References

Aksnes, K.: 1967, *Astroph. Norv.* **10**, 156.
Arsenault, J. L., Enright, J. D., and Purcell, C.: 1964, ASTIA Documents 437–75 and 437–76.
Bonavito, N. L., Watson, S., and Walden, H.: 1969, NASA TN D-5203.
Brouwer, D.: 1959, *Astron. J.* **64**, 378.
Brouwer, D. and Clemence, G. M.: 1961, *Methods of Celestial Mechanics*, Academic Press, New York, Ch. II, p. 79.
Brown, E. W. and Shook, C. A.: 1933, *Planetary Theory*, Cambridge University Press, Cambridge (G.B.), Ch. VI, p. 141 *et seq.*
Cain, B. J.: 1962, *Astron. J.* **67**, 391.
Delaunay, C.: 1867, *Mémoires de l'Académie des Sciences* **2g**, p. 801.
Deprit, A.: 1969, *Cel. Mech.* **1**, 12.
Deprit, A. and Rom, A.: 1967, *Bull. Astron.*, Série 3, **2**, 425.
Deprit, A. and Zahar, R.: 1966, *Zeit. angew. Math. Phys.* **17**, 425.
Lubowe, A. G.: 1966, *AIAA J.* **4**, 361–63.
Meffroy, J.: 1968, NASA Goddard Space Flight Center Memorandum X-641-68-320.
Moses, J.: 1969, Tutorial Session, July 2, Summer Institute in Dynamical Astronomy, M.I.T., Cambridge, Mass.
Poincaré, H.: 1893, *Les Méthodes nouvelles de la Mécanique céleste*, Gauthier-Villars, Paris, Tome II, Chapitre VIII, p. 17 *et seq.*
Rom, A.: 1969, Boeing Document D1-82-0840 submitted for publication in *Cel. Mech.*

ON THE COMPUTATION OF THE SPHERICAL HARMONIC TERMS NEEDED DURING THE NUMERICAL INTEGRATION OF THE ORBITAL MOTION OF AN ARTIFICIAL SATELLITE

LELAND E. CUNNINGHAM

Lockheed Missiles and Space Company, Sunnyvale, Calif., and
Astronomy Dept., University of California, Berkeley, Calif., U.S.A.

(Received 28 August, 1969)

Abstract. A method is presented for the accurate and efficient computation of the forces and their first derivatives arising from any number of zonal and tesseral terms in the Earth's gravitational potential. The basic formulae are recurrence relations between some solid spherical harmonics, $V_{n,m}$, associated with the standard polynomial ones.

The Earth's gravitational potential, $-V$, can be written

$$V = \sum_{n=0}^{\infty} \sum_{m=0}^{n} \frac{R_{\oplus}^{n} P_n^m(\sin\varphi)}{r^{n+1}} (C_{n,m} \cos m\lambda + S_{n,m} \sin m\lambda) \tag{1}$$

where $R_{\oplus}$ is the radius of the Earth; r is the geocentric distance of a point in the potential field, φ is its latitude, and λ is its geocentric longitude measured eastward from the meridian of Greenwich; $C_{n,m}$ and $S_{n,m}$ are physical constants; and P_n^m are associated Legendre polynomials.

Define

$$V_{n,m} \equiv \frac{P_n^m(\sin\varphi)(\cos m\lambda + i \sin m\lambda)}{r^{n+1}}. \tag{2}$$

Then

$$V = \text{Real} \sum_{n=0}^{\infty} \sum_{m=0}^{n} R_{\oplus}^{n} (C_{n,m} - iS_{n,m}) V_{n,m}. \tag{3}$$

For all zonal terms ($m=0$) we have

$$S_{n,0} = 0, \quad C_{n,0} = -J_n,$$

where J_n is the traditional notation used for a zonal coefficient. The $V_{n,m}$ are solid spherical harmonics of degree $-(n+1)$ and order m. They have properties useful to the problem at hand.

Let

$$\begin{aligned} x &= r \cos\varphi \cos\lambda \\ y &= r \cos\varphi \sin\lambda \\ z &= r \sin\varphi \end{aligned} \tag{4}$$

be body-fixed coordinates of a point in the potential field.
Then

$$V_{n,m} = \frac{P_n^m(\sin\varphi)\,(x+iy)^m}{r^{n+m+1}\cos^m\varphi}. \tag{5}$$

Define

$$Z_{n,m} \equiv \frac{r^{n-m}P_n^m(\sin\varphi)}{\cos^m\varphi}. \tag{6}$$

Then

$$V_{n,m} = \frac{(x+iy)^m Z_{n,m}}{r^{2n+1}}. \tag{7}$$

The explicit expression for an associated Legendre polynomial in $\sin\varphi$ is

$$P_n^m(\sin\varphi) = \frac{\cos^m\varphi}{2^n n!}\sum_{k=0}^{\mathrm{I}((n-m)/2)} (-1)^k \binom{n}{k}\frac{(2n-2k)!}{(n-m-2k)!}(\sin\varphi)^{n-m-2k},$$

where I denotes 'integral part of'. It follows that the explicit expression for $Z_{n,m}$ is

$$Z_{n,m}(z, r^2) = \frac{1}{2^n n!}\sum_{k=0}^{\mathrm{I}((n-m)/2)} (-1)^k \binom{n}{k}\frac{(2n-2k)!}{(n-m-2k)!} z^{n-m-2k} r^{2k}. \tag{8}$$

This can also be written in the form

$$Z_{n,m}(z, \sigma^2) = \frac{(n+m)!}{2^n n!}\sum_{k=0}^{\mathrm{I}((n-m)/2)} \frac{(-1)^k}{2^{2k}}\binom{m+2k}{k}\binom{n}{m+2k} z^{n-m-2k}\sigma^{2k}, \tag{9}$$

where

$$\sigma^2 \equiv x^2 + y^2.$$

The $V_{n,m}$ can be computed from Equations (7) and (8) or (9). Explicit expressions for a few small n and m are given in Table I. When the numerical values of several $V_{n,m}$ are required, it is simpler and more efficient to use the recurrence relations below, which produce accurate results.

The advantages gained by introducing the $V_{n,m}$ arise from the simple recurrence relations derived below, and from the fact that any derivative of $V_{n,m}$ is a simple linear combination of other $V_{n,m}$.

1. Two Recurrence Relations

Now

$$H_{n,m} \equiv (x+iy)^m Z_{n,m} \tag{10}$$

TABLE I

$V_{n,m}$

$(\sigma^2 \equiv x^2 + y^2)$

r	$V_{0,0} = 1$
r^3	$V_{1,0} = z$
r^3	$V_{1,1} = x+iy$
$2r^5$	$V_{2,0} = 2z^2 - \sigma^2$
r^5	$V_{2,1} = 3z(x+iy)$
r^5	$V_{2,2} = 3(x+iy)^2$
$2r^7$	$V_{3,0} = z(2z^2 - 3\sigma^2)$
$2r^7$	$V_{3,1} = 3(4z^2 - \sigma^2)(x+iy)$
r^7	$V_{3,2} = 15z(x+iy)^2$
r^7	$V_{3,3} = 15(x+iy)^3$
$8r^9$	$V_{4,0} = 8z^4 - 24z^2\sigma^2 + 3\sigma^4$
$2r^9$	$V_{4,1} = 5z(4z^2 - 3\sigma^2)(x+iy)$
$2r^9$	$V_{4,2} = 15(6z^2 - \sigma^2)(x+iy)^2$
r^9	$V_{4,3} = 105z(x+iy)^3$
r^9	$V_{4,4} = 105(x+iy)^4$
$8r^{11}$	$V_{5,0} = z(8z^4 - 40z^2\sigma^2 + 15\sigma^4)$
$8r^{11}$	$V_{5,1} = 15(8z^4 - 12z^2\sigma^2 + \sigma^4)$
$2r^{11}$	$V_{5,2} = 105z(2z^2 - \sigma^2)(x+iy)^2$
$2r^{11}$	$V_{5,3} = 105(8z^2 - \sigma^2)(x+iy)^3$
r^{11}	$V_{5,4} = 945z(x+iy)^4$
r^{11}	$V_{5,5} = 945(x+iy)^5$
$16r^{13}$	$V_{6,0} = 16z^6 - 90z^4\sigma^2 + 90z^2\sigma^4 - 5\sigma^6$
$8r^{13}$	$V_{6,1} = 21z(8z^4 - 20z^2\sigma^2 + 5\sigma^4)(x+iy)$
$8r^{13}$	$V_{6,2} = 105(16z^4 - 16z^2\sigma^2 + \sigma^4)(x+iy)^2$
$2r^{13}$	$V_{6,3} = 315z(8z^2 - 3\sigma^2)(x+iy)^3$
$2r^{13}$	$V_{6,4} = 945(10z^2 - \sigma^2)(x+iy)^4$
r^{13}	$V_{6,5} = 10395z(x+iy)^5$
r^{13}	$V_{6,6} = 10395(x+iy)^6$

is a standard solid spherical harmonic of degree n and order m, and is a homogeneous polynomial in x, y and z. It can be written in the operator form

$$H_{n,m} = \frac{(-1)^n}{(n-m)!} r^{2n+1}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^m \left(\frac{\partial}{\partial z}\right)^{n-m}\left(\frac{1}{r}\right).$$

Thus

$$V_{n,m} = \frac{(-1)^n}{(n-m)!}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^m \left(\frac{\partial}{\partial z}\right)^{n-m}\left(\frac{1}{r}\right). \tag{11}$$

For $m=n$,

$$V_{n,n} = (-1)^n\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^n\left(\frac{1}{r}\right). \tag{12}$$

Successive differentiations of $1/r$ yield

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)^n\left(\frac{1}{r}\right)=(-1)^n 1\cdot 3\cdot 5\cdot 7\cdots(2n-1)\frac{(x+iy)^n}{r^{2n+1}}.$$

Hence

$$V_{n,n}=+\frac{(2n)!}{2^n n!}\frac{(x+iy)^n}{r^{2n+1}}. \tag{13}$$

This leads immediately to the first recurrence relation

$$V_{n,n}=(2n-1)\frac{(x+iy)}{r^2}V_{n-1,n-1} \tag{14}$$

which needs only the single starting value

$$V_{0,0}=1/r. \tag{15}$$

This formula was programmed using both single and double precision floating-point arithmetic, and was applied successively to build up tables of $V_{n,n}$ through $n=28$. Comparison of the two results shows that no more than one significant decimal was lost.

A well-known recurrence formula for the associated Legendre polynomials is

$$(n-m)P_n^m(\sin\varphi)=(2n-1)\sin\varphi\, P_{n-1}^m(\sin\varphi)-(n+m-1)P_{n-2}^m(\sin\varphi).$$

Using Equation (5) this gives the second recurrence relation for $V_{n,m}$

$$(n-m)V_{n,m}=(2n-1)\frac{z}{r^2}V_{n-1,m}-\frac{(n+m-1)}{r^2}V_{n-2,m}. \tag{16}$$

From Equation (11)

$$V_{m+1,m}=\frac{-\partial}{\partial z}V_{m,m},$$

and from Equation (13)

$$\frac{\partial V_{m,m}}{\partial z}=-(2m+1)\frac{z}{r^2}V_{m,m}.$$

Hence for $n=m+1$,

$$(n-m)V_{n,m}=(2n-1)\frac{z}{r^2}V_{n-1,m} \tag{17}$$

which is precisely Equation (16) with the second term on the right-hand side set to zero. Thus the starting values needed for the second recurrence formula Equation (16) are given by Equations (14) and (17). These formulae were also programmed using both single and double precision floating-point arithmetic, and were applied successively to build up tables of $V_{n,m}$ through $n=m=28$. Comparison of the two results shows that no more than one significant decimal was lost.

2. Derivatives of $V_{n,m}$

Successive differentiations of Equation (11) with respect to z give

$$\frac{\partial^\gamma}{\partial z^\gamma}V_{n,m}=(-1)^\gamma\frac{(n-m+\gamma)!}{(n-m)!}V_{n+\gamma,m}. \tag{18}$$

Applying the operator

$$P\equiv\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}$$

successively to Equation (11) gives

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)^s V_{n,m}=(-1)^s V_{n+s,m+s}, \tag{19}$$

which holds for all non-negative values of n, m and s. Applying the operator

$$M\equiv\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}$$

to Equation (11), and using Laplace's equation in the form

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)V_{n,m}=-\frac{\partial^2}{\partial z^2}V_{n,m},$$

gives

$$\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)V_{n,m}=\frac{(n-m+2)!}{(n-m)!}V_{n+1,m-1},\quad(m\neq 0) \tag{20}$$

which does not hold for $m=0$ since $V_{n,m}$ has not been defined for negative m. Successive applications of M yield

$$\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)^q V_{n,m}=\frac{(n-m+2q)!}{(n-m)!}V_{n+q,m-q},\quad(m\geqslant q) \tag{21}$$

provided $m\geqslant q$. If $m\leqslant q$, we can write

$$\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)^q V_{n,m}=\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)^{q-m}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)^m V_{n,m}$$
$$=\frac{(n+m)!}{(n-m)!}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)^{q-m}V_{n+m,0}. \tag{22}$$

Now for any differentiable function $F(x+iy)$

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)F(x+iy)=0.$$

It follows that

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)^p f(z,r^2)=2^p(x+iy)^p\frac{\partial^p f}{\partial (r^2)^p}\equiv g(z,r^2).$$

Similarly

$$\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)^p f(z,r^2)=2^p(x-iy)^p\frac{\partial^p f}{\partial (r^2)^p}=g^*(z,r^2),$$

where the asterisk denotes the complex conjugate. Formula (7) shows that $V_{n+m,\,0}$ is a function only of z and r. Also, from Equation (19)

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)^{q-m} V_{n+m,\,0}=(-1)^{q-m}\,V_{n+q,\,q-m}.$$

Hence

$$\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)^{q} V_{n,\,m}=(-1)^{q-m}\frac{(n+m)!}{(n-m)!}\,V^*_{n+q,\,q-m}\qquad (q\geqslant m).$$

If we *define* for non-negative m

$$V_{n,\,-m}\equiv(-1)^m\frac{(n-m)!}{(n+m)!}\,V^*_{n,\,m},\tag{23}$$

then

$$\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right)^{q} V_{n,\,m}=\frac{(n-m+2q)!}{(n-m)!}\,V_{n+q,\,m-q}\tag{24}$$

holds for all non-negative values of n, m and q, provided that the definition (23) is used whenever $q>m$.

From the definitions of P and M it follows that

$$2\frac{\partial}{\partial x}=P+M,\qquad 2\frac{\partial}{\partial y}=-i(P-M).$$

Hence

$$2^{\alpha+\beta}\frac{\partial^{\alpha+\beta}}{\partial x^\alpha\,\partial y^\beta}=(-i)^\beta(P+M)^\alpha(P-M)^\beta,$$

or

$$2^{\alpha+\beta}\frac{\partial^{\alpha+\beta}}{\partial x^\alpha\,\partial y^\beta}=(-i)^\beta\sum_{j=0}^{\alpha+\beta} C_{\alpha,\,\beta,\,j}P^{\alpha+\beta-j}M^j,\tag{25}$$

where

$$C_{\alpha,\,\beta,\,j}\equiv\sum_k(-1)^k\binom{\alpha}{j-k}\binom{\beta}{k},\tag{26}$$

in which

$$\mathrm{Max}(0;\,j-\alpha)\leqslant k\leqslant \mathrm{Min}(\beta;\,j).\tag{27}$$

Apply operator (25) to (18):

$$\frac{2^{\alpha+\beta}\,\partial^{\alpha+\beta+\gamma}}{\partial x^\alpha\,\partial y^\beta\,\partial z^\gamma}V_{n,\,m}=(-1)^{\beta+\gamma}i^\beta\frac{(n-m+\gamma)!}{(n-m)!}\sum_{j=0}^{\alpha+\beta}C_{\alpha,\,\beta,\,j}M^jP^{\alpha+\beta-j}V_{n+\gamma,\,m}.$$

From Equation (19),

$$P^{\alpha+\beta-j}V_{n+\gamma,\,m}=(-1)^{\alpha+\beta-j}V_{n+\alpha+\beta+\gamma-j,\,m+\alpha+\beta-j}.$$

From Equation (24),

$$M^jP^{\alpha+\beta-j}V_{n+\gamma,\,m}=(-1)^{\alpha+\beta-j}\frac{(n-m+\gamma+2j)!}{(n-m+\gamma)!}V_{n+\alpha+\beta+\gamma,\,m+\alpha+\beta-2j}.$$

Finally,

$$\frac{\partial^{\alpha+\beta+\gamma}V_{n,\,m}}{\partial x^\alpha\,\partial y^\beta\,\partial z^\gamma}=i^\beta\sum_{j=0}^{\alpha+\beta}\frac{(-1)^{\alpha+\gamma-j}}{2^{\alpha+\beta}}\times\frac{(n-m+\gamma+2j)!}{(n-m)!}C_{\alpha,\,\beta,\,j}V_{n+\alpha+\beta+\gamma,\,m+\alpha+\beta-2j},\tag{28}$$

where $C_{\alpha,\beta,j}$ is given by Equations (26) and (27).

3. Differential Equations

Let Q denote the rotation matrix that transforms body-fixed coordinates into space-fixed coordinates. Then the differential equations of motion for a particle in the Earth's field are

$$\begin{pmatrix}\dfrac{d^2x_s}{dt^2}\\[2mm]\dfrac{d^2y_s}{dt^2}\\[2mm]\dfrac{d^2z_s}{dt^2}\end{pmatrix}=Q\begin{pmatrix}\mu\dfrac{\partial V}{\partial x}\\[2mm]\mu\dfrac{\partial V}{\partial y}\\[2mm]\mu\dfrac{\partial V}{\partial z}\end{pmatrix}\tag{29}$$

where μ is the gravitation constant, and x_s, y_s, z_s are the coordinates of the particle referred to space-fixed axes.

From Equation (3)

$$\frac{\partial V}{\partial}=\mathrm{Real}\sum_{n=0}^{\infty}\sum_{m=0}^{n}R^n_{\oplus}(C_{n,\,m}-iS_{n,\,m})\frac{\partial V_{n,\,m}}{\partial}.\tag{30}$$

Let E denote an initial coordinate, velocity or other quantity not contained in Q.

Then

$$\begin{pmatrix} \dfrac{d^2}{dt^2}\dfrac{\partial x_s}{\partial E} \\ \dfrac{d^2}{dt^2}\dfrac{\partial y_s}{\partial E} \\ \dfrac{d^2}{dt^2}\dfrac{\partial z_s}{\partial E} \end{pmatrix} = Q\mu \begin{pmatrix} \left(\dfrac{\partial^2 V}{\partial E\,\partial x}\right) + \dfrac{\partial^2 V}{\partial x^2}\dfrac{\partial x}{\partial E} + \dfrac{\partial^2 V}{\partial x\,\partial y}\dfrac{\partial y}{\partial E} + \dfrac{\partial^2 V}{\partial x\,\partial z}\dfrac{\partial z}{\partial E} \\ \left(\dfrac{\partial^2 V}{\partial E\,\partial y}\right) + \dfrac{\partial^2 V}{\partial x\,\partial y}\dfrac{\partial x}{\partial E} + \dfrac{\partial^2 V}{\partial y^2}\dfrac{\partial y}{\partial E} + \dfrac{\partial^2 V}{\partial y\,\partial z}\dfrac{\partial z}{\partial E} \\ \left(\dfrac{\partial^2 V}{\partial E\,\partial z}\right) + \dfrac{\partial^2 V}{\partial x\,\partial z}\dfrac{\partial x}{\partial E} + \dfrac{\partial^2 V}{\partial y\,\partial z}\dfrac{\partial y}{\partial E} + \dfrac{\partial^2 V}{\partial z^2}\dfrac{\partial z}{\partial E} \end{pmatrix} \tag{31}$$

where the parentheses denote derivatives with respect to E only as it appears *explicitly in* V. From (30)

$$\frac{\partial^2 V}{\partial^2} = \text{Real} \sum_{n=0}^{\infty} \sum_{m=0}^{n} R_{\oplus}^{n} (C_{n,m} - iS_{n,m}) \frac{\partial^2 V_{n,m}}{\partial^2}. \tag{32}$$

Equations (29) and (31) can be used numerically to integrate the orbital motion and the partial derivatives of x_s, y_s and z_s with respect to any number of parameters. Explicit expressions for the derivatives of $V_{n,m}$ needed in (30) and (32) can easily be written out from (28).

4. Summary of Formulae

The above formulae are applied as follows:

(1) The zonal terms through the largest n required are first computed from Equations (15), (16) and (17). There are no imaginary parts to these terms.

(2) The sectorial term $V_{1,1}$ is computed from Equations (14) and (15).

(3) All other terms required for $m=1$ are computed from Equations (16) and (17).

(4) Then $V_{2,2}$ is computed from Equation (14), and the $V_{n,2}$ are computed from Equations (16) and (17).

(5) Item 4 is repeated for $m=3,4,\ldots$ through the highest m required.

It is to be noted that except for the zonal terms each $V_{n,m}$ has distinct real and imaginary parts. Note also that one second derivative of $V_{n,m}$ can be computed from Laplace's equation, e.g.,

$$\frac{\partial^2 V_{n,m}}{\partial y^2} = -\frac{\partial^2 V_{n,m}}{\partial z^2} - \frac{\partial^2 V_{n,m}}{\partial x^2}. \tag{33}$$

Explicit expressions for the first and second partial derivatives of $V_{n,m}$ follow as an aid to analysis and checking. They should not be used for programming, which should always be based on formulae (14), (15) and (16).

$$\frac{\partial V_{n,m}}{\partial x} = -\frac{V_{n+1,m+1}}{2} + \frac{(n-m+2)!}{2(n-m)!} V_{n+1,m-1} \qquad m > 0$$

$$= -\frac{V_{n+1,1}}{2} - \frac{V_{n+1,1}^{*}}{2} \qquad m = 0$$

$$\frac{\partial V_{n,m}}{\partial y} = +\frac{iV_{n+1,m+1}}{2} + \frac{i(n-m+2)!}{2(n-m)!} V_{n+1,m-1} \qquad m > 0$$

$$= +\frac{iV_{n+1,1}}{2} - \frac{iV_{n+1,1}^{*}}{2} \qquad m = 0$$

$$\frac{\partial V_{n,m}}{\partial z} = -\frac{(n-m+1)!}{(n-m)!} V_{n+1,m} \qquad m \geqslant 0$$

$$\frac{\partial^2 V_{n,m}}{\partial x^2} = \frac{V_{n+2,m+2}}{4} - \frac{(n-m+2)!}{2(n-m)!} V_{n+2,m} + \frac{(n-m+4)!}{4(n-m)!} V_{n+2,m-2} \qquad m > 1$$

$$= \frac{V_{n+2,3}}{4} - \frac{(n+1)!}{2(n-1)!} V_{n+2,1} - \frac{(n+1)!}{4(n-1)!} V_{n+2,1}^{*} \qquad m = 1$$

$$= \frac{V_{n+2,2}}{4} - \frac{(n+2)!}{2n!} V_{n+2,0} + \frac{V_{n+2,2}^{*}}{4} \qquad m = 0$$

$$\frac{\partial^2 V_{n,m}}{\partial x\,\partial y} = -\frac{iV_{n+2,m+2}}{4} + \frac{i(n-m+4)!}{4(n-m)!} V_{n+2,m-2} \qquad m > 1$$

$$= -\frac{iV_{n+2,3}}{4} - \frac{i(n+1)!}{4(n-1)!} V_{n+2,1}^{*} \qquad m = 1$$

$$= -\frac{iV_{n+2,2}}{4} + \frac{iV_{n+2,2}^{*}}{4} \qquad m = 0$$

$$\frac{\partial^2 V_{n,m}}{\partial y^2} = -\frac{V_{n+2,m+2}}{4} - \frac{(n-m+2)!}{2(n-m)!} V_{n+2,m} - \frac{(n-m+4)!}{4(n-m)!} V_{n+2,m-2} \qquad m > 1$$

$$= -\frac{V_{n+2,3}}{4} - \frac{(n+1)!}{2(n-1)!} V_{n+2,1} + \frac{(n+1)!}{4(n-1)!} V_{n+2,1}^{*} \qquad m = 1$$

$$= -\frac{V_{n+2,2}}{4} - \frac{(n+2)!}{2n!} V_{n+2,0} - \frac{V_{n+2,2}^{*}}{4} \qquad m = 0$$

$$\frac{\partial^2 V_{n,m}}{\partial x\,\partial z} = \frac{(n-m+1)}{2} V_{n+2,m+1} - \frac{(n-m+3)!}{2(n-m)!} V_{n+2,m-1} \qquad m > 0$$

$$= \frac{(n+1)}{2} V_{n+2,1} + \frac{(n+1)}{2} V_{n+2,1}^{*} \qquad m = 0$$

$$\frac{\partial^2 V_{n,m}}{\partial y\,\partial z} = -\frac{i(n-m+1)}{2} V_{n+2,m+1} - \frac{i(n-m+3)!}{2(n-m)!} V_{n+2,m-1} \qquad m > 0$$

$$= -\frac{i(n+1)}{2} V_{n+2,1} + \frac{i(n+1)}{2} V_{n+2,1}^{*} \qquad m = 0$$

$$\frac{\partial^2 V_{n,m}}{\partial z^2} = +\frac{(n-m+2)!}{(n-m)!} V_{n+2,m}. \qquad m \geqslant 0$$

5. Possible Programming for a Large Calculator

The practical value of the above method for computing the accelerations and partial derivatives depends strongly on how well it lends itself to actual programming. This question was studied at some length, and several possible procedures were considered. On the one hand the greatest speed is attained by first computing all of the required $V_{n,m}$, after which the individual accelerations and derivatives are obtained and summed. There are several efficient ways of doing this. However, the relatively large space required to store all of the $V_{n,m}$ will ordinarily present a serious problem. On the other hand the least amount of storage space is expended by computing and recomputing each $V_{n,m}$ as it is needed. This latter procedure is too inefficient for general use. Thus a compromise somewhere between the two extremes is clearly necessary.

A procedure that is simple, easy to program, quite efficient and with no excessive space requirement has been written and tested. The basic idea is the use of a storage containing all of the $V_{n,m}$ for five adjacent values of m. As soon as the terms in the accelerations and partial derivatives have been computed and summed for the middle m of the five, the $V_{n,m}$ are moved to the locations they must have in the storage for $m+1$, and then the values of $V_{n;m+1}$ are computed and stored. The only complication arises from the slightly different formulae that must be used for $m=0$ and $m=1$.

6. Remarks

Recurrence formulae exist for each derivative separately, but their forms are not in general attractive for computation. It appears best to express each derivative in terms of the $V_{n,m}$, as is done above.

While this method was being tested my attention was called to a similar method proposed by DeWitt (1962).

Acknowledgements

It is a pleasure to acknowledge my indebtedness to J. B. VanVechten for several valuable discussions before and during the course of the work; to Rodney L. Jacobs for encouragement and valuable suggestions; and to Dr. V. P. Peline for the opportunity to undertake this study.

Reference

DeWitt, R. N.: 1962, Derivatives of Expressions Describing the Gravitational Field of the Earth, U.S. Naval Weapons Laboratory, Technical Memorandum No. K-35/62.

A SEPARABLE POTENTIAL IN TRIAXIALLY ELLIPSOIDAL COORDINATES SATISFYING THE LAPLACE EQUATION*

STEPHEN J. MADDEN, JR.

MIT Measurement Systems Laboratory, Cambridge, Mass., U.S.A.

(Received 8 November, 1969)

Abstract. This paper derives the most general potential function which allows separation of the Hamilton-Jacobi equation in orthogonal coordinates and which satisfies the Laplace equation. The resulting potential is then specialized to the case of interest for near-Earth satellites, where the proper behavior of the potential at infinity is obtained and singularities in the region of interest are eliminated. The Vinti potential is found as a special case.

1. Introduction

Various developments in the theory of artificial Earth satellites have made use of potential functions which allow separation of the Hamilton-Jacobi equation (Vinti, 1959–69; Garfinkel, 1958–64; Aksnes, 1965, 1967; Sterne, 1958; Izsak, 1960). The potentials contain free parameters which are chosen to make the separable potentials approximate closely the Earth's potential. Of special interest are those potentials which not only allow separation, but which also satisfy the Laplace equation so that they truly qualify as gravitational potentials (Vinti, 1959–69). A recent study (Cook, 1966) presented such potentials for various coordinate systems. Cook, however, did not give the most general orthogonal case, the triaxially ellipsoidal. He assumed that a certain condition which is sufficient for the existence of such a potential is also necessary. It was indeed a counter-example to this necessity that led to the writing of this paper.

Here we derive this general potential and specialize it for the case of satellite motion.

2. The Ellipsoidal Coordinates

Let there be a fixed Cartesian reference system with which each point, P, in space is assigned the coordinates (x, y, z). With P there are also associated ellipsoidal coordinates (λ, μ, ν), the three real solutions of the cubic equation in t resulting from the quadric surface formula

$$\frac{x^2}{a^2+t}+\frac{y^2}{b^2+t}+\frac{z^2}{c^2+t}=1, \tag{1}$$

(Kellogg, 1953). Without loss of generality we assume $a>b>c\geqslant 0$, and the three

* This paper was prepared under the sponsorship of the Electronics Research Center of the National Aeronautics and Space Administration through NASA Grant NGR-22-009-262.