1 Force model

So far we have only considered the gravitational force acting point masses. In reality, the Earth is not a point mass, neither a spherically symmetric mass distribution. In this section we will deep into the details of a more realistic model of the Earth's gravitational field.

1.1 Geopotential model

It is well-known that the gravity force $\mathbf{F}$ is a conservative force. This means $\mathbf{F}$ can be expressed as the gradient of a function $-V$, which we call the gravitational potential: $\mathbf{F} = -\boldsymbol{\nabla} V$. The minus sign is chosen according the convention that work done by the forces decreases the potential.

1.1.1 Poisson and Laplace equations

Theorem 1. Consider distribution of matter of density ρ in a region Ω. Then, the gravitational potential V satisfies the Poisson equation

$$\Delta V = 4\pi G\rho \tag{1}$$

Thus, at points outside Ω since we have $\rho = 0$, the potential V satisfies the Laplace equation:

$$\Delta V = 0 \tag{2}$$

Proof. TO-DO □

Hence, the gravitational potential V created by a distribution of mass in a region Ω is a solution of the following Dirichlet problem:

$$\begin{cases} \Delta V = 0 & \text{in } \Omega^c \\ V = f & \text{on } \partial\Omega \end{cases} \tag{3}$$

If Ω represents the Earth, then $f = f(\theta, \phi)$ represents is the boundary condition concerning the gravitational potential at the surface of the Earth as a function of the longitude θ and colatitude ϕ.

1.1.2 Expansion in spherical harmonics

We have just seen that V satisfies the Laplace equation. In **??** we have seen that the solution can be expressed as:

$$V(r,\theta,\phi) = \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (c_n^m r^n + d_n^m r^{-n-1}) P_n^{|m|}(\cos\phi) \mathrm{e}^{\mathrm{i}m\theta} \tag{4}$$

where $c_n^m, d_n^m \in \mathbb{C}$. Choosing the origin of potential the infinity, we must have $c_n^m = 0$. Thus, with a bit of algebra, our solution becomes:

$$V(r,\theta,\phi) = \frac{GM_\oplus}{r} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{{R_\oplus}^n}{r^n} P_n^m(\cos\phi)(C_n^m \cos(m\theta) + S_n^m \sin(m\theta)) \tag{5}$$

where $C_n^m, S_n^m \in \mathbb{R}$, G is the gravitational constant, $M_\oplus$ is the mass of the Earth and $R_\oplus$ is the reference radius of the Earth. The coefficients C_n^m, S_n^m describe the dependence on the Earth's internal structure. They are obtained from observation