

Tutorial 2

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Turbulence

M2 Applied and Theoretical Mathematics

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Exercise 1. Consider the hyper-viscous Navier-Stokes equation

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + (-1)^m \nu_m \nabla^{2m+2} \mathbf{u} + \mathbf{F} \quad (1)$$

where $\nabla^{2m} = \nabla^2 \nabla^2 \dots \nabla^2$.

- Write an expression for the energy dissipation and the energy balance relation and show that hyper-viscosity dissipates energy (i.e. leads to negative term).
- Predict the lengthscale ℓ_ν that dissipation becomes effective and dissipates the energy based on the energy injection rate and ν_m .

Resolution.

- We have to compute $\partial_t \mathcal{E} = \partial_t \langle \frac{1}{2} |\mathbf{u}|^2 \rangle = \langle \mathbf{u} \cdot \partial_t \mathbf{u} \rangle$. We have:

$$\begin{aligned} \langle \mathbf{u} \cdot \partial_t \mathbf{u} \rangle &= -\langle \mathbf{u} \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) \rangle - \langle \mathbf{u} \cdot \nabla P \rangle + (-1)^m \nu_m \langle \mathbf{u} \cdot \nabla^{2m+2} \mathbf{u} \rangle + \langle \mathbf{u} \cdot \mathbf{F} \rangle \\ &= -\left\langle \mathbf{u} \cdot \nabla \left(\frac{1}{2} |\mathbf{u}|^2 \right) \right\rangle - \langle \mathbf{u} \cdot \nabla P \rangle + (-1)^m \nu_m \langle \mathbf{u} \cdot \nabla^{2m+2} \mathbf{u} \rangle + \langle \mathbf{u} \cdot \mathbf{F} \rangle \\ &= (-1)^m \nu_m \langle \mathbf{u} \cdot \nabla^{2m+2} \mathbf{u} \rangle + \langle \mathbf{u} \cdot \mathbf{F} \rangle \end{aligned}$$

Now note that $\nabla^{2m+2} \mathbf{u}$ can be expressed in Einstein notation as $c_{a_1, a_2, a_3} \partial_x^{2a_1} \partial_y^{2a_2} \partial_z^{2a_3} u_i$ for $i = 1, 2, 3$, $a_1 + a_2 + a_3 = m + 1$ and $c_{a_1, a_2, a_3} \geq 0$ are constants. Then we have:

$$\begin{aligned} \langle \mathbf{u} \cdot \nabla^{2m+2} \mathbf{u} \rangle &= \langle c_{a_1, a_2, a_3} u_i \partial_x^{2a_1} \partial_y^{2a_2} \partial_z^{2a_3} u_i \rangle \\ &= (-1)^{a_1} \langle c_{a_1, a_2, a_3} \partial_x^{a_1} u_i \partial_x^{a_1} \partial_y^{2a_2} \partial_z^{2a_3} u_i \rangle \\ &= (-1)^{a_1 + a_2} \langle c_{a_1, a_2, a_3} \partial_x^{a_1} \partial_y^{a_2} u_i \partial_x^{a_1} \partial_y^{a_2} \partial_z^{2a_3} u_i \rangle \\ &= (-1)^{a_1 + a_2 + a_3} \langle c_{a_1, a_2, a_3} (\partial_x^{a_1} \partial_y^{a_2} \partial_z^{a_3} u_i)^2 \rangle \end{aligned}$$

where we used that the periodic boundary conditions. Thus:

$$\partial_t \mathcal{E} = (-1)^{-m} (-1)^{a_1 + a_2 + a_3} \nu_m \langle c_{a_1, a_2, a_3} (\partial_x^{a_1} \partial_y^{a_2} \partial_z^{a_3} u_i)^2 \rangle + \langle \mathbf{u} \cdot \mathbf{F} \rangle = -\nu_m \langle c_{a_1, a_2, a_3} (\partial_x^{a_1} \partial_y^{a_2} \partial_z^{a_3} u_i)^2 \rangle + \langle \mathbf{u} \cdot \mathbf{F} \rangle$$

whose first term is negative (because we assume $\nu_m \geq 0$), and thus the hyper-viscosity dissipates energy.

- Let $\mathcal{I} = \langle \mathbf{u} \cdot \mathbf{F} \rangle$ be the energy injection rate. The scaling for the dissipative term is:

$$\nu_m \langle c_{a_1, a_2, a_3} (\partial_x^{a_1} \partial_y^{a_2} \partial_z^{a_3} u_i)^2 \rangle \sim \nu_m \frac{1}{\ell^{2(a_1 + a_2 + a_3)}} u_\ell^2 \sim \nu_m \frac{1}{\ell^{2m+2}} u_\ell^2$$

Since $\partial_t \mathcal{E} \sim \frac{u_\ell^2}{\tau_\ell} = \frac{u_\ell^3}{\ell}$, and on the other hand $\partial_t \mathcal{E} \sim \mathcal{I}$, we have $u_\ell \sim (\ell \mathcal{I})^{1/3}$ and so:

$$\nu_m \frac{u_\ell^2}{\ell^{2m+2}} \sim \mathcal{I} \implies \nu_m \frac{(\ell \mathcal{I})^{2/3}}{\ell^{2m+2}} \sim \mathcal{I} \implies \ell_\nu \sim \left(\frac{\nu_m}{\mathcal{I}^{1/3}} \right)^{1/(2m+4/3)}$$

Exercise 2. Consider again the equation

$$\partial_t \mathbf{b} + \mathbf{v} \times \mathbf{b} = -\nabla P + (-1)^m \nu_m \nabla^{2m+2} \mathbf{b} + \mathbf{F} \quad (2)$$

$$(3)$$

in a periodic box of size L where m is an integer and \mathbf{F} is a forcing that injects energy at scale L at a rate \mathcal{I} . \mathbf{v} is related to \mathbf{b} as $\mathbf{v} = (\nabla \times)^n \mathbf{b}$ for some $n \in \mathbb{N}$. Recall that for any n , the energy $\mathcal{E} = \langle \frac{1}{2} |\mathbf{b}|^2 \rangle$ is conserved for $\nu_m = 0$ and $\alpha = 0$. Assuming:

- *Energy cascades to smaller scales*
- *Similar size eddies dominate the cascade*

show the following:

- Predict the energy spectrum of \mathbf{b} based on the assumptions above.*
- Predict the lengthscale ℓ_ν that dissipation becomes effective and dissipates the energy.*
- For which values of n and m , the viscosity will not be sufficient to dissipate the injected energy as $\nu_m \rightarrow 0$?*

Resolution.

- As in exercise 1 we have that:

$$\partial_t \mathcal{E} = -\nu_m \langle c_{a_1, a_2, a_3} (\partial_x^{a_1} \partial_y^{a_2} \partial_z^{a_3} b_i)^2 \rangle + \langle \mathbf{b} \cdot \mathbf{F} \rangle$$

Since the units of each term in the equation are the same, we have the following scalings:

$$\frac{b}{\tau} \sim \frac{b^2}{\ell^n} \implies \frac{1}{\tau} \sim \frac{b}{\ell^n}$$

Now, we have that $\epsilon \sim \Pi_{\mathbf{b}}$, where $\Pi_{\mathbf{b}}$ is the flux of energy of \mathbf{b} . Thus:

$$\epsilon \sim \Pi_{\mathbf{b}} \sim \frac{b_\ell^2}{\tau_\ell} \sim \frac{b_\ell^3}{\ell^n} \implies b_\ell \sim (\ell^n \epsilon)^{1/3}$$

Thus, the energy spectrum of \mathbf{b} is:

$$E(k) \sim \frac{b_\ell^2}{1/\ell} \sim \ell^{2n/3+1} \epsilon^{2/3} = \epsilon^{2/3} k^{-(3+2n)/3}$$

- When viscosity becomes effective, we have (using exercise 1):

$$\epsilon \sim \nu_m \frac{b_{\ell_\nu}^2}{\ell_\nu^{2m+2}} \sim \nu_m \frac{\ell_\nu^{2n/3} \epsilon^{2/3}}{\ell_\nu^{2m+2}} \sim \nu_m \ell_\nu^{2n/3-2m-2} \epsilon^{2/3} \implies \ell_\nu \sim \left(\frac{\nu_m}{\epsilon^{1/3}} \right)^{1/(2+2m-2n/3)}$$

- If $\nu_m \rightarrow 0$, then we must have $\ell_\nu \rightarrow \infty$ if we do not want the viscosity to dissipate the energy. This implies that the exponent of ℓ_ν must be negative, i.e. $2 + 2m - 2n/3 < 0$, which is equivalent to $n > 3(m+1)$.