Numerical propagation of trajectories of Earth-orbiting spacecraft

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1 Preliminaries

In this section we will review some basic concepts of linear algebra and vector calculus that will be used throughout the document.

1.1 Properties of cross and dot products

Proposition 1. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Then:

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} \tag{1}$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \tag{2}$$

Proof. Let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$. Then:

$$\begin{aligned}
((\mathbf{u} \times \mathbf{v}) \times \mathbf{w})_1 &= (u_3 v_1 - u_1 v_3) w_3 - (u_1 v_2 - u_2 v_1) w_2 \\
&= (u_2 w_2 + u_3 w_3) v_1 - (v_2 w_2 + v_3 w_3) u_1 \\
&= (u_1 w_1 + u_2 w_2 + u_3 w_3) v_1 - (v_1 w_1 + v_2 w_2 + v_3 w_3) u_1 \\
&= ((\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u})_1
\end{aligned}$$

The other components are treated similarly. The second equality follows in a similar way. $\square$

Proposition 2. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Then:

1. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} \cdot \mathbf{w} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} \cdot \mathbf{w}$

1.2 Conics in a nutshell

Definition 3. A conic is the curve obtained as the intersection of a plane with the surface of a double cone (a cone with two *nappes*).

In Fig. 1, we show the 3 types of conics: the ellipse, the parabola, and the hyperbola, which differ on their eccentricity, as we will see later. Note that the circle is a special case of the ellipse.

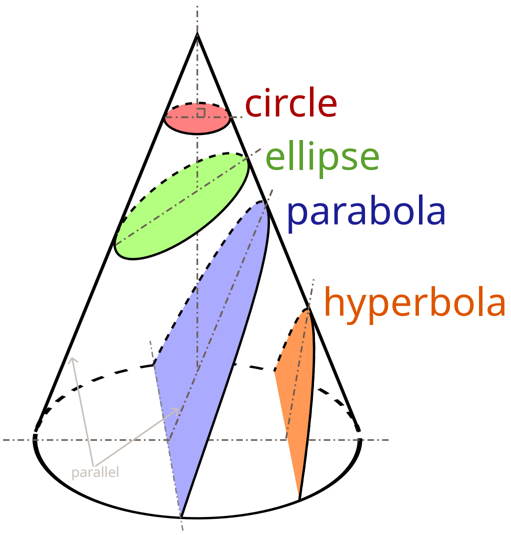

Figure 1: The black boundaries of the colored regions are conic sections. The other half of the hyperbola, which is not shown, is the other nappe of the double cone.

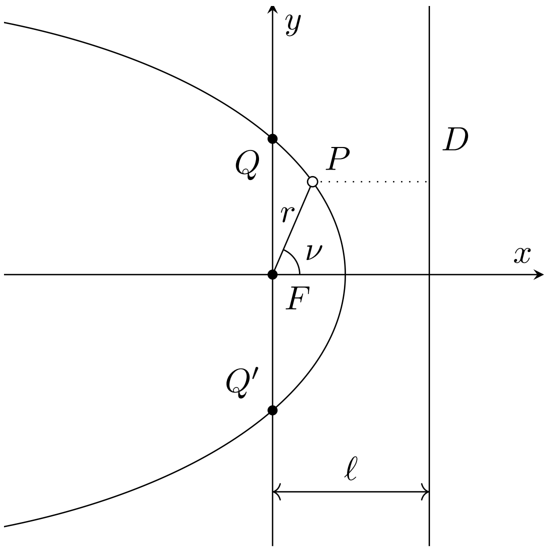

Figure 2: Reference frame centered at the focus of the conic and whose axes are such that the y-axis is parallel to the directrix and the x-axis is perpendicular to the directrix. The directions of the axes are chosen arbitrarily, subject to the constraint that a positive basis is chosen.

Definition 4. The *locus* is a set of points that satisfy a given condition.

The following proposition gives a characterization of the conics.

Proposition 5. A conic is the locus of all points P such that the distance from P to a fixed point F is a multiple of the distance from P to a fixed line D. Mathematically, this is expressed as:

$$d(P, F) = ed(P, D) \tag{3}$$

where d is the Euclidean distance. The point F is called the *focus*; the line D, *directrix*, and the constant of proportionality e, *eccentricity*.

Note that using the polar coordinates (r, ν) as in Fig. 2, we can rewrite Eq. (3) as:

$$r = e(\ell - r\cos\nu) \implies r = \frac{e\ell}{1 + e\cos\nu} =: \frac{p}{1 + e\cos\nu} \tag{4}$$

where we have defined $p := e\ell$.

Definition 6. Le C be a conic and e be its eccentricity. We say that C is

- an *ellipse* if $0 \leq e < 1$,
- a *parabola* if $e = 1$, and
- a *hyperbola* if $e > 1$.

If $e = 0$, the conic is a *circle*.

1.3 Spherical harmonics

1.3.1 Legendre polynomials, regularity and orthonormality

Definition 7. Consider the following second-order differential equation:

$$y'' + p_1(x)y' + p_0(x)y = 0 \tag{5}$$

We say that a is an *ordinary point* if p_1 and p_2 are analytic at $x = a$. We say that a is a *regular singular point* if p_1 has a pole up to order 1 at a and p_0 has a pole of order up to 2 at a. Otherwise we say that a is a *irregular singular point*.

Definition 8. Consider the following second-order differential equation called *Legendre differential equation*:

$$(1 - x^2)y'' - 2xy' + \lambda y = 0 \tag{6}$$

for $n \in \mathbb{N} \cup \{0\}$. This equation can be written as:

$$\left((1 - x^2)y'\right)' + \lambda y = 0 \tag{7}$$

If seek for analytic solutions of this equation using the power series method, i.e. looking for solutions of the form $y(x) = \sum_{j=0}^{\infty} a_j x^j$ one can check that we obtain this recursion:

$$a_{j+2} = \frac{j(j+1) - \lambda}{(j+1)(j+2)} a_j \quad j = 0, 1, 2, \ldots \tag{8}$$

From here we can obtain two independent solutions by setting the initial conditions a_0 and a_1 of the iteration. For example, setting $a_1 = 0$ we obtain a series that has only even powers of x. On the other hand, setting $a_0 = 0$ we obtain a series that has only odd powers of x. These two series converge on the interval $(-1, 1)$ by the ratio test (by looking at Eq. (8)) and can be expressed compactly as:

$$y_{\mathrm{e}}(x) = a_0 \sum_{j=0}^{\infty} \left[\prod_{k=0}^{j-1} (2k(2k+1) - \lambda)\right] \frac{x^{2j+1}}{(2j)!} \qquad y_{\mathrm{o}}(x) = a_1 \sum_{j=0}^{\infty} \left[\prod_{k=1}^{j} (2k(2k+1) - \lambda)\right] \frac{x^{2j+1}}{(2j+1)!} \tag{9}$$

However either for all $\lambda \in \mathbb{R}$ either one of these series diverge at $x = \pm 1$, as it behaves as the harmonic series in a neighbourhood of ± 1. We are interested, though, in the solutions that remain bounded on the whole interval $[-1, 1]$. Looking at the expressions of Eq. (9) one can check that the only possibility to make the series converge on $[-1, 1]$ is when $\lambda = n(n+1)$, $n \in \mathbb{N} \cup \{0\}$. In this case, either one of the

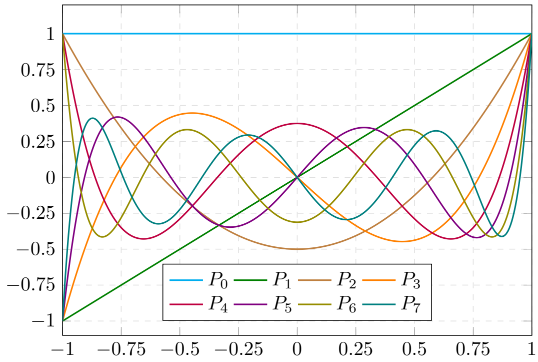

Figure 3: Graphic representation of the first eight Legendre polynomials.

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2-1)$
3	$\frac{1}{2}(5x^3-3x)$
4	$\frac{1}{8}(35x^4-30x^2+3)$
5	$\frac{1}{8}(63x^5-70x^3+15x)$
6	$\frac{1}{16}(231x^6-315x^4+105x^2-5)$
7	$\frac{1}{16}(429x^7-693x^5+315x^3-35x)$

Table 1: First eight Legendre polynomials

series is in fact a polynomial. In both cases it reduces to a polynomial of degree n. For each $n \in \mathbb{N} \cup \{0\}$ if we choose a_0 or a_1 be such that the polynomial evaluates to 1 at $x = 1$, these polynomials are called *Legendre polynomials* and they are denoted by $P_n(x)$. The other (divergent) series is usually denoted in the literature by $Q_n(x)$ (check [1]). And so the general solution of Eq. (7) for $\lambda = n(n+1)$ can be expressed as a linear combination of P_n and Q_n.

Proposition 9. Consider the function $g_x(t) = \frac{1}{\sqrt{1-2xt+t^2}}$ with $|x| \leq 1$. Then, the generating function of g is:

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n \tag{10}$$

Proof. Assume that formally $\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} Q_n(x)t^n$. We want to check that $Q_n(x) = P_n(x)$ for all $n \in \mathbb{N} \cup \{0\}$. Differentiating the equation with respect to x and with respect to t we obtain:

$$\frac{x-t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} nQ_n(x)t^{n-1} \qquad \frac{t}{(1-2xt+t^2)^{3/2}} = \sum_{n=0}^{\infty} {Q_n}'t^n \tag{11}$$

The second equation can be rewritten as:

$$t\sum_{n=0}^{\infty} Q_n t^n = (1-2xt+t^2)\sum_{n=0}^{\infty} {Q_n}'(x)t^{n-1} \tag{12}$$

So equating the coefficients of t^n we get:

$$Q_n = {Q_{n+1}}' - 2x{Q_n}' + {Q_{n-1}}' \tag{13}$$

Moreover, from Eq. (11) we have that:

$$t\sum_{n=0}^{\infty} nQ_n(x)t^{n-1} = (x-t)\sum_{n=0}^{\infty} {Q_n}'(x)t^n \tag{14}$$

Again equating the coefficients of t^n we get:

$$nQ_n = x{Q_n}' - {Q_{n-1}}' \tag{15}$$

Hence differentiating $(1-x^2){P_n}'$ we have:

$$\left((1-x^2){P_n}'\right)' = -2x{P_n}' + (1-x^2)P \tag{16}$$

NOT FINISHED!!!!!!!!!! □

Proposition 10. Let $y(x)$ be a solution to the Legendre differential equation. Then, $\forall m \in \mathbb{Z}$ the function

$$w_m(x) = (1-x^2)^{m/2} \frac{\mathrm{d}^m y(x)}{\mathrm{d}x^m} \tag{17}$$

solves the *general Legendre differential equation*:

$$(1-x^2)y'' - 2xy' + \left(\lambda - \frac{m^2}{1-x^2}\right) y = 0 \tag{18}$$

In particular if $\lambda = n(n+1)$ for $n \in \mathbb{N} \cup \{0\}$, then $w_(x)$ is denoted as

$$P_n^m(x) := (1-x^2)^{m/2} \frac{\mathrm{d}^m P_n}{\mathrm{d}x^m} \tag{19}$$

and it is called the *associated Legendre polynomial* of degree n and order m.

Note that although these functions P_n^m are referred to *polynomials*, they are only *real* polynomials if m is even. But we have opt to call them as it is the common practice in the literature (see [3]).
Moreover, from the definition of P_n^m, we can see $P_n^0 = P_n$ and that $P_n^m = 0$ if $m > n$. So we can restrict the domain of m to the set $\{0, 1, \ldots, n-1, n\}$.

Definition 11. Let $n \in \mathbb{N} \cup \{0\}$ and $m \in \{0, 1, \ldots, n-1, n\}$. We define the *spherical harmonic* Y_n^m as:

$$Y_n^m(\theta, \phi) = P_n^{|m|}(\cos\phi)\mathrm{e}^{\mathrm{i}m\theta} \tag{20}$$

1.3.2 Laplace equation in spherical coordinates

Definition 12. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a twice-differentiable function. The *Laplace equation* is the equation

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{21}$$

where Δ is the Laplace operator.

The next proposition gives the Laplace equation in spherical coordinates.

Proposition 13. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a twice-differentiable function. Then:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial f}{\partial \phi}\right) + \frac{1}{r^2 (\sin\phi)^2}\frac{\partial^2 f}{\partial \theta^2} = 0 \tag{22}$$

where r denotes the radial distance, θ denotes the azimuthal angle, and ϕ, the polar angle.

Recall that a solutions to the *Dirichlet problem* on a bounded domain $\Omega \subset \mathbb{R}^3$

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{23}$$

exists and is unique if g is sufficiently smooth. Theorem 14 gives them as a function of the so-called

Theorem 14. The regular solutions in a bounded region $\Omega \subseteq \mathbb{R}^3$ such that $0 \notin \overline{\Omega}$ to the Laplace equation in spherical coordinates are of the form

$$f(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (c_n^m r^n + d_n^m r^{-n-1}) P_n^{|m|}(\cos\phi)\mathrm{e}^{\mathrm{i}m\theta} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (c_n^m r^n + d_n^m r^{-n-1}) Y_n^m(\theta, \phi) \tag{24}$$

where $c_n^m, d_n^m \in \mathbb{C}$.

Proof. Let $f(r, \theta, \phi)$ be a solution of Eq. (22) Using separation variables $f(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$ one can write:

$$\frac{\Theta\Phi}{r^2}(r^2 R')' + \frac{R\Theta}{r^2 \sin\phi}(\sin\phi \Phi')' + \frac{R\Phi}{r^2 (\sin\phi)^2}\Theta'' = 0 \tag{25}$$

Isolating R from Θ and Φ yields:

$$\frac{\left(r^2R'\right)'}{R} = -\frac{1}{\sin\phi\Phi}\left(\sin\phi\Phi'\right)' - \frac{1}{\left(\sin\phi\right)^2\Theta}\Theta'' \tag{26}$$

Since the left-hand side depends entirely on r and the right-hand side does not, we must need that both sides are constant. Hence:

$$\frac{\left(r^2R'\right)'}{R} = \lambda \tag{27}$$

$$\frac{1}{\sin\phi\Phi}\left(\sin\phi\Phi'\right)' + \frac{1}{\left(\sin\phi\right)^2\Theta}\Theta'' = -\lambda \tag{28}$$

with $\lambda \in \mathbb{R}$. Similarly from Eq. (28) we obtain that the equations

$$\frac{1}{\Theta}\Theta'' = -m^2 \tag{29}$$

$$\frac{\sin\phi}{\Phi}\left(\sin\phi\Phi'\right)' + \lambda\left(\sin\phi\right)^2 = m^2 \tag{30}$$

must be constant with $m \in \mathbb{C}$ (a priori). The solution to Eq. (29) is a linear combination of the exponentials $\mathrm{e}^{im\theta}$, $\mathrm{e}^{-im\theta}$. Note, though, that since Θ must be a 2π-periodic function, that is satisfying $\Theta(\theta + 2\pi) = \Theta(\theta)\ \forall\theta \in \mathbb{R}$, m must be an integer. On the other hand making the change of variables $x = \cos\phi$ and $y = \Phi(\phi)$ in Eq. (30), that equation becomes:

$$(1 - x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\lambda - \frac{m^2}{1 - x^2}\right)y = 0 \tag{31}$$

which is the associate Legendre equation. We have argued in Proposition 10 that we need $\lambda = n(n+1)$ and $m \leq n$ in order to get regular solutions at $x = \cos\phi = \pm 1$. Moreover these solutions are $P_n^m(\cos\phi)$. Finally note that equation Eq. (27) is a Cauchy-Euler equation (check) and so it general solution is given by

$$R(r) = c_1r^n + c_2r^{-n-1} \tag{32}$$

because $\lambda = n(n+1)$. So the general solution becomes a linear combination of the each solution founded varying $n \in \mathbb{N} \cup \{0\}$ and $m \in \{0, 1, \ldots, n-1, n\}$:

$$f(r, \theta, \phi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\left(c_n^m r^n + d_n^m r^{-n-1}\right)P_n^{|m|}(\cos\phi)\mathrm{e}^{im\theta} \tag{33}$$

$\square$

From now we are not concerning of the singularity at $r = 0$ of Eq. (24) (see SECTION-POTENTIAL for more details).
With a bit of patience one can prove that $\forall n_1, n_2 \in \mathbb{N} \cup \{0\}$ and all $0 \leq m \leq \min\{n_1, n_2\}$:

$$\int_0^1 P_{n_1}^m(x)P_{n_2}^m(x)\,\mathrm{d}x = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{n_1,n_2} \tag{34}$$

where δ_{n_1,n_2} denotes the Kronecker delta. From here it's not hard to prove the spherical harmonics behave in a similar way:

$$\int_0^{2\pi}\int_0^{\pi} Y_{n_1}^{m_1}(\theta,\phi)\overline{Y_{n_2}^{m_2}(\theta,\phi)}\,\mathrm{d}\phi\,\mathrm{d}\theta = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}\delta_{n_1,n_2}\delta_{m_1,m_2} \tag{35}$$

Moreover, an important result in the Sturm-Liouville Theory of second order differential equations ([4, 2]) says that the family of spherical harmonics $\{Y_n^m(\theta,\phi) : n \in \mathbb{N} \cup \{0\}, |m| \leq n\}$ form a complete set in the sense that any smooth function defined on the sphere $f : S^2 \to \mathbb{R}$ can be expanded in a series of spherical harmonics:

$$f(\theta, \phi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} c_n^m Y_n^m(\theta, \phi) \tag{36}$$

2 Introduction to astrophysics and satellite tracking

2.1 The two body problem

2.1.1 Trajectory equation

We are interested in understanding the dynamics of a spacecraft in orbit around the Earth. These dynamics are governed by Newton's second law of motion, which assuming that both the Earth and the spacecraft are point masses (see Section 3 for a more realistic model), can be written as

$$\ddot{\mathbf{r}} = -\frac{GM_\oplus}{r^2}\mathbf{e}_r \tag{37}$$

where $\mathbf{r}$ is the position vector (also called *radius vector*) of the spacecraft with respect to the Earth, $r := \|\mathbf{r}\|$, $\mathbf{e}_r = \frac{\mathbf{r}}{r}$ is the unit vector in the direction of $\mathbf{r}$, $M_\oplus \simeq 5.972 \times 10^{24}$ kg is the mass of the Earth, and $G \simeq 6.674 \times 10^{-11}$ $\mathrm{m}^3 \cdot \mathrm{kg}^{-1} \cdot \mathrm{s}^{-2}$ is the gravitational constant. Note that the minus sign is due to the fact that the gravitational force is attractive, i.e. pointing towards the Earth. Here and along the document the notation $\ddot{\mathbf{r}}$ means that the derivative is taken with respect to time. Cross-multiplying Eq. (37) by $\mathbf{r}$, we obtain

$$\frac{\mathrm{d}(\mathbf{r} \times \dot{\mathbf{r}})}{\mathrm{d}t} = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = -\frac{GM_\oplus}{r^3}(\mathbf{r} \times \mathbf{r}) = 0 \tag{38}$$

Hence $\mathbf{r} \times \dot{\mathbf{r}} =: \mathbf{h}$ is constant. The physical intuition behind this is that the motion of the spacecraft around the Earth is confined to a plane, which is called the *orbital plane* because the position $\mathbf{r}$ and velocity $\mathbf{r}$ are always perpendicular to $\mathbf{h}$, which is the normal vector to the orbital planes and it relates to the *angular momentum* of the spacecraft.

We are interested now in what kind of curves may be described by a body orbiting the other one. That is, we want somehow isolate $\mathbf{r}$ (or r) from Eq. (37). In order to simplify the notation we will denote $\mu := GM_\oplus$.

Proposition 15 (Kepler's first law). The motion of a body orbiting another one is described by a conic. Hence it can be expressed in the form:

$$r(t) = \frac{p}{1 + e\cos(\nu(t))} \tag{39}$$

for some parameters p and e.

Proof. Cross-multiplying Eq. (37) by $\mathbf{h}$ we obtain

$$\frac{\mathrm{d}(\dot{\mathbf{r}} \times \mathbf{h})}{\mathrm{d}t} = \ddot{\mathbf{r}} \times \mathbf{h} = -\frac{\mu}{r^3}\mathbf{r} \times \mathbf{h} = -\frac{\mu}{r^3}\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}}) = -\frac{\mu}{r^3}[(\mathbf{r} \cdot \mathbf{r})\dot{\mathbf{r}} - (\mathbf{r} \cdot \dot{\mathbf{r}})\mathbf{r}] \tag{40}$$

where we have used Proposition 1. Now note that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right) = \frac{\dot{\mathbf{r}}}{r} - \frac{\dot{r}}{r^2}\mathbf{r} = \frac{1}{r^3}[(\mathbf{r} \cdot \mathbf{r})\dot{\mathbf{r}} - (\mathbf{r} \cdot \dot{\mathbf{r}})\mathbf{r}] \tag{41}$$

because $2r\dot{r} = \frac{\mathrm{d}(r^2)}{\mathrm{d}t} = \frac{(\mathbf{r} \cdot \mathbf{r})}{t} = 2\mathbf{r} \cdot \dot{\mathbf{r}}$[1]. Thus:

$$\frac{\mathrm{d}(\dot{\mathbf{r}} \times \mathbf{h})}{\mathrm{d}t} = \mu\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right) \tag{42}$$

Integrating with respect to the time yields

$$\dot{\mathbf{r}} \times \mathbf{h} = \frac{\mu}{r}\mathbf{r} + \mathbf{B} \tag{43}$$

where $\mathbf{B} \in \mathbb{R}^3$ is the constant of integration. Now dot-multiplying this last equation by $\mathbf{r}$ and using that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}\ \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ we obtain

$$h^2 = \mathbf{h} \cdot \mathbf{h} = (\mathbf{r} \times \dot{\mathbf{r}}) \cdot \mathbf{h} = \mathbf{r} \cdot (\dot{\mathbf{r}} \times \mathbf{h}) = \frac{\mu}{r}\mathbf{r} \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{B} = \mu r + rB\cos\nu \tag{44}$$

[1] Bear in mind that in general $\dot{r} \neq \|\dot{\mathbf{r}}\|$. Indeed, if β denotes the angle between $\mathbf{r}$ and $\dot{\mathbf{r}}$ we have that $\dot{r} = \|\dot{\mathbf{r}}\|\cos\beta$. In particular $\dot{r}$ may be negative.

where $h := \|\mathbf{h}\|$, $B := \|\mathbf{B}\|$ and ν denotes the angle between $\mathbf{r}$ and $\mathbf{B}$. Rearranging the terms we obtain finally the equation of a conic

$$r = \frac{h^2/\mu}{1 + (B/\mu)\cos(\nu)} \tag{45}$$

with $p := h^2/\mu$ and $e := B/\mu$. $\square$

Among the range of values that can r take, we are particularly interested in the minimum and maximum values, $r_{\min}$ and $r_{\max}$, that can be attained. Is easy to see that these are given by

$$r_{\min} = \frac{p}{1+e} \qquad \text{and} \qquad r_{\max} = \begin{cases} \dfrac{p}{1-e} & e < 1 \\ \infty & e \geq 1 \end{cases} \tag{46}$$

The points on the orbit of such distances are attained are called *apoapsis* and *periapsis* respectively. The line connecting both points is called *line of apsides*, and the half of the distance between them is the *semi-major axis* and is denoted by a:

$$a := \frac{r_{\max} + r_{\min}}{2} = \begin{cases} \dfrac{p}{1-e^2} & e < 1 \\ \infty & e \geq 1 \end{cases} = \begin{cases} \dfrac{h^2}{\mu(1-e^2)} & e < 1 \\ \infty & e \geq 1 \end{cases} \tag{47}$$

because we have considered the reference frame of Fig. 2 and so the line of apsides crosses the origin. Finally the angle ν is called *true anomaly*.

Definition 16. Let $\mathbf{r}(t)$, $\mathbf{r}(t+k)$ be the positions of the small body at times t, $t+k$ respectively. Let $A(t)$ be the area swept by the radius vector $\mathbf{r}(t)$ in the time interval $[0, t]$. We define the *areal velocity* as $\frac{\mathrm{d}A(t)}{\mathrm{d}t}$.

Proposition 17 (Kepler's second law). The areal velocity remains constant.

Proof. Recall that the area of a parallelogram generated by two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ is given by $\|\mathbf{u} \times \mathbf{v}\|$. Thus, approximating the area A by half of the parallelogram generated by $\mathbf{r}(t)$ and $\mathbf{r}(t+k)$ we obtain

$$\begin{aligned} \frac{\mathrm{d}A(t)}{\mathrm{d}t} = \lim_{k\to 0} \frac{A(t+k) - A(t)}{k} = \lim_{k\to 0} \frac{\|\mathbf{r}(t) \times \mathbf{r}(t+k)\|}{2k} = \lim_{h\to 0} \frac{\|\mathbf{r}(t) \times (\mathbf{r}(t+k) - \mathbf{r}(t))\|}{2k} = \\ = \frac{\|\mathbf{r}(t) \times \dot{\mathbf{r}}(t)\|}{2} = \frac{h}{2} \end{aligned} \tag{48}$$

where the penultimate equality is because the cross product is continuous and linear. $\square$

From now one we will suppose that the orbits are ellipses, which is the main case of interest.

2.1.2 Kepler's equation

So far we have been able to describe the geometry of motion of a body orbiting another one. However, we have not been concerned about the specific position of the body as a function of time. That is how to obtain $\nu(t)$ at each instant of time. In order to do this, we may think the area A as a function of ν, that measures the area swept by the radio vector from an initial instant ν_0. Thus, from differential calculus we know that:

$$A(\nu) = \int_{\nu_0}^{\nu} \int_0^{r(\theta)} r \,\mathrm{d}r \,\mathrm{d}\theta = \int_{\nu_0}^{\nu} \frac{r(\theta)^2}{2} \,\mathrm{d}\theta \implies \frac{\mathrm{d}A}{\mathrm{d}\nu} = \frac{r^2}{2} \tag{49}$$

And using the chain rule and Eq. (48) we obtain that:

$$\frac{h}{2} = \frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}\nu}\frac{\mathrm{d}\nu}{\mathrm{d}t} = \frac{r^2}{2}\dot{\nu} \tag{50}$$

So from Eqs. (45) and (50) we get the following differential equation that must satisfy ν:

$$\dot{\nu} = \frac{h}{r^2} = \frac{h}{p^2}(1 + e\cos\nu)^2 \tag{51}$$

which, when integrated with respect to the time, lead us to an elliptic integral. Our goal in this section is to find an easier way to compute exact position of the satellite ate each instant of time. This will lead us to the so-called *Kepler's equation*. For this purpose we are forced to introduce a new parameter, E, called *eccentric anomaly*. It is defined as the angle between the line of apsides and the line passing through the center of the ellipse and the point at the circle which is just above the position of the satellite (see Fig. 4 for a better understanding).

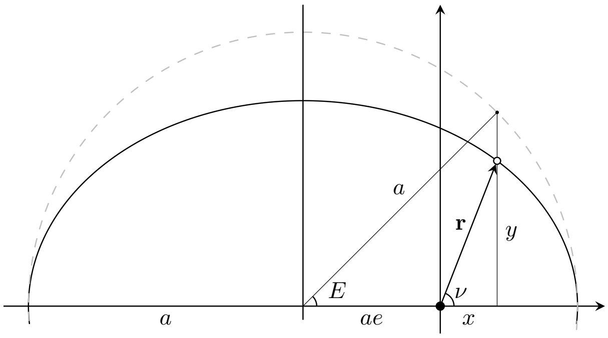

Figure 4: Ellipse orbit of the satellite together with an auxiliary circle of radius a needed to define the eccentric anomaly.

Clearly the position of the satellite is determined by $x = r\cos\nu$, $y = r\sin\nu$. But we would like to find an expression of x and y in terms of E rather than ν. To do this note that $a\cos E = ae + x$, so:

$$x = a(\cos E - e) \tag{52}$$

And so we can get an expression of r in terms of E by solving the equation:

$$r = \frac{p}{1 + e\cos\nu} = \frac{a(1-e^2)}{1 + e\frac{x}{r}} = \frac{ra(1-e^2)}{r + ae(\cos E - e)} \implies r = a(1 - e\cos E) \tag{53}$$

Finally from Eqs. (52) and (53) we get:

$$y^2 = r^2 - x^2 = a^2(1-e^2)(\sin E)^2 \implies y = a\sqrt{1-e^2}\sin E \tag{54}$$

Expressing now the areal velocity h as a function of E we have:

$$\begin{aligned} h &= x\dot{y} - y\dot{x} && (55)\\ &= a^2(\cos E - e)\sqrt{1-e^2}(\cos E)\dot{E} + a^2(\sin E)^2\dot{E}\sqrt{1-e^2} && (56)\\ &= a^2\sqrt{1-e^2}\dot{E}(1 - e\cos E) && (57)\end{aligned}$$

From Eq. (47) we know that $h = \sqrt{\mu a(1-e^2)}$. Thus substituting this in the latter equation we deduce that E must satisfy the following differential equation:

$$\dot{E}(1 - e\cos E) = \sqrt{\frac{\mu}{a^3}} =: n \tag{58}$$

where n is called the *mean motion*. Integrating this equation with respect to time yield the *Kepler's equation*:

$$E(t) - e\sin E(t) = n(t - t_0) \tag{59}$$

where t_0 is the time at which E vanishes. Using the reference frame of Fig. 4 this corresponds at the time at which the satellite is at the perigee. The value $M := n(t - t_0)$ is called *mean anomaly*.
Kepler's equation is the key to solve the problem of finding the position of the satellite at each instant of time. Later on we will discuss techniques to solve this equation for E knowing e and M.

2.2 Time and reference systems

3 Force model

3.1 Geopotential model

References

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