1 Preliminaries

In this section we will review some basic concepts of linear algebra and vector calculus that will be used throughout the document.

1.1 Properties of cross and dot products

Proposition 1. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Then:

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} \tag{1}$$
$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \tag{2}$$

Proof. Let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$. Then:

$$\begin{aligned}
((\mathbf{u} \times \mathbf{v}) \times \mathbf{w})_1 &= (u_3 v_1 - u_1 v_3) w_3 - (u_1 v_2 - u_2 v_1) w_2 \\
&= (u_2 w_2 + u_3 w_3) v_1 - (v_2 w_2 + v_3 w_3) u_1 \\
&= (u_1 w_1 + u_2 w_2 + u_3 w_3) v_1 - (v_1 w_1 + v_2 w_2 + v_3 w_3) u_1 \\
&= ((\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u})_1
\end{aligned}$$

The other components are treated similarly. The second equality follows in a similar way. □

Proposition 2. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Then:

1. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} \cdot \mathbf{w} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} \cdot \mathbf{w}$

1.2 Conics in a nutshell

Definition 3. A conic is the curve obtained as the intersection of a plane with the surface of a double cone (a cone with two *nappes*).

In Fig. 1, we show the 3 types of conics: the ellipse, the parabola, and the hyperbola, which differ on their eccentricity, as we will see later. Note that the circle is a special case of the ellipse.

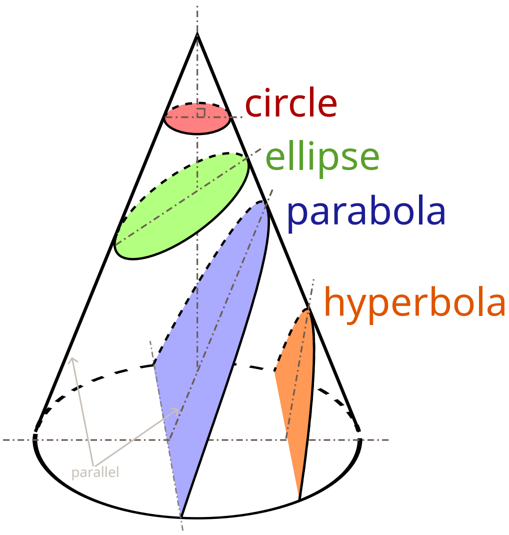


Figure 1: The black boundaries of the colored regions are conic sections. The other half of the hyperbola, which is not shown, is the other nappe of the double cone.

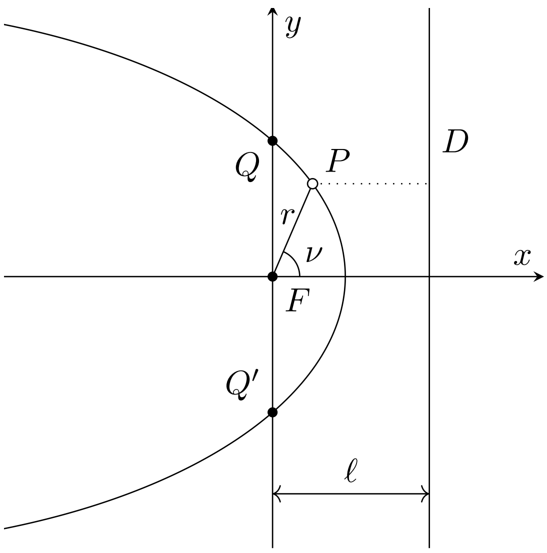


Figure 2: Reference frame centered at the focus of the conic and whose axes are such that the y-axis is parallel to the directrix and the x-axis is perpendicular to the directrix. The directions of the axes are chosen arbitrarily, subject to the constraint that a positive basis is chosen.

Definition 4. The *locus* is a set of points that satisfy a given condition.

The following proposition gives a characterization of the conics.

Proposition 5. A conic is the locus of all points P such that the distance from P to a fixed point F is a multiple of the distance from P to a fixed line D. Mathematically, this is expressed as:

$$d(P,F) = ed(P,D) \tag{3}$$

where d is the Euclidean distance. The point F is called the *focus*; the line D, *directrix*, and the constant of proportionality e, *eccentricity*.

Note that using the polar coordinates (r,ν) as in Fig. 2, we can rewrite Eq. (3) as:

$$r = e(\ell - r\cos\nu) \implies r = \frac{e\ell}{1+e\cos\nu} =: \frac{p}{1+e\cos\nu} \tag{4}$$

where we have defined $p := e\ell$.

Definition 6. Le C be a conic and e be its eccentricity. We say that C is

- an *ellipse* if $0 \le e < 1$,
- a *parabola* if $e = 1$, and
- a *hyperbola* if $e > 1$.

If $e = 0$, the conic is a *circle*.

1.3 Spherical harmonics

1.3.1 Legendre polynomials, regularity and orthonormality

Definition 7. Consider the following second-order differential equation:

$$y'' + p_1(x)y' + p_0(x)y = 0 \tag{5}$$

We say that a is an *ordinary point* if p_1 and p_2 are analytic at $x = a$. We say that a is a *regular singular point* if p_1 has a pole up to order 1 at a and p_0 has a pole of order up to 2 at a. Otherwise we say that a is a *irregular singular point.*

Definition 8. Consider the following second-order differential equation called *Legendre differential equation*:

$$(1-x^2)y'' - 2xy' + \lambda y = 0 \tag{6}$$

for $n \in \mathbb{N} \cup \{0\}$. This equation can be written as:

$$((1-x^2)y')' + \lambda y = 0 \tag{7}$$

If seek for analytic solutions of this equation using the power series method, i.e. looking for solutions of the form $y(x) = \sum_{j=0}^{\infty} a_j x^j$ one can check that we obtain this recursion:

$$a_{j+2} = \frac{j(j+1)-\lambda}{(j+1)(j+2)} a_j \quad j = 0,1,2,\dots \tag{8}$$

From here we can obtain two independent solutions by setting the initial conditions a_0 and a_1 of the iteration. For example, setting $a_1 = 0$ we obtain a series that has only even powers of x. On the other hand, setting $a_0 = 0$ we obtain a series that has only odd powers of x. These two series converge on the interval $(-1,1)$ by the ratio test (by looking at Eq. (8)) and can be expressed compactly as [Mez]:

$$y_{\mathrm{e}}(x) = a_0 \sum_{j=0}^{\infty} \left[\prod_{k=0}^{j-1} (2k(2k+1)-\lambda)\right] \frac{x^{2j+1}}{(2j)!} \qquad y_{\mathrm{o}}(x) = a_1 \sum_{j=0}^{\infty} \left[\prod_{k=1}^{j} (2k(2k+1)-\lambda)\right] \frac{x^{2j+1}}{(2j+1)!} \tag{9}$$

However for each $\lambda \in \mathbb{R}$ either one of these series diverge at $x = \pm 1$, as it behaves as the harmonic series in a neighbourhood of ± 1. We are interested, though, in the solutions that remain bounded on the whole interval $[-1,1]$. Looking at the expressions of Eq. (9) one can check that the only possibility to make the series converge on $[-1,1]$ is when $\lambda = n(n+1)$, $n \in \mathbb{N} \cup \{0\}$. In this case, either one of the series is in fact

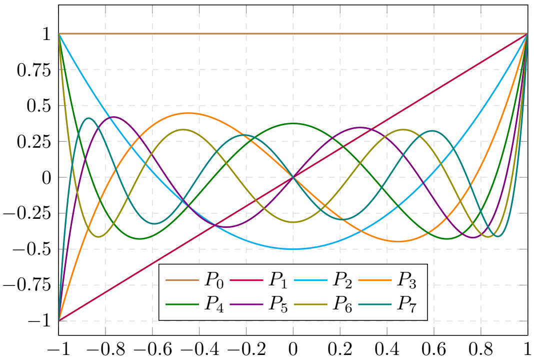


Figure 3: Graphic representation of the first eight Legendre polynomials.

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2-1)$
3	$\frac{1}{2}(5x^3-3x)$
4	$\frac{1}{8}(35x^4-30x^2+3)$
5	$\frac{1}{8}(63x^5-70x^3+15x)$
6	$\frac{1}{16}(231x^6-315x^4+105x^2-5)$
7	$\frac{1}{16}(429x^7-693x^5+315x^3-35x)$

Table 1: First eight Legendre polynomials

a polynomial. And in both cases the polynomial has degree n. For each $n \in \mathbb{N} \cup \{0\}$ if we choose a_0 or a_1 be such that the polynomial evaluates to 1 at $x = 1$, these polynomials are called *Legendre polynomials* and they are denoted by $P_n(x)$. The other (divergent) series is usually denoted in the literature by $Q_n(x)$ (check [RHB99]). And so the general solution of Eq. (7) for $\lambda = n(n+1)$ can be expressed as a linear combination of P_n and Q_n.

Next proposition gives and explicit formula for the Legendre polynomials. The following proposition is will be of our interest as in the next section [RHB99].

Proposition 9. Let $y(x)$ be a solution to the Legendre differential equation. Then, $\forall m \in \mathbb{N} \cup \{0\}$ the function

$$w_m(x) = (1-x^2)^{m/2} \frac{\mathrm{d}^m y(x)}{\mathrm{d}x^m} \tag{10}$$

solves the *general Legendre differential equation*:

$$(1-x^2)y'' - 2xy' + \left(\lambda - \frac{m^2}{1-x^2}\right) y = 0 \tag{11}$$

In particular if $\lambda = n(n+1)$ for $n \in \mathbb{N} \cup \{0\}$, then $w_m(x)$ is denoted as

$$P_{n,m}(x) := (1-x^2)^{m/2} \frac{\mathrm{d}^m P_n}{\mathrm{d}x^m} \tag{12}$$

and it is called the *associated Legendre polynomial* of degree n and order m.

Note that although these functions $P_{n,m}$ are referred to *polynomials*, they are only *true* polynomials if m is even. But we have opt to call them as it is the common practice in the literature (see [Wei]).

Moreover, from the definition of $P_{n,m}$, we can see $P_{n,0} = P_n$ and that $P_{n,m} = 0$ if $m > n$. So we can restrict the domain of m to the set $\{0, 1, \dots, n\}$.

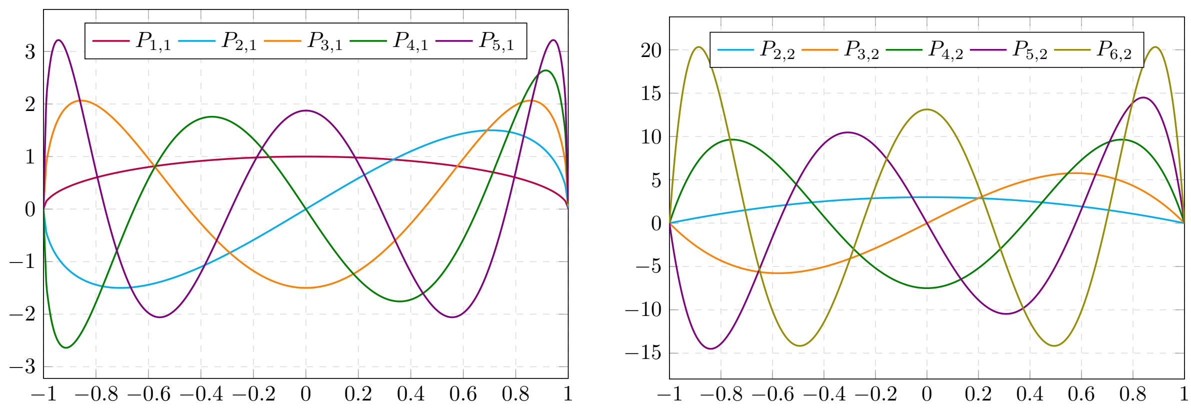
n	$P_{n,1}(x)$	n	$P_{n,2}(x)$
1	$\sqrt{1-x^2}$	2	$3(1-x^2)$
2	$3x\sqrt{1-x^2}$	3	$15x(1-x^2)$
3	$\frac{3}{2}(5x^2-1)\sqrt{1-x^2}$	4	$\frac{15}{2}(7x^2-1)(1-x^2)$
4	$\frac{5}{2}x(7x^2-3)\sqrt{1-x^2}$	5	$\frac{105}{2}x(3x^2-1)(1-x^2)$
5	$\frac{15}{8}(21x^4-14x^2+1)\sqrt{1-x^2}$	6	$\frac{105}{8}(33x^4-18x^2+1)(1-x^2)$

Table 2: First associated Legendre polynomials.

Definition 10. Let $n \in \mathbb{N} \cup \{0\}$ and $m \in \{0, 1, \dots, n\}$. We define the *real spherical harmonics* $Y^{\mathrm{c}}_{n,m}$ and $Y^{\mathrm{s}}_{n,m}$ as:

$$Y^{\mathrm{c}}_{n,m}(\theta,\phi) = \sqrt{(2-\delta_{0,m})(2n+1)\frac{(n-m)!}{(n+m)!}} P_{n,m}(\cos\phi)\cos m\theta \tag{13}$$

$$Y^{\mathrm{s}}_{n,m}(\theta,\phi) = \sqrt{(2-\delta_{0,m})(2n+1)\frac{(n-m)!}{(n+m)!}} P_{n,m}(\cos\phi)\sin m\theta \tag{14}$$



(a) $m = 1$

(b) $m = 2$

Figure 4: Graphic representation of the first associated Legendre polynomials for $m = 1$ and $m = 2$.

The factor $N_{n,m} := \sqrt{(2-\delta_{0,m})(2n+1)\frac{(n-m)!}{(n+m)!}}$ is called the *normalization factor* of the spherical harmonic $Y^{\mathrm{c}}_{n,m}$ and $\delta_{0,m}$ is the Kronecker delta.

n	m	$Y^{\mathrm{c}}_{n,1}(\theta,\phi)$	n	m	$Y^{\mathrm{c}}_{n,2}(\theta,\phi)$
0	0	1	2	2	$\frac{\sqrt{15}}{2}(\sin\phi)^2\cos 2\theta$
1	0	$\sqrt{3}\cos\phi$	3	0	$\frac{\sqrt{7}}{2}\cos\phi(5(\cos\phi)^2-3)$
1	1	$\sqrt{3}\sin\phi\cos\theta$	3	1	$\frac{\sqrt{42}}{4}(5(\cos\phi)^2-1)\sin\phi\cos\theta$
2	0	$\frac{\sqrt{5}}{2}(3(\cos\phi)^2-1)$	3	2	$\frac{\sqrt{105}}{2}(\sin\phi)^2\cos\phi\cos 2\theta$
2	1	$\sqrt{15}\sin\phi\cos\phi\cos\theta$	3	3	$\frac{\sqrt{70}}{4}(\sin\phi)^3\cos 3\theta$

Table 3: First cosine spherical harmonics.

1.3.2 Laplace equation in spherical coordinates

Definition 11. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a twice-differentiable function. The *Laplace equation* is the equation

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{15}$$

where Δ is the Laplace operator.

The next proposition gives the Laplace equation in spherical coordinates.

Proposition 12. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a twice-differentiable function. Then:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial f}{\partial\phi}\right) + \frac{1}{r^2(\sin\phi)^2}\frac{\partial^2 f}{\partial\theta^2} = 0 \tag{16}$$

where r denotes the radial distance, θ denotes the azimuthal angle, and ϕ, the polar angle.

We are now interested in solving the Laplace equation. Theorem 13 gives the solution of it as a function of the spherical harmonics.

Theorem 13. The regular solutions in a bounded region $\Omega \subseteq \mathbb{R}^3$ such that $0 \notin \overline{\Omega}$ to the Laplace equation in spherical coordinates are of the form

$$\begin{aligned} f(r,\theta,\phi) &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}(a_n r^n + b_n r^{-n-1})P_{n,m}(\cos\phi)(c_{n,m}\cos(m\theta) + s_{n,m}\sin(m\theta)) && (17)\\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n}(a_n r^n + b_n r^{-n-1})(\tilde{c}_{n,m}Y^{\mathrm{c}}_{n,m}(\theta,\phi) + \tilde{s}_{n,m}Y^{\mathrm{s}}_{n,m}(\theta,\phi)) && (18)\end{aligned}$$

where $a_n, b_n, c_{n,m}, d_{n,m}, \tilde{c}_{n,m}, \tilde{d}_{n,m} \in \mathbb{R}$.

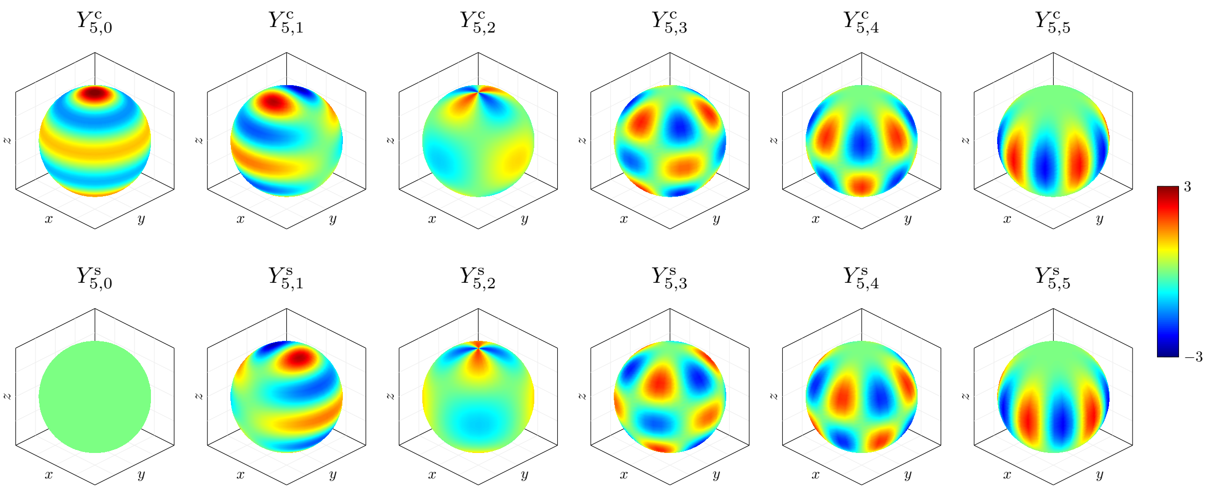


Figure 5: 3D heat map of the spherical harmonics of degree $n = 5$. The first row correspond to the cosine spherical harmonics and the second row to the sine spherical harmonics.

Proof. Let $f(r, \theta, \phi)$ be a solution of Eq. (16) Using separation variables $f(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$ one can write:

$$\frac{\Theta\Phi}{r^2}(r^2 R')' + \frac{R\Theta}{r^2 \sin\phi}(\sin\phi\Phi')' + \frac{R\Phi}{r^2(\sin\phi)^2}\Theta'' = 0 \tag{19}$$

Isolating R from Θ and Φ yields:

$$\frac{(r^2 R')'}{R} = -\frac{1}{\sin\phi\Phi}(\sin\phi\Phi')' - \frac{1}{(\sin\phi)^2\Theta}\Theta'' \tag{20}$$

Since the left-hand side depends entirely on r and the right-hand side does not, we must need that both sides are constant. Hence:

$$\frac{(r^2 R')'}{R} = \lambda \tag{21}$$

$$\frac{1}{\sin\phi\Phi}(\sin\phi\Phi')' + \frac{1}{(\sin\phi)^2\Theta}\Theta'' = -\lambda \tag{22}$$

with $\lambda \in \mathbb{R}$. Similarly from Eq. (22) we obtain that the equations

$$\frac{1}{\Theta}\Theta'' = -m^2 \tag{23}$$

$$\frac{\sin\phi}{\Phi}(\sin\phi\Phi')' + \lambda(\sin\phi)^2 = m^2 \tag{24}$$

must be constant with $m \in \mathbb{C}$ (a priori). The solution to Eq. (23) is a linear combination of the $\cos(m\theta)$ and $\sin(m\theta)$. Note, though, that since Θ must be a 2π-periodic function, that is satisfying $\Theta(\theta + 2\pi) = \Theta(\theta)\ \forall\theta \in \mathbb{R}$, m must be an integer. On the other hand making the change of variables $x = \cos\phi$ and $y = \Phi(\phi)$ in Eq. (24), that equation becomes:

$$(1 - x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\lambda - \frac{m^2}{1 - x^2}\right) y = 0 \tag{25}$$

which is the associate Legendre equation. We have argued in Proposition 9 that we need $\lambda = n(n+1)$ and $m \leq n$ in order to get regular solutions at $x = \cos\phi = \pm 1$. Moreover these solutions are $P_{n,m}(\cos\phi)$.

Finally note that equation Eq. (21) is a Cauchy-Euler equation (check [Wika]) and so the general solution of it is given by

$$R(r) = c_1 r^n + c_2 r^{-n-1} \tag{26}$$

because $\lambda = n(n+1)$. So the general solution becomes a linear combination of the each solution founded varying $n \in \mathbb{N} \cup \{0\}$ and $m \in \{0, 1, \ldots, n\}$:

$$f(r, \theta, \phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}(a_n r^n + b_n r^{-n-1})P_{n,m}(\cos\phi)(c_{n,m}\cos(m\theta) + s_{n,m}\sin(m\theta)) \tag{27}$$

□

From now we are not concerning of the singularity at $r = 0$ of Eq. (18) (see **??** for more details). The associated Legendre polynomials satisfy a orthogonality relation:

Lemma 14. Let $n_1, n_2 \in \mathbb{N} \cup \{0\}$ and $m \leq \min\{n_1, n_2\}$. Then:

$$\int_0^1 P_{n_1,m}(x) P_{n_2,m}(x)\,\mathrm{d}x = \frac{2}{2n_1+1}\frac{(n_1+m)!}{(n_1-m)!}\delta_{n_1,n_2} \tag{28}$$

where δ_{n_1,n_2} denotes the Kronecker delta.

Similarly it can be shown that the spherical harmonics from an orthonormal family of functions:

Proposition 15. The family of spherical harmonics $\{Y^{\mathrm{c}}_{n,m}(\theta,\phi), Y^{\mathrm{s}}_{n,m}(\theta,\phi) : n \in \mathbb{N} \cup \{0\}, m \leq n\}$ is orthonormal in the following sense:

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} Y^i_{n_1,m_1}(\theta,\phi) Y^j_{n_2,m_2}(\theta,\phi)\mathrm{d}\Omega = \delta_{n_1,n_2}\delta_{m_1,m_2}\delta_{i,j} \tag{29}$$

where $\mathrm{d}\Omega = \sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta$ is the solid angle element, which measures the element of area on a sphere of radius 1.

Proof. Let N_{n_1,m_1}, N_{n_2,m_2} be the normalization factors of the spherical harmonics Y_{n_1,m_1}, Y_{n_2,m_2} respectively. Note that we can separate the variables in the integral of Eq. (29). So if $i \neq j$, the integral over θ becomes $\int_0^{2\pi}\cos(m_1\theta)\sin(m_2\theta)\,\mathrm{d}\theta = 0$ regardless of the values of m_1 and m_2. So from now on assume that $i = j$. Due to the symmetry between the cosine and the sine we can suppose that $i = \mathrm{c}$. Thus:

$$\int_0^{2\pi}\int_0^{\pi} Y^i_{n_1,m_1}(\theta,\phi) Y^j_{n_2,m_2}(\theta,\phi)\mathrm{d}\Omega =$$

$$= N_{n_1,m_1}N_{n_2,m_2}\int_0^{\pi} P_{n_1,m_1}(\cos\phi)P_{n_2,m_2}(\cos\phi)\sin\phi\mathrm{d}\phi\int_0^{2\pi}\cos(m_1\theta)\cos(m_2\theta)\,\mathrm{d}\theta \tag{30}$$

An easy check shows that if $m_1 \neq m_2$ then the integral over θ is zero (and the same applies with sines). So suppose $m_1 = m_2 = m$. In that case, if $m \neq 0$ we have $\int_0^{2\pi}(\cos m\theta)^2\,\mathrm{d}\theta = \int_0^{2\pi}(\sin m\theta)^2\,\mathrm{d}\theta = \pi$ and if $m = 0$, the cosine integral evaluates to 2π whereas the sine integral is 0. We can omit this latter case because $Y^{\mathrm{s}}_{n,0}$ is identically zero. Thus:

$$\frac{2\pi}{2-\delta_{0,m}}N_{n_1,m}N_{n_2,m}\int_0^{\pi} P_{n_1,m}(\cos\phi)P_{n_2,m}(\cos\phi)\sin\phi\mathrm{d}\phi = \frac{2\pi}{2-\delta_{0,m}}N_{n_1,m}N_{n_2,m}\int_{-1}^{1} P_{n_1,m}(x)P_{n_2,m}(x)\,\mathrm{d}x \tag{31}$$

By Lemma 14 this latter integral is $\frac{2}{2n_1+1}\frac{(n_1+m)!}{(n_1-m)!}\delta_{n_1,n_2}$. Finally, if $n_1 = n_2 = n$, putting all normalization factors together we get:

$$\frac{2\pi}{2-\delta_{0,m}}N^m_n N^m_n \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!} = 4\pi \tag{32}$$

□

Moreover, an important result in the Sturm-Liouville Theory of second order differential equations ([Wikb; Wan+09]) says that the family of spherical harmonics $\{Y^{\mathrm{c}}_{n,m}(\theta,\phi), Y^{\mathrm{s}}_{n,m}(\theta,\phi) : n \in \mathbb{N} \cup \{0\}, m \leq n\}$ form a complete set in the sense that any smooth function defined on the sphere $f : S^2 \to \mathbb{R}$ can be expanded in a series of spherical harmonics:

$$f(\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}(c_{n,m}Y^{\mathrm{c}}_{n,m}(\theta,\phi) + s_{n,m}Y^{\mathrm{s}}_{n,m}(\theta,\phi)) \tag{33}$$