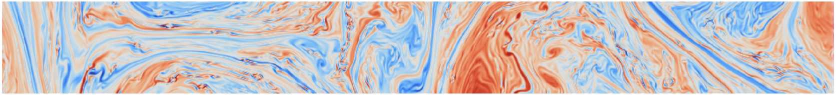
Turbulence
Kolmogorov Phenomenology



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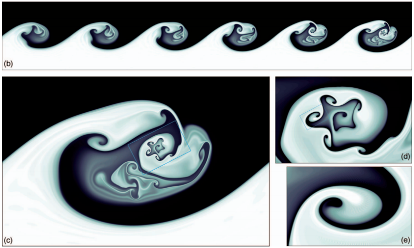
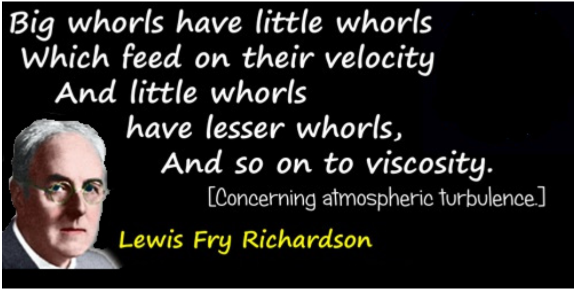
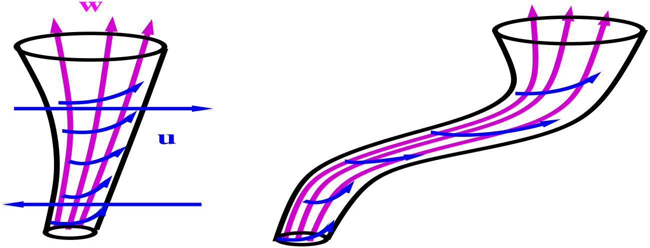
Richardson's Poem





Vortex stretching

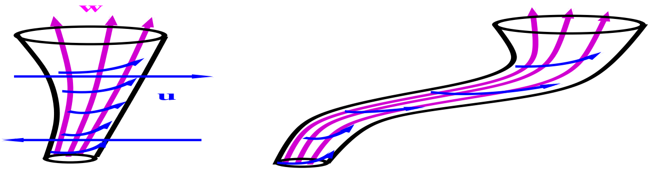
Consider an eddy of velocity u_ℓ and lengthscale ℓ being sheared by an eddy of velocity U_L lengthscale L



The rate energy moves to smaller scales is:

$$\frac{dE_\ell}{dt} \propto \frac{U_L}{L} u_\ell^2$$

Vortex stretching



The rate energy moves to smaller scales is:

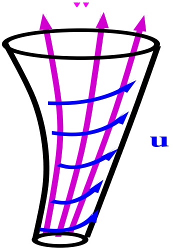
$$\frac{dE_\ell}{dt} \propto \frac{U_L}{L} u_\ell^2$$

Assuming

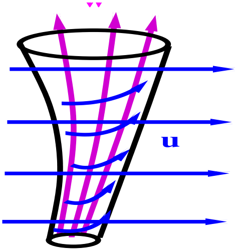
1. the flux of energy across scales is constant and equal to ϵ
2. the most effective interactions are among similar size eddies

$$\epsilon \propto \frac{u_\ell^3}{\ell} \quad \text{or} \quad u_\ell \propto \epsilon^{1/3} \ell^{1/3}$$

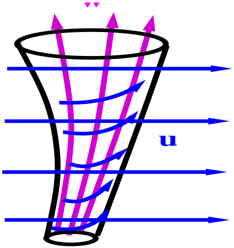
Vortex stretching

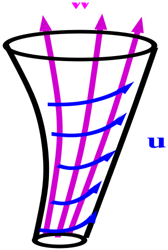


Vortex stretching



Vortex stretching





Vortex stretching

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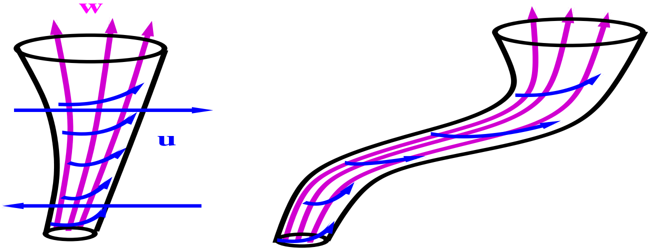
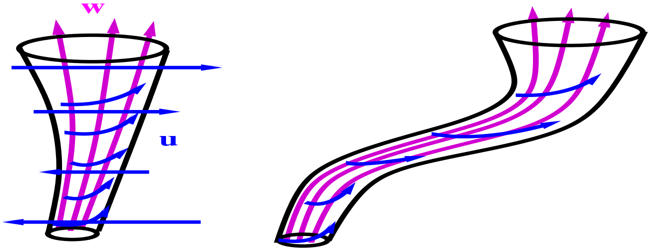


Vortex stretching



Energy in Scale space

$$\mathbf{u}(\mathbf{x},t) = \sum \tilde{\mathbf{u}}_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad \tilde{\mathbf{u}}_{\mathbf{k}}(t) = \left\langle \mathbf{u} e^{-i\mathbf{k}\cdot\mathbf{x}} \right\rangle$$

Energy Spectrum

$$E(k) = \frac{1}{2\delta k} \sum_{k \leq |\mathbf{k}| < k+\delta k} |\tilde{\mathbf{u}}_{\mathbf{k}}|^2 \sim \frac{\text{Energy}}{\text{per unit wavenumber}}$$

Energy at scale $\ell \sim 1/k$

$$u_\ell^2 \propto 2 \sum_{k \sim 1/\ell} E(k)\delta k \propto kE(k)$$

where the sum is over wavenumbers of magnitude $\sim 1/\ell$

$$\text{eg} 1/2\ell \leq k \leq 2/\ell$$

Energy Flux Π

$\mathcal{I}$ the rate energy is injected at large scales L
ϵ the rate energy is dissipated at small scales ℓ_ν
Π_ℓ the rate energy moves from scales of size ℓ to smaller scales.

$$\begin{array}{ccccc} \mathcal{I} & = & \Pi_\ell & = & \epsilon \\ \text{Energy injection} & = & \text{energy flux} & = & \text{Energy dissipation} \end{array}$$

$$\text{Flux estimate :} \qquad \Pi \propto \frac{u_\ell^2}{\tau_\ell} = \frac{u_\ell^3}{\ell}$$

$$\boxed{u_\ell = (\epsilon\ell)^{1/3}}$$

Fourier Space

$$\mathbf{u}(\mathbf{x},t) = \sum \tilde{\mathbf{u}}_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad \tilde{\mathbf{u}}_{\mathbf{k}}(t) = \left\langle \mathbf{u} e^{-i\mathbf{k}\cdot\mathbf{x}} \right\rangle$$

Energy Spectrum:

$$E(k) = \frac{1}{2\delta k} \sum_{k \le |\mathbf{k}| < k+\delta k} |\tilde{\mathbf{u}}_{\mathbf{k}}|^2 \sim \frac{\text{Energy}}{\text{per unit wavenumber}}$$

$$k \propto 1/\ell, \qquad E(k)k \propto u_\ell^2$$

$$\boxed{E(k) \propto \epsilon^{2/3} k^{-5/3}}$$

Kolmogorov's Spectrum!

Kolmogorov Spectrum and Kolmogorov scale

$$E(k) \propto \frac{u_\ell^2}{k} = \frac{(\epsilon\ell)^{2/3}}{1/\ell} = \frac{(\epsilon/k)^{2/3}}{k}$$

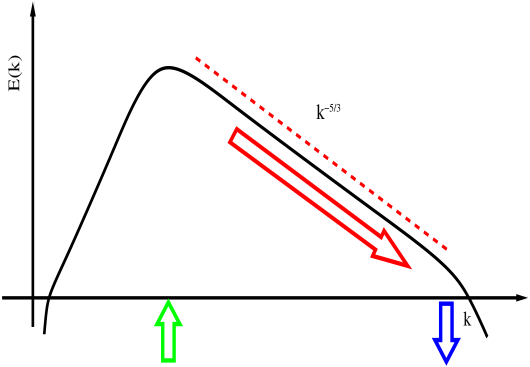
$$\boxed{E(k) \propto \epsilon^{2/3} k^{-5/3}}$$

Viscosity will become important when

$$\epsilon \propto \nu \frac{u_\nu^2}{\ell_\nu^2} \propto \nu \frac{(\epsilon^{1/3}\ell_\nu^{1/3})^2}{\ell_\nu^2} \propto \nu \frac{\epsilon^{2/3}}{\ell_\nu^{4/3}}$$

$$\boxed{\ell_\nu = \frac{\nu^{3/4}}{\epsilon^{1/4}}} \quad \text{or} \quad \boxed{k_\nu = \frac{\epsilon^{1/4}}{\nu^{3/4}}}$$

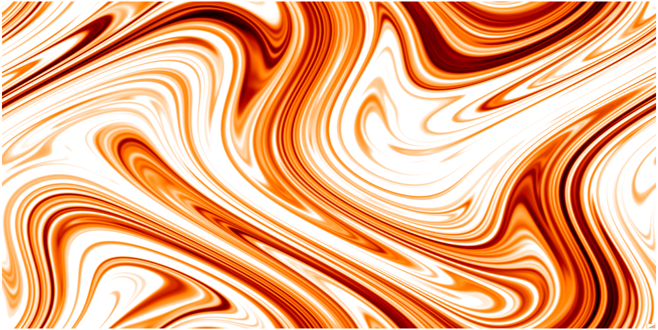
Kolmogorov Spectrum and Kolmogorov scale



$$\boxed{E(k) \propto \epsilon^{2/3} k^{-5/3}}$$

$$\boxed{k_\nu = \frac{\epsilon^{1/4}}{\nu^{3/4}}}$$

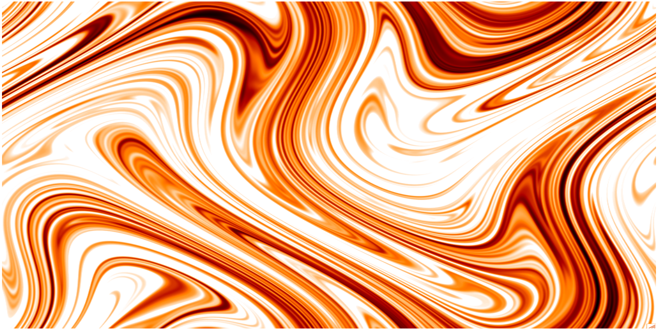
Passive scalar



$$\partial_t \phi + \mathbf{u} \nabla \phi = \kappa \phi + S(\mathbf{x}, t)$$

$\mathbf{u}$ is a divergence free ($\nabla \cdot \mathbf{u} = 0$) prescribed velocity field and S prescribed scalar source.

Passive scalar



$$\partial_t \phi + \mathbf{u} \cdot \nabla \phi = \kappa \phi + S(\mathbf{x}, t)$$

$$\frac{1}{2} \frac{d}{dt} \phi = -\kappa \langle \phi^2 \rangle + \langle \phi S \rangle$$

A) $\mathbf{u}$ a large scale (chaotic) flow

$$\partial_t \phi + \mathbf{u} \cdot \nabla \phi = \kappa \phi + S(\mathbf{x}, t)$$

Π_ℓ the rate variance $\langle \phi^2 \rangle$ moves to scales of smaller than ℓ.

$\mathcal{I}_\phi$	$=$	$\Pi_{\phi,\ell}$	$=$	ϵ_ϕ
Variance injection	$=$	variance flux	$=$	Variance dissipation

$$\text{Flux estimate :} \qquad \Pi \propto \frac{\phi_\ell^2}{\tau_L} = \frac{u_L \phi_\ell^2}{L}$$

$$\phi_\ell \propto (\epsilon_\phi L / u_L)^{1/2}$$

$$\boxed{E_\phi(k) \propto \frac{\phi_\ell^2}{1/\ell} \propto \frac{\epsilon_\phi L}{u_L} k^{-1}} \qquad \boxed{\ell_\kappa \propto \sqrt{\frac{\kappa L}{u_L}}}$$

B) $\mathbf{u}$ power law behavior $u_\ell \propto A\ell^\alpha$

$$\mathcal{I}_\phi \quad = \quad \Pi_{\phi,\ell} \quad = \quad \epsilon_\phi$$

Variance injection = variance flux = Variance dissipation

$$\text{Flux estimate :} \qquad \Pi \propto \frac{\phi_\ell^2}{\tau_\ell} = \frac{u_\ell \phi_\ell^2}{\ell} = A\ell^{\alpha-1}\phi_\ell^2$$

$$\phi_\ell \propto (\epsilon_\phi \ell^{1-\alpha}/A)^{1/2}$$

$$\boxed{E_\phi(k) \propto \frac{\phi_\ell^2}{1/\ell} \propto \frac{\epsilon_\phi \ell^{2-\alpha}}{A} \propto \frac{\epsilon_\phi}{A} k^{\alpha-2}} \qquad \boxed{\ell_\kappa \propto \left(\frac{\kappa}{A}\right)^{\frac{1}{3-\alpha}}}$$

C) Kolmogorov Turbulence $u_\ell \propto A\ell^{1/3}$, $L < \ell < \ell_\nu$

$\mathcal{I}_\phi$	$=$	$\Pi_{\phi,\ell}$	$=$	ϵ_ϕ
Variance injection	$=$	variance flux	$=$	Variance dissipation

$$\text{Flux estimate :} \qquad \Pi \propto \frac{\phi_\ell^2}{\tau_\ell} = \frac{u_\ell \phi_\ell^2}{\ell} = A\ell^{\alpha-1}\phi_\ell^2$$

$$\phi_\ell \propto (\epsilon_\phi \ell^{1-\alpha}/A)^{1/2}$$

$$\boxed{E_\phi(k) \propto \frac{\phi_\ell^2}{1/\ell} \propto \frac{\epsilon_\phi \ell^{2-\alpha}}{A} \propto \frac{\epsilon_\phi}{A} k^{\alpha-2}} \qquad \boxed{\ell_\kappa \propto \left(\frac{\kappa}{A}\right)^{\frac{1}{3-\alpha}}}$$