Exercises 02

1. Consider the hyper-viscous Navier-Stokes equation

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + (-1)^m \nu_m \nabla^{2m+2} \mathbf{u} + \mathbf{F} \tag{1}$$

   where $\nabla^{2m} = \nabla^2 \nabla^2 \dots (m - \text{times})$

   (a) Write an expression for the energy dissipation and the energy balance relation and show that hyperviscosity dissipates energy (ie leads to negative term)

   (b) Predict the lengthscale ℓ_ν that dissipation becomes effective and dissipates the energy based on the energy injection rate and ν_m.

2. Consider again the equation

$$\partial_t \mathbf{b} + \mathbf{v} \times \mathbf{b} = -\nabla P' + (-1)^m \nu_m \nabla^{2m+2} \mathbf{b} + \mathbf{F} \tag{2}$$

$$\nabla \cdot \mathbf{b} = 0 \tag{3}$$

   in a periodic box of size L where m is an integer and $\mathbf{F}$ is a forcing that injects energy at scale L at a rate $\mathcal{I}$.

   $\mathbf{v}$ is related to $\mathbf{b}$ as $\mathbf{v} = (\nabla \times)^n \mathbf{b}$ for some $n \in \mathbb{N}$ so that
   $(\nabla \times)^1 \mathbf{b} = \nabla \times \mathbf{b}$,
   $(\nabla \times)^2 \mathbf{b} = \nabla \times \nabla \times \mathbf{b}$,
   $(\nabla \times)^3 \mathbf{b} = \nabla \times \nabla \times \nabla \times \mathbf{b}$ and so on
   n, m are integers. For $n = 1$, $m = 0$ the system reduces to the Navier Stokes with $\mathbf{b} = \mathbf{u}$.
   For any n Energy $\mathcal{E} = \langle \frac{1}{2} |\mathbf{b}|^2 \rangle$ is conserved for $\nu_m = 0$ and $\alpha = 0$ (see last homework).
   Assuming :

   - energy cascades to smaller scales
   - similar size eddies dominate the cascade

   show the following:

   (a) Predict the energy spectrum of $\mathbf{b}$ based on the assumptions above.

   (b) Predict the lengthscale ℓ_ν that dissipation becomes effective and dissipates the energy.

   (c) For which values of n and m, the viscosity will not be sufficient to dissipate the injected energy as $\nu_m \to 0$?.