# Exercises 01

1. Consider the flow in a rectangular box of size $(L \times L \times H)$ with periodic boundary conditions along the horizontal directions $(x, y)$ and free slip boundary conditions along the vertical direction $(z)$:

$$\mathbf{u}(x + n_x L, y + n_y L, z) = \mathbf{u}(x, y, z), \qquad \text{and} \quad \partial_z u_x = \partial_z u_y = u_z = 0 \quad \text{at} \quad z = 0 \;\&\; z = H$$

   Are energy and helicity conserved for the incompressible Euler equations in this domain?

2. Consider the incompressible Navier-Stokes equations in infinite space in a rotating reference frame given by:

$$\partial_t \mathbf{u}' + \mathbf{u}' \cdot \nabla \mathbf{u}' + 2\boldsymbol{\Omega} \times \mathbf{u} = -\nabla P + \nu \nabla^2 \mathbf{u}' + \mathbf{f}$$

   where $\boldsymbol{\Omega}$ is a constant vector indicating the direction and amplitude of the rotation.
   Which of the aforementioned symmetries remain

   - Space translations: $x' = x + \ell$
   - Time translations: $t' = t + T$
   - Galilean transformations: $\mathbf{x}' = \mathbf{x} - \mathbf{c}t$, $t' = t$, $\mathbf{u}' = \mathbf{u} + \mathbf{c}$
   - Rotations $\mathbf{u}' = \mathcal{R}\mathbf{u}$, $\mathbf{x}' = \mathcal{R}^{-1}\mathbf{x}$
   - Parity (reflections): $\mathbf{u}' = -\mathbf{u}$, $\mathbf{x}' = -\mathbf{x}$
   - Scaling $\mathbf{x}' = \lambda \mathbf{x}$, $t' \to \lambda^{\alpha} t$, $\mathbf{u}' = \lambda^{\beta} \mathbf{u}$

3. Consider the equation

$$\partial_t \mathbf{a} + \mathbf{b} \times \mathbf{a} = -\nabla P' + \nu \nabla^2 \mathbf{a} \tag{1}$$

   where $\nabla \cdot \mathbf{a} = 0$.

   $\mathbf{b}$ is related to $\mathbf{a}$ as $\mathbf{b} = (\nabla \times)^n \mathbf{a}$ for some $n \in \mathbb{N}$ so that
   $(\nabla \times)^1 \mathbf{a} = \nabla \times \mathbf{a}$,
   $(\nabla \times)^2 \mathbf{a} = \nabla \times \nabla \times \mathbf{a}$,
   $(\nabla \times)^3 \mathbf{a} = \nabla \times \nabla \times \nabla \times \mathbf{a}$ and so on
   For $n = 1$ the system reduces to the Navier Stokes with $\mathbf{a} = \mathbf{u}$ and $\mathbf{b} = \mathbf{w}$.

   For $n \neq 1$:

   Are Energy $\langle \frac{1}{2} |\mathbf{a}|^2 \rangle$ & Helicity $\langle \mathbf{a} \cdot \mathbf{b} \rangle$ conserved for $\nu = 0$ for these systems (for smooth $\mathbf{a}, \mathbf{b}$)?
   What are the scaling symmetries they have for $\nu = 0$ and which one of these survives for $\nu \neq 0$?

4. Consider the map:

$$X_{n+1} = r X_n - (5 + 3r) X_n^3 + (6 + 2r) X_n^5$$

   - Construct (numerically) the bifurcation diagram for $r \in [0, 5]$.
   - Calculate the pdf of $X_n$ for r=5