1   Preliminaries

In this section we will review some basic concepts of linear algebra and vector calculus that will be used throughout the document.

1.1   Properties of cross and dot products

Proposition 1. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Then:

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} \tag{1}$$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w} \tag{2}$$

Proof. Let $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$. Then:

$$
\begin{aligned}
((\mathbf{u} \times \mathbf{v}) \times \mathbf{w})_1 &= (u_3 v_1 - u_1 v_3)w_3 - (u_1 v_2 - u_2 v_1)w_2 \\
&= (u_2 w_2 + u_3 w_3)v_1 - (v_2 w_2 + v_3 w_3)u_1 \\
&= (u_1 w_1 + u_2 w_2 + u_3 w_3)v_1 - (v_1 w_1 + v_2 w_2 + v_3 w_3)u_1 \\
&= ((\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u})_1
\end{aligned}
$$

The other components are treated similarly. The second equality follows in a similar way. $\square$

Proposition 2. Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$. Then:

1. $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} \cdot \mathbf{w} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u} \cdot \mathbf{w}$

1.2   Conics in a nutshell

1.2.1   General conics

Definition 3. A conic is the curve obtained as the intersection of a plane with the surface of a double cone (a cone with two *nappes*).

In Fig. 1, we show the 3 types of conics: the ellipse, the parabola, and the hyperbola, which differ on their eccentricity, as we will see later on. Note that the circle is a special case of the ellipse.

The following proposition gives a characterization of the conics.

Proposition 4. A conic is the set of all points P such that the distance from P to a fixed point F is a multiple of the distance from P to a fixed line D. Mathematically, this is expressed as:

$$d(P, F) = e\,d(P, D) \tag{3}$$

where d is the Euclidean distance. The point F is called the *focus*; the line D, *directrix*, and the constant of proportionality e, *eccentricity*.

Note that using the polar coordinates (r, ν) centered at F (as in Fig. 2), we can rewrite Eq. (3) as:

$$r = e(\ell - r \cos \nu) \implies r = \frac{e\ell}{1 + e \cos \nu} =: \frac{p}{1 + e \cos \nu} \tag{4}$$

where we have defined $p := e\ell$.

Definition 5. Le C be a conic and e be its eccentricity. We say that C is

- an *ellipse* if $0 \le e < 1$,

- a *parabola* if $e = 1$, and

- a *hyperbola* if $e > 1$.

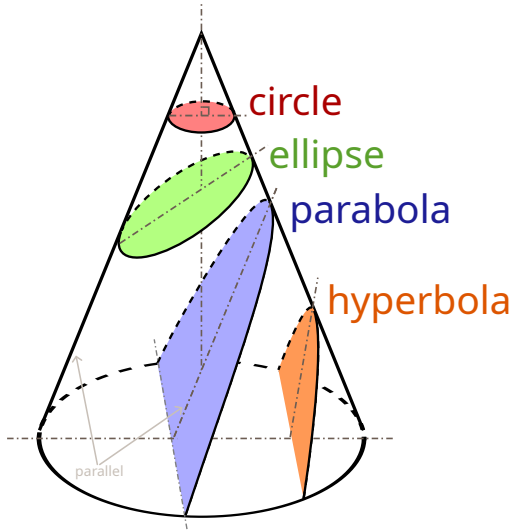If $e = 0$, the conic is called *circle*.

Figure 1: The black boundaries of the colored regions are conic sections. The other half of the hyperbola, which is not shown, is in the other nappe of the double cone. (based on [Mat12])
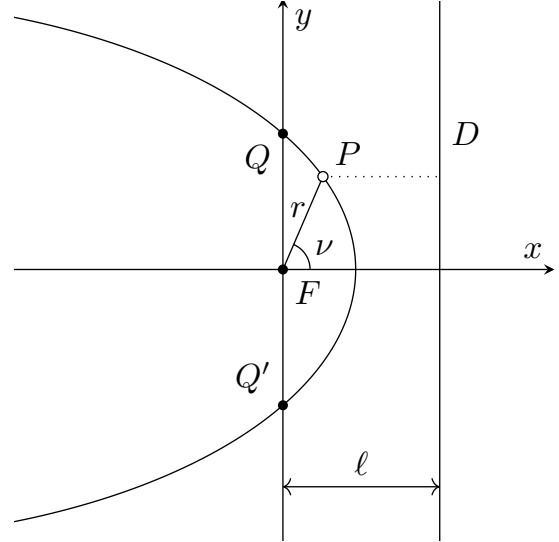

Figure 2: Reference frame centered at the focus of the conic and whose axes are such that the y-axis is parallel to the directrix and the x-axis is perpendicular to the directrix. The directions of the axes are chosen arbitrarily, subject to the constraint that a right-handed system (x, y) is obtained.

1.2.2 Ellipse

From now on we will focus on the study of the ellipse. From Eq. (4), since $e < 1$, it follows $r(\nu)$ is continuous. Therefore, the ellipse is bounded and closed conic section (and it is the only conic section satisfying these two properties).

Let's now study the extrema of $r(\nu)$. An easy check shows that the minimum is attained at $\nu = 0$ and the maximum at $\nu = \pi$ and these values are given by:

$$r_{\min} = \frac{p}{1+e} \qquad \text{and} \qquad r_{\max} = \frac{p}{1-e} \tag{5}$$

When considering orbits of bodies these points are called *periapsis* and *apoapsis*, respectively[1]. The line connecting both points is called *line of apsides*. Let's seek now the extrema of $x = r \cos \nu$ and $y = r \sin \nu$. Differentiating with respect to ν yields:

$$x' = -\frac{p \sin \nu}{(1 + e \cos \nu)^2} \qquad y' = \frac{p(e + \cos \nu)}{(1 + e \cos \nu)^2} \tag{6}$$

On the one hand, first expression vanishes at $\nu = 0, \pi$. Therefore, the extrema of x coincide with the periapsis and apoapsis points and at these points the y coordinate is equal to 0. This means that the line of apsides passes through the focus of the ellipse. On the other hand, y' vanishes at $\cos \nu = -e$. That is, at $\nu = \arccos(-e)$ and $\nu = 2\pi - \arccos(-e)$. Therefore, using that $\sin(\arccos x) = \sqrt{1 - x^2}$, the values of y at these extrema are:

$$y_{\min} = \frac{p}{1-e^2} \sin(\arccos(-e)) = \frac{p}{\sqrt{1-e^2}} \qquad y_{\max} = \frac{p}{1-e^2} \sin(2\pi - \arccos(-e)) = -\frac{p}{\sqrt{1-e^2}} \tag{7}$$

Note that the x coordinate at these two points is the same: $-\frac{pe}{1-e^2}$.

Definition 6. Consider the reference frame Fig. 3 centered at one focus. We define the *semi-major axis* a as half the segment that connects the two extrema of the x coordinate. The *semi-minor axis* b is defined

---

[1]Other names are used in the literature when the the central body and the orbiter are particular ones. For example for the system Sun-Earth, the words *perihelion* and *aphelion* are used, whereas for the system Earth-Moon, the words *perigee* and *apogee* are used instead.
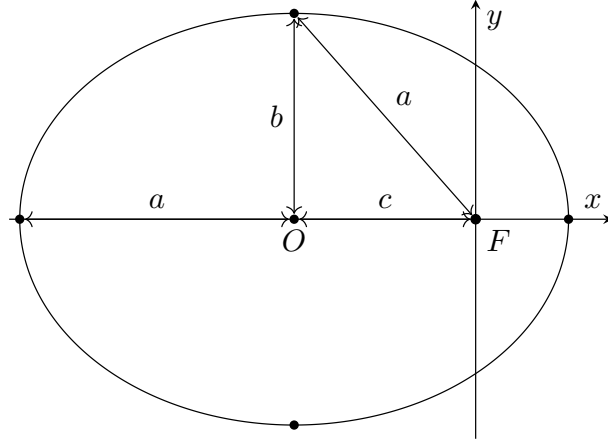
Figure 3: Ellipse

as half the segment that conects the two extrema of the y coordinate. The length of those segments are also denoted by a and b, respectively. Thus, these are given by the following expressions:

$$a := \frac{r_{\max} + r_{\min}}{2} = \frac{p}{1 - e^2} \qquad b := \frac{y_{\max} + y_{\min}}{2} = \frac{p}{\sqrt{1 - e^2}} \tag{8}$$

From here note that we can express b in terms of a and e as:

$$b = a\sqrt{1 - e^2} \tag{9}$$

Definition 7. We define the center O of the ellipse as the intersection of the semi-major axis and semi-minor axis.

Let's calculate the distance from the focus F to one of the extrema of the y coordinate.

$$d\left(F, \left(-\frac{pe}{1 - e^2}, \pm\frac{p}{\sqrt{1 - e^2}}\right)\right) = \frac{p}{\sqrt{1 - e^2}}\sqrt{\frac{e^2}{1 - e^2} + 1} = \frac{p}{1 - e^2} = a \tag{10}$$

Hence, the value of c in Fig. 3 can be simplified to:

$$c^2 = a^2 - b^2 = a^2 - a^2(1 - e^2) = a^2 e^2 \implies c = ae \tag{11}$$

Proposition 8. The area enclosed in a ellipse of semi-major axis a and semi-minor axis b is πab.

Proof. Consider the ellipse E centered at the origin and oriented as in Fig. 3. From Fig. 3 one can see that it can be parametrized by $(x, y) = (\frac{p \cos \nu}{1 - e^2} + ae, \frac{p \sin \nu}{1 + e \cos \nu})$ with $\nu \in [0, 2\pi)$. An easy check shows that this parametrization satisfies:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{12}$$

Hence, the area enclosed in the ellipse is can be parametrized by $(x, y) = (ar\cos\nu, br\sin\nu)$, with $r \in [0, 1]$ and $\nu \in [0, 2\pi]$. The Jacobian of this transformation is abr. Therefore, from the change of variable theorem we have that:

$$\text{Area}(E) = \iint\limits_{E} \mathrm{d}x\,\mathrm{d}y = \int\limits_{0}^{2\pi} \int\limits_{0}^{1} rab\,\mathrm{d}r\,\mathrm{d}\nu = \pi ab \tag{13}$$

$\square$

1.3 Spherical harmonics

1.3.1 Legendre polynomials, regularity and orthonormality

In this section we aim to introduce a class of functions that will appear later on in the general solution of the Laplace equation (see Section 1.3.2). To do this, we need first to introduce the Legendre polynomials.

There are several ways to define them, but the most convenient one for our purposes is from the following differential equation. Consider the following second-order differential equation called *Legendre differential equation*:

$$(1 - x^2)y'' - 2xy' + \lambda y = 0 \tag{14}$$

for $\lambda \in \mathbb{R}$. This equation can be rewritten as:

$$\left((1 - x^2)y'\right)' + \lambda y = 0 \tag{15}$$

Seeking for analytic solutions of this equation using the power series method [Mez], i.e. looking for solutions of the form $y(x) = \sum_{j=0}^{\infty} a_j x^j$, we see that:

$$0 = (1 - x^2)\sum_{j=0}^{\infty} a_{j+2}(j+1)(j+2)x^j - 2x\sum_{j=0}^{\infty} a_{j+1}(j+1)x^j + \lambda\sum_{j=0}^{\infty} a_j x^j = \sum_{j=0}^{\infty} a_{j+2}(j+1)(j+2)x^j -$$

$$- \sum_{j=0}^{\infty} a_j(j-1)jx^j - \sum_{j=0}^{\infty} 2a_j j x^j + \sum_{j=0}^{\infty} \lambda a_j x^j = \sum_{j=0}^{\infty} [a_{j+2}(j+1)(j+2) - a_j(j(j+1) - \lambda)]x^j \tag{16}$$

Equating the general term of the series equal to 0 we obtain this recursion:

$$a_{j+2} = \frac{j(j+1) - \lambda}{(j+1)(j+2)} a_j \quad j = 0, 1, 2, \ldots \tag{17}$$

From here we can obtain two independent solutions by setting the initial conditions a_0 and a_1 of the iteration. For example, setting $a_1 = 0$ we obtain a series that has only even powers of x. On the other hand, setting $a_0 = 0$ we obtain a series that has only odd powers of x. These two series converge on the interval $(-1, 1)$ by the ratio test (by looking at Eq. (17)) and can be expressed compactly as [Mez]:

$$y_{\mathrm{e}}(x) = a_0 \sum_{j=0}^{\infty} \left[\prod_{k=0}^{j-1}(2k(2k+1) - \lambda)\right] \frac{x^{2j}}{(2j)!} \quad y_{\mathrm{o}}(x) = a_1 \sum_{j=0}^{\infty} \left[\prod_{k=0}^{j-1}((2k+1)(2k+2) - \lambda)\right] \frac{x^{2j+1}}{(2j+1)!} \tag{18}$$

Here the empty product (that is, for instance when k ranges from 0 to -1) is defined to be 1. However for each $\lambda \in \mathbb{R}$ either one of these series or both diverge at $x = \pm 1$, as they behave as the harmonic series in a neighbourhood of $x = \pm 1$. We are interested, though, in the solutions that remain bounded on the whole interval $[-1, 1]$. Looking at the expressions of Eq. (18) one can check that the only possibility to make the series converge in $[-1, 1]$ is when $\lambda = n(n+1)$, $n \in \mathbb{N} \cup \{0\}$. In this case, for each $n \in \mathbb{N} \cup \{0\}$ exactly one of the series is in fact a polynomial of degree n. If, furthermore, we choose a_0 or a_1 be such that the polynomial evaluates to 1 at $x = 1$, these polynomials are called *Legendre polynomials* and they are denoted by $P_n(x)$. The other (divergent) series is usually denoted in the literature by $Q_n(x)$ (check [RHB99; Mez]). And so the general solution of Eq. (15) for $\lambda = n(n+1)$ can be expressed as a linear combination of P_n and Q_n, because the space of solutions form a vector space of dimension 2.
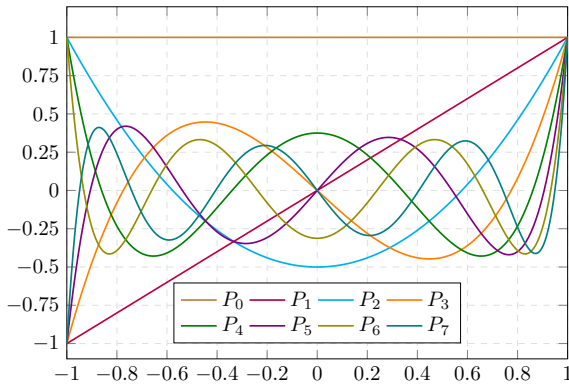

Figure 4: Graphic representation of the first eight Legendre polynomials.

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2}(3x^2 - 1)$
3	$\frac{1}{2}(5x^3 - 3x)$
4	$\frac{1}{8}(35x^4 - 30x^2 + 3)$
5	$\frac{1}{8}(63x^5 - 70x^3 + 15x)$
6	$\frac{1}{16}(231x^6 - 315x^4 + 105x^2 - 5)$
7	$\frac{1}{16}(429x^7 - 693x^5 + 315x^3 - 35x)$

Table 1: First eight Legendre polynomials

The following proposition will be of our interest in the next section [RHB99].

Proposition 9. Let $y(x)$ be a solution to the Legendre differential equation. Then, $\forall m \in \mathbb{N} \cup \{0\}$ the function

$$w_m(x) = (1 - x^2)^{m/2} \frac{\mathrm{d}^m y(x)}{\mathrm{d}x^m} \tag{19}$$

solves the *general Legendre differential equation*:

$$(1 - x^2)y'' - 2xy' + \left(\lambda - \frac{m^2}{1 - x^2}\right) y = 0 \tag{20}$$

In particular if $\lambda = n(n+1)$ for $n \in \mathbb{N} \cup \{0\}$, then $w_m(x)$ is denoted as

$$P_{n,m}(x) := (1 - x^2)^{m/2} \frac{\mathrm{d}^m P_n}{\mathrm{d}x^m} \tag{21}$$

and it is called the *associated Legendre polynomial* of degree n and order m.

Note that although we opted to call the functions $P_{n,m}$ as *polynomials*, they are only *true* polynomials when m is even. But we have opt to call them in that manner as it is the common practice in the literature (see [Wei; RHB99; Mez]).

Moreover, from the definition of $P_{n,m}$, we can see $P_{n,0} = P_n$ and that $P_{n,m} = 0$ if $m > n$. So we can restrict the domain of m to the set $\{0, 1, \ldots, n\}$.

n	$P_{n,1}(x)$	n	$P_{n,2}(x)$
1	$\sqrt{1 - x^2}$	2	$3(1 - x^2)$
2	$3x\sqrt{1 - x^2}$	3	$15x(1 - x^2)$
3	$\frac{3}{2}(5x^2 - 1)\sqrt{1 - x^2}$	4	$\frac{15}{2}(7x^2 - 1)(1 - x^2)$
4	$\frac{5}{2}x(7x^2 - 3)\sqrt{1 - x^2}$	5	$\frac{105}{2}x(3x^2 - 1)(1 - x^2)$
5	$\frac{15}{8}(21x^4 - 14x^2 + 1)\sqrt{1 - x^2}$	6	$\frac{105}{8}(33x^4 - 18x^2 + 1)(1 - x^2)$

Table 2: First associated Legendre polynomials for $m = 1$ and $m = 2$.
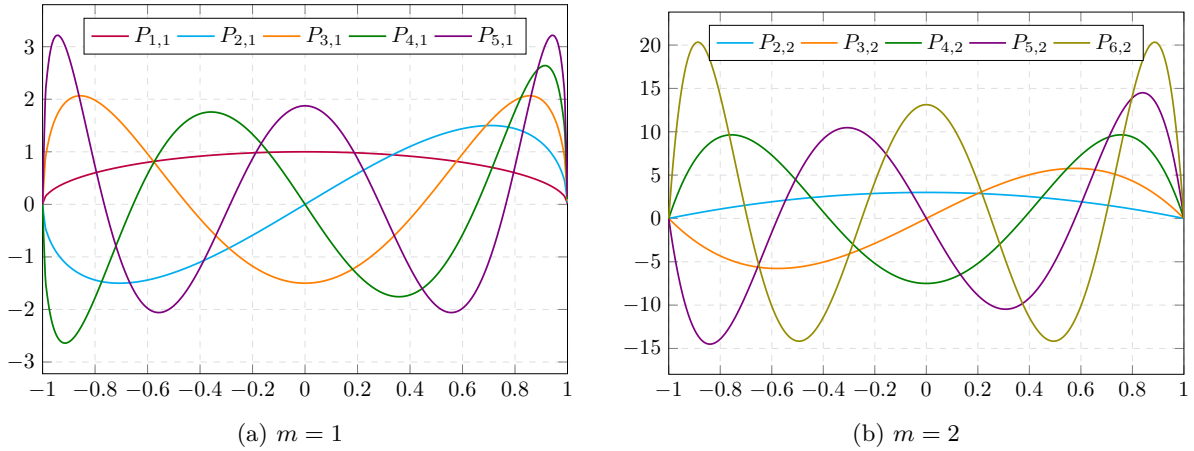

(a) $m = 1$ (b) $m = 2$

Figure 5: Graphic representation of the first associated Legendre polynomials for $m = 1$ and $m = 2$.

Definition 10. Let $n \in \mathbb{N} \cup \{0\}$ and $m \in \{0, 1, \ldots, n\}$. We define the *real spherical harmonics* $Y_{n,m}^{\mathrm{c}}$ and $Y_{n,m}^{\mathrm{s}}$ as:

$$Y_{n,m}^{\mathrm{c}}(\theta, \phi) = \sqrt{(2 - \delta_{0,m})(2n + 1)\frac{(n - m)!}{(n + m)!}} \, P_{n,m}(\cos\phi)\cos m\theta \tag{22}$$

$$Y_{n,m}^{\mathrm{s}}(\theta, \phi) = \sqrt{(2 - \delta_{0,m})(2n + 1)\frac{(n - m)!}{(n + m)!}} \, P_{n,m}(\cos\phi)\sin m\theta \tag{23}$$

The factor $N_{n,m} := \sqrt{(2 - \delta_{0,m})(2n + 1)\frac{(n - m)!}{(n + m)!}}$ is called the *normalization factor* of the spherical harmonics and $\delta_{0,m}$ is the Kronecker delta. The weird factor $2 - \delta_{0,m}$ in $N_{n,m}$ will become clear in the next section.

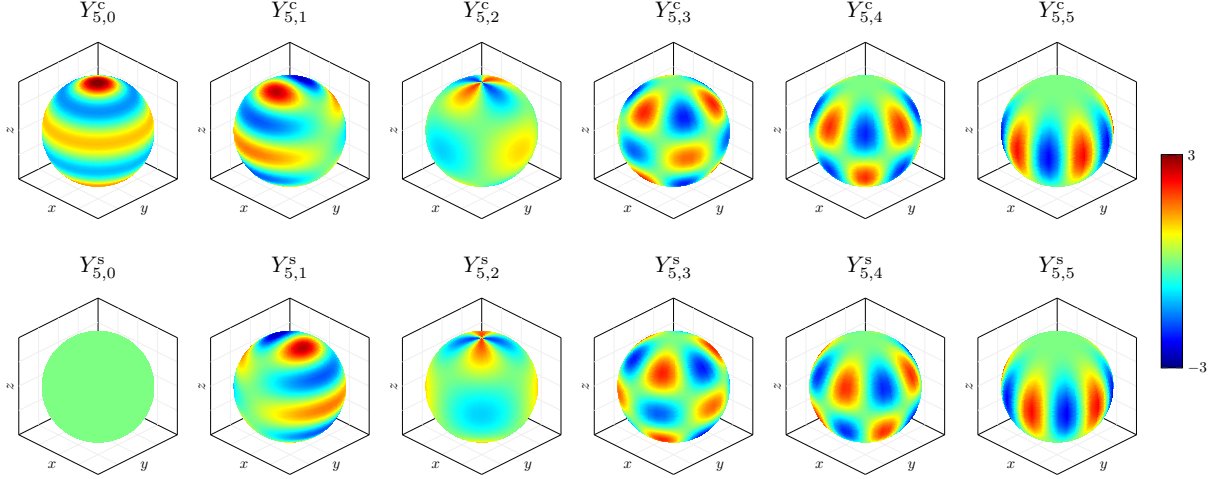| n	m	$Y^{\mathrm{c}}_{n,m}(\theta,\phi)$	n	m	$Y^{\mathrm{c}}_{n,m}(\theta,\phi)$
0	0	1	2	2	$\frac{\sqrt{15}}{2}(\sin\phi)^2\cos 2\theta$
1	0	$\sqrt{3}\cos\phi$	3	0	$\frac{\sqrt{7}}{2}\cos\phi(5(\cos\phi)^2-3)$
1	1	$\sqrt{3}\sin\phi\cos\theta$	3	1	$\frac{\sqrt{42}}{4}(5(\cos\phi)^2-1)\sin\phi\cos\theta$
2	0	$\frac{\sqrt{5}}{2}(3(\cos\phi)^2-1)$	3	2	$\frac{\sqrt{105}}{2}(\sin\phi)^2\cos\phi\cos 2\theta$
2	1	$\sqrt{15}\sin\phi\cos\phi\cos\theta$	3	3	$\frac{\sqrt{70}}{4}(\sin\phi)^3\cos 3\theta$

Table 3: First cosine spherical harmonics.

Figure 6: 3D heat map of the spherical harmonics of degree $n=5$. The first row correspond to the cosine spherical harmonics and the second row correspond to the sine spherical harmonics.

1.3.2 Laplace equation in spherical coordinates

Definition 11. Let $f:\mathbb{R}^3\to\mathbb{R}$ be a twice-differentiable function. The *Laplace equation* is the equation

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0 \tag{24}$$

where Δ is the Laplace operator.

The next proposition gives the Laplace equation in spherical coordinates.

Proposition 12. Let $f:\mathbb{R}^3\to\mathbb{R}$ be a twice-differentiable function. Then:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial f}{\partial\phi}\right) + \frac{1}{r^2(\sin\phi)^2}\frac{\partial^2 f}{\partial\theta^2} = 0 \tag{25}$$

where $r\in[0,\infty)$ denotes the radial distance, $\theta\in[-\pi,\pi)$ denotes the longitude, and $\phi\in[0,\pi]$, the colatitude:

$$\begin{aligned} x &= r\sin\phi\cos\theta \\ y &= r\sin\phi\sin\theta \\ z &= r\cos\phi \end{aligned} \tag{26}$$

We are now interested in solving the Laplace equation. Theorem 13 gives the solution of it as a function of the spherical harmonics.

Theorem 13. The regular solutions in a bounded region $\Omega\subseteq\mathbb{R}^3$ such that $0\notin\overline{\Omega}$ of the Laplace equation in spherical coordinates are of the form

$$f(r,\theta,\phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}(a_n r^n + b_n r^{-n-1})P_{n,m}(\cos\phi)(c_{n,m}\cos(m\theta) + s_{n,m}\sin(m\theta)) \tag{27}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{n}(a_n r^n + b_n r^{-n-1})(\tilde{c}_{n,m}Y^{\mathrm{c}}_{n,m}(\theta,\phi) + \tilde{s}_{n,m}Y^{\mathrm{s}}_{n,m}(\theta,\phi)) \tag{28}$$

where $a_n, b_n, c_{n,m}, s_{n,m}, \tilde{c}_{n,m}, \tilde{s}_{n,m}\in\mathbb{R}$.

Proof. Let $f(r, \theta, \phi)$ be a solution of Eq. (25) Using separation variables $f(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$ we can write:

$$\frac{\Theta\Phi}{r^2}(r^2 R')' + \frac{R\Theta}{r^2 \sin\phi}(\sin\phi\,\Phi')' + \frac{R\Phi}{r^2(\sin\phi)^2}\Theta'' = 0 \tag{29}$$

Here we are making and abuse of notation denoting all the derivative with a prime, but the reader should have no confusion with it. Isolating R from Θ and Φ yields:

$$\frac{(r^2 R')'}{R} = -\frac{1}{\sin\phi\,\Phi}(\sin\phi\,\Phi')' - \frac{1}{(\sin\phi)^2\Theta}\Theta'' \tag{30}$$

Since the left-hand side depends entirely on r and the right-hand side does not, it follows that both sides must be constant. Therefore:

$$\frac{(r^2 R')'}{R} = \lambda \tag{31}$$

$$\frac{1}{\sin\phi\,\Phi}(\sin\phi\,\Phi')' + \frac{1}{(\sin\phi)^2\Theta}\Theta'' = -\lambda \tag{32}$$

with $\lambda \in \mathbb{R}$. Similarly, separating variables from Eq. (32) we obtain that the equations

$$\frac{1}{\Theta}\Theta'' = -m^2 \tag{33}$$

$$\frac{\sin\phi}{\Phi}(\sin\phi\,\Phi')' + \lambda(\sin\phi)^2 = m^2 \tag{34}$$

must be constant with $m \in \mathbb{C}$ (a priori). The solution to the well-known Eq. (33) is a linear combination of the $\cos(m\theta)$ and $\sin(m\theta)$. Note, though, that since Θ must be a 2π-periodic function, that is satisfying $\Theta(\theta + 2\pi) = \Theta(\theta)\ \forall \theta \in \mathbb{R}$, m must be an integer. On the other hand making the change of variables $x = \cos\phi$ and $y = \Phi(\phi)$ in Eq. (34) and using the chain rule, that equation becomes:

$$(1 - x^2)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 2x\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \left(\lambda - \frac{m^2}{1 - x^2}\right)y = 0 \tag{35}$$

which is the associate Legendre equation. We have argued in Proposition 9 that we need $\lambda = n(n+1)$ and $m \leq n$ in order to get regular solutions at $x = \cos\phi = \pm 1$. Moreover these solutions are $P_{n,m}(\cos\phi)$.

Finally note that equation Eq. (31) is a Cauchy-Euler equation (check [Wika]) and so the general solution of it is given by

$$R(r) = c_1 r^n + c_2 r^{-n-1} \tag{36}$$

because $\lambda = n(n+1)$ (the reader may check that r^n and r^{-n-1} are indeed two independent solutions of Eq. (31)). So the general solution becomes a linear combination of the each solution founded varying $n \in \mathbb{N} \cup \{0\}$ and $m \in \{0, 1, \ldots, n\}$:

$$f(r, \theta, \phi) = \sum_{n=0}^{\infty}\sum_{m=0}^{n}(a_n r^n + b_n r^{-n-1})P_{n,m}(\cos\phi)(c_{n,m}\cos(m\theta) + s_{n,m}\sin(m\theta)) \tag{37}$$

$\square$

From now we are not concerning on the singularity at $r = 0$ of Eq. (28) (see **??** for more details).

The associated Legendre polynomials satisfy a orthogonality relation:

Lemma 14. Let $n_1, n_2 \in \mathbb{N} \cup \{0\}$ and $m \leq \min\{n_1, n_2\}$. Then:

$$\int_0^1 P_{n_1,m}(x)P_{n_2,m}(x)\,\mathrm{d}x = \frac{2}{2n_1 + 1}\frac{(n_1 + m)!}{(n_1 - m)!}\delta_{n_1,n_2} \tag{38}$$

where δ_{n_1,n_2} denotes the Kronecker delta.

Similarly it can be shown that the spherical harmonics from an orthonormal family of functions:

Proposition 15. The family of spherical harmonics $\{Y^{\mathrm{c}}_{n,m}(\theta,\phi), Y^{\mathrm{s}}_{n,m}(\theta,\phi) : n \in \mathbb{N} \cup \{0\}, m \leq n\}$ is orthonormal in the following sense:

$$\frac{1}{4\pi} \int\limits_0^{2\pi} \int\limits_0^{\pi} Y^i_{n_1,m_1}(\theta,\phi) Y^j_{n_2,m_2}(\theta,\phi) \mathrm{d}\Omega = \delta_{n_1,n_2} \delta_{m_1,m_2} \delta_{i,j} \tag{39}$$

where $\mathrm{d}\Omega = \sin\phi \, \mathrm{d}\phi \, \mathrm{d}\theta$ is the solid angle element, which measures the element of area on a sphere of radius 1.

Proof. Let N_{n_1,m_1}, N_{n_2,m_2} be the normalization factors of the spherical harmonics Y_{n_1,m_1}, Y_{n_2,m_2} respectively. Note that we can separate the variables in the integral of Eq. (39). So if $i \neq j$, the integral over θ becomes $\int_0^{2\pi} \cos(m_1\theta) \sin(m_2\theta) \, \mathrm{d}\theta$ which is equal to 0 regardless of the values of m_1 and m_2. So from now on assume that $i = j$. Due to the symmetry between the cosine and the sine we can suppose that $i = \mathrm{c}$. Thus:

$$\int\limits_0^{2\pi} \int\limits_0^{\pi} Y^i_{n_1,m_1}(\theta,\phi) Y^j_{n_2,m_2}(\theta,\phi) \mathrm{d}\Omega =$$

$$= N_{n_1,m_1} N_{n_2,m_2} \int\limits_0^{\pi} P_{n_1,m_1}(\cos\phi) P_{n_2,m_2}(\cos\phi) \sin\phi \mathrm{d}\phi \int\limits_0^{2\pi} \cos(m_1\theta) \cos(m_2\theta) \, \mathrm{d}\theta \tag{40}$$

An easy check shows that if $m_1 \neq m_2$ then the integral over θ is zero (and the same applies with sines). So suppose $m_1 = m_2 = m$. In that case, if $m \neq 0$ we have $\int_0^{2\pi} (\cos m\theta)^2 \, \mathrm{d}\theta = \int_0^{2\pi} (\sin m\theta)^2 \, \mathrm{d}\theta = \pi$ and if $m = 0$, the cosine integral evaluates to 2π whereas the sine integral is 0. We can omit this latter case because $Y^{\mathrm{s}}_{n,0}$ is identically zero. Thus:

$$\frac{2\pi}{2 - \delta_{0,m}} N_{n_1,m} N_{n_2,m} \int\limits_0^{\pi} P_{n_1,m}(\cos\phi) P_{n_2,m}(\cos\phi) \sin\phi \mathrm{d}\phi = \frac{2\pi}{2 - \delta_{0,m}} N_{n_1,m} N_{n_2,m} \int\limits_{-1}^{1} P_{n_1,m}(x) P_{n_2,m}(x) \, \mathrm{d}x$$
$$\tag{41}$$

By Lemma 14 this latter integral is $\frac{2}{2n_1+1} \frac{(n_1+m)!}{(n_1-m)!} \delta_{n_1,n_2}$. Finally, if $n_1 = n_2 = n$, putting all normalization factors together we get:

$$\frac{2\pi}{2 - \delta_{0,m}} N^m_n N^m_n \frac{2}{2n+1} \frac{(n+m)!}{(n-m)!} = 4\pi \tag{42}$$

$$\square$$

Moreover, an important result in the Sturm-Liouville Theory of second order differential equations ([Wikb; Wan+09]) says that the family of spherical harmonics $\{Y^{\mathrm{c}}_{n,m}(\theta,\phi), Y^{\mathrm{s}}_{n,m}(\theta,\phi) : n \in \mathbb{N} \cup \{0\}, m \leq n\}$ form a complete set in the sense that any smooth function defined on the sphere $f : S^2 \to \mathbb{R}$ can be expanded in a series of spherical harmonics:

$$f(\theta,\phi) = \sum_{n=0}^{\infty} \sum_{m=0}^{n} (c_{n,m} Y^{\mathrm{c}}_{n,m}(\theta,\phi) + s_{n,m} Y^{\mathrm{s}}_{n,m}(\theta,\phi)) \tag{43}$$

This will be useful in **??** when expanding the gravitational potential created by the Earth at some arbitrary point in spherical harmonics.