Universitat Autònoma
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BACHELOR THESIS

Numerical propagation of trajectories of Earth-orbiting spacecraft

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We are just an advanced breed of monkeys on a minor planet of a very average star. But we can understand the Universe. That makes us something very special.

Stephen Hawking

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1 Introduction

2 Preliminaries

In this section we will review some basic concepts of linear algebra and vector calculus that will be used throughout the document.

2.1 Properties of cross and dot products

Proposition 1. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Then:

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} \tag{1}$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \tag{2}$$

Proof. Let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$. Then:

$$\begin{aligned}
((\mathbf{u} \times \mathbf{v}) \times \mathbf{w})_1 &= (u_3 v_1 - u_1 v_3) w_3 - (u_1 v_2 - u_2 v_1) w_2 \\
&= (u_2 w_2 + u_3 w_3) v_1 - (v_2 w_2 + v_3 w_3) u_1 \\
&= (u_1 w_1 + u_2 w_2 + u_3 w_3) v_1 - (v_1 w_1 + v_2 w_2 + v_3 w_3) u_1 \\
&= ((\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u})_1
\end{aligned}$$

The other components are treated similarly. The second equality follows in a similar way. ☐

Proposition 2. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Then:

1. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} \cdot \mathbf{w} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} \cdot \mathbf{w}$

2.2 Conics in a nutshell

2.2.1 General conics

Definition 3. A conic is the curve obtained as the intersection of a plane with the surface of a double cone (a cone with two *nappes*).

In Fig. 1, we show the 3 types of conics: the ellipse, the parabola, and the hyperbola, which differ on their eccentricity, as we will see later on. Note that the circle is a special case of the ellipse.

The following proposition gives a characterization of the conics.

Proposition 4. A conic is the set of all points P such that the distance from P to a fixed point F is a multiple of the distance from P to a fixed line D. Mathematically, this is expressed as:

$$d(P, F) = e d(P, D) \tag{3}$$

where d is the Euclidean distance. The point F is called the *focus*; the line D, *directrix*, and the constant of proportionality e, *eccentricity*.

Note that using the polar coordinates (r, ν) centered at F (as in Fig. 2), we can rewrite Eq. (3) as:

$$r = e(\ell - r\cos\nu) \implies r = \frac{e\ell}{1 + e\cos\nu} =: \frac{p}{1 + e\cos\nu} \tag{4}$$

where we have defined $p := e\ell$.

Definition 5. Le C be a conic and e be its eccentricity. We say that C is

- an *ellipse* if $0 \leq e < 1$,
- a *parabola* if $e = 1$, and
- a *hyperbola* if $e > 1$.

If $e = 0$, the conic is called *circle*.

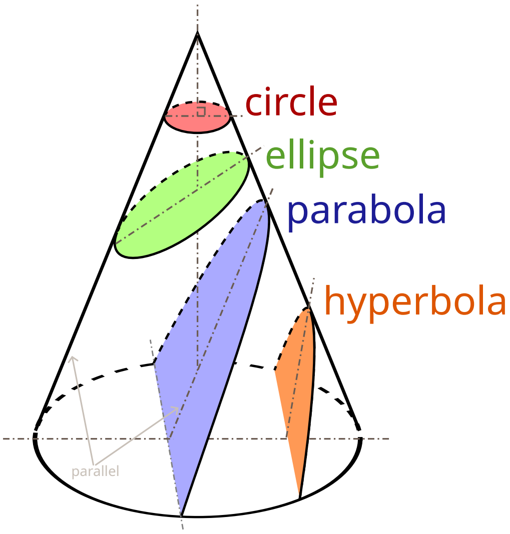

Figure 1: The black boundaries of the colored regions are conic sections. The other half of the hyperbola, which is not shown, is in the other nappe of the double cone. (based on [Mat12])

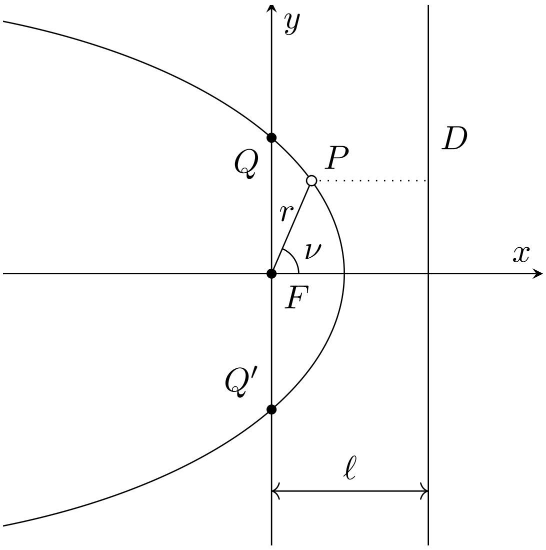

Figure 2: Reference frame centered at the focus of the conic and whose axes are such that the y-axis is parallel to the directrix and the x-axis is perpendicular to the directrix. The directions of the axes are chosen arbitrarily, subject to the constraint that a right-handed system (x, y) is obtained.

2.2.2 Ellipse

From now on we will focus on the study of the ellipse. From Eq. (4), since $e < 1$, it follows $r(\nu)$ is continuous. Therefore, the ellipse is bounded and closed conic section (and it is the only conic section satisfying these two properties).

Let's now study the extrema of $r(\nu)$. An easy check shows that the minimum is attained at $\nu = 0$ and the maximum at $\nu = \pi$ and these values are given by:

$$r_{\min} = \frac{p}{1+e} \qquad \text{and} \qquad r_{\max} = \frac{p}{1-e} \tag{5}$$

When considering orbits of bodies these points are called *periapsis* and *apoapsis*, respectively[1]. The line connecting both points is called *line of apsides*. Let's seek now the extrema of $x = r\cos\nu$ and $y = r\sin\nu$. Differentiating with respect to ν yields:

$$x' = -\frac{p\sin\nu}{(1+e\cos\nu)^2} \qquad y' = \frac{p(e+\cos\nu)}{(1+e\cos\nu)^2} \tag{6}$$

On the one hand, first expression vanishes at $\nu = 0, \pi$. Therefore, the extrema of x coincide with the periapsis and apoapsis points and at these points the y coordinate is equal to 0. This means that the line of apsides passes through the focus of the ellipse. On the other hand, y' vanishes at $\cos\nu = -e$. That is, at $\nu = \arccos(-e)$ and $\nu = 2\pi - \arccos(-e)$. Therefore, using that $\sin(\arccos x) = \sqrt{1-x^2}$, the values of y at these extrema are:

$$y_{\min} = \frac{p}{1-e^2}\sin(\arccos(-e)) = \frac{p}{\sqrt{1-e^2}} \qquad y_{\max} = \frac{p}{1-e^2}\sin(2\pi - \arccos(-e)) = -\frac{p}{\sqrt{1-e^2}} \tag{7}$$

Note that the x coordinate at these two points is the same: $-\frac{pe}{1-e^2}$.

Definition 6. Consider the reference frame Fig. 3 centered at one focus. We define the *semi-major axis* a as half the segment that connects the two extrema of the x coordinate. The *semi-minor axis* b is defined

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[1] Other names are used in the literature when the the central body and the orbiter are particular ones. For example for the system Sun-Earth, the words *perihelion* and *aphelion* are used, whereas for the system Earth-Moon, the words *perigee* and *apogee* are used instead.

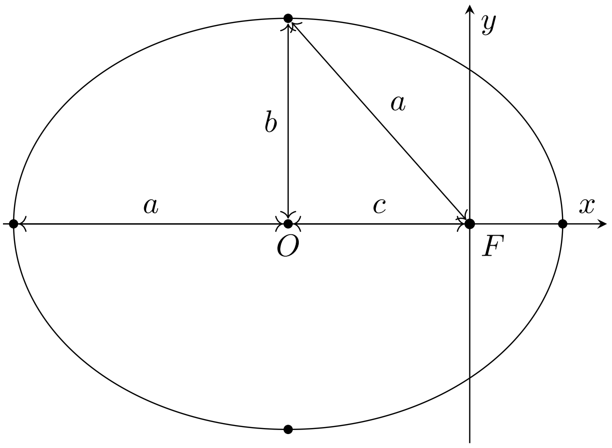


Figure 3: Ellipse

as half the segment that conects the two extrema of the y coordinate. The length of those segments are also denoted by a and b, respectively. Thus, these are given by the following expressions:

$$a := \frac{r_{\text{max}} + r_{\text{min}}}{2} = \frac{p}{1-e^2} \qquad b := \frac{y_{\text{max}} + y_{\text{min}}}{2} = \frac{p}{\sqrt{1-e^2}} \tag{8}$$

From here note that we can express b in terms of a and e as:

$$b = a\sqrt{1-e^2} \tag{9}$$

Definition 7. We define the center O of the ellipse as the intersection of the semi-major axis and semi-minor axis.

Let's calculate the distance from the focus F to one of the extrema of the y coordinate.

$$d\left(F, \left(-\frac{pe}{1-e^2}, \pm\frac{p}{\sqrt{1-e^2}}\right)\right) = \frac{p}{\sqrt{1-e^2}}\sqrt{\frac{e^2}{1-e^2}+1} = \frac{p}{1-e^2} = a \tag{10}$$

Hence, the value of c in Fig. 3 can be simplified to:

$$c^2 = a^2 - b^2 = a^2 - a^2(1-e^2) = a^2e^2 \implies c = ae \tag{11}$$

Proposition 8. The area enclosed in a ellipse of semi-major axis a and semi-minor axis b is πab.

Proof. Consider the ellipse E centered at the origin and oriented as in Fig. 3. From Fig. 3 one can see that it can be parametrized by $(x, y) = (\frac{p\cos\nu}{1-e^2} + ae, \frac{p\sin\nu}{1+e\cos\nu})$ with $\nu \in [0, 2\pi)$. An easy check shows that this parametrization satisfies:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{12}$$

Hence, the area enclosed in the ellipse is can be parametrized by $(x, y) = (ar\cos\nu, br\sin\nu)$, with $r \in [0, 1]$ and $\nu \in [0, 2\pi]$. The Jacobian of this transformation is abr. Therefore, from the change of variable theorem we have that:

$$\text{Area}(E) = \iint\limits_E \mathrm{d}x\,\mathrm{d}y = \int\limits_0^{2\pi}\int\limits_0^1 rab\,\mathrm{d}r\,\mathrm{d}\nu = \pi ab \tag{13}$$

□

2.3 Spherical harmonics

2.3.1 Legendre polynomials, regularity and orthonormality

Definition 9. Consider the following second-order differential equation called *Legendre differential equation*:

$$(1-x^2)y'' - 2xy' + \lambda y = 0 \tag{14}$$

for $\lambda \in \mathbb{R}$. This equation can be rewritten as:

$$((1-x^2)y')' + \lambda y = 0 \tag{15}$$

If seek for analytic solutions of this equation using the power series method [Mez], i.e. looking for solutions of the form $y(x) = \sum_{j=0}^{\infty} a_j x^j$, we see that:

$$0 = (1-x^2)\sum_{j=0}^{\infty} a_{j+2}(j+1)(j+2)x^j - 2x\sum_{j=0}^{\infty} a_{j+1}(j+1)x^j + \lambda\sum_{j=0}^{\infty} a_j x^j = \sum_{j=0}^{\infty} a_{j+2}(j+1)(j+2)x^j - \\ -\sum_{j=0}^{\infty} a_j(j-1)jx^j - \sum_{j=0}^{\infty} 2a_j j x^j + \sum_{j=0}^{\infty} \lambda a_j x^j = \sum_{j=0}^{\infty}[a_{j+2}(j+1)(j+2) - a_j(j(j+1)-\lambda)]x^j \tag{16}$$

Equating the general term of the series equal to 0 we obtain this recursion:

$$a_{j+2} = \frac{j(j+1)-\lambda}{(j+1)(j+2)}a_j \quad j = 0, 1, 2, \ldots \tag{17}$$

From here we can obtain two independent solutions by setting the initial conditions a_0 and a_1 of the iteration. For example, setting $a_1 = 0$ we obtain a series that has only even powers of x. On the other hand, setting $a_0 = 0$ we obtain a series that has only odd powers of x. These two series converge on the interval $(-1, 1)$ by the ratio test (by looking at Eq. (17)) and can be expressed compactly as [Mez]:

$$y_{\mathrm{e}}(x) = a_0 \sum_{j=0}^{\infty}\left[\prod_{k=0}^{j-1}(2k(2k+1)-\lambda)\right]\frac{x^{2j}}{(2j)!} \quad y_{\mathrm{o}}(x) = a_1 \sum_{j=0}^{\infty}\left[\prod_{k=0}^{j-1}((2k+1)(2k+2)-\lambda)\right]\frac{x^{2j+1}}{(2j+1)!} \tag{18}$$

Here the empty product (that is, for instance when k ranges from 0 to -1) is defined to be 1. However for each $\lambda \in \mathbb{R}$ either one of these series or both diverge at $x = \pm 1$, as they behave as the harmonic series in a neighbourhood of $x = \pm 1$. We are interested, though, in the solutions that remain bounded on the whole interval $[-1, 1]$. Looking at the expressions of Eq. (18) one can check that the only possibility to make the series converge in $[-1, 1]$ is when $\lambda = n(n+1)$, $n \in \mathbb{N} \cup \{0\}$. In this case, for each $n \in \mathbb{N} \cup \{0\}$ exactly one of the series is in fact a polynomial of degree n. If, furthermore, we choose a_0 or a_1 be such that the polynomial evaluates to 1 at $x = 1$, these polynomials are called *Legendre polynomials* and they are denoted by $P_n(x)$. The other (divergent) series is usually denoted in the literature by $Q_n(x)$ (check [RHB99; Mez]). And so the general solution of Eq. (15) for $\lambda = n(n+1)$ can be expressed as a linear combination of P_n and Q_n, because the space of solutions form a vector space of dimension 2.

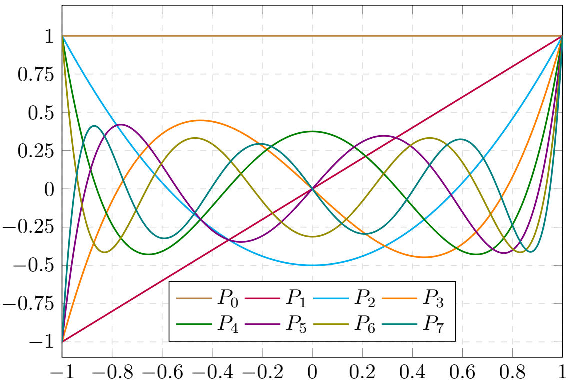


Figure 4: Graphic representation of the first eight Legendre polynomials.

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2-1)$
3	$\frac{1}{2}(5x^3-3x)$
4	$\frac{1}{8}(35x^4-30x^2+3)$
5	$\frac{1}{8}(63x^5-70x^3+15x)$
6	$\frac{1}{16}(231x^6-315x^4+105x^2-5)$
7	$\frac{1}{16}(429x^7-693x^5+315x^3-35x)$

Table 1: First eight Legendre polynomials

The following proposition will be of our interest in the next section [RHB99].

Proposition 10. Let $y(x)$ be a solution to the Legendre differential equation. Then, $\forall m \in \mathbb{N} \cup \{0\}$ the function

$$w_m(x) = (1-x^2)^{m/2}\frac{\mathrm{d}^m y(x)}{\mathrm{d}x^m} \tag{19}$$

solves the *general Legendre differential equation*:

$$(1-x^2)y'' - 2xy' + \left(\lambda - \frac{m^2}{1-x^2}\right) y = 0 \tag{20}$$

In particular if $\lambda = n(n+1)$ for $n \in \mathbb{N} \cup \{0\}$, then $w_m(x)$ is denoted as

$$P_{n,m}(x) := (1-x^2)^{m/2} \frac{\mathrm{d}^m P_n}{\mathrm{d}x^m} \tag{21}$$

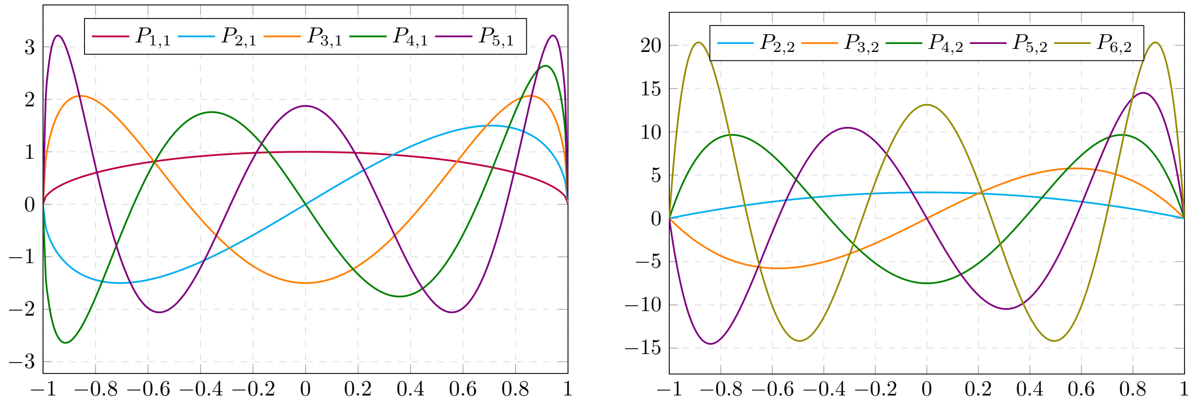
and it is called the *associated Legendre polynomial* of degree n and order m.

Note that although these functions $P_{n,m}$ are named as *polynomials*, they are only *true* polynomials when m is even. But we have opt to call them in that manner as it is the common practice in the literature (see [Wei; RHB99; Mez]).

Moreover, from the definition of $P_{n,m}$, we can see $P_{n,0} = P_n$ and that $P_{n,m} = 0$ if $m > n$. So we can restrict the domain of m to the set $\{0, 1, \ldots, n\}$.

n	$P_{n,1}(x)$	n	$P_{n,2}(x)$
1	$\sqrt{1-x^2}$	2	$3(1-x^2)$
2	$3x\sqrt{1-x^2}$	3	$15x(1-x^2)$
3	$\frac{3}{2}(5x^2-1)\sqrt{1-x^2}$	4	$\frac{15}{2}(7x^2-1)(1-x^2)$
4	$\frac{5}{2}x(7x^2-3)\sqrt{1-x^2}$	5	$\frac{105}{2}x(3x^2-1)(1-x^2)$
5	$\frac{15}{8}(21x^4-14x^2+1)\sqrt{1-x^2}$	6	$\frac{105}{8}(33x^4-18x^2+1)(1-x^2)$

Table 2: First associated Legendre polynomials for $m = 1$ and $m = 2$.



(a) $m = 1$

(b) $m = 2$

Figure 5: Graphic representation of the first associated Legendre polynomials for $m = 1$ and $m = 2$.

Definition 11. Let $n \in \mathbb{N} \cup \{0\}$ and $m \in \{0, 1, \ldots, n\}$. We define the *real spherical harmonics* $Y^{\mathrm{c}}_{n,m}$ and $Y^{\mathrm{s}}_{n,m}$ as:

$$Y^{\mathrm{c}}_{n,m}(\theta, \phi) = \sqrt{(2-\delta_{0,m})(2n+1)\frac{(n-m)!}{(n+m)!}} P_{n,m}(\cos\phi)\cos m\theta \tag{22}$$

$$Y^{\mathrm{s}}_{n,m}(\theta, \phi) = \sqrt{(2-\delta_{0,m})(2n+1)\frac{(n-m)!}{(n+m)!}} P_{n,m}(\cos\phi)\sin m\theta \tag{23}$$

The factor $N_{n,m} := \sqrt{(2-\delta_{0,m})(2n+1)\frac{(n-m)!}{(n+m)!}}$ is called the *normalization factor* of the spherical harmonics and $\delta_{0,m}$ is the Kronecker delta. The weird factor $2 - \delta_{0,m}$ in $N_{n,m}$ will become clear in the next section.

n	m	$Y^{\mathrm{c}}_{n,1}(\theta,\phi)$	n	m	$Y^{\mathrm{c}}_{n,2}(\theta,\phi)$
0	0	1	2	2	$\frac{\sqrt{15}}{2}(\sin\phi)^2\cos 2\theta$
1	0	$\sqrt{3}\cos\phi$	3	0	$\frac{\sqrt{7}}{2}\cos\phi(5(\cos\phi)^2-3)$
1	1	$\sqrt{3}\sin\phi\cos\theta$	3	1	$\frac{\sqrt{42}}{4}(5(\cos\phi)^2-1)\sin\phi\cos\theta$
2	0	$\frac{\sqrt{5}}{2}(3(\cos\phi)^2-1)$	3	2	$\frac{\sqrt{105}}{2}(\sin\phi)^2\cos\phi\cos 2\theta$
2	1	$\sqrt{15}\sin\phi\cos\phi\cos\theta$	3	3	$\frac{\sqrt{70}}{4}(\sin\phi)^3\cos 3\theta$

Table 3: First cosine spherical harmonics.

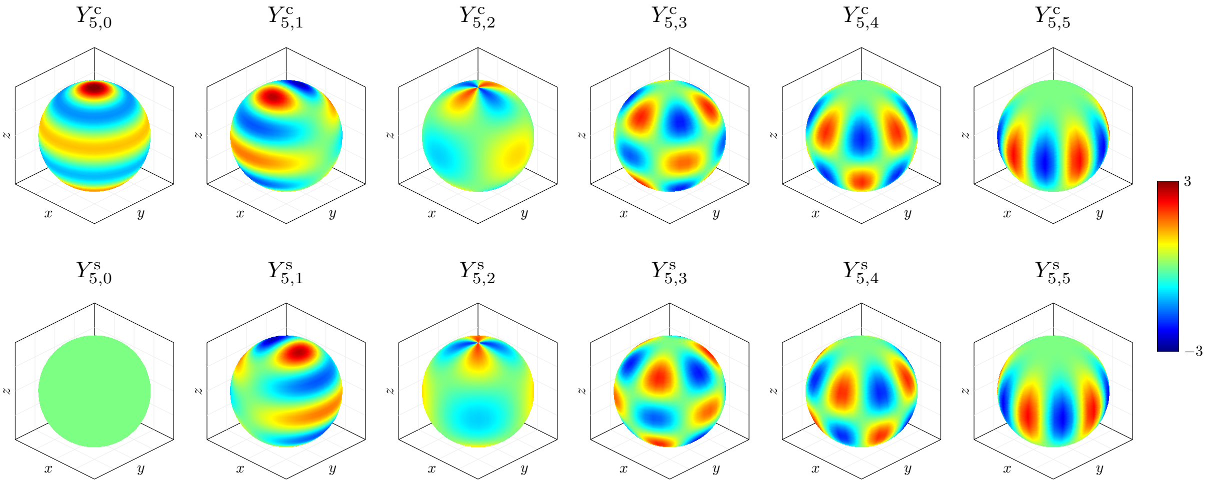


Figure 6: 3D heat map of the spherical harmonics of degree $n = 5$. The first row correspond to the cosine spherical harmonics and the second row correspond to the sine spherical harmonics.

2.3.2 Laplace equation in spherical coordinates

Definition 12. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a twice-differentiable function. The *Laplace equation* is the equation

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{24}$$

where Δ is the Laplace operator.

The next proposition gives the Laplace equation in spherical coordinates.

Proposition 13. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a twice-differentiable function. Then:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial f}{\partial\phi}\right) + \frac{1}{r^2(\sin\phi)^2}\frac{\partial^2 f}{\partial\theta^2} = 0 \tag{25}$$

where r denotes the radial distance, θ denotes the azimuthal angle, and ϕ, the polar angle (or colatitude).

We are now interested in solving the Laplace equation. Theorem 14 gives the solution of it as a function of the spherical harmonics.

Theorem 14. The regular solutions in a bounded region $\Omega \subseteq \mathbb{R}^3$ such that $0 \notin \overline{\Omega}$ to the Laplace equation in spherical coordinates are of the form

$$f(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}(a_n r^n + b_n r^{-n-1})P_{n,m}(\cos\phi)(c_{n,m}\cos(m\theta) + s_{n,m}\sin(m\theta)) \tag{26}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}(a_n r^n + b_n r^{-n-1})(\tilde{c}_{n,m}Y^{\mathrm{c}}_{n,m}(\theta,\phi) + \tilde{s}_{n,m}Y^{\mathrm{s}}_{n,m}(\theta,\phi)) \tag{27}$$

where $a_n, b_n, c_{n,m}, s_{n,m}, \tilde{c}_{n,m}, \tilde{s}_{n,m} \in \mathbb{R}$.

Proof. Let $f(r,\theta,\phi)$ be a solution of Eq. (25) Using separation variables $f(r,\theta,\phi)=R(r)\Theta(\theta)\Phi(\phi)$ we can write:

$$\frac{\Theta\Phi}{r^2}(r^2R')'+\frac{R\Theta}{r^2\sin\phi}(\sin\phi\Phi')'+\frac{R\Phi}{r^2(\sin\phi)^2}\Theta''=0 \tag{28}$$

Here we are making and abuse of notation denoting all the derivative with a prime, but the reader should have no confusion with it. Isolating R from Θ and Φ yields:

$$\frac{(r^2R')'}{R}=-\frac{1}{\sin\phi\Phi}(\sin\phi\Phi')'-\frac{1}{(\sin\phi)^2\Theta}\Theta'' \tag{29}$$

Since the left-hand side depends entirely on r and the right-hand side does not, it follows that both sides must be constant. Therefore:

$$\frac{(r^2R')'}{R}=\lambda \tag{30}$$

$$\frac{1}{\sin\phi\Phi}(\sin\phi\Phi')'+\frac{1}{(\sin\phi)^2\Theta}\Theta''=-\lambda \tag{31}$$

with $\lambda\in\mathbb{R}$. Similarly, separating variables from Eq. (31) we obtain that the equations

$$\frac{1}{\Theta}\Theta''=-m^2 \tag{32}$$

$$\frac{\sin\phi}{\Phi}(\sin\phi\Phi')'+\lambda(\sin\phi)^2=m^2 \tag{33}$$

must be constant with $m\in\mathbb{C}$ (a priori). The solution to the well-known Eq. (32) is a linear combination of the $\cos(m\theta)$ and $\sin(m\theta)$. Note, though, that since Θ must be a 2π-periodic function, that is satisfying $\Theta(\theta+2\pi)=\Theta(\theta)\ \forall\theta\in\mathbb{R}$, m must be an integer. On the other hand making the change of variables $x=\cos\phi$ and $y=\Phi(\phi)$ in Eq. (33) and using the chain rule, that equation becomes:

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2}+\left(\lambda-\frac{m^2}{1-x^2}\right)y=0 \tag{34}$$

which is the associate Legendre equation. We have argued in Proposition 10 that we need $\lambda=n(n+1)$ and $m\leq n$ in order to get regular solutions at $x=\cos\phi=\pm1$. Moreover these solutions are $P_{n,m}(\cos\phi)$.

Finally note that equation Eq. (30) is a Cauchy-Euler equation (check [Wika]) and so the general solution of it is given by

$$R(r)=c_1r^n+c_2r^{-n-1} \tag{35}$$

because $\lambda=n(n+1)$ (the reader may check that r^n and r^{-n-1} are indeed two independent solutions of Eq. (30)). So the general solution becomes a linear combination of the each solution founded varying $n\in\mathbb{N}\cup\{0\}$ and $m\in\{0,1,\dots,n\}$:

$$f(r,\theta,\phi)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}(a_nr^n+b_nr^{-n-1})P_{n,m}(\cos\phi)(c_{n,m}\cos(m\theta)+s_{n,m}\sin(m\theta)) \tag{36}$$

□

From now we are not concerning on the singularity at $r=0$ of Eq. (27) (see Section 4.1.3 for more details).

The associated Legendre polynomials satisfy a orthogonality relation:

Lemma 15. Let $n_1,n_2\in\mathbb{N}\cup\{0\}$ and $m\leq\min\{n_1,n_2\}$. Then:

$$\int_0^1 P_{n_1,m}(x)P_{n_2,m}(x)\,\mathrm{d}x=\frac{2}{2n_1+1}\frac{(n_1+m)!}{(n_1-m)!}\delta_{n_1,n_2} \tag{37}$$

where δ_{n_1,n_2} denotes the Kronecker delta.

Similarly it can be shown that the spherical harmonics from an orthonormal family of functions:

Proposition 16. The family of spherical harmonics $\{Y^{\mathrm{c}}_{n,m}(\theta,\phi), Y^{\mathrm{s}}_{n,m}(\theta,\phi) : n \in \mathbb{N}\cup\{0\}, m \leq n\}$ is orthonormal in the following sense:

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} Y^i_{n_1,m_1}(\theta,\phi)Y^j_{n_2,m_2}(\theta,\phi)\mathrm{d}\Omega = \delta_{n_1,n_2}\delta_{m_1,m_2}\delta_{i,j} \tag{38}$$

where $\mathrm{d}\Omega = \sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta$ is the solid angle element, which measures the element of area on a sphere of radius 1.

Proof. Let N_{n_1,m_1}, N_{n_2,m_2} be the normalization factors of the spherical harmonics Y_{n_1,m_1}, Y_{n_2,m_2} respectively. Note that we can separate the variables in the integral of Eq. (38). So if $i \neq j$, the integral over θ becomes $\int_0^{2\pi}\cos(m_1\theta)\sin(m_2\theta)\,\mathrm{d}\theta$ which is equal to 0 regardless of the values of m_1 and m_2. So from now on assume that $i = j$. Due to the symmetry between the cosine and the sine we can suppose that $i = \mathrm{c}$. Thus:

$$\int_0^{2\pi}\int_0^{\pi} Y^i_{n_1,m_1}(\theta,\phi)Y^j_{n_2,m_2}(\theta,\phi)\mathrm{d}\Omega =$$

$$= N_{n_1,m_1}N_{n_2,m_2}\int_0^{\pi} P_{n_1,m_1}(\cos\phi)P_{n_2,m_2}(\cos\phi)\sin\phi\mathrm{d}\phi\int_0^{2\pi}\cos(m_1\theta)\cos(m_2\theta)\,\mathrm{d}\theta \tag{39}$$

An easy check shows that if $m_1 \neq m_2$ then the integral over θ is zero (and the same applies with sines). So suppose $m_1 = m_2 = m$. In that case, if $m \neq 0$ we have $\int_0^{2\pi}(\cos m\theta)^2\,\mathrm{d}\theta = \int_0^{2\pi}(\sin m\theta)^2\,\mathrm{d}\theta = \pi$ and if $m = 0$, the cosine integral evaluates to 2π whereas the sine integral is 0. We can omit this latter case because $Y^{\mathrm{s}}_{n,0}$ is identically zero. Thus:

$$\frac{2\pi}{2-\delta_{0,m}}N_{n_1,m}N_{n_2,m}\int_0^{\pi} P_{n_1,m}(\cos\phi)P_{n_2,m}(\cos\phi)\sin\phi\mathrm{d}\phi = \frac{2\pi}{2-\delta_{0,m}}N_{n_1,m}N_{n_2,m}\int_{-1}^{1} P_{n_1,m}(x)P_{n_2,m}(x)\,\mathrm{d}x \tag{40}$$

By Lemma 15 this latter integral is $\frac{2}{2n_1+1}\frac{(n_1+m)!}{(n_1-m)!}\delta_{n_1,n_2}$. Finally, if $n_1 = n_2 = n$, putting all normalization factors together we get:

$$\frac{2\pi}{2-\delta_{0,m}}N^m_n N^m_n\frac{2}{2n+1}\frac{(n+m)!}{(n-m)!} = 4\pi \tag{41}$$

□

Moreover, an important result in the Sturm-Liouville Theory of second order differential equations ([Wikd; Wan+09]) says that the family of spherical harmonics $\{Y^{\mathrm{c}}_{n,m}(\theta,\phi), Y^{\mathrm{s}}_{n,m}(\theta,\phi) : n \in \mathbb{N}\cup\{0\}, m \leq n\}$ form a complete set in the sense that any smooth function defined on the sphere $f : S^2 \to \mathbb{R}$ can be expanded in a series of spherical harmonics:

$$f(\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}(c_{n,m}Y^{\mathrm{c}}_{n,m}(\theta,\phi) + s_{n,m}Y^{\mathrm{s}}_{n,m}(\theta,\phi)) \tag{42}$$

This will be useful in Section 4.1.3 when expanding the gravitational potential created by the Earth at some arbitrary point in spherical harmonics.

3 Introduction to astrophysics and satellite tracking

3.1 The two body problem

3.1.1 Trajectory equation

We are interested in understanding the dynamics of a spacecraft in orbit around the Earth. These dynamics are governed by Newton's second law of motion, which assuming that both the Earth and the spacecraft are point masses (see Section 4 for a more realistic model), this can be written as

$$\ddot{\mathbf{r}} = -\frac{GM_{\oplus}}{r^2}\mathbf{e}_r \tag{43}$$

where $\mathbf{r}$ is the position vector (also called *radius vector*) of the spacecraft with respect to the center of the Earth, $r := \|\mathbf{r}\|$, $\mathbf{e}_r = \frac{\mathbf{r}}{r}$ is the unit vector in the direction of $\mathbf{r}$, $M_{\oplus} \simeq 5.972 \times 10^{24}$ kg is the mass of the Earth, and $G \simeq 6.674 \times 10^{-11}\ \mathrm{m}^3 \cdot \mathrm{kg}^{-1} \cdot \mathrm{s}^{-2}$ is the universal gravitational constant. Note that the minus sign is due to the fact that the gravitational force is attractive, i.e. pointing towards the Earth. Here and along the document the notation $\ddot{\mathbf{r}}$ means that the derivative is taken with respect to time. Cross-multiplying Eq. (43) by $\mathbf{r}$, we obtain

$$\frac{\mathrm{d}(\mathbf{r} \times \dot{\mathbf{r}})}{\mathrm{d}t} = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = -\frac{GM_{\oplus}}{r^3}(\mathbf{r} \times \mathbf{r}) = 0 \tag{44}$$

Hence $\mathbf{r} \times \dot{\mathbf{r}} =: \mathbf{h}$ is constant. The physical intuition behind this is that the motion of the spacecraft around the Earth is confined to a plane, called the *orbital plane*, because the position $\mathbf{r}$ and velocity $\dot{\mathbf{r}}$ are always perpendicular to $\mathbf{h}$.

We are interested now in what kind of curves may be described by a spacecraft orbiting the Earth, when considered both objects as point masses. That is, we want somehow isolate $\mathbf{r}$ (or r) from Eq. (43). In order to simplify the notation we will denote $\mu := GM_{\oplus}$.

Proposition 17 (Kepler's first law). Consider to point-mass bodies. The motion of one body orbiting another can be described by a conic section. Hence, it can be expressed in the form:

$$r(t) = \frac{p}{1 + e\cos(\nu(t))} \tag{45}$$

for some parameters p and e.

Proof. Cross-multiplying Eq. (43) by $\mathbf{h}$ we obtain

$$\frac{\mathrm{d}(\dot{\mathbf{r}} \times \mathbf{h})}{\mathrm{d}t} = \ddot{\mathbf{r}} \times \mathbf{h} = -\frac{\mu}{r^3}\mathbf{r} \times \mathbf{h} = -\frac{\mu}{r^3}\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}}) = \frac{\mu}{r^3}[(\mathbf{r}\cdot\mathbf{r})\dot{\mathbf{r}} - (\mathbf{r}\cdot\dot{\mathbf{r}})\mathbf{r}] \tag{46}$$

where in the last equality we have used Proposition 1. Now note that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right) = \frac{\dot{\mathbf{r}}}{r} - \frac{\dot{r}}{r^2}\mathbf{r} = \frac{1}{r^3}[(\mathbf{r}\cdot\mathbf{r})\dot{\mathbf{r}} - (\mathbf{r}\cdot\dot{\mathbf{r}})\mathbf{r}] \tag{47}$$

because $2r\dot{r} = \frac{\mathrm{d}(r^2)}{\mathrm{d}t} = \frac{\mathrm{d}(\mathbf{r}\cdot\mathbf{r})}{\mathrm{d}t} = 2\mathbf{r}\cdot\dot{\mathbf{r}}$[2]. Thus:

$$\frac{\mathrm{d}(\dot{\mathbf{r}} \times \mathbf{h})}{\mathrm{d}t} = \mu\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right) \tag{48}$$

Integrating with respect to the time yields

$$\dot{\mathbf{r}} \times \mathbf{h} = \frac{\mu}{r}\mathbf{r} + \mathbf{B} \tag{49}$$

where $\mathbf{B} \in \mathbb{R}^3$ is the constant of integration. Observe that since $\dot{\mathbf{r}} \times \mathbf{h}$ is perpendicular to $\mathbf{h}$, it lies on the orbital plane and so does $\mathbf{r}$. Hence, $\mathbf{B}$ lies on the orbital plane. Now dot-multiplying this last equation by $\mathbf{r}$ and using that $\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v})\cdot\mathbf{w}\ \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ we obtain

$$h^2 = \mathbf{h}\cdot\mathbf{h} = (\mathbf{r} \times \dot{\mathbf{r}})\cdot\mathbf{h} = \mathbf{r}\cdot(\dot{\mathbf{r}} \times \mathbf{h}) = \frac{\mu}{r}\mathbf{r}\cdot\mathbf{r} + \mathbf{r}\cdot\mathbf{B} = \mu r + rB\cos\nu \tag{50}$$

---

[2] Bear in mind that in general $\dot{r} \neq \|\dot{\mathbf{r}}\|$. Indeed, if β denotes the angle between $\mathbf{r}$ and $\dot{\mathbf{r}}$ we have that $\dot{r} = \|\dot{\mathbf{r}}\|\cos\beta$. In particular $\dot{r}$ may be negative.

where $h := \|\mathbf{h}\|$, $B := \|\mathbf{B}\|$ and ν denotes the angle between $\mathbf{r}$ and $\mathbf{B}$. Rearranging the terms we obtain finally the equation of a conic section

$$r = \frac{h^2/\mu}{1 + (B/\mu)\cos(\nu)} \tag{51}$$

with $p := h^2/\mu$ and $e := B/\mu$. ☐

We've seen in Section 2.2.2 the range of values that can r take and we deduced an equation for the semi-major axis. Note that this latter quantity can also be expressed as:

$$a = \frac{r_{\max} + r_{\min}}{2} = \frac{p}{1 - e^2} = \frac{h^2}{\mu(1 - e^2)} \tag{52}$$

Finally the angle ν i astrophysics is called *true anomaly*. Note that at $\mathbf{r}_{\min}$, we have $\nu = 0$ and so $\mathbf{r} \parallel \mathbf{B}$. Hence $\mathbf{B}$ points towards the periapsis of the orbit.

Definition 18. Let $\mathbf{r}(t)$ be the position of the spacecraft at time t and $A(t)$ be the area swept by the radius vector $\mathbf{r}(t)$ in the time interval $[0, t]$. We define the *areal velocity* as $\frac{\mathrm{d}A(t)}{\mathrm{d}t}$.

Proposition 19 (Kepler's second law). The areal velocity remains constant.

Proof. Recall that the area of a parallelogram generated by two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ is given by $\|\mathbf{u} \times \mathbf{v}\|$. Thus, approximating the area A by half of the parallelogram generated by $\mathbf{r}(t)$ and $\mathbf{r}(t + k)$ (see Fig. 7) we obtain

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = \lim_{k\to 0} \frac{A(t+k) - A(t)}{k} = \lim_{k\to 0} \frac{\|\mathbf{r}(t) \times \mathbf{r}(t+k)\|}{2k} = \lim_{h\to 0} \frac{\|\mathbf{r}(t) \times (\mathbf{r}(t+k) - \mathbf{r}(t))\|}{2k} =$$

$$= \frac{\|\mathbf{r}(t) \times \dot{\mathbf{r}}(t)\|}{2} = \frac{h}{2} \tag{53}$$

where the penultimate equality is because the cross product is continuous and linear. ☐

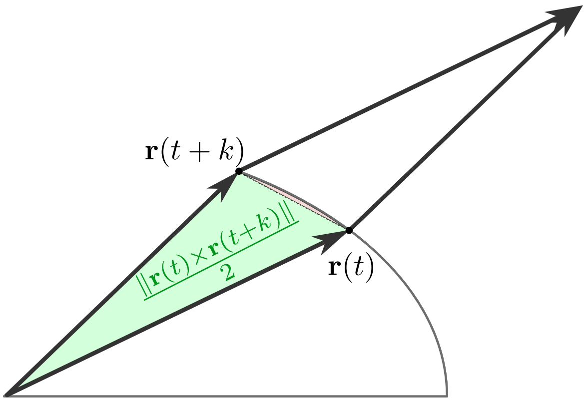


Figure 7: Graphic representation of the error committed (red region) when approximating the area swept by half the area of the parallelogram generated by $\mathbf{r}(t)$ and $\mathbf{r}(t + k)$ (green region).

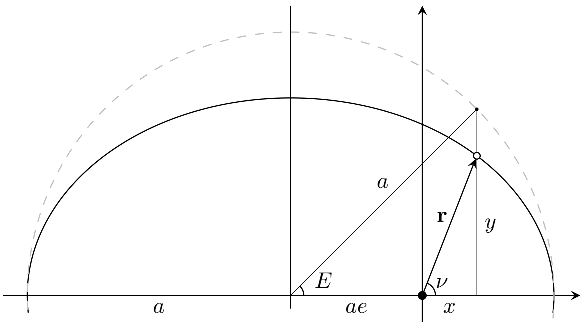


Figure 8: Ellipse orbit of the satellite together with an auxiliary circle of radius a needed to define the eccentric anomaly.

Definition 20. Let T be the orbital period of the satellite. We define the *mean motion* as $n := 2\pi/T$.

Proposition 21 (Kepler's third law). The mean motion is related to the semi-major axis by

$$n = \sqrt{\frac{\mu}{a^3}} \tag{54}$$

Proof. Integrating Eq. (53) with respect to time between and T (the period) yields:

$$\pi ab = \int_0^T A'(t)\,\mathrm{d}t = \int_0^T \frac{h}{2}\,\mathrm{d}t = \frac{hT}{2} \implies n = \frac{2\pi}{T} = \frac{h}{ab} = \frac{h}{a^2\sqrt{1 - e^2}} = \sqrt{\frac{\mu}{a^3}} \tag{55}$$

where we have used Eqs. (9) and (52). ☐

Finally we will need the following equation which relates the velocity of the satellite with the distance to the central of the Earth.

Proposition 22. We have that ([MG05]):

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \tag{56}$$

where $v := \|\dot{\mathbf{r}}\|$.

Proof. Using Eq. (49) we have that:

$$\|\mathbf{h}\times\dot{\mathbf{r}}\|^2 = \frac{\mu^2}{r^2}\mathbf{r}\cdot\mathbf{r} + 2\frac{\mu}{r}\mathbf{r}\cdot\mathbf{B} + \mathbf{B}\cdot\mathbf{B} = \mu^2(1+2e\cos\nu+e^2) = \mu^2(2(1+e\cos\nu)-(1-e^2)) \tag{57}$$

where we have used that $e\mu = B$ (see Eq. (51)). Now using Eqs. (51) and (52) we obtain that

$$2(1+e\cos\nu)-(1-e^2) = 2\frac{p}{r} - \frac{h^2}{\mu a} = 2\frac{h^2}{r\mu} - \frac{h^2}{\mu a} \tag{58}$$

Since $\mathbf{h}\perp\dot{\mathbf{r}}$, $\|\mathbf{h}\times\dot{\mathbf{r}}\| = hv$ and so:

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \tag{59}$$

□

From now one we will suppose that the orbits are ellipses, which is the main case of interest.

3.1.2 Kepler's equation

So far we have been able to describe the geometry of motion of a body orbiting another one. However, we have not concerned about the specific position of the body as a function of time. That is how to obtain $\nu(t)$ at each instant of time. In order to do this, we may think the area A as a function of ν, that measures the area swept by the radio vector from an initial instant ν_0. Thus, from differential calculus we know that:

$$A(\nu) = \int_{\nu_0}^{\nu}\int_{0}^{r(\theta)} r\,\mathrm{d}r\,\mathrm{d}\theta = \int_{\nu_0}^{\nu}\frac{r(\theta)^2}{2}\,\mathrm{d}\theta \implies \frac{\mathrm{d}A}{\mathrm{d}\nu} = \frac{r^2}{2} \tag{60}$$

And using the chain rule and Eq. (53) we obtain that:

$$\frac{h}{2} = \frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}\nu}\frac{\mathrm{d}\nu}{\mathrm{d}t} = \frac{r^2}{2}\dot{\nu} \tag{61}$$

So from Eqs. (51) and (61) we get the following differential equation that must satisfy ν:

$$\dot{\nu} = \frac{h}{r^2} = \frac{h}{p^2}(1+e\cos\nu)^2 \tag{62}$$

which, when integrated with respect to the time, lead us to an elliptic integral which can be very computationally expensive. Our goal in this section is to find an easier way to compute exact position of the satellite at each instant of time [MG05]. This will lead us to the so-called *Kepler's equation*. For this purpose we are forced to introduce a new parameter, E, called *eccentric anomaly*. It is defined as the angle between the segment from the origin of the ellipse and the periapsis, and the line passing through the center of the ellipse and the point a circle (of radius a and same center as the ellipse) which is just above the position of the satellite (see Fig. 8 for a better understanding).

Clearly, using the reference frame of Fig. 8, the position of the satellite is determined by $x = r\cos\nu$, $y = r\sin\nu$. But we would like to find an expression of x and y in terms of E rather than ν. To do this note that $a\cos E = ae + x$, so:

$$x = a(\cos E - e) \tag{63}$$

We can also get an expression of r in terms of E by solving the equation:

$$r = \frac{p}{1+e\cos\nu} = \frac{a(1-e^2)}{1+e\frac{x}{r}} = \frac{ra(1-e^2)}{r+ae(\cos E - e)} \implies r = a(1-e\cos E) \tag{64}$$

Finally from Eqs. (63) and (64) we get:

$$y^2 = r^2 - x^2 = a^2(1-e^2)(\sin E)^2 \implies y = a\sqrt{1-e^2}\sin E \tag{65}$$

Expressing now the areal velocity h as a function of E we have:

$$\begin{aligned} h &= x\dot{y} - y\dot{x} && (66) \\ &= a^2(\cos E - e)\sqrt{1-e^2}(\cos E)\dot{E} + a^2(\sin E)^2\dot{E}\sqrt{1-e^2} && (67) \\ &= a^2\sqrt{1-e^2}\dot{E}(1-e\cos E) && (68) \end{aligned}$$

From Eq. (52) we know that $h = \sqrt{\mu a(1-e^2)}$. Thus substituting this in the latter equation we deduce that E must satisfy the following differential equation:

$$\dot{E}(1-e\cos E) = \sqrt{\frac{\mu}{a^3}} = n \tag{69}$$

where the last equality follows from Proposition 21. Integrating this equation with respect to time yields the *Kepler's equation*:

$$E(t) - e\sin E(t) = n(t-t_0) \tag{70}$$

where t_0 is the time at which E vanishes. Using the reference frame of Fig. 8 (the Perifocal frame, see Definition 34) this corresponds at the time at which the satellite is at the perigee. The value $M := n(t-t_0)$ is called *mean anomaly*. Note that contrarily to E and ν, the mean anomaly increases linearly with time.

Kepler's equation is the key to solve the problem of finding the position of the satellite at each instant of time. Later on we will discuss techniques to solve this equation for E knowing e and M.

3.2 Time and reference systems

3.2.1 Time measurement

As human beings, we are naturally interested in how time passes and therefore the correct measure of it becomes an essential necessity for us. As it is the Sun that governs our daily activity, it is natural to define time from it. But first we need some preliminary definitions:

Definition 23. We define the *equatorial plane* as the plane in $\mathbb{R}^3$ that contains the Earth equator. We define the *ecliptic plane* as the orbital plane in $\mathbb{R}^3$ of the Earth around the Sun.

Definition 24. We define the *celestial sphere* as an abstract sphere of infinite radius centered at the center of mass of the Earth. All the celestial objects are thus, projected naturally on the celestial sphere, identifying them with two coordinates (longitude and latitude). The intersection of the equatorial plane with the celestial sphere is called *celestial equator*. The intersection of the ecliptic plane with the celestial sphere is called *ecliptic* (see Fig. 14 for a better understanding).

A first important thing to note is that, since the celestial sphere is centered at the Earth, the Sun moves along the ecliptic. Moreover, note that both the celestial equator and the ecliptic are two different great circles on the celestial sphere. Hence, they intersect at exactly two points.

Definition 25. Consider the two points of intersection between the celestial equator and the ecliptic. We define the *vernal equinox* as the point Υ, from these two, such that the Sun crosses the celestial equator from south to north.

The angle measured along the equator of any object on the celestial sphere from the vernal equinox is called *right ascension*, whereas the angle measured along the meridian of the object from the position of the object to the equator is called *declination* (see Fig. 9).

An *apparent solar day* is defined to be the time between two successive transits of the Sun across our local meridian. One should note that the Earth has to rotate on itself slightly more than one revolution in order to complete one solar day. The *apparent siderial day* is defined as the time it takes to the Earth to complete a rotation relative to very far away stars (see Fig. 10 for a better understanding). From the point of view of the celestial sphere, the aparent solar time is the angle (measured along the celestial equator) between the local meridian and the meridian of the Sun at that epoch, which is not constant because the Sun's right ascension increases about 1 degree per day [MG05].

The non-circular orbit of the Earth around the Sun causes some days to be shorter than others due to Kepler's second law. Thus, the real Sun is not well suited for precise time measurement. So the introduction of a *mean Sun* is necessary.

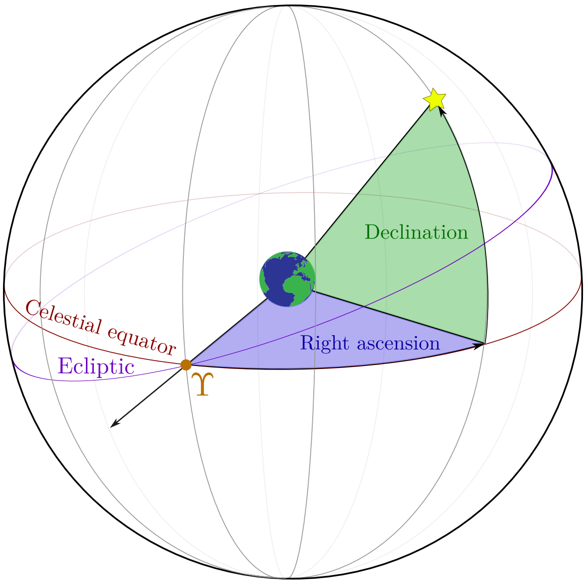


Figure 9: Right ascension and declination of a star in the celestial sphere

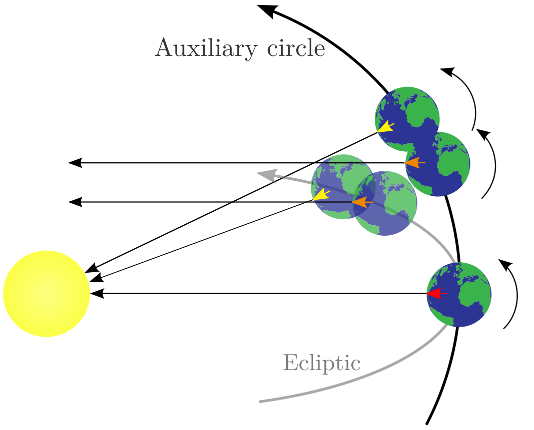


Figure 10: Solar and siderial days (not to scale)

Definition 26. The *mean Sun* is a fictitious Sun that moves along the celestial equator at a constant rate. This rate is determined such that the real Sun and the mean Sun coincide at the vernal equinox. We define the *mean solar day* as the hour angle (along the celestial equator) between the local meridian and the meridian of the mean Sun.

It is worth-noting that the mean Sun does not move around the ecliptic, but rather along the celestial equator.

Definition 27. We define the *prime meridian* or *zero meridian* as the meridian on the celestial sphere that passes through the Royal Observatory in Greenwich, England (when projected to the Earth).

Definition 28. The *Greenwich Mean Time* (GMT) or *Universal Time* (UT) is the hour angle of the mean solar day measured from the prime meridian and counted from midnight. That is, when the prime meridian and mean Sun meridians coincide, the GMT is 12:00.

The usa of two distinct names, namely GMT and UT, to refer to the same time can be attributed to historical reasons. Initially, GMT was defined as the mean solar time at the prime meridian with 00:00 GMT coinciding with the moment when the mean Sun was at the that meridian. On the other hand, UT was introduced as a 12-hour translation of GMT, intended for civilian purposes. Eventually, GMT was redefined to align with UT.

In the middle of the 20th century, *Ephemerides Time* (ET) was introduced to cope with the irregularities of the Earth's rotation (see POLAR MOTION). This time was defined from historical observations of planets in a Newtonian physics framework, isolating the time from the equations, and the origin was chosen accordingly to the GMT at January 1900. This time provided a uniform time, although it was more difficult to measure than the mean solar time. In the meantime, atomic clocks were invented and soon the *atomic time* (TAI, from French *Temps Athomique International*) was adopted as the SI unit of second. The origin was chosen such that the TAI matched UT at the 00:00:00 UT of January 1st, 1958, and at that time the ET was displaced from UT by 32.184 seconds. At the end of the century, the *Terrestrial Time* (TT) was introduced within a relativistic framework in order to succeed ET and provided a smooth and more accurate continuation of it yielding the relation ([MG05])

$$\mathrm{TT} = \mathrm{ET} = \mathrm{TAI} + 32.184\,\mathrm{s} \tag{71}$$

A representation of the siderial time is the *Greenwich Mean Siderial Time* (GMST) which is defined as the angle between the prime meridian and the mean vernal equinox of date (see Section 3.2.2). Due to unpredictable irregular changes on the rotation of the Earth (see POLAR motion), the GMST cannot be computed directly with a formula in terms of the TAI or TT.

The *Universal Time 1* (UT1) is the presently used form of Universal time and it is defined with the following deterministic formula given in [Aok+81]. For each day, the 00:00 UT1 is defined when the GMST has the value:

$$\mathrm{GMST}(0\mathrm{h\ UT1}) = 24110.54841 + 8640184.812866 T_{\mathrm{UT1},0} + 0.093104 {T_{\mathrm{UT1},0}}^2 - 6.2 \cdot 10^{-6} {T_{\mathrm{UT1},0}}^3 \tag{72}$$

where $T_{\mathrm{UT1},0} = \frac{\mathrm{JD}(0\mathrm{h\ UT1}) - 2451545}{36525}$ denotes the number of Julian centuries that have passed since January 2000, 1.5 UT1 at the beginning of the day. For any instant of time during the day, the following formula is used:

$$\begin{aligned}\mathrm{GMST}(\mathrm{UT1}) = 24110.54841'' + 8640184.812866'' T_{\mathrm{UT1},0} + 1.002737909350795\mathrm{UT1} + \\ + 0.093104'' {T_{\mathrm{UT1}}}^2 - 6.2 \cdot 10^{-6}{}'' {T_{\mathrm{UT1}}}^3\end{aligned} \tag{73}$$

where $T_{\mathrm{UT1}} = \frac{\mathrm{JD}(\mathrm{UT1}) - 2451545}{36525}$ and UT1 is measured in seconds. Similarly to the GMST, there is no simple conversion between the UT1 and the TT or TAI. Instead, the IERS (*International Earth Rotation and Reference Systems Service*) provides regularly a bulletin with the difference $\Delta T := \mathrm{TT} - \mathrm{UT1}$ at several dates. Interpolating these values we can obtain UT1 from TT at any epoch.

Finally, our everyday clock is based on the *Coordinated Universal Time* (UTC). It is defined to be as uniform as the TAI but always kept closer than 0.9 seconds to the UT1 in order to resemble a mean solar time (see Fig. 11). Scientists achieve this by introducing a *leap second* (see Fig. 12), which is an extra second added to UTC at irregular intervals. Fig. 11 summarizes all the time systems introduced in the document.

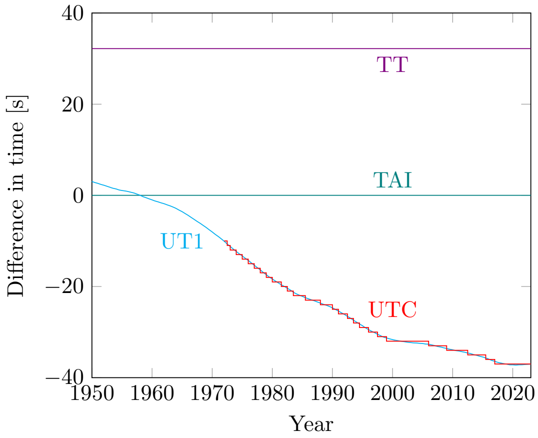


Figure 11: Evolution of times TT, UT1 and UTC in comparison with TAI. [Obs]

```
1998 December 31,      23h 59m 59s
1998 December 31,      23h 59m 60s
1999 January   1,      00h 00m 00s
1999 January   1,      00h 00m 01s
```

Figure 12: Leap second introduced to the UTC time at the end of the December 1998. [RS98]

We summarize here useful conversions between time systems:

$$\begin{aligned}\mathrm{GMST} &= 24110.54841 + 8640184.812866 T_{\mathrm{UT1},0} + 0.093104 (T_{\mathrm{UT1}})^2 - 6.2 \cdot 10^{-6} (T_{\mathrm{UT1}})^3 \\ \mathrm{UT1} &= \mathrm{TT} - \Delta T \\ \mathrm{TT} &= \mathrm{TAI} - 32.184 \\ \mathrm{TAI} &= \mathrm{UTC} + \delta(\mathrm{TAI})\end{aligned}$$

where ΔT is the difference between the TT and UT1, and $\delta(\mathrm{TAI})$ is a piecewise constant function that counts the number of leap seconds introduced since 1972, when they were introduced for the first time. All the numbers have units of second. Note that since the rotation of the Earth cannot be predicted accurately, ΔT can only be determined retrospectively, and is given by the *International Earth Rotation and Reference Systems Service* (IERS).

3.2.2 Reference systems

Newton's second law in only valid when applied to an *inertial reference frame*, that is, a frame of reference that is not undergoing any acceleration. In practise, however, almost any frame of reference is inertial. So in this chapter we will describe an almost-inertial frame of reference which will be used to integrate

Newton's second law. On the other hand, since the Earth is not a body with a homogeneous density of mass, there are zones with higher mass density than others, and therefore with higher gravitational pull (see Section 4.1.3). Therefore we will need the longitude and latitude of the satellite with respect to the Earth at each time of integration.

The first reference frame we must consider is the *celestial* one. On basis on the study of the satellite motion around the Earth, it is natural to locate all the origins of the reference frames considered along the document at the center of mass of the Earth.

In the celestial frame, the x-axis is defined as the line ℓ of intersection between the equatorial plane and the ecliptic plane. The positive direction is chosen to point towards the vernal equinox. The z-axis is chosen to be perpendicular to the equatorial plane and the y-axis is such that the triplet (x, y, z) is a right-handed system.

However, due to the presence of other solar system planets (and other smaller perturbations), the orbital plane of the Earth is not fixed in space, but is subjected to a small variation called *planetary precession*. Moreover, the gravitational attraction of the Sun and Moon on the Earth's equator cause Earth's axis of rotation to precess in a similar way to the action of a spinning top with a period of about 26000 years [MG05]. This motion is called *lunisolar precession*. On the other hand, smaller pertubations in amplitude ($<$ 18.6 years [Wikb]) with shorter period superposed with the precessional motion creates a motion called *nutation*. When this latter oscillations are averaged out, the vernal equinox, the ecliptic, and the equator are referred to *mean* values, rather than *true* values.

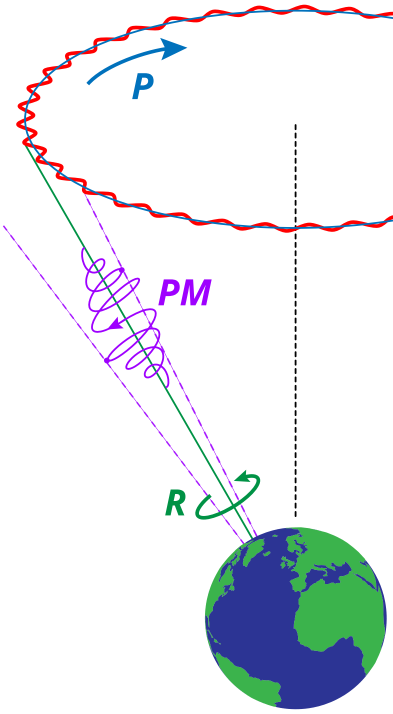


Figure 13: Graphical explanation of the perturbation by precession (blue), nutation (red) and polar motion (violet) of the Earth's axis of rotation (green).

In view of this time-dependent orientation of both the ecliptic and the equator, the standard-reference frame chosen is based on the mean equator, mean ecliptic and mean equinox of some fixed time, the beginning of the year 2000, namely at 12:00 TT on 1 January 2000, the so-called *J2000 epoch*.

Definition 29 (Earth-centered inertial frame). We define the *Earth-centered inertial* (ECI) reference frame or *J2000 reference frame* as the frame of reference whose x-axis is the intersection of the mean celestial equator and the mean ecliptic pointing at the mean vernal equinox, the z-axis is perpendicular to the mean equator and the y-axis is chosen such that the triplet (x, y, z) is a right-handed system. All these quantities are the one of the J2000 epoch.

Let's move on now to study an Earth-fixed reference frame.

Definition 30 (Earth-centered, Earth-fixed frame). We define the *Earth-centered, Earth-fixed frame of reference* (ECEF) as the frame of reference whose x-axis is pointing to the prime meridian, the z-axis is perpendicular to the Earth equator and the y-axis is chosen such that the triplet (x, y, z) is a right-handed system.

These two coordinate systems have, as mentioned earlier, the origin of this at the center of mass of the Earth.

3.2.3 Conversion between reference systems

As we noted in the previous section the angle ε between the celestial equator and ecliptic planes is not constant due to the planetary precession.

Our goal in this section is to transform the position of the satellite from the ECI system to the ECEF system and vice versa. This rotation transformation is given by a product of 4 rotations:

- The precession matrix $\mathbf{P}$,
- the nutation matrix $\mathbf{N}$,
- the Earth rotation matrix $\mathbf{\Theta}$, and
- the polar motion matrix $\mathbf{\Pi}$.

These matrix are such that:

$$\mathbf{r}_{\text{ECEF}}(t) = \mathbf{\Pi}(t)\mathbf{\Theta}(t)\mathbf{N}(t)\mathbf{P}(t)\mathbf{r}_{\text{ECI}}(t) \tag{74}$$

where $\mathbf{r}_{\text{ECEF}}(t)$ is the position vector of the satellite in the ECEF frame at time t and $\mathbf{r}_{\text{ECI}}(t)$ is the position vector of the satellite in the ECI frame at time t. From here on, we will omit the evaluation on the time t for the sake of simplify the lecture. Let's now argue why the transformation has this particular form.

The precession matrix is responsible for *eliminating* all the movement due to the planetary and lunisolar precession. Thus, $\mathbf{P}$ transforms the mean equator and mean equinox at time J2000 to the respective values at time t. With the help of Fig. 14 it's not hard to see that this transformation is given by:

$$\mathbf{P} = \mathbf{R}_z(-90 - z)\mathbf{R}_x(\theta)\mathbf{R}_z(90 - \zeta) \tag{75}$$

And with a bit of algebra it can be simplified to:

$$\mathbf{P} = \mathbf{R}_z(-z)\mathbf{R}_y(\theta)\mathbf{R}_z(-\zeta) \tag{76}$$

Recall that the fundamental rotation matrices $\mathbf{R}_x(\theta)$, $\mathbf{R}_y(\theta)$ and $\mathbf{R}_z(\theta)$ are with respect to the axis of the J2000 frame and they are given by:

$$\mathbf{R}_x(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix} \quad \mathbf{R}_y(\varphi) = \begin{pmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{pmatrix} \quad \mathbf{R}_z(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{77}$$

where we have used the convention of signs given by [GPS02]. The reader may wonder why we have used the notation $90 - z$ and $90 - \zeta$ instead of z and ζ (for example) for the angles in question. The reason is related to the precise definition of these angles from the pole of the celestial sphere rather than from where we have defined them, but we will not elaborate on this point here. Nonetheless, we have chosen this notation to maintain consistency with related articles.

The nutation perturbations are driven out by the nutation matrix $\mathbf{N}$. This matrix transforms the coordinates of the mean equator and equinox at epoch t to those of the true equator and equinox at the same epoch, respectively. Hence, from figure Fig. 15 we can see that the nutation matrix is given by:

$$\mathbf{N} = \mathbf{R}_x(-\varepsilon - \Delta\varepsilon)\mathbf{R}_z(-\Delta\psi)\mathbf{R}_x(\varepsilon) \tag{78}$$

In Section 3.2.1 we defined the GMST as the hour angle between the mean vernal equinox and the prime meridian. In a similar way, we define the *Greenwich Apparent Siderial Time* (GAST) as the hour angle between the true vernal equinox and the prime meridian.

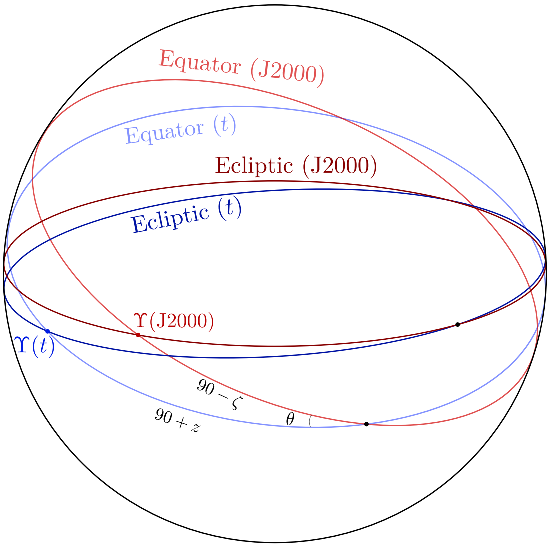


Figure 14: Celestial sphere showing the ecliptic and the equator of both the epoch J2000 and the current epoch t. Dark colors represent the ecliptic while light colors represent the equator. On the other hand, red colors represents the the J2000 epoch and blue colors represents the current epoch t.

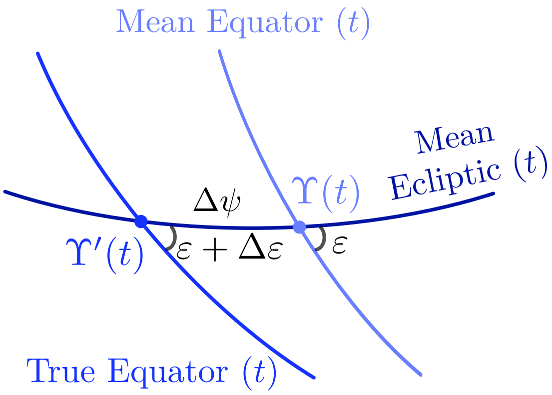


Figure 15: True equator and mean equator, and true equinox (Υ') and mean equinox (Υ) at a given epoch t together with the ecliptic at that time.

3.3 Keplerian orbital elements

3.3.1 Keplerian orbital elements from position and velocity

Definition 31. Consider a satellite orbiting the Earth. The *orbital plane* is the plane that contains the orbit of the satellite. The *line of nodes* is the line of intersection between the orbital plane and the equatorial plane. Finally, the *ascending node* is the point on the line of nodes and the orbit of the satellite where the satellite crosses the equatorial plane from south to north.

Definition 32 (Orbital elements). The *Keplerian orbital elements* of a satellite are five independent quantities that completely determine its orbit. If moreover the exact position of the satellite on the orbit is wanted, a sixth quantity is needed. The first five orbital elements are:

1. The *semi-major axis* a of the orbit.
2. The *eccentricity* e of the orbit.
3. The *inclination* i is the angle between the equatorial plane and the orbital plane.
4. The *longitude of the ascending node* Ω is the angle between the vernal equinox and the ascending node.
5. The *argument of perigee* ω is the angle between the ascending node and the periapsis.

The sixth quantity is the *true anomaly* ν which is the angle between the periapsis and the position of the satellite on the orbit.

The elements a, e and i are always well-defined. However, the elements Ω, ω and ν are not well-defined in general, namely for $e = 0$ or $i = 0$. We discuss this in more detail below. For the moment we suppose $e \neq 0$ and $i \neq 0$. In order to express these elements in terms of the position and velocity of the satellite, we need to introduce the following vectors:

Definition 33. Let $\hat{\mathbf{k}} = (0, 0, 1)$ be the unit vector perpendicular to the equatorial plane. We define the vector $\mathbf{n} := \hat{\mathbf{k}} \times \mathbf{h}$. We define the *eccentricity vector* $\mathbf{e}$ as $\mathbf{e} := \mathbf{B}/\mu$, whose norm is the eccentricity e.

Note that $\mathbf{n} \perp \hat{\mathbf{k}}$ and $\mathbf{n} \perp \mathbf{h}$ which imply that $\mathbf{n}$ must lie on the orbital plane and equatorial plane and therefore in the line of nodes pointing towards the ascending node, by the right-hand rule. On the other hand, since $\mathbf{B}$ points towards the periapsis, so does $\mathbf{e}$. From here and looking at Fig. 16 it's not hard to see that the angles i, Ω, ω and ν are given by:

$$i = \arccos\left(\frac{\mathbf{h} \cdot \hat{\mathbf{k}}}{h}\right) \quad \Omega = \arccos\left(\frac{\mathbf{n} \cdot \hat{\imath}}{n}\right) \quad \omega = \arccos\left(\frac{\mathbf{n} \cdot \mathbf{e}}{ne}\right) \quad \nu = \arccos\left(\frac{\mathbf{e} \cdot \mathbf{r}}{er}\right) \tag{79}$$

Here $\hat{\imath} = (1, 0, 0)$ denotes the basis unit vector that points towards the vernal equinox. By convention this angles are always given in the interval $[0, 2\pi)$, so in these formulas a small correction is needed when the angles are negative. The formulas for a and e were already given in Eqs. (52) and (56):

$$a = \left(\frac{2}{r} - \frac{v^2}{\mu}\right)^{-1} \quad p = \frac{h^2}{\mu} \quad e = \sqrt{1 - \frac{p}{a}} \tag{80}$$

Now we study the singular cases. If $e = 0$ (and therefore $\mathbf{e} = 0$) and $i \neq 0$, the orbit is called *circular inclined* [Val13]. In this case the elements ω, ν are not because there is no periapsis, or in other words, all the points lie at the same distance from the center of the Earth. To correct this we replace these variables by the *argument of latitude* u, which measures the angle between the ascending node and the position of the satellite on the orbit. This can be computed with the formula:

$$u = \arccos\left(\frac{\mathbf{n} \cdot \mathbf{r}}{nr}\right) \tag{81}$$

If $i = 0$ and $e \neq 0$, the orbit lies in the equatorial plane and it is called *elliptical equatorial*. Note that in this situation we have $\mathbf{n} = 0$ and the variables Ω and ω are undefined. By convention, we set $\Omega = 0$ and:

$$\omega = \arccos\left(\frac{\mathbf{e} \cdot \hat{\imath}}{e}\right) \tag{82}$$

If $e = 0$ and $i = 0$, the orbit is called *circular equatorial* and all these three variables are undefined. In this case Ω is set to 0 and the other two variables are replaced with the *true longitude* λ which is the angle between the vernal equinox and the position of the satellite on the orbit[3]:

$$\lambda = \arccos\left(\frac{\hat{\imath} \cdot \mathbf{r}}{r}\right) \tag{83}$$

3.3.2 TLE sets

The positions of satellites are recorded and stored in a particular way, called *Two Line Element* sets (TLEs). It is base on a two-line text with different data that facilitate the computation of position and velocity of the satellite at that specific instant of time. The following table summarizes all the information on it: Let's clarify the meaning of each data block. The classification (Class) of the satellite is divided into unclassified (U), classified (C) and secret (S). The international designator is splitted into the the launch year (Year), the launch number of the year (Launch num) and the piece of the launch (Piece). In the epoch year, the last to digits of it are specified and the day of the year is supposed to start from 0. The ephemerides type (Eph) is the orbital model used to generate the data [Kel; Wikc]. The element set number is incremented by one when a new TLE is generated for this satellite. In the second line, the number of revolutions at the time (Rev num at epoch) of the TLE are the number of revolutions around the Earth since it was launched. Finally the checksum (modulo 10) is used to verify the integrity of the data[4].

3.3.3 Position and velocity in terms of the TLEs' orbital elements

We are now interested in proceed the other way around, that is, given the orbital elements or more precisely the TLE data, we want to compute the position and velocity of the satellite. In order to do that, we need to introduce the basis $(\mathbf{P}, \mathbf{q}, \mathbf{W})$ linked to the orbit.

---

[3] In general the true longitude is defined as the angle between the ascending node and the position of the satellite on the orbit, that is $\lambda = \omega + \nu$.

[4] Taking into account that the negative sign is counted as 1, and all the other cells without a number as 0.

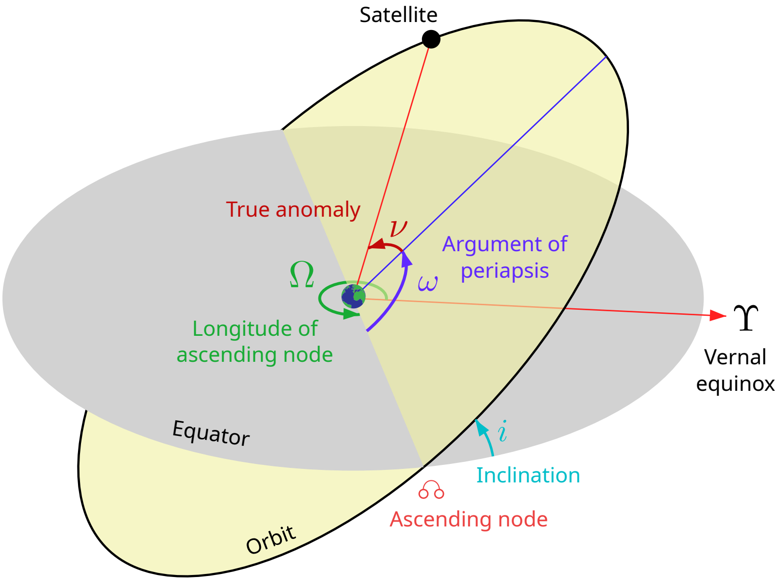


Figure 16: Orbital elements of a satellite.

Line		Satellite number	Class		International designator				Epoch (UTC)			$\dot{M}/2$ [rev/day]		$\ddot{M}/6$ [rev/day^2]		B^* (drag term)		Model		TLE number	Check
					Year	Launch	Piece		Year	Day of the year (as fraction)											
1	2	3–7	8	9	10–11	12–14	15–17	18	19–20	21–32	33	34–43	44	45–52	53	54–61	62	63	64	65–68	69
												S (34)		S. (45)		S. (54), S (60), E (61)					
1		55124	U		98	067	UR		23	086.16778994		.01540407		14102-2		26634-2		0		999	1

Line		Satellite number		Inclination i [deg]		Right ascension Ω [deg]		Eccentricity e		Argument perigee ω [deg]		Mean anomaly M [deg]		Mean motion n [rev/deg]	Number of revolutions	Check
1	2	3–7	8	9–16	17	18–25	26	27–33	34	35–42	43	44–51	52	53–63	64–68	69
								• (27)								
2		55124		51.6237		5.0017		0010757		188.3869		171.6963		16.01878199	1379	2

Table 4: TLE data set from the NUTSAT satellite. The white (empty) cells designate space characters and the green and yellow ones are used to distinguish consecutive data blocks. The cells labeled with S or E respresent cells reserved for the negative sign and the exponent a the number while the red dots denote that an implicit decimal point is assumed.

Definition 34 (Perifocal coordinate system). Consider the orbit of a satellite. We define its associated *perifocal coordinate system* $(\mathbf{P}, \mathbf{Q}, \mathbf{W})$ as follows. The center is on the Earth's center of mass. The unit vectors $\mathbf{P}$ and $\mathbf{Q}$ lie on the orbital plane and are such that $\mathbf{P}$ points towards the periapsis, that is $\mathbf{P} := \mathbf{B}/B$. The unit vector $\mathbf{W}$ is defined as $\mathbf{W} := \mathbf{h}/h$ and $\mathbf{Q} := \mathbf{W} \times \mathbf{P}$.

Recall that in Section 3.1.2 we have seen that the position of the satellite in the perifocal frame is given by:

$$\mathbf{r}_{\text{Peri}} = a(\cos E - e)\mathbf{P} + a\sqrt{1-e^2}\sin E\mathbf{Q} \tag{84}$$

Differentiating yields:

$$\dot{\mathbf{r}}_{\text{Peri}} = -a(\sin E)\dot{E}\mathbf{P} + a\sqrt{1-e^2}(\cos E)\dot{E}\mathbf{Q} \tag{85}$$

where $\dot{E} = \frac{n}{1-e\cos E}$ is given by Eq. (69). These two quantities depend on the two unknown variables: the semi-major axis a and the eccentric anomaly E because n and e are given in the TLE data set. The semi-major axis can be easily obtained from the definition of n in Eq. (69). For the eccentric anomaly, we need to solve the Kepler equation:

$$E - e\sin E = M \tag{86}$$

Lemma 35. Let $e \in [0, 1)$ and $M \in [0, 2\pi)$. Then, the function

$$f(E) = E - e\sin E - M \tag{87}$$

has a unique solution in the interval $[M, M+e]$ if $M \in [0, \pi)$ and in the interval $[M-e, M]$ if $M \in [\pi, 2\pi)$.

Proof. We prove first the uniqueness. Clearly $f \in \mathcal{C}^1(\mathbb{R})$ and $f'(E) = 1 - e\cos E > 0$ for all $E \in [0, 2\pi)$. Thus, f is strictly increasing and so it has at most one zero. Now, if $M \leq \pi$, then $f(M) = -e\sin M \leq 0$ and $f(M+e) = e(1 - \sin(M+e)) \geq 0$. So by Bolzano's theorem, f has a solution in $[M, M+e]$. If $M \geq \pi$, then $f(M) = -e\sin M \geq 0$ and $f(M-e) = -e(1 + \sin(M-e)) \leq 0$. So by Bolzano's theorem, f has a solution in $[M-e, M]$. ☐

We will use the Newton's method to find the zero of this non-linear equation. For small eccentricities e, the natural choice for the initial guess is $E_0 = M$. For large eccentricities ($e > 0.8$) the initial guess is $E_0 = \pi$ should be used in order to avoid convergency problems [MG05]. Once obtained E, the position and velocity of the satellite in the J2000 frame are given by:

$$\mathbf{r}ECI = \mathbf{A}\mathbf{r}_{\text{Peri}} \qquad \dot{\mathbf{r}}ECI = \mathbf{A}\dot{\mathbf{r}}_{\text{Peri}} \tag{88}$$

where $\mathbf{T}$ is the rotation matrix that transform one frame to another and is given by (look at Fig. 16):

$$\mathbf{T} = \mathbf{R}_z(-\Omega)\mathbf{R}_x(-i)\mathbf{R}_z(-\omega) \tag{89}$$

4 Force model

So far we have only considered the gravitational force acting point masses. In reality, the Earth is not a point mass, neither a spherically symmetric mass distribution. In this section we will deep into the details of a more realistic model of the Earth's gravitational field.

4.1 Geopotential model

4.1.1 Continuous distribution of mass

In Section 3.1 we have seen that the motion of a body orbiting another one can be described by a conic section. However, we have not been concerned about the mass distribution of the large body, in our case the Earth. In this section we will see that the motion of the smaller body, the satellite, is slightly perturbed by the mass distribution of the larger one as well as the precense of other forces such as atmospheric drag, solar radiation pressure, and the gravitational pull of the Moon and Sun, which we wil talk later on. Even though, the perturbations are relatively small and the orbits of the satellites are still approximating ellipses.

Consider a body confined in a compact region $\Omega \subseteq \mathbb{R}$ with a continuous density of mass $\rho : \Omega \to \mathbb{R}$. We would like to know the gravitational pull on a point mass m located at position $\mathbf{r}$ from the center of mass of the body. To do this we can discretize the body Ω in a set of cubes $m_{i,j,k}$ each of volume $\frac{1}{n_x n_y n_z}$ and density $\rho(\frac{i}{n_x}, \frac{j}{n_y}, \frac{k}{n_z}) =: \rho_{i,j,k}$, where n_x, n_y, and n_z are the number of cubes in the x, y, and z directions, respectively. The total gravitational acceleration $\mathbf{g}$ exerted on m is the sum of the contributions of all the forces and it is given by:

$$\mathbf{g} = -\sum_{i=0}^{n_x}\sum_{j=0}^{n_y}\sum_{k=0}^{n_z} \frac{m_{i,j,k}}{\|\mathbf{r}-\mathbf{s}_{i,j,k}\|^3}(\mathbf{r}-\mathbf{s}_{i,j,k}) = -\sum_{i=0}^{n_x}\sum_{j=0}^{n_y}\sum_{k=0}^{n_z} \frac{\rho_{i,j,k}}{\|\mathbf{r}-\mathbf{s}_{i,j,k}\|^3}(\mathbf{r}-\mathbf{s}_{i,j,k})\frac{1}{n_x n_y n_z} \tag{90}$$

where $\mathbf{s}_{i,j,k} = (\frac{i}{n_x}, \frac{j}{n_y}, \frac{k}{n_z})$ (in cartesian coordinates). Note that Eq. (90) is a Riemann sum and so letting $n_x, n_y, n_z \to \infty$ we get:

$$\mathbf{g} = -\int_\Omega \frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|^3}(\mathbf{r}-\mathbf{s})\mathrm{d}^3\mathbf{s} \tag{91}$$

where $\mathrm{d}^3\mathbf{s} := \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$, if $\mathbf{s} = (x', y', z')$.

Theorem 36. *Let Ω be a compact region in $\mathbb{R}^3$ with a continuous density of mass $\rho : \Omega \to \mathbb{R}$. Then, the gravitational acceleration field $\mathbf{g}$ is conservative. That is, there exists a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{g} = \boldsymbol{\nabla} f$.*

Proof. An easy computation shows that fixed $\mathbf{s} \in \mathbb{R}^3$ we have:

$$\boldsymbol{\nabla}\left(\frac{1}{\|\mathbf{r}-\mathbf{s}\|}\right) = -\frac{1}{\|\mathbf{r}-\mathbf{s}\|^3}(\mathbf{r}-\mathbf{s}) \tag{92}$$

So we need to justify if the following exchange of the gradient and the integral is correct:

$$\mathbf{g} = -\int_\Omega \frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|^3}(\mathbf{r}-\mathbf{s})\mathrm{d}^3\mathbf{s} = \int_\Omega \rho(\mathbf{s})\boldsymbol{\nabla}\left(\frac{1}{\|\mathbf{r}-\mathbf{s}\|}\right)\mathrm{d}^3\mathbf{s} = \boldsymbol{\nabla}\int_\Omega \frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|}\mathrm{d}^3\mathbf{s} \tag{93}$$

Without loss of generality it suffices to justify that

$$\frac{\partial}{\partial x}\int_\Omega \frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|}\mathrm{d}^3\mathbf{s} = \int_\Omega \frac{\partial}{\partial x}\left(\frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|}\right)\mathrm{d}^3\mathbf{s} \tag{94}$$

assuming $\mathbf{r} = (x, y, z)$ and $\mathbf{s} = (x', y', z')$. In order to apply the theorem of derivation under the integral sign we need to control $\frac{\partial}{\partial x}\left(\frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|}\right) = -\rho(\mathbf{s})\frac{x-x'}{\|\mathbf{r}-\mathbf{s}\|^3}$ by an integrable function $h(\mathbf{s})$. Using spherical coordinates centered at $\mathbf{r}$ and writing $(\mathbf{r}-\mathbf{s})_{\mathrm{sph}} = (\rho_{\mathbf{r}}, \theta, \phi)$, the integrand to bound becomes (in spherical coordinates):

$$\left|-\rho(\mathbf{s})\frac{x-x'}{\|\mathbf{r}-\mathbf{s}\|^3}{\rho_{\mathbf{r}}}^2\sin\phi\right| = |\rho(\mathbf{s})|\left|\frac{\rho_{\mathbf{r}}\cos\theta\sin\phi}{{\rho_{\mathbf{r}}}^3}{\rho_{\mathbf{r}}}^2\sin\phi\right| \leq |\rho(\mathbf{s})| \leq K \tag{95}$$

where the last inequality follows for certain $K \in \mathbb{R}$ by Weierstraß theorem (ρ is continuous and Ω is compact). Thus, since $h(\mathbf{s}) = K$ is integrable, because Ω is bounded, the equality of Eq. (94) is licit. ☐

Physically speaking, the gravitational force being conservative means that the work done by the force is independent of the path taken by the particle. Moreover, due to historical reasons, we will write $\mathbf{g} = -\boldsymbol{\nabla} V$ (with the minus sign) and call V the *gravitational potential*. The minus sign is chosen according the convention that work done by gravitational forces decreases the potential.

4.1.2 Laplace equations

Theorem 37. Consider distribution of matter of density ρ in a compact region Ω. Then, the gravitational potential V satisfies the Laplace equation

$$\Delta V = 0 \tag{96}$$

for all points outside Ω[5].

Proof. Recall that $\Delta V = \mathbf{div}(\boldsymbol{\nabla} V)$. So since $\mathbf{g} = -\boldsymbol{\nabla} V$ it suffices to prove that $\mathbf{div}(\mathbf{g}) = 0$. Note that if $\mathbf{r} \in \Omega^c$ and $\mathbf{s} \in \Omega$ then $\|\mathbf{r}-\mathbf{s}\| \geq \delta > 0$, so $\frac{\mathbf{r}-\mathbf{s}}{\|\mathbf{r}-\mathbf{s}\|^3}$ is differentiable and:

$$\mathbf{div}\left(\frac{\mathbf{r}-\mathbf{s}}{\|\mathbf{r}-\mathbf{s}\|^3}\right) = \frac{\partial}{\partial x}\left(\frac{x-x'}{\|\mathbf{r}-\mathbf{s}\|^3}\right) + \frac{\partial}{\partial y}\left(\frac{y-y'}{\|\mathbf{r}-\mathbf{s}\|^3}\right) + \frac{\partial}{\partial z}\left(\frac{z-z'}{\|\mathbf{r}-\mathbf{s}\|^3}\right) =$$
$$= \frac{\|\mathbf{r}-\mathbf{s}\|^2 - 3(x-x')^2}{\|\mathbf{r}-\mathbf{s}\|^5} + \frac{\|\mathbf{r}-\mathbf{s}\|^2 - 3(y-y')^2}{\|\mathbf{r}-\mathbf{s}\|^5} + \frac{\|\mathbf{r}-\mathbf{s}\|^2 - 3(z-z')^2}{\|\mathbf{r}-\mathbf{s}\|^5} = 0$$

Hence, as in Theorem 36, we have that for each $\mathbf{r} \in \Omega^c$ $\exists \varepsilon, \delta > 0$ such that $\forall \tilde{\mathbf{r}} \in B(\mathbf{r}, \varepsilon)$ we have:

$$\left|\rho(\mathbf{s}) \frac{\|\tilde{\mathbf{r}}-\mathbf{s}\|^2 - 3(\tilde{x}-x')^2}{\|\tilde{\mathbf{r}}-\mathbf{s}\|^5}\right| \leq \frac{4|\rho(\mathbf{s})|}{\|\tilde{\mathbf{r}}-\mathbf{s}\|^3} \leq \frac{4|\rho(\mathbf{s})|}{\delta^3}$$

which is integrable by Weierstraß theorem. Thus, by the theorem of derivation under the integral sign:

$$\mathbf{div}(\mathbf{g}) = -\,\mathbf{div} \int_\Omega \frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|^3}(\mathbf{r}-\mathbf{s})\mathrm{d}^3\mathbf{s} = -\int_\Omega \rho(\mathbf{s})\,\mathbf{div}\left(\frac{\mathbf{r}-\mathbf{s}}{\|\mathbf{r}-\mathbf{s}\|^3}\right)\mathrm{d}^3\mathbf{s} = 0 \tag{97}$$

□

So far we have seen that the gravitational potential V satisfies the Laplace equation. If moreover we choose the origin of potential to be at the infinity, that is, if we impose $\lim_{\|\mathbf{r}\|\to\infty} V = 0$, then the gravitational potential created by a distribution of mass in a compact region Ω is a solution of the following exterior Dirichlet problem:

$$\begin{cases} \Delta V = 0 & \text{in } \Omega^c \\ V = f & \text{on } \partial\Omega \\ \lim\limits_{\|\mathbf{r}\|\to\infty} V = 0 \end{cases} \tag{98}$$

If Ω represents the Earth, then $f = f(\theta, \phi)$ represents is the boundary condition concerning the gravitational potential at the surface of the Earth as a function of the longitude θ and colatitude ϕ.

We wil see now that Eq. (98) has uniqueness of solutions. To do that we invoke the maximum principle, which we will not prove (see [Eva10] for more details).

Theorem 38 (Maximum principle). *Let $U \subseteq \mathbb{R}^n$ be open and bounded and $u \in \mathcal{C}^2(U) \cap \mathcal{C}(\overline{U})$. Suppose that u is harmonic within U, that is, $\Delta u = 0$ in U. Then, $\max_{\overline{U}} u = \max_{\partial U} u$.*

Corollary 39. The Dirichlet problem of Eq. (98) has a unique solution.

Proof. Suppose we have two solutions V_1, V_2 of Eq. (98). Then, $W := V_1 - V_2$ is harmonic in Ω^c and $W = 0$ on $\partial\Omega$. Moreover, $\lim_{\|\mathbf{r}\|\to\infty} W = 0$. So $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ large enough such that $|W| \leq \varepsilon$ on $B(0, n)$. Thus, by the maximum principle, $|W| \leq \varepsilon$ on $\overline{B(0,n)} \cap \Omega^c$. Since the ε is arbitrary, we must have $W = 0$ on Ω^c, that is, $V_1 = V_2$. □

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[5] It can be seen that V satisfies in fact the *Poisson equation* $\Delta V = 4\pi G\rho$ for any point $\mathbf{r} \in \mathbb{R}^3$, which reduced to Laplace equation when $\mathbf{r} \in \Omega^c$, because there we have $\rho(\mathbf{r}) = 0$.

4.1.3 Expansion in spherical harmonics

We have just seen that V satisfies the exterior Dirichlet problem for the Laplace equation. In Section 2.3.2 we have seen that a solution to the Laplace equation can be expressed as:

$$V(r,\theta,\phi)=\sum_{n=0}^{\infty}\sum_{m=0}^{n}(a_n r^n+b_n r^{-n-1})(\tilde{c}_{n,m}Y^{\mathrm{c}}_{n,m}(\theta,\phi)+\tilde{s}_{n,m}Y^{\mathrm{s}}_{n,m}(\theta,\phi)) \tag{99}$$

where $a_n, b_n, \tilde{c}_{n,m}, \tilde{d}_{n,m}\in\mathbb{R}$. If we impose V to satisfy the third condition of Eq. (98), we must have $a_n=0$. Finally, if we choose $R_\oplus$ as a reference radius for a spherical model of the Earth, using the boundary condition on $\partial\Omega$

$$f(\theta,\phi)=\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\frac{b_n}{{R_\oplus}^{n+1}}(\tilde{c}_{n,m}Y^{\mathrm{c}}_{n,m}(\theta,\phi)+\tilde{s}_{n,m}Y^{\mathrm{s}}_{n,m}(\theta,\phi)) \tag{100}$$

and the orthogonality of the spherical harmonics we can deduce that the coefficients $b_n\tilde{c}_{n,m}$ and $b_n\tilde{s}_{n,m}$ are given by:

$$b_n\tilde{c}_{n,m}=\frac{{R_\oplus}^{n+1}}{4\pi}\int_0^{2\pi}\int_0^{\pi}f(\theta,\phi)Y^{\mathrm{c}}_{n,m}(\theta,\phi)\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta \tag{101}$$

$$b_n\tilde{s}_{n,m}=\frac{{R_\oplus}^{n+1}}{4\pi}\int_0^{2\pi}\int_0^{\pi}f(\theta,\phi)Y^{\mathrm{s}}_{n,m}(\theta,\phi)\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta \tag{102}$$

Hence introducing the gravitational constant G and the Earth's mass $M_\oplus$ to the equation, our final expression for the gravitational potential is

$$V(r,\theta,\phi)=\frac{GM_\oplus}{R_\oplus}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{R_\oplus}{r}\right)^{n+1}(C_{n,m}Y^{\mathrm{c}}_{n,m}(\theta,\phi)+S_{n,m}Y^{\mathrm{s}}_{n,m}(\theta,\phi)) \tag{103}$$

where the coefficients $C_{n,m}, S_{n,m}\in\mathbb{R}$ are given by the formulas:

$$C_{n,m}=\frac{1}{4\pi}\frac{R_\oplus}{GM_\oplus}\int_0^{2\pi}\int_0^{\pi}f(\theta,\phi)Y^{\mathrm{c}}_{n,m}(\theta,\phi)\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta \tag{104}$$

$$S_{n,m}=\frac{1}{4\pi}\frac{R_\oplus}{GM_\oplus}\int_0^{2\pi}\int_0^{\pi}f(\theta,\phi)Y^{\mathrm{s}}_{n,m}(\theta,\phi)\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta \tag{105}$$

here $\delta_{0,m}$ denotes the Kronecker delta. The coefficients $C_{n,m}$, $S_{n,m}$ describe the dependence on the Earth's internal structure. They are obtained from observation of the perturbations seen in the orbits of other satellites [MG05]. Other methods for obtaining such data are through surface gravimetry, which provides precise local and regional information about the gravity field, or through altimeter data, which can be used to provide a model for the geoid of the Earth, that is the shape that the ocean surface would take under the influence of the gravity of Earth, which in turn can be used to obtain the geopotential coefficients.

4.1.4 Numerical computation of the gravity acceleration

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5 Conclusions

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