Numerical propagation of errors of Earth-orbiting spacecraft

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Motivation

- Around 7000 satellites (active and inactive) orbit the Earth.
- Various unintentional collisions have occurred in the past.
- Point Earth model is not enough. We need a more accurate model.

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Goal:

- Develop a geopotential model of the Earth.
- Propagate the position of a satellite adding different perturbations.
- Estimate the error of the trajectory.

Equation for V

Let $\Omega \subseteq \mathbb{R}^3$ be the region that occupies the Earth.

$$\mathbf{g} = -\int_{\Omega} G \frac{\mathbf{r}-\mathbf{s}}{\|\mathbf{r}-\mathbf{s}\|^3} \rho(\mathbf{s}) \mathrm{d}^3\mathbf{s} = \boldsymbol{\nabla} V$$

$$V = \int_{\Omega} G \frac{\rho(\mathbf{s})}{\|\mathbf{r}-\mathbf{s}\|} \mathrm{d}^3\mathbf{s}$$

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Theorem

V satisfies the following exterior boundary problem:

$$\begin{cases} \Delta V = 0 & \textit{in } \Omega^c \\ V = f & \textit{on } \partial\Omega \\ \lim\limits_{\|\mathbf{r}\| \to \infty} V = 0 & \end{cases}$$

where $f : \partial\Omega \to \mathbb{R}$ is the gravitational potential on the surface of the Earth.

Separation of variables: $V = R(r)\Theta(\theta)\Phi(\phi)$

$$\begin{cases} \dfrac{\left(r^2 R'\right)'}{R} = n(n+1) \\ \dfrac{1}{\Theta}\Theta'' = -m^2 \\ \dfrac{\sin\phi}{\Phi}\left(\sin\phi\Phi'\right)' + n(n+1)\left(\sin\phi\right)^2 = m^2 \end{cases} \qquad n, m \in \mathbb{N} \cup \{0\}, m \leq n$$

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Imposing boundary conditions:

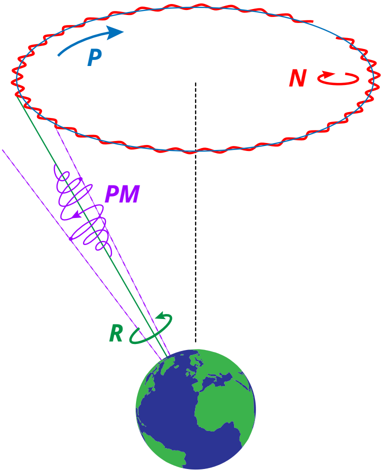
$$V = \frac{GM_\oplus}{R_\oplus} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{R_\oplus}{r}\right)^{n+1} (\bar{C}_{n,m} Y^{\mathrm{c}}_{n,m}(\theta, \phi) + \bar{S}_{n,m} Y^{\mathrm{s}}_{n,m}(\theta, \phi))$$

where

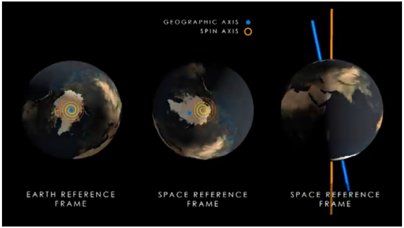
$$Y^{\mathrm{c}}_{n,m}(\theta, \phi) = N_{n,m} P_{n,m}(\cos\theta)\cos(m\phi)$$
$$Y^{\mathrm{s}}_{n,m}(\theta, \phi) = N_{n,m} P_{n,m}(\cos\theta)\sin(m\phi)$$

are the **spherical harmonics**.

Deviations of the Earth’s rotation axis



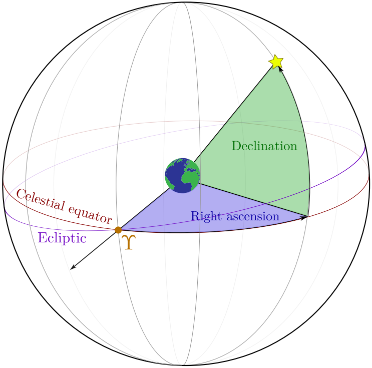
- Precession
- Nutation
- Rotation
- Polar motion



Source: NASA Earth Orientation Animations

Celestial sphere

- Abstract sphere of infinite radius centered at the Earth.
- All celestial objects are projected naturally onto it.



Mean axis of rotation: Axis of rotation when the nutation perturbation are averaged out.

Mean equator: plane perpendicular to the mean axis of rotation.

Mean vernal equinox ($\overline{\Upsilon}$): The point of intersection between the mean equator with the ecliptic where the Sun crosses the celestial equator from south to north.

Inertial and non-inertial reference frames

J2000 date: January 1, 2000 at 12:00 TT.

Quasi-inertial:

- x-axis: pointing towards the $\overline{\Upsilon}$ of the J2000 date
- z-axis: perpendicular to the mean equator of the J2000 date

Non-inertial (Earth fixed):

- z-axis: pointing towards the IRP (*International Reference Pole*)
- x-axis: pointing towards the zero meridian and in the plane perpendicular to the z-axis.

In both systems y-axis is chosen in order to complete a right-handed system.

Other perturbations

$\mathbf{r}$ = satellite position with respect to Earth's center of mass

- Third body perturbations (Moon and Sun):

$$\frac{GM}{\|\mathbf{s}-\mathbf{r}\|^2}(\mathbf{s}-\mathbf{r}) - \frac{GM}{\|\mathbf{s}\|^2}\mathbf{s}$$

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- Atmospheric drag:

$$-\frac{1}{2}C_{\mathrm{D}}\frac{A}{m}\rho v_{\mathrm{rel}}\mathbf{v}_{\mathrm{rel}}$$

where $\mathbf{v}_{\mathrm{rel}} = \dot{\mathbf{r}} - \boldsymbol{\omega}_{\oplus} \times \mathbf{r}$.

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- Solar radiation pressure:

$$-\nu P_{\odot} C_{\mathrm{R}} \frac{A_{\odot}}{m} \frac{\mathbf{s}_{\odot}-\mathbf{r}}{\|\mathbf{s}_{\odot}-\mathbf{r}\|}$$

Differential system

Using the Runge-Kutta-Fehlberg method of order 7(8) we will integrate the differential system:

$$\begin{cases} \dot{\mathbf{r}} = \mathbf{v} \\ \dot{\mathbf{v}} = \mathbf{a}_{\mathrm{GP}} + \delta_{\mathrm{D}}\mathbf{a}_{\mathrm{D}} + \delta_{\mathrm{R}}\mathbf{a}_{\mathrm{R}} + \delta_{\mathrm{sun}}\mathbf{a}_{\mathrm{sun}} + \delta_{\mathrm{moon}}\mathbf{a}_{\mathrm{moon}} \end{cases}$$

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\end{cases}
$$

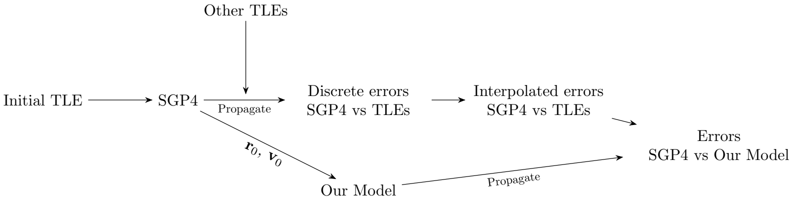
Initial conditions from **TLEs** (Two Line Elements sets).

Line		Satellite number	Class		International designator: Year	International designator: Launch	International designator: Piece		Epoch (UTC): Year	Epoch (UTC): Day of the year (as fraction)		$\dot{n}/2$ [rev/day^2]		$\ddot{n}/6$ [rev/day^3]		B^* (drag term)		Model		TLE number	Check
1	2	3–7	8	9	10–11	12–14	15–17	18	19–20	21–32	33	34–43	44	45–52	53	54–61	62	63	64	65–68	69
												S (34)		S. (45)		S. (54), S (60), E (61)					
1		55124	U		98	067	UR		23	086.16778994		.01540407		14102-2		26634-2		0		999	1

Line		Satellite number		Inclination i [deg]		Right ascension Ω [deg]		Eccentricity e		Argument periapsis ω [deg]		Mean anomaly M [deg]		Mean motion n [rev/day]	Number of revolutions	Check
1	2	3–7	8	9–16	17	18–25	26	27–33	34	35–42	43	44–51	52	53–63	64–68	69
2		55124		51.6237		5.0017		0010757		188.3869		171.6963		16.01878199	13792	2

Results

Scheme of our simulation:



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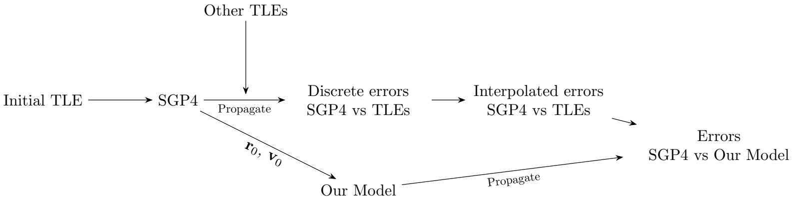
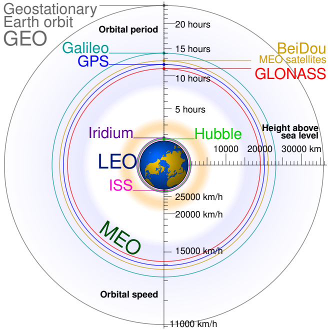


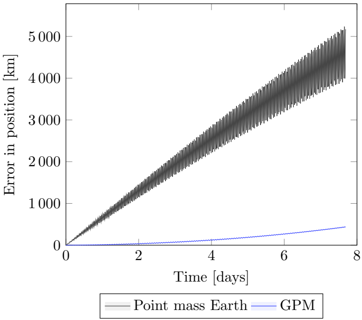


Zones that we will explore:

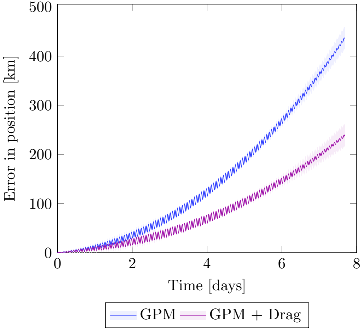
- **L**ow **E**arth **O**rbit satellites
- **M**edium **E**arth **O**rbit satellites
- **G**eostationary **E**arth **O**rbit satellites

Results - LEO

- ISS makes ~ 16 orbits per day.
- LEO satellites interact with the atmosphere.
- The atmospheric drag is difficult to predict.

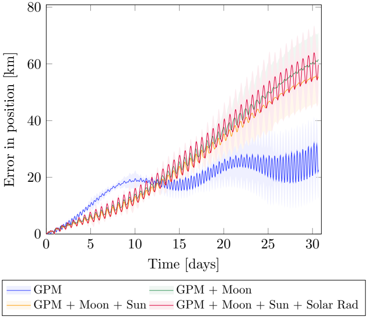


ISS

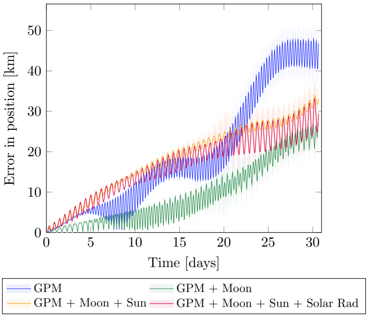


ISS

Results - MEO



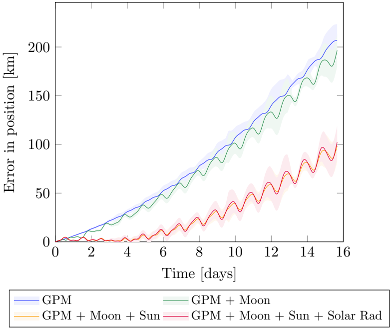
Sirius-3



Galileo-20

- GPM is very oscillatory.
- Sun and Moon reduce the oscillations.
- Solar radiation increases the oscillations.

Results - GEO



TDRS-3

- Adding the Sun and the Moon the errors are reduced.
- Solar radiation again increases the oscillations.
- Maneuver at around the 13th day.

Conclusions

- The point mass model is not enough to predict the orbit of a satellite.
- Adding both together the Sun and the Moon the variability of the errors is reduced.
- Solar radiation pressure increases the oscillations.

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Improvements:

- Atmospheric drag and solar radiation bad modelled.
- Study the influence of the inclination and eccentricity on the errors.