1 Proofs

1.1 Conics in a nutshell

Proposition 1. The area enclosed in an ellipse of semi-major axis a and semi-minor axis b is πab.

Proof. Consider the ellipse E centered at the origin and oriented as in Fig. 1.

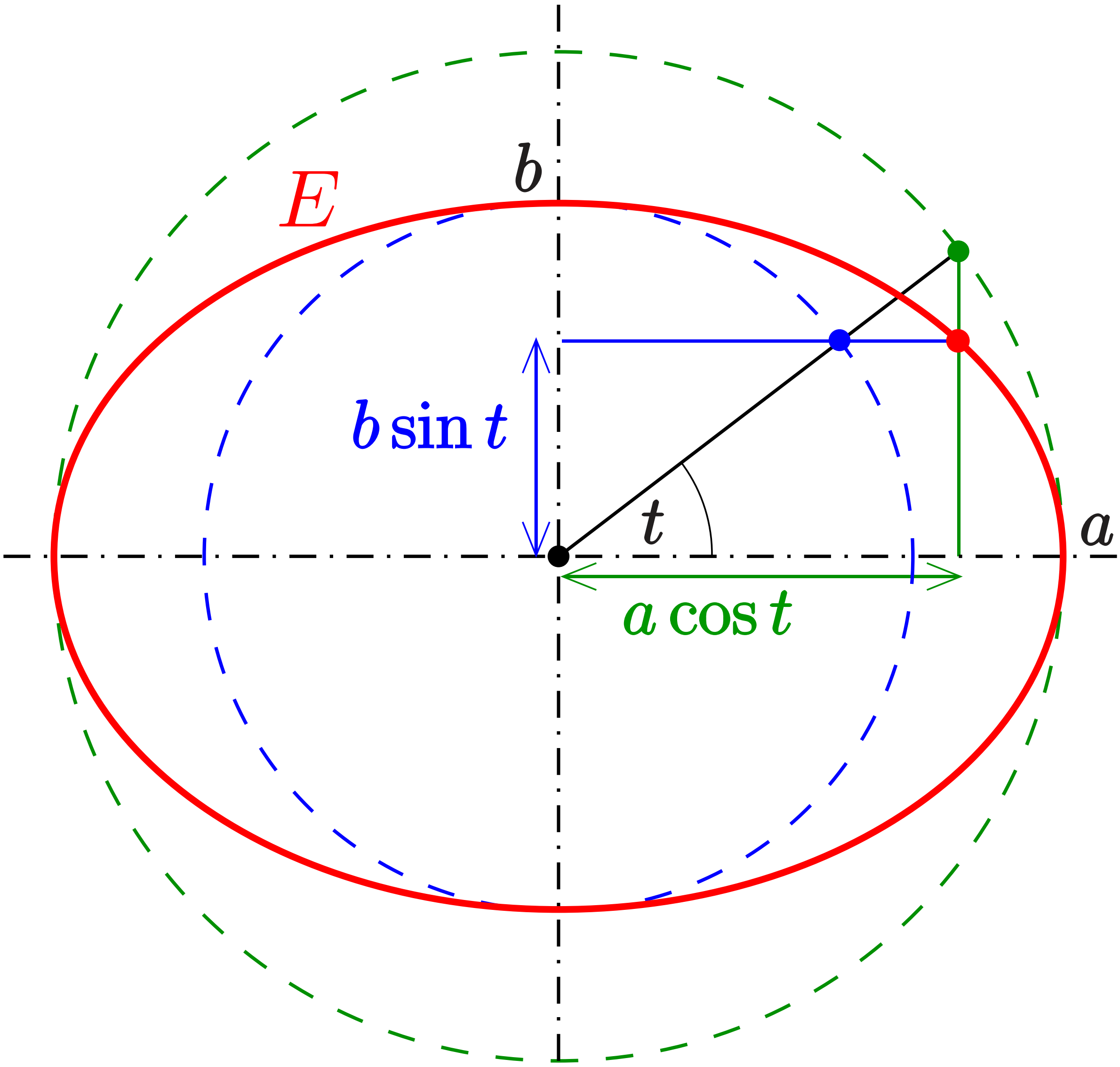

Figure 1: Reference frame centered at the center of the ellipse. Source: [Ag217].

From Fig. 1 one can check that it can be parametrized by $(x, y) = (a \cos t, b \sin t)$ with $t \in [0, 2\pi)$. This parametrization satisfies:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{1}$$

Hence, the area enclosed in the ellipse can be parametrized by $(x, y) = (ar \cos t, br \sin t)$, with $r \in [0, 1]$ and $t \in [0, 2\pi)$. The Jacobian of this transformation is abr. Therefore, from the change of variable theorem we have that:

$$\mathrm{Area}(E) = \iint_E \mathrm{d}x\,\mathrm{d}y = \int_0^{2\pi}\int_0^1 abr\,\mathrm{d}r\,\mathrm{d}t = \pi ab \tag{2}$$

$\square$

1.2 Introduction to astrodynamics and satellite tracking

Proposition 2 (Kepler's first law). Consider two point-mass bodies. The motion of one body orbiting the other can be described by a conic section. Hence, it can be expressed in the form:

$$r(t) = \frac{p}{1 + e\cos(\nu(t))} = \frac{h^2/\mu}{1 + (B/\mu)\cos(\nu)} \tag{3}$$

for some parameters $p = h^2/\mu$, $e = B/\mu$ and B.

Proof. Cross-multiplying **??** by $\mathbf{h}$ we obtain

$$\frac{\mathrm{d}(\dot{\mathbf{r}} \times \mathbf{h})}{\mathrm{d}t} = \ddot{\mathbf{r}} \times \mathbf{h} = -\frac{\mu}{r^3}\mathbf{r} \times \mathbf{h} = -\frac{\mu}{r^3}\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}}) = \frac{\mu}{r^3}[(\mathbf{r}\cdot\mathbf{r})\dot{\mathbf{r}} - (\mathbf{r}\cdot\dot{\mathbf{r}})\mathbf{r}] \tag{4}$$

where in the last equality we have used the vector equality $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u}\cdot\mathbf{w})\mathbf{v} - (\mathbf{u}\cdot\mathbf{v})\mathbf{w}$ for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Now note that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right) = \frac{\dot{\mathbf{r}}}{r} - \frac{\dot{r}}{r^2}\mathbf{r} = \frac{1}{r^3}[(\mathbf{r}\cdot\mathbf{r})\dot{\mathbf{r}} - (\mathbf{r}\cdot\dot{\mathbf{r}})\mathbf{r}] \tag{5}$$

because[1] $2r\dot{r} = \frac{\mathrm{d}(r^2)}{\mathrm{d}t} = \frac{\mathrm{d}(\mathbf{r}\cdot\mathbf{r})}{\mathrm{d}t} = 2\mathbf{r}\cdot\dot{\mathbf{r}}$. Thus:

$$\frac{\mathrm{d}(\dot{\mathbf{r}} \times \mathbf{h})}{\mathrm{d}t} = \mu\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right) \tag{6}$$

Integrating with respect to the time yields

$$\dot{\mathbf{r}} \times \mathbf{h} = \frac{\mu}{r}\mathbf{r} + \mathbf{B} \tag{7}$$

where $\mathbf{B} \in \mathbb{R}^3$ is the constant of integration. Observe that since $\dot{\mathbf{r}} \times \mathbf{h}$ is perpendicular to $\mathbf{h}$, $\dot{\mathbf{r}} \times \mathbf{h}$ lies on the orbital plane and so does $\mathbf{r}$. Hence, $\mathbf{B}$ lies on the orbital plane too. Now, dot-multiplying this last equation by $\mathbf{r}$ and using that $\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v})\cdot\mathbf{w}$ $\forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ we obtain

$$h^2 = \mathbf{h}\cdot\mathbf{h} = (\mathbf{r} \times \dot{\mathbf{r}})\cdot\mathbf{h} = \mathbf{r}\cdot(\dot{\mathbf{r}} \times \mathbf{h}) = \frac{\mu}{r}\mathbf{r}\cdot\mathbf{r} + \mathbf{r}\cdot\mathbf{B} = \mu r + rB\cos\nu \tag{8}$$

where $h := \|\mathbf{h}\|$, $B := \|\mathbf{B}\|$ and ν denotes the angle between $\mathbf{r}$ and $\mathbf{B}$, called *true anomaly*. Rearranging the terms we finally obtain the equation of a conic section

$$r = \frac{h^2/\mu}{1 + (B/\mu)\cos(\nu)} \tag{9}$$

with $p := h^2/\mu$ and $e := B/\mu$. $\square$

Proposition 3 (Kepler's second law). The areal velocity remains constant. In particular:

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = \frac{h}{2} \tag{10}$$

Proof. Recall that the area of a parallelogram generated by two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ is given by $\|\mathbf{u} \times \mathbf{v}\|$. Thus, approximating the difference $A(t+k) - A(t)$ by half of the area of the parallelogram generated by $\mathbf{r}(t)$ and $\mathbf{r}(t+k)$ (see Fig. 2) we obtain:

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = \lim_{k\to 0}\frac{A(t+k) - A(t)}{k} = \lim_{k\to 0}\frac{\|\mathbf{r}(t) \times \mathbf{r}(t+k)\|}{2k} = \lim_{k\to 0}\frac{\|\mathbf{r}(t) \times (\mathbf{r}(t+k) - \mathbf{r}(t))\|}{2k} =$$
$$= \frac{\|\mathbf{r}(t) \times \dot{\mathbf{r}}(t)\|}{2} = \frac{h}{2} \tag{11}$$

where the penultimate equality is due to the continuity and linearity of the cross product. $\square$

[1] Bear in mind that in general $\dot{r} \neq \|\dot{\mathbf{r}}\|$. Indeed, if β denotes the angle between $\mathbf{r}$ and $\dot{\mathbf{r}}$ we have that $\dot{r} = \|\dot{\mathbf{r}}\|\cos\beta$. In particular $\dot{r}$ may be negative.

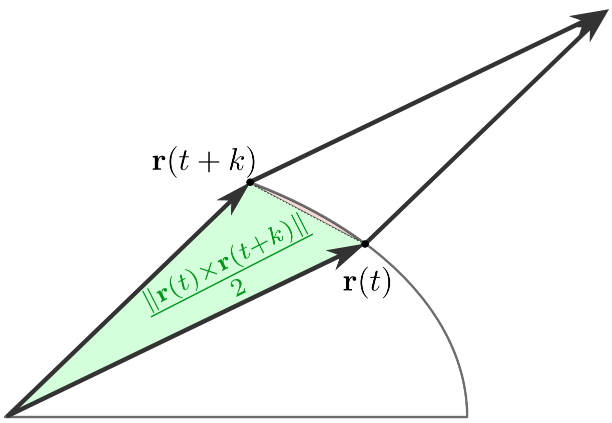

Figure 2: Graphical representation of the error made (red region) when approximating the area swept by the radius vector by half the area of the parallelogram generated by $\mathbf{r}(t)$ and $\mathbf{r}(t+k)$ (green region).

Proposition 4 (Kepler's third law). The mean motion is related to the semi-major axis by:

$$n = \sqrt{\frac{\mu}{a^3}} \tag{12}$$

Proof. Integrating **??** with respect to time between 0 and T (the period) yields:

$$\pi ab = A(T) = \int_0^T A'(t)\,\mathrm{d}t = \int_0^T \frac{h}{2}\,\mathrm{d}t = \frac{hT}{2} \implies n = \frac{2\pi}{T} = \frac{h}{ab} = \frac{h}{a^2\sqrt{1-e^2}} = \sqrt{\frac{\mu}{a^3}} \tag{13}$$

where we have used **????**. $\square$

Lemma 5. Let $e \in [0,1)$, $M \in \mathbb{R}$ and $\bar{M} = M \mod 2\pi$ be such that $\bar{M} \in [0, 2\pi)$. Then, the function

$$f(E) = E - e\sin E - M \tag{14}$$

has a unique solution in the interval $[M, M+e]$ if $\bar{M} \in [0,\pi)$ and in the interval $[M-e, M]$ if $\bar{M} \in [\pi, 2\pi)$.

Proof. We first prove the uniqueness. Clearly $f \in \mathcal{C}^1(\mathbb{R})$ and $f'(E) = 1 - e\cos E > 0$ for all $E \in [0, 2\pi)$ because $e < 1$. Thus, f is strictly increasing and so it has at most one zero. Now, if $0 \leq \bar{M} < \pi$, then:

$$f(M) = -e\sin M \leq 0 \qquad \text{and} \qquad f(M+e) = e(1-\sin(M+e)) \geq 0 \tag{15}$$

So by Bolzano's theorem, f has a solution in $[M, M+e]$. If $\pi \leq \bar{M} < 2\pi$, then:

$$f(M) = -e\sin M \geq 0 \qquad \text{and} \qquad f(M-e) = -e(1+\sin(M-e)) \leq 0 \tag{16}$$

So again by Bolzano's theorem, f has a solution in $[M-e, M]$. $\square$

1.3 Earth's gravitational field and other perturbations

Theorem 6. Let Ω be a compact region in $\mathbb{R}^3$ with a continuous density of mass $\rho : \Omega \to \mathbb{R}$. Then, the gravitational acceleration field $\mathbf{g}$ is conservative. That is, there exists a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{g} = \boldsymbol{\nabla} f$.

Proof. An easy computation shows that fixed $\mathbf{s} \in \mathbb{R}^3$ we have:

$$\boldsymbol{\nabla}\left(\frac{1}{\|\mathbf{r}-\mathbf{s}\|}\right) = -\frac{1}{\|\mathbf{r}-\mathbf{s}\|^3}(\mathbf{r}-\mathbf{s}) \tag{17}$$

So we need to justify whether the following exchange between the gradient and the integral is correct:

$$\mathbf{g} = -\int_\Omega \frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|^3}(\mathbf{r}-\mathbf{s})\mathrm{d}^3\mathbf{s} = \int_\Omega \rho(\mathbf{s})\boldsymbol{\nabla}\left(\frac{1}{\|\mathbf{r}-\mathbf{s}\|}\right)\mathrm{d}^3\mathbf{s} = \boldsymbol{\nabla}\int_\Omega \frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|}\mathrm{d}^3\mathbf{s} \tag{18}$$

Without loss of generality it suffices to justify that

$$\frac{\partial}{\partial x}\int\limits_{\Omega}\frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|}\mathrm{d}^3\mathbf{s}=\int\limits_{\Omega}\frac{\partial}{\partial x}\left(\frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|}\right)\mathrm{d}^3\mathbf{s} \tag{19}$$

assuming $\mathbf{r}=(x,y,z)$ and $\mathbf{s}=(x',y',z')$. In order to apply the theorem of derivation under the integral sign we need to control $\frac{\partial}{\partial x}\left(\frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|}\right) = -\rho(\mathbf{s})\frac{x-x'}{\|\mathbf{r}-\mathbf{s}\|^3}$ by an integrable function $h(\mathbf{s})$. Using spherical coordinates centered at $\mathbf{r}$ and writing $(\mathbf{r}-\mathbf{s})_{\text{sph}}=(\rho_{\mathbf{r}},\theta,\phi)$, the integrand to bound becomes (in spherical coordinates):

$$\left|-\rho(\mathbf{s})\frac{x-x'}{\|\mathbf{r}-\mathbf{s}\|^3}{\rho_{\mathbf{r}}}^2\sin\phi\right|=|\rho(\mathbf{s})|\left|\frac{\rho_{\mathbf{r}}\cos\theta\sin\phi}{{\rho_{\mathbf{r}}}^3}{\rho_{\mathbf{r}}}^2\sin\phi\right|\leq|\rho(\mathbf{s})|\leq K \tag{20}$$

where the last inequality follows for certain $K\in\mathbb{R}$ by Weierstrass theorem (ρ is continuous and Ω is compact). Thus, since $h(\mathbf{s})=K$ is integrable, because Ω is bounded, the equality of Eq. (19) is correct. □

Theorem 7. Consider a distribution of matter of density ρ in a compact region Ω. Then, the gravitational potential V satisfies the Laplace equation

$$\Delta V=0 \tag{21}$$

for all points outside Ω[2].

Proof. Recall that $\Delta V=\mathbf{div}(\boldsymbol{\nabla}V)$. So since $\mathbf{g}=-\boldsymbol{\nabla}V$ it suffices to prove that $\mathbf{div}(\mathbf{g})=0$. Note that if $\mathbf{r}\in\Omega^c$, then $\exists\delta>0$ such that $\|\mathbf{r}-\mathbf{s}\|\geq\delta>0\ \forall\mathbf{s}\in\Omega$ because Ω is closed. As a result, $\frac{\mathbf{r}-\mathbf{s}}{\|\mathbf{r}-\mathbf{s}\|^3}$ is differentiable and:

$$\begin{aligned}\mathbf{div}\left(\frac{\mathbf{r}-\mathbf{s}}{\|\mathbf{r}-\mathbf{s}\|^3}\right)&=\frac{\partial}{\partial x}\left(\frac{x-x'}{\|\mathbf{r}-\mathbf{s}\|^3}\right)+\frac{\partial}{\partial y}\left(\frac{y-y'}{\|\mathbf{r}-\mathbf{s}\|^3}\right)+\frac{\partial}{\partial z}\left(\frac{z-z'}{\|\mathbf{r}-\mathbf{s}\|^3}\right)=\\&=\frac{\|\mathbf{r}-\mathbf{s}\|^2-3(x-x')^2}{\|\mathbf{r}-\mathbf{s}\|^5}+\frac{\|\mathbf{r}-\mathbf{s}\|^2-3(y-y')^2}{\|\mathbf{r}-\mathbf{s}\|^5}+\frac{\|\mathbf{r}-\mathbf{s}\|^2-3(z-z')^2}{\|\mathbf{r}-\mathbf{s}\|^5}=0\end{aligned}$$

Hence, as in **??**, we have that for each $\mathbf{r}\in\Omega^c\ \exists\varepsilon,\delta>0$ such that $\forall\tilde{\mathbf{r}}\in B(\mathbf{r},\varepsilon)\subset\Omega^c$ we have:

$$\left|\rho(\mathbf{s})\frac{\|\tilde{\mathbf{r}}-\mathbf{s}\|^2-3(\tilde{x}-x')^2}{\|\tilde{\mathbf{r}}-\mathbf{s}\|^5}\right|\leq\frac{4|\rho(\mathbf{s})|}{\|\tilde{\mathbf{r}}-\mathbf{s}\|^3}\leq\frac{4|\rho(\mathbf{s})|}{\delta^3} \tag{22}$$

which is integrable by Weierstrass theorem. Thus, by the theorem of derivation under the integral sign:

$$\mathbf{div}(\mathbf{g})=-\,\mathbf{div}\int\limits_{\Omega}\frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|^3}(\mathbf{r}-\mathbf{s})\mathrm{d}^3\mathbf{s}=-\int\limits_{\Omega}\rho(\mathbf{s})\,\mathbf{div}\left(\frac{\mathbf{r}-\mathbf{s}}{\|\mathbf{r}-\mathbf{s}\|^3}\right)\mathrm{d}^3\mathbf{s}=0 \tag{23}$$

□

Corollary 8. The Dirichlet problem of **??** has a unique solution.

Proof. Suppose we have two solutions V_1, V_2 of **??**. Then, $W:=V_1-V_2$ is harmonic in Ω^c, $W=0$ on $\partial\Omega$ and $\lim\limits_{\|\mathbf{r}\|\to\infty}W=0$. So $\forall\varepsilon>0$, $\exists n\in\mathbb{N}$ large enough such that $\Omega\subseteq B(0,n)$ and $|W|\leq\varepsilon$ on $\mathbb{R}^3\setminus\overline{B(0,n)}$. Thus, by the maximum principle, $|W|\leq\varepsilon$ on $\overline{B(0,n)}\cap\Omega^c$. Since the ε is arbitrary, we must have $W=0$ on Ω^c, that is, $V_1=V_2$. □

Proposition 9. The family of spherical harmonics $\{Y^{\mathrm{c}}_{n,m}(\theta,\phi),Y^{\mathrm{s}}_{n,m}(\theta,\phi):n\in\mathbb{N}\cup\{0\},m\leq n\}$ is orthonormal in the following sense:

$$\frac{1}{4\pi}\int\limits_0^{2\pi}\int\limits_0^{\pi}Y^i_{n_1,m_1}(\theta,\phi)Y^j_{n_2,m_2}(\theta,\phi)\mathrm{d}\Omega=\delta_{n_1,n_2}\delta_{m_1,m_2}\delta_{i,j} \tag{24}$$

[2]It can be seen that V satisfies in fact the *Poisson equation* $\Delta V=4\pi G\rho$ for any point $\mathbf{r}\in\mathbb{R}^3$, which reduces to Laplace equation when $\mathbf{r}\in\Omega^c$, because there we have $\rho(\mathbf{r})=0$.

where $\mathrm{d}\Omega = \sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta$ is the solid angle element, which measures the element of area on a sphere of radius 1.

Proof. Let N_{n_1,m_1}, N_{n_2,m_2} be the normalization factors of the spherical harmonics Y_{n_1,m_1}, Y_{n_2,m_2} respectively. Note that we can separate the variables in the integral of Eq. (24). So if $i \neq j$, the integral over θ becomes $\int_0^{2\pi} \cos(m_1\theta)\sin(m_2\theta)\,\mathrm{d}\theta$ which is equal to 0 regardless of the values of m_1 and m_2. So from now on assume that $i = j$. Due to the symmetry between the cosine and the sine we can suppose that $i = \mathrm{c}$. Thus:

$$\int_0^{2\pi}\int_0^{\pi} Y^i_{n_1,m_1}(\theta,\phi)Y^j_{n_2,m_2}(\theta,\phi)\mathrm{d}\Omega =$$

$$= N_{n_1,m_1}N_{n_2,m_2}\int_0^{\pi} P_{n_1,m_1}(\cos\phi)P_{n_2,m_2}(\cos\phi)\sin\phi\mathrm{d}\phi\int_0^{2\pi}\cos(m_1\theta)\cos(m_2\theta)\,\mathrm{d}\theta \tag{25}$$

An easy check shows that if $m_1 \neq m_2$ then the integral over θ is zero (and the same applies with sines). So suppose $m_1 = m_2 = m$. In that case, if $m \neq 0$ we have $\int_0^{2\pi} (\cos m\theta)^2\,\mathrm{d}\theta = \int_0^{2\pi} (\sin m\theta)^2\,\mathrm{d}\theta = \pi$ and if $m = 0$, the cosine integral evaluates to 2π whereas the sine integral is 0. We can omit this latter case because $Y^{\mathrm{s}}_{n,0}$ is identically zero. Thus:

$$\frac{2\pi}{2-\delta_{0,m}}N_{n_1,m}N_{n_2,m}\int_0^{\pi} P_{n_1,m}(\cos\phi)P_{n_2,m}(\cos\phi)\sin\phi\mathrm{d}\phi = \frac{2\pi}{2-\delta_{0,m}}N_{n_1,m}N_{n_2,m}\int_{-1}^{1} P_{n_1,m}(x)P_{n_2,m}(x)\,\mathrm{d}x \tag{26}$$

By **??** this latter integral is $\frac{2}{2n_1+1}\frac{(n_1+m)!}{(n_1-m)!}\delta_{n_1,n_2}$. Finally, if $n_1 = n_2 = n$, putting all normalization factors together we get:

$$\frac{2\pi}{2-\delta_{0,m}}N_n^m N_n^m\frac{2}{2n+1}\frac{(n+m)!}{(n-m)!} = 4\pi \tag{27}$$

□

Theorem 10. The regular solutions in a bounded region $\Omega \subseteq \mathbb{R}^3$ such that $0 \notin \overline{\Omega}$ of the Laplace equation in spherical coordinates are of the form

$$f(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}(a_n r^n + b_n r^{-n-1})P_{n,m}(\cos\phi)(c_{n,m}\cos(m\theta) + s_{n,m}\sin(m\theta)) \tag{28}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}(a_n r^n + b_n r^{-n-1})(\tilde{c}_{n,m}Y^{\mathrm{c}}_{n,m}(\theta,\phi) + \tilde{s}_{n,m}Y^{\mathrm{s}}_{n,m}(\theta,\phi)) \tag{29}$$

where $a_n, b_n, c_{n,m}, s_{n,m}, \tilde{c}_{n,m}, \tilde{s}_{n,m} \in \mathbb{R}$.

Proof. Let $f(r,\theta,\phi)$ be a solution of **??**. Using separation variables $f(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$ we can write:

$$\frac{\Theta\Phi}{r^2}(r^2R')' + \frac{R\Theta}{r^2\sin\phi}(\sin\phi\Phi')' + \frac{R\Phi}{r^2(\sin\phi)^2}\Theta'' = 0 \tag{30}$$

Here, we are making and abuse of notation denoting all the derivatives with a prime, but the reader should have no confusion with it. Isolating R from Θ and Φ yields:

$$\frac{(r^2R')'}{R} = -\frac{1}{\sin\phi\Phi}(\sin\phi\Phi')' - \frac{1}{(\sin\phi)^2\Theta}\Theta'' \tag{31}$$

Since the left-hand side depends entirely on r and the right-hand side does not, it follows that both sides must be constant. Therefore:

$$\frac{(r^2R')'}{R} = \lambda \tag{32}$$

$$\frac{1}{\sin\phi\Phi}(\sin\phi\Phi')' + \frac{1}{(\sin\phi)^2\Theta}\Theta'' = -\lambda \tag{33}$$

with $\lambda \in \mathbb{R}$. Similarly, separating variables from Eq. (33) we obtain that the equations

$$\frac{1}{\Theta}\Theta'' = -m^2 \tag{34}$$

$$\frac{\sin\phi}{\Phi}(\sin\phi\Phi')' + \lambda(\sin\phi)^2 = m^2 \tag{35}$$

must be constant with $m \in \mathbb{C}$ (a priori). The solution to the well-known Eq. (34) is a linear combination of the $\cos(m\theta)$ and $\sin(m\theta)$. Note, though, that since Θ must be a 2π-periodic function, that is satisfying $\Theta(\theta + 2\pi) = \Theta(\theta)\ \forall\theta \in \mathbb{R}$, m must be an integer. On the other hand making the change of variables $x = \cos\phi$ and $y = \Phi(\phi)$ in Eq. (35) and using the chain rule, that equation becomes:

$$(1-x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\lambda - \frac{m^2}{1-x^2}\right) y = 0 \tag{36}$$

which is the associate Legendre equation. We have argued in **??** that we need $\lambda = n(n+1)$ and $m \leq n$ in order to obtain regular solutions at $x = \cos\phi = \pm 1$. Moreover, these solutions are $P_{n,m}(\cos\phi)$.

Finally, note that equation Eq. (32) is a Cauchy-Euler equation (check [Wik]) and so the general solution of it is given by

$$R(r) = c_1 r^n + c_2 r^{-n-1} \tag{37}$$

because $\lambda = n(n+1)$ (the reader may check that r^n and r^{-n-1} are indeed two independent solutions of Eq. (32)). So the general solution becomes a linear combination of each solution found varying $n \in \mathbb{N} \cup \{0\}$ and $m \in \{0, 1, \ldots, n\}$:

$$f(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}(a_n r^n + b_n r^{-n-1})P_{n,m}(\cos\phi)(c_{n,m}\cos(m\theta) + s_{n,m}\sin(m\theta)) \tag{38}$$

□