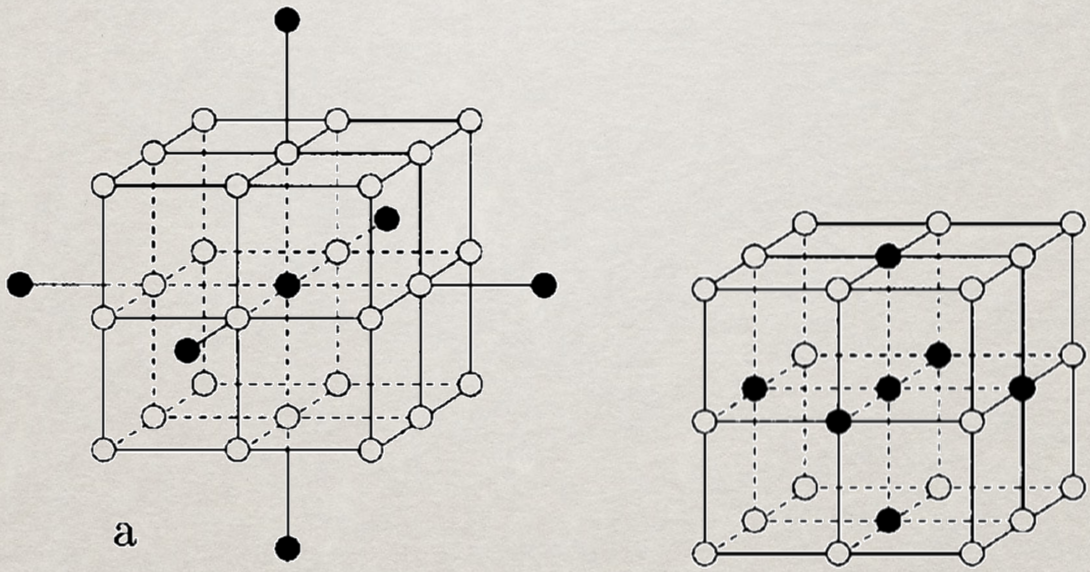
$$p_{j-2} - 2p_j + p_{j+2} = -(2\,\Delta x)^2\, f_j$$



$$p_{j-1} - 2p_j + p_{j+1} = -\Delta x^2\, f_j$$

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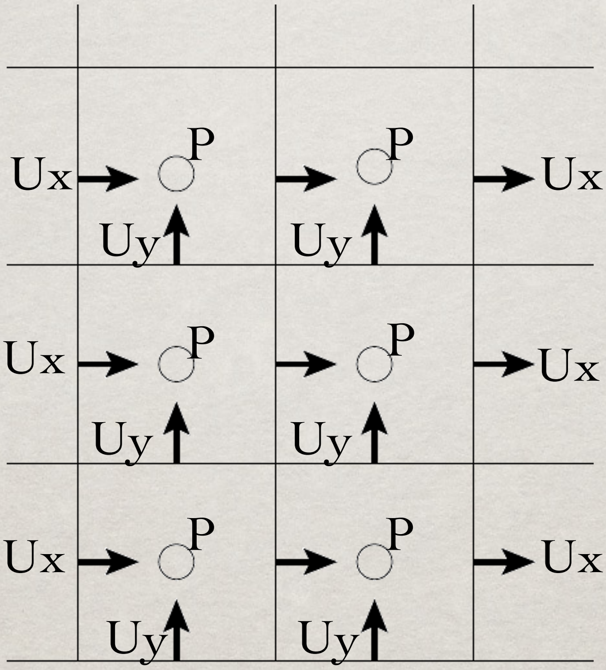

Staggered mesh

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The Hodge-Helmholtz decomposition:

$$\mathbf{w} = \mathbf{v} + \boldsymbol{\nabla}\Phi$$

$$\boldsymbol{\nabla} \cdot \mathbf{w} = \Delta\Phi$$

$$\mathbf{v} = \mathbf{w} - \boldsymbol{\nabla}\Phi$$

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The Leray projector

$$\mathbb{P}(\mathbf{u}) = \mathbf{u} - \boldsymbol{\nabla}\phi\,, \qquad \Delta\phi = \boldsymbol{\nabla}\cdot\mathbf{u}\,.$$

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$$\boldsymbol{\nabla}\cdot\mathbf{u} = 0\,,$$

$$\frac{\partial\mathbf{u}}{\partial t} = -\boldsymbol{\nabla}p + \Delta\mathbf{u}\,.$$

$$\frac{\partial\mathbf{u}}{\partial t} = \mathbb{P}(\Delta\mathbf{u})\,.$$

CFD

$$\nabla \cdot \mathbf{u} = 0\,,$$

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \Delta \mathbf{u}\,.$$

$$\boxed{\frac{\partial \mathbf{u}}{\partial t} = \mathbb{P}(\Delta \mathbf{u})\,.}$$

CFD

From the Stokes equations

$$\nabla \cdot \mathbf{u} = 0\,,$$

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla p + \Delta \mathbf{u}\,.$$

We get:

$$\Delta p = \nabla \cdot (\Delta \mathbf{u})$$

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$$\frac{\partial \mathbf{u}}{\partial t} = -\boldsymbol{\nabla} p + \Delta \mathbf{u}\,.$$

The fractional step method

$$\frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} = \Delta \mathbf{u}^n\,,$$

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^\star}{\Delta t} = -\boldsymbol{\nabla}_h p^{n+1}$$

(Temam-Chorin)

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The fractional step method

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^\star}{\Delta t} = -\boldsymbol{\nabla}_h p^{n+1}$$

$$\boldsymbol{\nabla}_h \cdot \boldsymbol{\nabla}_h\, p^{n+1} = \frac{1}{\Delta t} \boldsymbol{\nabla}_h \cdot \mathbf{u}^\star$$

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The fractional step method

$$\frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} = \Delta \mathbf{u}^n \,,$$

$$\mathbf{u}^\star \,|_{\partial\Omega} = \mathbf{g}$$

$$\nabla_h \cdot \nabla_h \, p^{n+1} = \frac{1}{\Delta t} \nabla_h \cdot \mathbf{u}^\star$$

$$\mathbf{n} \cdot \nabla_h p^{n+1} = 0$$

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The fractional step method

$$\frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} = \Delta \mathbf{u}^n \,,$$

$$\boxed{\mathbf{u}^\star \,|_{\partial\Omega} = \mathbf{g} + \Delta t \, \nabla_h p^n}$$

$$\nabla_h \cdot \nabla_h \, p^{n+1} = \frac{1}{\Delta t} \nabla_h \cdot \mathbf{u}^\star$$

$$\mathbf{n} \cdot \nabla_h p^{n+1} = 0$$

The fractional step method

Ideally, we would like to solve:

$$\frac{\mathbf{u}^* - \mathbf{u}^n}{\Delta t} = \Delta \mathbf{u}^n \,, \qquad \mathbf{u}^* = \mathbf{g} + \Delta t\, \boldsymbol{\nabla} p^{n+1} \quad \text{on} \quad \partial\Omega \,,$$

$$\Delta p^{n+1} = \frac{1}{\Delta t} \boldsymbol{\nabla} \cdot \mathbf{u}^* \,, \qquad \partial_n p^{n+1} = 0 \quad \text{on} \quad \partial\Omega \,,$$

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^*}{\Delta t} = -\boldsymbol{\nabla} p^{n+1} \,.$$

Alternative Formulations

Potentials

$$\boldsymbol{\nabla} \times \boldsymbol{\Psi} = \mathbf{u} \qquad \text{and} \qquad \boldsymbol{\nabla} \cdot \boldsymbol{\Psi} = 0$$

Stream functions (2D)

$$\boldsymbol{\Psi} = \mathit{\Psi}\, \mathbf{e}_z$$

Alternative Formulations

Potentials

$$\boldsymbol{\nabla} \times \boldsymbol{\Psi} = \mathbf{u} \qquad \text{and} \qquad \boldsymbol{\nabla} \cdot \boldsymbol{\Psi} = 0$$

Stream functions (2D)

$$\boldsymbol{\Psi} = \Psi \mathbf{e}_z$$

Poloidal-Toroidal decomposition (3D)

$$\mathbf{u} = \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\mathbf{r}\, u_p) + \boldsymbol{\nabla} \times (\mathbf{r}\, u_t)\,.$$

Alternative Formulations

Artificial compressibility

$$\boxed{\frac{1}{\beta}\frac{\partial p}{\partial t} +} \boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

$$\frac{\partial \mathbf{u}}{\partial t} = -\boldsymbol{\nabla} p + \Delta \mathbf{u}\,.$$

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Analysis of the splitting error:

$$\frac{\partial u}{\partial t} = (\mathcal{A} + \mathcal{B})u$$

$$\frac{\partial^2 u}{\partial t^2} = (\mathcal{A} + \mathcal{B})^2 u$$

$$\left(\frac{\partial}{\partial t}\right)^N u = (\mathcal{A} + \mathcal{B})^N u$$

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$$\begin{aligned} u(x,\Delta t) &= u(x,0) + \Delta t\ (\mathcal{A}+\mathcal{B})\ u(x,0) + \frac{1}{2}\Delta t^2\ (\mathcal{A}+\mathcal{B})^2\ u(x,0) + \ldots \\ &= \left(Id + \Delta t\ (\mathcal{A}+\mathcal{B}) + \frac{1}{2}\Delta t^2\ (\mathcal{A}+\mathcal{B})^2 + \ldots\right) u(x,0) \\ &= \sum_{j=0}^{\infty} \frac{\Delta t^j}{j!} (\mathcal{A}+\mathcal{B})^j\ u(x,0)\,. \end{aligned}$$

$$u(x,\Delta t) = \mathrm{e}^{\Delta t\,(\mathcal{A}+\mathcal{B})}\, u(x,0)$$

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$$u(x,\Delta t) = \mathrm{e}^{\Delta t\,(\mathcal{A}+\mathcal{B})}\,u(x,0)$$

$$u^\star(x,\Delta t) = \mathrm{e}^{\Delta t\,\mathcal{A}}\,u(x,0)\,,$$
$$u^{\star\star}(x,\Delta t) = \mathrm{e}^{\Delta t\,\mathcal{B}}\,u^\star(x,\Delta t)\,,$$
$$u^{\star\star}(x,\Delta t) = \mathrm{e}^{\Delta t\,\mathcal{B}}\,\mathrm{e}^{\Delta t\,\mathcal{A}}\,u(x,0)\,,$$

CFD

$$u(x,\Delta t) = \mathrm{e}^{\Delta t\,(\mathcal{A}+\mathcal{B})}\,u(x,0)$$

$$u^\star(x,\Delta t) = \mathrm{e}^{\Delta t\,\mathcal{A}}\,u(x,0)\,,$$
$$u^{\star\star}(x,\Delta t) = \mathrm{e}^{\Delta t\,\mathcal{B}}\,u^\star(x,\Delta t)\,,$$
$$u^{\star\star}(x,\Delta t) = \mathrm{e}^{\Delta t\,\mathcal{B}}\,\mathrm{e}^{\Delta t\,\mathcal{A}}\,u(x,0)\,,$$

$$u(x,\Delta t) - u^{\star\star}(x,\Delta t) = \left(\mathrm{e}^{\Delta t\,(\mathcal{A}+\mathcal{B})} - \mathrm{e}^{\Delta t\,\mathcal{B}}\,\mathrm{e}^{\Delta t\,\mathcal{A}}\right)\,u(x,0)$$

$$u^{\star\star}(x,\Delta t) = \left(Id + \Delta t\,(\mathcal{A}+\mathcal{B}) + \frac{1}{2}\,\Delta t^2\,\left(\mathcal{A}^2 + 2\mathcal{B}\mathcal{A} + \mathcal{B}^2\right) + \ldots\right)\,u(x,0)$$

$$u(x,\Delta t) - u^{\star\star}(x,\Delta t) = \frac{1}{2}\,\Delta t^2\,\left(\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}\right)\,u(x,0) + \mathcal{O}(\Delta t^3)$$

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$$u(x,\Delta t) - u^{\star\star}(x,\Delta t) = \frac{1}{2}\,\Delta t^2\ (\mathcal{AB} - \mathcal{BA})\ u(x,0) + \mathcal{O}(\Delta t^3)$$

$$\frac{\partial u}{\partial t} = -a\frac{\partial u}{\partial x} - \lambda(x)\,u$$

$$\mathcal{AB}\,u = a\frac{\partial}{\partial x}\,(\lambda(x)u) = a\,\lambda(x)\,\frac{\partial u}{\partial x} + a\,\lambda'(x)\,u\,,$$

$$\mathcal{BA}\,u = a\,\lambda(x)\,\frac{\partial u}{\partial x}\,.$$

$$u(x,\Delta t) - u^{\star\star}(x,\Delta t) = \frac{1}{2}\Delta t^2\,a\,\lambda'(x)\,u(x,0) + \mathcal{O}(\Delta t^3)$$