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GASTER TRANSFORMATION WITH THE EFFECTS OF HEAT TRANSFER

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ABSTRACT
The Gaster transformation, which determines the approximate relation between 2-D temporal and spatial amplification rates of wave disturbances, is extended to the cases with the effects of heat transfer. By applying the expansion method, a modified Gaster transformation is presented. The numerical results of plane Poiseuille flow show that the modified Gaster transformation works well when there are heat transfer effects.

Introduction

It is well known that disturbances can be classified with respect to temporal, spatial and temporal-spatial amplification. Generally, we can assume that the disturbances have the form

$$\tilde{\psi} = \hat{\psi}(y)exp[i\,\hat{\alpha}\,(x - \hat{c}\,t)] + c.c. = \hat{\psi}(y)exp[i(\hat{\alpha}\,x - \hat{\omega}\,t)] + c.c. \qquad (1)$$

where c.c. stands for the complex conjugate.

The temporal theory assumes $\hat{\alpha}$ is real and $\hat{\omega}$ is complex. The real part ω_r and the imaginary part ω_i represent the physical angle velocity and amplification rate respectively. The spatial theory assumes $\hat{\omega}$ is real and $\hat{\alpha}$ is complex. The real part α_r and the imaginary part α_i represent the physical wave number and amplification rate of the above disturbance. And the temporal-spatial amplification theory assumes $\hat{\omega}$, $\hat{\alpha}$ are all complex. This concept is applicable to wavepacket disturbances, see Gaster [1] for example.

For spatial theory, the dispersion relation $\hat{\alpha} = f(\omega, R)$ yields the unknown pair (α_r, α_i) when ω and R are specified; and for temporal theory, the dispersion relation $\hat{\omega} = f(\alpha, R)$ yields the unknown pair (ω_r, ω_i) when α and R are given. Gaster [2] showed that for a 2-D disturbance with a small amplification rate, the approximate relations between the spatial amplification rate α_i and the temporal amplification rate ω_i are given as:

$$\alpha_i = -\frac{\omega_i}{\frac{\partial \omega_r}{\partial \alpha}} = -\frac{\omega_i}{real(\hat{c}_g)}, \tag{2}$$

$$\omega_i = -\alpha_i \frac{\partial \omega}{\partial \alpha_r} = -\alpha_i * real(\hat{c}_g). \tag{3}$$

These two equations are called Gaster Transformation. Eq.(2) is used to calculate α_i if temporal theory results are known and Eq.(3) works in a converse way. Since the temporal stability results are always known, then Eq.(2) is the one often used. In this paper, we only discuss the transformation from temporal modes to spatial modes. The transformation from spatial modes to temporal modes can be disscussed in the same way. Recently, Arnal [3] pointed out numerical simulations indicate these transformations can be also applied for large amplification rates with confidence. Nayfeh and Padhey [4] found the relation between 3-D temporal and spatial stabilities.

Spatial theory corresponds more closely to many physical situations such as boundary layers. However, most researches still focus on temporal theory since ω appears linear in the stability equations. Moreover, the calculation of temporal stability is usually much easier than that of spatial stability for complex systems. Therefore Gaster transformation plays an important role to understand the growth rate relation between these two stability theories. With Gaster transformation, at least we can infer spatial stability results from the temporal stability ones.

Herwig and Schäfer [5] presented an asymptotic stability approach for small heat transfer rates, which can provide results that hold for all Newtonian fluids. This approach is called expansion method in later works [6]. By applying this method, Gaster transformation is extended to a modified Gaster transformation (MGT) which can be used under variable property effects. The key assumption to extend Gaster transformation is that the Eqs.(2) and (3) also hold when variable properties are involved. Our numerical results show the corresponding hypothesis obviously holds. Moreover, the MGT keeps the advantage of the expansion method. This means its results can hold for all Newtonian fluids. This point will be discussed in more detail later.

Gaster Transformation with the Effects of Heat Transfer

It is well known that heat transfer affects the stability of Laminar flow via temperature and pressure dependent properties such as density ρ^*, viscosity μ^*, thermal conductivity k^* and specific heat c_p^*. The basic idea behind the expansion method is to combine the Taylor series expansion of the above properties with respect to temperature or pressure by a regular perturbation procedure of the whole problem. The expansion method is described in [5].

With a representing one of the physical properties ρ^*, μ^*, k^* or c_p^*, the Taylor series expansion reads

$$a = \frac{a^*}{a_R^*} = 1 + \epsilon_T K_{aT} T + \epsilon_p K_{ap} p + O(\epsilon_T^2, \epsilon_T \epsilon_p, \epsilon_p^2) \tag{4}$$

where

$$\epsilon_T = \frac{q_w^* H^*}{k^* T_R^*}, \quad \epsilon_p = \frac{\rho_R^* U_R^{*2}}{p_R^*}, \quad T = \frac{k_R^*(T^* - T_R^*)}{q_w^* H^*}, \quad p = \frac{p^* - p_R^*}{\rho_R^* U_R^{*2}},$$

$$K_{aT} = \left(\frac{\partial a^*}{\partial T^*}\frac{T^*}{a^*}\right)_R \; and \; K_{ap} = \left(\frac{\partial a^*}{\partial p^*}\frac{p^*}{a^*}\right)_R .$$

Here ϵ_T and ϵ_p are the small (perturbation) parameters. The Taylor series are truncated after linear terms. But extension to higher orders with respect to ϵ_T and ϵ_R is straightforward. In this paper, only linear terms are taken into account. Hence, we call this a linear perturbation theory. All equations are non-dimensionalized with respect to a reference state R.

Within the common assumptions of stability theory, all quantities are decomposed into a mean value $\bar{a}$ and a disturbance $\tilde{a}$, i.e. $\tilde{a} = \bar{a} + \hat{a}\, exp[i\, \hat{\alpha}\, (x - \hat{c}\, t)]$. Eq.(4) can thus be rewritten as

$$\bar{a} = 1 + \epsilon_T K_{aT} \overline{T} + \epsilon_p K_{ap} \bar{p} \quad and \quad \hat{a} = \epsilon_T K_{aT} \hat{T} + \epsilon_p K_{ap} \hat{p} . \tag{5}$$

The mean quantity is affected by mean variable property through $\bar{\rho}$, $\bar{\mu}$, $\bar{k}$ and $\bar{c}_p$, whereas the disturbance is influenced by the mean as well as by the disturbance parts of the above properties.

Eqs.(4) and (5) suggests an expansion of all mean flow and disturbance quantities of the general form

$$b = b_0 + \epsilon_T \left(K_{\rho T} b_{\rho T} + K_{\mu T} b_{\mu T} + K_{kT} b_{kT} + K_{c_p T} b_{c_p T}\right) + \\ \epsilon_p \left(K_{\rho p} b_{\rho p} + K_{\mu p} b_{\mu p} + K_{kp} b_{kp} + K_{c_p p} b_{c_p p}\right) . \tag{6}$$

Here b represents either of the following: $\bar{u}$, $\hat{u}$, $\bar{v}$, $\hat{v}$, $\bar{p}$, $\hat{p}$, $\overline{T}$, $\hat{T}$, $\hat{c}$, $\hat{\omega}$ or $\hat{\alpha}$.

Based on Eq.(6), the group velocity is written as

$$c_{gr} = \frac{\partial \omega_r}{\partial \alpha} = \frac{\partial \omega_{0r}}{\partial \alpha} + \epsilon_T \left(K_{\rho T}\frac{\partial \omega_{\rho Tr}}{\partial \alpha} + K_{\mu T}\frac{\partial \omega_{\mu Tr}}{\partial \alpha} + K_{kT}\frac{\partial \omega_{kTr}}{\partial \alpha} + K_{c_p T}\frac{\partial \omega_{c_p Tr}}{\partial \alpha}\right) + \\ \epsilon_p \left(K_{\rho p}\frac{\partial \omega_{\rho pr}}{\partial \alpha} + K_{\mu p}\frac{\partial \omega_{\mu pr}}{\partial \alpha} + K_{kp}\frac{\partial \omega_{kpr}}{\partial \alpha} + K_{c_p p}\frac{\partial \omega_{c_p pr}}{\partial \alpha}\right) \\ = c_{g0r} + \epsilon_T \left(K_{\rho T} c_{g\rho Tr} + K_{\mu T} c_{g\mu Tr} + K_{kT} c_{gkTr} + K_{c_p T} c_{gc_p Tr}\right) + \\ \epsilon_p \left(K_{\rho p} c_{g\rho pr} + K_{\mu p} c_{g\mu pr} + K_{kp} c_{gkpr} + K_{c_p p} c_{gc_p pr}\right) . \tag{7}$$

Applying these expansions (see Eq.(6) and (7)) in Eq.(2), we have:

$$\alpha_i = -\frac{\omega_i}{c_{gr}} = -\frac{\omega_{0i}}{c_{g0r}}\left[1 + \epsilon_T K_{\rho T}\left(\frac{\omega_{\rho Ti}}{\omega_{0i}} - \frac{c_{g\rho Tr}}{c_{g0r}}\right) + \epsilon_T K_{\mu T}\left(\frac{\omega_{\mu Ti}}{\omega_{0i}} - \frac{c_{g\mu Tr}}{c_{g0r}}\right)\right]$$

$$-\frac{\omega_{0i}}{c_{g0r}}\left[\epsilon_T K_{kT}\left(\frac{\omega_{kTi}}{\omega_{0i}} - \frac{c_{gkTr}}{c_{g0r}}\right) + \epsilon_T K_{c_pT}\left(\frac{\omega_{c_pTi}}{\omega_{0i}} - \frac{c_{gc_pTr}}{c_{g0r}}\right) + \epsilon_p K_{\rho p}\left(\frac{\omega_{\rho pi}}{\omega_{0i}} - \frac{c_{g\rho pr}}{c_{g0r}}\right)\right]$$

$$-\frac{\omega_{0i}}{c_{g0r}}\left[\epsilon_p K_{\mu p}\left(\frac{\omega_{\mu pi}}{\omega_{0i}} - \frac{c_{g\mu pr}}{c_{g0r}}\right) + \epsilon_p K_{kp}\left(\frac{\omega_{kpi}}{\omega_{0i}} - \frac{c_{gkpr}}{c_{g0r}}\right) + \epsilon_p K_{c_pp}\left(\frac{\omega_{c_ppi}}{\omega_{0i}} - \frac{c_{gc_ppr}}{c_{g0r}}\right)\right] \tag{8}$$

with

$$\omega_r = \omega_{0r} + \epsilon_T\left(K_{\rho T}\omega_{\rho Tr} + K_{\mu T}\omega_{\mu Tr} + K_{kT}\omega_{kTr} + K_{c_pT}\omega_{c_pTr}\right) +$$
$$\epsilon_p\left(K_{\rho p}\omega_{\rho pr} + K_{\mu p}\omega_{\mu pr} + K_{kp}\omega_{kpr} + K_{c_pp}\omega_{c_ppr}\right) \tag{9}$$

Eqs.(8) and (9) are the modified Gaster transformation (MGT) under the influence of heat transfer. From Eq.(8), we find that all property effects on α_i are well separated as b in Eq.(6). Therefore we conclude that the MGT holds for all Newtonian fluids (see the discussion in [5]). The MGT will be explained and demonstrated numerically in the section Results and Discussion.

Temporal and Spatial Stability Equations

From either Eq.(6) or Eq.(8), it is obvious that the effects of different physical properties are independent. Within linear theory with respect to ϵ_T and ϵ_p, there are no mixed terms of temperature and pressure dependence. Therefore, a single property effect can be selected as an example. All other properties effects can be handled similarly without changing the previous solution, see Herwig and You [7]. In this paper, we only study the case having temperature dependent viscosity, which is a good approximation for water.

Mean Flow Field

The plane Poiseuille flow with constant heat flux boundary is chosen as a example in this study. The non-dimensional mean velocity and temperature field can be given analytically as

$$\overline{u}_0 = \left(1 - y^2\right), \quad \overline{v}_0 = 0, \quad \overline{T}_0 = \frac{3}{2}\left(-\frac{1}{12}y^4 + \frac{1}{2}y^2 - \frac{13}{140}\right) + \frac{3x}{2RePr},$$
$$\overline{u}_\mu = -\frac{1}{24}y^6 + \frac{3}{8}y^4 - \frac{111}{280}y^2 + \frac{53}{840}. \tag{10}$$

The reference point R is taken at $x = 0$ in all our calculations.

Disturbance Equations

In the stability theory, all quantities are decomposed into a mean value $\overline{a}$ and a superimposed disturbance $\tilde{a}$ ($\tilde{\psi}$, stream function; $\tilde{T}$, temperature fluctuation). The form of disturbance part can be assumed as that in Eq.(1). Then, from the Navier-Stokes equations (for the temperature dependent viscosity) and the thermal energy equation, the linear differential equations for $\hat{\psi}(y)$ and $\hat{T}(y)$ are deduced with $D = \frac{\partial}{\partial y}$:

$$\left[\left(\overline{u}-\hat{c}\right)\left(D^2-\hat{\alpha}^2\right)-D^2\overline{u}\right]\hat{\psi}=-\frac{i}{\hat{\alpha}\,Re}\left[\overline{\mu}\left(D^2-\hat{\alpha}^2\right)^2+2D\overline{\mu}D\left(D^2-\hat{\alpha}^2\right)\right]\hat{\psi}$$

$$-\frac{i}{\hat{\alpha}\,Re}\left\{\left[2i\,\hat{\alpha}\frac{\partial\overline{\mu}}{\partial x}(D^2-\hat{\alpha}^2)+D^2\overline{\mu}\left(D^2+\hat{\alpha}^2\right)\right]\hat{\psi}+\left[(D^3\overline{u}+\hat{\alpha}^2\,D\overline{u})+2D^2\overline{u}D+D\overline{u}D^2\right]\hat{\mu}\right\} \tag{11}$$

$$\left[\left(\overline{u}-\hat{c}\right)+\frac{i}{\hat{\alpha}\,Re\,\mathrm{Pr}}\left(D^2-\hat{\alpha}^2\right)\right]\hat{T}=(D\overline{T}+\frac{i}{\hat{\alpha}}\frac{\partial\overline{T}}{\partial x}D)\,\hat{\psi}\,. \tag{12}$$

The associated boundary conditions are

$$y=\pm1:\quad \hat{\psi}=D\,\hat{\psi}=D\,\hat{T}=0. \tag{13}$$

Temporal Disturbance Equations

Recall that in the temporal stability theory, the wave number α is a given real number and the wave angle velocity $\hat{\omega}=\alpha\,\hat{c}$ is a complex number. If we only consider the temperature dependent viscosity effects, then from Eq.(6) quantity b (i.e. $\overline{u},\overline{T},\hat{\psi},\hat{T},\ \hat{\mu},\ \hat{\omega}$ and $\hat{c}=c_r+ic_i$) is expanded as:

$$b=b_0+\epsilon K_\mu b_\mu. \tag{14}$$

Here $\epsilon_T, K_{\mu T}, b_{\mu T}$ are written as ϵ, K_μ, b_μ for convenience.

Inserting Eq.(14) into Eqs.(11)-(12) and collecting the terms of equal magnitude with respect to ϵK_μ, leads to the following set of stability equations:
zeroth order:

$$L_\psi\,\hat{\psi}_0=0,\quad L_T\,\hat{T}_0=(D\overline{T}_0+\frac{3i}{2\alpha RePr}D)\,\hat{\psi}_0, \tag{15}$$

first order:

$$L_\psi\,\hat{\psi}_\mu=M\,\hat{\psi}_0+N\,\hat{T}_0, \tag{16}$$

where the differential operators L_ψ, L_T, M and N are defined as

$$L_\psi=\left(\overline{u}_0-\hat{c}_0\right)\left(D^2-\alpha^2\right)-D^2\overline{u}_0+\frac{i}{\alpha Re}\left(D^2-\alpha^2\right)^2,$$

$$L_T=\left(\overline{u}_0-\hat{c}_0\right)+\frac{i}{\alpha Re\,\mathrm{Pr}}\left(D^2-\alpha^2\right),$$

$$M=-\left(\overline{u}_\mu-\hat{c}_\mu\right)\left(D^2-\alpha^2\right)+D^2\overline{u}_\mu+\frac{3}{Re^2Pr}(D^2-\alpha^2)$$

$$-\frac{i}{\alpha Re}\left[\overline{T}_0\left(D^2-\alpha^2\right)^2+2D\overline{T}_0\left(D^3-\alpha^2 D\right)+D^2\overline{T}_0\left(D^2+\alpha^2\right)\right],$$

$$N=-\frac{i}{\alpha Re}\left[(D^3\overline{u}_0+\alpha^2 D\overline{u}_0)+2D^2\overline{u}_0 D+D\overline{u}_0 D^2\right].$$

The associated boundary conditions are all homogeneous, c.f. Eq.(13).

Spatial Disturbance Equations

For spatial stability theory, the wave angle velocity ω is a given real number and the wave number $\hat{\alpha}$ is an unknown complex number. According to our perturbation approach, quantity b (i.e. $\overline{u}, \overline{T}, \hat{\psi}, \hat{T}, \hat{\mu}$ or $\hat{\alpha}=\alpha_r+i\alpha_i$) is expanded as Eq.(14). Following the same procedure to deduce the temporal stability equations, we obtain

zeroth order:

$$L_\psi\,\hat{\psi}_0=0,\quad L_T\,\hat{T}_0=(\hat{\alpha}_0\,D\overline{T}_0+\frac{3i}{2RePr}D)\,\hat{\psi}_0, \tag{17}$$

first order:

$$L_\psi\,\hat{\psi}_\mu=(\hat{\alpha}_\mu\,F_0+G_\mu)\,\hat{\psi}_0+M_0\,\hat{T}_0, \tag{18}$$

where

$$L_\psi=\left(\hat{\alpha}_0\,\overline{u}_0-\omega\right)\left(D^2-\hat{\alpha}_0^2\right)-\hat{\alpha}_0\,D^2\overline{u}_0+\frac{i}{Re}\left(D^2-\hat{\alpha}_0^2\right)^2,$$

$$L_T=\left(\hat{\alpha}_0\,\overline{u}_0-\omega\right)+\frac{i}{Re\,\mathrm{Pr}}\left(D^2-\hat{\alpha}_0^2\right),$$

$$F_0=-\left[\overline{u}_0\left(D^2-3\,\hat{\alpha}_0^2\right)+2\,\hat{\alpha}_0\,\omega-D^2\overline{u}_0\right]+\frac{\hat{\alpha}_0}{Re}(4i+\frac{3}{RePr})\left(D^2-\hat{\alpha}_0^{\,2}\right),$$

$$G_\mu=-\,\hat{\alpha}_0\left[\overline{u}_\mu\left(D^2-\hat{\alpha}_0^2\right)+D^2\overline{u}_\mu\right]$$

$$-\frac{i}{Re}\left[\overline{T}_0\left(D^2-\hat{\alpha}_0^2\right)^2+2D\overline{T}_0\left(D^3-\hat{\alpha}_0^2\,D\right)+D^2\overline{T}_0\left(D^2+\hat{\alpha}_0^2\right)\right],$$

$$M_0=-\frac{i}{Re}\left[(D^3\overline{u}_0+\hat{\alpha}_0^2\,D\overline{u}_0)+2D^2\overline{u}_0 D+D\overline{u}_0 D^2\right].$$

The associated boundary conditions are all homogeneous, c.f. Eq.(13).

Results and Discussion

The numerical method used to solve Eqs.(15)-(16) and Eqs.(17)-(18) are the Chebyshev tau method. All $\hat{\psi}_j$ and $\hat{T}_j$ are expanded in terms of Chebyshev polynomials. In our case, 35 Chebyshev polynomials are appropriate.

As we discussed before, only temperature dependent viscosity is considered in this paper. Then the MGT from Eq.(8) reads as:

$$\alpha_i = -\frac{\omega_{0i}}{c_{g0r}} \left[1 + \epsilon K_\mu \left(\frac{\omega_{\mu i}}{\omega_{0i}} - \frac{c_{g\mu r}}{c_{g0r}} \right) \right] \tag{19}$$

with

$$\omega_r = \omega_{0r} + \epsilon K_\mu \omega_{\mu r}. \tag{20}$$

Eqs.(19) and (20) are the MGT with the effects of temperature dependent viscosity. Reynolds number Re and Prandtl number Pr are constants during the transformation.

Let's consider the case without heat transfer effects. A disturbance wave $(\alpha_0, \omega_{0r} + i\omega_{0i})$ in temporal stability field corresponds to a disturbance wave $(\omega_{0r}, \alpha_{0r} + i\alpha_{0i})$ in the spatial stability field. Gaster transformation shows the approximation relation between the growth rates of these two disturbance waves, i.e. $\alpha_{0i} = -\omega_{0i}/\frac{\partial \omega_{0r}}{\partial \alpha_0}$.

When there are heat transfer effects, the situation becomes complex. For a disturbance wave $(\alpha_0, \omega_{0r} + i\omega_{0i})$ in the temporal stability field, by heating or cooling the flow field, the wave angle velocity ω_{0r} will change to ω_r due to heat transfer. Then the corresponding wave in the spatial stability field will be $(\omega_r, \alpha_r + i\alpha_i)$. The approximation growth rate relation between these two waves is given by the MGT, see Eqs.(19) and (20).

Many calculations have been done to check the correctness of the MGT. All numerical results prove that indeed it is right. Here we only report results of two cases. One is at subcritical region and the other is at supercritical region. Both these two cases are far from the neutral stability curve of plane Poiseuille flow without heat transfer effects. Table 1 and Table 2 show results for the angle velocities $\hat{\omega}$ (expanded according to Eq.(14)) and corresponding wave group velocities for the above two cases. These results hold for all fluids with a Prandtl number Pr = 7. However, the Prandtl number influence is extremely weak, see Herwig and You [8].

TABLE 1

Results of the Temporal Modes ($Re = 4000, Pr = 7.0$)

$No.$	α	$\hat{\omega}_0$	c_{g0r}	$\hat{\omega}_\mu \times 10$	$c_{g\mu r}$
1	$0.98 - 10^{-4}$	$0.270340 - 0.005458i$		$0.268473 + 0.211732i$	
2	0.98	$0.270381 - 0.005455i$	0.410000	$0.268533 + 0.211744i$	0.060500
3	$0.98 + 10^{-4}$	$0.270422 - 0.005452i$		$0.268594 + 0.211755i$	

TABLE 2

Results of the Temporal Modes ($Re = 10000, Pr = 7.0$)

$No.$	α	$\overset{\wedge}{\omega}_0$	c_{g0r}	$\overset{\wedge}{\omega}_\mu \times 10$	$c_{g\mu r}$
1	$1.0-10^{-4}$	$0.237492+0.003742i$		$0.294259+0.130382i$	
2	1.0	$0.237526+0.003740i$	0.345000	$0.294331+0.130363i$	0.072000
3	$1.0+10^{-4}$	$0.237561+0.003738i$		$0.294403+0.130344i$	

Here, from Eq.(7), we define $c_{g0r} = (\omega_{03r} - \omega_{01r})/(\alpha_3 - \alpha_1)$ and $c_{g\mu r} = (\omega_{\mu 3r} - \omega_{\mu 1r})/(\alpha_3 - \alpha_1)$ by the central finite difference scheme. Then from Eqs.(19)-(20), the MGT for $Re = 4000, \alpha = 0.98$ and $Pr = 7$ is

$$\alpha_i = 0.013305 - 0.053608\epsilon K_\mu \quad with \quad \omega_r = 0.270381 + 0.026853\epsilon K_\mu \tag{21}$$

and for $Re = 10000, \alpha = 1.0$ and $Pr = 7$,

$$\alpha_i = -0.010840 - 0.035524\epsilon K_\mu \quad with \quad \omega_r = 0.237526 + 0.029433\epsilon K_\mu \tag{22}$$

In order to check the correctness of the MGT, we need directly compute the spatial growth rate of the transformed wave by the spatial modes theory. Then we can compare these spatial modes results with those predicted by the MGT. If these two results are in a good agreement, the correctness of this transformation is thus demonstrated. In the MGT, a disturbance wave $(\alpha_0, \omega_{0r} + i\omega_{0i})$ in temporal stability field is transformed into a corresponding wave $(\omega_r, \alpha_r + i\alpha_i)$ in spatial stability field. In order to achieve the transformed wave, the following two steps are necessary.

(1) To heat or cool the spatial modes system with the wave $(\omega_{0r}, \alpha_{0r} + i\alpha_{0i})$ by the same heat transfer rate (i.e. the same ϵ) as that for temporal modes system. Then this wave will change to wave $(\omega_{0r}, \alpha_r^a + i\alpha_i^a)$. This new spatial growth rate is defined as α_i^a.

(2) To change the wave $(\omega_{0r}, \alpha_{0r} + i\alpha_{0i})$ to wave $(\omega_r, \alpha_r^b + i\alpha_i^b)$. The change of growth rate by this procedure is defined as $\Delta\alpha_i^b$.

After these two steps, we can approximately obtain the wave $(\omega_r, i\alpha_r + i\alpha_i)$ from the MGT, see Eqs.(19) and (20). Then the spatial growth rate should be $\alpha_i^a + \Delta\alpha_i^b$ after the above two steps. If we can show that $\alpha_i^a + \Delta\alpha_i^b$ is nearly equal to α_i, then the correctness of the MGT is proved. There are many routes to obtain the transformed wave $(\omega_r, \alpha_r + i\alpha_i)$ in spatial modes field from the wave $(\alpha_0, \omega_{0r} + i\omega_{0i})$ in temporal modes field. It is easy to show that all transformed waves are nearly the same within our linear stability theory. The difference between them is of the second order. From the above definition, we can write $\Delta\alpha_i^b$ as

$$\Delta\alpha_i^b = \alpha_i^b - \alpha_{0i}. \tag{23}$$

Then from Eq.(14), α_i^a is defined as:

$$\alpha_i^a = \alpha_{0i} + \epsilon K_\mu \alpha_{\mu i} \tag{24}$$

where α_{0i} and $\alpha_{\mu i}$ can be determined by spatial modes calculation based on Eqs.(17)-(18).

For the cases presented by Eqs.(21) and (22), Table 3 shows the spatial modes results of $\Delta\alpha_i^b$ with respect to different ϵK_μ. This corresponds to the step 2 given previously.

TABLE 3

Results of Spatial Modes ($Pr = 7$)

	$Re = 4000$			$Re = 10000$		
ϵK_μ	ω_r	α_i^b	$\Delta\alpha_i^b$	ω_r	α_i^b	$\Delta\alpha_i^b$
0.0	0.270381	0.013267	0.0	0.237526	-0.010942	0.0
0.05	0.271724	0.013037	-0.000230	0.238998	-0.010689	0.000253
0.1	0.273066	0.012815	-0.000452	0.240470	-0.010413	0.000529

Now we consider the first step. For the case $Re = 4000$ and $\alpha = 0.98$, the temporal modes yield $\omega_{0r} = 0.270381$. Solving Eqs.(17) and (18), we obtain the following spatial growth rate function, which depends on the heat transfer rate ϵ and viscosity parameter K_μ,

$$\alpha_i^a = \alpha_{0i} - 0.048543\epsilon K_\mu \quad with \quad \alpha_{0i} = 0.013267. \tag{25}$$

Similarly, for $Re = 10000$ and $\alpha = 1.0$, we have $\omega_{0r} = 0.237526$ and

$$\alpha_i^a = \alpha_{0i} - 0.040909\epsilon K_\mu \quad with \quad \alpha_{0i} = -0.010942. \tag{26}$$

Table 4 gives the results of the two steps and those α_i predicted by the MGT (Eq.(19)). It shows that our spatial modes results $\alpha_i^a + \Delta\alpha_i^b$ are in good agreement with those α_i predicted by the MGT. This proves the correctness of the MGT. Here we just present two cases of our calculation. The correctness of this transformation is proved by all cases in our computation. Thus it is reasonable to expect that the MGT can be applied to the cases with temperature dependent viscosity effects. Basically, the treatment for other variable properties effects such as density, heat conductivity and capacity heat effects is the same as that for temperature dependent viscosity. Thus correctness of the general modified Gaster transformation (Eqs.(8) and (9)) can be further proved without difficulty.

TABLE 4

The Results from the Spatial Modes and from the MGT

	$Re = 4000$			$Re = 10000$		
ϵK_μ	0.0	0.05	0.1	0.0	0.05	0.1
α_i^a	0.013267	0.010840	0.008413	-0.010942	-0.012987	-0.015033
$\Delta\alpha_i^a$	0.0	-0.002427	-0.004854	0.0	-0.002045	-0.004091
$\Delta\alpha_i^b$	0.0	-0.000230	-0.000452	0.0	0.000253	0.000529
$\alpha_i^a + \Delta\alpha_i^b$	0.013267	0.010610	0.007961	-0.010942	-0.012734	-0.014504
α_i	0.013305	0.010625	0.007944	-0.010840	-0.012616	-0.014392

Here $\Delta\alpha_i^a = \alpha_i^a - \alpha_{0i}$. α_i is from the MGT.

Table 4 also shows that $\Delta\alpha_i^a$ is always much larger than $\Delta\alpha_i^b$. This means that the spatial growth rate change of the first step is much larger than that of the second step. The growth rate change of the second step is an order smaller than that of the first step.

Conclusions

(1). The modified Gaster Transformation can correctly predict spatial growth rate for a large range of Reynolds number, wave number and heat transfer rate (not very small ϵ), at least in the case with temperature dependent viscosity effects.

(2). The correctness of the modified Gaster transformation is proved only in plane Poiseuille flow with temperature dependent viscosity effects and constant heat flux boundary condition system. We expect that this is also true for flow systems with all other variable properities effects and different boundary conditions. However, this is a subject for further studies.

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Nomenclature

a, b	general quantity
$\hat{a}$	shape function of a
$\hat{c}$	complex phase velocity
$\hat{c}_g$	complex group velocity
c_p	specific heat at constant pressure
H	half channel height
k	thermal conductivity
K_{aT}, K_{ap}	property gradient, Eq.(4)
K_μ	viscosity gradient, Eq.(14)
p	pressure
Pr	Prandtl number $\frac{\mu_R^* c_p^*}{k^*}$
q_w	wall heat flux
Re	Reynolds number $\frac{\rho^* U_R^* H^*}{\mu_R^*}$
t	time
T	temperature
$\hat{T}$	temperature shape function
u	streamwise velocity
U_R	reference velocity
v	velocity normal to the wall
x, y	Cartesian coordinates

Greek symbols

α	wave length parameter
ϵ_T, ϵ_p	pertubation parameter, Eq.(4)
μ	viscosity
ψ	stream function
ρ	density

Superscripts

*	dimensional quantity
–	mean value
~	disturbance quantity
∧	complex quantity

Subscripts

i	imaginary part
r	real part
R	reference state
w	wall
0	zero order
μ	first order viscosity effect

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