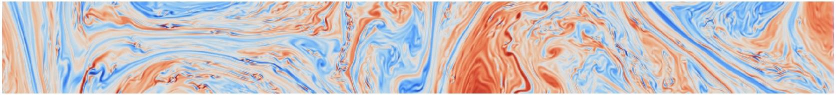
Turbulence
Equilibrium Dynamics



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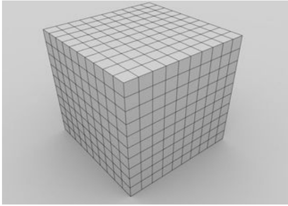
Phase space and Degrees of Freedom

Real space

Discretising space:



$$\mathbf{u}(\mathbf{x},t) \to \mathbf{u}(\mathbf{x_n},t) \quad \text{where} \quad \mathbf{x_n} = [x_{n_x}, y_{n_y}, z_{n_z}] = [n_x, n_y, n_z]\,\delta x,$$

$$\text{with} \quad \mathbf{n} \in \mathbb{Z}^3$$

$N = L/\delta x, \quad N^3$-points, 3-vector components

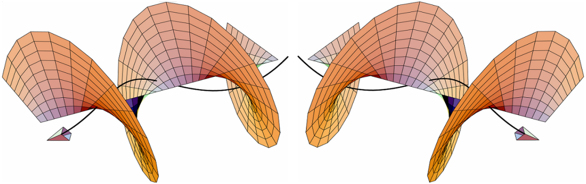
$+$

Incompressibility $\nabla \cdot \mathbf{u} = 0$

Fourier space

$$\tilde{\mathbf{u}}_{\mathbf{k}} = \frac{1}{L^3}\int e^{i\mathbf{k}\mathbf{x}}\mathbf{u}d\mathbf{x}^3, \quad \mathbf{u}(x) = \sum_{\mathbf{k}} e^{-i\mathbf{k}\mathbf{x}}\tilde{\mathbf{u}}_{\mathbf{k}}$$

$$\mathbf{k}\cdot\tilde{\mathbf{u}}_{\mathbf{k}} = \mathbf{0}$$

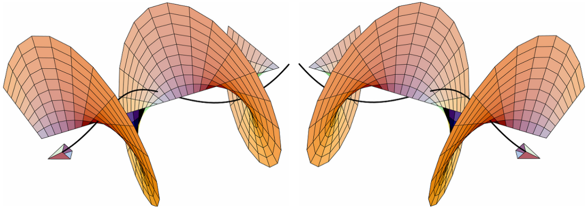


$$\boxed{\tilde{\mathbf{u}}_{\mathbf{k}} = u^+_{\mathbf{k}}\,\mathbf{h}^+_{\mathbf{k}} + u^-_{\mathbf{k}}\,\mathbf{h}^-_{\mathbf{k}}}$$

$$\mathbf{h}^{\pm}_{\mathbf{k}} = \frac{\mathbf{k}\times(\mathbf{k}\times\hat{\mathbf{e}})}{\sqrt{2}|\mathbf{k}\times(\mathbf{k}\times\hat{\mathbf{e}})|} \pm i\frac{\mathbf{k}\times\hat{\mathbf{e}}}{\sqrt{2}|\mathbf{k}\times\hat{\mathbf{e}}|}$$

Phase space and Degrees of Freedom

$$\boxed{\tilde{\mathbf{u}}_{\mathbf{k}} = u^{+}_{\mathbf{k}}\,\mathbf{h}^{+}_{\mathbf{k}} + u^{-}_{\mathbf{k}}\,\mathbf{h}^{-}_{\mathbf{k}}}$$



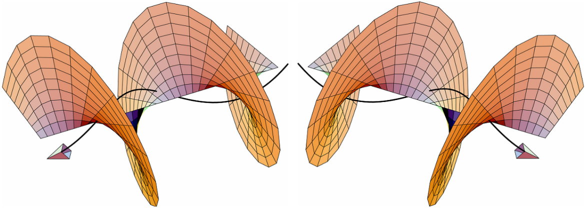
$$\mathbf{h}^{\pm}_{\mathbf{k}} = \frac{\mathbf{k}\times(\mathbf{k}\times\hat{\mathbf{e}})}{\sqrt{2}|\mathbf{k}\times(\mathbf{k}\times\hat{\mathbf{e}})|} \pm i\frac{\mathbf{k}\times\hat{\mathbf{e}}}{\sqrt{2}|\mathbf{k}\times\hat{\mathbf{e}}|}$$

$$i\mathbf{k}\times\mathbf{h}^{\pm}_{\mathbf{k}} = \pm k\mathbf{h}^{\pm}_{\mathbf{k}}, \quad i\mathbf{k}\cdot\mathbf{h}^{\pm}_{\mathbf{k}} = 0$$

$$\mathbf{h}^{s}_{\mathbf{k}}\cdot(\mathbf{h}^{s'}_{\mathbf{k}})^{*} = \mathbf{h}^{s}_{\mathbf{k}}\cdot\mathbf{h}^{-s'}_{\mathbf{k}} = \mathbf{h}^{s}_{\mathbf{k}}\cdot\mathbf{h}^{s'}_{-\mathbf{k}} = \delta_{s,s'}$$

Phase space and Degrees of Freedom

$$\boxed{\tilde{\mathbf{u}}_{\mathbf{k}} = u^{+}_{\mathbf{k}}\,\mathbf{h}^{+}_{\mathbf{k}} + u^{-}_{\mathbf{k}}\,\mathbf{h}^{-}_{\mathbf{k}}}$$



Energy:

$$\mathcal{E} = \frac{1}{2}\sum_{\mathbf{k}}\left[|u^{+}_{\mathbf{k}}|^2 + |u^{-}_{\mathbf{k}}|^2\right]$$

Helicity:

$$\mathcal{H} = \frac{1}{2}\sum_{\mathbf{k}} k\left[|u^{+}_{\mathbf{k}}|^2 - |u^{-}_{\mathbf{k}}|^2\right]$$

Navier Stokes in helical modes

Let $s_{\mathbf{k}} = \pm 1$, $s_{\mathbf{q}} = \pm 1$, $s_{\mathbf{p}} = \pm 1$.
Then

$$\partial_t \left\langle \mathbf{h}_{-\mathbf{k}}^{s_\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} \cdot \mathbf{w} \right\rangle = \left\langle \mathbf{h}_{-\mathbf{k}}^{s_\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{x}} \cdot \left(\nabla \times (\mathbf{u} \times \mathbf{w}) + \nu \nabla^2 \mathbf{w} + \nabla \times \mathbf{f}\right) \right\rangle$$

Navier Stokes:

$$\boxed{\partial_t u_{\mathbf{k}}^{s_\mathbf{k}} = \left(\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}, s_\mathbf{q}, s_\mathbf{p}} C_{\mathbf{k},\mathbf{q},\mathbf{p}}^{s_\mathbf{k}, s_\mathbf{q}, s_\mathbf{p}} u_{\mathbf{q}}^{s_\mathbf{q}} u_{\mathbf{p}}^{s_\mathbf{p}} \right) - \nu |\mathbf{k}|^2 u_{\mathbf{k}}^{s_\mathbf{k}} + f_{\mathbf{k}}^{s_\mathbf{k}}}$$

where

$$C_{\mathbf{k},\mathbf{q},\mathbf{p}}^{s_\mathbf{k}, s_\mathbf{q}, s_\mathbf{p}} u^{s_\mathbf{q}} u^{s_\mathbf{p}} = \frac{1}{2}(s_\mathbf{q} q - s_\mathbf{p} p) \left\langle \mathbf{h}_{-\mathbf{k}}^{s_\mathbf{k}} \cdot \mathbf{h}_{\mathbf{q}}^{s_\mathbf{q}} \times \mathbf{h}_{\mathbf{p}}^{s_\mathbf{p}} \right\rangle$$

Number of Degrees of Freedom of N wavenumber modes

$$\boxed{\tilde{\mathbf{u}}_{\mathbf{k}} = u_{\mathbf{k}}^{+}\,\mathbf{h}_{\mathbf{k}}^{+} + u_{\mathbf{k}}^{-}\,\mathbf{h}_{\mathbf{k}}^{-}}$$

$$\boxed{\partial_t u_{\mathbf{k}}^{s_{\mathbf{k}}} = \left(\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}, s_{\mathbf{q}}, s_{\mathbf{p}}} C_{\mathbf{k},\mathbf{q},\mathbf{p}}^{s_{\mathbf{k}}, s_{\mathbf{q}}, s_{\mathbf{p}}} u_{\mathbf{q}}^{s_{\mathbf{q}}} u_{\mathbf{p}}^{s_{\mathbf{p}}} \right) - \nu |\mathbf{k}|^2 u_{\mathbf{k}}^{s_{\mathbf{k}}} + f_{\mathbf{k}}^{s_{\mathbf{k}}}}$$

If N wavenumbers are sufficient to resolve a given problem then the number of degrees of freedom N_F are

- (2) $u_{\mathbf{k}}^{+}, u_{\mathbf{k}}^{-}$ two modes
- (2) $u_{\mathbf{k}}^{\pm}$ is complex
- (1/2) $u_{-\mathbf{k}}^{\pm} = (u_{\mathbf{k}}^{\pm})^{*}$ realizability condition

$$N_F = 2 \times 2 \times \frac{1}{2} \times N = 2N$$

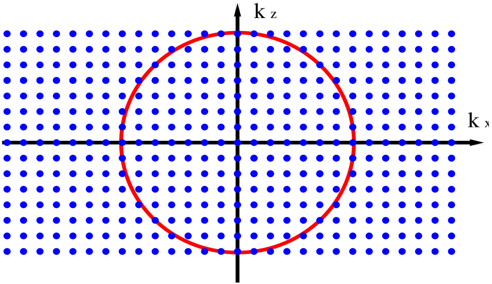
What if we forget dissipation?

Euler-equations

If $\nu = 0$ dissipation can still occur by the formation of singularities.

The Galerkin Truncated Euler-equations

$$\frac{d}{dt} u_{\mathbf{k}}^{s_{\mathbf{k}}} = \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}, s_{\mathbf{q}}, s_{\mathbf{p}}}^{|\mathbf{k}|,|\mathbf{q}|,|\mathbf{p}| \leq k_{\max}} C_{\mathbf{k},\mathbf{q},\mathbf{p}}^{s_{\mathbf{k}}, s_{\mathbf{q}}, s_{\mathbf{p}}} u_{\mathbf{q}}^{s_{\mathbf{q}}} u_{\mathbf{p}}^{s_{\mathbf{p}}}$$



Equilibrium Dynamics

(Lee 1952; Hopf 1952; Kraichnan 1967, 1973; Orszag 1977)

$$\frac{d}{dt} u_{\mathbf{k}}^{s_{\mathbf{k}}} = \sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}, s_{\mathbf{q}}, s_{\mathbf{p}}}^{|\mathbf{k}|,|\mathbf{q}|,|\mathbf{p}| \leq k_{\max}} C_{\mathbf{k},\mathbf{q},\mathbf{p}}^{s_{\mathbf{k}},s_{\mathbf{q}},s_{\mathbf{p}}} u_{\mathbf{q}}^{s_{\mathbf{q}}} u_{\mathbf{p}}^{s_{\mathbf{p}}}$$

Liouville's condition

$$\frac{\partial}{\partial u_{\mathbf{k}}^{s_{\mathbf{k}}}} \left(\sum_{\mathbf{p}+\mathbf{q}=\mathbf{k}, s_{\mathbf{q}}, s_{\mathbf{p}}}^{|\mathbf{k}|,|\mathbf{q}|,|\mathbf{p}| \leq k_{\max}} C_{\mathbf{k},\mathbf{q},\mathbf{p}}^{s_{\mathbf{k}},s_{\mathbf{q}},s_{\mathbf{p}}} u_{\mathbf{q}}^{s_{\mathbf{q}}} u_{\mathbf{p}}^{s_{\mathbf{p}}} \right) = 0$$

Density of states constant along trajectories

+Ergodicity assumption $\Rightarrow$

$\mathcal{P}[\mathbf{u}]$ **is determined by the invariants of the system** $(\mathcal{E}, \mathcal{H})$

Equilibrium Dynamics

Micro-Canonical Ensamble

$$\mathcal{P}[\mathbf{u}] = \frac{1}{Z}\delta\left[\mathcal{E} - \frac{1}{2}\sum_{s_{\mathbf{k}},|\mathbf{k}|\leq k_{\max}} |u_{\mathbf{k}}^{s_{\mathbf{k}}}|^2\right]\delta\left[\mathcal{H} - \frac{1}{2}\sum_{s_{\mathbf{k}},|\mathbf{k}|\leq k_{\max}} s_{\mathbf{k}} k |u_{\mathbf{k}}^{s_{\mathbf{k}}}|^2\right]$$

Canonical Ensemble

$$\mathcal{P}[\mathbf{u}] = \frac{1}{Z}\exp\left[\frac{1}{2}\beta\sum_{s_{\mathbf{k}},|\mathbf{k}|\leq k_{\max}} |u_{\mathbf{k}}^{s_{\mathbf{k}}}|^2 + \frac{1}{2}\gamma\sum_{s_{\mathbf{k}},|\mathbf{k}|\leq k_{\max}} s_{\mathbf{k}} k |u_{\mathbf{k}}^{s_{\mathbf{k}}}|^2\right]$$

Micro-Canonical Ensamble (Neglecting Helicity)

$$\mathcal{P}[\mathbf{u}] = \frac{1}{Z}\delta\left[\mathcal{E} - \frac{1}{2}\sum_{s_{\mathbf{k}},|\mathbf{k}|\leq k_{\max}} |u_{\mathbf{k}}^{s_{\mathbf{k}}}|^2\right]$$

$$\begin{aligned}
\left\langle|u_{\mathbf{k}}^{s_{\mathbf{k}}}|^2\right\rangle &= \frac{1}{Z}\int |u_{\mathbf{k}}^{s_{\mathbf{k}}}|^2\delta\left[\mathcal{E} - \frac{1}{2}\sum_{s_{\mathbf{q}},|\mathbf{q}|\leq k_{\max}} |u_{\mathbf{q}}^{s_{\mathbf{q}}}|^2\right]\prod_{\mathbf{q}} du_{\mathbf{q}}^{s_{\mathbf{q}}} \\
&= \frac{1}{NZ}\int \left(\sum_{\mathbf{k}|,s_{\mathbf{k}}} |u_{\mathbf{k}}^{s_{\mathbf{k}}}|^2\right)\delta\left[\mathcal{E} - \frac{1}{2}\sum_{s_{\mathbf{k}},|\mathbf{k}|\leq k_{\max}} |u_{\mathbf{k}}^{s_{\mathbf{k}}}|^2\right]\prod_{\mathbf{q},s_{\mathbf{q}}} du_{\mathbf{q}}^{s_{\mathbf{q}}} \\
&= \frac{2\mathcal{E}}{NZ}\int \delta\left[\mathcal{E} - \frac{1}{2}\sum_{s_{\mathbf{k}},|\mathbf{k}|\leq k_{\max}} |u_{\mathbf{k}}^{s_{\mathbf{k}}}|^2\right]\prod_{\mathbf{q}} du_{\mathbf{q}}^{s_{\mathbf{q}}} = 2\mathcal{E}/N
\end{aligned}$$

$$E(k) = \frac{1}{2}\sum_{k\leq|\mathbf{q}|<k+1} \left\langle|u_{\mathbf{k}}^{s_{\mathbf{k}}}|^2\right\rangle \propto k^2$$

Canonical Ensamble

$$\mathcal{P}[\mathbf{u}] = \frac{1}{Z}\exp\left[\frac{1}{2}\left[\sum_{s_{\mathbf{k}},|\mathbf{k}|\le k_{\max}}(\beta + s_{\mathbf{k}}\gamma k)|u_{\mathbf{k}}^{s_{\mathbf{k}}}|^2\right]\right]$$

$$\langle|u_{\mathbf{k}}^{s_{\mathbf{k}}}|^2\rangle = \frac{1}{Z}\int |u_{\mathbf{k}}^{s_{\mathbf{k}}}|^2\mathcal{P}[\mathbf{u}]\prod_{\mathbf{q}} du_{\mathbf{q}}^{s_{\mathbf{q}}}$$

$$= \frac{1}{(\beta + s_{\mathbf{k}}\gamma k)}, \qquad \left(\int_{-\infty}^{+\infty} e^{-ax^2}dx = \sqrt{\frac{\pi}{a}}\right)$$

$$\langle u_{-\mathbf{k}}^{s_{\mathbf{k}}}\cdot w_{\mathbf{k}}^{s_{\mathbf{k}}}\rangle = \frac{s_{\mathbf{k}}k}{(\beta + s_{\mathbf{k}}\gamma k)}, \qquad \left(\int_{-\infty}^{+\infty} x^2e^{-ax^2}dx = \sqrt{\frac{\pi}{4a^3}}\right)$$

$$\langle e_k\rangle = \frac{1}{2}\langle|u_{\mathbf{k}}^{+}|^2 + |u_{\mathbf{k}}^{-}|^2\rangle = \frac{\beta}{(\beta^2 - \gamma k^2)},$$

$$\langle h_k\rangle = \frac{k}{2}\langle|u_{\mathbf{k}}^{+}|^2 - |u_{\mathbf{k}}^{-}|^2\rangle = \frac{\gamma k^2}{(\beta^2 - \gamma^2 k^2)}$$

Canonical Ensamble

$$\langle e_k \rangle = \frac{1}{2}\langle |u^+_{\mathbf{k}}|^2 + |u^-_{\mathbf{k}}|^2 \rangle = \frac{\beta}{(\beta^2 - \gamma k^2)},$$

$$\langle h_k \rangle = \frac{k}{2}\langle |u^+_{\mathbf{k}}|^2 - |u^-_{\mathbf{k}}|^2 \rangle = \frac{\gamma k^2}{(\beta^2 - \gamma^2 k^2)}$$

$$E(k) = \frac{4\pi k^2}{(\beta^2 - \gamma^2 k^2)}, \quad H(k) = \frac{4\pi\gamma k^4}{(\beta^2 - \gamma^2 k^2)}$$

$$\mathcal{E} = \sum_{\mathbf{k}} \frac{\beta}{(\beta^2 - \gamma k^2)}, \qquad \mathcal{H} = \sum_{\mathbf{k}} \frac{\gamma k^2}{(\beta^2 - \gamma k^2)}$$

Canonical Ensamble

$$\langle e_k \rangle = \frac{1}{2}\big\langle |u_{\mathbf{k}}^{+}|^2 + |u_{\mathbf{k}}^{-}|^2 \big\rangle = \frac{\beta}{(\beta^2 - \gamma k^2)},$$

$$\langle h_k \rangle = \frac{k}{2}\big\langle |u_{\mathbf{k}}^{+}|^2 - |u_{\mathbf{k}}^{-}|^2 \big\rangle = \frac{\gamma k^2}{(\beta^2 - \gamma^2 k^2)}$$

$$E(k) = \frac{4\pi k^2}{(\beta^2 - \gamma^2 k^2)}, \quad H(k) = \frac{4\pi \gamma k^4}{(\beta^2 - \gamma^2 k^2)}$$

$$k_{\max} < |\beta/\gamma|$$

Canonical Ensamble

$$E(k) = \frac{4\pi k^2}{(\beta^2 - \gamma k^2)}, \quad H(k) = \frac{4\pi\gamma k^4}{(\beta^2 - \gamma k^2)}$$

Thermal equilibrium of large scales

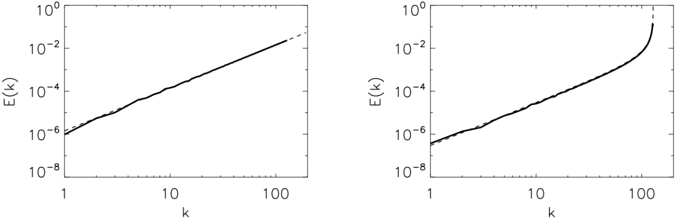
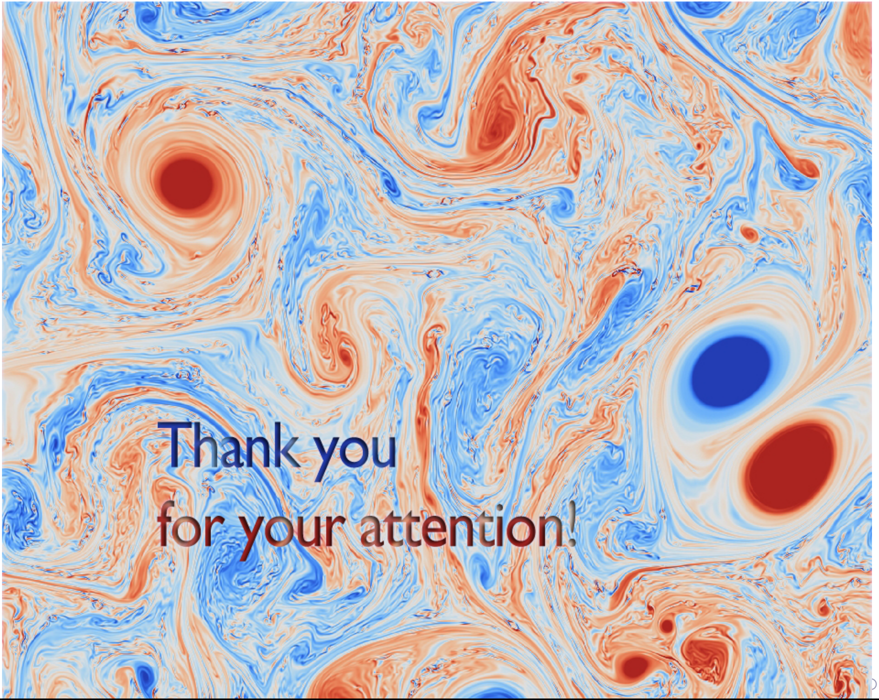


Figure 1: The energy spectra from two simulations of the truncated Euler equations with $k_{max} = 128$ and zero helicity (left) and $\mathcal{H}/\mathcal{E}k_{max} = 0.82$ (right). The dashed lines show the theoretical predictions given in eq. 1.2.



Thank you
for your attention!