

1 Proofs

1.1 Conics in a nutshell

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Proof. Consider the ellipse E centered at the origin and oriented as in ??.

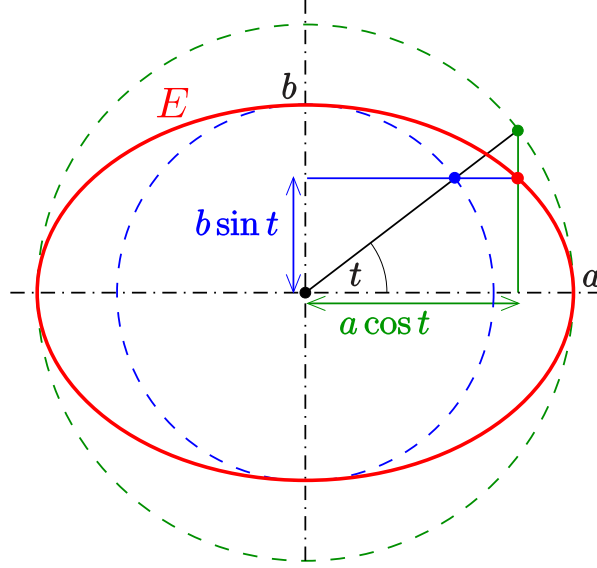


Figure 1: Reference frame centered at the center of the ellipse. Source: [Ag217].

From ?? one can check that it can be parametrized by $(x, y) = (a \cos t, b \sin t)$ with $t \in [0, 2\pi)$. This parametrization satisfies:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (1)$$

Hence, the area enclosed in the ellipse can be parametrized by $(x, y) = (ar \cos t, br \sin t)$, with $r \in [0, 1]$ and $t \in [0, 2\pi)$. The Jacobian of the transformation $(r, t) \rightarrow (x, y)$ transformation is abr . Therefore, from the change of variable theorem we have that:

$$\text{Area}(E) = \iint_E dx dy = \int_0^{2\pi} \int_0^1 abr dr dt = \pi ab \quad (2)$$

□

1.2 Introduction to astrodynamics and satellite tracking

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Proof. Cross-multiplying ?? by \mathbf{h} we obtain

$$\frac{d(\dot{\mathbf{r}} \times \mathbf{h})}{dt} = \dot{\mathbf{r}} \times \mathbf{h} = -\frac{\mu}{r^3} \mathbf{r} \times \mathbf{h} = -\frac{\mu}{r^3} \mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}}) = \frac{\mu}{r^3} [(\mathbf{r} \cdot \mathbf{r})\dot{\mathbf{r}} - (\mathbf{r} \cdot \dot{\mathbf{r}})\mathbf{r}] \quad (3)$$

where in the last equality we have used the vector equality $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$ for $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Now note that:

$$\frac{d}{dt} \left(\frac{\mathbf{r}}{r} \right) = \frac{\dot{\mathbf{r}}}{r} - \frac{\dot{r}}{r^2} \mathbf{r} = \frac{1}{r^3} [(\mathbf{r} \cdot \mathbf{r})\dot{\mathbf{r}} - (\mathbf{r} \cdot \dot{\mathbf{r}})\mathbf{r}] \quad (4)$$

because¹ $2r\dot{r} = \frac{d(r^2)}{dt} = \frac{d(\mathbf{r} \cdot \mathbf{r})}{dt} = 2\mathbf{r} \cdot \dot{\mathbf{r}}$. Thus:

$$\frac{d(\dot{\mathbf{r}} \times \mathbf{h})}{dt} = \mu \frac{d}{dt} \left(\frac{\mathbf{r}}{r} \right) \quad (5)$$

Integrating with respect to the time yields

$$\dot{\mathbf{r}} \times \mathbf{h} = \frac{\mu}{r} \mathbf{r} + \mathbf{B} \quad (6)$$

where $\mathbf{B} \in \mathbb{R}^3$ is the constant of integration. Observe that since $\dot{\mathbf{r}} \times \mathbf{h}$ is perpendicular to \mathbf{h} , $\dot{\mathbf{r}} \times \mathbf{h}$ lies on the orbital plane and so does \mathbf{r} . Hence, \mathbf{B} lies on the orbital plane too. Now, dot-multiplying this last equation by \mathbf{r} and using that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ we obtain

$$h^2 = \mathbf{h} \cdot \mathbf{h} = (\mathbf{r} \times \dot{\mathbf{r}}) \cdot \mathbf{h} = \mathbf{r} \cdot (\dot{\mathbf{r}} \times \mathbf{h}) = \frac{\mu}{r} \mathbf{r} \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{B} = \mu r + rB \cos \nu \quad (7)$$

where $h := \|\mathbf{h}\|$, $B := \|\mathbf{B}\|$ and ν denotes the angle between \mathbf{r} and \mathbf{B} , called *true anomaly*. Rearranging the terms we finally obtain the equation of a conic section

$$r = \frac{h^2/\mu}{1 + (B/\mu) \cos(\nu)} \quad (8)$$

with $p := h^2/\mu$ and $e := B/\mu$. □

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Proof. Recall that the area of a parallelogram generated by two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ is given by $\|\mathbf{u} \times \mathbf{v}\|$. Thus, approximating the difference $A(t+k) - A(t)$ by half of the area of the parallelogram generated by $\mathbf{r}(t)$ and $\mathbf{r}(t+k)$ (see ??) we obtain:

$$\begin{aligned} \frac{dA(t)}{dt} &= \lim_{k \rightarrow 0} \frac{A(t+k) - A(t)}{k} = \lim_{k \rightarrow 0} \frac{\|\mathbf{r}(t) \times \mathbf{r}(t+k)\|}{2k} = \lim_{k \rightarrow 0} \frac{\|\mathbf{r}(t) \times (\mathbf{r}(t+k) - \mathbf{r}(t))\|}{2k} = \\ &= \frac{\|\mathbf{r}(t) \times \dot{\mathbf{r}}(t)\|}{2} = \frac{h}{2} \end{aligned} \quad (9)$$

where the penultimate equality is due to the continuity and linearity of the cross product. □

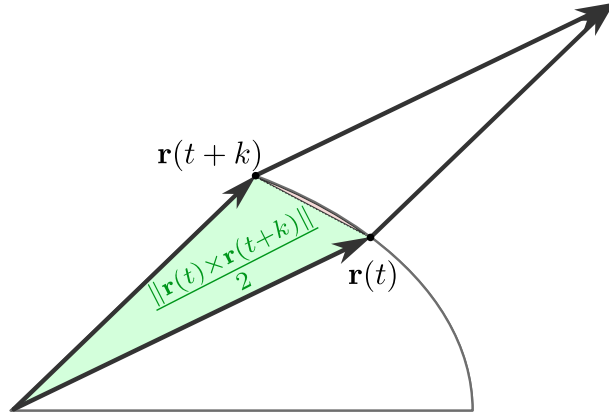


Figure 2: Graphical representation of the error made (red region) when approximating the area swept by the radius vector by half the area of the parallelogram generated by $\mathbf{r}(t)$ and $\mathbf{r}(t+k)$ (green region).

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¹Bear in mind that in general $\dot{r} \neq \|\dot{\mathbf{r}}\|$. Indeed, if β denotes the angle between \mathbf{r} and $\dot{\mathbf{r}}$ we have that $\dot{r} = \|\dot{\mathbf{r}}\| \cos \beta$. In particular \dot{r} may be negative.

Proof. Integrating ?? with respect to time between 0 and T (the period) yields:

$$\pi ab = A(T) = \int_0^T A'(t) dt = \int_0^T \frac{h}{2} dt = \frac{hT}{2} \implies n = \frac{2\pi}{T} = \frac{h}{ab} = \frac{h}{a^2 \sqrt{1-e^2}} = \sqrt{\frac{\mu}{a^3}} \quad (10)$$

where we have used ????. □

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Proof. We first prove the uniqueness. Clearly $f \in \mathcal{C}^1(\mathbb{R})$ and $f'(E) = 1 - e \cos E > 0$ for all $E \in [0, 2\pi)$ because $e < 1$. Thus, f is strictly increasing and so it has at most one zero. Now, if $0 \leq \bar{M} < \pi$, then:

$$f(M) = -e \sin M \leq 0 \quad \text{and} \quad f(M+e) = e(1 - \sin(M+e)) \geq 0 \quad (11)$$

So by Bolzano's theorem, f has a solution in $[M, M+e]$. If $\pi \leq \bar{M} < 2\pi$, then:

$$f(M) = -e \sin M \geq 0 \quad \text{and} \quad f(M-e) = -e(1 + \sin(M-e)) \leq 0 \quad (12)$$

So again by Bolzano's theorem, f has a solution in $[M-e, M]$. □

1.3 Earth's gravitational field and other perturbations

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Proof. An easy computation shows that fixed $\mathbf{s} \in \mathbb{R}^3$ we have:

$$\nabla \left(\frac{1}{\|\mathbf{r} - \mathbf{s}\|} \right) = -\frac{1}{\|\mathbf{r} - \mathbf{s}\|^3} (\mathbf{r} - \mathbf{s}) \quad (13)$$

So we need to justify whether the following exchange between the gradient and the integral is correct:

$$\mathbf{g} = - \int_{\Omega} \frac{\rho(\mathbf{s})}{\|\mathbf{r} - \mathbf{s}\|^3} (\mathbf{r} - \mathbf{s}) d^3\mathbf{s} = \int_{\Omega} \rho(\mathbf{s}) \nabla \left(\frac{1}{\|\mathbf{r} - \mathbf{s}\|} \right) d^3\mathbf{s} = \nabla \int_{\Omega} \frac{\rho(\mathbf{s})}{\|\mathbf{r} - \mathbf{s}\|} d^3\mathbf{s} \quad (14)$$

Without loss of generality it suffices to justify that

$$\frac{\partial}{\partial x} \int_{\Omega} \frac{\rho(\mathbf{s})}{\|\mathbf{r} - \mathbf{s}\|} d^3\mathbf{s} = \int_{\Omega} \frac{\partial}{\partial x} \left(\frac{\rho(\mathbf{s})}{\|\mathbf{r} - \mathbf{s}\|} \right) d^3\mathbf{s} \quad (15)$$

assuming $\mathbf{r} = (x, y, z)$ and $\mathbf{s} = (x', y', z')$. In order to apply the theorem of derivation under the integral sign we need to control $\frac{\partial}{\partial x} \left(\frac{\rho(\mathbf{s})}{\|\mathbf{r} - \mathbf{s}\|} \right) = -\rho(\mathbf{s}) \frac{x-x'}{\|\mathbf{r} - \mathbf{s}\|^3}$ by an integrable function $h(\mathbf{s})$. Using spherical coordinates centered at \mathbf{r} and writing $(\mathbf{r} - \mathbf{s})_{\text{sph}} = (\rho_{\mathbf{r}}, \theta, \phi)$, the integrand to bound becomes (in spherical coordinates):

$$\left| -\rho(\mathbf{s}) \frac{x-x'}{\|\mathbf{r} - \mathbf{s}\|^3} \rho_{\mathbf{r}}^2 \sin \phi \right| = |\rho(\mathbf{s})| \left| \frac{\rho_{\mathbf{r}} \cos \theta \sin \phi}{\rho_{\mathbf{r}}^3} \rho_{\mathbf{r}}^2 \sin \phi \right| \leq |\rho(\mathbf{s})| \leq K \quad (16)$$

where the last inequality follows for certain $K \in \mathbb{R}$ by Weierstrass theorem (ρ is continuous and Ω is compact). Thus, since $h(\mathbf{s}) = K$ is integrable, because Ω is bounded, the equality of ?? is correct. □

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Proof. Recall that $\Delta V = \text{div}(\nabla V)$. So since $\mathbf{g} = -\nabla V$ it suffices to prove that $\text{div}(\mathbf{g}) = 0$. Note that if $\mathbf{r} \in \Omega^c$, then $\exists \delta > 0$ such that $\|\mathbf{r} - \mathbf{s}\| \geq \delta > 0 \forall \mathbf{s} \in \Omega$ because Ω is closed. As a result, $\frac{\mathbf{r} - \mathbf{s}}{\|\mathbf{r} - \mathbf{s}\|^3}$ is differentiable and:

$$\begin{aligned} \text{div} \left(\frac{\mathbf{r} - \mathbf{s}}{\|\mathbf{r} - \mathbf{s}\|^3} \right) &= \frac{\partial}{\partial x} \left(\frac{x-x'}{\|\mathbf{r} - \mathbf{s}\|^3} \right) + \frac{\partial}{\partial y} \left(\frac{y-y'}{\|\mathbf{r} - \mathbf{s}\|^3} \right) + \frac{\partial}{\partial z} \left(\frac{z-z'}{\|\mathbf{r} - \mathbf{s}\|^3} \right) = \\ &= \frac{\|\mathbf{r} - \mathbf{s}\|^2 - 3(x-x')^2}{\|\mathbf{r} - \mathbf{s}\|^5} + \frac{\|\mathbf{r} - \mathbf{s}\|^2 - 3(y-y')^2}{\|\mathbf{r} - \mathbf{s}\|^5} + \frac{\|\mathbf{r} - \mathbf{s}\|^2 - 3(z-z')^2}{\|\mathbf{r} - \mathbf{s}\|^5} = 0 \end{aligned}$$

Hence, as in ??, we have that:

$$\left| \rho(\mathbf{s}) \frac{\|\mathbf{r} - \mathbf{s}\|^2 - 3(x - x')^2}{\|\mathbf{r} - \mathbf{s}\|^5} \right| \leq \frac{4|\rho(\mathbf{s})|}{\|\mathbf{r} - \mathbf{s}\|^3} \leq \frac{4|\rho(\mathbf{s})|}{\delta^3} \quad (17)$$

which is integrable by Weierstrass theorem. Thus, by the theorem of derivation under the integral sign:

$$\mathbf{div}(\mathbf{g}) = -\mathbf{div} \int_{\Omega} \frac{\rho(\mathbf{s})}{\|\mathbf{r} - \mathbf{s}\|^3} (\mathbf{r} - \mathbf{s}) d^3\mathbf{s} = -\int_{\Omega} \rho(\mathbf{s}) \mathbf{div} \left(\frac{\mathbf{r} - \mathbf{s}}{\|\mathbf{r} - \mathbf{s}\|^3} \right) d^3\mathbf{s} = 0 \quad (18)$$

□

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Proof. Suppose we have two solutions V_1, V_2 of ?. Then, $W := V_1 - V_2$ is harmonic in Ω^c , $W = 0$ on $\partial\Omega$ and $\lim_{\|\mathbf{r}\| \rightarrow \infty} W = 0$. So $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ large enough such that $\Omega \subseteq B(0, n)$ and $|W| \leq \varepsilon$ on $\mathbb{R}^3 \setminus \overline{B(0, n)}$.

Thus, by the maximum principle, $|W| \leq \varepsilon$ on $\overline{B(0, n)} \cap \Omega^c$. Since the ε is arbitrary, we must have $W = 0$ on Ω^c , that is, $V_1 = V_2$. □

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Proof. Let $N_{n_1, m_1}, N_{n_2, m_2}$ be the normalization factors of the spherical harmonics $Y_{n_1, m_1}, Y_{n_2, m_2}$ respectively. Note that we can separate the variables in the integral of ?. So if $i \neq j$, the integral over θ becomes $\int_0^{2\pi} \cos(m_1\theta) \sin(m_2\theta) d\theta$ which is equal to 0 regardless of the values of m_1 and m_2 . So from now on assume that $i = j$. Due to the symmetry between the cosine and the sine we can suppose that $i = c$. Thus:

$$\begin{aligned} \int_0^{2\pi} \int_0^{\pi} Y_{n_1, m_1}^i(\theta, \phi) Y_{n_2, m_2}^j(\theta, \phi) d\Omega &= \\ &= N_{n_1, m_1} N_{n_2, m_2} \int_0^{\pi} P_{n_1, m_1}(\cos \phi) P_{n_2, m_2}(\cos \phi) \sin \phi d\phi \int_0^{2\pi} \cos(m_1\theta) \cos(m_2\theta) d\theta \quad (19) \end{aligned}$$

An easy check shows that if $m_1 \neq m_2$ then the integral over θ is zero (and the same applies with sines). So suppose $m_1 = m_2 = m$. In that case, if $m \neq 0$ we have $\int_0^{2\pi} (\cos m\theta)^2 d\theta = \int_0^{2\pi} (\sin m\theta)^2 d\theta = \pi$ and if $m = 0$, the cosine integral evaluates to 2π whereas the sine integral is 0. We can omit this latter case because $Y_{n,0}^s$ is identically zero. Thus:

$$\frac{2\pi}{2 - \delta_{0,m}} N_{n_1, m} N_{n_2, m} \int_0^{\pi} P_{n_1, m}(\cos \phi) P_{n_2, m}(\cos \phi) \sin \phi d\phi = \frac{2\pi}{2 - \delta_{0,m}} N_{n_1, m} N_{n_2, m} \int_{-1}^1 P_{n_1, m}(x) P_{n_2, m}(x) dx \quad (20)$$

By ?? this latter integral is $\frac{2}{2n_1+1} \frac{(n_1+m)!}{(n_1-m)!} \delta_{n_1, n_2}$. Finally, if $n_1 = n_2 = n$, putting all normalization factors together we get:

$$\frac{2\pi}{2 - \delta_{0,m}} N_n^m N_n^m \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} = 4\pi \quad (21)$$

□

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Proof. Let $f(r, \theta, \phi)$ be a solution of ?. Using separation variables $f(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$ we can write:

$$\frac{\Theta\Phi}{r^2} (r^2 R')' + \frac{R\Theta}{r^2 \sin \phi} (\sin \phi \Phi')' + \frac{R\Phi}{r^2 (\sin \phi)^2} \Theta'' = 0 \quad (22)$$

Here, we are making and abuse of notation denoting all the derivatives with a prime, but the reader should have no confusion with it. Isolating R from Θ and Φ yields:

$$\frac{(r^2 R')'}{R} = -\frac{1}{\sin \phi \Phi} (\sin \phi \Phi')' - \frac{1}{(\sin \phi)^2 \Theta} \Theta'' \quad (23)$$

Since the left-hand side depends entirely on r and the right-hand side does not, it follows that both sides must be constant. Therefore:

$$\frac{(r^2 R')'}{R} = \lambda \quad (24)$$

$$\frac{1}{\sin \phi \Phi} (\sin \phi \Phi')' + \frac{1}{(\sin \phi)^2 \Theta} \Theta'' = -\lambda \quad (25)$$

with $\lambda \in \mathbb{R}$. Similarly, separating variables from ?? we obtain that the equations

$$\frac{1}{\Theta} \Theta'' = -m^2 \quad (26)$$

$$\frac{\sin \phi}{\Phi} (\sin \phi \Phi')' + \lambda (\sin \phi)^2 = m^2 \quad (27)$$

must be constant with $m \in \mathbb{C}$ (a priori). The solution to the well-known ?? is a linear combination of the $\cos(m\theta)$ and $\sin(m\theta)$. Note, though, that since Θ must be a 2π -periodic function, that is satisfying $\Theta(\theta + 2\pi) = \Theta(\theta) \forall \theta \in \mathbb{R}$, m must be an integer. On the other hand making the change of variables $x = \cos \phi$ and $y = \Phi(\phi)$ in ?? and using the chain rule, that equation becomes:

$$(1-x^2) \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + \left(\lambda - \frac{m^2}{1-x^2} \right) y = 0 \quad (28)$$

which is the associate Legendre equation. We have argued in ?? that we need $\lambda = n(n+1)$ and $m \leq n$ in order to obtain regular solutions at $x = \cos \phi = \pm 1$. Moreover, these solutions are $P_{n,m}(\cos \phi)$.

Finally, note that equation ?? is a Cauchy-Euler equation (check [Wik]) and so the general solution of it is given by

$$R(r) = c_1 r^n + c_2 r^{-n-1} \quad (29)$$

because $\lambda = n(n+1)$ (the reader may check that r^n and r^{-n-1} are indeed two independent solutions of ??). So the general solution becomes a linear combination of each solution found varying $n \in \mathbb{N} \cup \{0\}$ and $m \in \{0, 1, \dots, n\}$:

$$f(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=0}^n (a_n r^n + b_n r^{-n-1}) P_{n,m}(\cos \phi) (c_{n,m} \cos(m\theta) + s_{n,m} \sin(m\theta)) \quad (30)$$

□