# 1 Preliminaries

In this section we will review some basic concepts of linear algebra and vector calculus that will be used throughout the document.

## 1.1 Properties of cross and dot products

**Proposition 1.** Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Then:

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} \tag{1}$$
$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \tag{2}$$

*Proof.* Let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$. Then:

$$\begin{aligned}
((\mathbf{u} \times \mathbf{v}) \times \mathbf{w})_1 &= (u_3 v_1 - u_1 v_3) w_3 - (u_1 v_2 - u_2 v_1) w_2 \\
&= (u_2 w_2 + u_3 w_3) v_1 - (v_2 w_2 + v_3 w_3) u_1 \\
&= (u_1 w_1 + u_2 w_2 + u_3 w_3) v_1 - (v_1 w_1 + v_2 w_2 + v_3 w_3) u_1 \\
&= ((\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u})_1
\end{aligned}$$

The other components are treated similarly. The second equality follows in a similar way. $\square$

**Proposition 2.** Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Then:

1. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} \cdot \mathbf{w} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} \cdot \mathbf{w}$

## 1.2 Conics in a nutshell

**Definition 3.** A conic is the curve obtained as the intersection of a plane with the surface of a double cone (a cone with two *nappes*).

In Fig. 1, we show the 3 types of conics: the ellipse, the parabola, and the hyperbola, which differ on their eccentricity, as we will see later. Note that the circle is a special case of the ellipse.

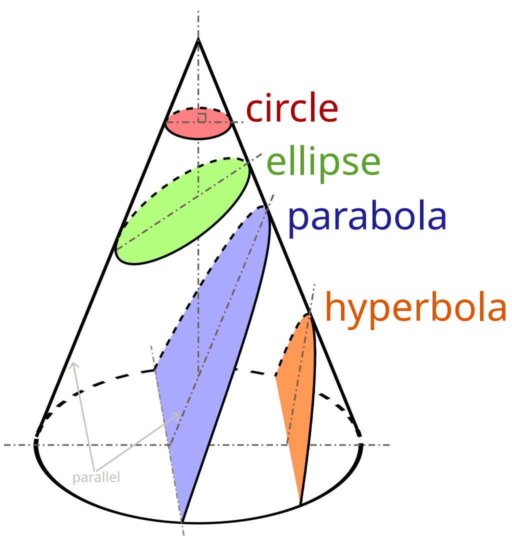


Figure 1: The black boundaries of the colored regions are conic sections. The other half of the hyperbola, which is not shown, is the other nappe of the double cone.

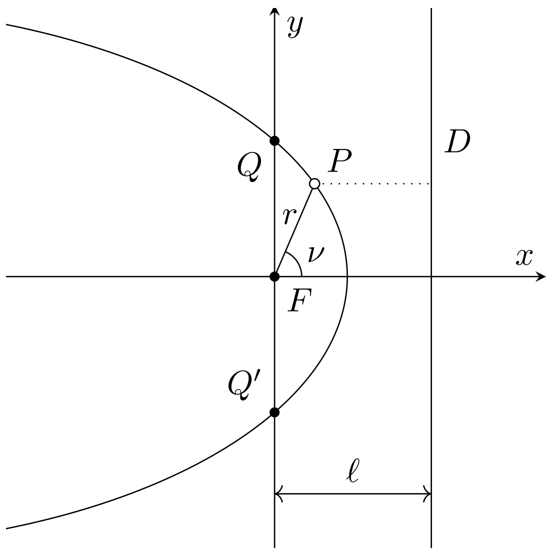


Figure 2: Reference frame centered at the focus of the conic and whose axes are such that the $y$-axis is parallel to the directrix and the $x$-axis is perpendicular to the directrix. The directions of the axes are chosen arbitrarily, subject to the constraint that a positive basis is chosen.

**Definition 4.** The *locus* is a set of points that satisfy a given condition.

The following proposition gives a characterization of the conics.

**Proposition 5.** A conic is the locus of all points $P$ such that the distance from $P$ to a fixed point $F$ is a multiple of the distance from $P$ to a fixed line $D$. Mathematically, this is expressed as:

$$d(P,F) = ed(P,D) \tag{3}$$

where $d$ is the Euclidean distance. The point $F$ is called the *focus*; the line $D$, *directrix*, and the constant of proportionality $e$, *eccentricity*.

Note that using the polar coordinates $(r,\nu)$ as in Fig. 2, we can rewrite Eq. (3) as:

$$r = e(\ell - r\cos\nu) \implies r = \frac{e\ell}{1+e\cos\nu} =: \frac{p}{1+e\cos\nu} \tag{4}$$

where we have defined $p := e\ell$.

**Definition 6.** Le $C$ be a conic and $e$ be its eccentricity. We say that $C$ is

- an *ellipse* if $0 \leq e < 1$,
- a *parabola* if $e = 1$, and
- a *hyperbola* if $e > 1$.

If $e = 0$, the conic is a *circle*.

## 1.3 Spherical harmonics

### 1.3.1 Legendre polynomials, regularity and orthonormality

**Definition 7.** Consider the following second-order differential equation:

$$y'' + p_1(x)y' + p_0(x)y = 0 \tag{5}$$

We say that $a$ is an *ordinary point* if $p_1$ and $p_2$ are analytic at $x = a$. We say that $a$ is a *regular singular point* if $p_1$ has a pole up to order 1 at $a$ and $p_0$ has a pole of order up to 2 at $a$. Otherwise we say that $a$ is a *irregular singular point.*

**Definition 8.** Consider the following second-order differential equation called *Legendre differential equation*:

$$(1-x^2)y'' - 2xy' + \lambda y = 0 \tag{6}$$

for $n \in \mathbb{N} \cup \{0\}$. This equation can be written as:

$$\left((1-x^2)y'\right)' + \lambda y = 0 \tag{7}$$

If seek for analytic solutions of this equation using the power series method, i.e. looking for solutions of the form $y(x) = \sum_{j=0}^{\infty} a_j x^j$ one can check that we obtain this recursion:

$$a_{j+2} = \frac{j(j+1)-\lambda}{(j+1)(j+2)} a_j \quad j = 0,1,2,\ldots \tag{8}$$

From here we can obtain two independent solutions by setting the initial conditions $a_0$ and $a_1$ of the iteration. For example, setting $a_1 = 0$ we obtain a series that has only even powers of $x$. On the other hand, setting $a_0 = 0$ we obtain a series that has only odd powers of $x$. These two series converge on the interval $(-1,1)$ by the ratio test (by looking at Eq. (8)) and can be expressed compactly as [?]:

$$y_{\mathrm{e}}(x) = a_0 \sum_{j=0}^{\infty} \left[\prod_{k=0}^{j-1} (2k(2k+1) - \lambda)\right] \frac{x^{2j+1}}{(2j)!} \qquad y_{\mathrm{o}}(x) = a_1 \sum_{j=0}^{\infty} \left[\prod_{k=1}^{j} (2k(2k+1) - \lambda)\right] \frac{x^{2j+1}}{(2j+1)!} \tag{9}$$

However either for all $\lambda \in \mathbb{R}$ either one of these series diverge at $x = \pm 1$, as it behaves as the harmonic series in a neighbourhood of $\pm 1$. We are interested, though, in the solutions that remain bounded on the whole interval $[-1,1]$. Looking at the expressions of Eq. (9) one can check that the only possibility to make the series converge on $[-1,1]$ is when $\lambda = n(n+1)$, $n \in \mathbb{N} \cup \{0\}$. In this case, either one of the

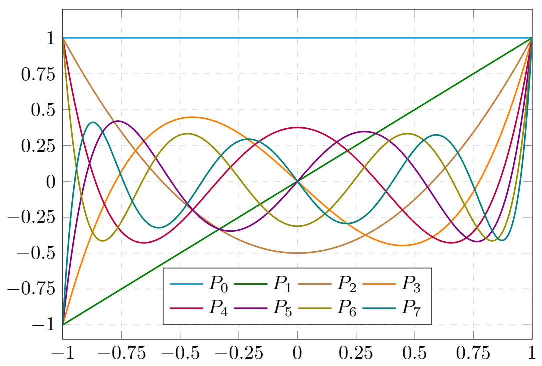


Figure 3: Graphic representation of the first eight Legendre polynomials.

| $n$ | $P_n(x)$ |
|---|---|
| 0 | $1$ |
| 1 | $x$ |
| 2 | $\frac{1}{2}(3x^2-1)$ |
| 3 | $\frac{1}{2}(5x^3-3x)$ |
| 4 | $\frac{1}{8}(35x^4-30x^2+3)$ |
| 5 | $\frac{1}{8}(63x^5-70x^3+15x)$ |
| 6 | $\frac{1}{16}(231x^6-315x^4+105x^2-5)$ |
| 7 | $\frac{1}{16}(429x^7-693x^5+315x^3-35x)$ |

Table 1: First eight Legendre polynomials

series is in fact a polynomial. In both cases it reduces to a polynomial of degree $n$. For each $n \in \mathbb{N} \cup \{0\}$ if we choose $a_0$ or $a_1$ be such that the polynomial evaluates to 1 at $x = 1$, these polynomials are called *Legendre polynomials* and they are denoted by $P_n(x)$. The other (divergent) series is usually denoted in the literature by $Q_n(x)$ (check [?]). And so the general solution of Eq. (7) for $\lambda = n(n+1)$ can be expressed as a linear combination of $P_n$ and $Q_n$.

**Proposition 9.** Consider the function $g_x(t) = \frac{1}{\sqrt{1-2xt+t^2}}$ with $|x| \leq 1$. Then, the generating function of $g$ is:

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n \tag{10}$$

*Proof.* Assume that formally $\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} Q_n(x)t^n$. We want to check that $Q_n(x) = P_n(x)$ for all $n \in \mathbb{N} \cup \{0\}$. Differentiating the equation with respect to $x$ and with respect to $t$ we obtain:

$$\frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} nQ_n(x)t^{n-1} \qquad \frac{t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} {Q_n}'t^n \tag{11}$$

The second equation can be rewritten as:

$$t\sum_{n=0}^{\infty} Q_n t^n = (1-2xt+t^2)\sum_{n=0}^{\infty} {Q_n}'(x)t^{n-1} \tag{12}$$

So equating the coefficients of $t^n$ we get:

$$Q_n = {Q_{n+1}}' - 2x{Q_n}' + {Q_{n-1}}' \tag{13}$$

Moreover, from Eq. (11) we have that:

$$t\sum_{n=0}^{\infty} nQ_n(x)t^{n-1} = (x-t)\sum_{n=0}^{\infty} {Q_n}'(x)t^n \tag{14}$$

Again equating the coefficients of $t^n$ we get:

$$nQ_n = x{Q_n}' - {Q_{n-1}}' \tag{15}$$

Hence differentiating $(1-x^2){P_n}'$ we have:

$$\left((1-x^2){P_n}'\right)' = -2x{P_n}' + (1-x^2)P \tag{16}$$

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**Proposition 10.** Let $y(x)$ be a solution to the Legendre differential equation. Then, $\forall m \in \mathbb{Z}$ the function

$$w_m(x) = (1-x^2)^{m/2} \frac{\mathrm{d}^m y(x)}{\mathrm{d}x^m} \tag{17}$$

solves the *general Legendre differential equation*:

$$(1-x^2)y'' - 2xy' + \left(\lambda - \frac{m^2}{1-x^2}\right) y = 0 \tag{18}$$

In particular if $\lambda = n(n+1)$ for $n \in \mathbb{N} \cup \{0\}$, then $w_(x)$ is denoted as

$$P_n^m(x) := (1-x^2)^{m/2} \frac{\mathrm{d}^m P_n}{\mathrm{d}x^m} \tag{19}$$

and it is called the *associated Legendre polynomial* of degree $n$ and order $m$.

Note that although these functions $P_n^m$ are referred to *polynomials*, they are only *real* polynomials if $m$ is even. But we have opt to call them as it is the common practice in the literature (see [**?**]).
Moreover, from the definition of $P_n^m$, we can see $P_n^0 = P_n$ and that $P_n^m = 0$ if $m > n$. So we can restrict the domain of $m$ to the set $\{0, 1, \dots, n-1, n\}$.

**Definition 11.** Let $n \in \mathbb{N} \cup \{0\}$ and $m \in \{0, 1, \dots, n-1, n\}$. We define the *spherical harmonic* $Y_n^m$ as:

$$Y_n^m(\theta, \phi) = P_n^{|m|}(\cos\phi) \mathrm{e}^{\mathrm{i}m\theta} \tag{20}$$

### 1.3.2 Laplace equation in spherical coordinates

**Definition 12.** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a twice-differentiable function. The *Laplace equation* is the equation

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{21}$$

where $\Delta$ is the Laplace operator.

The next proposition gives the Laplace equation in spherical coordinates.

**Proposition 13.** Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a twice-differentiable function. Then:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial f}{\partial \phi}\right) + \frac{1}{r^2 (\sin\phi)^2}\frac{\partial^2 f}{\partial \theta^2} = 0 \tag{22}$$

where $r$ denotes the radial distance, $\theta$ denotes the azimuthal angle, and $\phi$, the polar angle.

Recall that a solutions to the *Dirichlet problem* on a bounded domain $\Omega \subset \mathbb{R}^3$

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{23}$$

exists and is unique if $g$ is sufficiently smooth. Theorem 14 gives them as a function of the so-called

**Theorem 14.** The regular solutions in a bounded region $\Omega \subseteq \mathbb{R}^3$ such that $0 \notin \overline{\Omega}$ to the Laplace equation in spherical coordinates are of the form

$$f(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (c_n^m r^n + d_n^m r^{-n-1}) P_n^{|m|}(\cos\phi) \mathrm{e}^{\mathrm{i}m\theta} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (c_n^m r^n + d_n^m r^{-n-1}) Y_n^m(\theta, \phi) \tag{24}$$

where $c_n^m, d_n^m \in \mathbb{C}$.

*Proof.* Let $f(r, \theta, \phi)$ be a solution of Eq. (22) Using separation variables $f(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$ one can write:

$$\frac{\Theta\Phi}{r^2}(r^2 R')' + \frac{R\Theta}{r^2 \sin\phi}(\sin\phi \Phi')' + \frac{R\Phi}{r^2 (\sin\phi)^2}\Theta'' = 0 \tag{25}$$

Isolating $R$ from $\Theta$ and $\Phi$ yields:

$$\frac{(r^2 R')'}{R} = -\frac{1}{\sin\phi\Phi}(\sin\phi\Phi')' - \frac{1}{(\sin\phi)^2\Theta}\Theta'' \tag{26}$$

Since the left-hand side depends entirely on $r$ and the right-hand side does not, we must need that both sides are constant. Hence:

$$\frac{(r^2 R')'}{R} = \lambda \tag{27}$$

$$\frac{1}{\sin\phi\Phi}(\sin\phi\Phi')' + \frac{1}{(\sin\phi)^2\Theta}\Theta'' = -\lambda \tag{28}$$

with $\lambda \in \mathbb{R}$. Similarly from Eq. (28) we obtain that the equations

$$\frac{1}{\Theta}\Theta'' = -m^2 \tag{29}$$

$$\frac{\sin\phi}{\Phi}(\sin\phi\Phi')' + \lambda(\sin\phi)^2 = m^2 \tag{30}$$

must be constant with $m \in \mathbb{C}$ (a priori). The solution to Eq. (29) is a linear combination of the exponentials $\mathrm{e}^{\mathrm{i}m\theta}$, $\mathrm{e}^{-\mathrm{i}m\theta}$. Note, though, that since $\Theta$ must be a $2\pi$-periodic function, that is satisfying $\Theta(\theta+2\pi) = \Theta(\theta)$ $\forall\theta \in \mathbb{R}$, $m$ must be an integer. On the other hand making the change of variables $x = \cos\phi$ and $y = \Phi(\phi)$ in Eq. (30), that equation becomes:

$$(1-x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\lambda - \frac{m^2}{1-x^2}\right) y = 0 \tag{31}$$

which is the associate Legendre equation. We have argued in Proposition 10 that we need $\lambda = n(n+1)$ and $m \leq n$ in order to get regular solutions at $x = \cos\phi = \pm 1$. Moreover these solutions are $P_n^m(\cos\phi)$. Finally note that equation Eq. (27) is a Cauchy-Euler equation (check [?]) and so it general solution is given by

$$R(r) = c_1 r^n + c_2 r^{-n-1} \tag{32}$$

because $\lambda = n(n+1)$. So the general solution becomes a linear combination of the each solution founded varying $n \in \mathbb{N} \cup \{0\}$ and $m \in \{0, 1, \ldots, n-1, n\}$:

$$f(r,\theta,\phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (c_n^m r^n + d_n^m r^{-n-1}) P_n^{|m|}(\cos\phi)\mathrm{e}^{\mathrm{i}m\theta} \tag{33}$$

□

From now we are not concerning of the singularity at $r = 0$ of Eq. (24) (see SECTION-POTENTIAL for more details).
With a bit of patience one can prove that $\forall n_1, n_2 \in \mathbb{N} \cup \{0\}$ and all $0 \leq m \leq \min\{n_1, n_2\}$:

$$\int_0^1 P_{n_1}^m(x) P_{n_2}^m(x)\,\mathrm{d}x = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{n_1,n_2} \tag{34}$$

where $\delta_{n_1,n_2}$ denotes the Kronecker delta. From here it's not hard to prove the spherical harmonics behave in a similar way:

$$\int_0^{2\pi}\int_0^{\pi} Y_{n_1}^{m_1}(\theta,\phi)\overline{Y_{n_2}^{m_2}(\theta,\phi)}\,\mathrm{d}\phi\,\mathrm{d}\theta = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{n_1,n_2}\delta_{m_1,m_2} \tag{35}$$

Moreover, an important result in the Sturm-Liouville Theory of second order differential equations ([?, ?]) says that the family of spherical harmonics $\{Y_n^m(\theta,\phi) : n \in \mathbb{N} \cup \{0\}, |m| \leq n\}$ form a complete set in the sense that any smooth function defined on the sphere $f : S^2 \to \mathbb{R}$ can be expanded in a series of spherical harmonics:

$$f(\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} c_n^m Y_n^m(\theta,\phi) \tag{36}$$