Tutorial 2

Víctor Ballester Ribó

Turbulence
M2 Applied and Theoretical Mathematics
Université Paris-Dauphine
Februrary 2024

Exercice 1. *Consider the hyper-viscous Navier-Stokes equation*

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + (-1)^m \nu_m \nabla^{2m+2} \mathbf{u} + \mathbf{F} \tag{1}$$

where $\nabla^{2m} = \nabla^2 \nabla^2 \overset{m}{\cdots} \nabla^2$.

a. Write an expression for the energy dissipation and the energy balance relation and show that hyper-viscosity dissipates energy (i.e. leads to negative term).

b. Predict the lengthscale ℓ_ν that dissipation becomes effective and dissipates the energy based on the energy injection rate and ν_m.

Resolution.

a. We have to compute $\partial_t \mathcal{E} = \partial_t \langle \frac{1}{2} |\mathbf{u}|^2 \rangle = \langle \mathbf{u} \cdot \partial_t \mathbf{u} \rangle$. We have:

$$\begin{aligned}
\langle \mathbf{u} \cdot \partial_t \mathbf{u} \rangle &= -\langle \mathbf{u} \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) \rangle - \langle \mathbf{u} \cdot \nabla P \rangle + (-1)^m \nu_m \langle \mathbf{u} \cdot \nabla^{2m+2} \mathbf{u} \rangle + \langle \mathbf{u} \cdot \mathbf{F} \rangle \\
&= -\left\langle \mathbf{u} \cdot \nabla \left(\frac{1}{2} |\mathbf{u}|^2 \right) \right\rangle - \langle \mathbf{u} \cdot \nabla P \rangle + (-1)^m \nu_m \langle \mathbf{u} \cdot \nabla^{2m+2} \mathbf{u} \rangle + \langle \mathbf{u} \cdot \mathbf{F} \rangle \\
&= (-1)^m \nu_m \langle \mathbf{u} \cdot \nabla^{2m+2} \mathbf{u} \rangle + \langle \mathbf{u} \cdot \mathbf{F} \rangle
\end{aligned}$$

Now note that $\nabla^{2m+2} \mathbf{u}$ can be expressed in Einstein notation as $c_{a_1,a_2,a_3} \partial_x^{2a_1} \partial_y^{2a_2} \partial_z^{2a_3} u_i$ for $i = 1, 2, 3$, $a_1 + a_2 + a_3 = m + 1$ and some constants $c_{a_1,a_2,a_3} \geq 0$. Then we have:

$$\begin{aligned}
\langle \mathbf{u} \cdot \nabla^{2m+2} \mathbf{u} \rangle &= \langle c_{a_1,a_2,a_3} u_i \partial_x^{2a_1} \partial_y^{2a_2} \partial_z^{2a_3} u_i \rangle \\
&= (-1)^{a_1} \langle c_{a_1,a_2,a_3} \partial_x^{a_1} u_i \partial_x^{a_1} \partial_y^{2a_2} \partial_z^{2a_3} u_i \rangle \\
&= (-1)^{a_1+a_2} \langle c_{a_1,a_2,a_3} \partial_x^{a_1} \partial_y^{a_2} u_i \partial_x^{a_1} \partial_y^{a_2} \partial_z^{2a_3} u_i \rangle \\
&= (-1)^{a_1+a_2+a_3} \langle c_{a_1,a_2,a_3} (\partial_x^{a_1} \partial_y^{a_2} \partial_z^{a_3} u_i)^2 \rangle
\end{aligned}$$

where we used that the periodic boundary conditions. Thus:

$$\partial_t \mathcal{E} = (-1)^{-m} (-1)^{a_1+a_2+a_3} \nu_m \langle c_{a_1,a_2,a_3} (\partial_x^{a_1} \partial_y^{a_2} \partial_z^{a_3} u_i)^2 \rangle + \langle \mathbf{u} \cdot \mathbf{F} \rangle = -\nu_m \langle c_{a_1,a_2,a_3} (\partial_x^{a_1} \partial_y^{a_2} \partial_z^{a_3} u_i)^2 \rangle + \langle \mathbf{u} \cdot \mathbf{F} \rangle$$

whose first term is negative (because we assume $\nu_m \geq 0$), and thus the hyper-viscosity dissipates energy.

b. Let $\mathcal{I} = \langle \mathbf{u} \cdot \mathbf{F} \rangle$ be the energy injection rate. The scaling for the dissipative term is:

$$\nu_m \langle c_{a_1,a_2,a_3} (\partial_x^{a_1} \partial_y^{a_2} \partial_z^{a_3} u_i)^2 \rangle \sim \nu_m \frac{1}{\ell^{2(a_1+a_2+a_3)}} {u_\ell}^2 \sim \nu_m \frac{1}{\ell^{2m+2}} {u_\ell}^2$$

Since $\partial_t \mathcal{E} \sim \frac{{u_\ell}^2}{\tau_\ell} = \frac{{u_\ell}^3}{\ell} = \epsilon$ (assumed constant across scales), thus we have $u_\ell \sim (\ell \epsilon)^{1/3}$, $\epsilon \sim \mathcal{I}$ and when dissipation becomes important:

$$\nu_m \frac{{u_\ell}^2}{\ell^{2m+2}} \sim \epsilon \implies \nu_m \frac{(\ell \epsilon)^{2/3}}{\ell^{2m+2}} \sim \epsilon \implies \ell_\nu \sim \left(\frac{\nu_m}{\epsilon^{1/3}} \right)^{1/(2m+4/3)}$$

Exercice 2. *Consider again the equation*

$$\partial_t \mathbf{b} + \mathbf{v} \times \mathbf{b} = -\nabla P + (-1)^m \nu_m \nabla^{2m+2} \mathbf{b} + \mathbf{F} \tag{2}$$
$$\tag{3}$$

in a periodic box of size L where m is an integer and $\mathbf{F}$ is a forcing that injects energy at scale L at a rate $\mathcal{I}$. $\mathbf{v}$ is related to $\mathbf{b}$ as $\mathbf{v} = (\boldsymbol{\nabla} \times)^n \mathbf{b}$ for some $n \in \mathbb{N}$. Recall that for any n, the energy $\mathcal{E} = \langle \frac{1}{2} |\mathbf{b}|^2 \rangle$ is conserved for $\nu_m = 0$ and $\alpha = 0$. Assuming:

- *Energy cascades to smaller scales*
- *Similar size eddies dominate the cascade*

show the following:

a. Predict the energy spectrum of **b** *based on the assumptions above.*

b. Predict the lengthscale ℓ_ν that dissipation becomes effective and dissipates the energy.

c. For which values of n and m, the viscosity will not be sufficient to dissipate the injected energy as $\nu_m \to 0$?

Resolution.

a. As in exercise 1 we have that:

$$\partial_t \mathcal{E} = -\nu_m \langle c_{a_1,a_2,a_3} (\partial_x^{a_1} \partial_y^{a_2} \partial_z^{a_3} b_i)^2 \rangle + \langle \mathbf{b} \cdot \mathbf{F} \rangle$$

Since the units of each term in the equation are the same, we have the following scalings:

$$\frac{b}{\tau} \sim \frac{b^2}{\ell^n} \implies \frac{1}{\tau} \sim \frac{b}{\ell^n}$$

Now, we have that $\epsilon \sim \partial_t \mathcal{E}$ (assumed constant across scales) and since the interactions are done between similar eddies we have: Thus:

$$\epsilon \sim \frac{b_\ell^2}{\tau_\ell} \sim \frac{b_\ell^3}{\ell^n} \sim \implies b_\ell \sim (\ell^n \epsilon)^{1/3}$$

Thus, the energy spectrum of **b** is (whenever the injection rate of energy and dissipation can be neglected):

$$E(k) \sim \frac{b_\ell^2}{1/\ell} \sim \ell^{2n/3+1} \epsilon^{2/3} = \epsilon^{2/3} k^{-(3+2n)/3}$$

b. When viscosity becomes effective, we have (using exercise 1):

$$\epsilon \sim \nu_m \frac{{b_{\ell_\nu}}^2}{\ell_\nu^{2m+2}} \sim \nu_m \frac{\ell_\nu^{2n/3} \epsilon^{2/3}}{\ell_\nu^{2m+2}} \sim \nu_m \ell_\nu^{2n/3-2m-2} \epsilon^{2/3} \implies \ell_\nu \sim \left(\frac{\nu_m}{\epsilon^{1/3}} \right)^{1/(2+2m-2n/3)}$$

c. If $\nu_m \to 0$, then we must have $k_\nu = \frac{1}{\ell_\nu} \to \infty$ if we do not want the viscosity to dissipate the energy. This implies that the exponent of ℓ_ν must be positive, i.e. $2 + 2m - 2n/3 > 0$, which is equivalent to $3(m+1) > n$.