

1 Introduction to astrophysics and satellite tracking

1.1 The two body problem

1.1.1 Trajectory equation

We are interested in understanding the dynamics of a spacecraft in orbit around the Earth. These dynamics are governed by Newton's second law of motion, which assuming that both the Earth and the spacecraft are point masses (see ?? for a more realistic model), this can be written as

$$\ddot{\mathbf{r}} = -\frac{GM_{\oplus}}{r^2}\mathbf{e}_r \quad (1)$$

where \mathbf{r} is the position vector (also called *radius vector*) of the spacecraft with respect to the center of the Earth, $r := \|\mathbf{r}\|$, $\mathbf{e}_r = \frac{\mathbf{r}}{r}$ is the unit vector in the direction of \mathbf{r} , $M_{\oplus} \simeq 5.972 \times 10^{24}$ kg is the mass of the Earth, and $G \simeq 6.674 \times 10^{-11} \text{ m}^3 \cdot \text{kg}^{-1} \cdot \text{s}^{-2}$ is the universal gravitational constant. Note that the minus sign is due to the fact that the gravitational force is attractive, i.e. pointing towards the Earth. Here and along the document the notation $\dot{\mathbf{r}}$ means that the derivative is taken with respect to time. Cross-multiplying Eq. (1) by \mathbf{r} , we obtain

$$\frac{d(\mathbf{r} \times \dot{\mathbf{r}})}{dt} = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = -\frac{GM_{\oplus}}{r^3}(\mathbf{r} \times \mathbf{r}) = 0 \quad (2)$$

Hence $\mathbf{r} \times \dot{\mathbf{r}} =: \mathbf{h}$ is constant. The physical intuition behind this is that the motion of the spacecraft around the Earth is confined to a plane, called the *orbital plane*, because the position \mathbf{r} and velocity $\dot{\mathbf{r}}$ are always perpendicular to \mathbf{h} .

We are interested now in what kind of curves may be described by a spacecraft orbiting the Earth, when considered both objects as point masses. That is, we want somehow isolate \mathbf{r} (or r) from Eq. (1). In order to simplify the notation we will denote $\mu := GM_{\oplus}$.

Proposition 1 (Kepler's first law). Consider to point-mass bodies. The motion of one body orbiting another can be described by a conic section. Hence, it can be expressed in the form:

$$r(t) = \frac{p}{1 + e \cos(\nu(t))} \quad (3)$$

for some parameters p and e .

Proof. Cross-multiplying Eq. (1) by \mathbf{h} we obtain

$$\frac{d(\dot{\mathbf{r}} \times \mathbf{h})}{dt} = \ddot{\mathbf{r}} \times \mathbf{h} = -\frac{\mu}{r^3}\mathbf{r} \times \mathbf{h} = -\frac{\mu}{r^3}\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}}) = \frac{\mu}{r^3}[(\mathbf{r} \cdot \mathbf{r})\dot{\mathbf{r}} - (\mathbf{r} \cdot \dot{\mathbf{r}})\mathbf{r}] \quad (4)$$

where in the last equality we have used ??. Now note that:

$$\frac{d}{dt} \left(\frac{\mathbf{r}}{r} \right) = \frac{\dot{\mathbf{r}}}{r} - \frac{\dot{r}}{r^2}\mathbf{r} = \frac{1}{r^3}[(\mathbf{r} \cdot \mathbf{r})\dot{\mathbf{r}} - (\mathbf{r} \cdot \dot{\mathbf{r}})\mathbf{r}] \quad (5)$$

because $2r\dot{r} = \frac{d(r^2)}{dt} = \frac{d(\mathbf{r} \cdot \mathbf{r})}{dt} = 2\mathbf{r} \cdot \dot{\mathbf{r}}$ ¹. Thus:

$$\frac{d(\dot{\mathbf{r}} \times \mathbf{h})}{dt} = \mu \frac{d}{dt} \left(\frac{\mathbf{r}}{r} \right) \quad (6)$$

Integrating with respect to the time yields

$$\dot{\mathbf{r}} \times \mathbf{h} = \frac{\mu}{r}\mathbf{r} + \mathbf{B} \quad (7)$$

where $\mathbf{B} \in \mathbb{R}^3$ is the constant of integration. Observe that since $\dot{\mathbf{r}} \times \mathbf{h}$ is perpendicular to \mathbf{h} , it lies on the orbital plane and so does \mathbf{r} . Hence, \mathbf{B} lies on the orbital plane. Now dot-multiplying this last equation by \mathbf{r} and using that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ we obtain

$$h^2 = \mathbf{h} \cdot \mathbf{h} = (\mathbf{r} \times \dot{\mathbf{r}}) \cdot \mathbf{h} = \mathbf{r} \cdot (\dot{\mathbf{r}} \times \mathbf{h}) = \frac{\mu}{r}\mathbf{r} \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{B} = \mu r + rB \cos \nu \quad (8)$$

¹Bear in mind that in general $\dot{r} \neq \|\dot{\mathbf{r}}\|$. Indeed, if β denotes the angle between \mathbf{r} and $\dot{\mathbf{r}}$ we have that $\dot{r} = \|\dot{\mathbf{r}}\| \cos \beta$. In particular \dot{r} may be negative.

where $h := \|\mathbf{h}\|$, $B := \|\mathbf{B}\|$ and ν denotes the angle between \mathbf{r} and \mathbf{B} . Rearranging the terms we obtain finally the equation of a conic section

$$r = \frac{h^2/\mu}{1 + (B/\mu) \cos(\nu)} \quad (9)$$

with $p := h^2/\mu$ and $e := B/\mu$. \square

We've seen in ?? the range of values that can r take and we deduced an equation for the semi-major axis. Note that this latter quantity can also be expressed as:

$$a = \frac{r_{\max} + r_{\min}}{2} = \frac{p}{1 - e^2} = \frac{h^2}{\mu(1 - e^2)} \quad (10)$$

Finally the angle ν in astrophysics is called *true anomaly*. Note that at \mathbf{r}_{\min} , we have $\nu = 0$ and so $\mathbf{r} \parallel \mathbf{B}$. Hence \mathbf{B} points towards the periapsis of the orbit.

Definition 2. Let $\mathbf{r}(t)$ be the position of the spacecraft at time t and $A(t)$ be the area swept by the radius vector $\mathbf{r}(t)$ in the time interval $[0, t]$. We define the *areal velocity* as $\frac{dA(t)}{dt}$.

Proposition 3 (Kepler's second law). The areal velocity remains constant.

Proof. Recall that the area of a parallelogram generated by two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ is given by $\|\mathbf{u} \times \mathbf{v}\|$. Thus, approximating the area A by half of the parallelogram generated by $\mathbf{r}(t)$ and $\mathbf{r}(t+k)$ (see Fig. 1) we obtain

$$\begin{aligned} \frac{dA(t)}{dt} &= \lim_{k \rightarrow 0} \frac{A(t+k) - A(t)}{k} = \lim_{k \rightarrow 0} \frac{\|\mathbf{r}(t) \times \mathbf{r}(t+k)\|}{2k} = \lim_{h \rightarrow 0} \frac{\|\mathbf{r}(t) \times (\mathbf{r}(t+k) - \mathbf{r}(t))\|}{2k} = \\ &= \frac{\|\mathbf{r}(t) \times \dot{\mathbf{r}}(t)\|}{2} = \frac{h}{2} \quad (11) \end{aligned}$$

where the penultimate equality is because the cross product is continuous and linear. \square

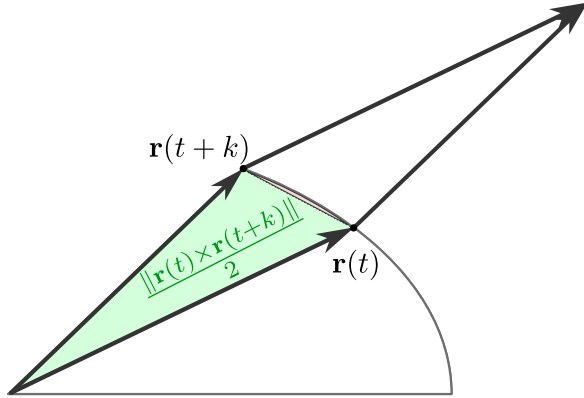


Figure 1: Graphic representation of the error committed (red region) when approximating the area swept by half the area of the parallelogram generated by $\mathbf{r}(t)$ and $\mathbf{r}(t+k)$ (green region).

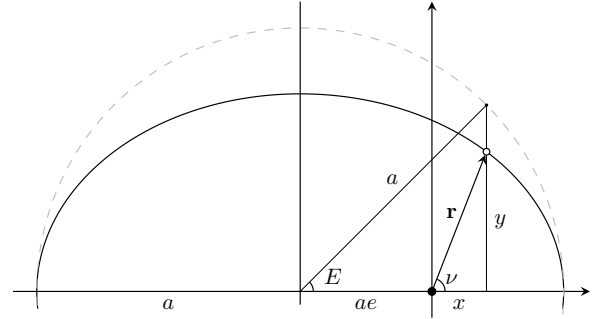


Figure 2: Ellipse orbit of the satellite together with an auxiliary circle of radius a needed to define the eccentric anomaly.

Definition 4. Let T be the orbital period of the satellite. We define the *mean motion* as $n := 2\pi/T$.

Proposition 5 (Kepler's third law). The mean motion is related to the semi-major axis by

$$n = \sqrt{\frac{\mu}{a^3}} \quad (12)$$

Proof. Integrating Eq. (11) with respect to time between 0 and T (the period) yields:

$$\pi ab = A(T) = \int_0^T A'(t) dt = \int_0^T \frac{h}{2} dt = \frac{hT}{2} \implies n = \frac{2\pi}{T} = \frac{h}{ab} = \frac{h}{a^2 \sqrt{1 - e^2}} = \sqrt{\frac{\mu}{a^3}} \quad (13)$$

where we have used Eq. (10) and ??. \square

From now on, we will assume that the orbits are ellipses, as this is the primary case of interest.

1.1.2 Kepler's equation

So far we have been able to describe the geometry of motion of a body orbiting another one. However, we have not concerned about the specific position of the body as a function of time. That is how to obtain $\nu(t)$ at each instant of time. In order to do this, we may think the area A as a function of ν , that measures the area swept by the radio vector from an initial instant ν_0 . Thus, from differential calculus we know that:

$$A(\nu) = \int_{\nu_0}^{\nu} \int_0^{r(\theta)} r \, dr \, d\theta = \int_{\nu_0}^{\nu} \frac{r(\theta)^2}{2} d\theta \implies \frac{dA}{d\nu} = \frac{r^2}{2} \quad (14)$$

Using the chain rule and Eq. (11) we obtain that:

$$\frac{h}{2} = \frac{dA}{dt} = \frac{dA}{d\nu} \frac{d\nu}{dt} = \frac{r^2}{2} \dot{\nu} \quad (15)$$

So from Eqs. (9) and (15) we get the following differential equation that must satisfy ν :

$$\dot{\nu} = \frac{h}{r^2} = \frac{h}{p^2} (1 + e \cos \nu)^2 \quad (16)$$

which, when integrated with respect to the time, lead us to an elliptic integral which can be very computationally expensive. Our goal in this section is to find an easier way to compute the exact position of the satellite at each instant of time [MG05]. This will lead us to the so-called *Kepler's equation*. For this purpose we are forced to introduce a new parameter, E , called *eccentric anomaly*. It is defined as the angle between the segment from the origin of the ellipse and the periapsis, and the line passing through the center of the ellipse and the point in a circle (of radius a and same center as the ellipse) which is just above the position of the satellite (see Fig. 2 for a better understanding).

Clearly, using the reference frame of Fig. 2, the position of the satellite is determined by $x = r \cos \nu$, $y = r \sin \nu$. But we would like to find an expression of x and y in terms of E rather than ν . To do this note that $a \cos E = ae + x$, so:

$$x = a(\cos E - e) \quad (17)$$

We can also get an expression of r in terms of E by solving the equation:

$$r = \frac{p}{1 + e \cos \nu} = \frac{a(1 - e^2)}{1 + e \frac{x}{r}} = \frac{ra(1 - e^2)}{r + ae(\cos E - e)} \implies r = a(1 - e \cos E) \quad (18)$$

Finally from Eqs. (17) and (18) we get:

$$y^2 = r^2 - x^2 = a^2(1 - e^2)(\sin E)^2 \implies y = a\sqrt{1 - e^2} \sin E \quad (19)$$

Expressing now the areal velocity h as a function of E we have:

$$h = x\dot{y} - y\dot{x} \quad (20)$$

$$= a^2(\cos E - e)\sqrt{1 - e^2}(\cos E)\dot{E} + a^2(\sin E)^2\dot{E}\sqrt{1 - e^2} \quad (21)$$

$$= a^2\sqrt{1 - e^2}\dot{E}(1 - e \cos E) \quad (22)$$

From Eq. (10) we know that $h = \sqrt{\mu a(1 - e^2)}$. Thus substituting this in the latter equation we deduce that E must satisfy the following differential equation:

$$\dot{E}(1 - e \cos E) = \sqrt{\frac{\mu}{a^3}} = n \quad (23)$$

where the last equality follows from Proposition 5. Integrating this equation with respect to time yields the *Kepler's equation*:

$$E(t) - e \sin E(t) = n(t - t_0) \quad (24)$$

where t_0 is the time at which E vanishes. Using the reference frame of Fig. 2 (the Perifocal frame, see Definition 18) this corresponds to the time at which the satellite is at the perigee. The value $M := n(t - t_0)$ is called *mean anomaly*. Note that contrarily to E and ν , the mean anomaly increases linearly with time.

Kepler's equation is the key to solve the problem of finding the position of the satellite at each instant of time. Later on we will discuss techniques to solve this equation for E knowing e and M .

1.2 Time and reference systems

1.2.1 Julian day

The way humans measure the time has changed a lot over the centuries and still varies between different cultures. Astronomy, however, has always needed a universal time measurement system and easy-to-work with. In view of this, the so-called *Julian date* is defined:

Definition 6 (Julian date). The *Julian date* (JD) is the number of days elapsed since the beginning of the *Julian period*, that is, since January 1st, 4713 BC at 12:00 (noon) in the Julian calendar². A *Julian year* is defined as 365.25 days and therefore a *Julian century* as 36525 days.

It's clear that the number of Julian dates will vary depending on the time system that we use (UT1, TT, UTC, ...) (see [Section 1.2.2](#)). We will distinguish them by adding a subscript to the JD (e.g. JD_{UT1}, JD_{TT}, JD_{UTC}, ...).

As we will see later on in this section, the number of Julian centuries elapsed since the January 1st, 2000 at 12:00 (noon) in the Julian calendar will be of our interest. This number is given by:

$$T = \frac{\text{JD} - 2451545}{36525} \quad (25)$$

Here 2451545 corresponds to the Julian date of January 1st, 2000 at 12:00 (noon).

1.2.2 Time measurement

As human beings, we are naturally interested in how time passes and therefore the correct measure of it becomes an essential necessity for us. As it is the Sun that governs our daily activity, it is natural to define time from it. But first we need some preliminary definitions:

Definition 7. We define the *equatorial plane* as the plane in \mathbb{R}^3 that contains the Earth equator. We define the *ecliptic plane* as the orbital plane in \mathbb{R}^3 of the Earth around the Sun.

Definition 8. We define the *celestial sphere* as an abstract sphere of infinite radius centered at the center of mass of the Earth. All the celestial objects are thus, projected naturally on the celestial sphere, identifying them with two coordinates (longitude and latitude). The intersection of the equatorial plane with the celestial sphere is called *celestial equator*. The intersection of the ecliptic plane with the celestial sphere is called *ecliptic* (see [Fig. 8](#) for a better understanding).

A first important thing to note is that, since the celestial sphere is centered at the Earth, the Sun moves along the ecliptic. Moreover, note that both the celestial equator and the ecliptic are two different great circles on the celestial sphere. Hence, they intersect at exactly two points.

Definition 9. Consider the two points of intersection between the celestial equator and the ecliptic. We define the *vernal equinox* as the point Υ , from these two, such that the Sun crosses the celestial equator from south to north.

The angle measured along the equator of any object on the celestial sphere from the vernal equinox is called *right ascension*, whereas the angle measured along the meridian of the object from the position of the object to the equator is called *declination* (see [Fig. 3](#)).

An *apparent solar day* is defined to be the time between two successive transits of the Sun across our local meridian. One should note that the Earth has to rotate on itself slightly more than one revolution in order to complete one solar day. The *apparent sidereal day* is defined as the time it takes to the Earth to complete a rotation relative to very far away stars (see [Fig. 4](#) for a better understanding). From the point of view of the celestial sphere, the apparent solar time is the angle (measured along the celestial equator) between the local meridian and the meridian of the Sun at that epoch, which is not constant because the Sun's right ascension increases about 1 degree per day [\[MG05\]](#).

Due to Kepler's second law, the Earth's non-circular orbit around the Sun results in some days being shorter than others. As a result, the actual position of the Sun is not ideal for precise time measurement. The introduction of a *mean Sun* is, thus, a necessity.

²From [\[Val13\]](#), the convention to start the JD at noon each day benefits astronomers (who often work at night) because they can make all their observations on a single day.

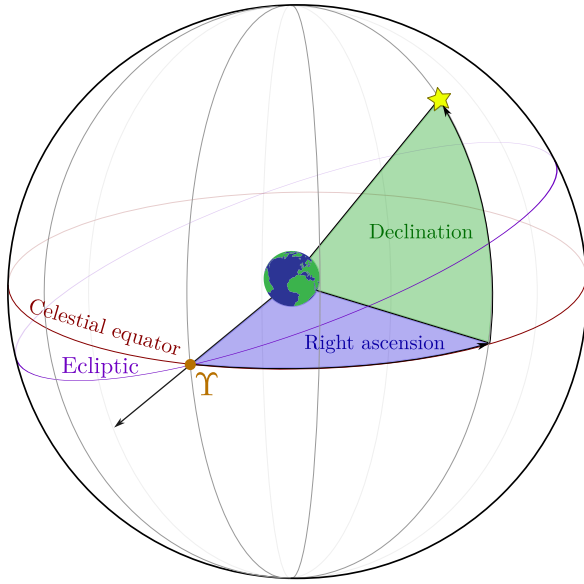


Figure 3: Right ascension and declination of a star in the celestial sphere

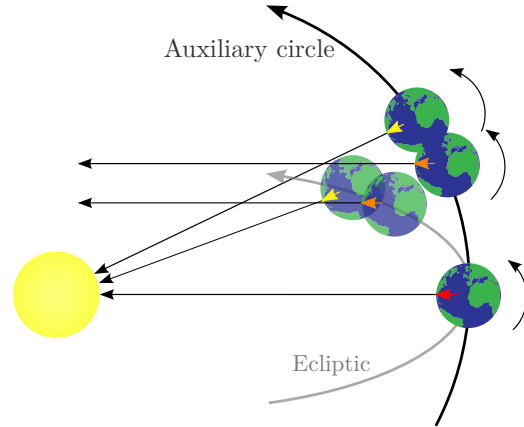


Figure 4: Solar and sidereal days (not to scale)

Definition 10. The *mean Sun* is a fictitious Sun that moves along the celestial equator at a constant rate. This rate is determined such that the real Sun and the mean Sun coincide at the vernal equinox. We define the *mean solar day* as the hour angle (along the celestial equator) between the local meridian and the meridian of the mean Sun.

It is worth noting that unlike the real Sun, the mean Sun moves along the celestial equator, rather than the ecliptic.

Definition 11. We define the *prime meridian* or *zero meridian* as the meridian on the celestial sphere that passes through the Royal Observatory in Greenwich, England (when projected to the Earth).

Definition 12. The *Greenwich Mean Time* (GMT) or *Universal Time* (UT) is the hour angle of the mean solar day measured from the prime meridian and counted from midnight. That is, when the prime meridian and mean Sun meridians coincide, the GMT is 12:00.

The use of two distinct names, namely GMT and UT, to refer to the same time can be attributed to historical reasons. Initially, GMT was defined as the mean solar time at the prime meridian with 00:00 GMT coinciding with the moment when the mean Sun was at the that meridian. On the other hand, UT was introduced as a 12-hour translation of GMT, intended for civilian purposes. Eventually, GMT was redefined to align with UT.

In the middle of the 20th century, *Ephemerides Time* (ET) was introduced to cope with the irregularities of the Earth's rotation (see [Section 1.2.3](#)). This time was defined from historical observations of planets in a Newtonian physics framework, isolating the time from the equations, and the origin was chosen accordingly to the GMT at January 1900. This time provided a uniform time, although it was more difficult to measure than the mean solar time. In the meantime, atomic clocks were invented and soon the *atomic time* (TAI, from French *Temps Atomique International*) was adopted as the SI unit of second. The origin was chosen such that the TAI matched UT at the 00:00:00 UT of January 1st, 1958, and at that time the ET was displaced from UT by 32.184 seconds. At the end of the century, the *Terrestrial Time* (TT) was introduced within a relativistic framework in order to succeed ET and provided a smooth and more accurate continuation of it yielding the relation ([MG05])

$$TT = ET = TAI + 32.184s \quad (26)$$

A representation of the sidereal time is the *Greenwich Mean Sidereal Time* (GMST) which is defined as the angle between the prime meridian and the mean vernal equinox of date (see [Section 1.2.3](#)). Due to unpredictable irregular changes on the rotation of the Earth (see POLAR motion), the GMST cannot be computed directly with a formula in terms of the TAI or TT.

The *Universal Time 1* (UT1) is the presently used form of Universal time and it is defined with the following deterministic formula given in [Aok+81]. For each day, the 00:00 UT1 is defined when the GMST has the value:

$$\text{GMST}(0\text{h UT1}) = 24110.54841 + 8640184.812866T_{\text{UT1},0} + 0.093104T_{\text{UT1},0}^2 - 6.2 \cdot 10^{-6}T_{\text{UT1},0}^3 \quad (27)$$

where $T_{\text{UT1},0} = \frac{\text{JD}(0\text{h UT1}) - 2451545}{36525}$ denotes the number of Julian centuries that have passed since January 2000, 1.5 UT1 at the beginning of the day. For any instant of time during the day, the following formula is used:

$$\begin{aligned} \text{GMST}(\text{UT1}) = & 24110.54841'' + 8640184.812866''T_{\text{UT1},0} + 1.002737909350795\text{UT1} + \\ & + 0.093104''T_{\text{UT1}}^2 - 6.2 \cdot 10^{-6}''T_{\text{UT1}}^3 \quad (28) \end{aligned}$$

where $T_{\text{UT1}} = \frac{\text{JD}(\text{UT1}) - 2451545}{36525}$ and UT1 is measured in seconds. Similarly to the GMST, there is no simple conversion between the UT1 and the TT or TAI. Instead, the IERS (*International Earth Rotation and Reference Systems Service*) provides regularly a bulletin with the difference $\Delta T := \text{TT} - \text{UT1}$ at several dates. Interpolating these values we can obtain UT1 from TT at any epoch.

Finally, our everyday clock is based on the *Coordinated Universal Time* (UTC). It is defined to be as uniform as the TAI but always kept closer than 0.9 seconds to the UT1 in order to resemble a mean solar time (see Fig. 5). Scientists achieve this by introducing a *leap second* (see Fig. 6), which is an extra second added to UTC at irregular intervals. Fig. 5 summarizes all the time systems introduced in the document.

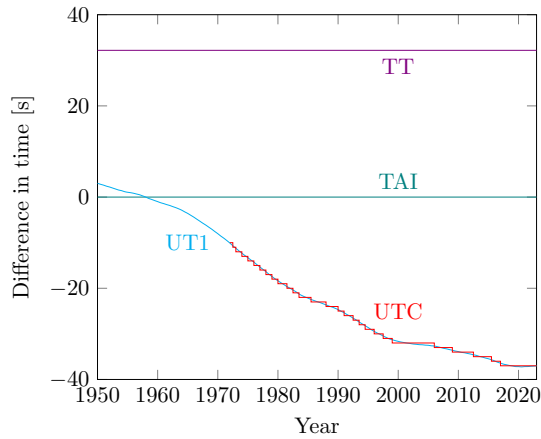


Figure 5: Evolution of times TT, UT1 and UTC in comparison with TAI. [Obs]

1998 December 31,	23h 59m 59s
1998 December 31,	23h 59m 60s
1999 January 1,	00h 00m 00s
1999 January 1,	00h 00m 01s

Figure 6: Leap second introduced to the UTC time at the end of the December 1998. [RS98b]

We summarize here useful conversions between time systems:

$$\begin{aligned} \text{GMST} &= 24110.54841 + 8640184.812866T_{\text{UT1},0} + 0.093104(T_{\text{UT1}})^2 - 6.2 \cdot 10^{-6}(T_{\text{UT1}})^3 \\ \text{UT1} &= \text{TT} - \Delta T \\ \text{TT} &= \text{TAI} - 32.184 \\ \text{TAI} &= \text{UTC} + \delta(\text{TAI}) \end{aligned}$$

where ΔT is the difference between the TT and UT1, and $\delta(\text{TAI})$ is a piecewise constant function that counts the number of leap seconds introduced since 1972, when they were introduced for the first time. All the numbers have units of second. Note that since the rotation of the Earth cannot be predicted accurately, ΔT can only be determined retrospectively, and is given by the *International Earth Rotation and Reference Systems Service* (IERS).

1.2.3 Reference systems

Newton's second law is only valid when applied to an *inertial reference frame*, that is, a frame of reference that is not undergoing any acceleration. In practise, however, almost any frame of reference is non-inertial. So in this chapter we will describe an almost-inertial frame of reference which will be used to integrate

Newton's second law. On the other hand, since the Earth is not a body with a homogeneous density of mass, there are zones with higher mass density than others, and therefore with higher gravitational pull (see ??). Therefore we will need the longitude and latitude of the satellite with respect to the Earth at each time of integration in order to place it in the correct position with respect to the Earth.

On basis on the study of the satellite motion around the Earth, it is natural to locate all the origins of the reference frames considered along the document at the center of mass of the Earth.

The first reference frame we must consider is the *celestial* one. In the celestial frame, the x -axis is defined as the line of intersection between the equatorial plane and the ecliptic plane. The positive direction is chosen to point towards the vernal equinox. The z -axis is chosen to be perpendicular to the equatorial plane and the y -axis is such that the triplet (x, y, z) is a right-handed system.

However, due to the presence of other solar system planets (and other smaller perturbations), the orbital plane of the Earth is not fixed in space, but is subjected to a small variation called *planetary precession*. Moreover, the gravitational attraction of the Sun and Moon on the Earth's equator cause Earth's axis of rotation to precess in a similar way to the action of a spinning top with a period of about 26000 years [MG05]. This motion is called *lunisolar precession*. On the other hand, smaller perturbations in amplitude (< 18.6 years [Wika]) with shorter period superposed with the precessional motion creates a motion called *nutation*. When this latter oscillations are averaged out, the vernal equinox, the ecliptic, and the equator are referred to *mean* values, rather than *true* values.

In addition, the Earth's axis of rotation undergoes a slight periodic motion around a reference axis of rotation, that passes through the *IRS* (*IERS Reference Pole*). This motion is called *polar motion* and is caused by the redistribution of the Earth's mass due to the seasonal variations of the atmosphere and the oceans. The polar motion is usually described by the *CEP* (*Celestial Ephemeris Pole*), which is the intersection of the Earth's axis of rotation with the celestial sphere. In Fig. 7 we summarize the different types of perturbations on the Earth's axis of rotation.

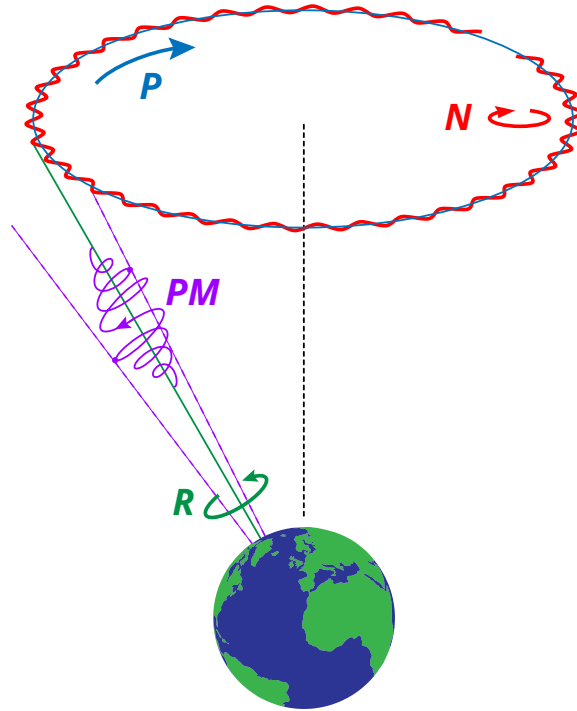


Figure 7: Graphical explanation of the perturbation by precession (blue), nutation (red) and polar motion (violet) of the Earth's axis of rotation (green).

In view of this time-dependent orientation of both the ecliptic and the equator, the standard-reference frame chosen is based on the mean equator, mean ecliptic and mean equinox of some fixed time, the beginning of the year 2000, namely at 12:00 TT on 1 January 2000, the so-called *J2000 epoch*.

Definition 13 (Earth-centered inertial frame). We define the *Earth-centered inertial* (ECI) reference frame or *J2000 reference frame* as the frame of reference whose x -axis is the intersection of the mean celestial equator and the mean ecliptic pointing at the mean vernal equinox, the z -axis is perpendicular

to the mean equator and the y -axis is chosen such that the triplet (x, y, z) is a right-handed system. All these quantities are the respective ones of the J2000 epoch.

Let's define now to study an Earth-fixed reference frame.

Definition 14 (Earth-centered, Earth-fixed frame). We define the *Earth-centered, Earth-fixed frame of reference* (ECEF) as the frame of reference whose x -axis is pointing to the prime meridian, the z -axis is perpendicular to the Earth's equator and the y -axis is chosen such that the triplet (x, y, z) is a right-handed system.

These two coordinate systems have, as mentioned earlier, the origin at the center of mass of the Earth.

1.2.4 Conversion between reference systems

As we noted in the previous section the angle ε between the celestial equator and ecliptic planes is not constant due to the planetary precession.

Our goal in this section is to transform the position of the satellite from the ECI system to the ECEF system and vice versa. This rotation transformation is given by a product of 4 rotations:

- The precession matrix \mathbf{P} ,
- the nutation matrix \mathbf{N} ,
- the Earth rotation matrix $\mathbf{\Theta}$, and
- the polar motion matrix $\mathbf{\Pi}$.

These matrix are such that:

$$\mathbf{r}_{\text{ECEF}}(t) = \mathbf{\Pi}(t)\mathbf{\Theta}(t)\mathbf{N}(t)\mathbf{P}(t)\mathbf{r}_{\text{ECI}}(t) \quad (29)$$

where $\mathbf{r}_{\text{ECEF}}(t)$ is the position vector of the satellite in the ECEF frame at time t and $\mathbf{r}_{\text{ECI}}(t)$ is the position vector of the satellite in the ECI frame at time t . From here on, we will omit the evaluation on the time t for the sake of simplify the lecture. Let's now argue why the transformation has this particular form.

The precession matrix is responsible for *eliminating* all the movement due to the planetary and lunisolar precession. Thus, \mathbf{P} transforms the mean equator and mean equinox at time J2000 to the respective values at time t . With the help of [Fig. 8](#) it's not hard to see that this transformation is given by:

$$\mathbf{P} = \mathbf{R}_z(-90 - z)\mathbf{R}_x(\theta)\mathbf{R}_z(90 - \zeta) \quad (30)$$

And with a bit of algebra it can be simplified to:

$$\mathbf{P} = \mathbf{R}_z(-z)\mathbf{R}_y(\theta)\mathbf{R}_z(-\zeta) \quad (31)$$

Recall that the fundamental rotation matrices $\mathbf{R}_x(\theta)$, $\mathbf{R}_y(\theta)$ and $\mathbf{R}_z(\theta)$ are with respect to the axis of the J2000 frame and they are given by:

$$\mathbf{R}_x(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & \sin \varphi \\ 0 & -\sin \varphi & \cos \varphi \end{pmatrix} \quad \mathbf{R}_y(\varphi) = \begin{pmatrix} \cos \varphi & 0 & -\sin \varphi \\ 0 & 1 & 0 \\ \sin \varphi & 0 & \cos \varphi \end{pmatrix} \quad \mathbf{R}_z(\varphi) = \begin{pmatrix} \cos \varphi & \sin \varphi & 0 \\ -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (32)$$

where we have used the convention of signs given by [\[GPS02\]](#). The reader may wonder why we have used the notation $90 - z$ and $90 - \zeta$ instead of z and ζ (for example) for the angles in question. The reason is related to the precise definition of these angles from the pole of the celestial sphere rather than from where we have defined them, but we will not elaborate on this point here. Nonetheless, we have chosen this notation to maintain consistency with related articles [\[Lie+77\]](#).

The nutation perturbations are driven out by the nutation matrix \mathbf{N} . This matrix transforms the coordinates of the mean equator and equinox at epoch t to those of the true equator and equinox at the same epoch, respectively. Hence, from figure [Fig. 9](#) we can see that the nutation matrix is given by:

$$\mathbf{N} = \mathbf{R}_x(-\varepsilon - \Delta\varepsilon)\mathbf{R}_z(-\Delta\psi)\mathbf{R}_x(\varepsilon) \quad (33)$$

In [Section 1.2.2](#) we defined the GMST as the hour angle between the mean vernal equinox and the prime meridian. In a similar way, we define the *Greenwich Apparent Siderial Time* (GAST) as the hour

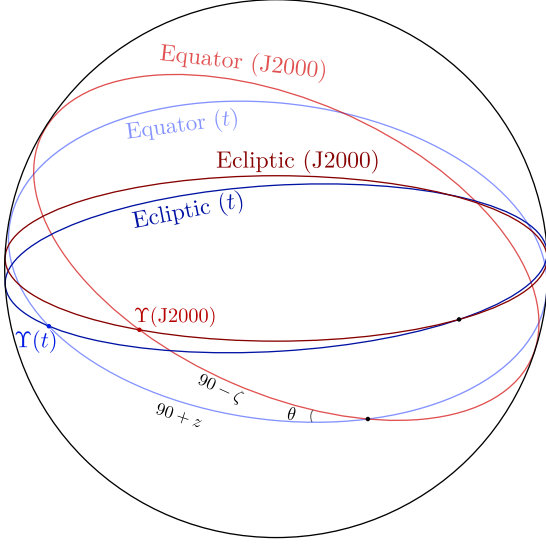


Figure 8: Celestial sphere showing the ecliptic and the equator of both the epoch J2000 and the current epoch t . Dark colors represent the ecliptic while light colors represent the equator. On the other hand, red colors represents the the J2000 epoch and blue colors represents the current epoch t .

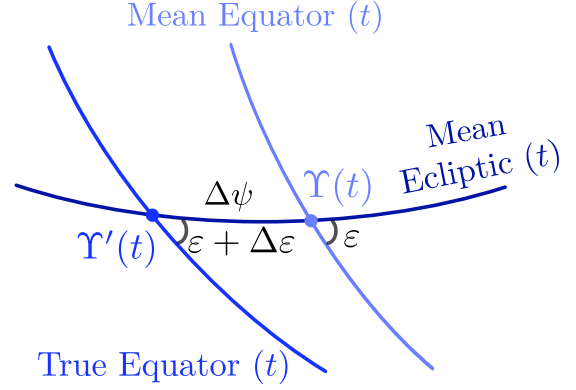


Figure 9: True equator and mean equator, and true equinox (Υ') and mean equinox (Υ) at a given epoch t together with the ecliptic at that time.

angle between the true vernal equinox and the prime meridian. The difference between these two times is given by the so called *equation of the equinoxes*, which up to first order in the nutation angles is given by:

$$\text{GAST} - \text{GMST} \simeq \Delta\psi \cos \varepsilon \quad (34)$$

The rotation matrix is responsible for aligning the meridian of the satellite with the local meridian at the moment of the observation. This matrix is given by:

$$\mathbf{\Theta} = \mathbf{R}_z(\text{GAST}) \quad (35)$$

Finally the polar motion movement around the CEP axis is modelled with the matrix $\mathbf{\Pi}$ given by:

$$\mathbf{\Pi} = \mathbf{R}_y(-x_p)\mathbf{R}_x(-y_p) \quad (36)$$

In Fig. 10 we can see a graphical representation of these two transformation matrices. The parameters x_p and y_p are constantly updated in the IERS bulletins [RS98a]. All the other parameters are given in [Lie+77] and we provide here a summary of them:

$$\varepsilon = 23.4392911^\circ - 46.8150''T - 0.00059''T^2 + 0.001813''T^3 \quad (37)$$

$$\zeta = 2306.2181'' + 0.30188''T + 0.017998''T^2 \quad (38)$$

$$\theta = 2004.3109''T - 0.42665''T^2 - 0.041833''T^3 \quad (39)$$

$$z = \zeta + 0.79280''T^2 + 0.000205''T^2 \quad (40)$$

where $T = \frac{\text{JD}_{TT} - 2451545}{36525}$ are the Julian centuries that have elapsed since the J2000 epoch. The values for nutations $\Delta\psi$ and $\Delta\varepsilon$ can be expanded into [MG05]:

$$\Delta\psi = \sum_{i=1}^{106} (\Delta\psi)_i \sin \phi_i \quad \Delta\varepsilon = \sum_{i=1}^{106} (\Delta\varepsilon)_i \cos \phi_i \quad (41)$$

where $\phi_i = p_{\ell,i}\ell + p_{\ell',i}\ell' + p_{F,i}F + p_{D,i}D + p_{\Omega,i}\Omega$ are integer coefficients and the other variables are the Moon's mean anomaly (ℓ), the Sun's mean anomaly (ℓ'), the mean distance of the Moon from the ascending node (F), the difference between the mean longitudes of the Sun and the Moon (D), and the

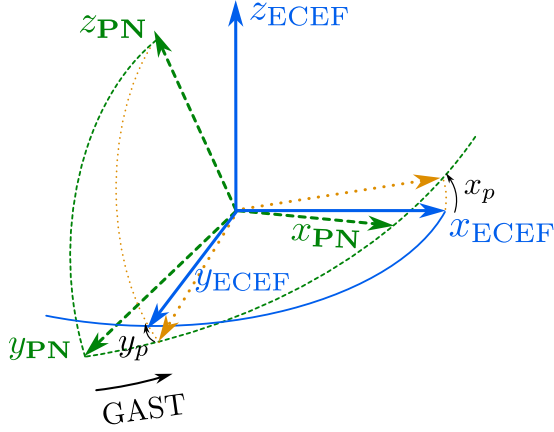


Figure 10: Three reference frames used to transform from the ECI frame to the ECEF frame. In green, the ECI frame once applied the precession and nutation transformations. In orange, the transformation of the green system once applied the rotation matrix Θ . Finally, in blue, the transformation of the orange system once applied the polar motion matrix Π .

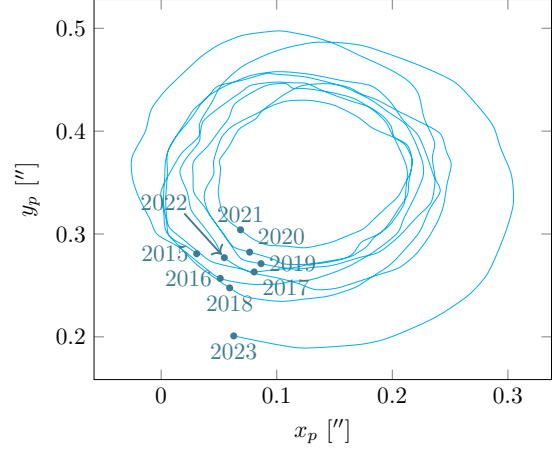


Figure 11: Graphical representation of evolution of the CEP with respect to IRS (placed at the origin of the graphic) due to polar motion. The blue dots indicate the position of the CEP at the beginning of the respective years.

mean longitude of the ascending node of the Moon (Ω). We will not go into detail about these variables, there are explicit expressions for them in [MG05] as a function of $T = \frac{JD_{TT} - 2451545}{36525}$:

$$\ell = 134^\circ 57' 46.733'' + 477198^\circ 52' 02.633''T + 31.310''T^2 + 0.064''T^3 \quad (42)$$

$$\ell' = 357^\circ 31' 39.804'' + 35999^\circ 03' 01.224''T - 0.577''T^2 - 0.012''T^3 \quad (43)$$

$$F = 93^\circ 16' 18.877'' + 483202^\circ 01' 03.137''T - 13.257''T^2 - 0.011''T^3 \quad (44)$$

$$D = 297^\circ 51' 01.307'' + 445267^\circ 06' 41.328''T - 6.891''T^2 + 0.019''T^3 \quad (45)$$

$$\Omega = 125^\circ 02' 40.280'' - 1934^\circ 08' 10.539''T + 7.455''T^2 + 0.008''T^3 \quad (46)$$

Some of these integer coefficients together with the values of the amplitudes $(\Delta\psi)_i$ and $(\Delta\epsilon)_i$ are given in Table 1.

i	$p_{\ell,i}$	$p_{\ell',i}$	$p_{F,i}$	$p_{D,i}$	$p_{\Omega,i}$	$(\Delta\psi)_i$ [0.0001'']	$(\Delta\epsilon)_i$ [0.0001'']
1	0	0	0	0	1	$-171996 - 174.2T$	$92025 + 8.9T$
2	0	0	0	0	2	$2062 + 0.2T$	$-895 + 0.5T$
3	-2	0	2	0	1	46	-24
4	2	0	-2	0	0	11	0
5	-2	0	2	0	2	-3	1
6	1	-1	0	-1	0	-3	0
7	0	-2	2	-2	2	-2	1
8	2	0	-2	0	1	1	0
9	0	0	2	-2	2	$-13187 - 1.6T$	$5736 - 3.1T$
\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots	\vdots

Table 1: First nutation coefficients

1.3 Keplerian orbital elements

1.3.1 Keplerian orbital elements from position and velocity

Definition 15. Consider a satellite orbiting the Earth. The *orbital plane* is the plane that contains the orbit of the satellite. The *line of nodes* is the line of intersection between the orbital plane and the

equatorial plane. Finally, the *ascending node* is the point on the line of nodes and the orbit of the satellite where the satellite crosses the equatorial plane from south to north.

Definition 16 (Orbital elements). The *Keplerian orbital elements* of a satellite are five independent quantities that completely determine its orbit. If moreover the exact position of the satellite on the orbit is wanted, a sixth quantity is needed. The first five orbital elements are:

1. The *semi-major axis* a of the orbit.
2. The *eccentricity* e of the orbit.
3. The *inclination* i , which is the angle between the equatorial plane and the orbital plane.
4. The *longitude of the ascending node* Ω , which is the angle between the vernal equinox and the ascending node.
5. The *argument of perigee* ω , which is the angle between the ascending node and the periapsis.

The sixth quantity is the *true anomaly* ν which is the angle between the periapsis and the position of the satellite on the orbit.

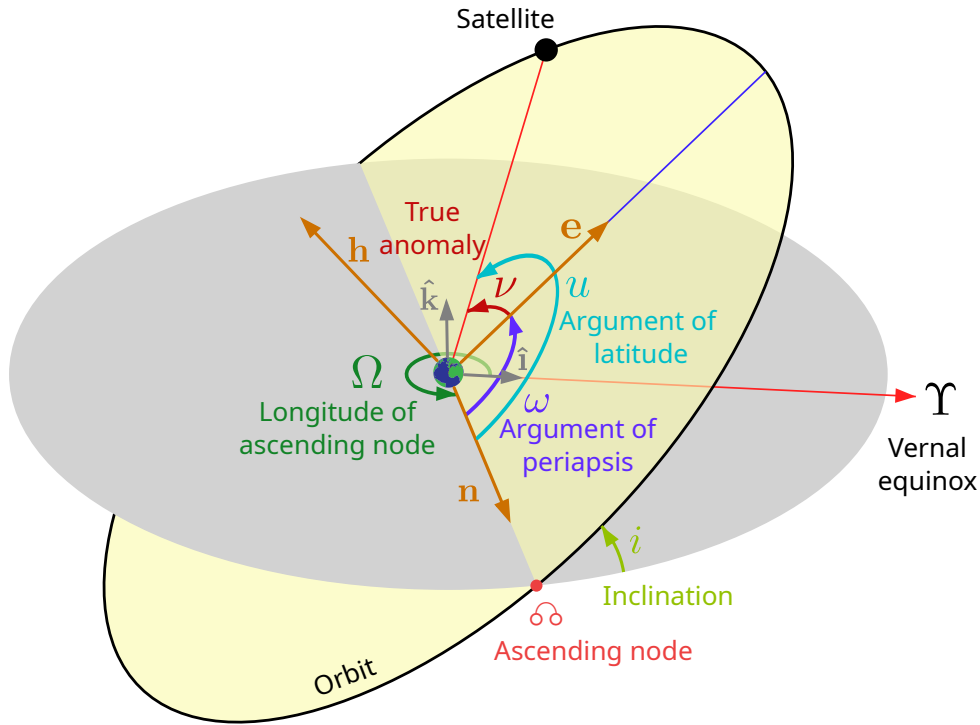


Figure 12: Orbital elements of a satellite.

Fig. 12 shows a schematic representation of these elements. The elements a , e and i are always well-defined. However, the elements Ω , ω and ν are not well-defined in general, namely for $e = 0$ or $i = 0$. We discuss this in more detail below. For the moment assume $e \neq 0$ and $i \neq 0$. In order to express these elements in terms of the position and velocity of the satellite, we need to introduce the following vectors:

Definition 17. Let $\hat{\mathbf{k}} = (0, 0, 1)$ be the unit vector perpendicular to the equatorial plane. We define the vector $\mathbf{n} := \hat{\mathbf{k}} \times \mathbf{h}$ and the *eccentricity vector* \mathbf{e} as $\mathbf{e} := \mathbf{B}/\mu$, whose norm is the eccentricity e .

Note that $\mathbf{n} \perp \hat{\mathbf{k}}$ and $\mathbf{n} \perp \mathbf{h}$ which imply that \mathbf{n} must lie on the orbital plane and equatorial plane and therefore in the line of nodes pointing towards the ascending node, by the right-hand rule. On the other hand, since \mathbf{B} points towards the periapsis, so does \mathbf{e} . From here, let's obtain the orbital elements in terms of \mathbf{r} and $\dot{\mathbf{r}}$.

First of all, we calculate $\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}$ and note that the vectors \mathbf{n} and \mathbf{B} can be computed from \mathbf{r} , $\dot{\mathbf{r}}$ and \mathbf{h} . Thus, we can get e by taking the norm of \mathbf{e} . From here and looking at Fig. 12 it's not hard to see

$$i = \arccos\left(\frac{\mathbf{n} \cdot \hat{\mathbf{k}}}{h}\right) \quad \Omega = \arccos\left(\frac{\mathbf{n} \cdot \hat{\mathbf{r}}}{n}\right) \quad \omega = \arccos\left(\frac{\mathbf{n} \cdot \mathbf{e}}{ne}\right) \quad \nu = \arccos\left(\frac{\mathbf{e} \cdot \mathbf{r}}{er}\right) \quad (47)$$
$$a = \frac{h^2}{\mu(1 - e^2)} \quad (48)$$
$$u = \arccos \left(\frac{\mathbf{n} \cdot \mathbf{r}}{nr} \right) \quad (49)$$
$$\omega = \arccos \left(\frac{\mathbf{e} \cdot \hat{\mathbf{1}}}{e} \right) \quad (50)$$
$$\lambda = \arccos \left(\frac{\mathbf{r} \cdot \hat{\mathbf{1}}}{r} \right) \quad (51)$$

Line	Satellite number							Class	International designerator						Epoch (UTC)								$\dot{n}/2$ [rev/day ²]								$\ddot{n}/6$ [rev/day ³]								B^* (drag term)								Model	TLE number						Check															
									Year	Launch	Piece	Day of the year (as fraction)	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67		68	69																				
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69	
1	5	5	1	2	4	U			9	8	0	6	6	7	U	R		2	3	0	8	6	.	1	6	7	7	8	9	9	9	4	S	.	0	1	5	4	0	4	0	7	S	.	1	4	1	0	2	-	2	S	.	2	6	6	3	4	-	2	E	0				9	9	9	1
Line	Satellite number							Inclination i [deg]						Right ascension Ω [deg]						Eccentricity e						Argument perigee ω [deg]						Mean anomaly M [deg]						Mean motion n [rev/day]						Number of revolutions						Check																			
1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48	49	50		51	52	53	54	55	56	57	58	59	60	61	62	63	64	65	66	67	68	69
2	5	5	1	2	4				5	1	.	6	2	3	7		5	.	0	0	1	7			0	0	1	0	7	5	7		1	8	8	.	3	8	6	9		1	7	1	.	6	9	6	3		1	6	.	0	1	8	7	8	1	9	9		1	3	7	9			

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the orbital model used to generate the data, as specified in [Kel; Wikb]. The element set number (TLE number) is incremented by one when a new TLE (Two-Line Element) is generated for this satellite. On the second line, the number of revolutions indicates the number of times the satellite has orbited the Earth since its launch. Finally, the checksum (modulo 10) is used to verify the integrity of the data³.

1.3.3 Position and velocity in terms of the TLEs' orbital elements

We are now interested in proceed the other way around, that is, given the orbital elements or more precisely the TLE data, we want to compute the position and velocity of the satellite. In order to do that, we need to introduce the basis $(\mathbf{P}, \mathbf{Q}, \mathbf{W})$ linked to the orbit.

Definition 18 (Perifocal coordinate system). Consider the orbit of a satellite. We define its associated *perifocal coordinate system* $(\mathbf{P}, \mathbf{Q}, \mathbf{W})$ as follows. The center is on the Earth's center of mass. The unit vectors \mathbf{P} and \mathbf{Q} lie on the orbital plane and are such that \mathbf{P} points towards the periapsis, that is $\mathbf{P} := \mathbf{B}/B$. The unit vector \mathbf{W} is defined as $\mathbf{W} := \mathbf{h}/h$ and $\mathbf{Q} := \mathbf{W} \times \mathbf{P}$.

Recall that in Section 1.1.2 we have seen that the position of the satellite in the perifocal frame is given by:

$$\mathbf{r}_{\text{Peri}} = a(\cos E - e)\mathbf{P} + a\sqrt{1 - e^2} \sin E \mathbf{Q} \quad (52)$$

Differentiating yields:

$$\dot{\mathbf{r}}_{\text{Peri}} = -a(\sin E)\dot{E}\mathbf{P} + a\sqrt{1 - e^2}(\cos E)\dot{E}\mathbf{Q} \quad (53)$$

where $\dot{E} = \frac{n}{1 - e \cos E}$ is given by Eq. (23). These two quantities depend on the two unknown variables a and E , because n and e are given in the TLE data set. The semi-major axis can be easily obtained from n due to the Kepler's third law (Proposition 5). For the eccentric anomaly, we need to solve the Kepler equation:

$$E - e \sin E = M \quad (54)$$

Lemma 19. Let $e \in [0, 1)$ and $M \in [0, 2\pi)$. Then, the function

$$f(E) = E - e \sin E - M \quad (55)$$

has a unique solution in the interval $[M, M + e]$ if $M \in [0, \pi)$ and in the interval $[M - e, M]$ if $M \in [\pi, 2\pi)$.

Proof. We prove first the uniqueness. Clearly $f \in \mathcal{C}^1(\mathbb{R})$ and $f'(E) = 1 - e \cos E > 0$ for all $E \in [0, 2\pi)$ because $e < 1$. Thus, f is strictly increasing and so it has at most one zero. Now, if $0 \leq M < \pi$, then $f(M) = -e \sin M \leq 0$ and $f(M + e) = e(1 - \sin(M + e)) \geq 0$. So by Bolzano's theorem, f has a solution in $[M, M + e]$. If $\pi \leq M < 2\pi$, then $f(M) = -e \sin M \geq 0$ and $f(M - e) = -e(1 + \sin(M - e)) \leq 0$. So by Bolzano's theorem, f has a solution in $[M - e, M]$. \square

We will use the Newton's method to find the zero of this non-linear equation. For small eccentricities e , the natural choice for the initial guess is $E_0 = M$. For large eccentricities ($e > 0.8$) the initial guess $E_0 = \pi$ should be used in order to avoid convergency problems [MG05]. Once obtained E , the position and velocity of the satellite in the J2000 frame are given by:

$$\mathbf{r}_{\text{ECI}} = \mathbf{T}\mathbf{r}_{\text{Peri}} \quad \dot{\mathbf{r}}_{\text{ECI}} = \mathbf{T}\dot{\mathbf{r}}_{\text{Peri}} \quad (56)$$

where \mathbf{T} is the rotation matrix that transform one frame to another and is given by (look at Fig. 12):

$$\mathbf{T} = \mathbf{R}_z(-\Omega)\mathbf{R}_x(-i)\mathbf{R}_z(-\omega) \quad (57)$$

³Taking into account that the negative sign is counted as 1, and all the other cells without a number as 0.