Tutorial 3

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Exercise 1: In this homework, we will explore some properties of Burgers equation.

$$\partial_t V_i + V_j \partial_j V_i = \nu \Delta V_i \tag{1}$$

$$V_i = -\partial_i \Psi \tag{2}$$

1. Use Eq. (2) to find the equation for Ψ.
2. 1D case: Check that the "Khokhlov" velocity field $u^\nu(x,t) = \frac{x - L\tanh\left(\frac{Lx}{2\nu t}\right)}{t}$ is a solution of Eq. (1). Draw it at several times for $L = 1$, and $\nu = 1, 10^{-2}, 10^{-6}$.
3. Find the limit of the Khokhlov solution as $\nu \to 0$. This solution represents a shock.

Solution:

1. Inserting Eq. (2) into Eq. (1), we get:

$$-\partial_t \partial_i \Psi + \partial_j \Psi \partial_j \partial_i \Psi = -\nu \partial_j^2 \partial_i \Psi$$
$$\partial_i \left(\partial_t \Psi - \frac{1}{2} \left(\partial_j \Psi\right)^2 \right) = \partial_i (\nu \Delta \Psi)$$
$$\partial_t \Psi - \frac{1}{2} \left(\partial_j \Psi\right)^2 = \nu \Delta \Psi + f(t) \tag{3}$$

   Note that the constant $f(t)$ does not depend on any spatial variable x_i, because for each i, the equation is the same for Ψ. Thus, it can only depend on time.

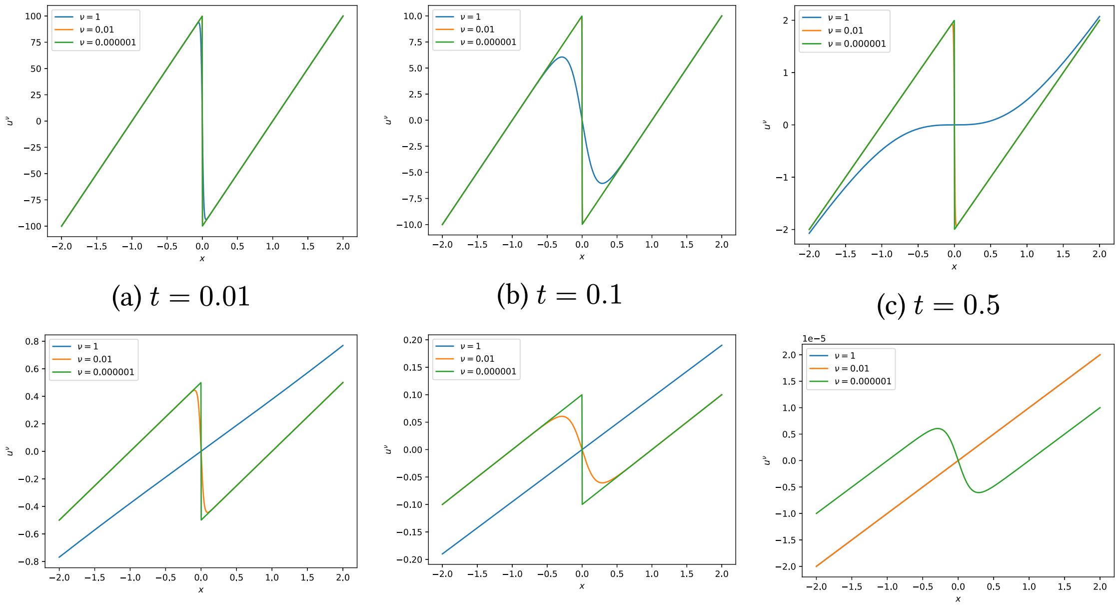
2. We need to check that:

$$\partial_t u^\nu + u^\nu \partial_x u^\nu = \nu \partial_{xx} u^\nu$$

   We have that:

$$
\begin{aligned}
\partial_t u^\nu &= -\frac{x - L\tanh\left(\frac{Lx}{2\nu t}\right)}{t^2} + \frac{L^2 x}{2\nu t^3}\left(\operatorname{sech}\left(\frac{Lx}{2\nu t}\right)\right)^2 \\
\partial_x u^\nu &= \frac{1}{t} - \frac{L^2}{2\nu t^2}\left(\operatorname{sech}\left(\frac{Lx}{2\nu t}\right)\right)^2 \\
u^\nu \partial_x u^\nu &= \frac{x}{t^2} - \frac{L^2 x}{2\nu t^3}\left(\operatorname{sech}\left(\frac{Lx}{2\nu t}\right)\right)^2 - \frac{L}{t^2}\tanh\left(\frac{Lx}{2\nu t}\right) + \frac{L^3 x}{2\nu t^3}\left(\operatorname{sech}\left(\frac{Lx}{2\nu t}\right)\right)^2 \tanh\left(\frac{Lx}{2\nu t}\right) \\
-\nu \partial_{xx} u^\nu &= -\frac{L^3}{2\nu t^3}\left(\operatorname{sech}\left(\frac{Lx}{2\nu t}\right)\right)^3 \sinh\left(\frac{Lx}{2\nu t}\right)
\end{aligned}
$$

Adding all the terms (except the second one), we get 0. Thus, the equation is satisfied. In



(a) $t = 0.01$ (b) $t = 0.1$ (c) $t = 0.5$

(c) $t = 0.2$ (c) $t = 10$ (c) $t = 100000$

Figure 1: Khokhlov velocity field for different values of t and ν

3. Fix $x \in \mathbb{R}$ and $t > 0$. Then, since $\lim_{y \to \pm\infty} \tanh(y) = \pm 1$, we have that:

$$
u(x,t) := \lim_{\nu \to 0} u^\nu(x,t) = \lim_{\nu \to 0} \frac{x - L\tanh\left(\frac{Lx}{2\nu t}\right)}{t} = \frac{x - L\operatorname{sgn}(x)}{t}
$$

where we have assumed that $L > 0$. This function is not continuous at $x = 0$ for any $t > 0$, which is the signature of a shock.

Exercise 2: The Hopf-Cole transformation is defined as:

$$
V_i = -2\nu \partial_i \log(\Phi) \tag{4}
$$

1. Find the link between Ψ and Φ.
2. Show that the equation for Φ is linear (it is the heat equation).
3. Consider the 1D case. Find the solution of the heat equation in case of periodic boundary conditions.

Solution:

1. For each i, we have that $-\partial_i \Psi = -2\nu \partial_i \log(\Phi)$. Thus, integrating with respect to x_i, we get $\Psi = 2\nu \log(\Phi) + g(t)$, and again, the constant $g(t)$ does not depend on any spatial variable

x_i (by the same argument as in the previous exercise). We can in fact determine g from Eq. (3). Since $V_i = -\partial_i(2\nu \log(\Phi))$, then by Exercise 1 $2\nu \log(\Phi)$ satisfies also Eq. (3) and thus by the linearity of the derivatives we get that $g'(t) = f(t)$.

2. From Eq. (3) we get:

$$\partial_t \Psi - \frac{1}{2}(\partial_j \Psi)^2 = \nu \Delta \Psi + f(t)$$

$$\partial_t(2\nu \log(\Phi) + g(t)) - \frac{1}{2}(2\nu \partial_j \log(\Phi))^2 = \nu \Delta(2\nu \log(\Phi)) + f(t)$$

$$\frac{\partial_t \Phi}{\Phi} - \nu \left(\frac{\partial_j \Phi}{\Phi}\right)^2 = \nu \left[\frac{\partial_j^2 \Phi}{\Phi} - \left(\frac{\partial_j \Phi}{\Phi}\right)^2\right]$$

$$\partial_t \Phi = \nu \Delta \Phi$$

   And this last equation is the heat equation, which is linear.

3. The 1D heat equation is $\partial_t \Phi = \nu \partial_{xx} \Phi$. We assume it is defined in a domain $\left[-\frac{L}{2}, \frac{L}{2}\right]$ and we equip it with periodic boundary conditions. Now we express the solution in Fourier series:

$$\Phi(x,t) = \sum_{n \in \mathbb{Z}} \hat{\Phi}_n(t) e^{\frac{2\pi i n x}{L}}$$

   Plugging this formula into the equation we get:

$$\partial_t \hat{\Phi}_n(t) = -\nu \left(\frac{2\pi n}{L}\right)^2 \hat{\Phi}_n(t)$$

   This is a linear ODE, whose solution is $\hat{\Phi}_n(t) = \hat{\Phi}_n(0) e^{-\nu\left(\frac{2\pi n}{L}\right)^2 t}$. Thus, the solution is:

$$\Phi(x,t) = \sum_{n \in \mathbb{Z}} \hat{\Phi}_n(0) e^{-\nu\left(\frac{2\pi n}{L}\right)^2 t} e^{\frac{2\pi i n x}{L}}$$

   The coefficients $\hat{\Phi}_n(0)$ are determined by the initial condition.

Exercise 3: At very large scale, the Universe is described by Newton equations in a flat, expanding geometry. The equations are:

$$\begin{aligned} \partial_t u_i + \frac{\dot{a}}{a} u_i + \frac{1}{a} u_j \partial_j u_i &= -\frac{1}{a} \partial_i \Phi \\ \partial_t \rho + 3\frac{\dot{a}}{a}\rho + \frac{1}{a}\partial_j(\rho u_j) &= 0 \\ \Delta\Phi &= 4\pi G a^2 (\rho - \rho_b) \end{aligned} \tag{5}$$

where $a(t)$ is the expansion factor, Φ is the gravitational potential, ρ is the density and $\boldsymbol{u}$ us the velocity of the gas. Show that these equations can be mapped into the inviscid Burgers equation ($\nu = 0$) by using Zeldovich transformation:

$$\begin{aligned} V_i &= \frac{u_i}{a\dot{b}} = -\partial_i \tilde{\Psi} \\ \left(\partial_t + 2\frac{\dot{a}}{a}\right)\partial_t b &= 4\pi G \rho_b(t) b \\ \tilde{\Phi} &= \frac{\Phi}{4\pi G \rho_b a^2 b} \\ \tilde{\Phi} &= \tilde{\Psi} \end{aligned}$$

Solution: We have that $u_i = -a\dot{b}\partial_i\left(\tilde{\Psi}\right)$. Then, introducing this into the equation for u_i, we have:

$$\begin{aligned} \partial_t u_i + \frac{\dot{a}}{a} u_i + \frac{1}{a} u_j \partial_j u_i &= -\frac{1}{a}\partial_i \Phi \\ -\dot{a}\dot{b}\partial_i\left(\tilde{\Psi}\right) - a\ddot{b}\partial_i\tilde{\Psi} - a\dot{b}\partial_t\partial_i\left(\tilde{\Psi}\right) - \dot{a}\dot{b}\partial_i\tilde{\Psi} + a\dot{b}^2\partial_j\tilde{\Psi}\partial_j\partial_i\tilde{\Psi} &= -4\pi G\rho_b a b \partial_i \tilde{\Phi} \\ -a\dot{b}\partial_t\partial_i\tilde{\Psi} - 4\pi G\rho_b a b\partial_i\tilde{\Phi} + a\dot{b}^2\partial_j\tilde{\Psi}\partial_j\partial_i\tilde{\Psi} &= -4\pi G\rho_b a b \partial_i\tilde{\Phi} \\ -a\dot{b}\partial_t\partial_i\tilde{\Psi} + a\dot{b}^2\partial_j\tilde{\Psi}\partial_j\partial_i\tilde{\Psi} &= 0 \\ -\partial_b\partial_i\tilde{\Psi} + \partial_j\tilde{\Psi}\partial_j\partial_i\tilde{\Psi} &= 0 \\ \partial_b V_i + V_j\partial_j V_i &= 0 \end{aligned}$$

which is the inviscid Burgers equation with time variable b.

Exercise 4: Burgers equation develop finite time singularities. Let us study this in the 1D case.

1. Use Eq. (2) to write an equation for $A = \partial_x u$.
2. Introduce the Lagrangian derivative $\mathrm{D}_t A = \partial_t A + u\partial_x A$. Use 1) to find the ordinary differential equation that links A and its Lagrangian derivative.
3. Integrate this equation in the case $\nu = 0$, and discuss in which condition there is a finite time blow up of A.
4. Use this discussion to explain the features of the Khokhlov solution at $\nu \to 0$ (presence of positive ramps and no negative ramps).
5. Can this method be used to study potential blow-up in Euler equation?

Solution:

1. Taking ∂_x to 1D Burgers equation $\partial_t u + u\partial_x u = \nu\partial_{xx} u$, we get:

$$\partial_t\partial_x u + \partial_x(u\partial_x u) = \nu\partial_{xxx} u$$

Thus:

$$\partial_t A + u\partial_x A = \nu\partial_{xx} A \tag{6}$$

2. From Eq. (6) we get:

$$\mathrm{D}_t A + A^2 = \nu\partial_{xx} A$$

3. For $\nu = 0$ we have:

$$\mathrm{D}_t A + A^2 = 0$$

   Separating variables, we get:

$$\frac{\mathrm{d}A}{A^2} = -\,\mathrm{d}t$$

   Integrating between $s = t_0$ and $s = t$, we get:

$$\frac{1}{A(t, x(t))} - \frac{1}{A_0} = t - t_0$$

$$A(t, x(t)) = \frac{A_0}{1 + (t - t_0)A_0}$$

   where $A_0 := A(t_0, x(t_0))$. We have a finite time blow up of A at $t = t_0 - \frac{1}{A_0}$, provided that $A_0 \neq 0$.

4. For simplicity we take $t_0 = 0$. The derivative of the Khokhlov solution is defined everywhere (expect for $x = 0$) for all positive times. Thus, since the blow up is at $t = -\frac{1}{A_0}$, we must have $A_0 > 0$, which implies that $A(t, x(t)) > 0\ \forall t > 0$, that is, the solution develops positive ramps and no negative ramps.

5. The 1D Euler equation is $\partial_t u + u\partial_x u = -\partial_x p$. We can use the same method to study potential blow-up in the Euler equation. However, the pressure term will make the analysis more complicated.