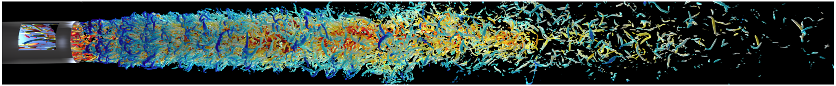
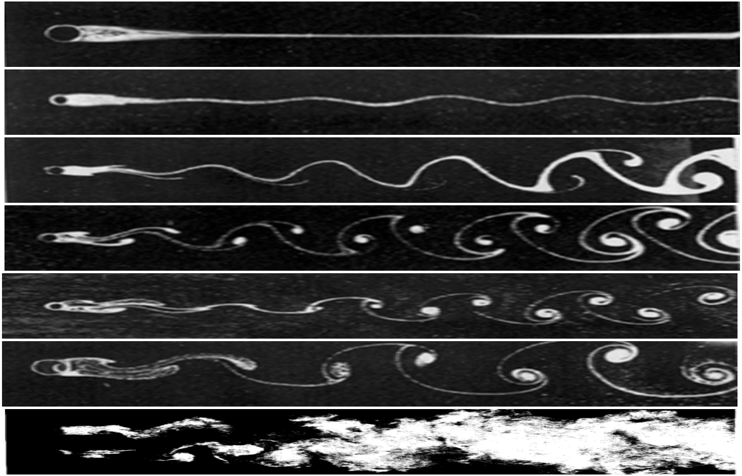
Turbulence
From Deterministic Dynamics to Randomness



Alexandros ALEXAKIS
alexakis@phys.ens.fr
Dep. Physique ENS Ulm

Flow behind a cylinder



Re= 32

Re= 55

Re= 60

Re= 73

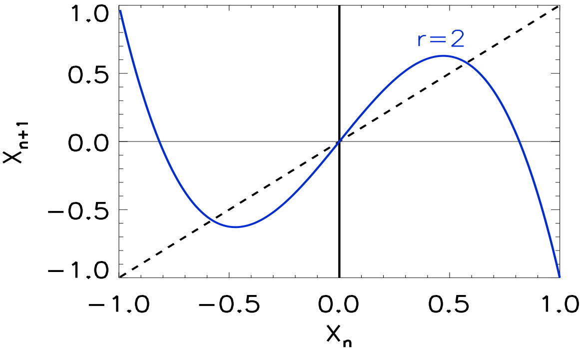
Re=102

Re=161

Re=1800

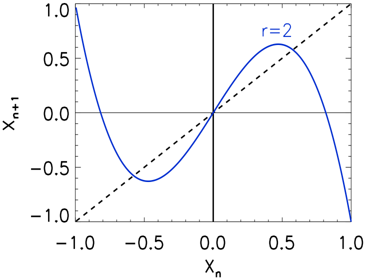
A deterministic map

$$\mathbf{X_{n+1} = X_n \left[r - (r+1) X_n^2 \right]}$$



Properties

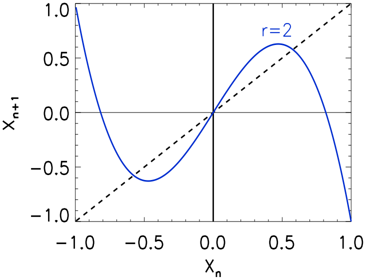
$$\mathbf{X_{n+1} = X_n \left[r - (r+1) X_n^2 \right]}$$



- It is has $X \to -X$ symmetry
  - if $\mathbf{X} = [\ X_0,\ X_1,\ X_2, \ldots]$ is a solution
  - then $\mathbf{X}' = [-X_0, -X_1, -X_2, \ldots]$ is a solution

Properties

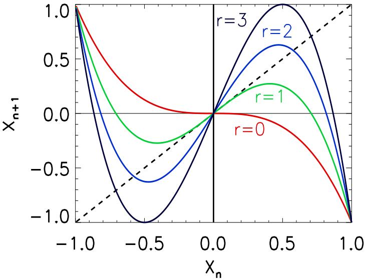
$$\mathbf{X_{n+1} = X_n \left[r - (r+1) X_n^2 \right]}$$



- It is has $X \to -X$ symmetry
- $\mathbf{X = 0}$ is always a solution
- $\mathbf{X = [+1, -1, +1, -1, +1, -1, \dots]}$ is also a solution

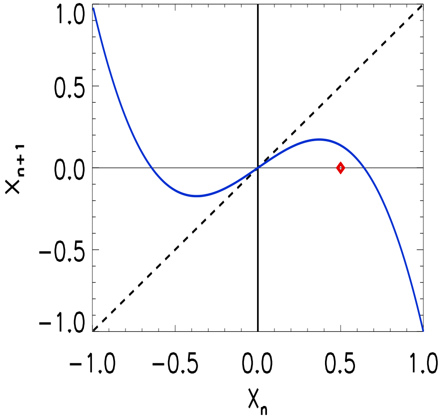
Properties

$$\mathbf{X_{n+1} = X_n \left[r - (r+1)X_n^2\right]}$$



- It is has $X \to -X$ symmetry
- $\mathbf{X = 0}$ is always a solution
- $\mathbf{X = [+1, -1, +1, -1, +1, -1, \ldots]}$ is also a solution
- Maps $[\mathbf{-1, 1}] \to [\mathbf{-1, 1}]$ for $0 \leq r \leq 3$

$$\mathbf{X_{n+1} = X_n \left[r - (r+1) X_n^2 \right]}, \qquad r = 0.7$$



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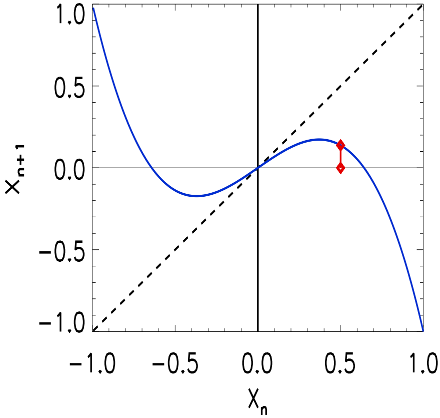

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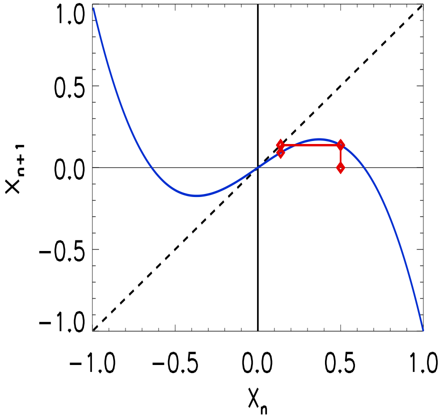

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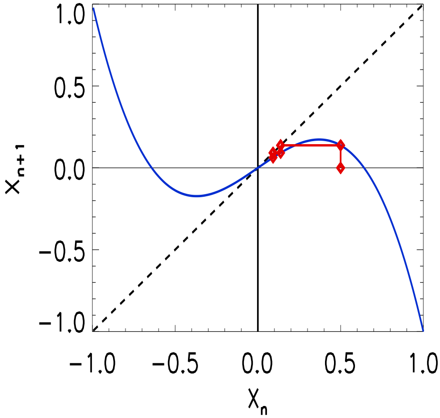

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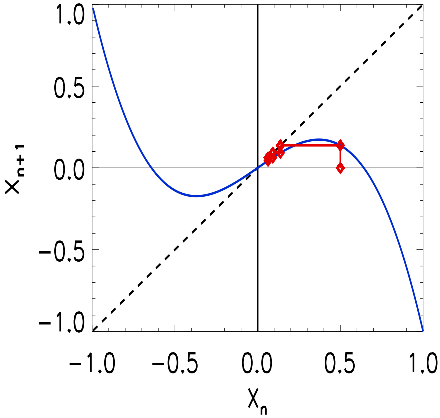

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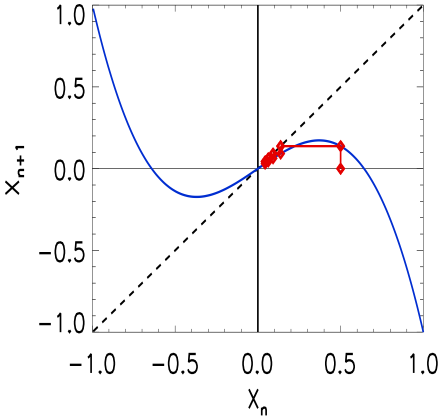

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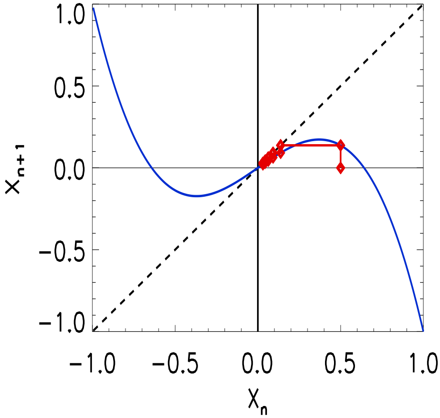

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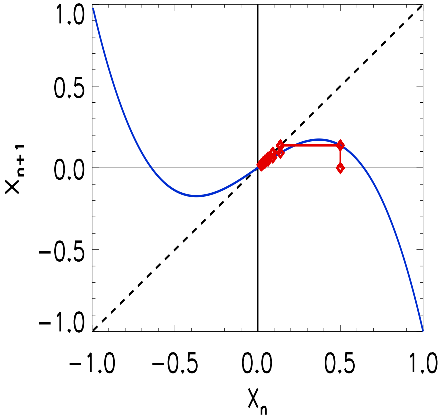

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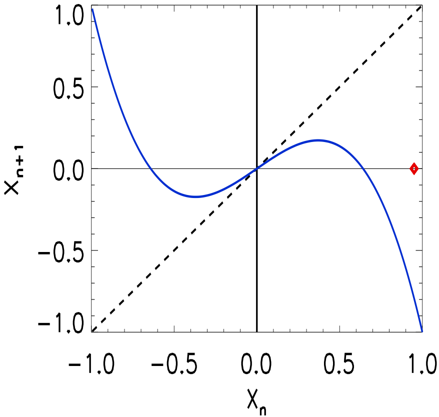

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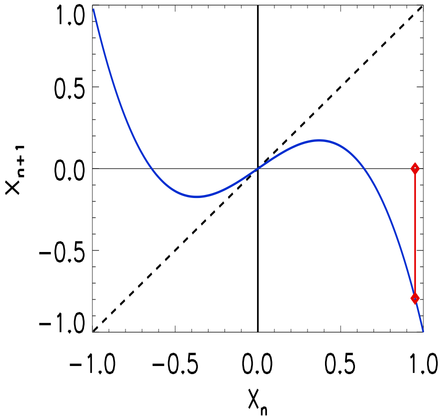

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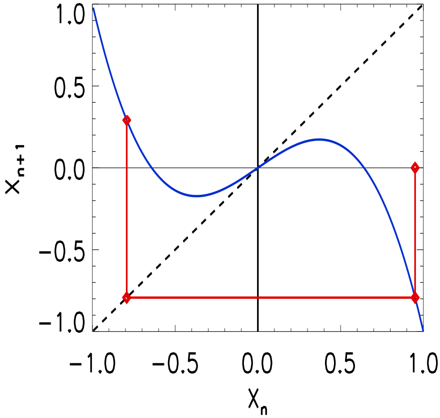

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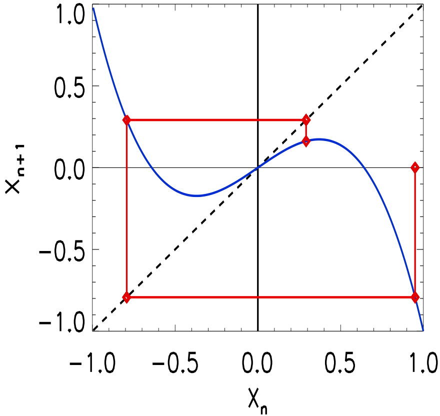

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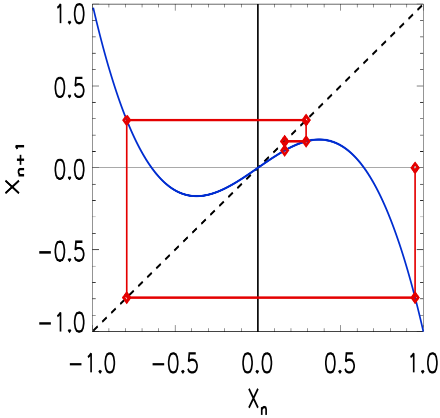

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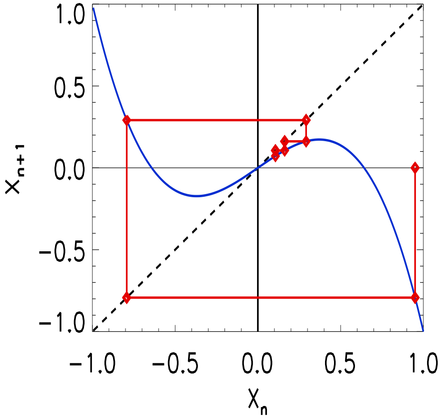

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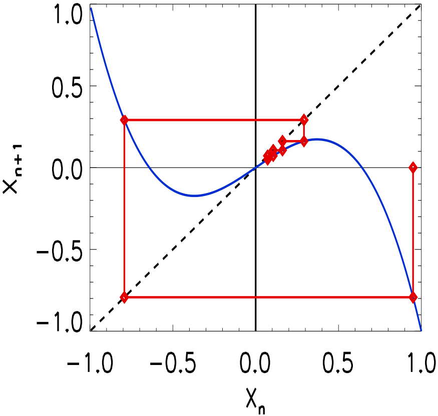

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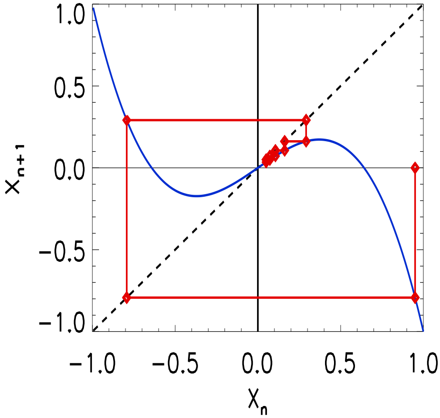

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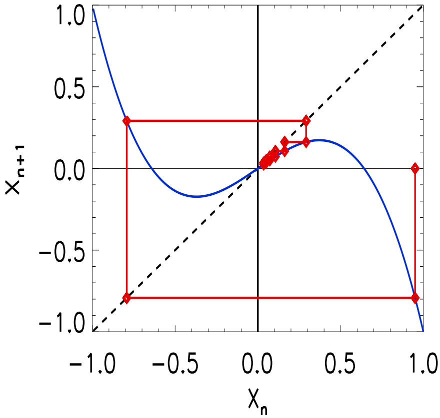


Examples

$$\mathbf{X_{n+1} = X_n \left[r - (r+1) X_n^2 \right]}$$

For $X_0 \ll 1$

$$X_1 \simeq r X_0$$

$$X_2 \simeq r X_1 \simeq r^2 X_0$$

$$\dots$$

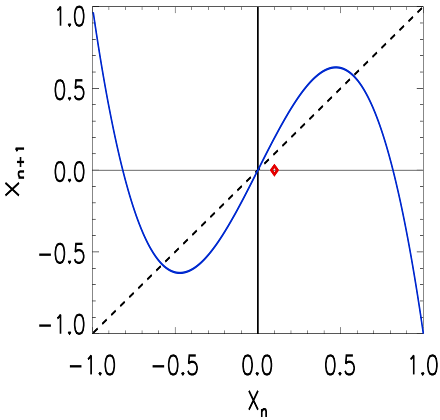
$$X_n = r^n X_0$$

if $r < 1$ when $n \to \infty$

$$X_n \to 0$$

Examples

$$\mathbf{X_{n+1} = X_n \left[r - (r+1)X_n^2\right]}, \qquad r = 2.0$$



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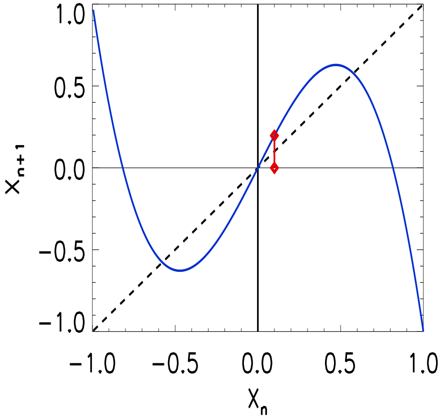

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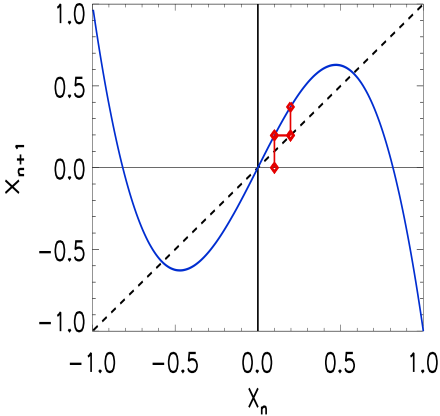

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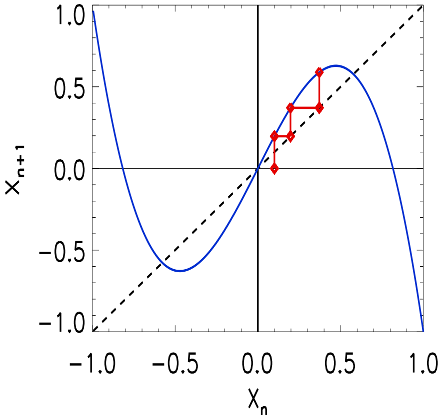

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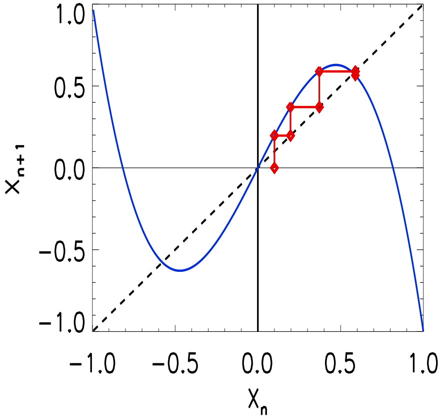


Examples

For $n \to \infty$

$$\mathbf{X_n = X_n \left[r - (r+1) X_n^2\right]}$$

$$X_n - X_n \left[r - (r+1) X_n^2\right] = 0$$

$$X_n \left[1 - r + (r+1) X_n^2\right] = 0$$

$$X_n = 0 \qquad \text{or} \qquad X_n^2 = (r-1)/(r+1)$$

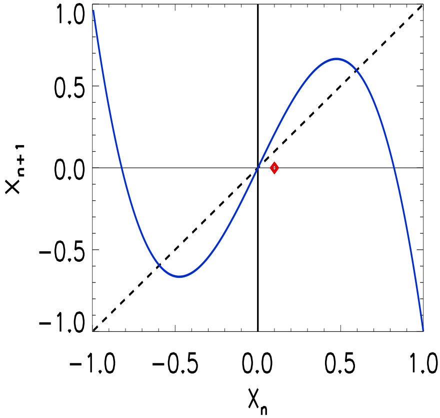
New solution

$$X_n = \pm\sqrt{\frac{r-1}{r+1}}$$

Valid only for $r \geq 1$

Examples

$$\mathbf{X_{n+1} = X_n \left[r - (r+1)X_n^2\right]}, \qquad r = 2.2$$



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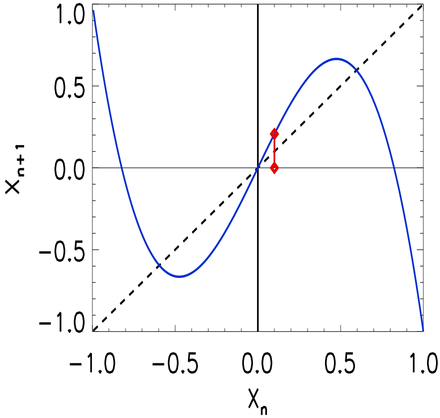

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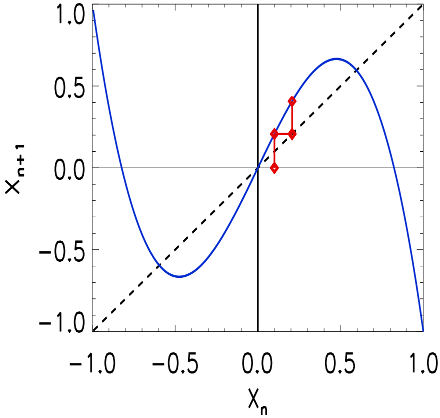

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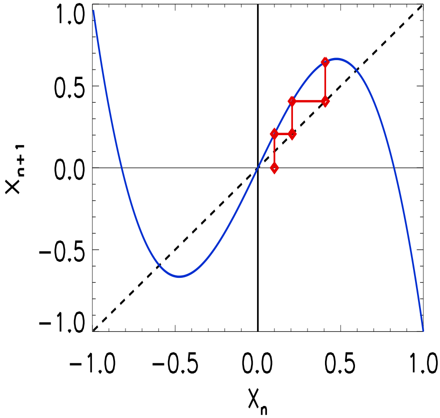

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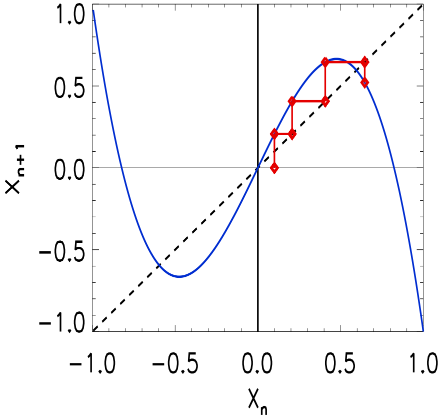

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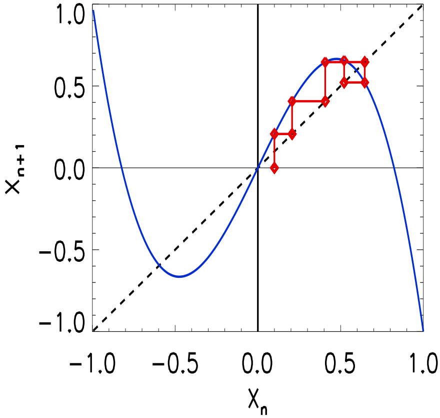

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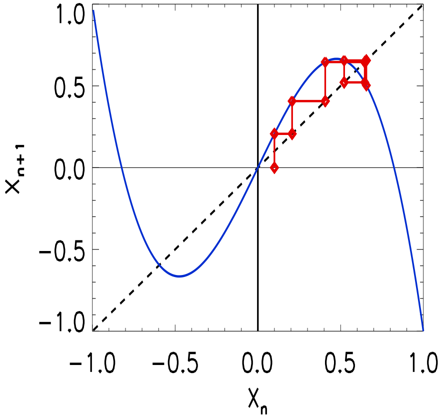

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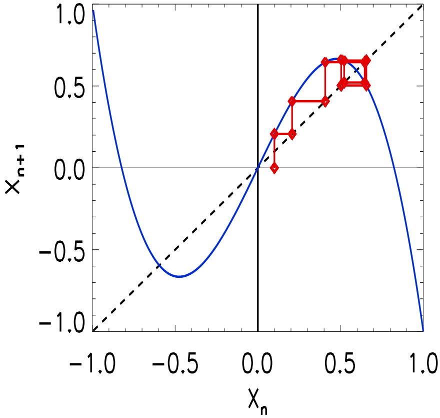

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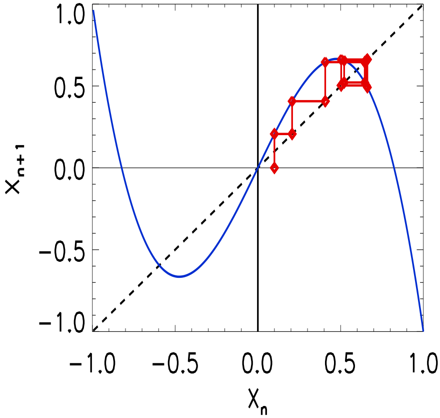

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$$\mathbf{X_{n+2} = X_{n+1} \left[r - (r+1)X_{n+1}^2\right]}$$

Examples

$$\mathbf{X_{n+1} = X_n \left[r - (r+1)X_n^2\right]}$$

$$\mathbf{X_{n+2} = X_{n+1} \left[r - (r+1)X_{n+1}^2\right]}$$

$$X_{n+2} = (X_n \left[r - (r+1)X_n^2\right]) \left[r - (r+1)(X_n \left[r - (r+1)X_n^2\right])^2\right]$$

For $n \to \infty$

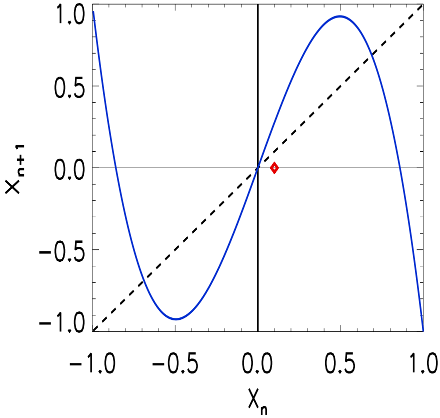
$$X_n = (X_n \left[r - (r+1)X_n^2\right]) \left[r - (r+1)(X_n \left[r - (r+1)X_n^2\right])^2\right]$$

$$\dots$$

$$X_n =$$

Examples

$$\mathbf{X_{n+1} = X_n \left[r - (r+1)X_n^2\right]}, \qquad r = 2.8$$



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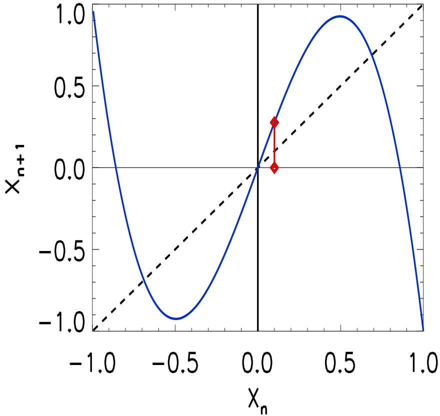

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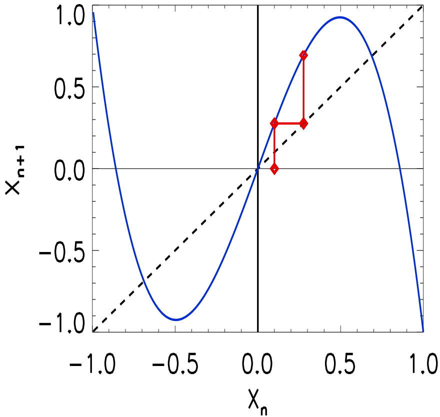

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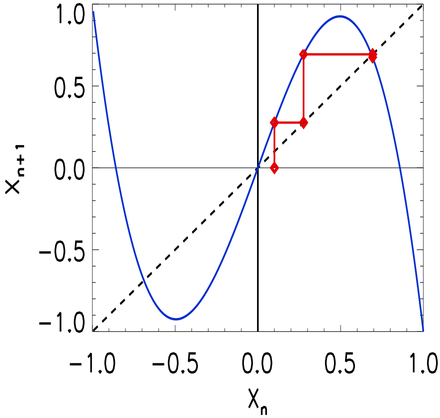

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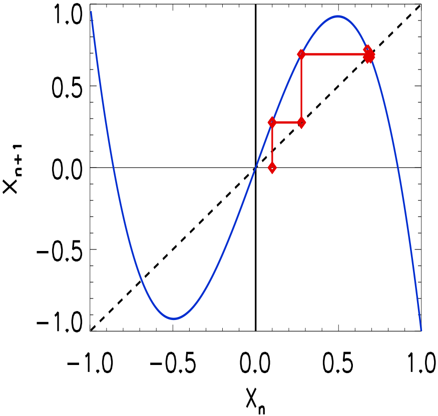

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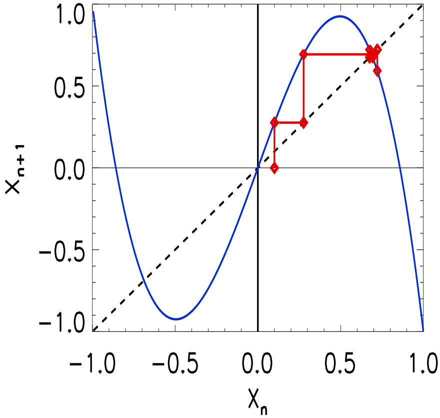

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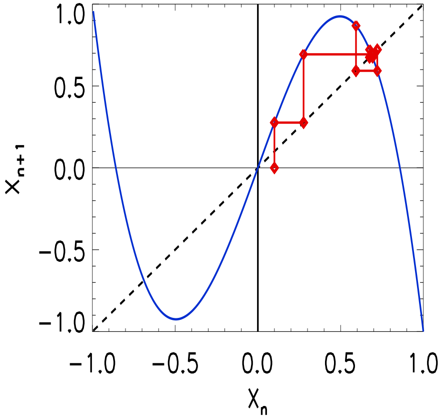

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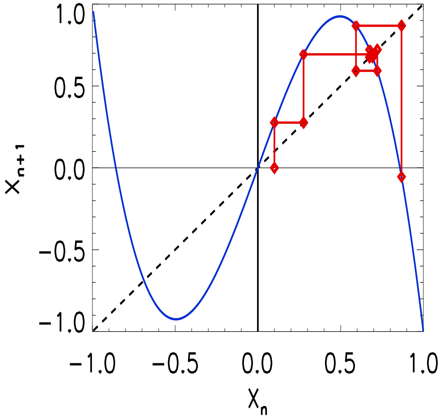

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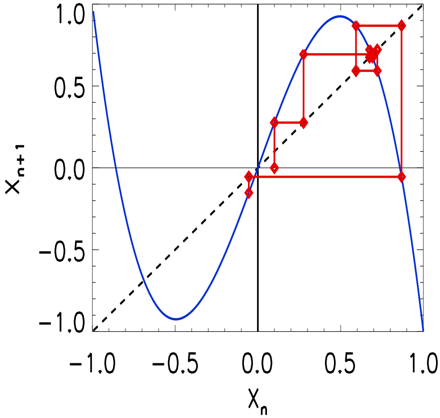

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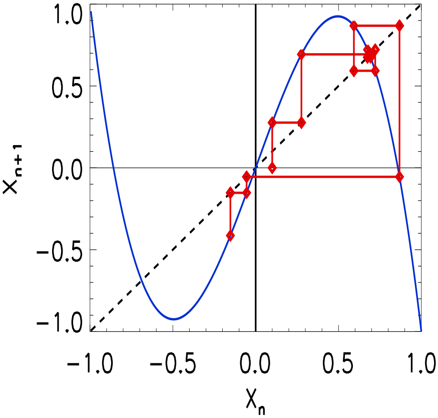

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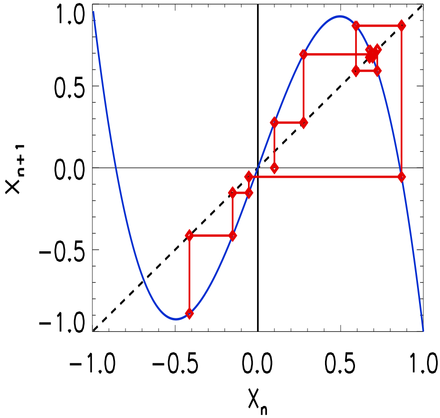

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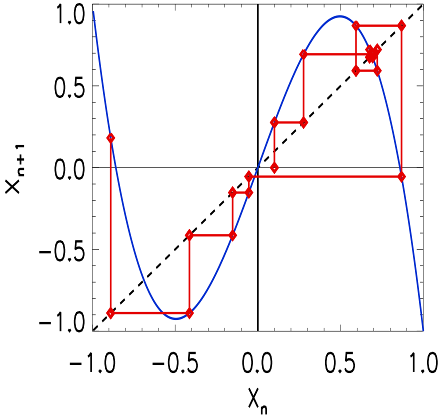

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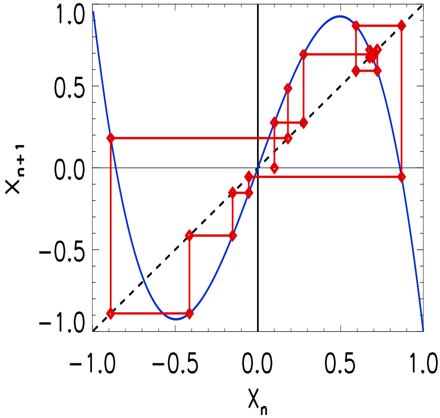

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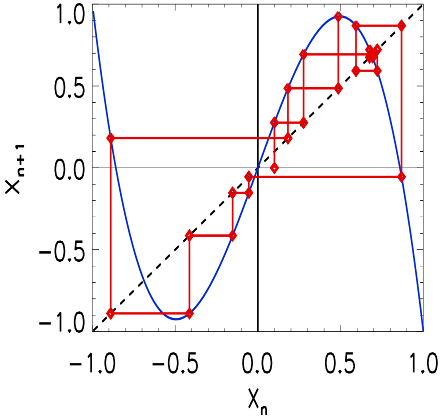

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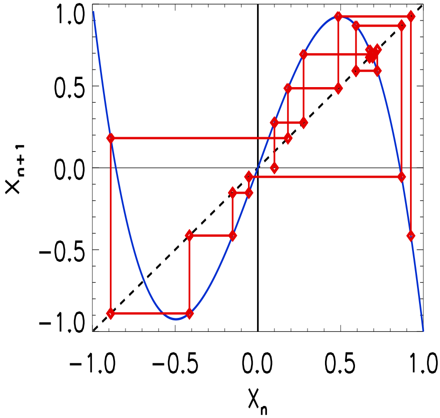

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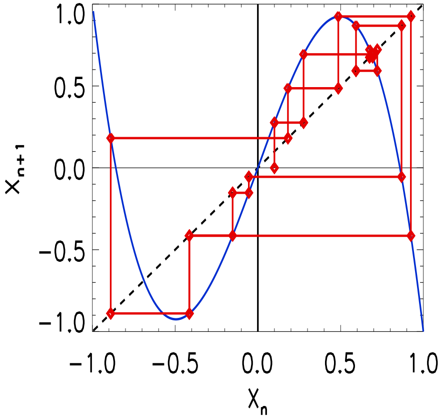

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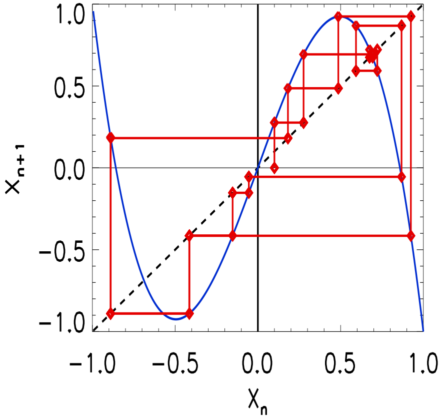

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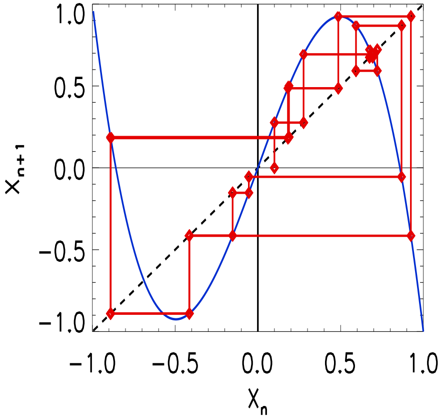

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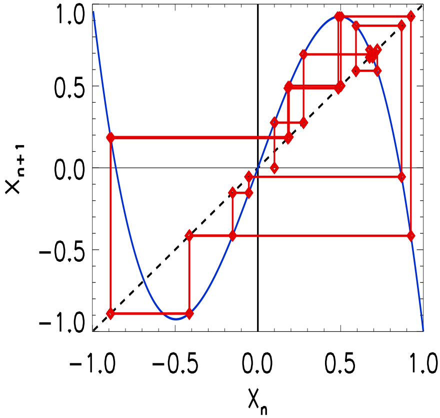

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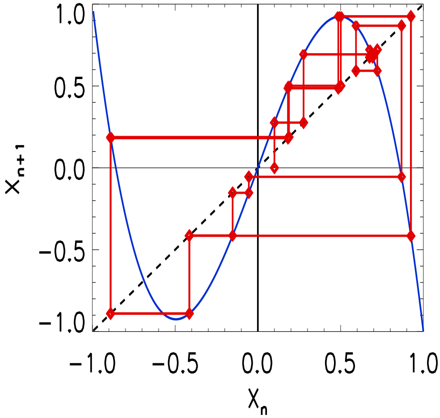

Examples

$$\mathbf{X_{n+1} = X_n \left[r - (r+1)X_n^2\right]},$$

for $r = 3$

$$\mathbf{X_{n+1} = 3X_n - 4X_n^3},$$

Examples

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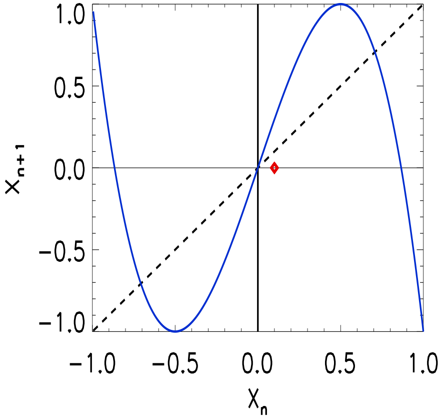

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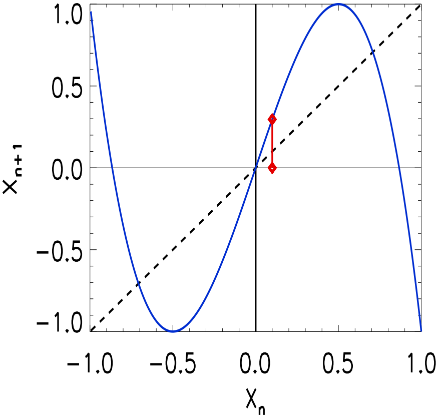

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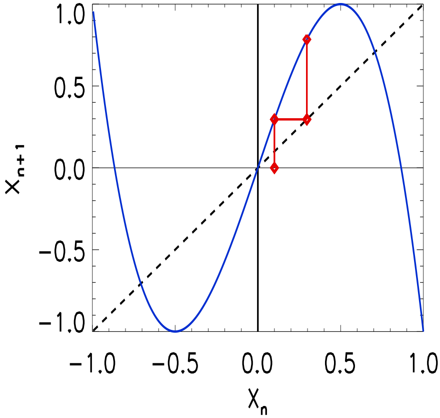

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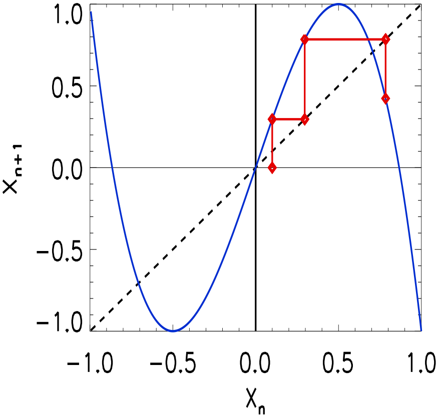

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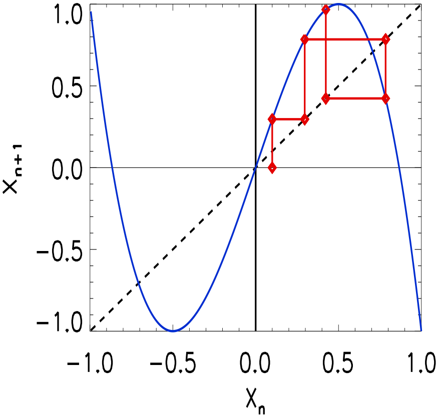

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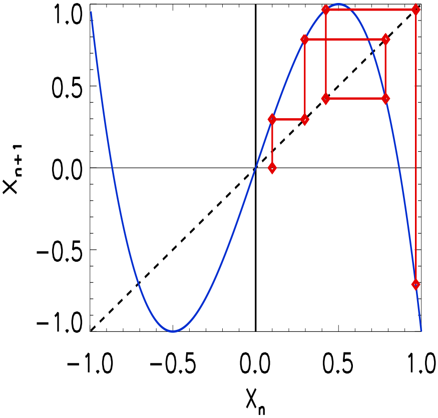

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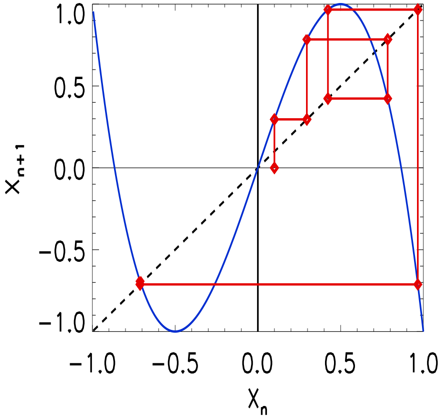

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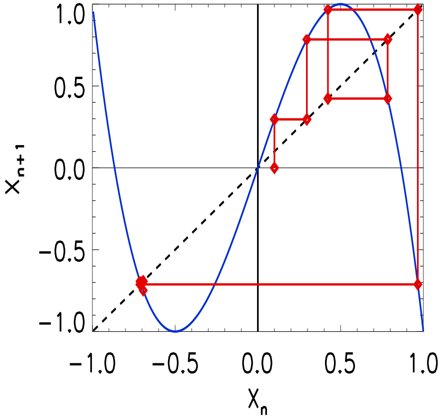

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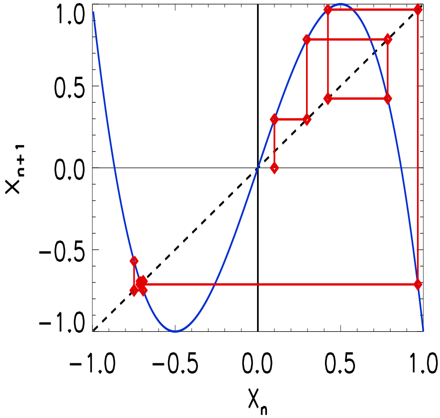

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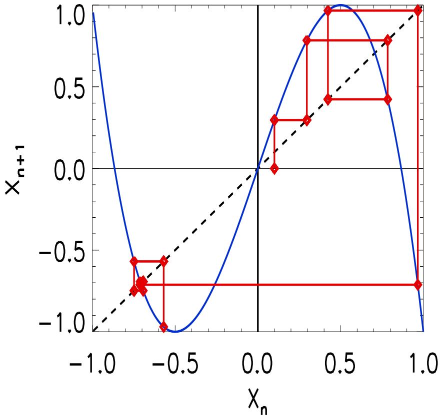

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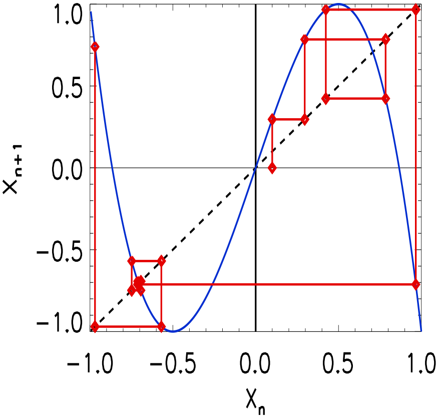

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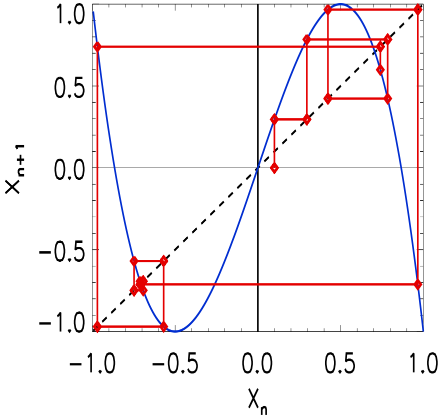

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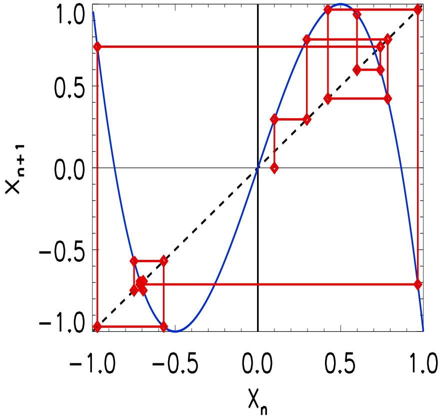

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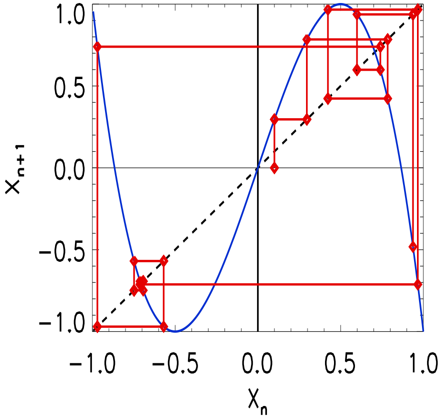

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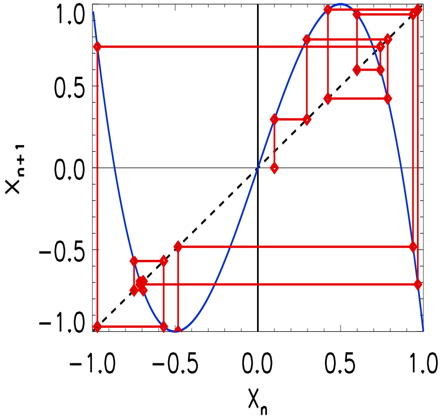

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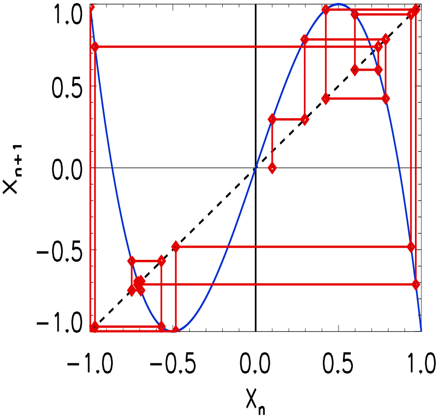

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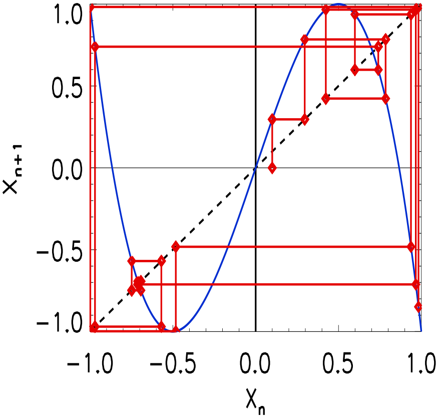

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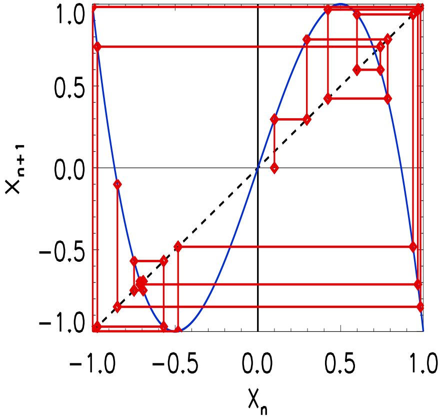

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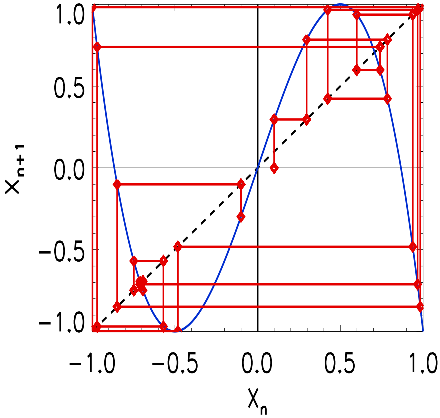

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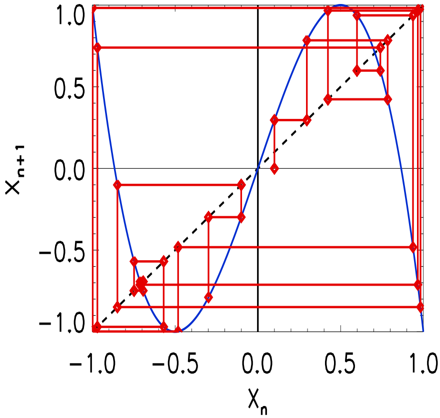
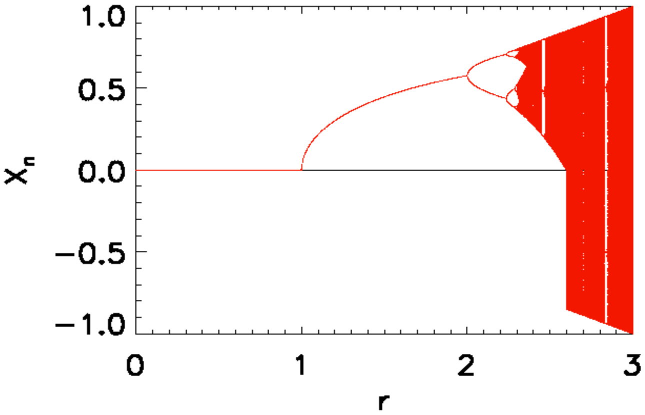


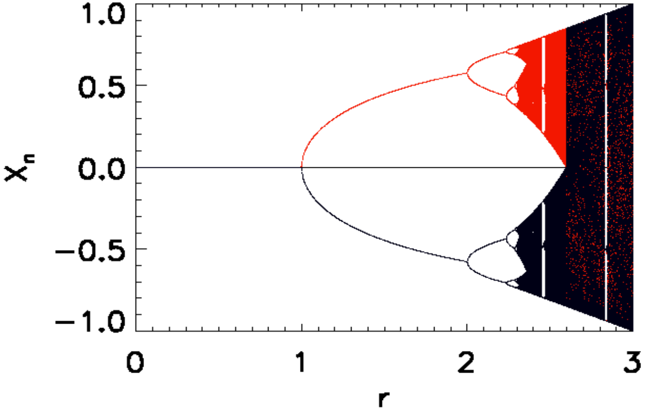
Bifurcation Diagram

$$\mathbf{X_{n+1} = X_n \left[r - (r+1)X_n^2\right]}$$

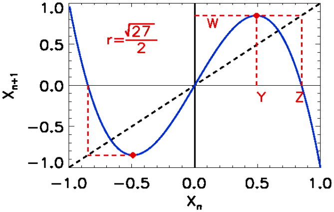


Bifurcations Diagram

$$\mathbf{X_{n+1} = X_n \left[r - (r+1) X_n^2 \right]}$$



The r=3 case



$$\mathbf{X_{n+1} = X_n(r - (1+r)X_n^2}$$

- The map changes sign at $Z = \sqrt{r/(1+r)}$
- Local maximum at: $dX_{n+1}/dX_n = 0$ at $Y = \sqrt{r/3(r+1)}$
- Maximum value $W = Y(r - (r+1)Y^2) = \sqrt{2r^3/27(r+1)}$

The transition to the symmetric behavior comes when $W(r) > Z(r)$ where

$$r > \frac{\sqrt{27}}{2}$$

Bifurcation Diagram

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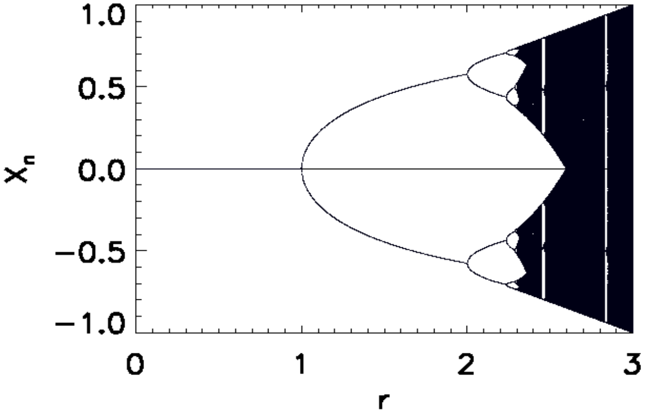

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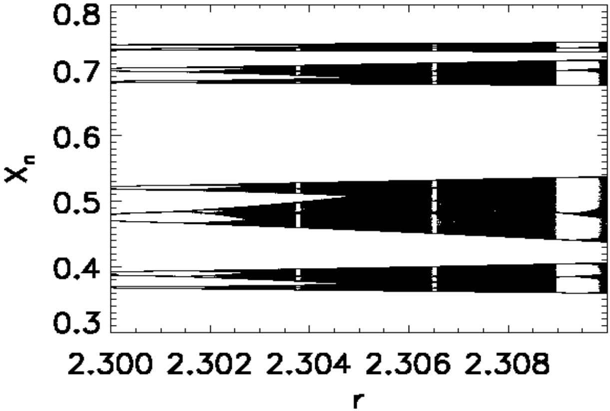


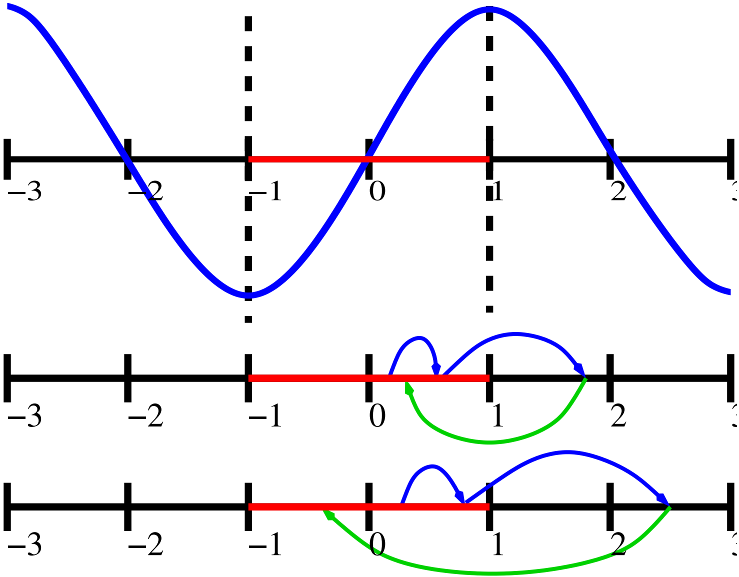
$$\mathbf{X_{n+1} = 3X_n - 4X_n^3}$$

A change of variables: $X_n = \sin(\pi\theta_n/2)$

$$\begin{aligned} \sin\left(\frac{\pi\theta_{n+1}}{2}\right) &= 3\sin\left(\frac{\pi\theta_n}{2}\right) - 4\sin^3\left(\frac{\pi\theta_n}{2}\right) \\ &= \sin\left(3\frac{\pi\theta_n}{2}\right) \end{aligned}$$

$$\theta_{n+1} = 3\theta_n$$

$\theta_n \to 3\theta_n$



Probability Distribution function

$$
\begin{aligned}
P_X(X)dX &= \text{Probability of } X_n &\in [X, X+dX] \\
P_\theta(\theta)d\theta &= \text{Probability of } \theta_n &\in [\theta, \theta + d\theta_n]
\end{aligned}
$$

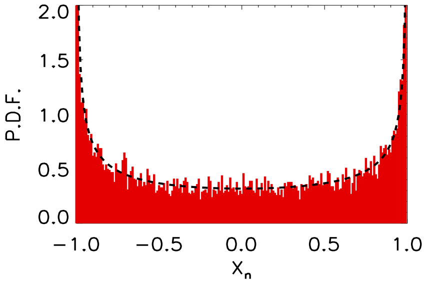
$$P_\theta(\theta)d\theta = P_X(X)dX$$

$$P_\theta(\theta) = 1/2$$

$$
\begin{aligned}
P_X &= \left(\frac{dX}{d\theta}\right)^{-1} P_\theta \\
&= \left(\frac{\pi}{2}\cos(\pi\theta/2)\right)^{-1} P_\theta
\end{aligned}
$$

$$\boxed{P_X(X) = \frac{2}{\pi\sqrt{1-X^2}}}$$

Probability Distribution function



PDF respects the $X \to -X$ symmetry!

Statistical Recovery of Symmetries

We will say that a field $\mathbf{u}(\mathbf{x}, t)$ is statistically invariant under a transformation $\mathcal{T}$ or that it has a statistical $\mathcal{T}$-symmetry if:

$$P(\mathcal{T}[\mathbf{u}]) = P(\mathbf{u})$$

eg

$$P[\mathbf{u}(\mathbf{x}, t)] = P[\mathbf{u}(\mathbf{x} + \ell, t)]$$

Lyapunov Exponents

$$\mathbf{X_{n+1} = X_n \left[r - (r+1)X_n^2\right]}$$

$$X'_n = X_n + \delta X_n, \qquad , \qquad \delta X_n \ll X_n$$

$$\delta X_{n+1} \simeq [r - 3(r+1)X_n^2]\delta X_n$$

$$\left(\frac{\delta X_{n+1}}{\delta X_n}\right) = [r - 3(r+1)X_n^2]$$

Lyapunov Exponents

$$\mathbf{X_{n+1} = X_n \left[r - (r+1)X_n^2\right]}$$

$$\begin{aligned}
\delta X_{n+1} &= \delta X_0 \left(\frac{\delta X_1}{\delta X_0}\right) \left(\frac{\delta X_2}{\delta X_1}\right) \dots \left(\frac{\delta X_{n+1}}{\delta X_n}\right) \\
&= \delta X_0 \left[r - 3(r+1)X_0^2\right] \left[r - 3(r+1)X_1^2\right] \dots \left[r - 3(r+1
\end{aligned}$$

$$\ln\left|\frac{\delta X_{n+1}}{\delta X_n}\right| = \sum_n \ln\left|r - 3(r+1)X_n^2\right|$$

For $n = N \gg 1$

$$\ln\left|\frac{\delta X_{n+1}}{\delta X_n}\right| = n\lambda, \qquad \left(or \quad |\delta X_{n+1}| = |\delta X_0|\, e^{n\lambda}\right)$$

where

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N} \ln\left|r - 3(r+1)X_n^2\right|$$

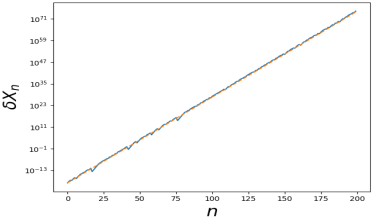
Lyapunov Exponents

$$\mathbf{X_{n+1} = X_n \left[r - 3(r+1)X_n^2\right]}$$

$$\lambda = \int_{-1}^{+1} P(X) \ln\left|r - 3(r+1)X_n^2\right| dX$$

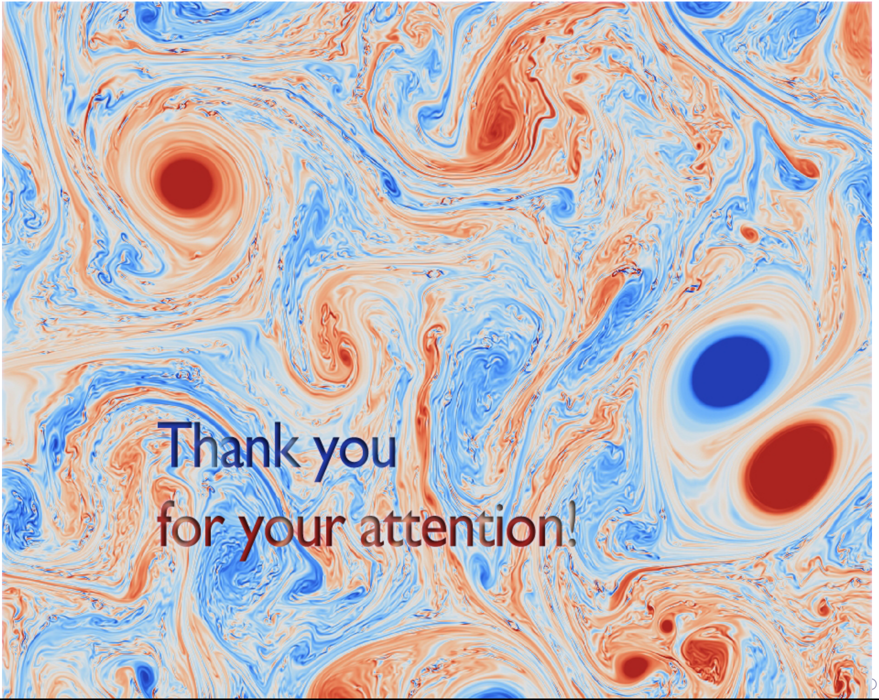
for $r = 3$

$$\lambda = \int_{-1}^{+1} \frac{\ln\left|3 - 12X_n^2\right|}{\pi\sqrt{1 - X^2}} dX$$



Some key points

- At very small r, (Re) symmetric solutions are observed
- When r, (Re) is increased symmetries are broken: ie observed individual solutions do not satisfy them
- Symmetric solutions still exist but are unstable
- Symmetries are recovered in a statistical sense
- Individual solution convey little information
- The system needs to be described in a statistical way: $P(\mathbf{u})$


Thank you
for your attention!