Numerical propagation of trajectories of Earth-orbiting spacecraft

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We are just an advanced breed of monkeys on a minor planet of a very average star. But we can understand the Universe. That makes us something very special.

Stephen Hawking

Contents

1 Introduction

2 Preliminaries

In this section we will review some basic concepts of linear algebra and vector calculus that will be used throughout the document.

2.1 Properties of cross and dot products

Proposition 1. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Then:

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} \tag{1}$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \tag{2}$$

Proof. Let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$. Then:

$$\begin{aligned}
((\mathbf{u} \times \mathbf{v}) \times \mathbf{w})_1 &= (u_3 v_1 - u_1 v_3) w_3 - (u_1 v_2 - u_2 v_1) w_2 \\
&= (u_2 w_2 + u_3 w_3) v_1 - (v_2 w_2 + v_3 w_3) u_1 \\
&= (u_1 w_1 + u_2 w_2 + u_3 w_3) v_1 - (v_1 w_1 + v_2 w_2 + v_3 w_3) u_1 \\
&= ((\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u})_1
\end{aligned}$$

The other components are treated similarly. The second equality follows in a similar way. □

Proposition 2. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Then:

1. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} \cdot \mathbf{w} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} \cdot \mathbf{w}$

2.2 Conics in a nutshell

Definition 3. A conic is the curve obtained as the intersection of a plane with the surface of a double cone (a cone with two *nappes*).

In Fig. 1, we show the 3 types of conics: the ellipse, the parabola, and the hyperbola, which differ on their eccentricity, as we will see later. Note that the circle is a special case of the ellipse.

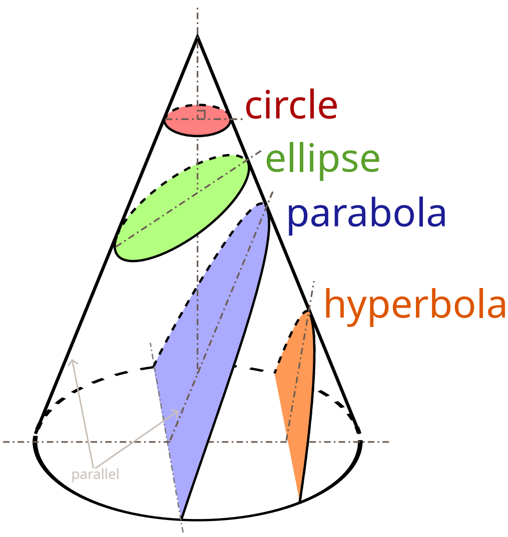

Figure 1: The black boundaries of the colored regions are conic sections. The other half of the hyperbola, which is not shown, is the other nappe of the double cone.

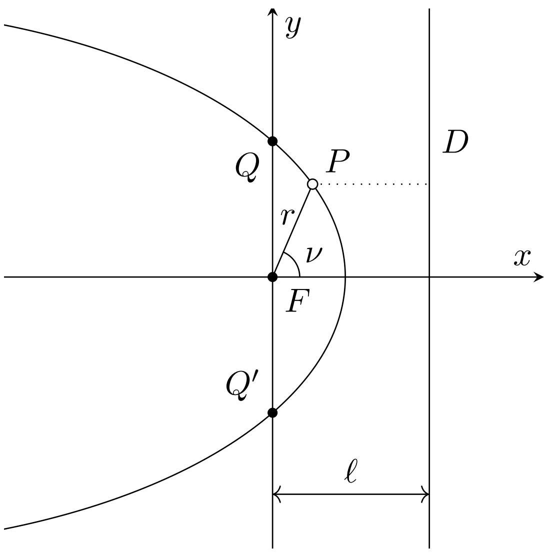

Figure 2: Reference frame centered at the focus of the conic and whose axes are such that the y-axis is parallel to the directrix and the x-axis is perpendicular to the directrix. The directions of the axes are chosen arbitrarily, subject to the constraint that a positive basis is chosen.

Definition 4. The *locus* is a set of points that satisfy a given condition.

The following proposition gives a characterization of the conics.

Proposition 5. A conic is the locus of all points P such that the distance from P to a fixed point F is a multiple of the distance from P to a fixed line D. Mathematically, this is expressed as:

$$d(P,F) = ed(P,D) \tag{3}$$

where d is the Euclidean distance. The point F is called the *focus*; the line D, *directrix*, and the constant of proportionality e, *eccentricity*.

Note that using the polar coordinates (r, ν) as in Fig. 2, we can rewrite Eq. (3) as:

$$r = e(\ell - r\cos\nu) \implies r = \frac{e\ell}{1 + e\cos\nu} =: \frac{p}{1 + e\cos\nu} \tag{4}$$

where we have defined $p := e\ell$.

Definition 6. Le C be a conic and e be its eccentricity. We say that C is

- an *ellipse* if $0 \le e < 1$,
- a *parabola* if $e = 1$, and
- a *hyperbola* if $e > 1$.

If $e = 0$, the conic is a *circle*.

2.3 Spherical harmonics

2.3.1 Legendre polynomials, regularity and orthonormality

Definition 7. Consider the following second-order differential equation:

$$y'' + p_1(x)y' + p_0(x)y = 0 \tag{5}$$

We say that a is an *ordinary point* if p_1 and p_2 are analytic at $x = a$. We say that a is a *regular singular point* if p_1 has a pole up to order 1 at a and p_0 has a pole of order up to 2 at a. Otherwise we say that a is a *irregular singular point*.

Definition 8. Consider the following second-order differential equation called *Legendre differential equation*:

$$(1 - x^2)y'' - 2xy' + \lambda y = 0 \tag{6}$$

for $n \in \mathbb{N} \cup \{0\}$. This equation can be written as:

$$\left((1 - x^2)y'\right)' + \lambda y = 0 \tag{7}$$

If seek for analytic solutions of this equation using the power series method, i.e. looking for solutions of the form $y(x) = \sum_{j=0}^{\infty} a_j x^j$ one can check that we obtain this recursion:

$$a_{j+2} = \frac{j(j+1) - \lambda}{(j+1)(j+2)} a_j \quad j = 0, 1, 2, \ldots \tag{8}$$

From here we can obtain two independent solutions by setting the initial conditions a_0 and a_1 of the iteration. For example, setting $a_1 = 0$ we obtain a series that has only even powers of x. On the other hand, setting $a_0 = 0$ we obtain a series that has only odd powers of x. These two series converge on the interval $(-1, 1)$ by the ratio test (by looking at Eq. (8)) and can be expressed compactly as [Mez]:

$$y_{\text{e}}(x) = a_0 \sum_{j=0}^{\infty} \left[\prod_{k=0}^{j-1} (2k(2k+1) - \lambda) \right] \frac{x^{2j+1}}{(2j)!} \qquad y_{\text{o}}(x) = a_1 \sum_{j=0}^{\infty} \left[\prod_{k=1}^{j} (2k(2k+1) - \lambda) \right] \frac{x^{2j+1}}{(2j+1)!} \tag{9}$$

However for each $\lambda \in \mathbb{R}$ either one of these series diverge at $x = \pm 1$, as it behaves as the harmonic series in a neighbourhood of ± 1. We are interested, though, in the solutions that remain bounded on the whole

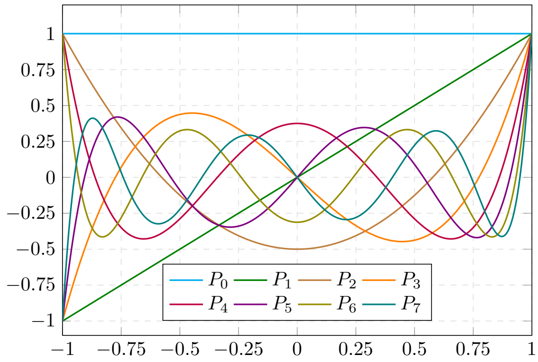

Figure 3: Graphic representation of the first eight Legendre polynomials.

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2-1)$
3	$\frac{1}{2}(5x^3-3x)$
4	$\frac{1}{8}(35x^4-30x^2+3)$
5	$\frac{1}{8}(63x^5-70x^3+15x)$
6	$\frac{1}{16}(231x^6-315x^4+105x^2-5)$
7	$\frac{1}{16}(429x^7-693x^5+315x^3-35x)$

Table 1: First eight Legendre polynomials

interval $[-1,1]$. Looking at the expressions of Eq. (9) one can check that the only possibility to make the series converge on $[-1,1]$ is when $\lambda = n(n+1)$, $n \in \mathbb{N} \cup \{0\}$. In this case, either one of the series is in fact a polynomial. And in both cases the polynomial has degree n. For each $n \in \mathbb{N} \cup \{0\}$ if we choose a_0 or a_1 be such that the polynomial evaluates to 1 at $x = 1$, these polynomials are called *Legendre polynomials* and they are denoted by $P_n(x)$. The other (divergent) series is usually denoted in the literature by $Q_n(x)$ (check [RHB99]). And so the general solution of Eq. (7) for $\lambda = n(n+1)$ can be expressed as a linear combination of P_n and Q_n.

Proposition 9 (Rodrigues' formula). Let $n \in \mathbb{N} \cup \{0\}$. Then, $\forall x \in [-1,1]$:

$$P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n} (x^2-1)^n \tag{10}$$

Proposition 10. Let $y(x)$ be a solution to the Legendre differential equation. Then, $\forall m \in \mathbb{N} \cup \{0\}$ the function

$$w_m(x) = (1-x^2)^{m/2} \frac{\mathrm{d}^m y(x)}{\mathrm{d}x^m} \tag{11}$$

solves the *general Legendre differential equation*:

$$(1-x^2)y'' - 2xy' + \left(\lambda - \frac{m^2}{1-x^2}\right) y = 0 \tag{12}$$

In particular if $\lambda = n(n+1)$ for $n \in \mathbb{N} \cup \{0\}$, then $w_m(x)$ is denoted as

$$P_n^m(x) := (1-x^2)^{m/2} \frac{\mathrm{d}^m P_n}{\mathrm{d}x^m} \tag{13}$$

and it is called the *associated Legendre polynomial* of degree n and order m.

Note that although these functions P_n^m are referred to *polynomials*, they are only *real* polynomials if m is even. But we have opt to call them as it is the common practice in the literature (see [Wei]).

Moreover, from the definition of P_n^m, we can see $P_n^0 = P_n$ and that $P_n^m = 0$ if $m > n$. So we can restrict the domain of m to the set $\{0, 1, \ldots, n-1, n\}$.

Definition 11. Let $n \in \mathbb{N} \cup \{0\}$ and $m \in \{-n, -(n-1), \ldots, 0, \ldots, n-1, n\}$. We define the *spherical harmonic* Y_n^m as the following function:

$$Y_n^m(\theta, \phi) = P_n^{|m|}(\cos\phi)\mathrm{e}^{\mathrm{i}m\theta} \tag{14}$$

2.3.2 Laplace equation in spherical coordinates

Definition 12. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a twice-differentiable function. The *Laplace equation* is the equation

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{15}$$

where Δ is the Laplace operator.

The next proposition gives the Laplace equation in spherical coordinates.

Proposition 13. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a twice-differentiable function. Then:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial f}{\partial\phi}\right) + \frac{1}{r^2(\sin\phi)^2}\frac{\partial^2 f}{\partial\theta^2} = 0 \tag{16}$$

where r denotes the radial distance, θ denotes the azimuthal angle, and ϕ, the polar angle.

Recall that a solutions to the *Dirichlet problem* on a bounded domain $\Omega \subset \mathbb{R}^3$

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases} \tag{17}$$

exists and is unique if g is sufficiently smooth. Theorem 14 gives them as a function of the so-called

Theorem 14. The regular solutions in a bounded region $\Omega \subseteq \mathbb{R}^3$ such that $0 \notin \overline{\Omega}$ to the Laplace equation in spherical coordinates are of the form

$$f(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(c_n^m r^n + d_n^m r^{-n-1})P_n^{|m|}(\cos\phi)\mathrm{e}^{im\theta} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}(c_n^m r^n + d_n^m r^{-n-1})Y_n^m(\theta,\phi) \tag{18}$$

where $c_n^m, d_n^m \in \mathbb{C}$.

Proof. Let $f(r,\theta,\phi)$ be a solution of Eq. (16) Using separation variables $f(r,\theta,\phi) = R(r)\Theta(\theta)\Phi(\phi)$ one can write:

$$\frac{\Theta\Phi}{r^2}(r^2R')' + \frac{R\Theta}{r^2\sin\phi}(\sin\phi\Phi')' + \frac{R\Phi}{r^2(\sin\phi)^2}\Theta'' = 0 \tag{19}$$

Isolating R from Θ and Φ yields:

$$\frac{(r^2R')'}{R} = -\frac{1}{\sin\phi\Phi}(\sin\phi\Phi')' - \frac{1}{(\sin\phi)^2\Theta}\Theta'' \tag{20}$$

Since the left-hand side depends entirely on r and the right-hand side does not, we must need that both sides are constant. Hence:

$$\frac{(r^2R')'}{R} = \lambda \tag{21}$$

$$\frac{1}{\sin\phi\Phi}(\sin\phi\Phi')' + \frac{1}{(\sin\phi)^2\Theta}\Theta'' = -\lambda \tag{22}$$

with $\lambda \in \mathbb{R}$. Similarly from Eq. (22) we obtain that the equations

$$\frac{1}{\Theta}\Theta'' = -m^2 \tag{23}$$

$$\frac{\sin\phi}{\Phi}(\sin\phi\Phi')' + \lambda(\sin\phi)^2 = m^2 \tag{24}$$

must be constant with $m \in \mathbb{C}$ (a priori). The solution to Eq. (23) is a linear combination of the exponentials $\mathrm{e}^{im\theta}$, $\mathrm{e}^{-im\theta}$. Note, though, that since Θ must be a 2π-periodic function, that is satisfying $\Theta(\theta + 2\pi) = \Theta(\theta)\ \forall\theta \in \mathbb{R}$, m must be an integer. On the other hand making the change of variables $x = \cos\phi$ and $y = \Phi(\phi)$ in Eq. (24), that equation becomes:

$$(1-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\lambda - \frac{m^2}{1-x^2}\right)y = 0 \tag{25}$$

which is the associate Legendre equation. We have argued in Proposition 10 that we need $\lambda = n(n+1)$ and $m \leq n$ in order to get regular solutions at $x = \cos\phi = \pm 1$. Moreover these solutions are $P_n^m(\cos\phi)$.

Finally note that equation Eq. (21) is a Cauchy-Euler equation (check [Wika]) and so it general solution is given by

$$R(r) = c_1r^n + c_2r^{-n-1} \tag{26}$$

because $\lambda = n(n+1)$. So the general solution becomes a linear combination of the each solution founded varying $n \in \mathbb{N} \cup \{0\}$ and $m \in \{0, 1, \dots, n-1, n\}$:

$$f(r,\theta,\phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (c_n^m r^n + d_n^m r^{-n-1}) P_n^{|m|}(\cos\phi) \mathrm{e}^{\mathrm{i}m\theta} \tag{27}$$

□

From now we are not concerning of the singularity at $r = 0$ of Eq. (18) (see SECTION-POTENTIAL for more details).

With a bit of patience one can prove that $\forall n_1, n_2 \in \mathbb{N} \cup \{0\}$ and all $0 \leq m \leq \min\{n_1, n_2\}$:

$$\int_0^1 P_{n_1}^m(x) P_{n_2}^m(x) \, \mathrm{d}x = \frac{2}{2n_1+1} \frac{(n_1+m)!}{(n_1-m)!} \delta_{n_1,n_2} \tag{28}$$

where δ_{n_1,n_2} denotes the Kronecker delta. From here it's not hard to prove that the spherical harmonics behave in a similar way:

$$\int_0^{2\pi} \int_0^{\pi} Y_{n_1}^{m_1}(\theta,\phi) \overline{Y_{n_2}^{m_2}(\theta,\phi)} \, \mathrm{d}\phi \, \mathrm{d}\theta = \frac{2}{2n_1+1} \frac{(n_1+m_1)!}{(n_1-m_1)!} \delta_{n_1,n_2} \delta_{m_1,m_2} \tag{29}$$

Moreover, an important result in the Sturm-Liouville Theory of second order differential equations ([Wikc; Wan+09]) says that the family of spherical harmonics $\{Y_n^m(\theta,\phi) : n \in \mathbb{N} \cup \{0\}, |m| \leq n\}$ form a complete set in the sense that any smooth function defined on the sphere $f : S^2 \to \mathbb{R}$ can be expanded in a series of spherical harmonics:

$$f(\theta,\phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} c_n^m Y_n^m(\theta,\phi) \tag{30}$$

3 Introduction to astrophysics and satellite tracking

3.1 The two body problem

3.1.1 Trajectory equation

We are interested in understanding the dynamics of a spacecraft in orbit around the Earth. These dynamics are governed by Newton's second law of motion, which assuming that both the Earth and the spacecraft are point masses (see Section 4 for a more realistic model), can be written as

$$\ddot{\mathbf{r}} = -\frac{GM_\oplus}{r^2} \mathbf{e}_r \tag{31}$$

where $\mathbf{r}$ is the position vector (also called *radius vector*) of the spacecraft with respect to the Earth, $r := \|\mathbf{r}\|$, $\mathbf{e}_r = \frac{\mathbf{r}}{r}$ is the unit vector in the direction of $\mathbf{r}$, $M_\oplus \simeq 5.972 \times 10^{24}$ kg is the mass of the Earth, and $G \simeq 6.674 \times 10^{-11}$ $\mathrm{m}^3 \cdot \mathrm{kg}^{-1} \cdot \mathrm{s}^{-2}$ is the gravitational constant. Note that the minus sign is due to the fact that the gravitational force is attractive, i.e. pointing towards the Earth. Here and along the document the notation $\ddot{\mathbf{r}}$ means that the derivative is taken with respect to time. Cross-multiplying Eq. (31) by $\mathbf{r}$, we obtain

$$\frac{\mathrm{d}(\mathbf{r} \times \dot{\mathbf{r}})}{\mathrm{d}t} = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}} = -\frac{GM_\oplus}{r^3} (\mathbf{r} \times \mathbf{r}) = 0 \tag{32}$$

Hence $\mathbf{r} \times \dot{\mathbf{r}} =: \mathbf{h}$ is constant. The physical intuition behind this is that the motion of the spacecraft around the Earth is confined to a plane, which is called the *orbital plane* because the position $\mathbf{r}$ and velocity $\mathbf{r}$ are always perpendicular to $\mathbf{h}$, which is the normal vector to the orbital planes and it relates to the *angular momentum* of the spacecraft.

We are interested now in what kind of curves may be described by a body orbiting the other one. That is, we want somehow isolate $\mathbf{r}$ (or r) from Eq. (31). In order to simplify the notation we will denote $\mu := GM_\oplus$.

Proposition 15 (Kepler's first law). The motion of a body orbiting another one is described by a conic. Hence it can be expressed in the form:

$$r(t) = \frac{p}{1 + e\cos(\nu(t))} \tag{33}$$

for some parameters p and e.

Proof. Cross-multiplying Eq. (31) by $\mathbf{h}$ we obtain

$$\frac{\mathrm{d}(\dot{\mathbf{r}} \times \mathbf{h})}{\mathrm{d}t} = \ddot{\mathbf{r}} \times \mathbf{h} = -\frac{\mu}{r^3}\mathbf{r} \times \mathbf{h} = -\frac{\mu}{r^3}\mathbf{r} \times (\mathbf{r} \times \dot{\mathbf{r}}) = -\frac{\mu}{r^3}[(\mathbf{r} \cdot \mathbf{r})\dot{\mathbf{r}} - (\mathbf{r} \cdot \dot{\mathbf{r}})\mathbf{r}] \tag{34}$$

where we have used Proposition 1. Now note that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right) = \frac{\dot{\mathbf{r}}}{r} - \frac{\dot{r}}{r^2}\mathbf{r} = \frac{1}{r^3}[(\mathbf{r} \cdot \mathbf{r})\dot{\mathbf{r}} - (\mathbf{r} \cdot \dot{\mathbf{r}})\mathbf{r}] \tag{35}$$

because $2r\dot{r} = \frac{\mathrm{d}(r^2)}{\mathrm{d}t} = \frac{(\mathbf{r}\cdot\mathbf{r})}{t} = 2\mathbf{r} \cdot \dot{\mathbf{r}}$[1]. Thus:

$$\frac{\mathrm{d}(\dot{\mathbf{r}} \times \mathbf{h})}{\mathrm{d}t} = \mu\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathbf{r}}{r}\right) \tag{36}$$

Integrating with respect to the time yields

$$\dot{\mathbf{r}} \times \mathbf{h} = \frac{\mu}{r}\mathbf{r} + \mathbf{B} \tag{37}$$

where $\mathbf{B} \in \mathbb{R}^3$ is the constant of integration. Observe that since $\dot{\mathbf{r}} \times \mathbf{h}$ is perpendicular to $\mathbf{h}$, it lies in the orbital plane and so does $\mathbf{r}$. Hence, $\mathbf{B}$ lies on the orbital plane. Now dot-multiplying this last equation by $\mathbf{r}$ and using that $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}\ \forall \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ we obtain

$$h^2 = \mathbf{h} \cdot \mathbf{h} = (\mathbf{r} \times \dot{\mathbf{r}}) \cdot \mathbf{h} = \mathbf{r} \cdot (\dot{\mathbf{r}} \times \mathbf{h}) = \frac{\mu}{r}\mathbf{r} \cdot \mathbf{r} + \mathbf{r} \cdot \mathbf{B} = \mu r + rB\cos\nu \tag{38}$$

where $h := \|\mathbf{h}\|$, $B := \|\mathbf{B}\|$ and ν denotes the angle between $\mathbf{r}$ and $\mathbf{B}$. Rearranging the terms we obtain finally the equation of a conic

$$r = \frac{h^2/\mu}{1 + (B/\mu)\cos(\nu)} \tag{39}$$

with $p := h^2/\mu$ and $e := B/\mu$. ☐

Among the range of values that can r take, we are particularly interested in the minimum and maximum values, $r_{\min}$ and $r_{\max}$, that can be attained. Is easy to see that these are given by

$$r_{\min} = \frac{p}{1+e} \qquad \text{and} \qquad r_{\max} = \begin{cases} \dfrac{p}{1-e} & e < 1 \\ \infty & e \geq 1 \end{cases} \tag{40}$$

The points on the orbit of such distances are attained are called *apoapsis* and *periapsis* respectively. The line connecting both points is called *line of apsides*, and the half of the distance between them is the *semi-major axis* and is denoted by a:

$$a := \frac{r_{\max} + r_{\min}}{2} = \begin{cases} \dfrac{p}{1-e^2} & e < 1 \\ \infty & e \geq 1 \end{cases} = \begin{cases} \dfrac{h^2}{\mu(1-e^2)} & e < 1 \\ \infty & e \geq 1 \end{cases} \tag{41}$$

because we have considered the reference frame of Fig. 2 and so the line of apsides crosses the origin. Finally the angle ν is called *true anomaly*. Note that at $\mathbf{r}_{\min}$, we have $\nu = 0$ and so $\mathbf{r} \parallel \mathbf{B}$. Hence $\mathbf{B}$ points towards the periapsis of the orbit.

Definition 16. Let $\mathbf{r}(t)$, $\mathbf{r}(t+k)$ be the positions of the small body at times t, $t+k$ respectively. Let $A(t)$ be the area swept by the radius vector $\mathbf{r}(t)$ in the time interval $[0, t]$. We define the *areal velocity* as $\frac{\mathrm{d}A(t)}{\mathrm{d}t}$.

[1]Bear in mind that in general $\dot{r} \neq \|\dot{\mathbf{r}}\|$. Indeed, if β denotes the angle between $\mathbf{r}$ and $\dot{\mathbf{r}}$ we have that $\dot{r} = \|\dot{\mathbf{r}}\|\cos\beta$. In particular $\dot{r}$ may be negative.

Proposition 17 (Kepler's second law). The areal velocity remains constant.

Proof. Recall that the area of a parallelogram generated by two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ is given by $\|\mathbf{u} \times \mathbf{v}\|$. Thus, approximating the area A by half of the parallelogram generated by $\mathbf{r}(t)$ and $\mathbf{r}(t+k)$ we obtain

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = \lim_{k\to 0} \frac{A(t+k) - A(t)}{k} = \lim_{k\to 0} \frac{\|\mathbf{r}(t) \times \mathbf{r}(t+k)\|}{2k} = \lim_{h\to 0} \frac{\|\mathbf{r}(t) \times (\mathbf{r}(t+k) - \mathbf{r}(t))\|}{2k} = \\ = \frac{\|\mathbf{r}(t) \times \dot{\mathbf{r}}(t)\|}{2} = \frac{h}{2} \tag{42}$$

where the penultimate equality is because the cross product is continuous and linear. □

Finally we will need the following equation which relates the velocity of the satellite with the distance to the central of the Earth.

Proposition 18. We have that ([MG05]):

$$v^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right) \tag{43}$$

where $v := \|\dot{\mathbf{r}}\|$.

Proof. Using Eq. (37) we have that $\mathbf{h} \times \mathbf{r} = (\mathbf{r} \cdot \dot{\mathbf{r}})\mathbf{r} - (\mathbf{r} \cdot \mathbf{r})\dot{\mathbf{r}}$ and so:

$$\|\mathbf{h} \times \dot{\mathbf{r}}\|^2 = \frac{\mu^2}{r^2}\mathbf{r} \cdot \mathbf{r} + 2\frac{\mu}{r}\mathbf{r} \cdot \mathbf{B} + \mathbf{B} \cdot \mathbf{B} = \mu^2(1 + 2e\cos\nu + e^2) = \mu^2(2(1 + e\cos\nu) - (1 - e^2)) \tag{44}$$

where we have used that $e\mu = B$ (see Eq. (39)). Now using Eqs. (39) and (41) we obtain that

$$2(1 + e\cos\nu) - (1 - e^2) = 2\frac{p}{r} - \frac{h^2}{\mu a} = 2\frac{h^2}{r\mu} - \frac{h^2}{\mu a} \tag{45}$$

Since $\mathbf{h} \perp \dot{\mathbf{r}}$, $\|\mathbf{h} \times \dot{\mathbf{r}}\| = hv$ and so:

$$v^2 = \mu \left(\frac{2}{r} - \frac{1}{a} \right) \tag{46}$$

□

From now one we will suppose that the orbits are ellipses, which is the main case of interest.

3.1.2 Kepler's equation

So far we have been able to describe the geometry of motion of a body orbiting another one. However, we have not been concerned about the specific position of the body as a function of time. That is how to obtain $\nu(t)$ at each instant of time. In order to do this, we may think the area A as a function of ν, that measures the area swept by the radio vector from an initial instant ν_0. Thus, from differential calculus we know that:

$$A(\nu) = \int_{\nu_0}^{\nu} \int_{0}^{r(\theta)} r \,\mathrm{d}r \,\mathrm{d}\theta = \int_{\nu_0}^{\nu} \frac{r(\theta)^2}{2} \,\mathrm{d}\theta \implies \frac{\mathrm{d}A}{\mathrm{d}\nu} = \frac{r^2}{2} \tag{47}$$

And using the chain rule and Eq. (42) we obtain that:

$$\frac{h}{2} = \frac{\mathrm{d}A}{\mathrm{d}t} = \frac{\mathrm{d}A}{\mathrm{d}\nu}\frac{\mathrm{d}\nu}{\mathrm{d}t} = \frac{r^2}{2}\dot{\nu} \tag{48}$$

So from Eqs. (39) and (48) we get the following differential equation that must satisfy ν:

$$\dot{\nu} = \frac{h}{r^2} = \frac{h}{p^2}(1 + e\cos\nu)^2 \tag{49}$$

which, when integrated with respect to the time, lead us to an elliptic integral. Our goal in this section is to find an easier way to compute exact position of the satellite ate each instant of time. This will lead us to the so-called *Kepler's equation*. For this purpose we are forced to introduce a new parameter, E, called *eccentric anomaly*. It is defined as the angle between the line of apsides and the line passing through the

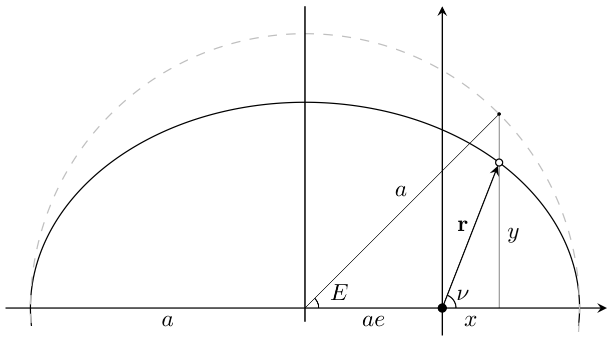

Figure 4: Ellipse orbit of the satellite together with an auxiliary circle of radius a needed to define the eccentric anomaly.

center of the ellipse and the point at the circle which is just above the position of the satellite (see Fig. 4 for a better understanding).

Clearly the position of the satellite is determined by $x = r\cos\nu$, $y = r\sin\nu$. But we would like to find an expression of x and y in terms of E rather than ν. To do this note that $a\cos E = ae + x$, so:

$$x = a(\cos E - e) \tag{50}$$

And so we can get an expression of r in terms of E by solving the equation:

$$r = \frac{p}{1 + e\cos\nu} = \frac{a(1-e^2)}{1 + e\frac{x}{r}} = \frac{ra(1-e^2)}{r + ae(\cos E - e)} \implies r = a(1 - e\cos E) \tag{51}$$

Finally from Eqs. (50) and (51) we get:

$$y^2 = r^2 - x^2 = a^2(1-e^2)(\sin E)^2 \implies y = a\sqrt{1-e^2}\sin E \tag{52}$$

Expressing now the areal velocity h as a function of E we have:

$$h = x\dot{y} - y\dot{x} \tag{53}$$

$$= a^2(\cos E - e)\sqrt{1-e^2}(\cos E)\dot{E} + a^2(\sin E)^2\dot{E}\sqrt{1-e^2} \tag{54}$$

$$= a^2\sqrt{1-e^2}\dot{E}(1 - e\cos E) \tag{55}$$

From Eq. (41) we know that $h = \sqrt{\mu a(1-e^2)}$. Thus substituting this in the latter equation we deduce that E must satisfy the following differential equation:

$$\dot{E}(1 - e\cos E) = \sqrt{\frac{\mu}{a^3}} =: n \tag{56}$$

where n is called the *mean motion*. Integrating this equation with respect to time yield the *Kepler's equation*:

$$E(t) - e\sin E(t) = n(t - t_0) \tag{57}$$

where t_0 is the time at which E vanishes. Using the reference frame of Fig. 4 this corresponds at the time at which the satellite is at the perigee. The value $M := n(t - t_0)$ is called *mean anomaly*.

Kepler's equation is the key to solve the problem of finding the position of the satellite at each instant of time. Later on we will discuss techniques to solve this equation for E knowing e and M.

3.2 Time and reference systems

3.2.1 Time measurement

As human beings, we are naturally interested in how time passes and so the correct measure of it becomes an essential necessity for us. As it is the sun that governs our daily activity, it is natural to define time from it.

An *apparent solar day* is defined to be the time between two successive transits of the sun across our local meridian. One should note that the Earth has to rotate on itself slightly more than one revolution in order to complete one solar day. The *apparent siderial day* is defined as the time it take to the Earth to complete a rotation relative to very far away stars (see Fig. 5 for a beter understanding). The non-circular orbit of the Earth around the sun causes some days to be shorter than others due to Kepler's second law. So the introduction of a *mean solar day* and *mean siderial day* are necessary, which are defined as if the Earth was in a circular orbit of the same period as the true one.

Definition 19. The *Greenwich mean time* (GMT) or *Universal time* (UT or UT1 for disambiguation[2]) is defined as the mean solar time at the Greenwich meridian. The *Greenwich mean siderial time* (GMST) is a representation of the *mean siderial day* and it is defined as the angle between the Greenwich meridian and th emean vernal equinox of date. In a similar way, the *Greenwich aparent siderial time* (GAST) is a representation of the *siderial day* and it is defined as the angle between the Greenwich meridian and the true vernal equinox of date.

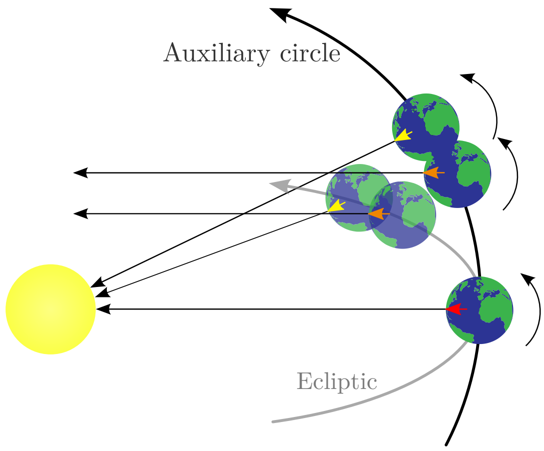

Figure 5: Solar and siderial days (not to scale)

In the middle of the 20th century, *Ephemerides time* (ET) was introduced to cope with the irregularities of the Earth's rotation (see POLAR MOTION). This time was defined from historical observations of planets in a Newtonian physics isolating the time from the equations. At the end of the century, the *Terrestrial time* (TT) succeed ET in a relativistic frame. In the meantime, in 1955 *atomic time* (TAI, from French *Temps Athomique International*) was adopted as the SI unit of second. The origin was adopted such that the TAI matches UT at the 00:00:00 UT of January 1st, 1958, yielding the relation

$$\mathrm{TT} = \mathrm{ET} = \mathrm{TAI} + 32.184\,\mathrm{s} \tag{58}$$

Finally, our everyday clock is based on the *Coordinated Universal Time* (UTC). It is defined to be as uniform as the TAI but always kept closer than 0.9 seconds to the UT1 (see GRAPH-TIME). Scientists achieve this by introducing a *leap second* which is an extra second added to UTC at irregular intervals. The last leap second was added on June 30th, 2012. The next one is expected to be added on December 31st, 2016. Fig. 6 summarizes all the time systems introduced in the document.

The folowing conversions between time systems are useful:

$$\begin{aligned}
\mathrm{GMST} &= 24110.54841 + 8640184.812866 T_{\mathrm{UT1}} + 0.093104 (T_{\mathrm{UT1}})^2 - 6.2 \cdot 10^{-6} (T_{\mathrm{UT1}})^3 \\
\mathrm{UT1} &= \mathrm{TT} - \Delta T \\
\mathrm{TT} &= \mathrm{TAI} - 32.184 \\
\mathrm{TAI} &= \mathrm{UTC} + \delta(\mathrm{TAI})
\end{aligned}$$

where $T = \frac{\mathrm{JD(UT1)} - 2451545}{36525}$ is the number of Julian centuries since the epoch J2000.0, ΔT is the difference between the TT and UT1, $\delta(\mathrm{TAI})$ is a piecewise constant function that counts the number of leap seconds introduced since 1972 (when they were introduced for the first time). All the numbers have units of second. Note that since the rotation of the Earth cannot be predicted accurately, ΔT can only be determined retrospectively, and is given by the *International Earth Rotation and Reference Systems Service* (IERS).

[2]There are other times also referred as *universal*, such as UT1R or UT12 but we will not talk about them in the document.

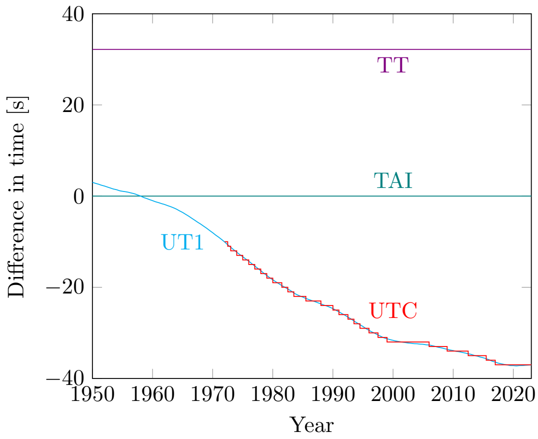

Figure 6: Evolution of times TT, UT1 and UTC in comparison with TAI. [Obs]

```
1998 December 31,      23h 59m 59s
1998 December 31,      23h 59m 60s
1999 January   1,      00h 00m 00s
1999 January   2,      00h 00m 01s
```

Figure 7: Leap second introduced to the UTC time at the end of the December 1998. [RS98]

3.2.2 Reference systems

Newton's second law in only valid when applied to an *inertial reference frame*, that is, a frame of reference that is not undergoing any acceleration. In practise, however, almost any frame of reference is not inertial. So in this chapter we will describe an almost-inertial frame of reference which will be used to integrate Newton's second law. On the other hand, since the earth is not a perfect sphere and there are zones which higher mass density than others, and therefore with higher gravitational field (see SPHERICAL-HARMONICS), we would need the longitude and latitude of the satellite with respect to the Earth at each time of integration. Hence, a Earth-centered system will be needed too.

The first reference frame we must consider is the *celestial*one. On basis of the study of the solar system a natural origin for this frame is the center of mass of the solar system. In order to define the x-, y-, and z-axes, we will need the following definitions.

Definition 20. We define the *celestial equator* as the plane on $\mathbb{R}^3$ that contains the Earth equator. We define the *celestial ecliptic* as the plane on $\mathbb{R}^3$ that contains the ecliptic, that is, the orbit of the Earth around the sun. In order to simplify the language, we will call them *equator* and *ecliptic* respectively.

Definition 21. We define the *celestial sphere* as an abstract sphere of infiniteradius concentric with the Earth. All the celestial objects are thus, projected naturally on the celestial sphere, identifying them with two coordinates (longitude and latitude) (see Fig. 8 for a better understanding).

Definition 22. Consider the line ℓ of intersection between the celestial equator and the celestial ecliptic. We define the *vernal equinox* as the point Υ on the ecliptic that lies on the line ℓ and is such that the Earth is crossing the celestial equator from south to north.

From here, we can define the x-axis as the line ℓ and pointing to the vernal equinox, the z-axis perpendicular to the celestial equator and the y-axis chosen such that the triplet (x, y, z) is a right-handed system.

However, due to the presence of other solar system planets (and other smaller perturbations), the orbital plane of the Earth is not fixed in space, but is subjected to a small variation called *planetary precession*. Moreover, the gravitational attraction of the Sun and Moon on the Earth's equator cause Earth's axis of rotation to precess in a similar way to the action of a spinning top with a period of about 26000 years [MG05]. This motion is called *lunisolar precession*. On the other hand, smaller pertubations in amplitude ($<$ 18.6 years [Wikb]) with shorter period superposed with the precessional motion creates a motion called *nutation*. When this latter oscillations are averaged out, the Earth's axis of rotation, the ecliptic, and the equator are referred to *mean* values, rather than *true* values.

In view of this time-dependent orientation of both the ecliptic and the equator, the standard-reference frame chosen is based on the mean equator, mean ecliptic and mean equinox of some fixed time, the beginning of the year 2000, namely at 12:00 TT on 1 January 2000, the so-called *J2000 epoch*.

Definition 23 (J2000 frame of reference). We define the *J2000 frame of reference* as the frame of reference whose x-axis is the intersection of the mean celestial equator and the mean celestial ecliptic pointing at the mean vernal equinox, the z-axis is perpendicular to the mean ecliptic plane and the y-axis is chosen such that the triplet (x, y, z) is a right-handed system. The origin of this system is chosen to be at the center of mass of the solar system.

Let's move on now to study an Earth-fixed reference frame.

Definition 24. We define the *prime meridian* or *zero meridian* is the meridian that passes through the Royal Observatory in Greenwich, England, and is the reference meridian for longitude measurements.

Definition 25 (Earth-fixed frame of reference). We define the *Earth-fixed frame of reference* at time t as the frame of reference whose x-axis is pointing to the prime meridian, the z-axis is perpendicular to the Earth equator at time t and the y-axis is chosen such that the triplet (x, y, z) is a right-handed system. The origin of this system is chosen to be at the center of mass of the Earth.

3.2.3 Conversion between reference systems

As we noted in the previous section the angle ε between the celestial equator and celestial ecliptic planes is not constant due to the planetary precession.

We would like to transform the position of the satellite from the J2000 frame of reference to the Earth-fixed frame of reference and vice versa. This rotation transformation is given by a product of 4 rotations:

- The precession matrix $\mathbf{P}$,
- the nutation matrix $\mathbf{N}$,
- the Earth rotation matrix $\mathbf{\Theta}$,
- and the polar motion matrix $\mathbf{\Pi}$.

These matrix are such that:

$$\mathbf{r}_{\mathrm{EF}}(t) = \mathbf{\Pi}(t)\mathbf{\Theta}(t)\mathbf{N}(t)\mathbf{P}(t)\mathbf{r}_{J2000}(t) \tag{59}$$

where $\mathbf{r}_{\mathrm{EF}}(t)$ is the position vector of the satellite in the Earth-fixed frame of reference at time t and $\mathbf{r}_{J2000}(t)$ is the position vector of the satellite in the J2000 frame of reference at time t. From here on, we will omit the evaluation on the time t for the sake of simplify the lecture.

The precession matrix is responsible for *eliminating* all the movement due to the planetary and lunisolar precession. Thus, $\mathbf{P}$ transforms the mean equator and mean equinox at time J2000 to the respective values at time t. With the help of Fig. 8 it's not easy to see that this transformation is given by:

$$\mathbf{P} = \mathbf{R}_z(-90 - z)\mathbf{R}_x(\theta)\mathbf{R}_z(90 - \zeta) \tag{60}$$

which with a bit of algebra can be simplified to:

$$\mathbf{P} = \mathbf{R}_z(-z)\mathbf{R}_y(\theta)\mathbf{R}_z(-\zeta) \tag{61}$$

Recall that the fundamental rotation matrices $\mathbf{R}_x(\theta)$, $\mathbf{R}_y(\theta)$ and $\mathbf{R}_z(\theta)$ are with respect to the axis of the J2000 frame and they are given by:

$$\mathbf{R}_x(\varphi) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & \sin\varphi \\ 0 & -\sin\varphi & \cos\varphi \end{pmatrix} \quad \mathbf{R}_y(\varphi) = \begin{pmatrix} \cos\varphi & 0 & -\sin\varphi \\ 0 & 1 & 0 \\ \sin\varphi & 0 & \cos\varphi \end{pmatrix} \quad \mathbf{R}_z(\varphi) = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{62}$$

where we have used the convention of signs given by [GPS02]. The reader may wonder why we have used the notation $90 - z$ and $90 - \zeta$ instead of z and ζ (for example) for the angles in question. The reason is related to the precise definition of these angles from the pole of the celestial sphere rather than from where we have defined them, but we will not elaborate on this point here. Nonetheless, we have chosen this notation to maintain consistency with related articles.

The nutation perturbations are driven out by the nutation matrix $\mathbf{N}$. This matrix transforms the coordinates of the mean equator and equinox at epoch t to those of the true equator and equinox at the same epoch, respectively. Hence, from figure Fig. 9 we can see that the nutation matrix is given by:

$$\mathbf{N} = \mathbf{R}_x(-\varepsilon - \Delta\varepsilon)\mathbf{R}_z(-\Delta\psi)\mathbf{R}_x(\varepsilon) \tag{63}$$

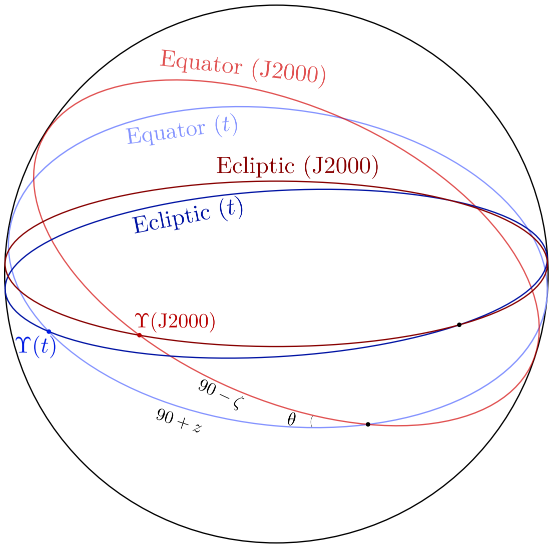

Figure 8: Celestial sphere showing the ecliptic and the equator of both the epoch J2000 and the current epoch t. Dark colors represent the ecliptic while light colors represent the equator. On the other hand, red colors represents the the J2000 epoch and blue colors represents the current epoch t.

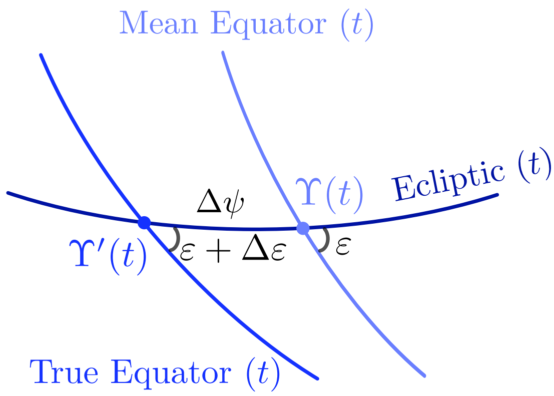

Figure 9: True equator and mean equator, and true equinox (Υ') and mean equinox (Υ) at a given epoch t together with the ecliptic at that time.

3.3 Orbital elements

3.3.1 Orbital elements from position and velocity

Definition 26. Consider a satellite orbiting the Earth. The *orbital plane* is the plane that contains the orbit of the satellite. The *line of nodes* of intersection between the orbital plane and the equator. Finally, the *ascending node* is the point on the line of nodes and the orbit of the satellite where the satellite crosses the equatorial plane from south to north.

Definition 27 (Orbital elements). The *orbital elements* of a satellite are five independent quantities that completely determine its orbit. If moreover the exact position of the satellite on the orbit is wanted, a sixth quantity is needed. The first five orbital elements are:

1. The *semi-major axis* a of the orbit.
2. The *eccentricity* e of the orbit.
3. The *inclination* i is the angle between the equatorial plane and the orbital plane.
4. The *longitude of the ascending node* Ω is the angle between the vernal equinox and the ascending node.
5. The *argument of perigee* ω is the angle between the ascending node and the periapsis.

The sixth quantity is the *true anomaly* ν which is the angle between the periapsis and the position of the satellite on the orbit.

The elements a, e and i are always well-defined. However, the elements Ω, ν are not well-defined in the case of $e = 0$ because there is no periapsis, because all the points lie at the same distance from the center of the Earth. Note that $i \in [0, \pi/2]$ and by convention we will impose the angles Ω, ω, ν to be in $[0, 2\pi)$ In order to properly define these elements in terms of the position $\mathbf{r}$ and the velocity $\dot{\mathbf{r}}$ of the satellite, we need to introduce the basis $(\mathbf{p}, \mathbf{q}, \mathbf{w})$ linked to the orbit.

Definition 28 (Perifocal coordinate system). Consider the orbit of a satellite. We define its associated *perifocal coordinate system* $(\mathbf{p}, \mathbf{q}, \mathbf{w})$ as follows. The center is on the Earth's center of mass. The unit vectors $\mathbf{p}$ and $\mathbf{q}$ lie on the orbital plane and are such that $\mathbf{p}$ points towards the periapsis, that is $\mathbf{p} := \mathbf{B}/B$. The unit vector $\mathbf{w}$ is defined as $\mathbf{w} := \mathbf{h}/h$ and $\mathbf{q} := \mathbf{w} \times \mathbf{p}$.

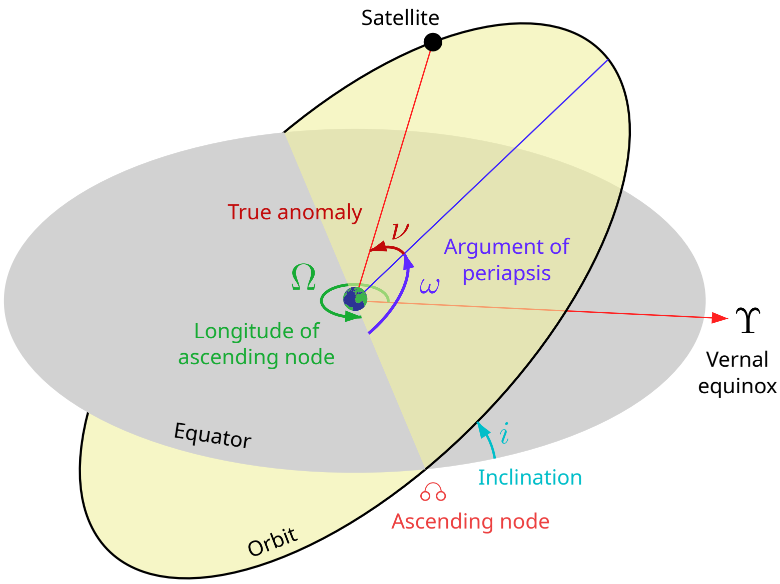

Figure 10: Orbital elements of a satellite.

Theorem 29. Consider a satellite orbiting the Earth at position $\mathbf{r}$ and velocity $\dot{\mathbf{r}}$ from the Earth-fixed frame. The orbital elements of the satellite are given by:

$$a = \left(\frac{2}{r} - \frac{v^2}{\mu}\right)^{-1} \qquad e = \sqrt{1 - \frac{p}{a}} \qquad i = \arccos\left(\frac{h_z}{h}\right)$$
$$\Omega = \arctan\left(\frac{h_x}{-h_y}\right) \mod 2\pi \quad \omega = \arctan\left(\frac{B_z h}{h_x B_y - h_y B_x}\right) \mod 2\pi \quad \nu = \arccos\left(\frac{p-r}{re}\right) \mod 2\pi$$

where $\mathbf{h} = (h_x, h_y, h_z)$ and $\mathbf{B} = (B_x, B_y, B_z)$.

Proof. First note that we can know $\mathbf{h}$ and p from $\mathbf{h} = \mathbf{r} \times \dot{\mathbf{r}}$ and $p = h^2/\mu$. Moreover $\mathbf{B} = \dot{\mathbf{r}} \times \mathbf{h} - \frac{\mu}{r}\mathbf{r}$ form Eq. (37). The semi-major axis a is given in Eq. (43) and the eccentricity formula follows from Eq. (41). Isolating ν from the equation of the ellipse (Eq. (39)) we can get the true anomaly. Moreover, one can note (looking at Fig. 10) that the matrix transformation from the perifocal frame to the Earth-fixed frame is given by:

$$\mathbf{A} = \mathbf{R}_z(-\Omega)\mathbf{R}_x(-i)\mathbf{R}_z(-\omega) \tag{64}$$

Now, from linear algebra we know that $(\mathbf{p}, \mathbf{q}, \mathbf{w})$ are just the column vectors of $\mathbf{A}$. From here, computing $\mathbf{A}$ we get that:

$$\mathbf{A} = \begin{pmatrix} \cos(\omega)\cos(\Omega) - \sin(\omega)\sin(\Omega)\cos(i) & -\sin(\omega)\cos(\Omega) - \cos(\omega)\sin(\Omega)\cos(i) & \sin(\Omega)\sin(i) \\ \cos(\omega)\sin(\Omega) + \sin(\omega)\cos(\Omega)\cos(i) & -\sin(\omega)\sin(\Omega) + \cos(\omega)\cos(\Omega)\cos(i) & -\cos(\Omega)\sin(i) \\ \sin(\omega)\sin(i) & \cos(\omega)\sin(i) & \cos(i) \end{pmatrix} \tag{65}$$

From the definition we can get $\mathbf{p}$ and $\mathbf{w}$ by:

$$\mathbf{p} = \begin{pmatrix} B_x/B \\ B_y/B \\ B_z/B \end{pmatrix} \qquad \mathbf{w} = \begin{pmatrix} h_x/h \\ h_y/h \\ h_z/h \end{pmatrix} \tag{66}$$

From here and looking at the third column of $\mathbf{A}$ we deduce the expressions for i and Ω. Now, if $\mathbf{q} = (q_x, q_y, q_z)$, we have that $q_z = w_x p_y - w_y p_x = \frac{h_x B_y - h_y B_x}{hB}$ and so:

$$\frac{B_z/B}{(h_x B_y - h_y B_x)/hB} = \tan(\omega) \implies \omega = \arctan\left(\frac{B_z h}{h_x B_y - h_y B_x}\right) \tag{67}$$

□

Està relament be?????????????????????????? Pensar en que els TLE estan referits respecte el j2000 o no?????

3.3.2 TLE sets and determining position and velocity from orbital elements

The positions of satellites at each instant of time

4 Force model

So far we have only considered the gravitational force acting point masses. In reality, the Earth is not a point mass, neither a spherically symmetric mass distribution. In this section we will deep into the details of a more realistic model of the Earth's gravitational field.

4.1 Geopotential model

4.1.1 Continuous distribution of mass

In Section 3.1 we have seen that the motion of a body orbiting another one can be described by a conic section. However, we have not been concerned about the mass distribution of the large body, in our case the Earth. In this section we will see that the motion of the smaller body, the satellite, is slightly perturbed by the mass distribution of the larger one as well as the precense of other forces such as atmospheric drag, solar radiation pressure, and the gravitational pull of the Moon and Sun, which we wil talk later on. Even though, the perturbations are relatively small and the orbits of the satellites are still approximating ellipses.

Consider a body confined in a compact region $\Omega \subseteq \mathbb{R}$ with a continuous density of mass $\rho : \Omega \to \mathbb{R}$. We would like to know the gravitational pull on a point mass m located at position $\mathbf{r}$ from the center of mass of the body. To do this we can discretize the body Ω in a set of cubes $m_{i,j,k}$ each of volume $\frac{1}{n_x n_y n_z}$ and density $\rho(\frac{i}{n_x}, \frac{j}{n_y}, \frac{k}{n_z}) =: \rho_{i,j,k}$, where n_x, n_y, and n_z are the number of cubes in the x, y, and z directions, respectively. The total gravitational acceleration $\mathbf{g}$ exerted on m is the sum of the contributions of all the forces and it is given by:

$$\mathbf{g} = -\sum_{i=0}^{n_x}\sum_{j=0}^{n_y}\sum_{k=0}^{n_z} \frac{m_{i,j,k}}{\|\mathbf{r}-\mathbf{s}_{i,j,k}\|^3}(\mathbf{r}-\mathbf{s}_{i,j,k}) = -\sum_{i=0}^{n_x}\sum_{j=0}^{n_y}\sum_{k=0}^{n_z} \frac{\rho_{i,j,k}}{\|\mathbf{r}-\mathbf{s}_{i,j,k}\|^3}(\mathbf{r}-\mathbf{s}_{i,j,k})\frac{1}{n_x n_y n_z} \tag{68}$$

where $\mathbf{s}_{i,j,k} = (\frac{i}{n_x}, \frac{j}{n_y}, \frac{k}{n_z})$ (in cartesian coordinates). Note that Eq. (68) is a Riemann sum and so letting $n_x, n_y, n_z \to \infty$ we get:

$$\mathbf{g} = -\int_\Omega \frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|^3}(\mathbf{r}-\mathbf{s})\mathrm{d}^3\mathbf{s} \tag{69}$$

where $\mathrm{d}^3\mathbf{s} := \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'$, if $\mathbf{s} = (x', y', z')$.

Theorem 30. Let Ω be a compact region in $\mathbb{R}^3$ with a continuous density of mass $\rho : \Omega \to \mathbb{R}$. Then, the gravitational acceleration field $\mathbf{g}$ is conservative. That is, there exists a function $f : \mathbb{R}^3 \to \mathbb{R}$ such that $\mathbf{g} = \boldsymbol{\nabla} f$.

Proof. An easy computation shows that fixed $\mathbf{s} \in \mathbb{R}^3$ we have:

$$\boldsymbol{\nabla}\left(\frac{1}{\|\mathbf{r}-\mathbf{s}\|}\right) = -\frac{1}{\|\mathbf{r}-\mathbf{s}\|^3}(\mathbf{r}-\mathbf{s}) \tag{70}$$

So we need to justify if the following exchange of the gradient and the integral is correct:

$$\mathbf{g} = -\int_\Omega \frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|^3}(\mathbf{r}-\mathbf{s})\mathrm{d}^3\mathbf{s} = \int_\Omega \rho(\mathbf{s})\boldsymbol{\nabla}\left(\frac{1}{\|\mathbf{r}-\mathbf{s}\|}\right)\mathrm{d}^3\mathbf{s} = \boldsymbol{\nabla}\int_\Omega \frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|}\mathrm{d}^3\mathbf{s} \tag{71}$$

Without loss of generality it suffices to justify that

$$\frac{\partial}{\partial x}\int_\Omega \frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|}\mathrm{d}^3\mathbf{s} = \int_\Omega \frac{\partial}{\partial x}\left(\frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|}\right)\mathrm{d}^3\mathbf{s} \tag{72}$$

assuming $\mathbf{r} = (x, y, z)$ and $\mathbf{s} = (x', y', z')$. In order to apply the theorem of derivation under the integral sign we need to control $\frac{\partial}{\partial x}\left(\frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|}\right) = -\rho(\mathbf{s})\frac{x-x'}{\|\mathbf{r}-\mathbf{s}\|^3}$ by an integrable function $h(\mathbf{s})$. Using spherical coordinates centered at $\mathbf{r}$ and writing $(\mathbf{r}-\mathbf{s})_{\text{sph}} = (\rho_{\mathbf{r}}, \theta, \phi)$, the integrand to bound becomes (in spherical coordinates):

$$\left|-\rho(\mathbf{s})\frac{x-x'}{\|\mathbf{r}-\mathbf{s}\|^3}{\rho_{\mathbf{r}}}^2 \sin\phi\right| = |\rho(\mathbf{s})|\left|\frac{\rho_{\mathbf{r}}\cos\theta\sin\phi}{{\rho_{\mathbf{r}}}^3}{\rho_{\mathbf{r}}}^2\sin\phi\right| \leq |\rho(\mathbf{s})| \leq K \tag{73}$$

where the last inequality follows for certain $K \in \mathbb{R}$ by Weierstraß theorem (ρ is continuous and Ω is compact). Thus, since $h(\mathbf{s}) = K$ is integrable, because Ω is bounded, the equality of Eq. (72) is licit. □

Physically speaking, the gravitational force being conservative means that the work done by the force is independent of the path taken by the particle. Moreover, due to historical reasons, we will write $\mathbf{g} = -\boldsymbol{\nabla}V$ (with the minus sign) and call V the *gravitational potential*. The minus sign is chosen according the convention that work done by gravitational forces decreases the potential.

4.1.2 Laplace equations

Theorem 31. Consider distribution of matter of density ρ in a compact region Ω. Then, the gravitational potential V satisfies the Laplace equation

$$\Delta V = 0 \tag{74}$$

for all points outside Ω[3].

Proof. Recall that $\Delta V = \mathbf{div}(\boldsymbol{\nabla}V)$. So since $\mathbf{g} = -\boldsymbol{\nabla}V$ it suffices to prove that $\mathbf{div}(\mathbf{g}) = 0$. Note that if $\mathbf{r} \in \Omega^c$ and $\mathbf{s} \in \Omega$ then $\|\mathbf{r}-\mathbf{s}\| \geq \delta > 0$, so $\frac{\mathbf{r}-\mathbf{s}}{\|\mathbf{r}-\mathbf{s}\|^3}$ is differentiable and:

$$\begin{aligned}\mathbf{div}\left(\frac{\mathbf{r}-\mathbf{s}}{\|\mathbf{r}-\mathbf{s}\|^3}\right) = \frac{\partial}{\partial x}\left(\frac{x-x'}{\|\mathbf{r}-\mathbf{s}\|^3}\right) + \frac{\partial}{\partial y}\left(\frac{y-y'}{\|\mathbf{r}-\mathbf{s}\|^3}\right) + \frac{\partial}{\partial z}\left(\frac{z-z'}{\|\mathbf{r}-\mathbf{s}\|^3}\right) = \\ = \frac{\|\mathbf{r}-\mathbf{s}\|^2 - 3(x-x')^2}{\|\mathbf{r}-\mathbf{s}\|^5} + \frac{\|\mathbf{r}-\mathbf{s}\|^2 - 3(y-y')^2}{\|\mathbf{r}-\mathbf{s}\|^5} + \frac{\|\mathbf{r}-\mathbf{s}\|^2 - 3(z-z')^2}{\|\mathbf{r}-\mathbf{s}\|^5} = 0\end{aligned}$$

Hence, as in Theorem 30, we have that for each $\mathbf{r} \in \Omega^c$ $\exists \varepsilon, \delta > 0$ such that $\forall \tilde{\mathbf{r}} \in B(\mathbf{r}, \varepsilon)$ we have:

$$\left|\rho(\mathbf{s})\frac{\|\tilde{\mathbf{r}}-\mathbf{s}\|^2 - 3(\tilde{x}-x')^2}{\|\tilde{\mathbf{r}}-\mathbf{s}\|^5}\right| \leq \frac{4|\rho(\mathbf{s})|}{\|\tilde{\mathbf{r}}-\mathbf{s}\|^3} \leq \frac{4|\rho(\mathbf{s})|}{\delta^3}$$

which is integrable by Weierstraß theorem. Thus, by the theorem of derivation under the integral sign:

$$\mathbf{div}(\mathbf{g}) = -\,\mathbf{div}\int_\Omega \frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|^3}(\mathbf{r}-\mathbf{s})\mathrm{d}^3\mathbf{s} = -\int_\Omega \rho(\mathbf{s})\,\mathbf{div}\left(\frac{\mathbf{r}-\mathbf{s}}{\|\mathbf{r}-\mathbf{s}\|^3}\right)\mathrm{d}^3\mathbf{s} = 0 \tag{75}$$

□

So far we have seen that the gravitational potential V satisfies the Laplace equation. If moreover we choose the origin of potential to be at the infinity, that is, if we impose $\lim\limits_{\|\mathbf{r}\|\to\infty} V = 0$, then the gravitational potential created by a distribution of mass in a compact region Ω is a solution of the following exterior Dirichlet problem:

$$\begin{cases}\Delta V = 0 & \text{in } \Omega^c \\ V = f & \text{on } \partial\Omega \\ \lim\limits_{\|\mathbf{r}\|\to\infty} V = 0 & \end{cases} \tag{76}$$

If Ω represents the Earth, then $f = f(\theta, \phi)$ represents is the boundary condition concerning the gravitational potential at the surface of the Earth as a function of the longitude θ and colatitude ϕ.

We wil see now that Eq. (76) has uniquenes of solutions. To do that we invoke the maximum principle, which we will not prove (see [Eva10] for more details).

[3]It can be seen that V satisfies in fact the *Poisson equation* $\Delta V = 4\pi G\rho$ for any point $\mathbf{r} \in \mathbb{R}^3$, which reduced to Laplace equation when $\mathbf{r} \in \Omega^c$, because there we have $\rho(\mathbf{r}) = 0$.

Theorem 32 (Maximum principle). Let $U \subseteq \mathbb{R}^n$ be open and bounded and $u \in \mathcal{C}^2(U) \cap \mathcal{C}(\overline{U})$. Suppose that u is harmonic within U, that is, $\Delta u = 0$ in U. Then, $\max_{\overline{U}} u = \max_{\partial U} u$.

Corollary 33. The Dirichlet problem of Eq. (76) has a unique solution.

Proof. Suppose we have two solutions V_1, V_2 of Eq. (76). Then, $W := V_1 - V_2$ is harmonic in Ω^c and $W = 0$ on $\partial\Omega$. Moreover, $\lim_{\|\mathbf{r}\|\to\infty} W = 0$. So $\forall \varepsilon > 0$, $\exists n \in \mathbb{N}$ large enough such that $|W| \leq \varepsilon$ on $B(0, n)$. Thus, by the maximum principle, $|W| \leq \varepsilon$ on $\overline{B(0, n)} \cap \Omega^c$. Since the ε is arbitrary, we must have $W = 0$ on Ω^c, that is, $V_1 = V_2$. □

4.1.3 Expansion in spherical harmonics

We have just seen that V satisfies the exterior Dirichlet problem for the Laplace equation. In Section 2.3.2 we have seen that a solution to the Laplace equation can be expressed as:

$$V(r, \theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (c_n^m r^n + d_n^m r^{-n-1}) P_n^{|m|}(\cos\phi) \mathrm{e}^{im\theta} = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (c_n^m r^n + d_n^m r^{-n-1}) Y_n^m(\theta, \phi) \tag{77}$$

where $c_n^m, d_n^m \in \mathbb{C}$. If we impose V to satisfy the third condition of Eq. (76), we must have $c_n^m = 0$. Finally, if we choose $R_\oplus$ as a reference radius for a spherical model of the Earth, using the boundary condition on $\partial\Omega$

$$f(\theta, \phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} \frac{d_n^m}{{R_\oplus}^{n+1}} Y_n^m(\theta, \phi) \tag{78}$$

and the orthogonality of the spherical harmonics we can deduce that the coefficients d_n^m are given by:

$$d_n^m = \frac{2n+1}{2} \frac{(n-m)!}{(n+m)!} {R_\oplus}^{n+1} \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) \overline{Y_n^m(\theta, \phi)} \, \mathrm{d}\phi \, \mathrm{d}\theta \tag{79}$$

Thus, with a bit of algebra, namely changing the exponential with sines and cosines and introducing the gravitational constant G and the Earth's mass $M_\oplus$ to the equation, our solution can be expressed as:

$$V(r, \theta, \phi) = \frac{GM_\oplus}{R_\oplus} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{{R_\oplus}^{n+1}}{r^{n+1}} P_n^m(\cos\phi)(C_n^m \cos(m\theta) + S_n^m \sin(m\theta)) \tag{80}$$

where the coefficients $C_n^m, S_n^m \in \mathbb{R}$ are given by the formulas:

$$C_n^m = \frac{1}{GM_\oplus {R_\oplus}^n}(d_n^m + d_n^{-m}) = (2n+1)\frac{(n-m)!}{(n+m)!} \frac{R_\oplus}{GM_\oplus} \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) P_n^m(\cos\phi) \cos(m\theta) \, \mathrm{d}\phi \, \mathrm{d}\theta \tag{81}$$

$$S_n^m = \frac{\mathrm{i}}{GM_\oplus {R_\oplus}^n}(d_n^m - d_n^{-m}) = (2n+1)\frac{(n-m)!}{(n+m)!} \frac{R_\oplus}{GM_\oplus} \int_0^{2\pi} \int_0^{\pi} f(\theta, \phi) P_n^m(\cos\phi) \sin(m\theta) \, \mathrm{d}\phi \, \mathrm{d}\theta \tag{82}$$

In order to use a more uniform model in magnitude for the coefficients C_n^m, S_n^m and avoid large oscillations which may provoke a loss of data in double-precision arithmetic, the following normalization is used:

$$\bar{P}_n^m = \frac{P_n^m}{\sqrt{\frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}}} \qquad \bar{C}_n^m = \sqrt{\frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}} C_n^m \qquad \bar{S}_n^m = \sqrt{\frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}} S_n^m \tag{83}$$

Hence, our potential at the coordinate (r, θ, ϕ) outside the Earth is given by:

$$V(r, \theta, \phi) = \frac{GM_\oplus}{R_\oplus} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{{R_\oplus}^{n+1}}{r^{n+1}} \bar{P}_n^m(\cos\phi)(\bar{C}_n^m \cos(m\theta) + \bar{S}_n^m \sin(m\theta)) \tag{84}$$

The coefficients $\bar{C}_n^m$, $\bar{S}_n^m$ describe the dependence on the Earth's internal structure. They are obtained from observation of the perturbations seen in the orbits of other satellites [MG05]. Other methods for obtaining such data are through surface gravimetry, which provides precise local and regional information about the gravity field, or through altimeter data, which can be used to provide a model for the geoid of the Earth, that is the shape that the ocean surface would take under the influence of the gravity of Earth, which in turn can be used to obtain the geopotential coefficients.

4.1.4 Numerical computation of the gravity acceleration

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5 Conclusions

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