

1 Force model

So far we have only considered the gravitational force acting point masses. In reality, the Earth is not a point mass, neither a spherically symmetric mass distribution. In this section we will deep into the details of a more realistic model of the Earth's gravitational field.

1.1 Geopotential model

It is well-known that the gravity force \mathbf{F} is a conservative force. This means \mathbf{F} can be expressed as the gradient of a function $-V$, which we call the gravitational potential: $\mathbf{F} = -\nabla V$. The minus sign is chosen according the convention that work done by the forces decreases the potential.

1.1.1 Poisson and Laplace equations

Theorem 1. Consider distribution of matter of density ρ in a region Ω . Then, the gravitational potential V satisfies the Poisson equation

$$\Delta V = 4\pi G\rho \quad (1)$$

Thus, at points outside Ω since we have $\rho = 0$, the potential V satisfies the Laplace equation:

$$\Delta V = 0 \quad (2)$$

Proof. TO-DO □

Hence, the gravitational potential V created by a distribution of mass in a region Ω is a solution of the following Dirichlet problem:

$$\begin{cases} \Delta V = 0 & \text{in } \Omega^c \\ V = f & \text{on } \partial\Omega \end{cases} \quad (3)$$

If Ω represents the Earth, then $f = f(\theta, \phi)$ represents is the boundary condition concerning the gravitational potential at the surface of the Earth as a function of the longitude θ and colatitude ϕ .

1.1.2 Expansion in spherical harmonics

We have just seen that V satisfies the Laplace equation. In ?? we have seen that the solution can be expressed as:

$$V(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^n (c_n^m r^n + d_n^m r^{-n-1}) P_n^{|m|}(\cos \phi) e^{im\theta} \quad (4)$$

where $c_n^m, d_n^m \in \mathbb{C}$. Choosing the origin of potential the infinity, we must have $c_n^m = 0$. Thus, with a bit of algebra, our solution becomes:

$$V(r, \theta, \phi) = \frac{GM_{\oplus}}{r} \sum_{n=0}^{\infty} \sum_{m=0}^n \frac{R_{\oplus}^n}{r^n} P_n^m(\cos \phi) (C_n^m \cos(m\theta) + S_n^m \sin(m\theta)) \quad (5)$$

where $C_n^m, S_n^m \in \mathbb{R}$, G is the gravitational constant, M_{\oplus} is the mass of the Earth and R_{\oplus} is the reference radius of the Earth. The coefficients C_n^m, S_n^m describe the dependence on the Earth's internal structure. They are obtained from observation