Tutorial 1

Víctor Ballester Ribó

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Exercice 1. *Consider the flow in a rectangular box of size $(L \times L \times H)$ with periodic boundary conditions along the horizontal directions (x, y) and free slip boundary conditions along the vertical direction (z):*

$$u(x + nxL, y + nyL, z) = u(x, y, z), \qquad \text{and} \qquad \partial_z u_x = \partial_z u_y = u_z = 0 \quad \text{at } z = 0 \text{ and } z = H.$$

Are energy and helicity conserved for the incompressible Euler equations in this domain?

Resolution. The incompressible Euler equations are:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p \\ \boldsymbol{\nabla} \cdot \mathbf{u} = 0 \end{cases}$$

We study first the energy $\langle \frac{1}{2}|\mathbf{u}|^2 \rangle$. Note that $\frac{\mathrm{d}}{\mathrm{d}t}\langle \frac{1}{2}|\mathbf{u}|^2 \rangle = \langle \mathbf{u} \cdot \partial_t \mathbf{u} \rangle$. Then:

$$\begin{aligned}
\langle \mathbf{u} \cdot \partial_t \mathbf{u} \rangle &= -\langle \mathbf{u} \cdot (\mathbf{u} \cdot \nabla)\mathbf{u} \rangle - \langle \mathbf{u} \cdot \nabla p \rangle \\
&= -\frac{1}{2}\langle \mathbf{u} \cdot \nabla |\mathbf{u}|^2 \rangle - \langle \mathbf{u} \cdot \nabla p \rangle \\
&= -\frac{1}{2}\left[\langle \boldsymbol{\nabla} \cdot (\mathbf{u}|\mathbf{u}|^2) \rangle - \langle |\mathbf{u}|^2 \, \boldsymbol{\nabla} \cdot \mathbf{u} \rangle\right] - \left[\langle \boldsymbol{\nabla} \cdot (\mathbf{u}p) \rangle - \langle p \, \boldsymbol{\nabla} \cdot \mathbf{u} \rangle\right] \\
&= -\frac{1}{2}\langle \boldsymbol{\nabla} \cdot (\mathbf{u}|\mathbf{u}|^2) \rangle - \langle \boldsymbol{\nabla} \cdot (\mathbf{u}p) \rangle \\
&= -\frac{1}{2}\langle \partial_z (u_z|\mathbf{u}|^2) \rangle - \langle \partial_z (u_z p) \rangle
\end{aligned}$$

where we have used the incompressibility condition $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ twice in the penultimate equality and the periodic boundary conditions in x and y in the last equality. Now, note that from the hypothesis $u_z p\big|_{z=0}^{z=H} = 0$ and so the second term of the last equation vanishes. Similarly, for the first term we have $u_z|\mathbf{u}|^2\big|_{z=0}^{z=H} = 0$ and so the first term also vanishes. Therefore, $\frac{\mathrm{d}}{\mathrm{d}t}\langle \frac{1}{2}|\mathbf{u}|^2 \rangle = 0$ and so energy is conserved.
Now let's check the helicity $\langle \mathbf{u} \cdot \mathbf{w} \rangle$. Note that $\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{u} \cdot \mathbf{w} \rangle = \langle \partial_t \mathbf{u} \cdot \mathbf{w} \rangle + \langle \mathbf{u} \cdot \partial_t \mathbf{w} \rangle$. We recall the equation for the vorticity $\mathbf{w} = \nabla \times \mathbf{u}$:

$$\partial_t \mathbf{w} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{w})$$

with $\boldsymbol{\nabla} \cdot \mathbf{w} = 0$. So:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle \mathbf{u} \cdot \mathbf{w} \rangle &= \langle \partial_t \mathbf{u} \cdot \mathbf{w} \rangle + \langle \mathbf{u} \cdot \partial_t \mathbf{w} \rangle \\
&= \langle \mathbf{w} \cdot (\mathbf{u} \times \mathbf{w}) \rangle - \langle \mathbf{w} \cdot \boldsymbol{\nabla} p' \rangle + \langle \mathbf{u} \cdot \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{w}) \rangle \\
&= -\langle \boldsymbol{\nabla} \cdot (p'\mathbf{w}) \rangle + \langle p' \, \boldsymbol{\nabla} \cdot \mathbf{w} \rangle + \langle (\mathbf{u} \times \mathbf{w}) \cdot \boldsymbol{\nabla} \times \mathbf{u} \rangle \\
&= -\langle \partial_z (p' w_z) \rangle + \langle (\mathbf{u} \times \mathbf{w}) \cdot \mathbf{w} \rangle \\
&= -\langle \partial_z (p' w_z) \rangle
\end{aligned}$$

with $p' = p + \frac{1}{2}|\mathbf{u}|^2$ and the third and last equality follows from $\mathbf{w} \perp \mathbf{u} \times \mathbf{w}$. Now note at $z = 0, H$ we have:

$$w_z = \partial_x u_y - \partial_y u_x \tag{1}$$

but the quantity $p' w_z$ is not necessarily zero at $z = 0, H$. Thus, the helicity is not necessarily conserved.

Exercice 2. *Consider the incompressible Navier-Stokes equations in infinite space in a rotating reference frame given by:*

$$\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + 2\mathbf{\Omega}\times\mathbf{u} = -\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{f}$$

where Ω is a constant vector indicating the direction and amplitude of the rotation. Which of the aforementioned symmetries remain?

- *Space translations*
- *Time translations*
- *Galilean transformations*
- *Rotations*
- *Parity (reflections)*
- *Scaling*

Resolution. In what follows we assume that if we are studying a symmetry $\mathcal{S}$, then $\mathbf{f}$ is invariant under $\mathcal{S}$. Otherwise, we would not have any symmetry.

- Space translations: Since we keep the time fixed and $\boldsymbol{\nabla}' = \boldsymbol{\nabla}$ (assuming $\mathbf{x}' = \mathbf{x} + \boldsymbol{\ell}$), the equation is invariant under space translations.
- Time translations: Since we keep the space fixed and $\partial_{t'} = \partial_t$ (assuming $t' = t + T$), the equation is invariant under time translations.
- Galilean transformations: We need to check whether $\mathbf{u}'(\mathbf{x}', t') = \mathbf{u}(\mathbf{x}' - \mathbf{c}t', t') + \mathbf{c}$ is a solution provided that $\mathbf{u}(\mathbf{x}, t)$ is a solution.

  Using the chain rule, we have that: $\partial_{t'}\mathbf{u}' = \partial_t\mathbf{u} - \mathbf{c}\cdot\boldsymbol{\nabla}\mathbf{u}$ and $\boldsymbol{\nabla}' = \boldsymbol{\nabla}$. Thus:

$$\begin{aligned}
\partial_{t'}\mathbf{u}' &= \partial_t\mathbf{u} - \mathbf{c}\cdot\boldsymbol{\nabla}\mathbf{u}\\
(\mathbf{u}'\cdot\boldsymbol{\nabla}')\mathbf{u}' &= (\mathbf{u}+\mathbf{c})\cdot\boldsymbol{\nabla}\mathbf{u}\\
2\mathbf{\Omega}\times\mathbf{u}' &= 2\mathbf{\Omega}\times\mathbf{u} + 2\mathbf{\Omega}\times\mathbf{c}\\
\boldsymbol{\nabla}'p' &= \boldsymbol{\nabla}p\\
\nu\nabla'^2\mathbf{u}' &= \nu\nabla^2\mathbf{u}\\
\mathbf{f}' &= \mathbf{f}
\end{aligned}$$

  Summing the terms, we have that the equation is invariant under Galilean transformations if and only if $\mathbf{\Omega}\times\mathbf{c} = 0$, that is, $\mathbf{c}$ is parallel to $\mathbf{\Omega}$.

- Rotations: We need to check whether $\mathbf{u}'(\mathbf{x}', t') = \mathbf{R}\mathbf{u}(\mathbf{R}^{-1}\mathbf{x}', t')$ is a solution provided that $\mathbf{u}(\mathbf{x}, t)$ is a solution.

  We will use Einstein's notation to compute the nonlinear and dispersive terms. Let $v(\mathbf{x}', t') = u(\mathbf{R}^{-1}\mathbf{x}', t')$. Then:

$$\partial_j(v_i(\mathbf{x}', t')) = \partial_j(u_i(r^{-1}_{k\ell}x_\ell\mathbf{e}_k, t')) = (\partial_k u_i)|_{(\mathbf{x},t)=(\mathbf{R}^{-1}\mathbf{x}',t')}\partial_j(r^{-1}_{k\ell}x_\ell) = (\partial_k u_i)|_{(\mathbf{x},t)=(\mathbf{R}^{-1}\mathbf{x}',t')}r^{-1}_{kj}$$

  Thus, $\partial_j(v_i(\mathbf{x}', t')) = r^{-1}_{kj}(\partial_k u_i)(\mathbf{R}^{-1}\mathbf{x}', t')$. Taking another derivative:

$$\begin{aligned}
\partial_j^2(v_i(\mathbf{x}', t')) = \partial_j[r^{-1}_{kj}(\partial_k u_i)(\mathbf{R}^{-1}\mathbf{x}', t')] = r_{jk}(\partial_m\partial_k u_i)(\mathbf{R}^{-1}\mathbf{x}', t')\partial_j(r^{-1}_{m\ell}x_\ell) =&\\
= r_{jk}r^{-1}_{mj}(\partial_m\partial_k u_i)(\mathbf{R}^{-1}\mathbf{x}', t') = \delta_{km}(\partial_m\partial_k u_i)(\mathbf{R}^{-1}\mathbf{x}', t') = (\partial_k\partial_k u_i)(\mathbf{R}^{-1}\mathbf{x}', t')&
\end{aligned}$$

  Moreover since $\mathbf{D}_\mathbf{x} f = \mathbf{D}_{\mathbf{x}'} f \circ \mathbf{R}$, taking transpose we have that $\boldsymbol{\nabla}' f = \mathbf{R}\boldsymbol{\nabla} f$. Thus, using the linearity of the derivative we conclude:

$$\begin{aligned}
\partial_{t'}\mathbf{u}' &= \mathbf{R}\partial_t\mathbf{u}\\
(\mathbf{u}'\cdot\boldsymbol{\nabla}')\mathbf{u}' &= \mathbf{R}(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\\
2\mathbf{\Omega}\times\mathbf{u}' &= 2\mathbf{R}(\mathbf{\Omega}\times\mathbf{u}) - 2(\mathbf{R}\mathbf{\Omega})\times(\mathbf{R}\mathbf{u}) + 2\mathbf{\Omega}\times(\mathbf{R}\mathbf{u})\\
\boldsymbol{\nabla}'p' &= \mathbf{R}\boldsymbol{\nabla}p\\
\nu\nabla'^2\mathbf{u}' &= \nu\mathbf{R}\nabla^2\mathbf{u}\\
\mathbf{f}' &= \mathbf{R}\mathbf{f}
\end{aligned}$$

where we have used the identity $(\mathbf{R}\mathbf{a}) \times (\mathbf{R}\mathbf{b}) = \mathbf{R}(\mathbf{a} \times \mathbf{b})$ for rotations. So in order to be invariant under rotations we need $(\boldsymbol{\Omega} - \mathbf{R}\boldsymbol{\Omega}) \times (\mathbf{R}\mathbf{u}) = 0$. So if $\boldsymbol{\Omega} = \mathbf{R}\boldsymbol{\Omega}$, that is, $\boldsymbol{\Omega}$ is parallel to the rotation axis, then the equation is invariant under rotations.

- Parity: The parity relations are: $\mathbf{u}' = -\mathbf{u}$, $\mathbf{x}' = -\mathbf{x}$. Since $\boldsymbol{\nabla}' = -\boldsymbol{\nabla}$ and $\mathbf{u}' = -\mathbf{u}$ and we keep the same temporal variable, each term of the equation changes sign. Thus, the equation is invariant under parity.

- Scaling: We need to check whether $\mathbf{u}'(\mathbf{x}', t') = \lambda^\beta \mathbf{u}(\lambda \mathbf{x}', \lambda^\alpha t')$ is a solution provided that $\mathbf{u}(\mathbf{x}, t)$ is a solution. We have that:

$$
\begin{aligned}
\partial_{t'} \mathbf{u}' &= \lambda^{\beta+\alpha} \partial_t \mathbf{u} \\
(\mathbf{u}' \cdot \boldsymbol{\nabla}')\mathbf{u}' &= \lambda^{2\beta+1} (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} \\
2\boldsymbol{\Omega} \times \mathbf{u}' &= 2\lambda^\beta \boldsymbol{\Omega} \times \mathbf{u} \\
\boldsymbol{\nabla}' p' &= \lambda^{2\beta+1} \boldsymbol{\nabla} p \\
\nu \nabla'^2 \mathbf{u}' &= \lambda^{\beta+2} \nabla^2 \mathbf{u}
\end{aligned}
$$

Here we have used that $\boldsymbol{\nabla} p$ has the same scaling as $\mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u}$ even in this rotating frame. But we can clearly see that there is no way to match the scalings β and $\beta + 2$. Thus, the equation is not invariant under scaling.

Exercice 3. *Consider the equation:*

$$
\partial_t \mathbf{a} + \mathbf{b} \times \mathbf{a} = -\nabla P' + \nu \nabla^2 \mathbf{a}
$$

where $\boldsymbol{\nabla} \cdot \mathbf{a} = 0$*.* $\mathbf{b}$ *is related to* $\mathbf{a}$ *as* $\mathbf{b} = (\nabla \times)^n \mathbf{a}$ *for some* $n \in \mathbb{N}$*. For* $n = 1$ *the system reduces to the Navier-Stokes equations with* $\mathbf{a} = \mathbf{u}$ *and* $\mathbf{b} = \mathbf{w}$*. For* $n \neq 1$*, are energy* $\langle \frac{1}{2} |\mathbf{a}|^2 \rangle$ *and helicity* $\langle \mathbf{a} \cdot \mathbf{b} \rangle$ *conserved for* $\nu = 0$ *for these systems (for smooth* $\mathbf{a}$ *and* $\mathbf{b}$*)? What are the scaling symmetries they have for* $\nu = 0$ *and which one of these survives for* $\nu \neq 0$*?*

Resolution. Throughout the resolution we assume that we have periodic boundary conditions on our domain. We first assume $\nu = 0$. Note that $\frac{\mathrm{d}}{\mathrm{d}t} \langle \frac{1}{2} |\mathbf{a}|^2 \rangle = \langle \mathbf{a} \cdot \partial_t \mathbf{a} \rangle$. Then:

$$
\langle \mathbf{a} \cdot \partial_t \mathbf{a} \rangle = -\langle \mathbf{a} \cdot (\mathbf{b} \times \mathbf{a}) \rangle - \langle \mathbf{a} \cdot \nabla P' \rangle = 0
$$

The first term vanishes since $\mathbf{a} \perp \mathbf{b} \times \mathbf{a}$ and the second term vanishes since $\mathbf{a}$ is divergence-free and the fact that we have periodic boundary conditions. Thus, energy is conserved. For the helicity we first need to find the PDE that $\mathbf{b}$ satisfies. Taking $(\boldsymbol{\nabla} \times)^n$ to the initial equation we have:

$$
\partial_t \mathbf{b} + (\boldsymbol{\nabla} \times)^n (\mathbf{b} \times \mathbf{a}) = 0
$$

with $\boldsymbol{\nabla} \cdot \mathbf{b} = 0$. Thus:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \langle \mathbf{a} \cdot \mathbf{b} \rangle &= \langle \partial_t \mathbf{a} \cdot \mathbf{b} \rangle + \langle \mathbf{a} \cdot \partial_t \mathbf{b} \rangle \\
&= -\langle (\mathbf{b} \times \mathbf{a}) \cdot \mathbf{b} \rangle - \langle \boldsymbol{\nabla} P' \cdot \mathbf{b} \rangle - \langle \mathbf{a} \cdot (\boldsymbol{\nabla} \times)^n (\mathbf{b} \times \mathbf{a}) \rangle \\
&= -\langle ((\boldsymbol{\nabla} \times)^n \mathbf{a}) \cdot (\mathbf{b} \times \mathbf{a}) \rangle \\
&= -\langle \mathbf{b} \cdot (\mathbf{b} \times \mathbf{a}) \rangle \\
&= 0
\end{aligned}
$$

where in the third equality we have used the fact that $\mathbf{b} \perp \mathbf{b} \times \mathbf{a}$, $\boldsymbol{\nabla} \cdot \mathbf{b} = 0$ and the periodic boundary conditions. Thus, helicity is conserved.
We now study the scaling symmetries. We need to check whether $\mathbf{a}'(\mathbf{x}', t') = \lambda^\beta \mathbf{a}(\lambda \mathbf{x}', \lambda^\alpha t')$ is a solution provided that $\mathbf{a}(\mathbf{x}, t)$ is a solution. We have that $\mathbf{b}' = (\boldsymbol{\nabla}' \times)^n \mathbf{a}' = \lambda^{n+\beta} \mathbf{b}$. Thus:

$$
\begin{aligned}
\partial_{t'} \mathbf{a}' &= \lambda^{\beta+\alpha} \partial_t \mathbf{a} \\
\mathbf{b}' \times \mathbf{a}' &= \lambda^{n+2\beta} \mathbf{b} \times \mathbf{a} \\
\boldsymbol{\nabla}' P' &= \lambda^{n+2\beta} \boldsymbol{\nabla} P' \\
\nu \nabla'^2 \mathbf{a}' &= \lambda^{\beta+2} \nu \nabla^2 \mathbf{a}
\end{aligned}
$$

If $\nu = 0$, from $\beta + \alpha = n + 2\beta$ we have the family of invariant scaling $\alpha = n + \beta$. If $\nu \neq 0$, then for each n we only have one scaling: $\beta = 2 - n$ and $\alpha = 2$.