1 Earth's gravitational field

So far we have only considered the gravitational force acting between point masses. In reality, the Earth is not a point mass, neither a spherically symmetric mass distribution. In this section we will delve into the details of a more realistic model of the Earth's gravitational field.

1.1 Geopotential model

1.1.1 Continuous distribution of mass

In **??** we saw that the motion of a body orbiting another one can be described by a conic section. However, we have not been concerned about the mass distribution of the large body, in our case the Earth. In this section we will see that the motion of the smaller body, the satellite, is slightly perturbed by the mass distribution of the Earth as well as the presence of other forces such as atmospheric drag, solar radiation pressure, or the gravitational pull of the Moon and Sun, which we will discuss later on. Nevertheless, the perturbations are relatively small and the orbits of the satellites are still approximating ellipses, but as we will corroborate experimentally in **??** they are essential to obtain accurate results.

Consider a body confined in a compact region $\Omega \subseteq \mathbb{R}$ with a continuous density of mass $\rho : \Omega \to \mathbb{R}$. We would like to know the gravitational pull on a point mass m located at position $\mathbf{r}$ from the center of mass of the body. To do this, consider a covering of Ω in a set of disjoint cubes Q_i, $i = 1, \ldots, N$, small enough to be considered as point masses and let $R_i := Q_i \cap \Omega$. Then, $\bigsqcup_{i=1}^{N} R_i = \Omega$. If each R_i has mass m_i, volume V_i, density ρ_i, and its center is located at $\mathbf{s}_i \in \mathbb{R}^3$, then the total gravitational acceleration $\mathbf{g}$ exerted on m is the sum of the contributions of all the forces exerted by the N point masses, and it is given by:

$$\mathbf{g} = -\sum_{i=0}^{N} \frac{m_i}{\|\mathbf{r}-\mathbf{s}_i\|^3}(\mathbf{r}-\mathbf{s}_i) = -\sum_{i=0}^{N} \frac{\rho_i}{\|\mathbf{r}-\mathbf{s}_i\|^3}(\mathbf{r}-\mathbf{s}_i)V_i \tag{1}$$

Note that Eq. (1) is a Riemann sum and so letting $N \to \infty$ we get:

$$\mathbf{g} = -\int_{\Omega} \frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|^3}(\mathbf{r}-\mathbf{s})\mathrm{d}^3\mathbf{s} \tag{2}$$

where $\mathrm{d}^3\mathbf{s} := \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$, if $\mathbf{s} = (x', y', z')$.

Theorem 1. *Let Ω be a compact region in $\mathbb{R}^3$ with a continuous density of mass $\rho : \Omega \to \mathbb{R}$. Then, the gravitational acceleration field $\mathbf{g}$ is conservative. That is, there exists a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{g} = \boldsymbol{\nabla} f$.*

Proof. An easy computation shows that fixed $\mathbf{s} \in \mathbb{R}^3$ we have:

$$\boldsymbol{\nabla}\left(\frac{1}{\|\mathbf{r}-\mathbf{s}\|}\right) = -\frac{1}{\|\mathbf{r}-\mathbf{s}\|^3}(\mathbf{r}-\mathbf{s}) \tag{3}$$

So we need to justify whether the following exchange between the gradient and the integral is correct:

$$\mathbf{g} = -\int_{\Omega} \frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|^3}(\mathbf{r}-\mathbf{s})\mathrm{d}^3\mathbf{s} = \int_{\Omega} \rho(\mathbf{s})\boldsymbol{\nabla}\left(\frac{1}{\|\mathbf{r}-\mathbf{s}\|}\right)\mathrm{d}^3\mathbf{s} = \boldsymbol{\nabla}\int_{\Omega} \frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|}\mathrm{d}^3\mathbf{s} \tag{4}$$

Without loss of generality it suffices to justify that

$$\frac{\partial}{\partial x}\int_{\Omega} \frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|}\mathrm{d}^3\mathbf{s} = \int_{\Omega} \frac{\partial}{\partial x}\left(\frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|}\right)\mathrm{d}^3\mathbf{s} \tag{5}$$

assuming $\mathbf{r} = (x, y, z)$ and $\mathbf{s} = (x', y', z')$. In order to apply the theorem of derivation under the integral sign we need to control $\frac{\partial}{\partial x}\left(\frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|}\right) = -\rho(\mathbf{s})\frac{x-x'}{\|\mathbf{r}-\mathbf{s}\|^3}$ by an integrable function $h(\mathbf{s})$. Using spherical coordinates centered at $\mathbf{r}$ and writing $(\mathbf{r}-\mathbf{s})_{\text{sph}} = (\rho_{\mathbf{r}}, \theta, \phi)$, the integrand to bound becomes (in spherical coordinates):

$$\left|-\rho(\mathbf{s})\frac{x-x'}{\|\mathbf{r}-\mathbf{s}\|^3}{\rho_{\mathbf{r}}}^2\sin\phi\right| = |\rho(\mathbf{s})|\left|\frac{\rho_{\mathbf{r}}\cos\theta\sin\phi}{{\rho_{\mathbf{r}}}^3}{\rho_{\mathbf{r}}}^2\sin\phi\right| \leq |\rho(\mathbf{s})| \leq K \tag{6}$$

where the last inequality follows for certain $K \in \mathbb{R}$ by Weierstrass theorem (ρ is continuous and Ω is compact). Thus, since $h(\mathbf{s}) = K$ is integrable, because Ω is bounded, the equality of Eq. (5) is correct. $\square$

Physically speaking, the gravitational force $\mathbf{F}$ being conservative means that the work W done by the force along a path C

$$W = \int_C \mathbf{F} \cdot \mathrm{d}\mathbf{s} \tag{7}$$

depends only on the initial and final positions of it. Moreover, due to historical reasons, we will write $\mathbf{g} = -\boldsymbol{\nabla} V$ (with the minus sign) and call V the *gravitational potential.* The minus sign is chosen according the convention that work done by gravitational forces decreases the potential.

1.1.2 Laplace's equation for V

Theorem 2. Consider a distribution of matter of density ρ in a compact region Ω. Then, the gravitational potential V satisfies the Laplace equation

$$\Delta V = 0 \tag{8}$$

for all points outside Ω[1].

Proof. Recall that $\Delta V = \mathbf{div}(\boldsymbol{\nabla} V)$. So since $\mathbf{g} = -\boldsymbol{\nabla} V$ it suffices to prove that $\mathbf{div}(\mathbf{g}) = 0$. Note that if $\mathbf{r} \in \Omega^c$, then $\exists \delta > 0$ such that $\|\mathbf{r} - \mathbf{s}\| \geq \delta > 0$ $\forall \mathbf{s} \in \Omega$ because Ω is closed. As a result, $\frac{\mathbf{r}-\mathbf{s}}{\|\mathbf{r}-\mathbf{s}\|^3}$ is differentiable and:

$$\begin{aligned}\mathbf{div}\left(\frac{\mathbf{r}-\mathbf{s}}{\|\mathbf{r}-\mathbf{s}\|^3}\right) &= \frac{\partial}{\partial x}\left(\frac{x-x'}{\|\mathbf{r}-\mathbf{s}\|^3}\right) + \frac{\partial}{\partial y}\left(\frac{y-y'}{\|\mathbf{r}-\mathbf{s}\|^3}\right) + \frac{\partial}{\partial z}\left(\frac{z-z'}{\|\mathbf{r}-\mathbf{s}\|^3}\right) = \\ &= \frac{\|\mathbf{r}-\mathbf{s}\|^2 - 3(x-x')^2}{\|\mathbf{r}-\mathbf{s}\|^5} + \frac{\|\mathbf{r}-\mathbf{s}\|^2 - 3(y-y')^2}{\|\mathbf{r}-\mathbf{s}\|^5} + \frac{\|\mathbf{r}-\mathbf{s}\|^2 - 3(z-z')^2}{\|\mathbf{r}-\mathbf{s}\|^5} = 0\end{aligned}$$

Hence, as in Theorem 1, we have that for each $\mathbf{r} \in \Omega^c$ $\exists \varepsilon, \delta > 0$ such that $\forall \tilde{\mathbf{r}} \in B(\mathbf{r}, \varepsilon) \subset \Omega^c$ we have:

$$\left|\rho(\mathbf{s})\frac{\|\tilde{\mathbf{r}}-\mathbf{s}\|^2 - 3(\tilde{x}-x')^2}{\|\tilde{\mathbf{r}}-\mathbf{s}\|^5}\right| \leq \frac{4|\rho(\mathbf{s})|}{\|\tilde{\mathbf{r}}-\mathbf{s}\|^3} \leq \frac{4|\rho(\mathbf{s})|}{\delta^3}$$

which is integrable by Weierstrass theorem. Thus, by the theorem of derivation under the integral sign:

$$\mathbf{div}(\mathbf{g}) = -\,\mathbf{div}\int_\Omega \frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|^3}(\mathbf{r}-\mathbf{s})\mathrm{d}^3\mathbf{s} = -\int_\Omega \rho(\mathbf{s})\,\mathbf{div}\left(\frac{\mathbf{r}-\mathbf{s}}{\|\mathbf{r}-\mathbf{s}\|^3}\right)\mathrm{d}^3\mathbf{s} = 0 \tag{9}$$

$\square$

So far we have seen that the gravitational potential V satisfies the Laplace equation. If, moreover, we choose the origin of potential to be at the infinity, that is, if we impose $\lim\limits_{\|\mathbf{r}\|\to\infty} V = 0$, then the gravitational potential created by a distribution of mass in a compact region Ω is a solution of the following exterior Dirichlet problem:

$$\begin{cases}\Delta V = 0 & \text{in } \Omega^c \\ V = f & \text{on } \partial\Omega \\ \lim\limits_{\|\mathbf{r}\|\to\infty} V = 0\end{cases} \tag{10}$$

If Ω represents the Earth, then $f = f(\theta, \phi)$ is the boundary condition concerning the gravitational potential at the surface of the Earth as a function of the longitude θ and colatitude ϕ.

We will see now that Eq. (10) has a unique solution. To do that we invoke the maximum principle, which we will not prove here (see [Eva10] for more details).

[1] It can be seen that V satisfies in fact the *Poisson equation* $\Delta V = 4\pi G\rho$ for any point $\mathbf{r} \in \mathbb{R}^3$, which reduces to Laplace equation when $\mathbf{r} \in \Omega^c$, because there we have $\rho(\mathbf{r}) = 0$.

Theorem 3 (Maximum principle). Let $U \subset \mathbb{R}^n$ be open and bounded and $u \in \mathcal{C}^2(U) \cap \mathcal{C}(\overline{U})$. Suppose that u is harmonic within U, that is, $\Delta u = 0$ in U. Then, $\max_{\overline{U}} u = \max_{\partial U} u$.

Corollary 4. The Dirichlet problem of Eq. (10) has a unique solution.

Proof. Suppose we have two solutions V_1, V_2 of Eq. (10). Then, $W := V_1 - V_2$ is harmonic in Ω^c, $W = 0$ on $\partial\Omega$ and $\lim_{\|\mathbf{r}\| \to \infty} W = 0$. So $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ large enough such that $\Omega \subseteq B(0, n)$ and $|W| \leq \varepsilon$ on $\mathbb{R}^3 \setminus \overline{B(0, n)}$. Thus, by the maximum principle, $|W| \leq \varepsilon$ on $\overline{B(0, n)} \cap \Omega^c$. Since the ε is arbitrary, we must have $W = 0$ on Ω^c, that is, $V_1 = V_2$. $\square$

Now that we know that the Dirichlet problem has a unique solution, we can proceed to find it. In the next section we will construct an explicit solution for the gravitational potential of created by the Earth.

1.2 Spherical harmonics

1.2.1 Legendre polynomials, regularity and orthonormality

In this section we aim to introduce a class of functions that will appear later on in the general solution of the Laplace equation (see Section 1.2.2). To accomplish this, we need first to introduce the Legendre polynomials. There are several ways to define them, but the most convenient one for our purposes is from the following differential equation. Consider the following second-order differential equation called *Legendre differential equation*:

$$(1 - x^2)y'' - 2xy' + \lambda y = 0 \tag{11}$$

for $\lambda \in \mathbb{R}$. This equation can be rewritten as:

$$\left((1 - x^2)y'\right)' + \lambda y = 0 \tag{12}$$

Seeking for analytic solutions of this equation using the power series method [Mez], i.e. looking for solutions of the form $y(x) = \sum_{j=0}^{\infty} a_j x^j$, we see that:

$$\begin{aligned} 0 = (1 - x^2)\sum_{j=0}^{\infty} a_{j+2}(j+1)(j+2)x^j - 2x\sum_{j=0}^{\infty} a_{j+1}(j+1)x^j + \lambda\sum_{j=0}^{\infty} a_j x^j = \sum_{j=0}^{\infty} a_{j+2}(j+1)(j+2)x^j - \\ - \sum_{j=0}^{\infty} a_j(j-1)jx^j - \sum_{j=0}^{\infty} 2a_j jx^j + \sum_{j=0}^{\infty} \lambda a_j x^j = \sum_{j=0}^{\infty}[a_{j+2}(j+1)(j+2) - a_j(j(j+1) - \lambda)]x^j \end{aligned} \tag{13}$$

Equating the general term of the series to 0 we obtain this recursion:

$$a_{j+2} = \frac{j(j+1) - \lambda}{(j+1)(j+2)} a_j \quad j = 0, 1, 2, \ldots \tag{14}$$

From here we can obtain two independent solutions by setting the initial conditions a_0 and a_1 of the iteration. For example, setting $a_1 = 0$ we obtain a series that has only even powers of x. On the other hand, setting $a_0 = 0$ we obtain a series that has only odd powers of x. These two series converge on the interval $(-1, 1)$ by the ratio test (by looking at Eq. (14)) and can be expressed compactly as [Mez]:

$$y_{\mathrm{e}}(x) = a_0 \sum_{j=0}^{\infty} \left[\prod_{k=0}^{j-1} (2k(2k+1) - \lambda)\right] \frac{x^{2j}}{(2j)!} \quad y_{\mathrm{o}}(x) = a_1 \sum_{j=0}^{\infty} \left[\prod_{k=0}^{j-1} ((2k+1)(2k+2) - \lambda)\right] \frac{x^{2j+1}}{(2j+1)!} \tag{15}$$

Here, the empty product (that is, when k ranges from 0 to -1) is defined to be 1. However, for each $\lambda \in \mathbb{R}$ either one of these series or both diverge at $x = \pm 1$, as they behave as the harmonic series in a neighborhood of $x = \pm 1$. We are interested, though, in the solutions that remain bounded on the whole interval $[-1, 1]$. Looking at the expressions of Eq. (15) one can check that the only possibility to make the series converge in $[-1, 1]$ is when $\lambda = n(n+1)$, $n \in \mathbb{N} \cup \{0\}$. In this case, for each $n \in \mathbb{N} \cup \{0\}$ exactly one of the series is in fact a polynomial of degree n. If, furthermore, we choose a_0 or a_1 be such that the polynomial evaluates to 1 at $x = 1$, these polynomials are called *Legendre polynomials*, and they are denoted by $P_n(x)$. The other (divergent) series is usually denoted in the literature by $Q_n(x)$ (check [RHB99; Mez]) and it is independent of $P_n(x)$. Thus, the general solution of Eq. (12) for $\lambda = n(n+1)$ can be expressed as a linear combination of P_n and Q_n, because the space of solutions form a vector space of dimension 2. The following proposition will be of our interest in the next section [RHB99].

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2-1)$
3	$\frac{1}{2}(5x^3-3x)$
4	$\frac{1}{8}(35x^4-30x^2+3)$
5	$\frac{1}{8}(63x^5-70x^3+15x)$
6	$\frac{1}{16}(231x^6-315x^4+105x^2-5)$
7	$\frac{1}{16}(429x^7-693x^5+315x^3-35x)$

Table 1: First eight Legendre polynomials

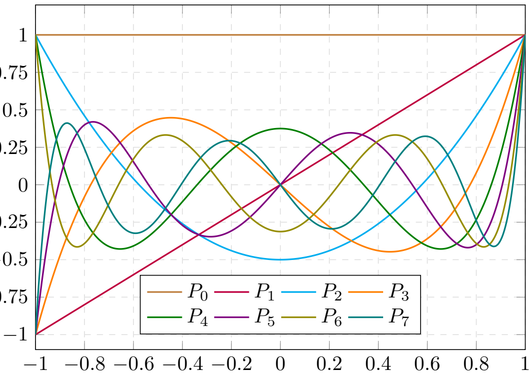


Figure 1: Graphic representation of the first eight Legendre polynomials.

Proposition 5. Let $y(x)$ be a solution to the Legendre differential equation. Then, $\forall m \in \mathbb{N} \cup \{0\}$ the function

$$w_m(x) = (1-x^2)^{m/2}\frac{\mathrm{d}^m y(x)}{\mathrm{d}x^m} \tag{16}$$

solves the *general Legendre differential equation*:

$$(1-x^2)y'' - 2xy' + \left(\lambda - \frac{m^2}{1-x^2}\right) y = 0 \tag{17}$$

In particular if $\lambda = n(n+1)$ for $n \in \mathbb{N} \cup \{0\}$, then $w_m(x)$ is denoted as

$$P_{n,m}(x) := (1-x^2)^{m/2}\frac{\mathrm{d}^m P_n}{\mathrm{d}x^m} \tag{18}$$

and it is called the *associated Legendre polynomial* of degree n and order m.

Note that although we opted to call the functions $P_{n,m}$ as polynomials, they are only true polynomials when m is even. But we have opted to call them in that manner as it is the common practice in the literature (see [Wei; RHB99; Mez]).

Moreover, from the definition of $P_{n,m}$, we can see that $P_{n,0} = P_n$ and that $P_{n,m} = 0$ if $m > n$. So we can restrict the domain of m to the set $\{0, 1, \ldots, n\}$.

n	$P_{n,1}(x)$	n	$P_{n,2}(x)$
1	$\sqrt{1-x^2}$	2	$3(1-x^2)$
2	$3x\sqrt{1-x^2}$	3	$15x(1-x^2)$
3	$\frac{3}{2}(5x^2-1)\sqrt{1-x^2}$	4	$\frac{15}{2}(7x^2-1)(1-x^2)$
4	$\frac{5}{2}x(7x^2-3)\sqrt{1-x^2}$	5	$\frac{105}{2}x(3x^2-1)(1-x^2)$
5	$\frac{15}{8}(21x^4-14x^2+1)\sqrt{1-x^2}$	6	$\frac{105}{8}(33x^4-18x^2+1)(1-x^2)$

Table 2: First associated Legendre polynomials for $m = 1$ and $m = 2$.

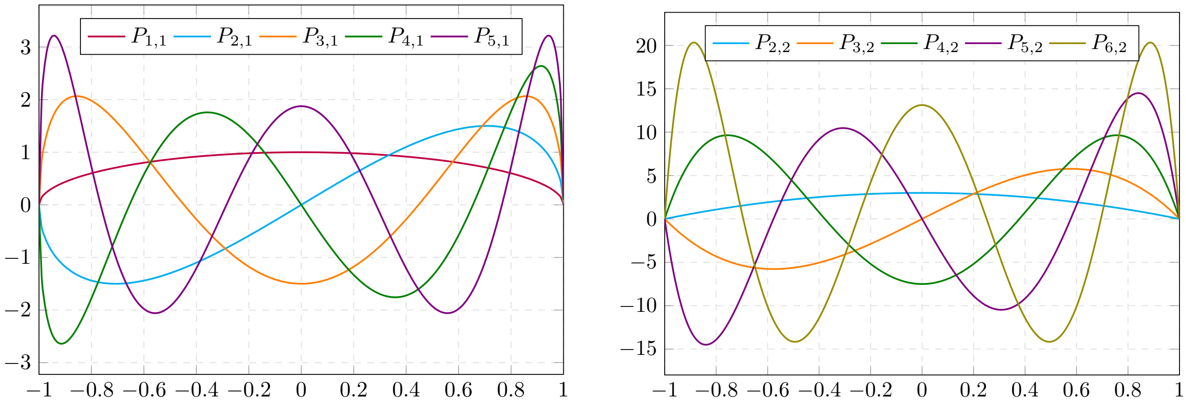
Definition 6. Let $n \in \mathbb{N} \cup \{0\}$ and $m \in \{0, 1, \ldots, n\}$. We define the *real spherical harmonics* $Y^{\mathrm{c}}_{n,m}$ and $Y^{\mathrm{s}}_{n,m}$ as:

$$Y^{\mathrm{c}}_{n,m}(\theta,\phi) = \sqrt{(2-\delta_{0,m})(2n+1)\frac{(n-m)!}{(n+m)!}}P_{n,m}(\cos\phi)\cos m\theta \tag{19}$$

$$Y^{\mathrm{s}}_{n,m}(\theta,\phi) = \sqrt{(2-\delta_{0,m})(2n+1)\frac{(n-m)!}{(n+m)!}}P_{n,m}(\cos\phi)\sin m\theta \tag{20}$$

The factor $N_{n,m} := \sqrt{(2-\delta_{0,m})(2n+1)\frac{(n-m)!}{(n+m)!}}$ is called the *normalization factor* of the spherical harmonics and $\delta_{0,m}$ is the Kronecker delta and will become clear in the next section.

The associated Legendre polynomials satisfy an orthogonality relation:



(a) $m=1$ **(b)** $m=2$

Figure 2: Graphic representation of the first associated Legendre polynomials for $m=1$ and $m=2$.

n	m	$Y^{c}_{n,m}(\theta,\phi)$	n	m	$Y^{c}_{n,m}(\theta,\phi)$
0	0	1	2	2	$\frac{\sqrt{15}}{2}(\sin\phi)^2\cos 2\theta$
1	0	$\sqrt{3}\cos\phi$	3	0	$\frac{\sqrt{7}}{2}\cos\phi(5(\cos\phi)^2-3)$
1	1	$\sqrt{3}\sin\phi\cos\theta$	3	1	$\frac{\sqrt{42}}{4}(5(\cos\phi)^2-1)\sin\phi\cos\theta$
2	0	$\frac{\sqrt{5}}{2}(3(\cos\phi)^2-1)$	3	2	$\frac{\sqrt{105}}{2}(\sin\phi)^2\cos\phi\cos 2\theta$
2	1	$\sqrt{15}\sin\phi\cos\phi\cos\theta$	3	3	$\frac{\sqrt{70}}{4}(\sin\phi)^3\cos 3\theta$

Table 3: First cosine spherical harmonics.

Lemma 7. Let $n_1,n_2\in\mathbb{N}\cup\{0\}$ and $m\le\min\{n_1,n_2\}$. Then:

$$\int_0^1 P_{n_1,m}(x)P_{n_2,m}(x)\,\mathrm{d}x=\frac{2}{2n_1+1}\frac{(n_1+m)!}{(n_1-m)!}\delta_{n_1,n_2} \tag{21}$$

where δ_{n_1,n_2} denotes the Kronecker delta.

Similarly, it can be shown that the spherical harmonics from an orthonormal family of functions:

Proposition 8. The family of spherical harmonics $\{Y^{c}_{n,m}(\theta,\phi),Y^{s}_{n,m}(\theta,\phi):n\in\mathbb{N}\cup\{0\},m\le n\}$ is orthonormal in the following sense:

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}Y^{i}_{n_1,m_1}(\theta,\phi)Y^{j}_{n_2,m_2}(\theta,\phi)\mathrm{d}\Omega=\delta_{n_1,n_2}\delta_{m_1,m_2}\delta_{i,j} \tag{22}$$

where $\mathrm{d}\Omega=\sin\phi\,\mathrm{d}\phi\,\mathrm{d}\theta$ is the solid angle element, which measures the element of area on a sphere of radius 1.

Proof. Let N_{n_1,m_1}, N_{n_2,m_2} be the normalization factors of the spherical harmonics Y_{n_1,m_1}, Y_{n_2,m_2} respectively. Note that we can separate the variables in the integral of Eq. (22). So if $i\ne j$, the integral over θ becomes $\int_0^{2\pi}\cos(m_1\theta)\sin(m_2\theta)\,\mathrm{d}\theta$ which is equal to 0 regardless of the values of m_1 and m_2. So from now on assume that $i=j$. Due to the symmetry between the cosine and the sine we can suppose that $i=\mathrm{c}$. Thus:

$$\int_0^{2\pi}\int_0^{\pi}Y^{i}_{n_1,m_1}(\theta,\phi)Y^{j}_{n_2,m_2}(\theta,\phi)\mathrm{d}\Omega=$$

$$=N_{n_1,m_1}N_{n_2,m_2}\int_0^{\pi}P_{n_1,m_1}(\cos\phi)P_{n_2,m_2}(\cos\phi)\sin\phi\mathrm{d}\phi\int_0^{2\pi}\cos(m_1\theta)\cos(m_2\theta)\,\mathrm{d}\theta \tag{23}$$

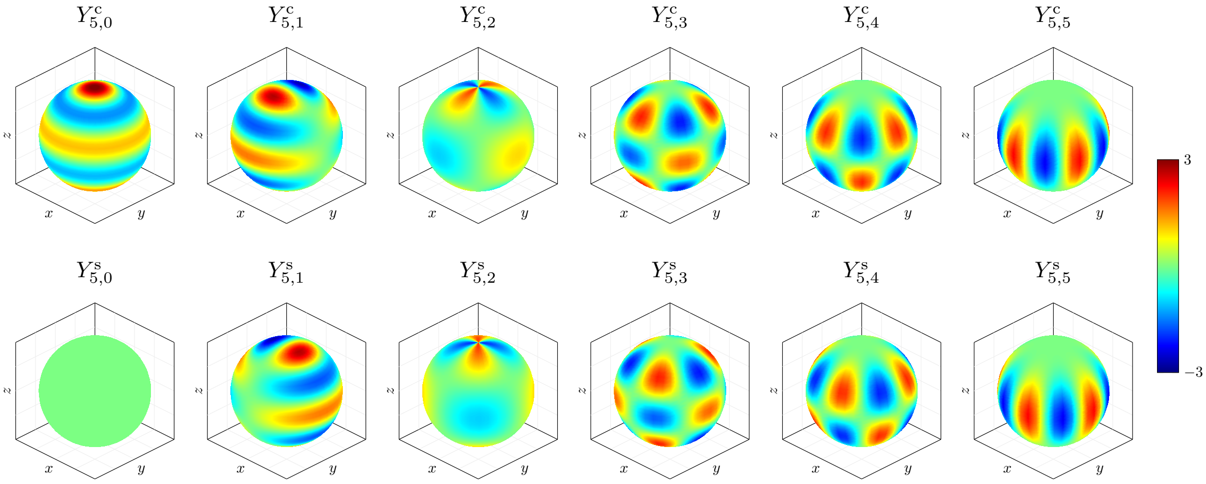


Figure 3: 3D heat map of the spherical harmonics of degree $n = 5$. The first row correspond to the cosine spherical harmonics and the second row correspond to the sine spherical harmonics.

An easy check shows that if $m_1 \neq m_2$ then the integral over θ is zero (and the same applies with sines). So suppose $m_1 = m_2 = m$. In that case, if $m \neq 0$ we have $\int_0^{2\pi} (\cos m\theta)^2 \,\mathrm{d}\theta = \int_0^{2\pi} (\sin m\theta)^2 \,\mathrm{d}\theta = \pi$ and if $m = 0$, the cosine integral evaluates to 2π whereas the sine integral is 0. We can omit this latter case because $Y^{\mathrm{s}}_{n,0}$ is identically zero. Thus:

$$\frac{2\pi}{2-\delta_{0,m}} N_{n_1,m} N_{n_2,m} \int_0^{\pi} P_{n_1,m}(\cos\phi) P_{n_2,m}(\cos\phi) \sin\phi \mathrm{d}\phi = \frac{2\pi}{2-\delta_{0,m}} N_{n_1,m} N_{n_2,m} \int_{-1}^{1} P_{n_1,m}(x) P_{n_2,m}(x) \,\mathrm{d}x \tag{24}$$

By Lemma 7 this latter integral is $\frac{2}{2n_1+1}\frac{(n_1+m)!}{(n_1-m)!}\delta_{n_1,n_2}$. Finally, if $n_1 = n_2 = n$, putting all normalization factors together we get:

$$\frac{2\pi}{2-\delta_{0,m}} N_n^m N_n^m \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} = 4\pi \tag{25}$$

□

Moreover, an important result in the Sturm-Liouville Theory of second order differential equations ([Wikb; Wan+09]) says that the family of spherical harmonics $\{Y^{\mathrm{c}}_{n,m}(\theta,\phi), Y^{\mathrm{s}}_{n,m}(\theta,\phi) : n \in \mathbb{N} \cup \{0\}, m \leq n\}$ form a complete set in the sense that any smooth function defined on the sphere $f : \mathrm{S}^2 \to \mathbb{R}$ can be expanded in a series of spherical harmonics:

$$f(\theta,\phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (c_{n,m} Y^{\mathrm{c}}_{n,m}(\theta,\phi) + s_{n,m} Y^{\mathrm{s}}_{n,m}(\theta,\phi)) \tag{26}$$

This will be useful in Section 1.2.3 when expanding the gravitational potential created by the Earth at some arbitrary point in spherical harmonics.

1.2.2 Laplace's equation in spherical coordinates

Definition 9. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a twice-differentiable function. The *Laplace equation* is the equation

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{27}$$

where Δ is the Laplace operator.

The next proposition gives the Laplace equation in spherical coordinates.

Proposition 10. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a twice-differentiable function. Then:

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial f}{\partial \phi}\right) + \frac{1}{r^2 (\sin\phi)^2}\frac{\partial^2 f}{\partial \theta^2} \tag{28}$$

where $r \in [0, \infty)$ denotes the radial distance, $\theta \in [-\pi, \pi)$ denotes the longitude, and $\phi \in [0, \pi]$, the colatitude:

$$\begin{aligned} x &= r \sin\phi \cos\theta \\ y &= r \sin\phi \sin\theta \\ z &= r \cos\phi \end{aligned} \tag{29}$$

We are now interested in solving the Laplace equation. Theorem 11 gives the solution of it as a function of the spherical harmonics.

Theorem 11. The regular solutions in a bounded region $\Omega \subseteq \mathbb{R}^3$ such that $0 \notin \overline{\Omega}$ of the Laplace equation in spherical coordinates are of the form

$$f(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (a_n r^n + b_n r^{-n-1}) P_{n,m}(\cos\phi)(c_{n,m}\cos(m\theta) + s_{n,m}\sin(m\theta)) \tag{30}$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{n} (a_n r^n + b_n r^{-n-1})(\tilde{c}_{n,m} Y^{\mathrm{c}}_{n,m}(\theta,\phi) + \tilde{s}_{n,m} Y^{\mathrm{s}}_{n,m}(\theta,\phi)) \tag{31}$$

where $a_n, b_n, c_{n,m}, s_{n,m}, \tilde{c}_{n,m}, \tilde{s}_{n,m} \in \mathbb{R}$.

Proof. Let $f(r, \theta, \phi)$ be a solution of Eq. (28). Using separation variables $f(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$ we can write:

$$\frac{\Theta\Phi}{r^2}(r^2 R')' + \frac{R\Theta}{r^2 \sin\phi}(\sin\phi \Phi')' + \frac{R\Phi}{r^2(\sin\phi)^2}\Theta'' = 0 \tag{32}$$

Here, we are making and abuse of notation denoting all the derivatives with a prime, but the reader should have no confusion with it. Isolating R from Θ and Φ yields:

$$\frac{(r^2 R')'}{R} = -\frac{1}{\sin\phi \Phi}(\sin\phi \Phi')' - \frac{1}{(\sin\phi)^2\Theta}\Theta'' \tag{33}$$

Since the left-hand side depends entirely on r and the right-hand side does not, it follows that both sides must be constant. Therefore:

$$\frac{(r^2 R')'}{R} = \lambda \tag{34}$$

$$\frac{1}{\sin\phi \Phi}(\sin\phi \Phi')' + \frac{1}{(\sin\phi)^2\Theta}\Theta'' = -\lambda \tag{35}$$

with $\lambda \in \mathbb{R}$. Similarly, separating variables from Eq. (35) we obtain that the equations

$$\frac{1}{\Theta}\Theta'' = -m^2 \tag{36}$$

$$\frac{\sin\phi}{\Phi}(\sin\phi \Phi')' + \lambda(\sin\phi)^2 = m^2 \tag{37}$$

must be constant with $m \in \mathbb{C}$ (a priori). The solution to the well-known Eq. (36) is a linear combination of the $\cos(m\theta)$ and $\sin(m\theta)$. Note, though, that since Θ must be a 2π-periodic function, that is satisfying $\Theta(\theta + 2\pi) = \Theta(\theta)$ $\forall \theta \in \mathbb{R}$, m must be an integer. On the other hand making the change of variables $x = \cos\phi$ and $y = \Phi(\phi)$ in Eq. (37) and using the chain rule, that equation becomes:

$$(1 - x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\lambda - \frac{m^2}{1 - x^2}\right) y = 0 \tag{38}$$

which is the associate Legendre equation. We have argued in Proposition 5 that we need $\lambda = n(n+1)$ and $m \leq n$ in order to obtain regular solutions at $x = \cos\phi = \pm 1$. Moreover, these solutions are $P_{n,m}(\cos\phi)$.

Finally, note that equation Eq. (34) is a Cauchy-Euler equation (check [Wika]) and so the general solution of it is given by

$$R(r) = c_1 r^n + c_2 r^{-n-1} \tag{39}$$

because $\lambda = n(n+1)$ (the reader may check that r^n and r^{-n-1} are indeed two independent solutions of Eq. (34)). So the general solution becomes a linear combination of each solution found varying $n \in \mathbb{N} \cup \{0\}$ and $m \in \{0, 1, \ldots, n\}$:

$$f(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (a_n r^n + b_n r^{-n-1}) P_{n,m}(\cos\phi)(c_{n,m}\cos(m\theta) + s_{n,m}\sin(m\theta)) \tag{40}$$

$\square$

We ignore the singularity at $r = 0$ of Eq. (31) from now. See Section 1.2.3 for more details.

1.2.3 Expansion in spherical harmonics

We have just seen that if V satisfies the exterior Dirichlet problem for the Laplace equation, then, by uniqueness of solutions, it can be expressed as:

$$V(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (a_n r^n + b_n r^{-n-1})(\tilde{c}_{n,m} Y^{\mathrm{c}}_{n,m}(\theta, \phi) + \tilde{s}_{n,m} Y^{\mathrm{s}}_{n,m}(\theta, \phi)) \tag{41}$$

where $a_n, b_n, \tilde{c}_{n,m}, \tilde{s}_{n,m} \in \mathbb{R}$. If we impose V to satisfy the third condition of Eq. (10), we must have $a_n = 0$. Finally, if we choose $R_\oplus$ as a reference radius for a spherical model of the Earth, using the boundary condition on $\partial\Omega$

$$f(\theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{b_n}{{R_\oplus}^{n+1}} (\tilde{c}_{n,m} Y^{\mathrm{c}}_{n,m}(\theta, \phi) + \tilde{s}_{n,m} Y^{\mathrm{s}}_{n,m}(\theta, \phi)) \tag{42}$$

and the orthogonality of the spherical harmonics, we can deduce that the coefficients $b_n \tilde{c}_{n,m}$ and $b_n \tilde{s}_{n,m}$ are given by:

$$b_n \tilde{c}_{n,m} = \frac{{R_\oplus}^{n+1}}{4\pi} \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) Y^{\mathrm{c}}_{n,m}(\theta, \phi) \sin\phi \, \mathrm{d}\phi \, \mathrm{d}\theta \tag{43}$$

$$b_n \tilde{s}_{n,m} = \frac{{R_\oplus}^{n+1}}{4\pi} \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) Y^{\mathrm{s}}_{n,m}(\theta, \phi) \sin\phi \, \mathrm{d}\phi \, \mathrm{d}\theta \tag{44}$$

Hence, introducing the gravitational constant G and the Earth's mass $M_\oplus$ into the equation, our final expression for the gravitational potential is

$$V(r, \theta, \phi) = \frac{GM_\oplus}{R_\oplus} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{R_\oplus}{r}\right)^{n+1} (\bar{C}_{n,m} Y^{\mathrm{c}}_{n,m}(\theta, \phi) + \bar{S}_{n,m} Y^{\mathrm{s}}_{n,m}(\theta, \phi)) \tag{45}$$

where the coefficients $\bar{C}_{n,m}, \bar{S}_{n,m} \in \mathbb{R}$ are given by the formulas:

$$\bar{C}_{n,m} = \frac{1}{4\pi} \frac{R_\oplus}{GM_\oplus} \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) Y^{\mathrm{c}}_{n,m}(\theta, \phi) \sin\phi \, \mathrm{d}\phi \, \mathrm{d}\theta \tag{46}$$

$$\bar{S}_{n,m} = \frac{1}{4\pi} \frac{R_\oplus}{GM_\oplus} \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) Y^{\mathrm{s}}_{n,m}(\theta, \phi) \sin\phi \, \mathrm{d}\phi \, \mathrm{d}\theta \tag{47}$$

The coefficients $\bar{C}_{n,m}$, $\bar{S}_{n,m}$ are called *geopotential coefficients*, and they describe the dependence on the Earth's internal structure. They are obtained from observation of the perturbations seen in the orbits of other satellites [MG05], because it is not possible to measure the Earth's density directly. Other methods for obtaining such data are through surface gravimetry, which provides precise local and regional information about the gravity field, or through altimeter data, which can be used to provide a model for the geoid of the Earth (which is the shape that the ocean surface takes under the influence of the gravity of Earth) which in turn can be used to obtain the geopotential coefficients.

1.3 Numerical computation

1.3.1 Gravitational acceleration

Up to this point, we have only studied the gravitational potential exerted by the non-homogeneous Earth on a satellite. But, in order to integrate the equations of motion of the satellite, we need to compute the gravitational acceleration $\mathbf{g} = -\boldsymbol{\nabla} V$ instead. In order to do this efficiently, we will make use of the following formulas given in [MG05; Cun70]. First, let

$$V_{n,m}(\theta,\phi) = \left(\frac{R_\oplus}{r}\right)^{n+1} P_{n,m}(\cos\phi)\cos(m\theta) \qquad W_{n,m}(\theta,\phi) = \left(\frac{R_\oplus}{r}\right)^{n+1} P_{n,m}(\cos\phi)\sin(m\theta) \tag{48}$$

Thus, we can write:

$$V = \frac{GM_\oplus}{R_\oplus}\sum_{n=0}^{\infty}\sum_{m=0}^{n}(\bar{C}_{n,m}N_{n,m}V_{n,m} + \bar{S}_{n,m}N_{n,m}W_{n,m}) \tag{49}$$

Let $C_{n,m} := \bar{C}_{n,m}N_{n,m}$ and $S_{n,m} := \bar{S}_{n,m}N_{n,m}$. If $\mathbf{g} = (\ddot{x},\ddot{y},\ddot{z})$, then:

$$\ddot{x} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\ddot{x}_{n,m} \qquad \ddot{y} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\ddot{y}_{n,m} \qquad \ddot{z} = \sum_{n=0}^{\infty}\sum_{m=0}^{n}\ddot{z}_{n,m} \tag{50}$$

where the *partial* accelerations $\ddot{x}_{n,m}$, $\ddot{y}_{n,m}$, $\ddot{z}_{n,m}$ are given by:

$$\ddot{x}_{n,m} = \begin{cases} -\dfrac{GM_\oplus}{{R_\oplus}^2}C_{n,0}V_{n+1,1} & \text{if } m=0 \\ -\dfrac{GM_\oplus}{{R_\oplus}^2}\cdot\dfrac{1}{2}\Big[C_{n,m}V_{n+1,m+1} + S_{n,m}W_{n+1,m+1} - \\ \qquad -\dfrac{(n-m+2)!}{(n-m)!}\left(C_{n,m}V_{n+1,m-1} + S_{n,m}W_{n+1,m-1}\right)\Big] & \text{if } m>0 \end{cases} \tag{51}$$

$$\ddot{y}_{n,m} = \begin{cases} -\dfrac{GM_\oplus}{{R_\oplus}^2}C_{n,0}W_{n+1,1} & \text{if } m=0 \\ -\dfrac{GM_\oplus}{{R_\oplus}^2}\cdot\dfrac{1}{2}\Big[C_{n,m}W_{n+1,m+1} - S_{n,m}V_{n+1,m+1} - \\ \qquad -\dfrac{(n-m+2)!}{(n-m)!}\left(C_{n,m}W_{n+1,m-1} - S_{n,m}V_{n+1,m-1}\right)\Big] & \text{if } m>0 \end{cases} \tag{52}$$

$$\ddot{z}_{n,m} = -\frac{GM_\oplus}{{R_\oplus}^2}(n-m+1)\left(C_{n,m}V_{n+1,m} + S_{n,m}W_{n+1,m}\right) \tag{53}$$

and the functions $V_{n,m}$, $W_{n,m}$ are calculated using the following recurrence relations:

$$\begin{cases} \begin{aligned} V_{n,m} &= \frac{2n-1}{n-m}\frac{R_\oplus}{r}\cos\phi V_{n-1,m} - \frac{n+m-1}{n-m}\frac{{R_\oplus}^2}{r^2}V_{n-2,m} \\ W_{n,m} &= \frac{2n-1}{n-m}\frac{R_\oplus}{r}\cos\phi W_{n-1,m} - \frac{n+m-1}{n-m}\frac{{R_\oplus}^2}{r^2}W_{n-2,m} \end{aligned} & \text{if } 0 \le m \le n-2 \\[2ex] \begin{aligned} V_{n,n-1} &= (2n-1)\frac{R_\oplus}{r}\cos\phi V_{n-1,n-1} \\ W_{n,n-1} &= (2n-1)\frac{R_\oplus}{r}\cos\phi W_{n-1,n-1} \end{aligned} & \text{if } m = n-1 \\[2ex] \begin{aligned} V_{n,n} &= (2m-1)\frac{R_\oplus}{r}\sin\phi[\cos\theta V_{n-1,n-1} - \sin\theta W_{n-1,n-1}] \\ W_{n,n} &= (2m-1)\frac{R_\oplus}{r}\sin\phi[\cos\theta W_{n-1,n-1} + \sin\theta V_{n-1,n-1}] \end{aligned} & \text{if } m = n \end{cases}$$

starting from the initial quantities $V_{00} = \frac{R_\oplus}{r}$ and $W_{00} = 0$ and using the following scheme [MG05]:

$$
\begin{tikzcd}
V_{0,0},\ W_{0,0} \arrow[d] \arrow[dr] & & & & \\
V_{1,0},\ W_{1,0} \arrow[d] & V_{1,1},\ W_{1,1} \arrow[d] \arrow[dr] & & & \\
V_{2,0},\ W_{2,0} \arrow[d] & V_{2,1},\ W_{2,1} \arrow[d] & V_{2,2},\ W_{2,2} \arrow[d] \arrow[dr] & & \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \ddots \arrow[dr] & \\
V_{n_{\max},0},\ W_{n_{\max},0} & V_{n_{\max},1},\ W_{n_{\max},1} & V_{n_{\max},2},\ W_{n_{\max},2} & \cdots & V_{n_{\max},n_{\max}},\ W_{n_{\max},n_{\max}}
\end{tikzcd}
$$

Figure 4: Scheme for the calculation of the coefficients $V_{n,m}$ and $W_{n,m}$ for $0 \leq m \leq n \leq n_{\max}$. The red arrows indicate that the coefficients are calculated using the first of the above recursions; the green arrows indicate that they are calculated using the second recursion; and the blue arrows indicate that they are calculated using the third recursion.

1.3.2 Position and velocity

With all the ingredients in place, the position and velocity of the satellite can be calculated from the following system of differential equations:

$$
\ddot{\mathbf{r}} = \mathbf{g} \implies \begin{cases} x' = v_x \\ y' = v_y \\ z' = v_z \\ {v_x}' = \ddot{x} \\ {v_y}' = \ddot{y} \\ {v_z}' = \ddot{z} \end{cases}
$$

where $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ are given in Eq. (50). The initial conditions will be the position and velocity obtained from the TLE.

In order to solve this system of differential equations, we have opted to use the Runge-Kutta-Fehlberg of order 78.

1.4 Other perturbations

atmospheric drag, solar radiation pressure, and the gravitational pull of the Moon and Sun, POSAR-HO SI AL FINAL FAIG SIMULACIO AMB AIXO, SI NO, NOOOO.