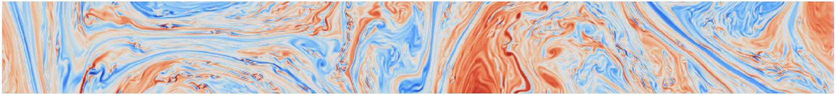
Turbulence:
Scale by Scale energy balance



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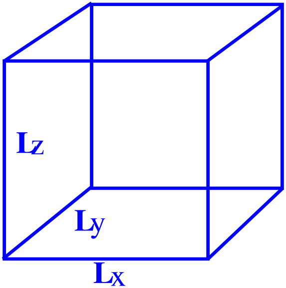
Key points from last week

- **Finite dissipation at the $\nu \to 0$ limit**
- **Transfer of energy from large scales to small scales**
- **Vorticity stretching**
- **Power-law behavior of structure function** $\left\langle |\delta \mathbf{u}(r)|^2 \right\rangle \propto r^{2/3}$
- **Power-law behavior of Energy spectra** $E(k) \propto k^{-5/3}$

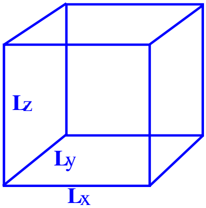
Navier - Stokes



$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = -\nabla P + \nu \nabla^2 \mathbf{u} + \mathbf{f}$$

$$\nabla \cdot \mathbf{u} = 0$$

Navier-Stokes in Fourier Space



$$\mathbf{u}(\mathbf{x},t) = \sum \tilde{\mathbf{u}}_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad \tilde{\mathbf{u}}_{\mathbf{k}}(t) = \left\langle \mathbf{u} e^{-i\mathbf{k}\cdot\mathbf{x}} \right\rangle$$

$$\partial_t \tilde{\mathbf{u}}_{\mathbf{k}} = -\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} i \left((\mathbf{p}\cdot\tilde{\mathbf{u}}_{\mathbf{q}})\tilde{\mathbf{u}}_{\mathbf{p}} - \mathbf{k}\frac{(\mathbf{k}\cdot\tilde{\mathbf{u}}_{\mathbf{p}})(\mathbf{p}\cdot\tilde{\mathbf{u}}_{\mathbf{q}})}{|\mathbf{k}|^2} \right) - \nu|\mathbf{k}|^2\tilde{\mathbf{u}}_{\mathbf{k}} + \tilde{\mathbf{f}}_{\mathbf{k}}$$

Defining the notion of scale $\ell = k^{-1}$

The filtering operators $\mathcal{P}_k^<$ and $\mathcal{P}_k^>$

Let

$$\mathbf{u}_k^<(\mathbf{x}) = \mathcal{P}_k^<[\mathbf{u}(\mathbf{x})] \qquad \text{and} \qquad \mathbf{u}_k^>(\mathbf{x}) = \mathcal{P}_k^>[\mathbf{u}(\mathbf{x})]$$

where

$$\mathcal{P}_k^<[\mathbf{u}(\mathbf{x})] = \sum_{|\mathbf{k}|\leq k} \tilde{\mathbf{u}}_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}}$$

and

$$\mathcal{P}_k^>[\mathbf{u}(\mathbf{x})] = \sum_{|\mathbf{k}|> k} \tilde{\mathbf{u}}_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}}$$

The field $\mathbf{u}_k^<$ contains scales larger than $\ell = k^{-1}$
The field $\mathbf{u}_k^>$ contains scales smaller than $\ell = k^{-1}$

$$\mathbf{u} = \mathbf{u}_k^< + \mathbf{u}_k^>$$

Defining the notion of scale $\ell = k^{-1}$

$$\mathcal{P}_k^<[\mathbf{u}(\mathbf{x})] = \sum_{|\mathbf{k}|\le k} \tilde{\mathbf{u}}_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad \mathcal{P}_k^>[\mathbf{u}(\mathbf{x})] = \sum_{|\mathbf{k}|> k} \tilde{\mathbf{u}}_{\mathbf{k}}(t)e^{i\mathbf{k}\cdot\mathbf{x}}$$

Properties:

- $\mathcal{P}_k^< \left[\mathcal{P}_k^<[\mathbf{u}]\right] = \mathcal{P}_k^<[\mathbf{u}(\mathbf{x})]$, (ie it is a projector)
- $\mathcal{P}_k^> \left[\mathcal{P}_k^>[\mathbf{u}]\right] = \mathcal{P}_k^>[\mathbf{u}(\mathbf{x})]$, (ie it is a projector)
- $\mathcal{P}_k^< \left[\mathcal{P}_k^>[\mathbf{u}]\right] = 0$ $\mathbf{u}_k^>$ and $\mathbf{u}_k^<$ are orthogonal
- $\mathcal{P}_k^<[\nabla\mathbf{u}] = \nabla\mathcal{P}_k^<[\mathbf{u}(\mathbf{x})]$, it commutes with derivatives
- $\left\langle \mathbf{v}\mathcal{P}_k^<[\mathbf{u}]\right\rangle = \left\langle \mathcal{P}_k^<[\mathbf{v}]\mathbf{u}\right\rangle = \left\langle \mathcal{P}_k^<[\mathbf{v}]\mathcal{P}_k^<[\mathbf{u}]\right\rangle$
- $\left\langle \mathcal{P}_k^<[\mathbf{v}]\mathcal{P}_k^>[\mathbf{u}]\right\rangle = 0$

Scale by scale energy balance

$$\partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla P + \nu\nabla^2\mathbf{u} + \mathbf{f}$$

$$\langle \mathbf{u}\cdot\partial_t\mathbf{u}\rangle + \langle \mathbf{u}\cdot(\mathbf{u}\cdot\nabla\mathbf{u})\rangle = -\langle \mathbf{u}\cdot\nabla P\rangle + \nu\big\langle \mathbf{u}\cdot\nabla^2\mathbf{u}\big\rangle + \langle \mathbf{u}\cdot\mathbf{f}\rangle$$

$$\langle \mathbf{u}\cdot\partial_t\mathbf{u}\rangle + \frac{1}{2}\big\langle \mathbf{u}\cdot\nabla|\mathbf{u}|^2\big\rangle = -\langle \mathbf{u}\cdot\nabla P\rangle + \nu\big\langle \mathbf{u}\cdot\nabla^2\mathbf{u}\big\rangle + \langle \mathbf{u}\cdot\mathbf{f}\rangle$$

$$\frac{1}{2}\frac{d}{dt}\big\langle|\mathbf{u}|^2\big\rangle = -\nu\big\langle|\nabla\mathbf{u}|^2\big\rangle + \langle \mathbf{u}\cdot\mathbf{f}\rangle$$

$$\boxed{\frac{d}{dt}\mathcal{E} = -\epsilon + \mathcal{I}}$$

Scale by scale energy balance

For the filtered field $\mathbf{u}_k^<$

$$\langle \mathbf{u}_k^< \cdot \partial_t \mathbf{u} \rangle + \langle \mathbf{u}_k^< \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) \rangle = -\langle \mathbf{u}_k^< \cdot \nabla P \rangle + \nu \langle \mathbf{u}_k^< \cdot \nabla^2 \mathbf{u} \rangle + \langle \mathbf{u}_k^< \cdot \mathbf{f} \rangle$$

$$\frac{1}{2}\frac{d}{dt}\langle |\mathbf{u}_k^<|^2 \rangle + \langle \mathbf{u}_k^< \cdot (\mathbf{u} \cdot \nabla \mathbf{u}) \rangle = +\nu \langle |\nabla \mathbf{u}_k^>|^2 \rangle + \langle \mathbf{u}_k^< \cdot \mathbf{f} \rangle$$

$$\frac{1}{2}\frac{d}{dt}\sum_{|\mathbf{k}|\le k} |\tilde{\mathbf{u}}_\mathbf{k}|^2 + i \sum_{|\mathbf{k}|\le k} \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} (\tilde{\mathbf{u}}_\mathbf{k}^* \cdot [(\mathbf{p} \cdot \tilde{\mathbf{u}}_\mathbf{q})\tilde{\mathbf{u}}_\mathbf{p}]) =$$

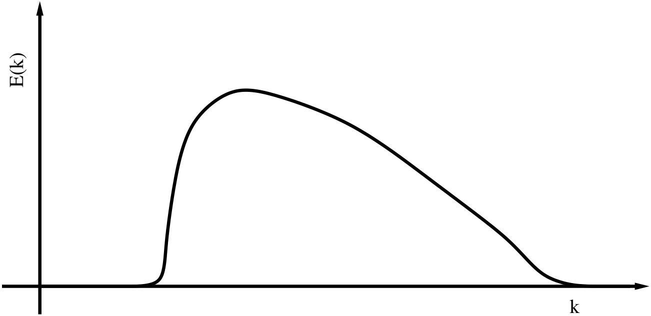
$$- \sum_{|\mathbf{k}|\le k} \nu |\mathbf{k}|^2 |\tilde{\mathbf{u}}_\mathbf{k}|^2 + \sum_{|\mathbf{k}|\le k} \tilde{\mathbf{u}}_\mathbf{k} \tilde{\mathbf{f}}_\mathbf{k}$$

$$\boxed{\frac{d}{dt}\mathcal{E}_k^< + \Pi_k = -\epsilon_k^< + \mathcal{I}_k^<}$$

Scale by scale energy balance

$$\boxed{\frac{d}{dt}\mathcal{E}_k^< + \Pi_k = -\epsilon_k^< + \mathcal{I}_k^<}$$

$$\mathcal{E}_k^< = \sum_0^k E(q)$$



Scale by scale energy balance

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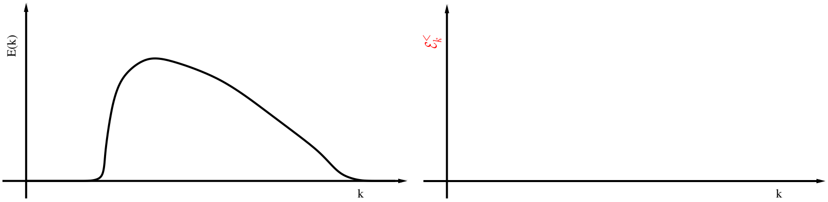
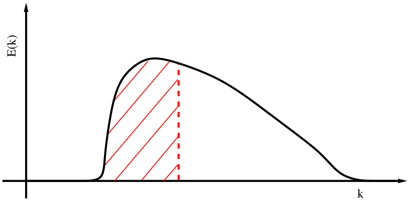


Scale by scale energy balance

$$\boxed{\frac{d}{dt}\mathcal{E}_k^< + \Pi_k = -\epsilon_k^< + \mathcal{I}_k^<}$$

$$\mathcal{E}_k^< = \sum_0^k E(q)$$



Scale by scale energy balance

$$\boxed{\frac{d}{dt}\mathcal{E}_k^< + \Pi_k = -\epsilon_k^< + \mathcal{I}_k^<}$$

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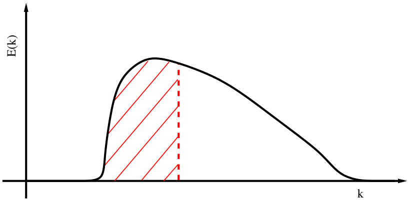
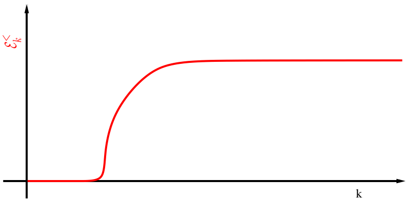




Scale by scale energy balance

$$\boxed{\frac{d}{dt}\mathcal{E}_k^< + \Pi_k = -\epsilon_k^< + \mathcal{I}_k^<}$$

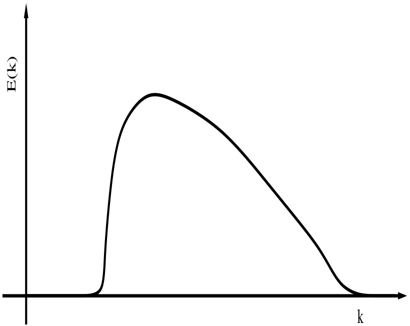
if we time average

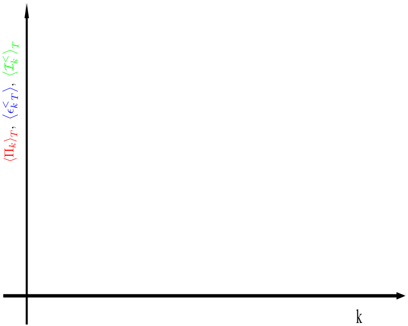
$$\begin{aligned}
\left\langle \frac{d}{dt}\mathcal{E}_k^< \right\rangle_T &= \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{d}{dt}\mathcal{E}_k^< dt \\
&= \lim_{T\to\infty} \frac{\mathcal{E}_k^<(T) - \mathcal{E}_k^<(0)}{T} \\
&= 0
\end{aligned}$$

$$\boxed{\langle \Pi_k \rangle_T = -\left\langle \epsilon_k^< \right\rangle_T + \left\langle \mathcal{I}_k^< \right\rangle}_T$$

Scale by scale energy balance

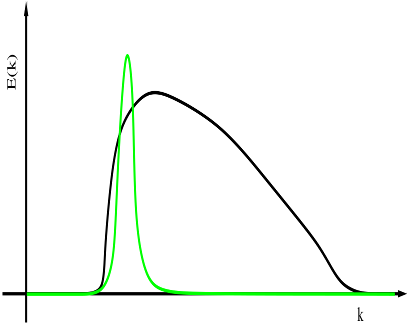
$$\boxed{\langle \Pi_k \rangle_T = -\left\langle {\epsilon^{<}_k}_T \right\rangle + \left\langle \mathcal{I}^{<}_k \right\rangle_T}$$

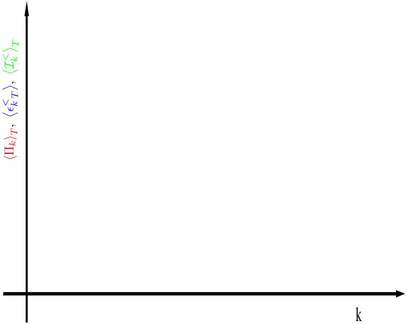




Scale by scale energy balance

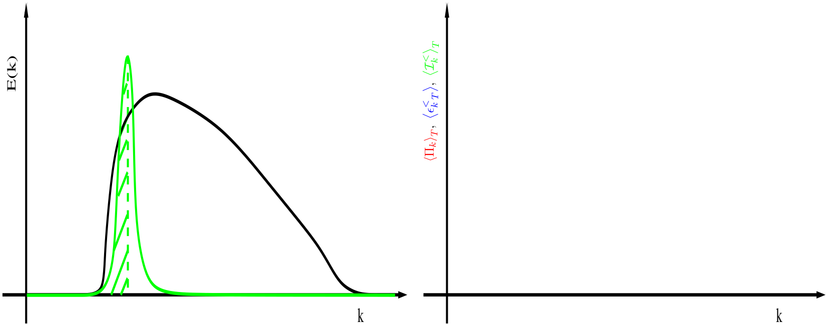
$$\langle \mathcal{I}^{<}_{\mathbf{k}} \rangle_{\mathbf{T}} = \left\langle \sum_{|\mathbf{k}|\leq \mathbf{k}} \tilde{\mathbf{u}}_{\mathbf{k}} \tilde{\mathbf{f}}_{\mathbf{k}} \right\rangle_{\mathbf{T}}$$





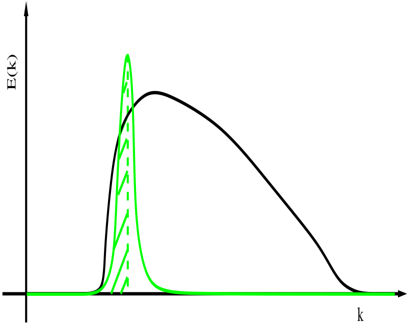
Scale by scale energy balance

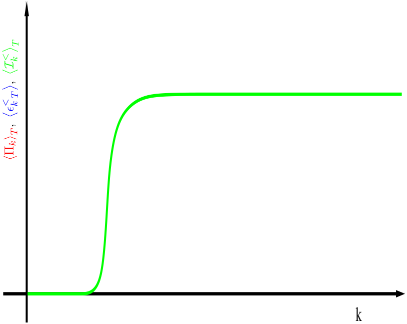
$$\langle \mathcal{I}^{<}_{\mathbf{k}} \rangle_{\mathbf{T}} = \left\langle \sum_{|\mathbf{k}| \leq \mathbf{k}} \tilde{\mathbf{u}}_{\mathbf{k}} \tilde{\mathbf{f}}_{\mathbf{k}} \right\rangle_{\mathbf{T}}$$



Scale by scale energy balance

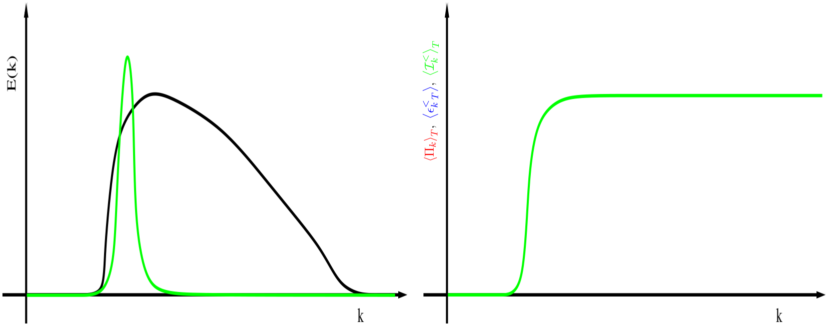
$$\langle \mathcal{I}_{\mathbf{k}}^{<} \rangle_{\mathbf{T}} = \left\langle \sum_{|\mathbf{k}| \leq \mathbf{k}} \tilde{\mathbf{u}}_{\mathbf{k}} \tilde{\mathbf{f}}_{\mathbf{k}} \right\rangle_{\mathbf{T}}$$





Scale by scale energy balance

$$\langle \epsilon_{\mathbf{k}}^{<} \rangle_{\mathbf{T}} = \nu \left\langle \sum_{|\mathbf{k}| \leq \mathbf{k}} |\mathbf{k}|^{\mathbf{2}} |\tilde{\mathbf{u}}_{\mathbf{k}}|^{\mathbf{2}} \right\rangle_{\mathbf{T}}$$



Large scale filtered Dissipation

$$\langle \epsilon_k^< \rangle_T = \nu \sum_{q=0}^{k} q^2 E(q) \simeq \nu \int_0^k q^2 E(q) dq$$

if $E(k) \propto k^{-\alpha}$ for k such that $k_f < k < k_\nu$ then

$$\begin{array}{rcll}
\langle \epsilon_k^< \rangle_T \propto & \nu \int_0^k q^{2-\alpha} dq & & \\
= & \frac{\nu}{3-\alpha}[k^{3-\alpha} - k_f^{3-\alpha}] & \text{if} & \alpha \neq 3 \\
\propto & \nu k^{3-\alpha}, & \text{if} & \alpha < 3 \\
\propto & \nu \log(k), & \text{if} & \alpha = 3 \\
\propto & \epsilon & \text{if} & \alpha > 3
\end{array}$$

Large scale filtered Dissipation

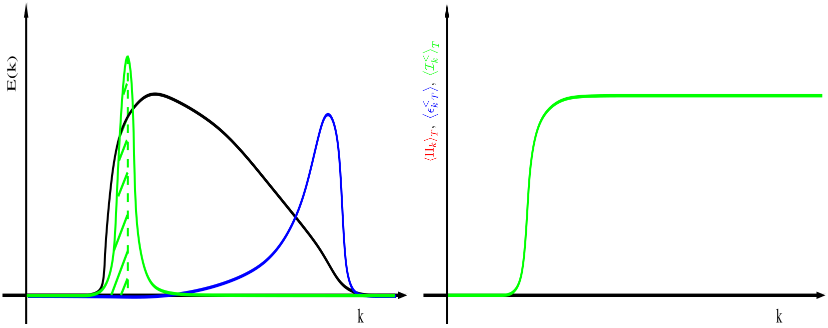
$$\lim_{\nu\to 0}\lim_{k\to\infty}\left\langle \epsilon_k^< \right\rangle_T = \lim_{\nu\to 0}\left(\lim_{k\to\infty}\left\langle \epsilon_k^< \right\rangle_T\right) = \lim_{\nu\to 0}\epsilon = \epsilon$$

$$\begin{aligned}
\lim_{k\to\infty}\lim_{\nu\to 0}\left\langle \epsilon_k^< \right\rangle_T &= \lim_{k\to\infty}\left(\lim_{\nu\to 0}\nu\sum_{q=0}^{k} q^2 E(q)\right) \\
&\le \lim_{k\to\infty}\left(\lim_{\nu\to 0}\nu k^2\sum_{q=0}^{k} E(q)\right) \\
&\le \lim_{k\to\infty}\left(\lim_{\nu\to 0}\nu k^2\mathcal{E}\right) \\
&= \lim_{k\to\infty} 0 \qquad\qquad \text{(if } \mathcal{E} \text{ remains finite)} \\
&= 0
\end{aligned}$$

$$\lim_{\nu\to 0}\lim_{k\to\infty}\left\langle \epsilon_k^< \right\rangle_T \neq \lim_{k\to\infty}\lim_{\nu\to 0}\left\langle \epsilon_k^< \right\rangle_T$$

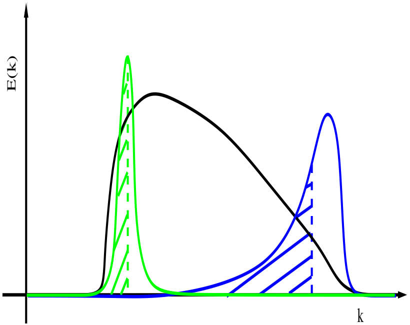
Scale by scale energy balance

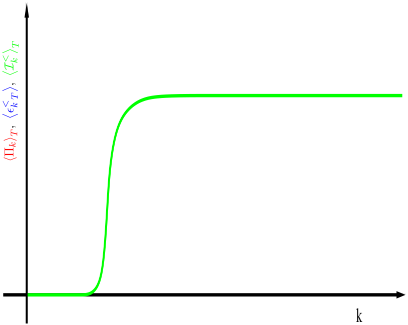
$$\langle \epsilon_{\mathbf{k}}^{<} \rangle_{\mathbf{T}} = \nu \left\langle \sum_{|\mathbf{k}| \leq \mathbf{k}} |\mathbf{k}|^{\mathbf{2}} |\tilde{\mathbf{u}}_{\mathbf{k}}|^{\mathbf{2}} \right\rangle_{\mathbf{T}}$$



Scale by scale energy balance

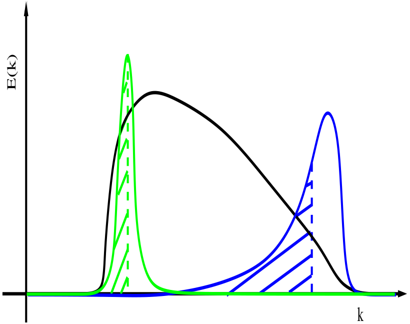
$$\langle \epsilon_{\mathbf{k}}^{<} \rangle_{\mathbf{T}} = \nu \left\langle \sum_{|\mathbf{k}| \leq \mathbf{k}} |\mathbf{k}|^{\mathbf{2}} |\tilde{\mathbf{u}}_{\mathbf{k}}|^{\mathbf{2}} \right\rangle_{\mathbf{T}}$$

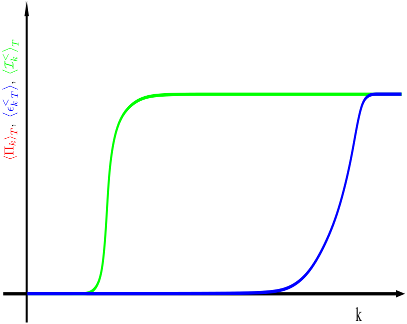




Scale by scale energy balance

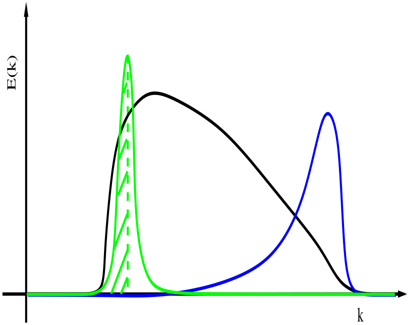
$$\langle \epsilon_{\mathbf{k}}^{<} \rangle_{\mathbf{T}} = \nu \left\langle \sum_{|\mathbf{k}| \leq \mathbf{k}} |\mathbf{k}|^2 |\tilde{\mathbf{u}}_{\mathbf{k}}|^2 \right\rangle_{\mathbf{T}}$$

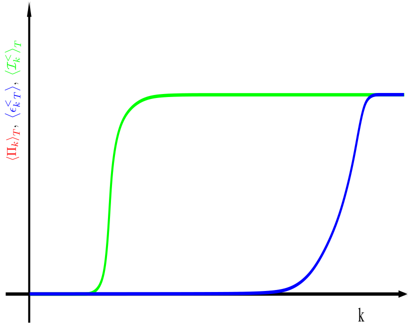




Scale by scale energy balance

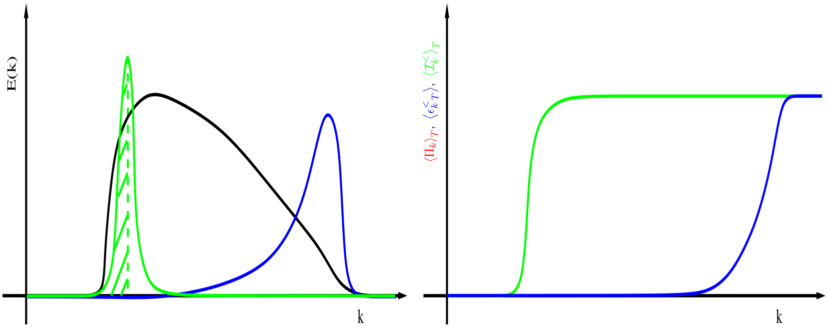
$$\langle \mathbf{\Pi_k} \rangle_\mathbf{T} = \left\langle \mathbf{i} \sum_{|\mathbf{k}| \leq \mathbf{k}} \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \left(\tilde{\mathbf{u}}^*_\mathbf{k} \cdot [(\mathbf{p} \cdot \tilde{\mathbf{u}}_\mathbf{q}) \tilde{\mathbf{u}}_\mathbf{p}] \right) \right\rangle$$





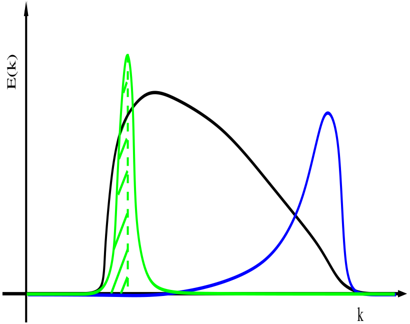
Scale by scale energy balance

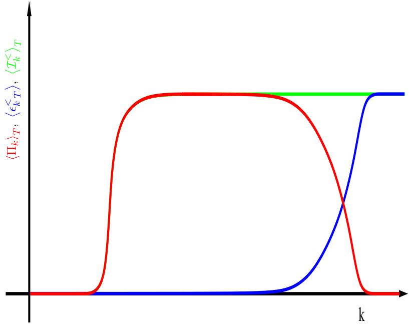
$$\langle \boldsymbol{\Pi_k} \rangle_{\mathbf{T}} = -\langle \epsilon^{<}_{\mathbf{k\,T}} \rangle + \langle \mathcal{I}^{<}_{\mathbf{k}} \rangle_{\mathbf{T}}$$



Scale by scale energy balance

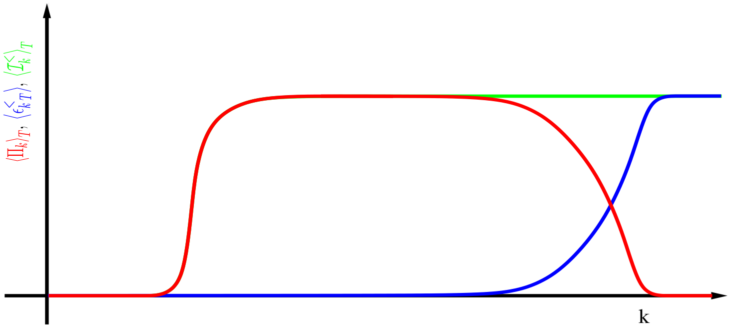
$$\color{red}{\langle \boldsymbol{\Pi_k} \rangle_{\mathbf{T}}} = -\color{blue}{\langle \epsilon^{<}_{\mathbf{k}\,\mathbf{T}} \rangle} + \color{green}{\langle \mathcal{I}^{<}_{\mathbf{k}} \rangle_{\mathbf{T}}}$$





Scale by scale energy balance

$$\langle \mathbf{\Pi_k} \rangle_{\mathbf{T}} = -\left\langle \epsilon^{<}_{\mathbf{k}\,\mathbf{T}} \right\rangle + \left\langle \mathcal{I}^{<}_{\mathbf{k}} \right\rangle_{\mathbf{T}}$$



Scale by scale energy balance

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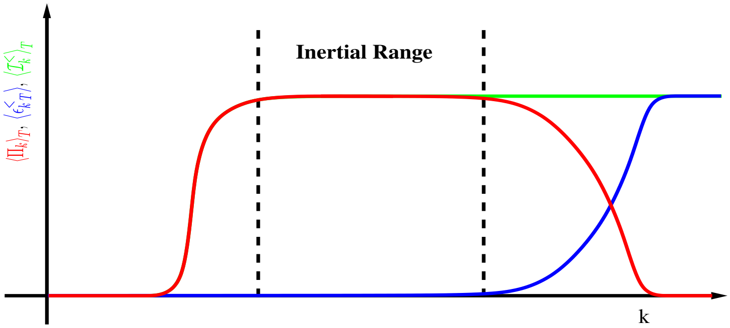
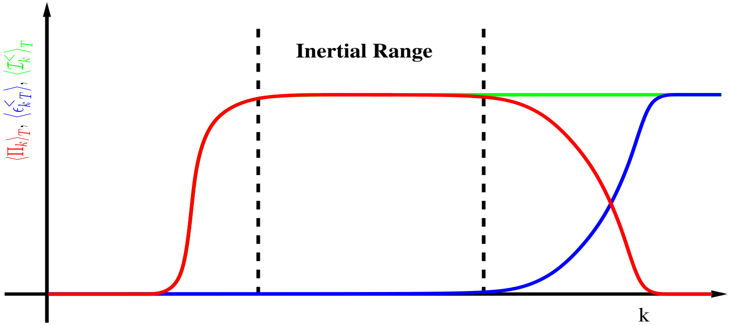


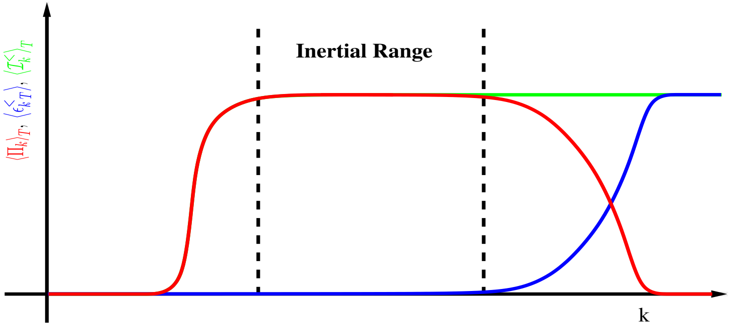
Scale by scale energy balance

$$\langle \mathbf{\Pi_k} \rangle_\mathbf{T} = \left\langle \mathbf{i} \sum_{|\mathbf{k}| \leq \mathbf{k}} \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} (\tilde{\mathbf{u}}^*_\mathbf{k} \cdot [(\mathbf{p} \cdot \tilde{\mathbf{u}}_\mathbf{q})\tilde{\mathbf{u}}_\mathbf{p}]) \right\rangle$$

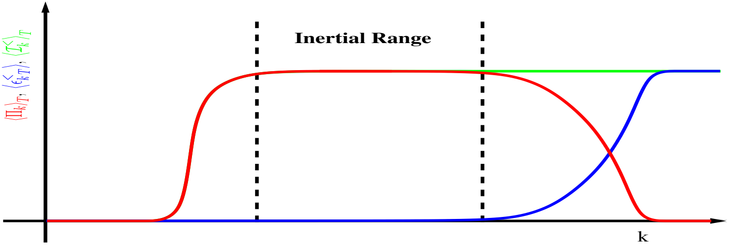


Scale by scale energy balance

$$\langle \boldsymbol{\Pi_k} \rangle_{\mathbf{T}} = \left\langle \mathbf{i} \sum_{|\mathbf{k}| \le \mathbf{k}} \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}} \left(\tilde{\mathbf{u}}^*_{\mathbf{k}} \cdot [(\mathbf{p} \cdot \tilde{\mathbf{u}}_{\mathbf{q}}) \tilde{\mathbf{u}}_{\mathbf{p}}] \right) \right\rangle = \epsilon$$



Scale by scale energy balance



In the limit $\nu \to 0$ a range of scales develops where

- forcing effects can be neglected
- viscosity effects can be neglected
- Energy flows in a constant rate from large to small scales
- Independent of how the system is forced in the large scales and how energy is dissipated in the small scales it Fourier modes are restricted to satisfy the constant energy flux relation

Break

Back to Real Space

Correlation Functions

$$\Gamma^{i,j}(r) = \left\langle u^i(\mathbf{x}+\mathbf{r}) \cdot u^j(\mathbf{x}) \right\rangle$$

- Spatial Average

$$\langle f \rangle_V = \frac{1}{V} \int_V f(\mathbf{x}) dx^3$$

- Temporal average

$$\langle f \rangle_T = \lim_{T \to \infty} \frac{1}{T} \int f(t) dt$$

- Ensemble average

$$\langle f \rangle_S = \lim_{N \to \infty} \frac{1}{N} \sum_i^{\infty} f_i(t)$$

- Homogeneous

$$\langle f \rangle_V = \langle f \rangle_S$$

  System is statistically invariant under space translations

- Ergodic

$$\langle f \rangle_T = \langle f \rangle_S$$

  System is statistically invariant under time translations

In general we will assume statistically homogeneous and ergodic flows

$$\langle f \rangle_V = \langle f \rangle_T = \langle f \rangle_S = \langle f \rangle$$

Real Space Proprties

Implications of Homogeneity and time invariance

- Ergodicity time invariance

$$\langle \mathbf{u}(\mathbf{x}, t_1) \cdot \mathbf{u}(\mathbf{x}, t_2) \rangle = f(t_1 - t_2)$$

- Homogeneity

$$\langle \mathbf{u}(\mathbf{x}_1, t) \cdot \mathbf{u}(\mathbf{x}_2, t) \rangle = f(\mathbf{x}_1 - \mathbf{x}_2)$$

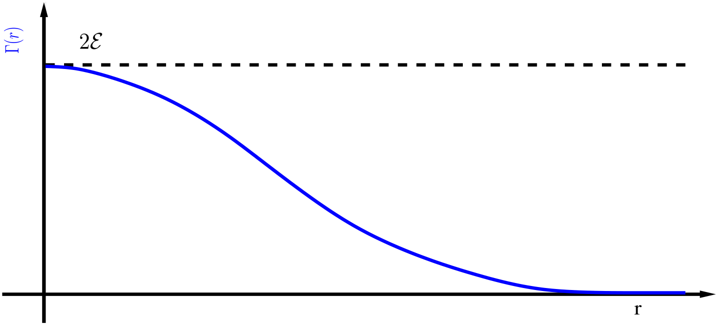
- Isotropy

$$\langle \mathbf{u}(\mathbf{x}_1, t) \cdot \mathbf{u}(\mathbf{x}_2, t) \rangle = f(|\mathbf{x}_1 - \mathbf{x}_2|)$$

Correlation Function

$$\Gamma(r) = \langle \mathbf{u}(\mathbf{x}+\mathbf{r}) \cdot \mathbf{u}(\mathbf{x}) \rangle$$

$$\lim_{r\to 0} \Gamma(r) = \left\langle |\mathbf{u}|^2 \right\rangle = 2\mathcal{E}, \qquad \lim_{r\to\infty} \Gamma(r) = 0$$



Correlation Function & Second order Structure function

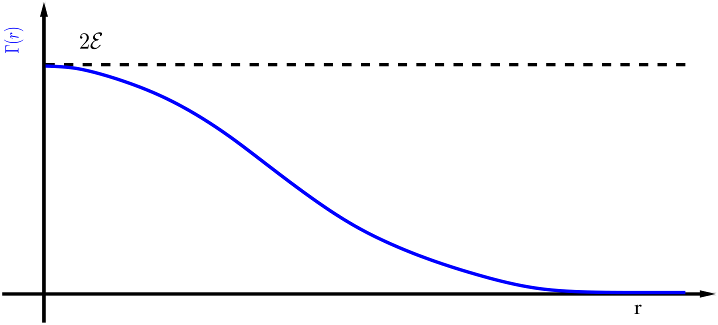
$$
\begin{aligned}
S_2(r) &= \left\langle |\delta \mathbf{u}|^2 \right\rangle \\
&= \left\langle |\mathbf{u}(\mathbf{x}+\mathbf{r}) - \mathbf{u}(\mathbf{x})|^2 \right\rangle \\
&= \left\langle |\mathbf{u}(\mathbf{x}+\mathbf{r})|^2 - 2\mathbf{u}(\mathbf{x}+\mathbf{r}) \cdot \mathbf{u}(\mathbf{x}) + |\mathbf{u}(\mathbf{x})|^2 \right\rangle \\
&= 2\left\langle |\mathbf{u}(\mathbf{x}+\mathbf{r})|^2 \right\rangle - 2\langle \mathbf{u}(\mathbf{x}+\mathbf{r}) \cdot \mathbf{u}(\mathbf{x}) \rangle \\
&= 4\mathcal{E} - 2\langle \mathbf{u}(\mathbf{x}+\mathbf{r}) \cdot \mathbf{u}(\mathbf{x}) \rangle
\end{aligned}
$$

$$S_2(r) = 4\mathcal{E} - 2\Gamma(r)$$

Correlation Function & Structure function

$$\Gamma(r) = \langle \mathbf{u}(\mathbf{x}+\mathbf{r}) \cdot \mathbf{u}(\mathbf{x}) \rangle$$

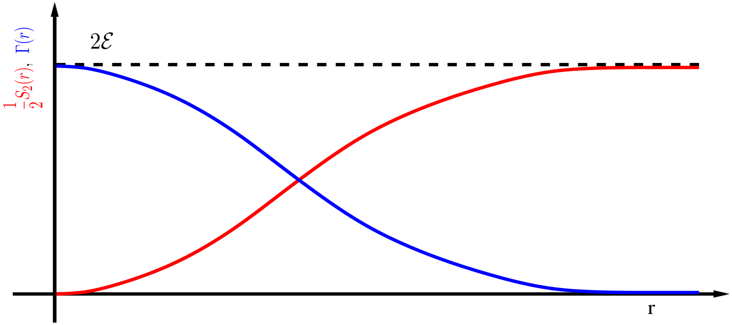
$$\lim_{r \to 0} \Gamma(r) = \left\langle |\mathbf{u}|^2 \right\rangle = 2\mathcal{E}, \qquad \lim_{r \to \infty} \Gamma(r) = 0$$



Correlation Function & Structure function

$$\Gamma(r) = \langle \mathbf{u}(\mathbf{x}+\mathbf{r}) \cdot \mathbf{u}(\mathbf{x}) \rangle, \qquad S_2(r) = \left\langle |\delta \mathbf{u}|^2 \right\rangle = 4\mathcal{E} - 2\Gamma(r)$$

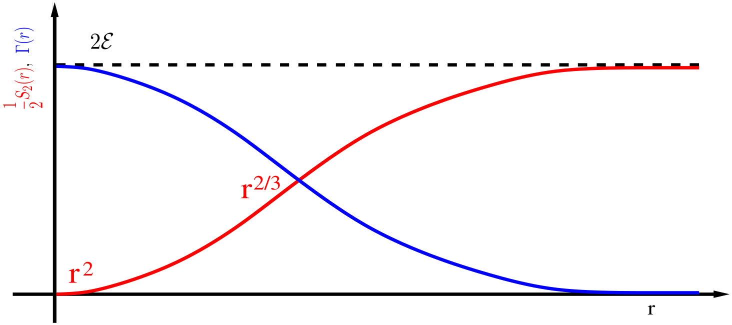
$$\lim_{r\to 0} \Gamma(r) = \left\langle |\mathbf{u}|^2 \right\rangle = 2\mathcal{E}, \qquad \lim_{r\to\infty} \Gamma(r) = 0$$



Correlation Function & Structure function

$$\Gamma(r) = \langle \mathbf{u}(\mathbf{x}+\mathbf{r}) \cdot \mathbf{u}(\mathbf{x}) \rangle, \qquad S_2(r) = \left\langle |\delta \mathbf{u}|^2 \right\rangle = 4\mathcal{E} - 2\Gamma(r)$$

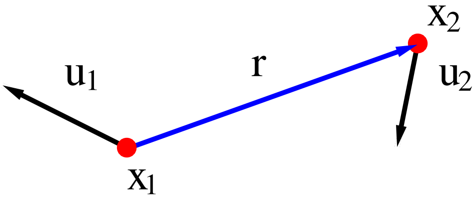
$$r \ll \ell_\nu \to \delta \mathbf{u} \propto r \to S_2(r) \propto r^2, \qquad \ell_\nu \ll r \ll \ell_f \to S_2 \propto r^{2/3}$$



Generalized Structure functions

Longitutonal and transverse Structure functions

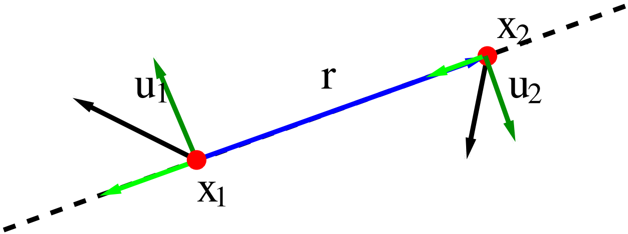
$$S_2(r) = \langle |\delta \mathbf{u}|^2 \rangle$$



Generalized Structure functions

Longitutonal and transverse Structure functions

$$S_2^{\parallel}(r) = \left\langle |\delta \mathbf{u}_{\parallel}|^2 \right\rangle \qquad S_2^{\perp}(r) = \left\langle |\delta \mathbf{u}_{\perp}|^2 \right\rangle$$

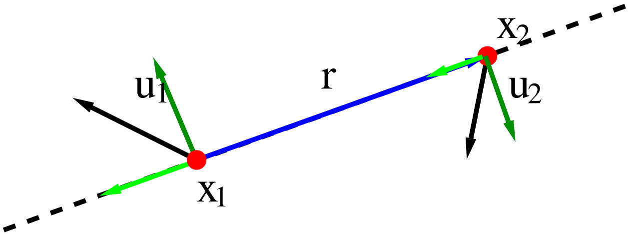


Generalized Structure functions

Longitutonal and transverse Structure functions

$$S_2^{\|}(r) = \left\langle |\delta \mathbf{u}_{\|}|^2 \right\rangle \qquad S_2^{\perp}(r) = \left\langle |\delta \mathbf{u}_{\perp}|^2 \right\rangle$$

$$S_2^{\|}(r) = \left\langle |\delta \mathbf{u} \cdot \hat{\mathbf{r}}|^2 \right\rangle \qquad S_2^{\perp}(r) = \left\langle |\delta \mathbf{u} \times \hat{\mathbf{r}}|^2 \right\rangle$$

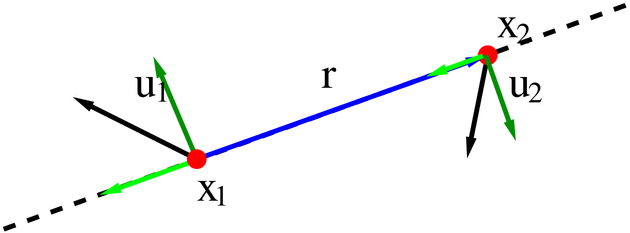


Generalized Structure functions

Higher order Structure functions

$$S_n^{\|}(r) = \left\langle |\delta \mathbf{u}_{\|}|^n \right\rangle \qquad S_n^{\perp}(r) = \langle |\delta \mathbf{u}_{\perp}|^n \rangle$$

$$S_n^{\|}(r) = \langle |\delta \mathbf{u} \cdot \hat{\mathbf{r}}|^n \rangle \qquad S_n^{\perp}(r) = \langle |\delta \mathbf{u} \times \hat{\mathbf{r}}|^n \rangle$$



Fourier Space and Real Space

- **Energy Spectrum:**

$$E(k) = \frac{L}{2} \sum_{k \le |\mathbf{k}| < k+1} |\tilde{\mathbf{u}}_{\mathbf{k}}|^2$$

- **Correlation Function**

$$\Gamma(r) = \langle \mathbf{u}(\mathbf{x} + \mathbf{r}) \cdot \mathbf{u}(\mathbf{x}) \rangle$$

Fourier Space and Real Space

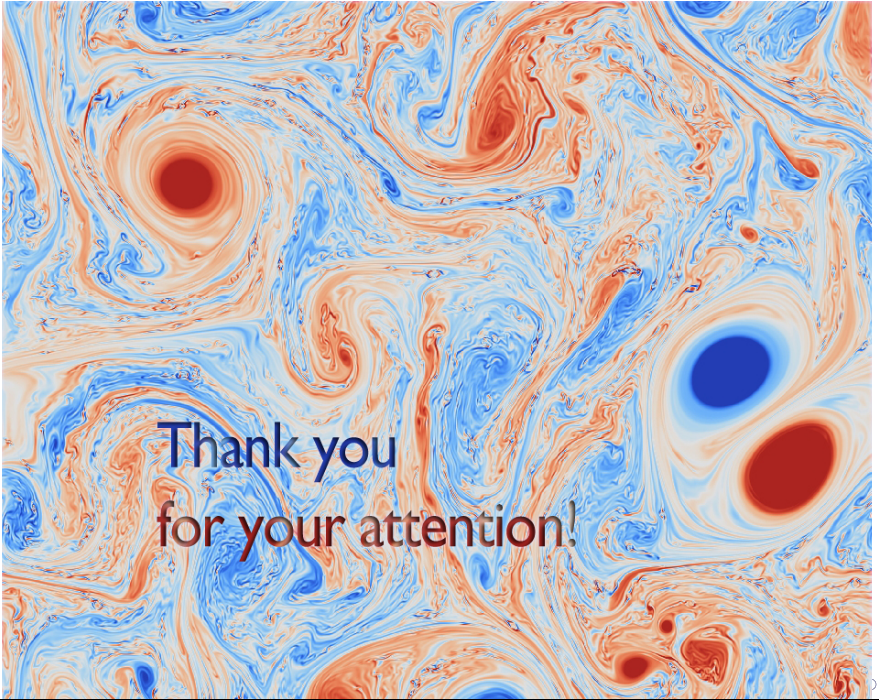
$$
\begin{aligned}
\left\langle e^{-i\mathbf{k}\mathbf{r}}\Gamma(\mathbf{r})\right\rangle &= \left\langle e^{-i\mathbf{k}\mathbf{r}}\langle \mathbf{u}(\mathbf{x}+\mathbf{r})\cdot\mathbf{u}(\mathbf{x})\rangle\right\rangle \\
&= \frac{1}{V^2}\int\int e^{-i\mathbf{k}\mathbf{r}}\mathbf{u}(\mathbf{x}+\mathbf{r})\cdot\mathbf{u}(\mathbf{x})dr^3dx^3 \\
&= \frac{1}{V^2}\int\int\sum_{\mathbf{q}}\sum_{\mathbf{p}}\tilde{\mathbf{u}}_{\mathbf{p}}\tilde{\mathbf{u}}_{\mathbf{q}}e^{i(\mathbf{p}(\mathbf{r}+\mathbf{x})+\mathbf{q}\mathbf{x}-\mathbf{k}\mathbf{r})}dx^3dr^3 \\
&= \frac{1}{V^2}\int\int\sum_{\mathbf{q}}\sum_{\mathbf{p}}\tilde{\mathbf{u}}_{\mathbf{p}}\tilde{\mathbf{u}}_{\mathbf{q}}e^{i(\mathbf{p}+\mathbf{q})\mathbf{x}+(\mathbf{p}-\mathbf{k})\mathbf{r})}dx^3dr^3 \\
&= \sum_{\mathbf{q}}\sum_{\mathbf{p}}\tilde{\mathbf{u}}_{\mathbf{p}}\tilde{\mathbf{u}}_{\mathbf{q}}\delta_{\mathbf{p},-\mathbf{q}}\delta_{\mathbf{p},\mathbf{k}} \\
&= \sum_{\mathbf{p}}\tilde{\mathbf{u}}_{\mathbf{p}}\tilde{\mathbf{u}}_{-\mathbf{p}}\delta_{\mathbf{p},\mathbf{k}} \\
&= |\tilde{\mathbf{u}}_{\mathbf{k}}|^2 = 2\times \text{Energy density in Fourier Space}
\end{aligned}
$$

Fourier Space and Real Space

$$
\begin{aligned}
E(k) &= \frac{L}{2} \sum_{k \le |\mathbf{k}| < k+1} |\tilde{\mathbf{u}}_{\mathbf{k}}|^2 \\
&\simeq 2\pi |\mathbf{k}|^2 |\tilde{\mathbf{u}}_{\mathbf{k}}|^2 \\
&= 2\pi |\mathbf{k}|^2 \left\langle e^{-i\mathbf{kr}} \Gamma(\mathbf{r}) \right\rangle \\
&= \frac{2\pi |\mathbf{k}|^2}{V} \int e^{-i\mathbf{kr}} \Gamma(\mathbf{r}) d\mathbf{r}^3 \\
&= \frac{2\pi |\mathbf{k}|^2}{V} \int e^{-ikr\cos\theta} \Gamma(r) r^2 dr d\phi \sin(\theta) d\theta
\end{aligned}
$$

$$
\boxed{E(k) = \frac{4\pi^2}{V} \int \Gamma(r) kr \sin(kr) dr}
$$

Wiener-Khinchin Formula


Thank you
for your attention!