M2 Soft Matter/Bio
Homework Turbulence 2
Some properties of Burgers equation

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1 Introduction

In this homework, we will explore some properties of Burgers equation.

$$\begin{aligned} \partial_t V_i + V_j \partial_j V_i &= \nu \Delta V_i, \\ V_i &= -\nabla_i \Psi, \end{aligned} \tag{1}$$

a) Use the second equation of Eq. (1) to find the equation for Ψ.

b) 1D case: Check that the "Khokhlov" velocity field $u^\nu(x,t) = (x - L\tanh(Lx/2\nu t))/t$ is a solution of Burgers equation Eq. (1). Draw it at several time for $L = 1$, and $\nu = 1$, $\nu = 1e-2$ and $\nu = 1e-6$.

c) Find the limit of the Khokhlov solution when $\nu \to 0$. This solution represents a shock.

2 Link with the heat equation

The Hopf-Cole transformation is defined as:

$$V_i = -2\nu \nabla_i \log(\Phi). \tag{2}$$

a) Find the link between Ψ and Φ;

b) Show that the equation for Φ is linear. It is called the heat equation, that has the interesting property of having simple solutions.

c) Consider the 1D Case. Find a solution of the heat equation in case of periodic boundary conditions. (Trick: use Fourier transform).

3 Burgers and the large scale of the universe

At very large scale, the Universe is described by Newton equations in a flat, expanding geometry. The equations are:

$$\partial_t u_i + \frac{\dot{a}}{a} u_i + \frac{1}{a} u_j \partial_j u_i = -\frac{1}{a} \partial_i \Phi,$$

$$
\begin{aligned}
\partial_t \rho + 3\frac{\dot a}{a}\rho + \frac{1}{a}\partial_j \rho u_j &= 0, \\
\Delta \Phi = 4\pi G a^2 \left(\rho - \rho_b\right), & \qquad (3)
\end{aligned}
$$

where $a(t)$ is the expansion factor, Φ is the gravitational potential, ρ is the density and u is the velocity of the gaz.

Q: Show that -these equations can be mapped into inviscid Burgers equation ($\nu = 0$) by using Zeldovich transfromation:

$$
\begin{aligned}
V &= \frac{u}{a\dot b} = -\nabla \tilde\Psi, \\
\left(\partial_t + 2\frac{\dot a}{a}\right)\partial_t b &= 4\pi G \rho_b(t) b \\
\tilde\Phi &= \frac{\Phi}{4\pi G \rho_b a^2 b}, \\
\tilde\Phi &= \tilde\Psi. \qquad (4)
\end{aligned}
$$

4 Singularities of Burgers

Burgers equation develop finite time singularities. Let us study this in the 1D case.

a) Use Eq. (1) to write an equation for $A = \partial_x u$.

b) Introduce the Lagrangian derivative $D_t A = \partial_t A + u\partial_x A$. Use (a) to find the ordinary differential equation that links A and its Lagrangian derivative.

c) Integrate this equation in the case $\nu = 0$, and discuss in which condition there is a finite time blow up of A.

d) Use this discussion to explain the features of the Khokhlov solution at $\nu \to 0$ (presence of positive ramps and no negative ramps).

e) BONUS question: Can this method be used to study potential blow-up in Euler equation?