Llista de problemes 6 (Seminari)

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Exercici 1.

a. Calculeu la transformada de Fourier de la distribució $\partial^\alpha \delta$, $\alpha \in (\mathbb{N} \cup \{0\})^n$.

b. Calculeu $\langle \widehat{P}, \varphi \rangle$, on $P(x) = x^2 - 2x + 1$.

c. Calculeu $\langle \widehat{P}, \varphi \rangle$, on $P(x)$ és un polinomi de $\mathbb{R}^n$.

Resolució.

a. Tenim que:
$$\langle \widehat{\partial^\alpha \delta}, \varphi \rangle = \langle (2\pi \mathrm{i}\xi)^\alpha \widehat{\delta}, \varphi \rangle = \langle (2\pi \mathrm{i}\xi)^\alpha, \varphi \rangle \implies \widehat{\partial^\alpha \delta} = (2\pi \mathrm{i}\xi)^\alpha$$
ja que $\widehat{\delta} = 1$.

b. Tenim que:
$$\langle \widehat{P}, \varphi \rangle = \langle P, \widehat{\varphi} \rangle = \left\langle \frac{(2\pi \mathrm{i}x)^2}{(2\pi \mathrm{i})^2} - 2\frac{(2\pi \mathrm{i}x)}{(2\pi \mathrm{i})} + 1, \widehat{\varphi} \right\rangle = \left\langle \frac{\widehat{\delta''}}{(2\pi \mathrm{i})^2} - 2\frac{\widehat{\delta'}}{(2\pi \mathrm{i})} + \widehat{\delta}, \widehat{\varphi} \right\rangle = \left\langle \frac{\delta''}{(2\pi \mathrm{i})^2} - 2\frac{\delta'}{(2\pi \mathrm{i})} + \delta, \mathcal{F}^2\varphi \right\rangle$$
on hem utilitzat l'apartat anterior en la tercera igualtat. Ara bé, com que les deltes centrades en el 0 són simètriques respecte tenim que l'última igualtat és igual a:
$$\left\langle \frac{\delta''}{(2\pi \mathrm{i})^2} - 2\frac{\delta'}{(2\pi \mathrm{i})} + \delta, \varphi \right\rangle$$
Per tant, $\widehat{P} = \frac{\delta''}{(2\pi \mathrm{i})^2} - 2\frac{\delta'}{(2\pi \mathrm{i})} + \delta$.

c. Sigui $P(x) = \sum_{|\alpha| \leq n} a_\alpha x^\alpha$. Tenim que:
$$\langle \widehat{P}, \varphi \rangle = \sum_{|\alpha| \leq n} a_\alpha \langle x^\alpha, \widehat{\varphi} \rangle = \sum_{|\alpha| \leq n} a_\alpha \left\langle \frac{(2\pi \mathrm{i}x)^\alpha}{(2\pi \mathrm{i})^\alpha}, \widehat{\varphi} \right\rangle = \sum_{|\alpha| \leq n} \frac{a_\alpha}{(2\pi \mathrm{i})^\alpha} \left\langle \widehat{\partial^\alpha \delta}, \widehat{\varphi} \right\rangle = \sum_{|\alpha| \leq n} \frac{a_\alpha}{(2\pi \mathrm{i})^\alpha} \langle \partial^\alpha \delta, \varphi \rangle =$$
$$= \left\langle \sum_{|\alpha| \leq n} \frac{a_\alpha}{(2\pi \mathrm{i})^\alpha} \partial^\alpha \delta, \varphi \right\rangle$$
on en la penúltima igualtat hem fet servir la simetria respecte el 0 de la δ. Per tant:
$$\widehat{P} = \sum_{|\alpha| \leq n} \frac{a_\alpha}{(2\pi \mathrm{i})^\alpha} \partial^\alpha \delta$$

Exercici 2. *Calculeu les transformades de Fourier de les distribucions temperades $T_1 = \mathrm{p.v.}\left(\frac{1}{x}\right)$, $T_2 = \mathrm{sgn}(x)$ i $T_3 = x^2 \sin(2\pi x)$.*

Resolució. Calculem $\widehat{T_1}$. De teoria sabem que $\widehat{\mathbf{1}_{(0,\infty)}} = \frac{1}{2\pi \mathrm{i}}\mathrm{p.v.}\left(\frac{1}{x}\right) + \frac{1}{2}\delta$. Per tant, com que $\mathcal{F}^2(\mathbf{1}_{(0,\infty)}) = \mathbf{1}_{(-\infty,0)}$, tenim que:
$$\widehat{\mathrm{p.v.}\left(\frac{1}{x}\right)} = \pi \mathrm{i}(2\mathcal{F}^2(\mathbf{1}_{(0,\infty)}) - \widehat{\delta}) = \pi \mathrm{i}(2\mathbf{1}_{(-\infty,0)} - 1) = -\pi \mathrm{i}\,\mathrm{sgn}$$

Fem ara $\widehat{T_2}$. Tenim que si $\varphi \in \mathcal{S}(\mathbb{R})$:

$$\langle \widehat{\operatorname{sgn}(x)}, \varphi \rangle = \frac{-1}{\pi \mathrm{i}} \left\langle \mathcal{F}^2 \left(\text{p.v.} \left(\frac{1}{x} \right) \right), \varphi \right\rangle = \frac{-1}{\pi \mathrm{i}} \left\langle \text{p.v.} \left(\frac{1}{x} \right), \mathcal{F}^2 \varphi \right\rangle = \frac{-1}{\pi \mathrm{i}} \left\langle \text{p.v.} \left(\frac{1}{x} \right), \varphi(-x) \right\rangle =$$

$$= \frac{-1}{\pi \mathrm{i}} \int_0^\infty \frac{\varphi(-x) - \varphi(x)}{x} \, \mathrm{d}x = \frac{1}{\pi \mathrm{i}} \int_0^\infty \frac{\varphi(x) - \varphi(-x)}{x} \, \mathrm{d}x = \left\langle \frac{\text{p.v.} \left(\frac{1}{x} \right)}{\pi \mathrm{i}}, \varphi \right\rangle$$

Per tant, $\widehat{\operatorname{sgn}(x)} = \frac{\text{p.v.}\left(\frac{1}{x}\right)}{\pi \mathrm{i}}$. Finalment, calculem $\widehat{T_3}$. Tenim que:

$$\langle \mathcal{F}(x^2 \sin(2\pi x)), \varphi \rangle = \left\langle \mathcal{F} \left(\frac{(-2\pi \mathrm{i} x)^2}{(-2\pi \mathrm{i})^2} \sin(2\pi x) \right), \varphi \right\rangle = \frac{-1}{4\pi^2} \left\langle (\mathcal{F}(\sin(2\pi x)))'', \varphi \right\rangle = \frac{-1}{8\pi^2 \mathrm{i}} \left\langle \mathcal{F}(\mathrm{e}^{2\pi \mathrm{i} x} - \mathrm{e}^{-2\pi \mathrm{i} x}), \varphi'' \right\rangle =$$

$$= \frac{-1}{8\pi^2 \mathrm{i}} \left\langle \mathcal{F}^2(\delta_{-1}) - \mathcal{F}^2(\delta_1), \varphi'' \right\rangle = \frac{-1}{8\pi^2 \mathrm{i}} \left\langle \delta_1 - \delta_{-1}, \varphi'' \right\rangle = \frac{-1}{8\pi^2 \mathrm{i}} \left\langle {\delta_1}'' - {\delta_{-1}}'', \varphi \right\rangle$$

on en la quarta igualtat hem fet servir que $\widehat{\delta_a} = \mathrm{e}^{-2\pi \mathrm{i} a x}$. Per tant, $\widehat{T_3} = \frac{-1}{8\pi^2 \mathrm{i}} \left({\delta_1}'' - {\delta_{-1}}'' \right)$.

Exercici 3.

a. Per $m \geq 1$, sigui $T_m = \frac{(-1)^{m-1}}{(m-1)!} \partial^m \log|x| \in \mathcal{D}^*(\mathbb{R})$. Demostreu que $x^m T_m = 1$.

b. Demostreu que si $T \in \mathcal{D}^*(\mathbb{R})$ és tal que $x^m T = 0$, llavors $T = \sum_{j=0}^{m-1} C_j \partial^j \delta$.

c. Deduïu que, en general, si $x^m T = 1$, llavors $T = T_m + \sum_{j=0}^{m-1} C_j \partial^j \delta$.

Resolució.

a. Ho fem per inducció sobre m. El cas $m = 1$, ja el sabem perquè $(\log|x|)' = \text{p.v.}\left(\frac{1}{x}\right)$. Suposem que ho tenim per $m - 1$. Aleshores:

$$x^m T_m = x^m \frac{(-1)}{m-1} \partial T_{m-1} = \frac{1}{1-m} [\partial(x^m T_{m-1}) - m x^{m-1} T_{m-1}] = \frac{1}{1-m} [\partial(x \cdot 1) - m] = 1$$

on en la segona igualtat hem fet servir la regla del producte: $\partial(x^m T_{m-1}) = x^m \partial T_{m-1} + m x^{m-1} T_{m-1}$.

b. És clar que $x\delta^{(k)} = 0 \ \forall k \in \mathbb{N} \cup \{0\}$. Per tant:

$$\left(x\delta^{(k)}\right)' = \delta^{(k)} + x\delta^{(k+1)} = 0 \implies \delta^{(k)} = -x\delta^{(k+1)}$$

A més, recordem que si $xS = 0$ per $S \in \mathcal{D}^*(\mathbb{R})$, aleshores $S = C\delta$. Així doncs, si T és tal que $x^m T = 0$, aleshores:

$$x^m T = x(x^{m-1} T) = 0 \implies x^{m-1} T = C_1 \delta \implies x^{m-1} T + C_1 x \delta' = 0 \implies x(x^{m-2} T + C_1 \delta') = 0 \implies$$

$$\implies x^{m-2} T + C_1 \delta' = C_2 \delta \implies x(x^{m-3} T - C_1 \delta'' + C_2 \delta') = 0 \implies x^{m-3} T - C_1 \delta'' + C_2 \delta' = C_3 \delta \implies \cdots \implies$$

$$\implies T = \sum_{j=0}^{m-1} \tilde{C}_j \partial^j \delta$$

c. Dels apartats anteriors deduïm que $x^m T = 1$ aleshores:

$$x^m (T - T_m) = 0 \implies T - T_m = \sum_{j=0}^{m-1} C_j \partial^j \delta$$