Two Body Problem Fundamentals

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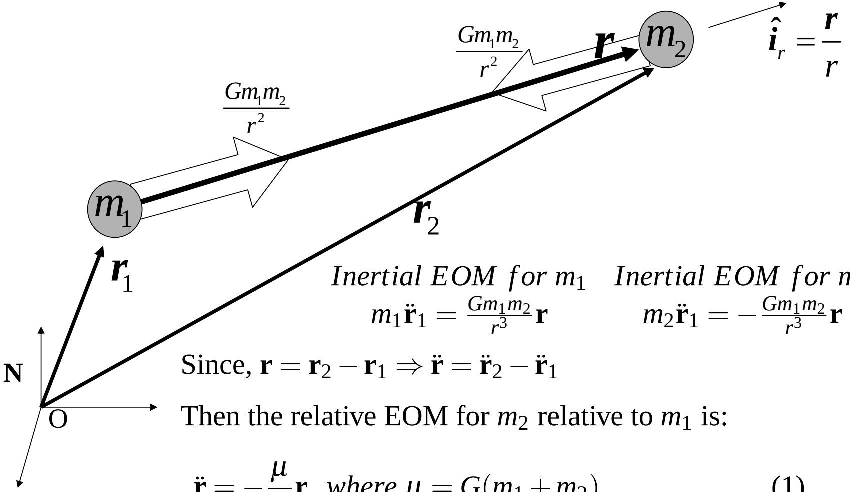
Objective

- Fundamental Integrals
  - Angular Momentum
  - Eccentricity Integral
  - Energy Integral
  - Orientation of the Orbit Plane
  - Kepler's Equation
  - $\cdots$
- Classical Solution for Elliptic Orbits
- Universal Solutions



Inertial EOM for m_1

$$m_1\ddot{\mathbf{r}}_1 = \frac{Gm_1m_2}{r^3}\mathbf{r}$$

Inertial EOM for m_2

$$m_2\ddot{\mathbf{r}}_1 = -\frac{Gm_1m_2}{r^3}\mathbf{r}$$

Since, $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1 \Rightarrow \ddot{\mathbf{r}} = \ddot{\mathbf{r}}_2 - \ddot{\mathbf{r}}_1$

Then the relative EOM for m_2 relative to m_1 is:

$$\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}, \; where \; \mu = G(m_1 + m_2) \tag{1}$$

This vector differential equation is the most important equation in Celestial Mechanics!

- It is possible to integrate two body equations of motion in a closed form.
- When the approximation to the actual force field is more complex it will usually be necessary to employ series expansion or numerical methods for integration.
- The analytical solution for two-body problem may be useful if departure from them are small enough.
  - J-2 problem.
- We will develop 2-body problem solution, to establish not only Kepler's law but also many other *integrals and equations of motion* that are useful both in *calculation and in further theoretical developments* including perturbation theory.

Conservation of Angular Momentum and Kepler's 2^{nd} Law

- Recall, EOM for 2-body problem: $\ddot{\mathbf{r}} = -\frac{\mu}{r^3}\mathbf{r}$
- Note: There is no force that would cause the secondary body to depart from a fixed plane that passes through the primary body.
- Also, path will be a straight line iff. the initial velocity vector is directed along the initial position vector.
- Since acceleration vector is radial, therefore, $\mathbf{r} \times \ddot{\mathbf{r}} = 0$
- Define, Ang. Momentum Vector/unit mass: $\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}} \Rightarrow \dot{\mathbf{h}} = 0$
- Also, $\mathbf{r}.\mathbf{h} = xh_x + yh_y + zh_z = 0$, which is equation of a plane passing through origin and confirms that *motion occurs in an initially fixed plane.*
- Introducing *polar coordinates*, $\mathbf{h} = (r\hat{\mathbf{i}}_r) \times (\dot{r}\hat{\mathbf{i}}_r + r\dot{\theta}\hat{\mathbf{i}}_\theta) = r^2\dot{\theta}\hat{\mathbf{i}}_h$
  $\Rightarrow h = r^2\dot{\theta} = 2A$
- Thus, *Kepler's 2^{nd} Law* has been proven analytically and is simply a geometrical property of *conservation of angular momentum.*

- We seek a function $\mathscr{F}$ whose 2^{nd} derivative has the same form as those for $\mathbf{r}$ i.e. $\ddot{\mathscr{F}} = -\mu\frac{\mathscr{F}}{r^3}$
- It is natural to consider $\mathscr{F}$ in following derivatives:

$$r^2 = \mathbf{r}.\mathbf{r} \tag{2}$$
$$r\dot{r} = \mathbf{r}.\dot{\mathbf{r}} \tag{3}$$

- Let us consider higher order time derivatives of Eq. (3):

$$\frac{d}{dt}(r\dot{r}) = \dot{\mathbf{r}}.\dot{\mathbf{r}} - \frac{\mu}{r} \tag{4}$$
$$\frac{d^2}{dt^2}(r\dot{r}) = 2\dot{\mathbf{r}}.\ddot{\mathbf{r}} + \frac{\mu\dot{r}}{r^2} = -\frac{\mu}{r^3}\underbrace{(r\dot{r})}_{\mathscr{F}} \tag{5}$$

$$\Rightarrow \mathscr{F} = r\dot{r} \;\&\; \ddot{\mathscr{F}} = -\mu\frac{\mathscr{F}}{r^3} \tag{6}$$

- $\ddot{\mathscr{F}} = -\mu \frac{\mathscr{F}}{r^3} \Rightarrow \ddot{\mathscr{F}}\mathbf{r} - \mathscr{F}\ddot{\mathbf{r}} = 0 \Rightarrow \dot{\mathscr{F}}\mathbf{r} - \mathscr{F}\dot{\mathbf{r}} = \mathbf{c}$
- $\mathbf{c}$ is a constant and it is a linear combination of $\mathbf{r}$ and $\dot{\mathbf{r}}$.
  - $\mathbf{c}$ lies in the orbital plane.
- Let us consider: $\mathbf{r}.\mathbf{c} = \dot{\mathscr{F}} r^2 - \mathscr{F} r\dot{r} = r|\mathbf{c}|\cos(\angle \mathbf{a}, \mathbf{r})$

$$r|\mathbf{c}|\cos(\angle \mathbf{a}, \mathbf{r}) = \left(\dot{\mathbf{r}}.\dot{\mathbf{r}} - \frac{\mu}{r}\right) r^2 - r^2\dot{r}^2 = h^2 - \mu r$$

$$\Rightarrow \quad \frac{h^2}{\mu} = r\left[1 + \frac{|\mathbf{c}|}{\mu}\cos(\angle \mathbf{a}, \mathbf{r})\right] \tag{7}$$

- So, we have established Kepler's first law, that the path is a conic with origin at one focus.

$$h^2 = \mu p, \quad \angle \mathbf{a}, \mathbf{r} = f, \quad \& \quad \mathbf{c} = \mu e \hat{\mathbf{i}}_e$$

- Note, $\mathbf{c}$ is directed to the perigee.

- Consider the relative kinetic energy/ mass: $2T = \dot{\mathbf{r}}.\dot{\mathbf{r}} = v^2$.
- Take the time derivative:

$$\dot{T} = \dot{\mathbf{r}}.\ddot{\mathbf{r}} = -\frac{\mu}{r^3}(\dot{\mathbf{r}}.\mathbf{r}) = -\frac{\mu}{r^2}\dot{r} = \frac{d}{dt}\left(\frac{\mu}{r}\right) \tag{8}$$

$$\Rightarrow T = \frac{\mu}{r} + constant \Leftarrow \ Energy\ Equation \tag{9}$$

- Making use of $T = \frac{v^2}{2}$, we have: $v^2 = 2\frac{\mu}{r} + \ const. = \mu\left(\frac{2}{r} - \alpha\right)$
- Evaluation of Energy const. (α) at Perigee:

$$r_p = a(1-e) \ \ v_p^2 = r_p^2\dot{\theta}_p^2 = \frac{h^2}{r_p^2} = \frac{\mu p}{r_p^2} = \frac{\mu(1+e)}{a(1-e)}$$

$$\alpha = \frac{2}{r_p} - \frac{v_p^2}{\mu} = \frac{2}{a(1-e)} - \frac{1+e}{a(1-e)} = \frac{1}{a} \tag{10}$$

- *Energy Integral:* $v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$

MOTION AS A FUNCTION OF TIME: A FRONTAL ASSAULT

Let's begin with *conservation of angular momentum*:

$$r^2\dot{\theta} = r^2\dot{f} = h = constant \tag{11}$$

$$hdt = r^2 df,\ h^2 = \mu p \ \&\ r = \frac{p}{1 + e\cos f} \tag{12}$$

$$\frac{\sqrt{\mu}}{p^{3/2}} dt = \frac{df}{(1 + e\cos f)^2} \tag{13}$$

$$\frac{\sqrt{\mu}}{p^{3/2}}(t - t_0) = \int_{f_0}^{f} \frac{d\phi}{(1 + e\cos\phi)^2} = not\ much\ fun!! \tag{14}$$

Question: Is there a way to "duck" this non-standard elliptic integral?
Answer: Yes, we need to use eccentric anomaly instead of true anomaly as angle variable.

Motion as a Function of Time: Kepler's Equation

Go another route: $\Longleftrightarrow$ Use E as the angle variable

$\boldsymbol{h} = \boldsymbol{r} \times \dot{\boldsymbol{r}} = \text{constant}$ $\Leftarrow \boldsymbol{r} = x\boldsymbol{i}_e + y\boldsymbol{i}_m, \quad \dot{\boldsymbol{r}} = \dot{x}\boldsymbol{i}_e + \dot{y}\boldsymbol{i}_m$

$$\boldsymbol{h} = (x\,\dot{y} - y\,\dot{x})\boldsymbol{i}_h = h\,\boldsymbol{i}_h \quad \Leftarrow \left\{ \begin{array}{l} x = a(\cos E - e), \quad y = a\sqrt{1-e^2}\sin E \\ \dot{x} = -a\sin E\,\dfrac{dE}{dt}, \quad \dot{y} = a\sqrt{1-e^2}\cos E\,\dfrac{dE}{dt} \end{array} \right\}$$

$$h = x\,\dot{y} - y\,\dot{x}$$

$$h = a^2\sqrt{1-e^2}\left(\cos^2 E + \sin^2 E - e\cos E\right)\frac{dE}{dt} \quad \Leftarrow \quad h = \sqrt{\mu p} = \sqrt{\mu a(1-e^2)}$$

$$\frac{\sqrt{\mu}}{a^{3/2}}dt = (1 - e\cos E)\,dE$$ $\Leftarrow$ Integrate this (much easier) equation

$$\frac{\sqrt{\mu}}{a^{3/2}}(t - t_0) = E - e\sin E)\Big|_{E_0}^{E}$$ $\Leftarrow$ This is "Kepler's Equation"...

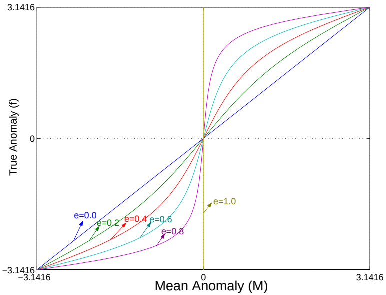
Classical form of Kepler's Equation:

$$\boxed{M = E - e\sin E}$$

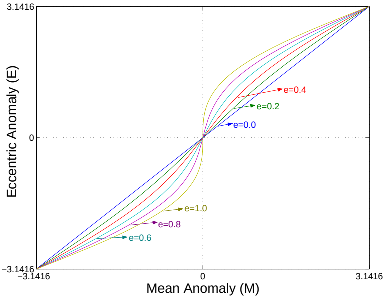
where:

$$M = \text{"mean anomaly"} = M_0 + \frac{\sqrt{\mu}}{a^{3/2}}(t - t_0) = M_0 + n(t - t_0)$$

note that: $0 \le M \le 2\pi, \quad n = \dfrac{2\pi}{P} = \dfrac{\sqrt{\mu}}{a^{3/2}} = \text{"mean angular motion'}$



f vs M



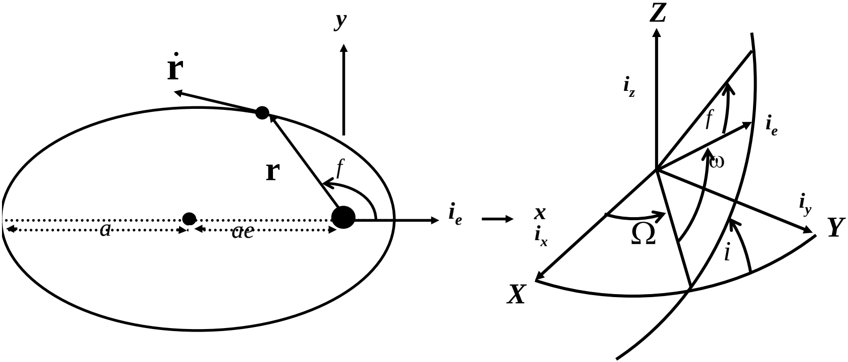
E vs M

Note, as the eccentricity increases, the graph of true anomaly vs mean anomaly varies dramatically as a function of e, but less dramatic changes occur in eccentric anomaly.

"orbit elements"

$$\left\{\mathbf{r}(t_0),\dot{\mathbf{r}}(t_0)\right\} \Rightarrow \left\{a,e,i,\Omega,\omega,t_p,\ldots\right\} \Rightarrow \left\{\mathbf{r}(t),\dot{\mathbf{r}}(t)\right\}$$

size & shape *orientation angles* *time of perigee*



Two Body Solution as a Function of $(h, e, i, \Omega, \omega, f)$

$$\mathbf{r} = r(\cos\Omega\cos\theta - \sin\Omega\sin\theta\cos i)\hat{\mathbf{i}}_x$$
$$+ \; r(\sin\Omega\cos\theta + \cos\Omega\sin\theta\cos i)\hat{\mathbf{i}}_y$$
$$+ \; r(\sin\theta \sin i)\hat{\mathbf{i}}_z$$

and

$$\dot{\mathbf{r}} = \mathbf{v} = -\frac{\mu}{h}[\cos\Omega(\sin\theta + e\sin\omega) + \sin\Omega\left(\cos\theta + e\cos\omega\right)\cos i]\hat{\mathbf{i}}_x$$
$$-\frac{\mu}{h}[\sin\Omega(\sin\theta + e\sin\omega) - \cos\Omega\left(\cos\theta + e\cos\omega\right)\cos i]\hat{\mathbf{i}}_y$$
$$+-\frac{\mu}{h}\left(\cos\theta + e\cos\omega\right)\sin i \; \hat{\mathbf{i}}_z$$

where

$$\theta = \omega + f \qquad r = \frac{h^2/\mu}{1 + e\cos f}$$

$$r^2\dot{\theta} = h = \text{constant} \; \Rightarrow \dot{\theta} = h/r^2$$

Projection of Orbital Unit Vectors onto Inertial Axes: Orientation of the Orbit Plane

$$\begin{Bmatrix} \hat{\boldsymbol{i}}_e \\ \hat{\boldsymbol{i}}_m \\ \hat{\boldsymbol{i}}_h \end{Bmatrix} = [C] \begin{Bmatrix} \hat{\boldsymbol{i}}_x \\ \hat{\boldsymbol{i}}_y \\ \hat{\boldsymbol{i}}_z \end{Bmatrix} \quad \Leftrightarrow \quad \begin{Bmatrix} \hat{\boldsymbol{i}}_x \\ \hat{\boldsymbol{i}}_y \\ \hat{\boldsymbol{i}}_z \end{Bmatrix} = [C]^{\mathrm{T}} \begin{Bmatrix} \hat{\boldsymbol{i}}_e \\ \hat{\boldsymbol{i}}_m \\ \hat{\boldsymbol{i}}_h \end{Bmatrix}$$

where the direction cosine matrix is

$$[C] = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{31} & C_{33} \end{bmatrix} = \begin{bmatrix} c\omega & s\omega & 0 \\ -s\omega & c\omega & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & ci & si \\ 0 & -si & ci \end{bmatrix} \begin{bmatrix} c\Omega & s\Omega & 0 \\ -s\Omega & c\Omega & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{Bmatrix} where \\ c \equiv cos \\ s \equiv sin \end{Bmatrix}$$

$$= \begin{bmatrix} c\omega c\Omega - s\omega ci\ \mathrm{s}\Omega & c\omega s\Omega - s\omega ci\ c\Omega & s\omega\ si \\ s\omega c\Omega - c\omega ci\ \mathrm{s}\Omega & -s\omega c\Omega - c\omega ci\ c\Omega & c\omega\ \mathrm{si} \\ si\ s\Omega & -si\ c\Omega & ci \end{bmatrix}$$

The inverse relationships are

$$\Omega = tan^{-1}\left(\frac{C_{31}}{-C_{32}}\right), \quad \omega = tan^{-1}\left(\frac{C_{13}}{-C_{23}}\right), \quad i = cos^{-1}\left(C_{33}\right)$$

Also, the same direction cosine matrix accomplishes the coordinate transformation:

$$\begin{Bmatrix} x \\ y \\ 0 \end{Bmatrix} = C \begin{Bmatrix} \mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \end{Bmatrix} \quad \Leftrightarrow \quad \begin{Bmatrix} \mathrm{X} \\ \mathrm{Y} \\ \mathrm{Z} \end{Bmatrix} = [C]^{T} \begin{Bmatrix} x \\ y \\ 0 \end{Bmatrix}$$

Transformation from Rectangular Coordinates to Orbit Elements

Rectangular Coordinates	Orbital Elements
$r_0^2 = \mathbf{r}_0.\mathbf{r}_0 = X_0^2 + Y_0^2 + Z_0^2$	$\hat{\mathbf{i}}_h = \frac{\mathbf{h}}{h} = C_{31}\hat{\mathbf{i}}_x + C_{32}\hat{\mathbf{i}}_y + C_{33}\hat{\mathbf{i}}_z$
$v_0^2 = \dot{\mathbf{r}}_0.\dot{\mathbf{r}}_0 = \dot{X}_0^2 + \dot{Y}_0^2 + \dot{Z}_0^2$	$\hat{\mathbf{i}}_e = \frac{\mathbf{c}}{\mu e} = C_{11}\hat{\mathbf{i}}_x + C_{12}\hat{\mathbf{i}}_y + C_{13}\hat{\mathbf{i}}_z$
$\mathbf{h} = \mathbf{r}_0 \times \dot{\mathbf{r}}_0 = h_{x0}\hat{\mathbf{i}}_x + h_{y0}\hat{\mathbf{i}}_y + h_{z0}\hat{\mathbf{i}}_z$	$\hat{\mathbf{i}}_m = \hat{\mathbf{i}}_h \times \hat{\mathbf{i}}_e = C_{21}\hat{\mathbf{i}}_x + C_{22}\hat{\mathbf{i}}_y + C_{23}\hat{\mathbf{i}}_z$
$\frac{1}{a} = \frac{2}{r_0} - \frac{v_0^2}{\mu}$	$i = \cos^{-1}(C_{33}),\ 0 \le i < \pi$
$p = \frac{h^2}{\mu}$	$\Omega = \tan^{-1}\left(\frac{C_{31}}{-C_{32}}\right),\ 0 \le \Omega < 2\pi$
$\dot{\mathscr{F}}\mathbf{r} - \mathscr{F}\dot{\mathbf{r}} = \mathbf{c}$	$\omega = \tan^{-1}\left(\frac{C_{13}}{-C_{23}}\right),\ 0 \le \omega < 2\pi$
$e = \frac{\lvert\mathbf{c}\rvert}{\mu}$	$M_0 = E_0 - e\sin E_0,\ t_p = t_0 - \frac{M_0}{\sqrt{\mu}a^{3/2}}$

Note: $\sigma = \frac{\mathscr{F}}{\sqrt{\mu}} = \frac{\mathbf{r}.\dot{\mathbf{r}}}{\sqrt{\mu}} = \frac{r\dot{r}}{\sqrt{\mu}} = \frac{rae\sin E\dot{E}}{\sqrt{\mu}},\ \dot{E} = \sqrt{\frac{\mu}{a}}\frac{1}{r}$

$\Rightarrow e\sin E = \frac{\sigma}{\sqrt{a}},\ also, e\cos E = 1 - \frac{r}{a}$

$\Rightarrow E = \tan^{-1}\left(\frac{\sigma/\sqrt{a}}{1-r/a}\right)$

$$\{a, e, i\ \Omega, \omega, M_0, (t - t_0)\} \Rightarrow Kepler's\ Eqn. \Rightarrow \{\mathbf{r}(t), \dot{\mathbf{r}}(t)\}$$

$$n = \frac{\sqrt{\mu}}{a^{3/2}}, \qquad M = M_0 + n(t - t_0)$$

Solve Kepler's Eqn. for E (e.g., via Newton's Method)

$$M = E - e\ \sin E \quad \Rightarrow E$$

$$x = a(\cos E - e), \qquad y = a\sqrt{1 - e^2}\ \sin E$$

$$\dot{x} = -\frac{\sqrt{\mu a}}{r}\sin E, \qquad \dot{y} = \frac{\sqrt{\mu a(1 - e^2)}}{r}\cos E$$

$$r = a(1 - e\cos E)$$

$$\begin{Bmatrix} X \\ Y \\ Z \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T \begin{Bmatrix} x \\ y \\ 0 \end{Bmatrix}, \begin{Bmatrix} \dot{X} \\ \dot{Y} \\ \dot{Z} \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T \begin{Bmatrix} \dot{x} \\ \dot{y} \\ 0 \end{Bmatrix}$$