Numerical Methods for Fluid Dynamics

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CFD

Webpage for the course:

http://www.math.ens.fr/~dormy/CFD

Login: Fluid
Passwd: Flow

3. Integration in time

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3. Integration in time

3.1 Introduction

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$$
\begin{cases}
\dfrac{\partial T}{\partial t} = \kappa \dfrac{\partial^2 T}{\partial x^2}\,, \\
T = 0\,, \qquad \text{at } x = 0 \text{ and } x = 1,\ \forall t\,, \\
T = T^0\,, \qquad \text{at } t = 0\,.
\end{cases}
$$

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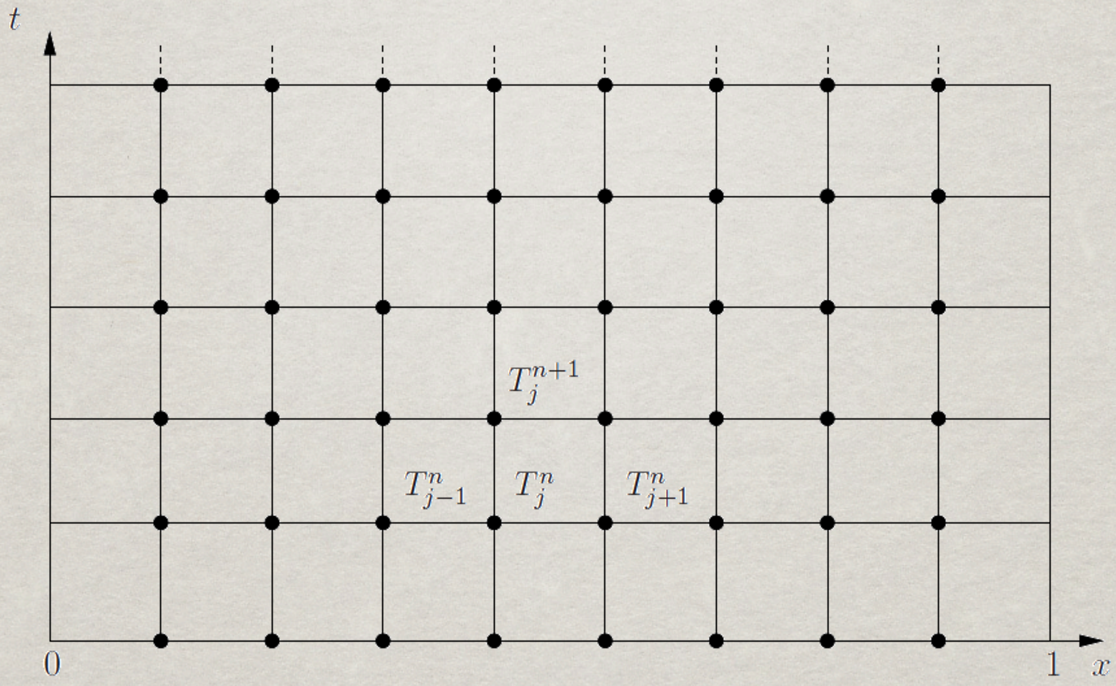

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$$\begin{cases} L(\boldsymbol{u}) = \boldsymbol{g} & \text{in} \quad \Omega\,, \\ B(\boldsymbol{u}) = \boldsymbol{\gamma} & \text{on} \quad \partial\Omega\,, \end{cases}$$

$$\begin{cases} L_h(\boldsymbol{u}_h) = \boldsymbol{g} & \text{in} \quad \Omega\,, \\ B_h(\boldsymbol{u}_h) = \boldsymbol{\gamma} & \text{on} \quad \partial\Omega\,, \end{cases}$$

$$L_h \underset{h\to 0}{\longrightarrow} L \qquad\qquad \boldsymbol{u}_h \underset{h\to 0}{\longrightarrow} \boldsymbol{u}$$

Consistancy — Convergence

$$R_h(\boldsymbol{u}) = L_h(\boldsymbol{u}) - L(\boldsymbol{u}) = L_h(\boldsymbol{u}) - \boldsymbol{g}$$

Truncation error

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$$L(T) = \frac{\partial T}{\partial t} - \kappa \frac{\partial^2 T}{\partial x^2} = 0\,,$$

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$$L(T) = \frac{\partial T}{\partial t} - \kappa \frac{\partial^2 T}{\partial x^2} = 0\,,$$

$$L_h(T) = \frac{T(x_j, t_{n+1}) - T(x_j, t_n)}{\Delta t}$$

$$-\kappa \frac{T(x_{j-1}, t_n) - 2T(x_j, t_n) + T(x_{j+1}, t_n)}{\Delta x^2}\,.$$

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$$L(T) = \frac{\partial T}{\partial t} - \kappa \frac{\partial^2 T}{\partial x^2} = 0\,,$$

$$L_h(T) = \frac{T(x_j, t_{n+1}) - T(x_j, t_n)}{\Delta t}$$

$$-\kappa \frac{T(x_{j-1}, t_n) - 2T(x_j, t_n) + T(x_{j+1}, t_n)}{\Delta x^2}\,.$$

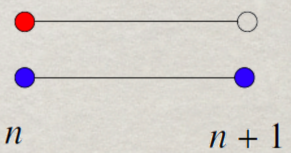
$$R_h(T) = \frac{\Delta t}{2} \left. \frac{\partial^2 T}{\partial t^2} \right|_j^n - \kappa \frac{\Delta x^2}{12} \left. \frac{\partial^4 T}{\partial x^4} \right|_j^n + \mathcal{O}(\Delta t^2) + \mathcal{O}(\Delta x^4)\,.$$

3. Integration in time

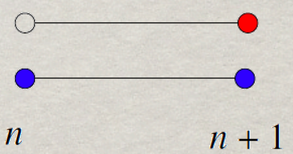
3.2 Ordinary Differential Equations

$$\frac{\mathrm{d}u}{\mathrm{d}t} = f(t,u) \qquad \text{with} \qquad u(0) = u0\,.$$

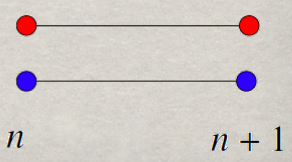
$$u^{n+1} = u^n + \Delta t\, f^n\,,$$



$$u^{n+1} = u^n + \Delta t\, f^{n+1}\,,$$



$$u^{n+1} = u^n + \Delta t\, \frac{f^{n+1} + f^n}{2}\,.$$



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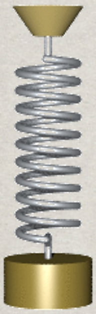

The harmonic oscillator

$$\ddot{X} + \omega^2 X = 0\,.$$

$$\dot{X} = \omega Y\,, \qquad \text{and} \qquad \dot{Y} = -\omega X\,.$$

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The harmonic oscillator

$$\dot{X} = \omega Y\,, \qquad \text{and} \qquad \dot{Y} = -\omega X\,.$$

Explicit:

$$X^{n+1} = X^n + \Delta t\,\omega\,Y^n\,, \qquad Y^{n+1} = Y^n - \Delta t\,\omega\,X^n\,.$$

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The harmonic oscillator

$$\dot{X} = \omega Y\,, \qquad \text{and} \qquad \dot{Y} = -\omega X\,.$$

Explicit:

$$X^{n+1} = X^n + \Delta t\,\omega\,Y^n\,, \qquad Y^{n+1} = Y^n - \Delta t\,\omega\,X^n\,.$$

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The harmonic oscillator

$$\dot{X} = \omega Y\,, \qquad \text{and} \qquad \dot{Y} = -\omega X\,.$$

Explicit:

$$X^{n+1} = X^n + \Delta t\,\omega\,Y^n\,, \qquad Y^{n+1} = Y^n - \Delta t\,\omega\,X^n\,.$$

Implicit:

$$X^{n+1} = X^n + \Delta t\,\omega\,Y^{n+1}\,, \quad Y^{n+1} = Y^n - \Delta t\,\omega\,X^{n+1}\,.$$

Crank-Nicholson:

$$X^{n+1} = X^n + \Delta t\,\omega\,\left(Y^n + Y^{n+1}\right)/2\,,$$

$$Y^{n+1} = Y^n - \Delta t\,\omega\,\left(X^n + X^{n+1}\right)/2\,.$$

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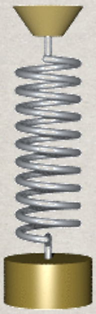
The harmonic oscillator

$$\dot{X} = \omega Y \,, \qquad \text{and} \qquad \dot{Y} = -\omega X \,.$$



$$\mathscr{H} = X^2 + Y^2$$

$$\dot{\mathscr{H}} = 2\omega XY - 2\omega XY = 0$$

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The harmonic oscillator

$$\dot{X} = \omega Y \,, \qquad \text{and} \qquad \dot{Y} = -\omega X \,.$$

Implicit:

$$X^{n+1} = X^n + \Delta t\, \omega\, Y^{n+1} \,, \quad Y^{n+1} = Y^n - \Delta t\, \omega\, X^{n+1} \,.$$

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The harmonic oscillator

$$\dot{X} = \omega Y, \qquad \text{and} \qquad \dot{Y} = -\omega X .$$

Implicit:

$$X^{n+1} = X^n + \Delta t\, \omega\, Y^{n+1}, \quad Y^{n+1} = Y^n - \Delta t\, \omega\, X^{n+1} .$$

$$\mathcal{V} = X + iY$$

$$\dot{\mathcal{V}} = -i\omega \mathcal{V} .$$

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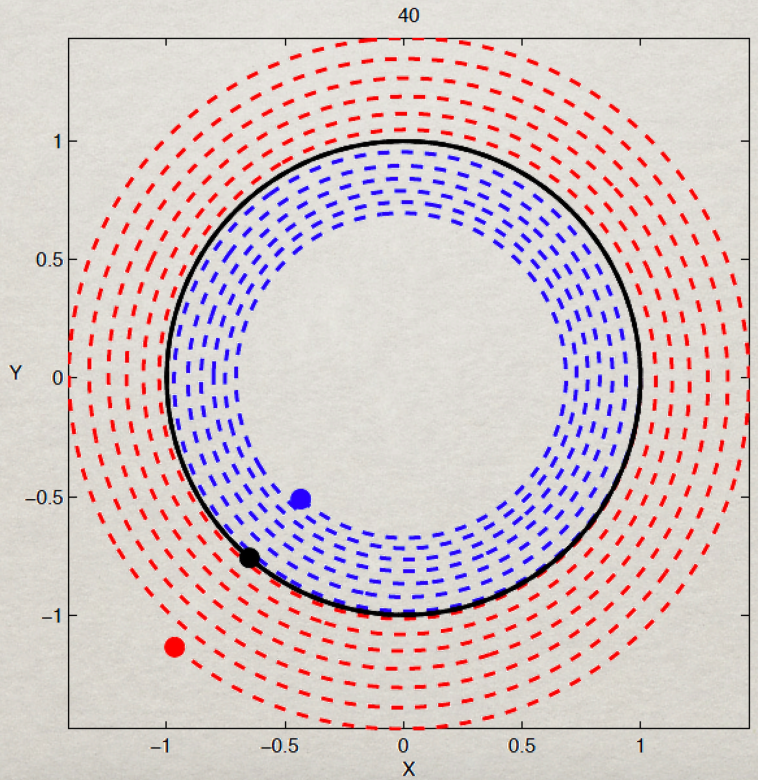
The harmonic oscillator



3. Integration in time

3.2 Ordinary Differential Equations

3.2.1 Linear Multistep methods

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Linear multistep methods

$$u^{n+1-s}, \ldots, u^{n-1}, u^n$$

$$\sum_{j=0}^{s} \alpha_j \, u^{n+1-j} = \Delta t \sum_{j=0}^{s} \beta_j \, f^{n+1-j} \, .$$

$$f^{n+1-j} = f(t^{n+1-j}, u^{n+1-j})$$

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$$u^{n+1} = \sum_{j=1}^{s} \left(-\alpha_j \, u^{n+1-j} + \Delta t \, \beta_j f_{n+1-j}\right)$$

Adams family

$$\alpha_0 = 1\,, \quad \alpha_1 = -1\,, \quad \text{and}\, \alpha_j = 0 \quad \forall j > 1\,.$$

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Adams-Bashforth

$$u^{n+1} = u^n + \Delta t \, f^n$$

$$u^{n+1} = u^n + \Delta t \left(\frac{3}{2} f_n - \frac{1}{2} f_{n-1}\right)$$

Adams-Bashforth

$$(u^{n+1} - u^n)/\Delta t = f^n + \mathcal{O}(\Delta t), \tag{3.25}$$

$$(u^{n+1} - u^n)/\Delta t = (3/2)\, f^n - (1/2)\, f^{n-1} + \mathcal{O}(\Delta t^2), \tag{3.26}$$

$$(u^{n+1} - u^n)/\Delta t = (23/12)\, f^n - (16/12)\, f^{n-1} + (5/12)\, f^{n-2} + \mathcal{O}(\Delta t^3)\,. \tag{3.27}$$

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Adams-Moulton

$$u^{n+1} = u^n + \Delta t\, f^{n+1}\,,$$

$$u^{n+1} = u^n + \Delta t\, \frac{(f^n + f^{n+1})}{2}$$

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Adams-Moulton

$$(u^{n+1} - u^n)/\Delta t = f^{n+1} + \mathcal{O}(\Delta t), \tag{3.29}$$

$$(u^{n+1} - u^n)/\Delta t = (1/2)\, f^{n+1} + (1/2)\, f^n + \mathcal{O}(\Delta t^2), \tag{3.30}$$

$$(u^{n+1} - u^n)/\Delta t = (5/12)\, f^{n+1} + (8/12)\, f^n - (1/12)\, f^{n-1} + \mathcal{O}(\Delta t^3)\,. \tag{3.31}$$

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Backward differentiation formula (BDF)

$$(u^{n+1} - u^n)/\Delta t = f^{n+1} + \mathcal{O}(\Delta t), \tag{3.34}$$

$$((3/2)\, u^{n+1} - 2\, u^n + (1/2)\, u^{n-1})/\Delta t = f^{n+1} + \mathcal{O}(\Delta t^2), \tag{3.35}$$

$$((11/6)\, u^{n+1} - 3\, u^n + (3/2)\, u^{n-1} - (1/3)\, u^{n-2})/\Delta t = f^{n+1} + \mathcal{O}(\Delta t^3)\,. \tag{3.36}$$

3. Integration in time

3.2 Ordinary Differential Equations

3.2.2 Multistage Methods

An alternative : multi-stage formulas

 

Runge & Kutta

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Runge-Kutta (RK2)

$$U_1 = u^n , \quad U_2 = u^n + \Delta t\, f(t^n, U_1)$$

$$u^{n+1} = u^n + \frac{\Delta t}{2}\left(f(t^n, U_1) + f(t^{n+1}, U_2)\right)$$

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Runge-Kutta (RK4)

$$\begin{aligned}
U_1 &= \Delta t\, f(t^n, u^n)\,, \\
U_2 &= \Delta t\, f(t^n + \frac{\Delta t}{2}, u^n + \frac{U_1}{2})\,, \\
U_3 &= \Delta t\, f(t^n + \frac{\Delta t}{2}, u^n + \frac{U_2}{2})\,, \\
U_4 &= \Delta t\, f(t^n + \Delta t, u^n + U_3)\,, \\
u^{n+1} &= u^n + \frac{U_1}{6} + \frac{U_2}{3} + \frac{U_3}{3} + \frac{U_4}{6}\,.
\end{aligned}$$

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3. Integration in time

3.2 Ordinary Differential Equations

3.2.3 Stability

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There are two different stability concepts:

• Zero stability: If $t > 0$ is held constant, do the values $u(t)$ remain bounded as $\Delta t \to 0$?
(i.e. vanishing time-step for fixed time)

• Absolute stability: If Δt is kept fixed, do the computed values $u(t)$ remain bounded as $t \to \infty$?
(i.e. finite time-step, infinite time horizon)

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Absolute stability

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \lambda\, u \qquad \text{with} \qquad u(0) = u0$$

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Absolute stability

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \lambda\, u \qquad \text{with} \qquad u(0) = u0$$

$$|\, u(t)\, | \leq |\, u(0)\, |, \quad \forall t > 0 \quad \leftrightarrow \quad Re(\lambda) \leq 0$$

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Absolute stability

$$\frac{\mathrm{d}u}{\mathrm{d}t} = \lambda\, u \qquad \text{with} \qquad u(0) = u0$$

$$\mid u(t) \mid \leq \mid u(0) \mid, \quad \forall t > 0 \quad \leftrightarrow \quad Re(\lambda) \leq 0$$

$$\mid u^{n+1} \mid \leq \mid u^{n} \mid, \quad n = 0, 1, 2, \ldots$$

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$$\mu = \lambda \Delta t$$

$$u^{n+1} = \mathcal{Z} u^{n} \qquad \mid \mathcal{Z} \mid \leq 1$$

Explicit Euler:

$$\mathcal{Z} = 1 + \mu$$

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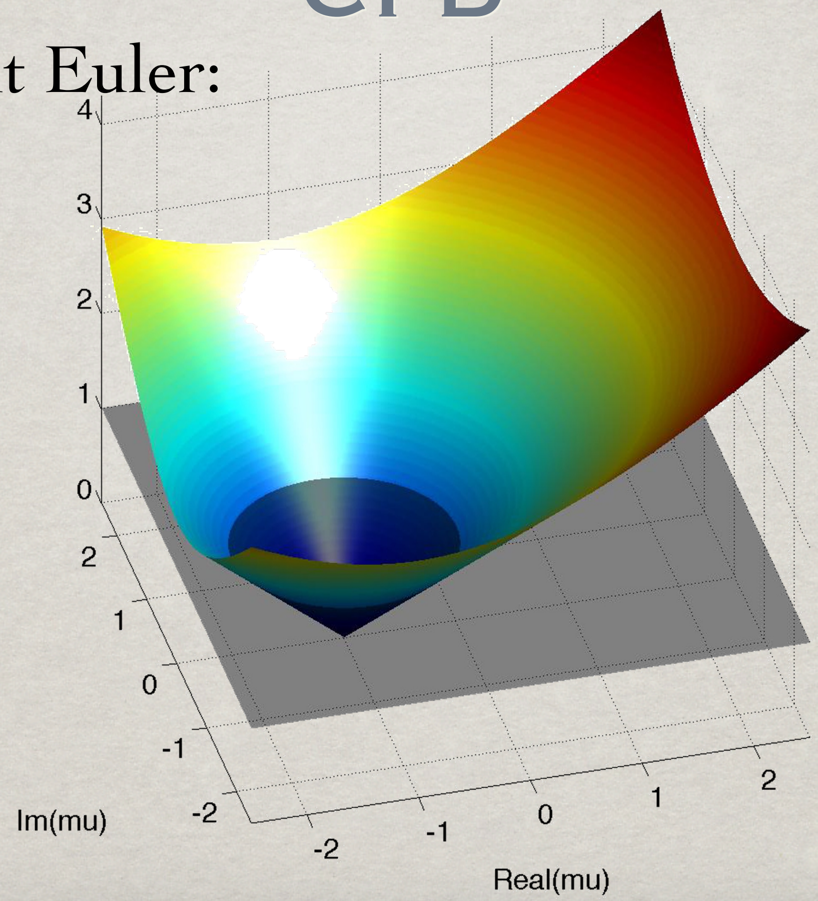
Explicit Euler:



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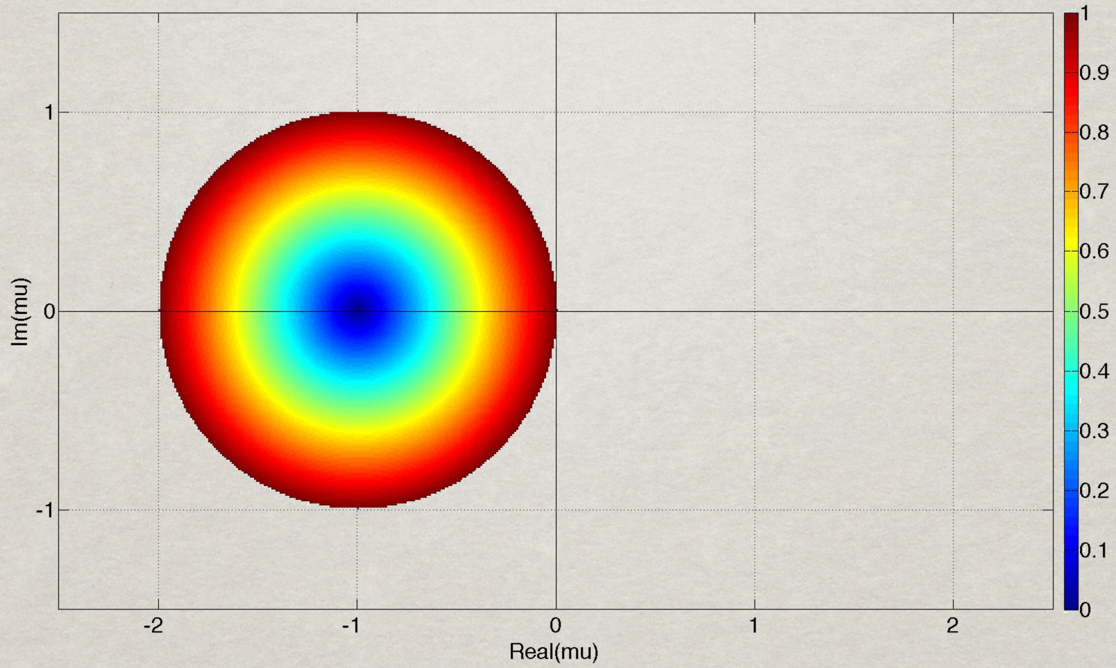
Explicit Euler:



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$$\mu = \lambda \Delta t$$

$$u^{n+1} = \mathcal{Z} u^n \qquad | \mathcal{Z} | \leq 1$$

Implicit Euler:

$$\mathcal{Z} = \frac{1}{1-\mu}$$

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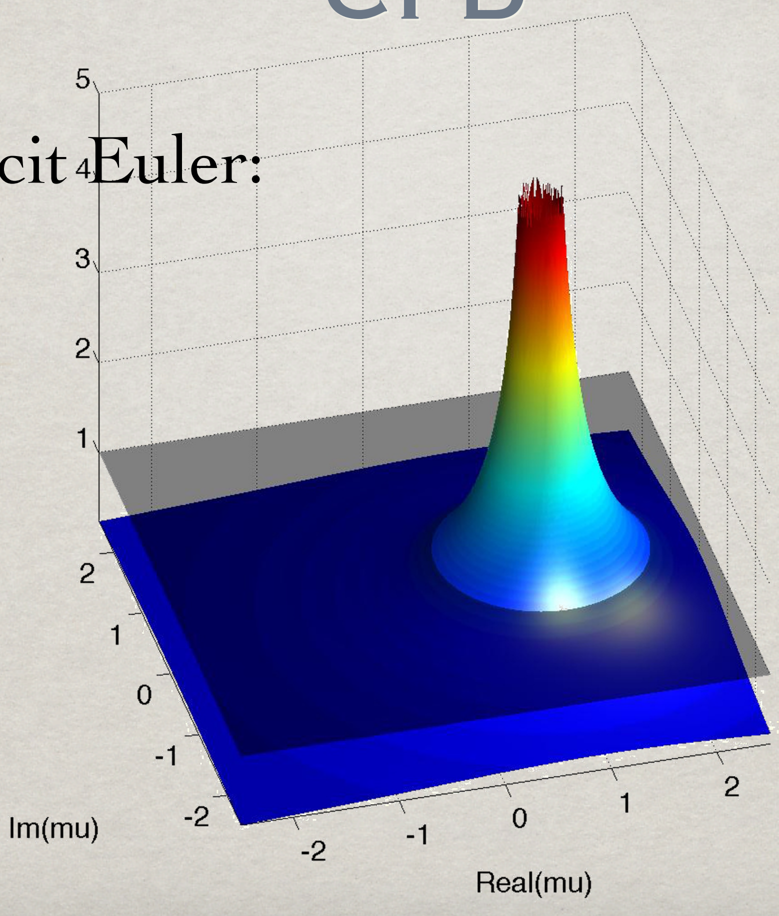
Implicit Euler:



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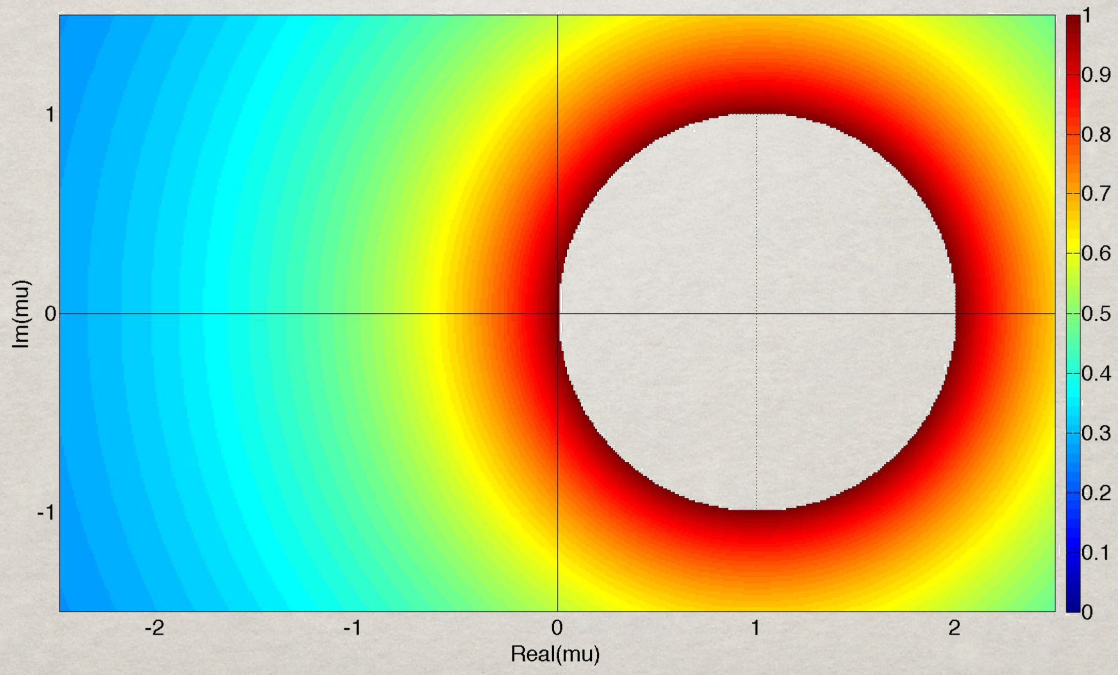
Implicit Euler:



CFD

$$\mu = \lambda \Delta t$$

$$u^{n+1} = \mathcal{Z} u^n \qquad |\, \mathcal{Z} \,| \leq 1$$

Midpoint formula

$$\mathcal{Z} = \frac{1 + \mu/2}{1 - \mu/2}$$

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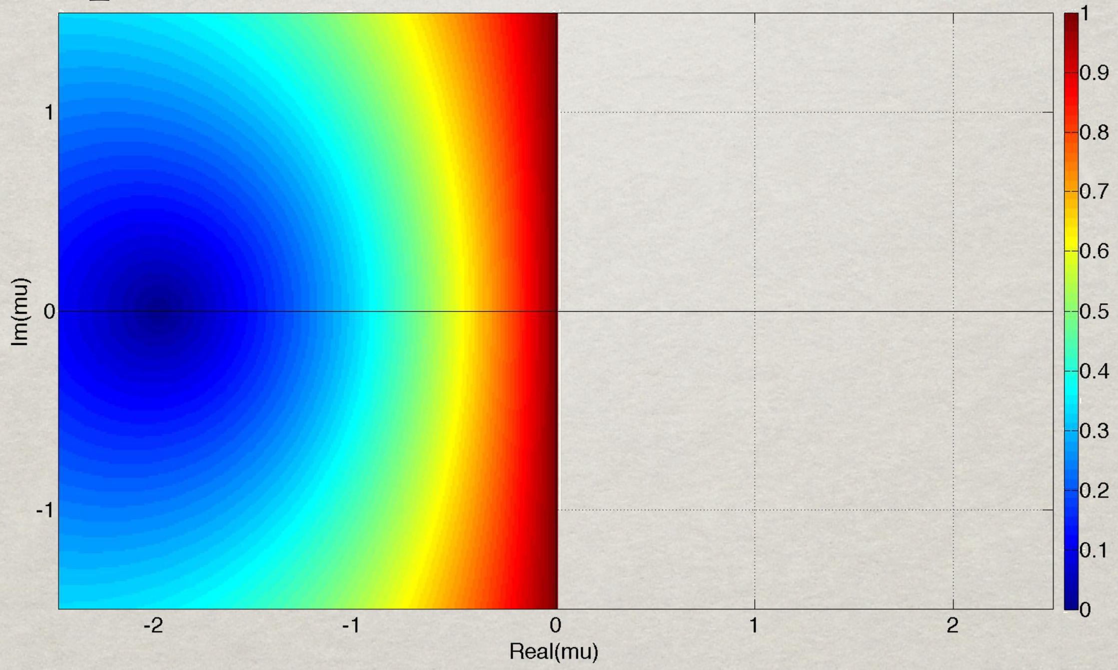
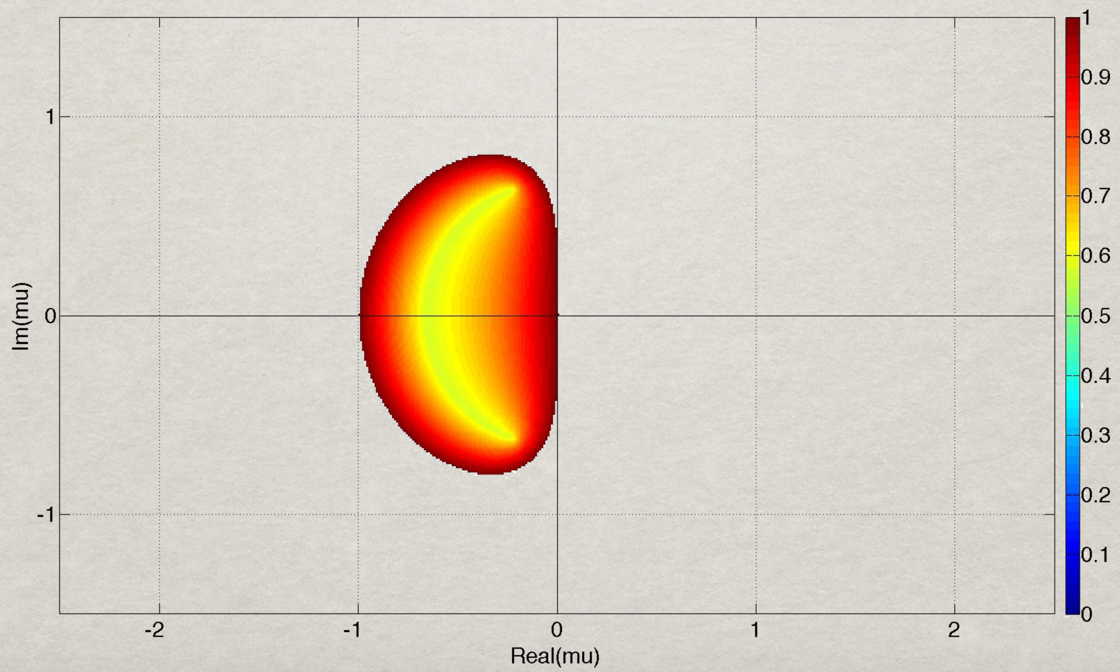
Midpoint formula



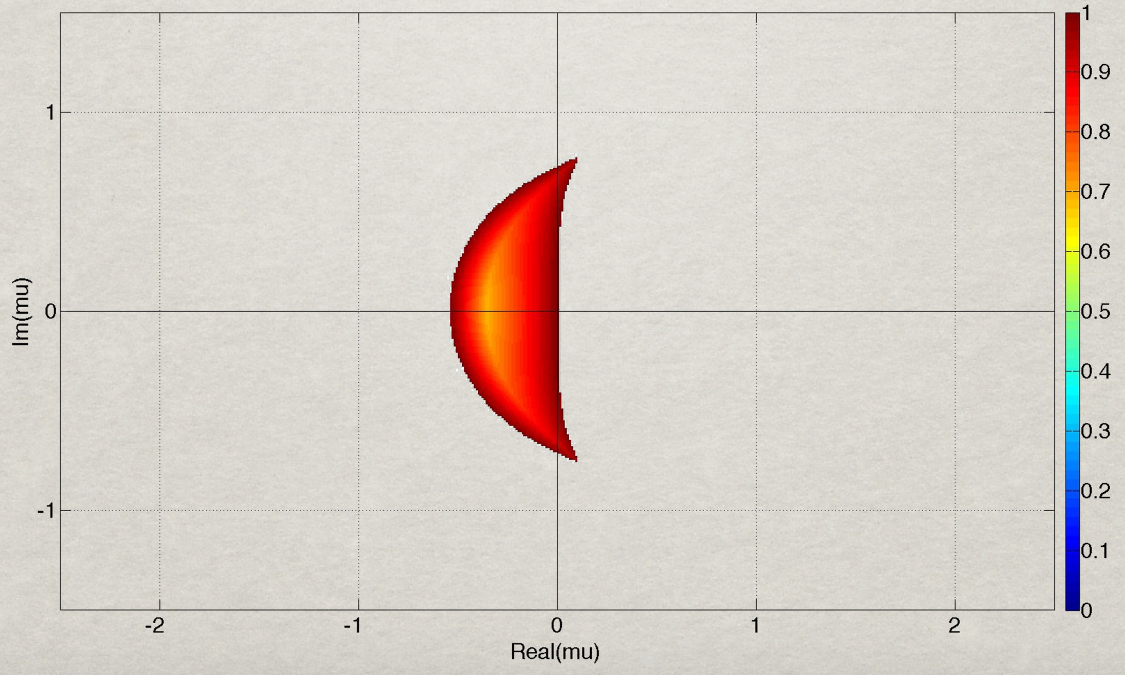
CFD

Adams-Bashforth 2



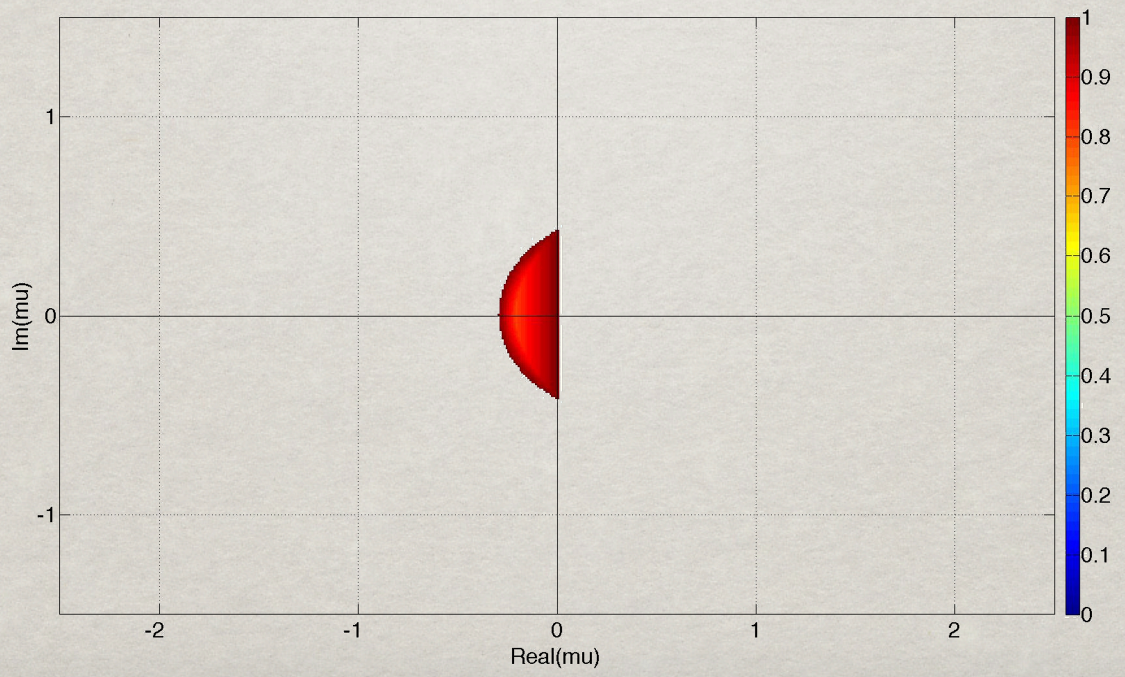
CFD

Adams-Bashforth 3



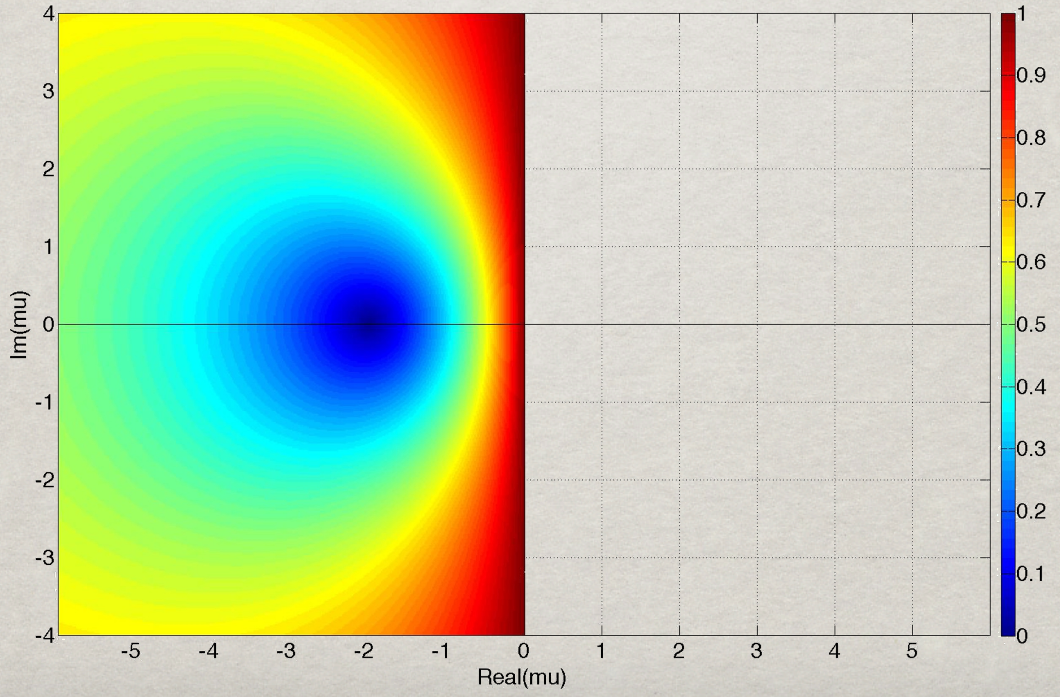
CFD

Adams-Bashforth 4



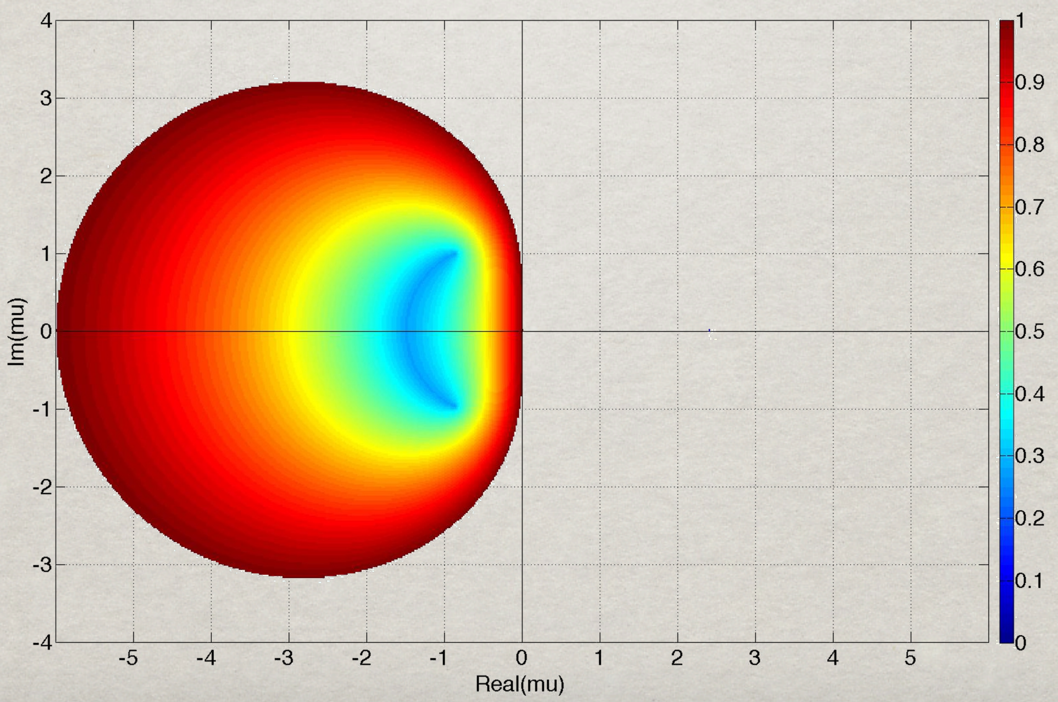
CFD

Adams-Moulton 2



CFD

Adams-Moulton 3



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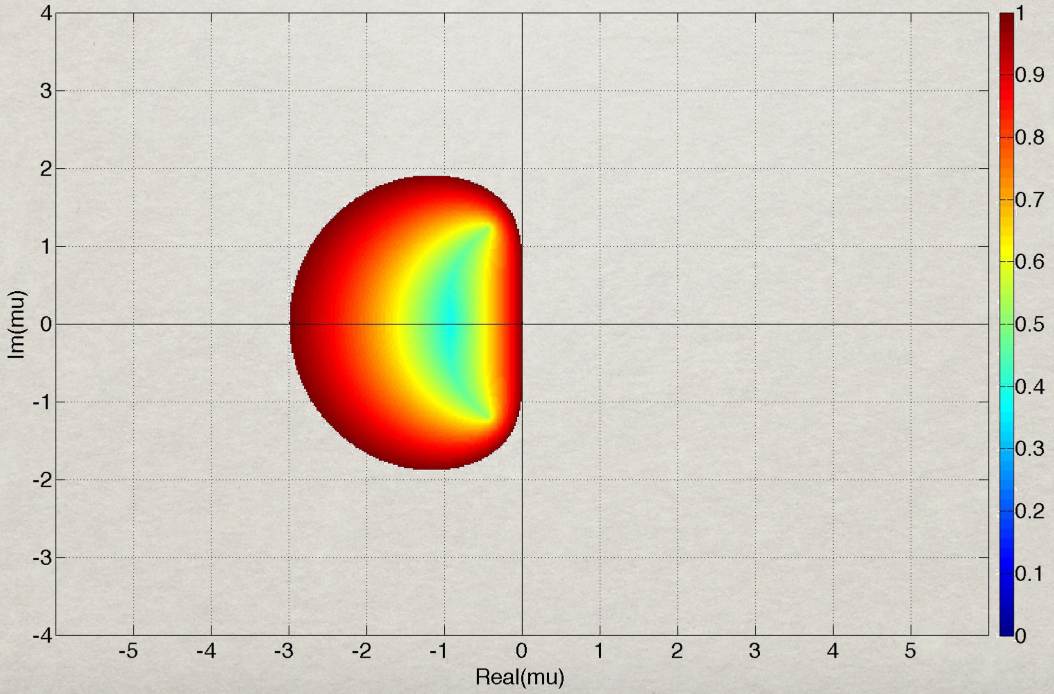
Adams-Moulton 4



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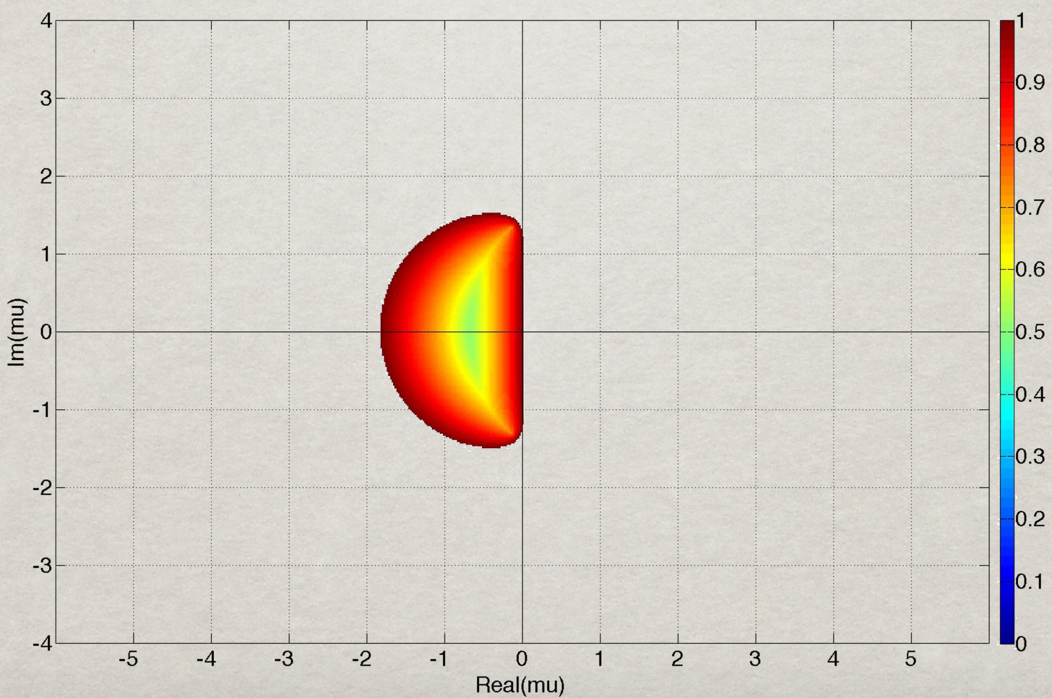
Adams-Moulton 5



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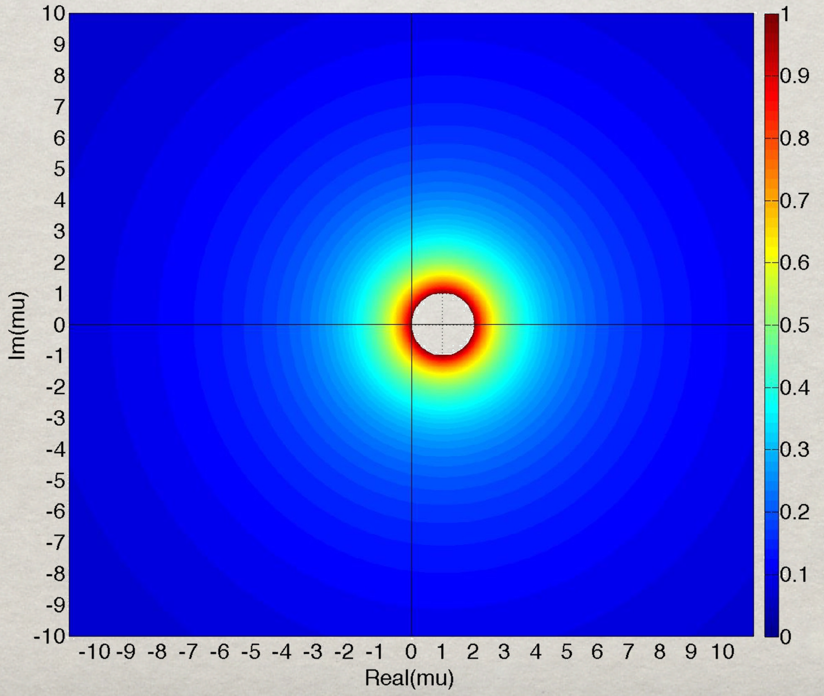
BDF1:



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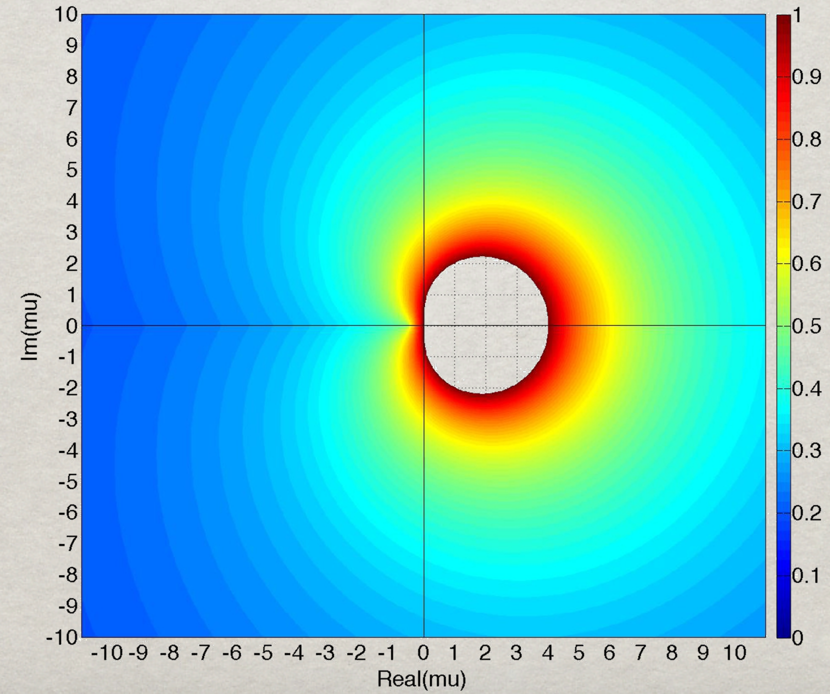
BDF2:



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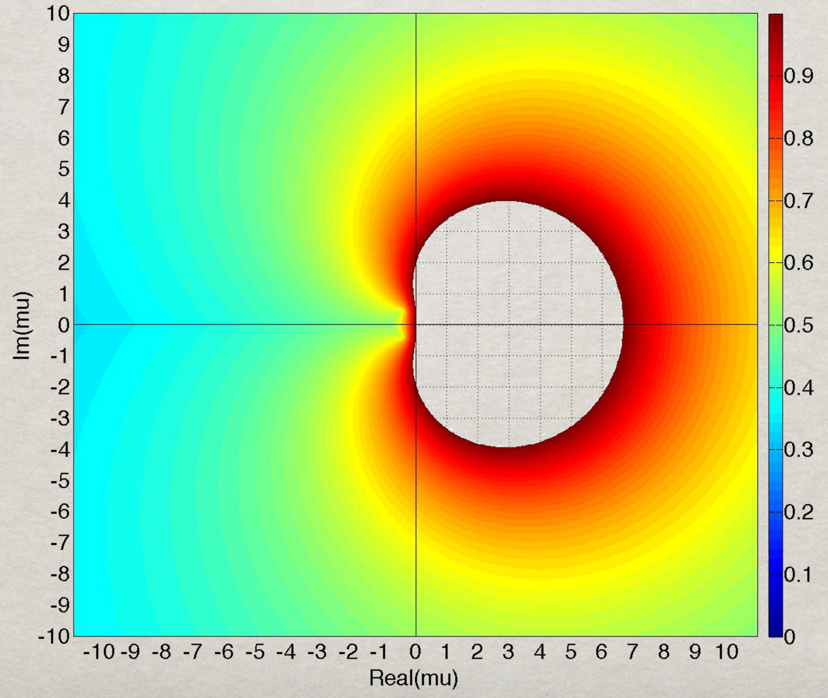
BDF3:



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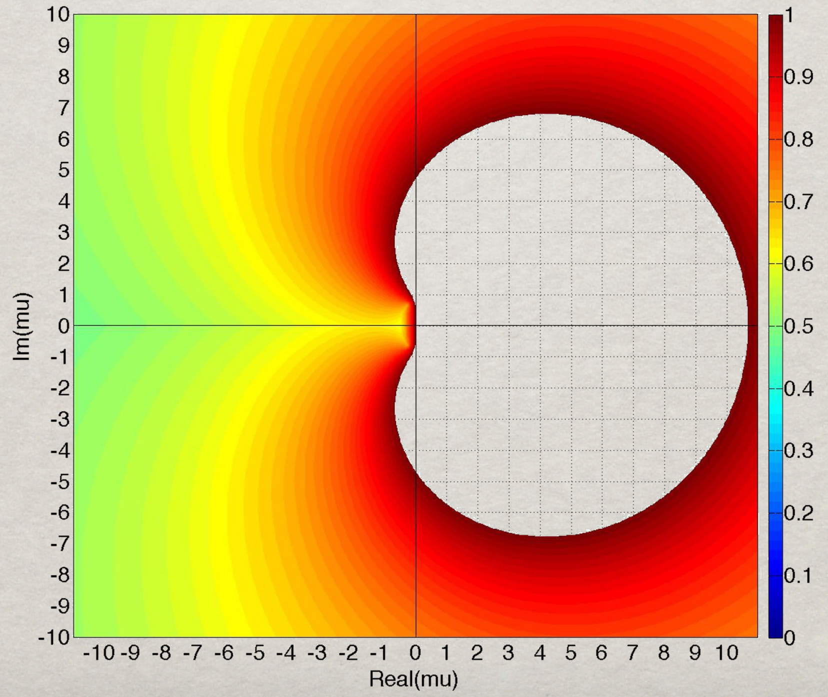
BDF4:



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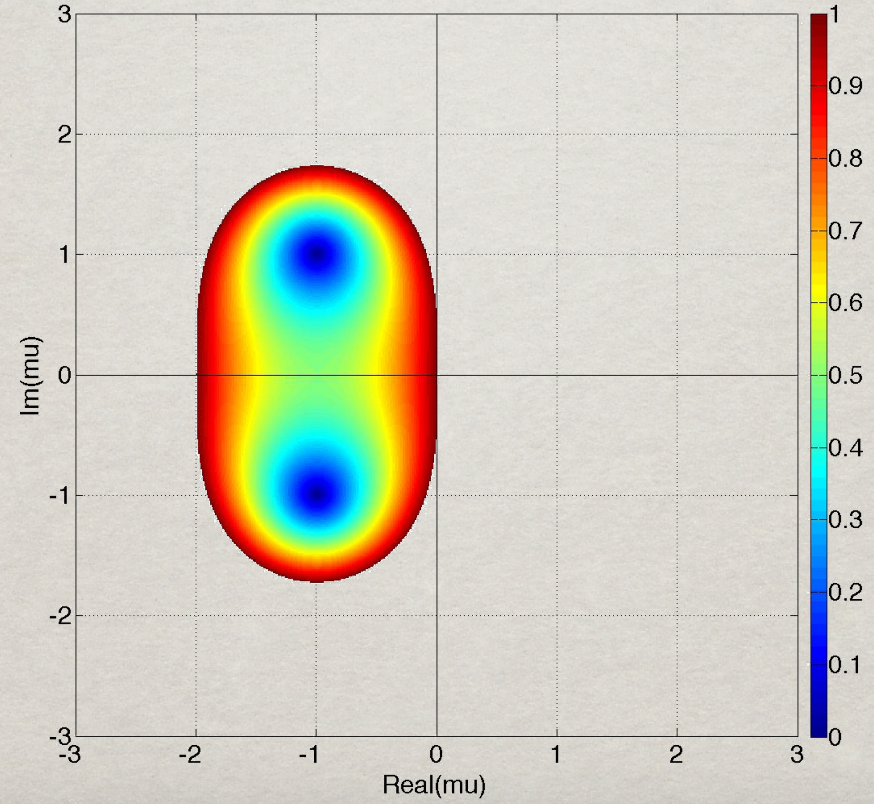
RK2:



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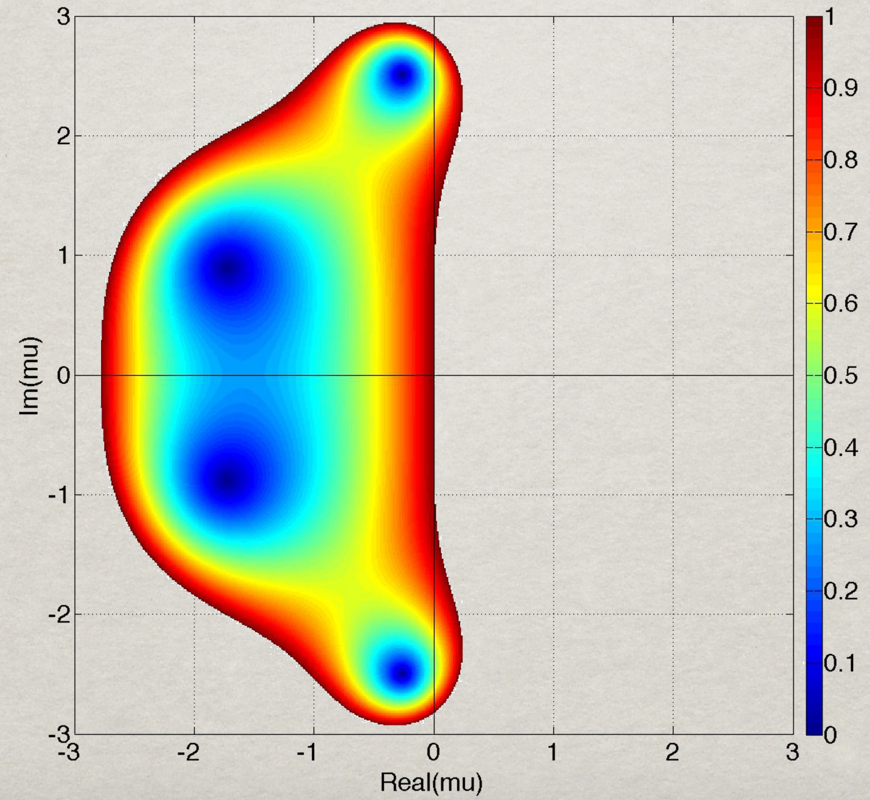
RK4:



3. Integration in time

3.3 Partial Differential Equations

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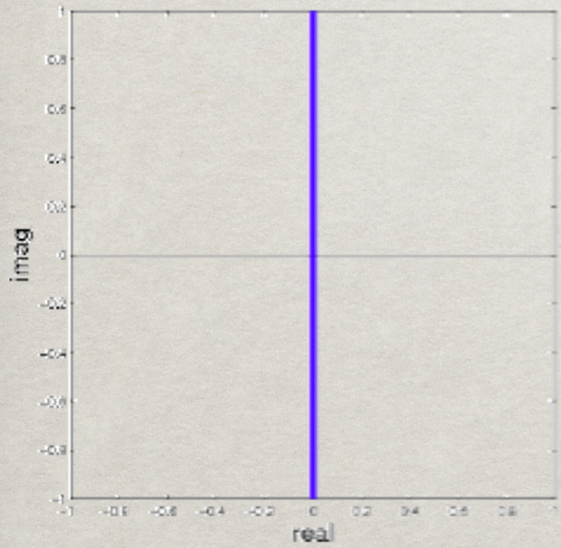
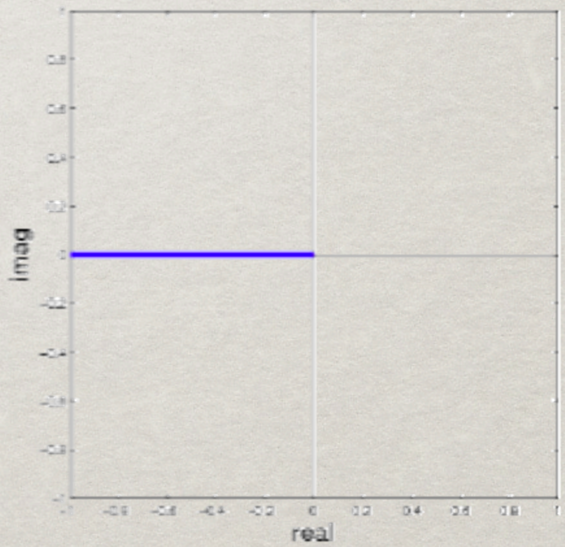
$$u_t = u_x$$



$$u_t = u_{xx}$$



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What is the spectrum of $\mathbf{A} = \begin{pmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{pmatrix}$

or, more generally,

What is the spectrum of $\mathbf{A} = \begin{pmatrix} a_0 & a_{-1} & & \cdots & a_{1-N} \\ a_1 & a_0 & & & \vdots \\ & \ddots & \ddots & \ddots & \\ \vdots & & & a_0 & a_{-1} \\ a_{N-1} & \cdots & & a_1 & a_0 \end{pmatrix}$

Toeplitz matrix

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Advection-Diffusion Problem

$$\frac{\partial u}{\partial t} = c\,\frac{\partial u}{\partial x} + \nu\,\frac{\partial^2 u}{\partial x^2} \qquad u(t=0,x) = u_0(x)$$

Semi-discrete expression

$$\frac{\partial u}{\partial t} = c\,\frac{u(x+\Delta x) - u(x-\Delta x)}{2\Delta x} + \nu\,\frac{u(x-\Delta x) - 2u(x) + u(x+\Delta x)}{\Delta x^2}$$

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Discrete Spectrum

$$\lambda_j = c\, i\, \sin(k_j \Delta x)/\Delta x + 2\, \nu\, \frac{\cos(k_j \Delta x) - 1}{\Delta x^2}$$

$$\text{with} \quad k_j = j - \frac{N}{2} \qquad j = 1 \ldots N$$

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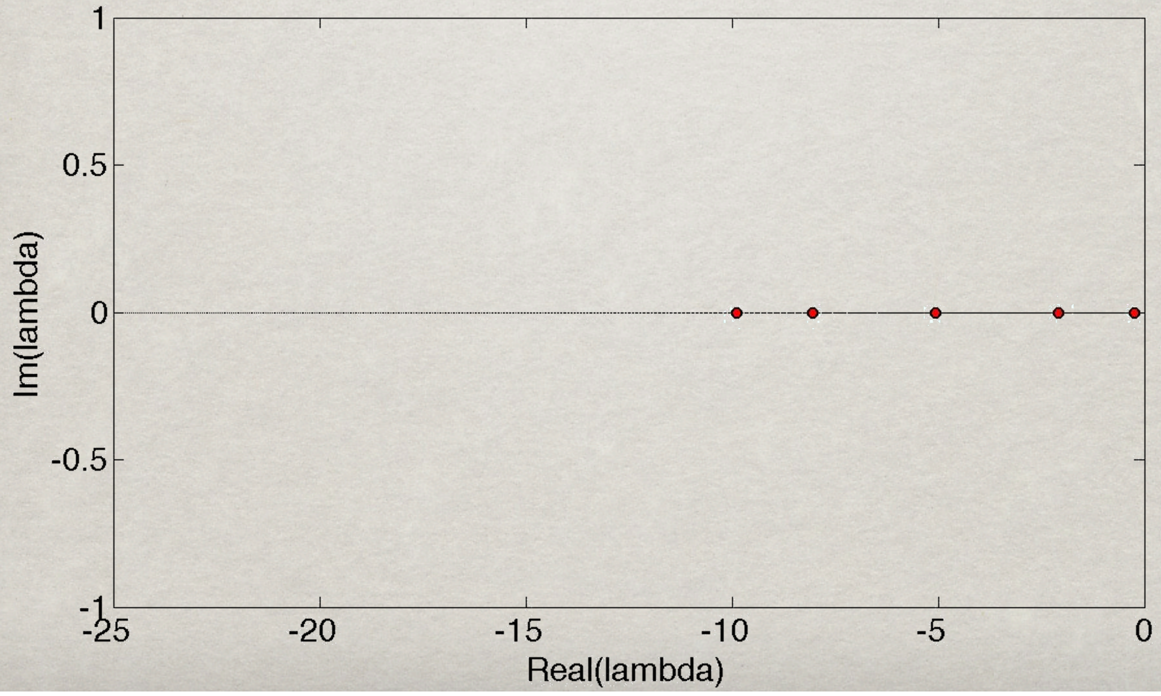

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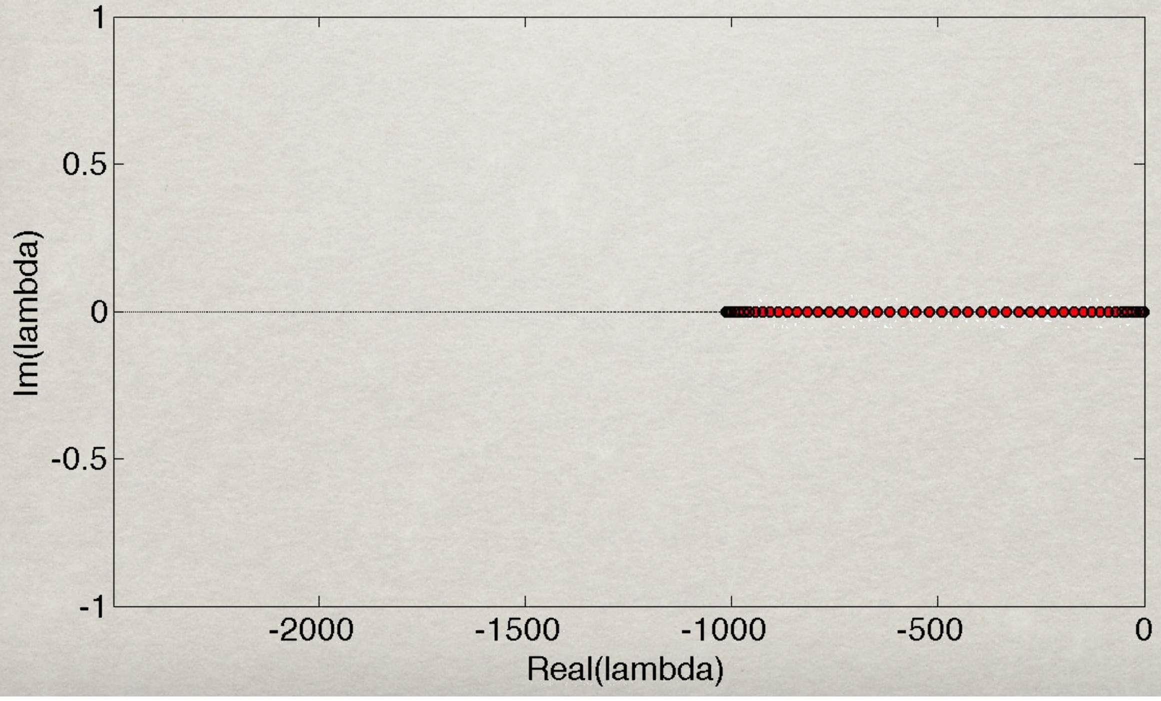
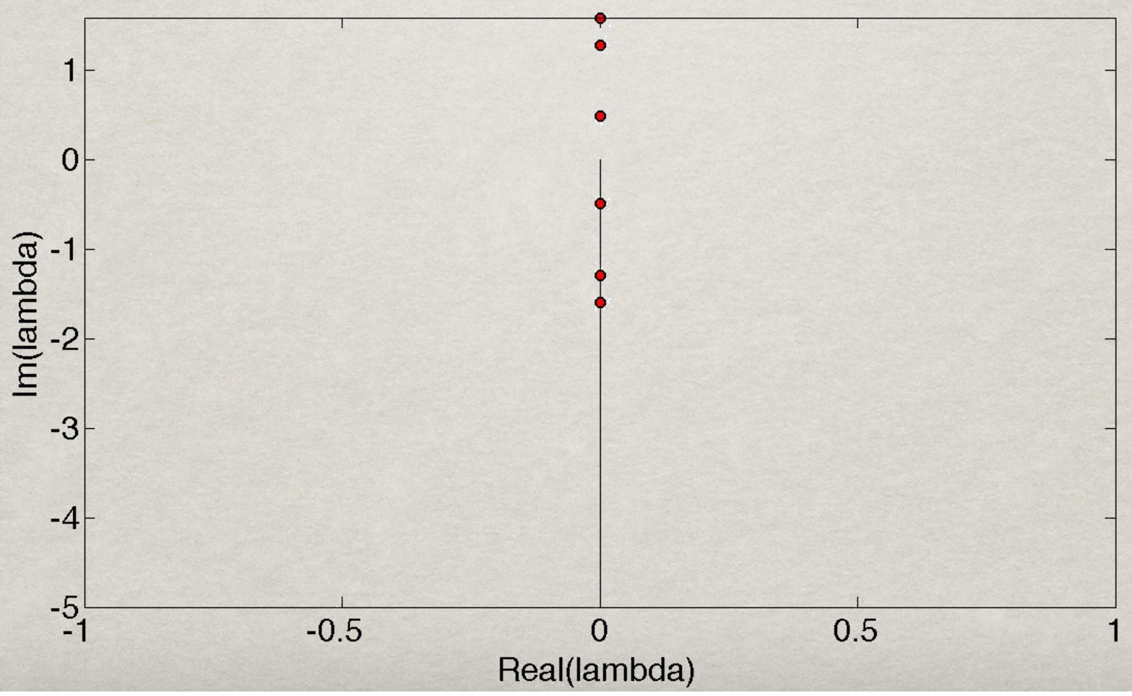


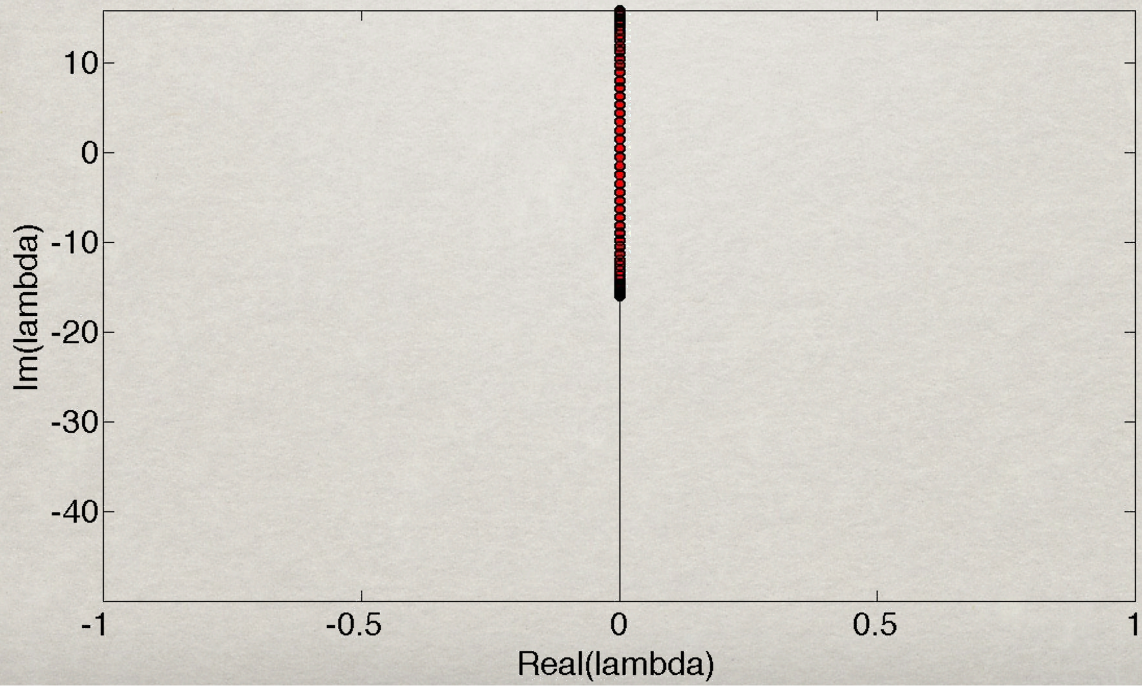
Im(lambda)
Real(lambda)
1
0.5
0
-0.5
-1
-2000
-1500
-1000
-500
0

CFD



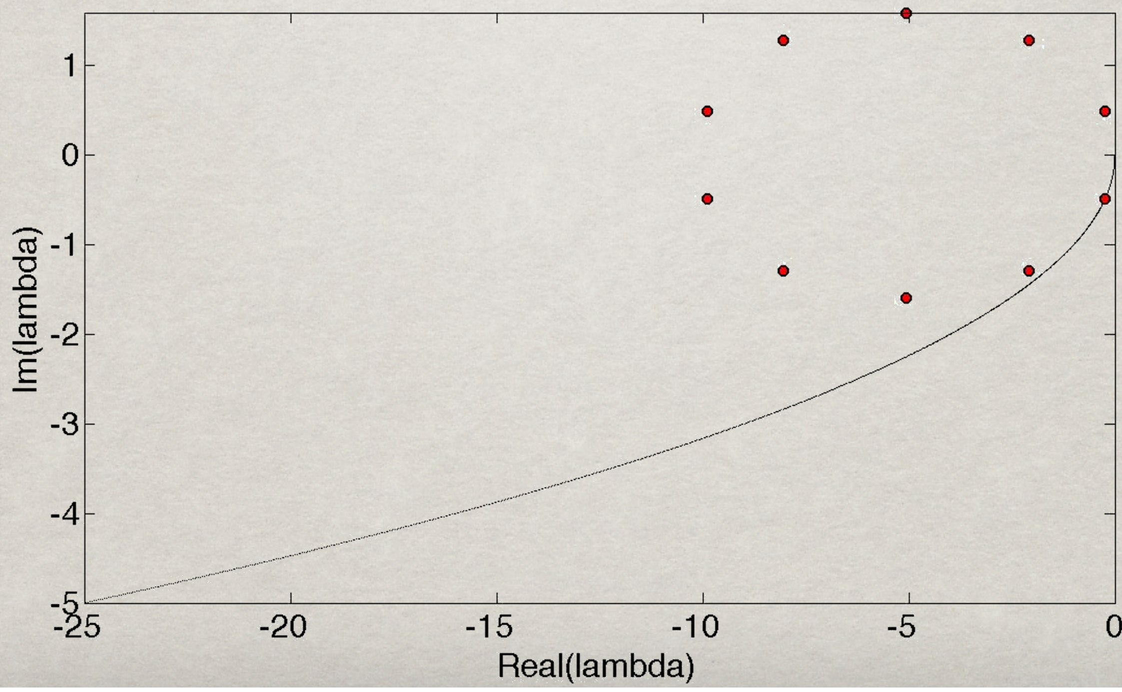
Im(lambda)
Real(lambda)
1
0
-1
-2
-3
-4
-5
-1
-0.5
0
0.5
1

CFD



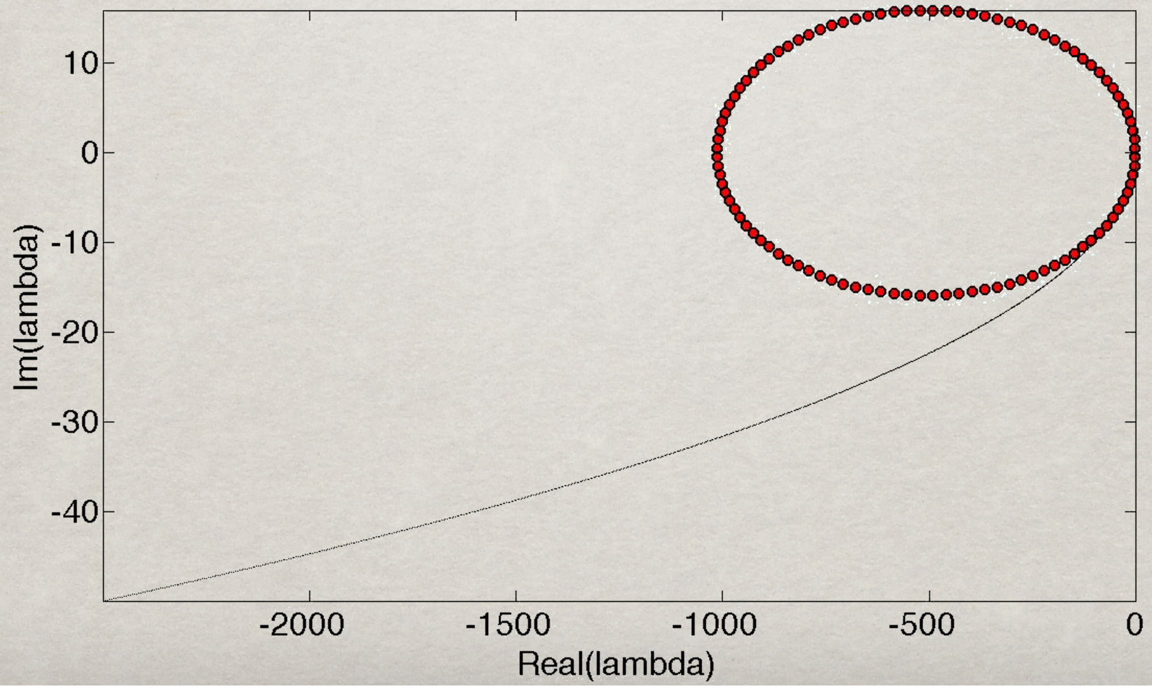
10
0
-10
-20
-30
-40
Im(lambda)
-1
-0.5
0
0.5
1
Real(lambda)

CFD



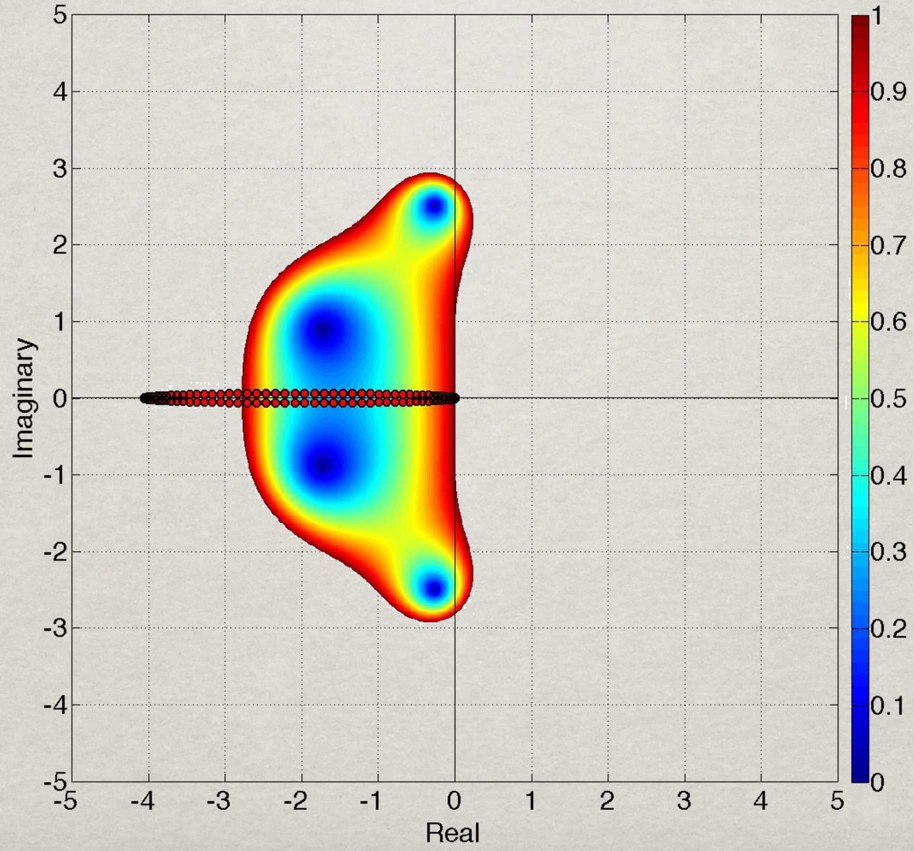
1
0
-1
-2
-3
-4
-5
Im(lambda)
-25
-20
-15
-10
-5
0
Real(lambda)

CFD



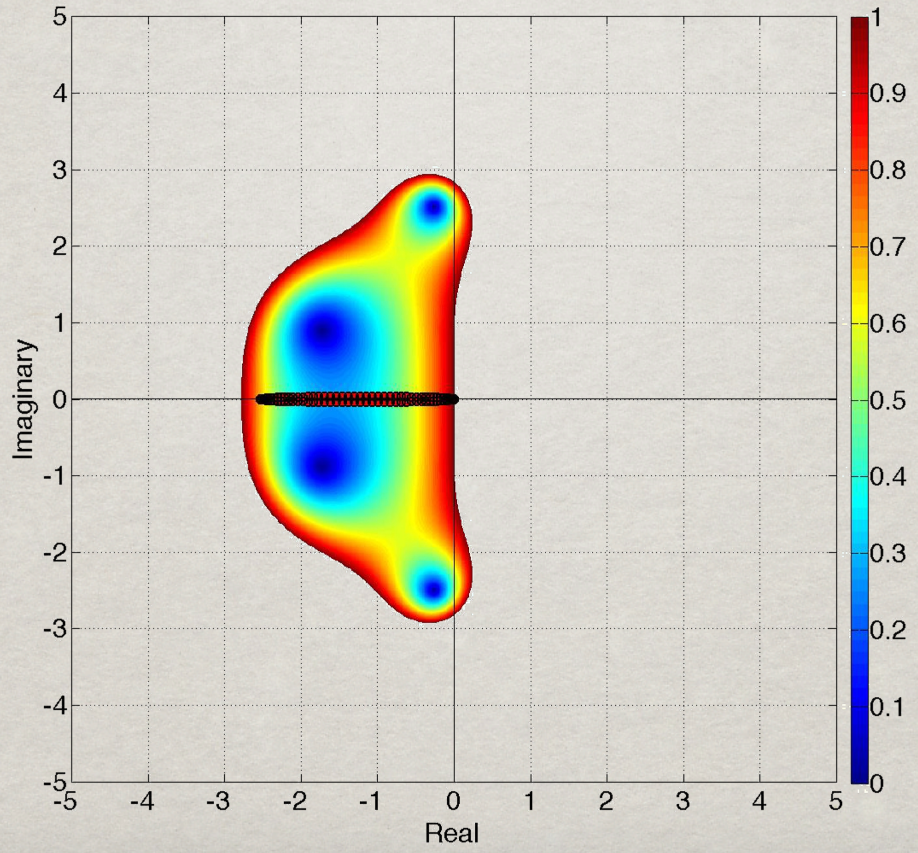
10
0
-10
-20
-30
-40
Im(lambda)
-2000
-1500
-1000
-500
0
Real(lambda)

CFD



5
4
3
2
1
0
-1
-2
-3
-4
-5
Imaginary
-5
-4
-3
-2
-1
0
1
2
3
4
5
Real
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0

CFD



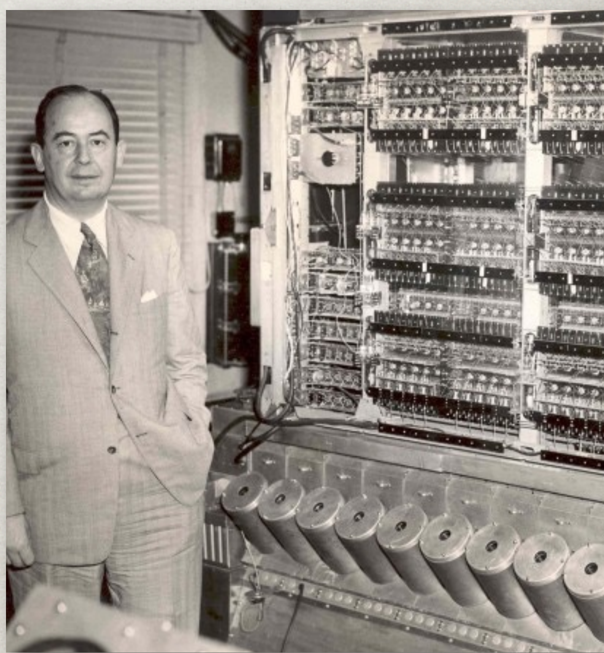
CFD

Von Neumann

$$u' = e^{ikx}$$

Amplification factor:

$$\xi = \frac{u'^{n+1}}{u'^{n}}$$



John Von Neumann
(1903-1957)

CFD

Von Neumann

$$L_h(T) = \frac{T(x_j, t_{n+1}) - T(x_j, t_n)}{\Delta t}$$

$$-\kappa \frac{T(x_{j-1}, t_n) - 2T(x_j, t_n) + T(x_{j+1}, t_n)}{\Delta x^2} .$$

$$\xi = 1 + c\,(\mathrm{e}^{ik\Delta x} - 2 + \mathrm{e}^{-ik\Delta x})$$

CFD

Von Neumann

$$\xi = 1 + c\,(\mathrm{e}^{ik\Delta x} - 2 + \mathrm{e}^{-ik\Delta x})$$

$$\xi = 1 - 2\,c\,[1 - \cos k\Delta x]$$

$$\xi = 1 - 4\,c\,\sin^2\left(\frac{k\Delta x}{2}\right)$$

CFD

Von Neumann

$$\xi = 1 - 4\,c\,\sin^2\left(\frac{k\Delta x}{2}\right)$$

$$-1 \leq 1 - 4c\,,$$

$$c \equiv \nu\,\frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}\,.$$