Tutorial 2

Víctor Ballester Ribó

Turbulence
M2 Applied and Theoretical Mathematics
Université Paris-Dauphine
Februrary 2024

Exercice 1. *In this homework, we will explore some properties of Burgers equation.*

$$\partial_t V_i + V_j \partial_j V_i = \nu \Delta V_i, \tag{1}$$

$$V_i = -\partial_i \Psi, \tag{2}$$

a. Use equation (2) to find the equation for Ψ.

b. 1D case: Check that the "Khokhlov" velocity field $u^\nu(x,t) = (x - L\tanh(Lx/(2\nu t)))/t$ is a solution of Burgers equation (1). Draw it at several times for $L = 1$, and $\nu = 1, \nu = 10^{-2}$ and $\nu = 10^{-6}$.

c. Find the limit of the Khokhlov solution when $\nu \to 0$. This solution represents a shock.

Resolution.

a. Putting (2) into (1) we get:

$$\begin{aligned} -\partial_t\partial_i\Psi + \partial_j\Psi\partial_j\partial_i\Psi &= -\nu\partial_j^2\partial_i\Psi \\ \partial_i\left(\partial_t\Psi - \frac{1}{2}(\partial_j\Psi)^2\right) &= \partial_i(\nu\Delta\Psi) \\ \partial_t\Psi - \frac{1}{2}(\partial_j\Psi)^2 &= \nu\Delta\Psi + f(t) \end{aligned} \tag{3}$$

Note that the constant $f(t)$ does not depend on any spatial variable i, because for each i, we have the same equation for Ψ. Thus, it can only depend on time.

b. We need to check that:

$$\partial_t u^\nu + u^\nu \partial_x u^\nu = \nu \partial_{xx} u^\nu$$

We have that:

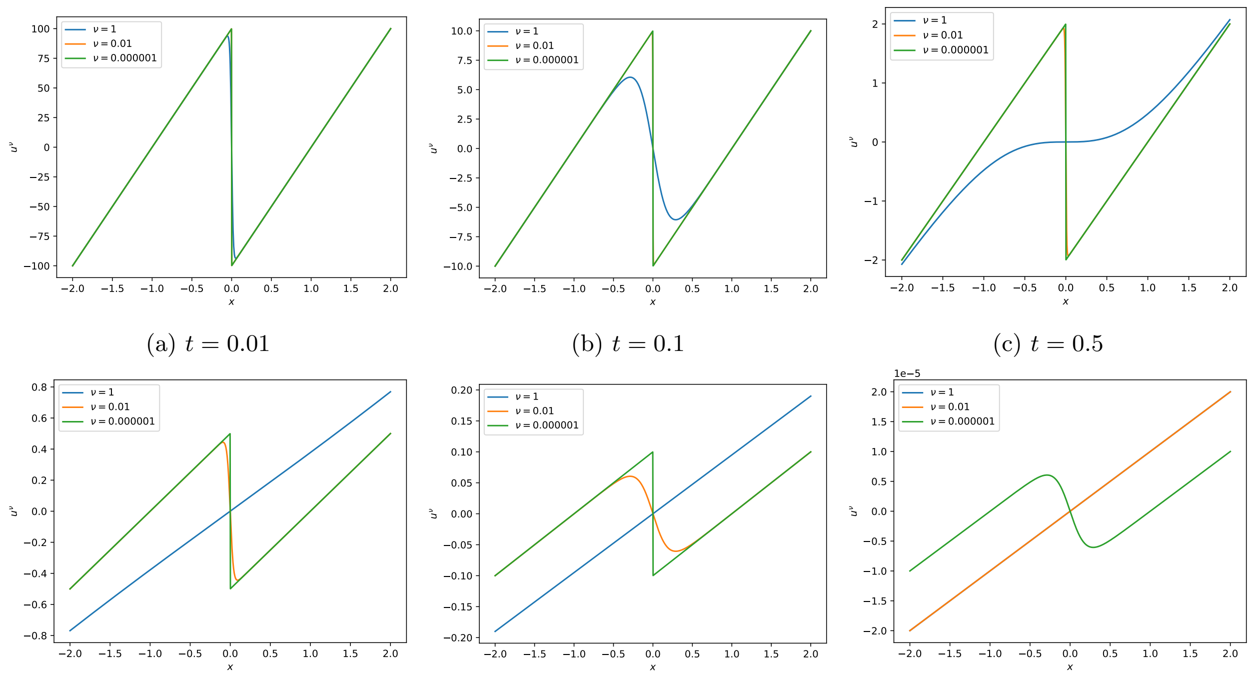
$$\begin{aligned} \partial_t u^\nu &= -\frac{\left(x - L\tanh\left(\frac{Lx}{2\nu t}\right)\right)}{t^2} + \frac{L^2 x}{2\nu t^3}\operatorname{sech}^2\left(\frac{Lx}{2\nu t}\right) \\ \partial_x u^\nu &= \frac{1}{t} - \frac{L^2}{2\nu t^2}\operatorname{sech}^2\left(\frac{Lx}{2\nu t}\right) \\ u^\nu\partial_x u^\nu &= \frac{x}{t^2} - \frac{L^2 x}{2\nu t^3}\operatorname{sech}^2\left(\frac{Lx}{2\nu t}\right) - \frac{L}{t^2}\tanh\left(\frac{Lx}{2\nu t}\right) + \frac{L^3}{2\nu t^3}\operatorname{sech}^2\left(\frac{Lx}{2\nu t}\right)\tanh\left(\frac{Lx}{2\nu t}\right) \\ -\nu\partial_{xx}u^\nu &= -\frac{L^3}{2\nu t^3}\operatorname{sech}^3\left(\frac{Lx}{2\nu t}\right)\sinh\left(\frac{Lx}{2\nu t}\right) \end{aligned}$$

Adding all the terms (except the second one), we get 0. Thus, the equation is satisfied. In fig. 1, we represent the solution for different values of t and ν.

c. Fix $x \in \mathbb{R}$ and $t > 0$. Then, since $\lim_{y\to\pm\infty} \tanh(y) = \pm 1$, we have that:

$$u(x,t) := \lim_{\nu\to 0} u^\nu(x,t) = \lim_{\nu\to 0} \frac{x - L\tanh(Lx/(2\nu t))}{t} = \frac{x - L\lim_{\nu\to 0}\tanh(Lx/(2\nu t))}{t} = \frac{x - L\operatorname{sgn}(x)}{t}.$$

Which is not continuous at $x = 0$ for any $t > 0$. This is the signature of a shock.



(a) $t = 0.01$ (b) $t = 0.1$ (c) $t = 0.5$

(d) $t = 2$ (e) $t = 10$ (f) $t = 10^5$

Figure 1: Khokhlov velocity field for different values of t and ν.

Exercice 2. *The Hopf-Cole transformation is defined as:*

$$V_i = -2\nu\partial_i \log(\Phi). \tag{4}$$

a. *Find the link between Ψ and Φ.*

b. *Show that the equation for Φ is linear. It is called the heat equation, that has the interesting property of having simple solutions.*

c. *Consider the 1D Case. Find a solution of the heat equation in case of periodic boundary conditions. (Trick: use Fourier transform).*

Resolution.

a. For each i, we have that $-\partial_i\Psi = -2\nu\partial_i \log(\Phi)$. Thus, integrating with respect to x_i we get $\Psi = 2\nu\log(\Phi) + g(t)$, and again the constant $g(t)$ does not depend on any spatial variable i (by the same argument as in exercice 1). We can in fact determine g from eq. (3). Since $V_i = -\partial_i(2\nu\log(\Phi))$, then by exercice 1, $2\nu\log\Phi$ satisfies also eq. (3) and thus by the linearity of the derivatives, we get that $g'(t) = f(t)$.

b. From eq. (3) we get:

$$\partial_t\Psi - \frac{1}{2}(\partial_j\Psi)^2 = \nu\Delta\Psi + f(t)$$
$$\partial_t(2\nu\log(\Phi) + g(t)) - \frac{1}{2}(2\nu\partial_j\log(\Phi))^2 = \nu\Delta(2\nu\log(\Phi)) + f(t)$$
$$\frac{\partial_t\Phi}{\Phi} - \nu\left(\frac{\partial_j\Phi}{\Phi}\right)^2 = \nu\left(\frac{\partial_j^2\Phi}{\Phi} - \left(\frac{\partial_j\Phi}{\Phi}\right)^2\right)$$
$$\partial_t\Phi = \nu\Delta\Phi$$

And this equation is linear in Φ.

c. The 1D heat equation is $\partial_t\Phi = \nu\partial_{xx}\Phi$. We assume it is defined in a domain $[-L/2, L/2]$, and we equip it with periodic boundary conditions. Now we express the solution in Fourier series

$$\Phi(x,t) = \sum_{n\in\mathbb{Z}} \widehat{\Phi}_n(t)\mathrm{e}^{\frac{2\pi i n x}{L}}$$

Plugging this formula into the equation we get:

$$\partial_t \widehat{\Phi}_n = -\nu \left(\frac{2\pi n}{L}\right)^2 \widehat{\Phi}_n$$

This is a linear ode, whose solution is $\widehat{\Phi}_n(t) = \widehat{\Phi}_n(0)\mathrm{e}^{-\nu\left(\frac{2\pi n}{L}\right)^2 t}$. Thus, the solution $\Phi(x,t)$ is:

$$\Phi(x,t) = \sum_{n\in\mathbb{Z}} \widehat{\Phi}_n(0)\mathrm{e}^{\frac{2\pi i n x}{L}}\mathrm{e}^{-\nu\left(\frac{2\pi n}{L}\right)^2 t}$$

Exercice 3. *At very large scale, the Universe is described by Newton equations in a flat, expanding geometry. The equations are:*

$$\partial_t u_i + \frac{\dot{a}}{a}u_i + \frac{1}{a}u_j\partial_j u_i = -\frac{1}{a}\partial_i\Phi, \tag{5}$$

$$\partial_t\rho + 3\frac{\dot{a}}{a}\rho + \frac{1}{a}\partial_j\rho u_j = 0, \tag{6}$$

$$\Delta\Phi = 4\pi G a^2(\rho - \rho_b), \tag{7}$$

where $a(t)$ is the expansion factor, Φ is the gravitational potential, ρ is the density and u is the velocity of the gas. Show that these equations can be mapped into inviscid Burgers equation ($\nu = 0$) by using Zeldovich transformation:

$$V = \frac{u}{a\dot{b}} = -\nabla\tilde{\Psi} \tag{8}$$

$$\left(\partial_t + 2\frac{\dot{a}}{a}\right)\partial_t b = 4\pi G\rho_b(t)b \tag{9}$$

$$\tilde{\Phi} = \frac{\Phi}{4\pi G\rho_b a^2 b} \tag{10}$$

$$\tilde{\Phi} = \tilde{\Psi} \tag{11}$$

Resolution. We have that $u_i = -a\dot{b}\partial_i\tilde{\Psi}$. Then, introducing this into eq. (5) we get:

$$\begin{aligned}
\partial_t u_i + \frac{\dot{a}}{a}u_i + \frac{1}{a}u_j\partial_j u_i &= -\frac{1}{a}\partial_i\Phi \\
-\dot{a}\dot{b}\partial_i\tilde{\Psi} - a\ddot{b}\partial_i\tilde{\Psi} - a\dot{b}\partial_i\partial_t\tilde{\Psi} - \dot{a}\dot{b}\partial_i\tilde{\Psi} - a\dot{b}^2\partial_j\tilde{\Psi}\partial_j\partial_i\tilde{\Psi} &= -4\pi G\rho_b ab\partial_i\tilde{\Psi} \\
-a\dot{b}\partial_i\partial_t\tilde{\Psi} - 4\pi G\rho_b ab\partial_i\tilde{\Psi} - a\dot{b}^2\partial_j\tilde{\Psi}\partial_j\partial_i\tilde{\Psi} &= -4\pi G\rho_b ab\partial_i\tilde{\Psi} \\
-a\dot{b}\partial_i\partial_t\tilde{\Psi} - a\dot{b}^2\partial_j\tilde{\Psi}\partial_j\partial_i\tilde{\Psi} &= 0
\end{aligned}$$

Exercice 4. *Burgers equation develop finite time singularities. Let us study this in the 1D case.*

a. *Use (2) to write an equation for $A = \partial_x u$.*

b. *Introduce the Lagrangian derivative $D_t A = \partial_t A + u\partial_x A$. Use a to find the ordinary differential equation that links A and its Lagrangian derivative.*

c. *Integrate this equation in the case $\nu = 0$, and discuss in which condition there is a finite time blow up of A.*

d. *Use this discussion to explain the features of the Khokhlov solution at $\nu \to 0$ (presence of positive ramps and no negative ramps).*

e. *BONUS question: Can this method be used to study potential blow-up in Euler equation?*

Resolution.

a. Taking ∂_x to the 1D Burgers equation $\partial_t u + u\partial_x u = \nu\partial_{xx}u$, we get:

$$\partial_t\partial_x u + \partial_x(u\partial_x u) = \nu\partial_{xxx}u$$

Thus:

$$\partial_t A + u\partial_x A + A^2 = \nu\partial_{xx}A \tag{12}$$

b. From eq. (12) we get:

$$D_t A + A^2 = \nu \partial_{xx} A$$

c. For $\nu = 0$ we have:

$$D_t A + A^2 = 0$$

Separating variables we get:

$$\frac{dA}{A^2} = -dt$$

Integrating between $s = t_0$ and $s = t$ we get:

$$\frac{1}{A(t, x(t))} - \frac{1}{A(t_0, x(t_0))} = t - t_0$$

Thus:

$$A(t, x(t)) = \frac{A(t_0, x(t_0))}{1 + A(t_0, x(t_0))(t - t_0)}$$

We have a finite time blow up of A at $t = t_0 - 1/A(t_0, x(t_0))$.

d. We take $t_0 = 0$

e.