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Cite this article: Ceci S, Seis C. 2022 On the dynamics of point vortices for the two-dimensional Euler equation with L^p vorticity. *Phil. Trans. R. Soc. A* **380**: 20210046. <https://doi.org/10.1098/rsta.2021.0046>

Received: 27 July 2021

Accepted: 11 October 2021

One contribution of 14 to a theme issue 'Mathematical problems in physical fluid dynamics (part 2)'.

Subject Areas:

applied mathematics, analysis, differential equations

Keywords:

Euler equations, vortex dynamics, stability estimates

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On the dynamics of point vortices for the two-dimensional Euler equation with L^p vorticity

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We study the evolution of solutions to the two-dimensional Euler equations whose vorticity is sharply concentrated in the Wasserstein sense around a finite number of points. Under the assumption that the vorticity is merely L^p integrable for some $p > 2$, we show that the evolving vortex regions remain concentrated around points, and these points are close to solutions to the Helmholtz–Kirchhoff point vortex system.

This article is part of the theme issue 'Mathematical problems in physical fluid dynamics (part 2)'.

1. Introduction

This article is concerned with the study of the interaction of vortices in an ideal incompressible fluid contained in the two-dimensional plane. We examine the evolution of sharply concentrated vortex regions that are driven by the Euler equations and we analyse their convergence to a point vortex system if the vortex regions shrink to a collection of points.

The study of the relation between the PDE description of vortex dynamics given by the Euler equations and the ODE description given by a point vortex system has a long history. The earliest research in this direction goes back to Helmholtz [1], who in the middle of the nineteenth century derived a model for the interaction of point vortices from the Euler equations via formal asymptotics. His work was complemented by that of Kirchhoff [2], who showed that this point vortex system is Hamiltonian. We will refer to this model accordingly as the Helmholtz–Kirchhoff point vortex system.

On a rigorous level, a derivation of the point vortex system from the Euler equation was achieved for the first time by Marchioro & Pulvirenti in the 1980s [3] and was improved in many subsequent works, e.g. [4–6]. In these works, the authors consider bounded vorticity solutions to the Euler equations whose support is confined in separated regions of small diameter, and prove that the vortex centres are during their evolution close to the point vortices described by the Helmholtz–Kirchhoff system.

In a recent work [7], we extended the Marchioro–Pulvirenti method in two ways: first, we derived the point vortex system from the Euler equations for vorticity fields that admit a small *unbounded* part, in the sense that the L^p norm, $p \in (2, \infty)$, of this vorticity part is sufficiently small. Second, we weakened the concentration requirement of the vortex regions. In the present study, we remove the smallness assumption on the unbounded part of the vorticity field.

Considering unbounded vorticities fields is particularly interesting from an analytical point of view for the reason that, opposed to the situation of bounded vorticities [8], uniqueness of solutions to the two-dimensional Euler equations is presently unknown. On the contrary, there is some evidence of the existence of more than one solution to the associated initial-value problem [9–12]. Our result now shows that even in the case that the two-dimensional Euler equations were ill-posed, in the setting under consideration the Euler equations would inherit a certain degree of uniqueness from the limiting Kirchhoff–Helmholtz point vortex system. We will discuss this property later in §2 in more detail.

As in [7], we choose to work with a mild notion of vorticity concentration by considering as a measure of concentration the Wasserstein distance W_2 between the vorticity field and the empirical measure associated with the point vortices. Considering weak measures of concentration is less restrictive than supposing concentration in terms of the support if we envision chaotic or turbulent fluid motions. Moreover, since Wasserstein distances metrize weak convergence, see, e.g. theorem 7.12 in [13], the main result of this article can be understood as a bound on the rate of weak convergence of Euler solutions towards the Helmholtz–Kirchhoff point vortex system. Measures of weak concentration were used earlier in a related study of vortex filaments in three-dimensional inviscid fluids [14]. The Wasserstein distance that we consider here was already present in the original approach [3]; it was, however, interpreted as a second moment function (cf. (2.7)) and played a role only on the level of intermediate results.

The method of Marchioro and Pulvirenti relies crucially on certain symmetry properties of the two-dimensional Biot–Savart kernel and it breaks down in situations in which these fail to be satisfied. We are aware of two alternative approaches. One is an energy expansion method pioneered by Turkington [15], which applies, for instance, to the axisymmetric Euler equations [16,17], which describe the evolution of vortex rings, but also to the lake equations [18], which represent the motion of fluids whose velocity varies on scales much larger than its depth and which is smaller than the speed of gravitational waves. More recently, the so-called gluing method was applied successfully to the context of Euler vortex dynamics by Davila *et al.* [19,20]. The authors of these papers construct solutions in the planar two-dimensional case and for three-dimensional flows with a helical symmetry. This approach had previously proved to be effective in other desingularization problems, see for example [21,22] for the Schrödinger and the Allen–Cahn equations, respectively.

Our article is organized as follows. In §2, we delineate the rigorous mathematical setting and present our main results. Section 3 is devoted to the proofs.

2. Mathematical setting

We consider an incompressible fluid moving in the two-dimensional plane \mathbb{R}^2 . The motion of the fluid can be described by either the scalar vorticity field $\omega = \omega(t, x) \in \mathbb{R}$, which measures the tendency to rotate, or the vectorial velocity field $u = u(t, x) \in \mathbb{R}^2$. As we are interested in vortex dynamics, it is convenient to consider the two-dimensional Euler equations in vorticity

formulation, that is,

$$\partial_t \omega + u \cdot \nabla \omega = 0 \quad \text{in } (0, +\infty) \times \mathbb{R}^2. \quad (2.1)$$

This transport equation states that the vorticity is advected along particle trajectories. The incompressibility of the fluid is mathematically encoded in the condition that the velocity is divergence-free,

$$\nabla \cdot u = 0. \quad (2.2)$$

Both vorticity and velocity depend on each other via the relation $\omega = \nabla \times u = \partial_1 u_2 - \partial_2 u_1$, which can be reversed with the help of the Biot–Savart law

$$u(t, x) = K * \omega(t, x) = \int_{\mathbb{R}^2} K(x - y) \omega(t, y) dy \quad (2.3)$$

(the symbol of convolution always being meant in space), and allows to recover the velocity from the vorticity. The function K is the so-called Biot–Savart kernel, which is defined as the rotated gradient of the Newtonian potential $G(z) = -\frac{1}{2\pi} \log |z|$, so that

$$K(z) = \frac{1}{2\pi} \frac{z^\perp}{|z|^2}, \quad \text{where } z^\perp = (-z_2, z_1).$$

We equip the vorticity equation (2.1) with a compactly supported and possibly unbounded initial datum $\bar{\omega} \in L^p(\mathbb{R}^2)$ and study the evolution of distributional solutions ω in the class $C^0((0, \infty); L^p(\mathbb{R}^2))$. In this paper, we have to assume that $p > 2$, which guarantees that the corresponding velocity fields, cf. (2.3), are (Hölder) continuous. Indeed, if $\omega \in L^p(\mathbb{R}^2)$ is compactly supported, maximal regularity estimates imply that u belongs to $W^{1,p}(\mathbb{R}^2)$, which in turn embeds into a Hölder space. The existence of distributional solutions in our setting is known since the work of DiPerna & Majda [23]. We remark that these solutions are renormalized in the sense of DiPerna & Lions [24] as noticed, for instance, in [25]. This guarantees, in particular, the preservation in time of the Lebesgue norms. Moreover, solutions to the transport equation are Lagrangian, that is, they are advected by the Lagrangian flow associated with the fluid velocity, as in the classical setting.

We suppose that the initial vorticity can be decomposed into N separate compactly supported components of definite sign

$$\bar{\omega} = \sum_{i=1}^N \bar{\omega}_i,$$

where either $\bar{\omega}_i \geq 0$ or $\bar{\omega}_i \leq 0$, and

$$\min_{i \neq j} \text{dist}(\text{spt} \bar{\omega}_i, \text{spt} \bar{\omega}_j) \geq \delta, \quad (2.4)$$

for some $\delta > 0$. Each individual component is now advected by the total fluid flow, and thus, its motion is described by the linear transport equation

$$\left. \begin{aligned} \partial_t \omega_i + u \cdot \nabla \omega_i &= 0 \\ \omega_i(0, \cdot) &= \bar{\omega}_i(\cdot). \end{aligned} \right\} \quad (2.5)$$

Thanks to the Sobolev regularity of the velocity field discussed above, solutions to the linear advection equations are renormalized and unique (cf. [24]) and thus, the sign of each component ω_i is preserved over time and the solution ω to (2.1) is still described by the sum

$$\omega = \sum_{i=1}^N \omega_i.$$

We notice that, thanks to the incompressibility assumption (2.2), the evolution equation in (2.5) is conservative, and thus, the total vorticity of each individual component, the so-called *intensity*,

is constant in time,

$$a_i = \int_{\mathbb{R}^2} \bar{\omega}_i(x) dx = \int_{\mathbb{R}^2} \omega_i(t, x) dx.$$

We suppose that each vortex component is initially weakly concentrated around a single point, in the sense that there exist N distinct points $\bar{Y}_i, \dots, \bar{Y}_N \in \mathbb{R}^2$ such that

$$W_2\left(\frac{\bar{\omega}_i}{a_i}, \delta_{\bar{Y}_i}\right) \leq \varepsilon, \quad (2.6)$$

for any $i = 1, \dots, N$. Here, $\varepsilon > 0$ is the concentration scale, which we assume to be small, and W_2 is the 2-Wasserstein distance, which takes a particularly simple expression if one of the measures is atomic as it is in our setting,

$$W_2\left(\frac{\bar{\omega}_i}{a_i}, \delta_{\bar{Y}_i}\right) = \sqrt{\frac{1}{a_i} \int_{\mathbb{R}^2} |x - \bar{Y}_i|^2 \bar{\omega}_i(x) dx}. \quad (2.7)$$

We refer to Villani's monograph [13] for a comprehensive introduction to Wasserstein distances. At this point, we only want to make three observations.

First, we remark that the representation (2.7) particularly implies that the weak concentration assumption (2.6) holds true whenever the vortex components are strongly concentrated in the sense that the diameter of each component is at most of the size 2ε . This observation was already exploited in Marchioro & Puvirenti's original work [3]. In their work and in ours, the second moment function plays a central role in the analysis.

Second, an elementary computation shows that the second moment function is minimized at the centre of vorticity

$$\bar{X}_i = \frac{1}{a_i} \int_{\mathbb{R}^2} x \bar{\omega}_i(x) dx,$$

and thus, the weak concentration assumption in (2.6) entails that

$$W_2\left(\frac{\bar{\omega}_i}{a_i}, \delta_{\bar{X}_i}\right) \leq W_2\left(\frac{\bar{\omega}_i}{a_i}, \delta_{\bar{Y}_i}\right) \leq \varepsilon. \quad (2.8)$$

Third, since Wasserstein distances metrize weak convergence, see theorem 7.12 in [13], the concentration assumption (2.6) for the initial vorticity implies that the (rescaled) vortex component converges weakly to an atomic measure provided that $\varepsilon \rightarrow 0$. In particular, if the intensities a_i are chosen independently of ε , for any $p > 1$, the quantities $\|\bar{\omega}\|_{L^p}$ have to diverge in the $\varepsilon \rightarrow 0$ limit. In this paper, we suppose that this divergence occurs at most at an algebraic rate,

$$\|\bar{\omega}\|_{L^p} \leq \Lambda \varepsilon^{-\gamma}, \quad (2.9)$$

where γ is some positive number and $\Lambda > 0$ is a constant. In our earlier work [7], we had to restrict to exponents $\gamma \leq 2(p-2)/p$, while Caprini and Marchioro considered (2.9) with arbitrary γ but $p = \infty$ [5]. Our new condition brings together the strengths of both papers. Note that since ω and ω_i satisfy (2.1) and (2.5), respectively, in the sense of a renormalized solution [24], the scaling assumption on the initial datum (2.9) applies to any vortex component during the evolution

$$\|\omega_i(t)\|_{L^p} = \|\bar{\omega}_i\|_{L^p} \leq \|\bar{\omega}\|_{L^p} = \|\omega(t)\|_{L^p}. \quad (2.10)$$

We have to make an additional assumption on the geometry of the vortex components under consideration. In order to rule out the initial vorticity to have long tentacles away from the vortex centre, we suppose that there exists a radius R (significantly) smaller than the separation distance δ , and large enough that the support of each initial component lies in a (open) ball of radius R around its vortex centre

$$\text{spt} \bar{\omega}_i \subset B_R(\bar{X}_i), \quad (2.11)$$

for any $i \in \{1, \dots, N\}$. Such a radius can always be chosen independently of ε .

In our first main result, we show that under these concentration and scaling assumptions, the vortex component solutions are close to the Helmholtz–Kirchhoff point-vortex system, which is given by a collection of points $Y_1, \dots, Y_N \in \mathbb{R}^2$, satisfying the initial-value problem

$$\begin{cases} \frac{d}{dt} Y_i(t) = \sum_{j \neq i} a_j K(Y_i(t) - Y_j(t)), \\ Y_i(0) = \tilde{Y}_i. \end{cases} \quad \forall i = 1, \dots, N. \quad (2.12)$$

To be more specific, our result shows that the vortex components remain concentrated in the Wasserstein sense around the Helmholtz–Kirchhoff point vortices at the scale ε .

Theorem 2.1. *Let $i \in \{1, \dots, N\}$ be given and suppose that*

$$\varepsilon \ll 1 \quad \text{and} \quad R \ll \delta.$$

Then there exists a time $T > 0$ independent of ε and a positive constant C such that

$$W_2 \left(\frac{\omega_i(t)}{a_i}, \delta_{Y_i(t)} \right) \leq C e^{Ct} \varepsilon$$

for any $t \in [0, T]$.

In the hypothesis of the theorem, we assume that the size of the vortex components is initially small compared with the distance of the components. This assumption is necessary in order to guarantee that the vortex components remain well separated on order-one time scales. In particular, both geometric scales R and δ are supposed to be of order one in ε , which has to be sufficiently small (dependent on γ and p).

For initial data that are sharply concentrated in terms of the support of the vortex components, a variation of the analysis in [5] (along the lines of lemma 3.3) would also show that the vorticity remains strongly concentrated around the point vortex system. As in [5], however, the radius of concentration would increase from ε up to $\varepsilon^{1/2-}$. We will not pursue this direction here but focus on a rather weak notion of vorticity concentration.

The result from theorem 2.1 translates into an estimate on the distance between the full solution ω and the singular vorticity field $\sum_i a_i \delta_{Y_i}$ if the 2-Wasserstein distance is traded for the 1-Wasserstein distance. The reason for choosing W_1 over W_2 is that the former can be extended to configurations that are not necessarily non-negative. Indeed, thanks to the dual Kantorovich–Rubinstein representation

$$W_1(f, g) = \sup \left\{ \int_{\mathbb{R}^2} (f - g) \zeta \, dx : \|\nabla \zeta\|_{L^\infty} \leq 1 \right\},$$

see theorem 1.14 in [13], the 1-Wasserstein distance is well defined if f and g are integrable functions of the same global average.

Hence, we can obtain the following corollary.

Corollary 2.2. *Under the same assumptions of theorem 2.1, there exists a time $T > 0$ independent of ε and a positive constant C such that*

$$W_1 \left(\omega(t), \sum_i a_i \delta_{Y_i(t)} \right) \leq C e^{Ct} \varepsilon,$$

for any $t \in [0, T]$.

In particular, since Wasserstein distances metrize weak convergence in the sense of measures, the Euler vorticity distribution $\omega(t)$ converges weakly to the point vortex measure $\sum_i a_i \delta_{Y_i(t)}$. Our result provides in addition an estimate on the convergence rate, which, up to an exponentially in time diverging prefactor, remains of the order of ε as for the initial datum, see (2.6). Furthermore, the estimate holds true for *any* solution ω satisfying our initial assumptions. As we have written earlier, this is a remarkable property in light of the fact that it is currently not known whether solutions to the Euler equations with vorticity in L^p are unique, if $p < \infty$. As solutions to the point

vortex system (2.12) are unique, our result shows that in any case uniqueness is in some sense recovered in the limit as $\varepsilon \rightarrow 0$.

We remark that the estimate in corollary 2.2 can be interpreted as a stability estimate for the Euler equation. Indeed, as observed by Schochet [26], the point vortex system can be considered as a very weak solution to the Euler equation (in the sense of Delort [27]). Adopting this point of view, our result provides a stability estimate between a distributional solution and an even weaker solution, which, however, carries more structure. Notice that general stability estimates for the Euler equation are currently still missing. The works closest to this direction include that by Loeper [28], who also works with Wasserstein distances, see also [29]. In the context of linear transport equations with general Sobolev vector fields, stability estimates were obtained only recently [30,31].

We finally translate the estimates on the vortex components into an estimate of the vorticity centres

$$X_i(t) = \frac{1}{a_i} \int_{\mathbb{R}^2} x \omega_i(t, x) dx$$

and their velocities.

Theorem 2.3. *Under the same assumptions of theorem 2.1, there exists a time $T > 0$ independent of ε and a positive constant C such that*

$$|X_i(t) - Y_i(t)| \leq C e^{Ct} \varepsilon, \quad \text{and} \quad \left| \frac{d}{dt} X_i(t) - \frac{d}{dt} Y_i(t) \right| \leq C e^{Ct} \varepsilon,$$

for any $t \in [0, T]$.

The result thus shows that the centres of vorticity and their velocities remain ε -close to the point vortex solutions.

3. Proofs

The strategy of our proof is heavily inspired by Marchioro & Pulvirenti's original work [3], Carprini & Marchioro's improvement [5] and builds up on our own earlier contribution [7].

We start by introducing some notation.

In the following, we write $A \lesssim B$ if an inequality $A \leq CB$ holds for some constant $C > 0$ that is independent of t and ε . We furthermore write $A \sim B$ if both $A \lesssim B$ and $B \lesssim A$ hold. This way, for instance, the vortex intensities a_i will be neglected in our analysis. In situations in which we choose to explicitly introduce a constant C , this constant will always be independent of t and ε , and its value may change from line to line.

We shall also introduce $\Omega_i(t)$ as the support of the i th vortex at time t , that is, $\Omega_i(t) = \text{spt} \omega_i(t)$. With this notation, for any $i \neq j$, it holds that $\text{dist}(\Omega_i(0), \Omega_j(0)) \geq \delta$, see (2.4). In our main results, we claim that there exists a time T that is independent of ε up to which solutions to the Euler equations are close to those of the point vortex system. Here, we make this choice more precise. In fact, we choose T as the time at which the distance between the vortex components has decreased by a factor of two

$$T = \sup \left\{ t \in (0, +\infty) : \text{dist}(\Omega_i(s), \Omega_j(s)) \geq \frac{\delta}{2}, \forall i \neq j, s \leq t \right\}. \quad (3.1)$$

Because the fluid velocity u is continuous, it is clear that the vortex components cannot collide instantaneously, thus, $T > 0$. We will see later that this time is bounded from below uniformly in ε , that is, $T \gtrsim 1$.

Without loss of generality, we may restrict our attention to time intervals such that

$$\min_{i \neq j} |Y_i(t) - Y_j(t)| \geq \frac{\delta}{2}, \quad (3.2)$$

for any $t \leq T$. Indeed, if this is not the case, we select a minimal time $\tilde{T} < T$ at which (3.2) holds with an equality. Since the point vortex system $\{Y_1, \dots, Y_N\}$ is independent of ε , the time \tilde{T} must also be independent of ε , and we can work on the time interval $[0, \tilde{T}]$.

We finally define by $u_i = K * \omega_i$ the velocity that is induced by the i th vortex component. For every fixed i , we may then decompose the total velocity into the field induced by the i th vortex component and that induced by the other components. That is, we write

$$u = \sum_{j=1}^N K * \omega_j = u_i + F_i, \quad (3.3)$$

where $F_i = \sum_{j \neq i} u_j$ is the far field.

Our first step in the proof is essentially identical to that in [3]. We show that until time T , the vorticity remains sharply concentrated in the Wasserstein sense. From this statement, our main estimates on the relationship between the Euler equations and the Helmholtz–Kirchhoff system can be directly deduced. This is the content of the following §3a. In the final §3b, we will show that T is indeed bounded from below uniformly in ε , which complements the proof of our results.

(a) Concentration of vorticity and proofs of the main results under the assumption $T \gtrsim 1$

The concentration estimate relies on elementary bounds on the far field.

Lemma 3.1. *Let $i \in \{1, \dots, N\}$ be given. Then for any $j \neq i$ it holds that*

$$\|u_j(t)\|_{L^\infty(\Omega_i(t))} \lesssim 1 \quad \text{and} \quad \|\nabla u_j(t)\|_{L^\infty(\Omega_i(t))} \lesssim 1,$$

for any $t \in [0, T]$.

Proof. The result is an immediate consequence of the scaling properties of the Biot–Savart kernel and the separation of the vortex components assumed up to time T , see (3.1). For instance, if $x \in \Omega_i(t)$ and $z \in \Omega_j(t)$, it holds that $|x - z| \gtrsim 1$ and thus

$$|u_j(t, x)| \lesssim \int \frac{1}{|x - z|} |\omega_j(t, z)| \, dz \lesssim 1.$$

Similarly, since K is Lipschitz on $B_{\delta/2}(0)^c$ with a constant of the order of $1/\delta^2 \lesssim 1$, for any $x, y \in \Omega_i(t)$, we find that

$$|u_j(t, x) - u_j(t, y)| \leq \int |K(x - z) - K(y - z)| |\omega_j(t, z)| \, dz \lesssim |x - y|,$$

which is what we claimed. ■

Lemma 3.1 establishes the Lipschitz property of the far-field, which is already sufficient to show that the vortex components remain concentrated near their vorticity centres with respect to the 2-Wasserstein distance.

Lemma 3.2. *Let $i \in \{1, \dots, N\}$ be given. There exists a universal constant $C < \infty$ such that*

$$W_2\left(\frac{\omega_i(t)}{a_i}, \delta_{X_i(t)}\right) \leq e^{Ct} \varepsilon,$$

for any $t \in [0, T]$.

This lemma is the key estimate in the Marchioro–Pulvirenti method. We recall the simple proof for the convenience of the reader.

Proof. For notational convenience, we suppose that $a_i = 1$ and we set $W_i(t) = W_2(\omega_i(t), \delta_{X_i(t)})$. Using the definition of the vortex centres X_i and the splitting of the velocity field in (3.3), we

compute the rate of change of W_i ,

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} W_i^2 &= \int (x - X_i) \cdot u(x) \, dx \\ &= \int (x - X_i) \cdot (F_i(x) - F_i(X_i)) \omega_i(x) \, dx + \int (x - X_i) \cdot u_i(x) \omega_i(x) \, dx. \end{aligned}$$

The second term on the right-hand side vanishes by the symmetry properties of the Biot–Savart kernel K . Using the Lipschitz bound for the far field from lemma 3.1, we thus find the differential inequality

$$\frac{d}{dt} W_i^2 \lesssim W_i^2,$$

which can be solved via a Gronwall argument, and uses the concentration of the initial datum in (2.8). ■

We will now show that the concentration estimate implies all the estimates of our main results. We start with the proof of theorem 2.3, that is, the bound on the distance between vortex centres and point vortices and on the difference of the corresponding velocities.

Proof of theorem 2.3 supposing that $T \gtrsim 1$. We compute how the distance between the point vortices and the vortex centres changes under the evolutions (2.5) and (2.12). We start by noticing that

$$\frac{d}{dt} X_i = \frac{1}{a_i} \int u \omega_i \, dx = \frac{1}{a_i} \sum_{j \neq i} \iint K(x - y) \omega_i(x) \omega_j(y) \, dx \, dy, \quad (3.4)$$

thanks to the symmetry properties of the Biot–Savart kernel, and thus

$$\begin{aligned} \left| \frac{d}{dt} |X_i - Y_i| \right| &\leq \left| \frac{dX_i}{dt} - \frac{dY_i}{dt} \right| \\ &\leq \sum_{j \neq i} \left| \frac{1}{a_i} \iint K(x - y) \omega_i(x) \omega_j(y) \, dy \, dx - a_j K(Y_i - Y_j) \right| \\ &\leq \frac{1}{|a_i|} \sum_{j \neq i} \iint |K(x - y) - K(Y_i - Y_j)| |\omega_i(x)| |\omega_j(y)| \, dx \, dy, \end{aligned}$$

as a consequence of the triangle inequality. Now, since both $|x - y| \gtrsim 1$ and $|Y_i - Y_j| \gtrsim 1$ in the integrand, we may use the Lipschitz property of the Biot–Savart kernel away from the origin and the triangle inequality to deduce that

$$\begin{aligned} \left| \frac{d}{dt} |X_i - Y_i| \right| &\lesssim \int |x - Y_i| |\omega_i(x)| \, dx + \sum_{j \neq i} \int |y - Y_j| |\omega_j(y)| \, dy \\ &\lesssim \sum_j \int_{\mathbb{R}^2} |x - X_j| |\omega_j(x)| \, dx + \sum_j |X_j - Y_j|. \end{aligned}$$

We use Jensen’s inequality and lemma 3.2 to bound the first term on the right-hand side. Summing over i , we then find the differential inequality

$$\left| \frac{d}{dt} \sum_i |X_i - Y_i| \right| \lesssim e^{Ct} \varepsilon + \sum_j |X_j - Y_j|,$$

and applying a standard Gronwall argument, we deduce both statements of the theorem. ■

From what we have established so far, theorem 2.1 is an immediate consequence.

Proof of theorem 2.1 supposing that $T \gtrsim 1$. The statement of theorem 2.1 follows from lemma 3.2 and theorem 2.3 via the triangle inequality. ■

It remains to provide the proof of corollary 2.2.

Proof of corollary 2.2 supposing that $T \gtrsim 1$. Keeping in mind that $\omega = \sum_i \omega_i$, we have by the triangle and Jensen's inequalities that

$$\begin{aligned} W_1 \left(\omega(t), \sum_i a_i \delta_{Y_i} \right) &\leq \sum_i W_1(\omega_i(t), a_i \delta_{Y_i(t)}) \\ &\leq \sum_i |a_i| W_1 \left(\frac{\omega_i(t)}{a_i}, \delta_{Y_i(t)} \right) \leq \sum_i |a_i| W_2 \left(\frac{\omega_i(t)}{a_i}, \delta_{Y_i(t)} \right), \end{aligned}$$

and thus, the statement follows from theorem 2.1. ■

(b) Proof of $T \gtrsim 1$

Our goal in this subsection is to show that T is bounded from below, uniformly in ε .

Before proceeding, we have to introduce some further notation. For any $\rho > 0$ and $t \in [0, T]$, we define by

$$m_i(t, \rho) = \frac{1}{a_i} \int_{B_\rho(X_i(t))^c} \omega_i(t, x) \, dx,$$

the vorticity portion of the i th component at time t lying at least at distance ρ from its centre $X_i(t)$. In the view of the choice of R in (2.11), it thus holds that $m_i(0, R) = 0$. We also remark that as a consequence of lemma 3.2, this outer vorticity is always small in the sense that

$$m_i(t, \rho) \leq \frac{1}{\rho^2} W_2^2 \left(\frac{\omega_i(t)}{a_i}, \delta_{X_i(t)} \right) \leq \frac{e^{CT}}{\rho^2} \varepsilon^2. \quad (3.5)$$

In our earlier paper [7], we made use of this estimate in order to conclude the uniform finiteness of T in the case $\gamma \leq 2(p-2)/p$.

In the proof of $T \gtrsim 1$ for larger exponents γ , we distinguish two cases.

In the *first case*, we assume that

$$m_i(t, 2R) < \varepsilon^\alpha, \quad \text{for any } t \in (0, T), i \in \{1, \dots, N\}, \quad (3.6)$$

where $\alpha = p\gamma/(p-2)$. Notice that $\alpha \in (2, \infty)$ for exponents $\gamma > 2(p-2)/p$. Therefore, the condition in (3.6) is stronger than the one inferred above from the concentration estimate.

We show that under (3.6), the velocity is bounded away from the vortex centre.

Lemma 3.3. *Suppose that (3.6) holds and let $i \in \{1, \dots, N\}$ be given. Then it holds*

$$|u_i(t, x)| \lesssim 1,$$

for any $t \in [0, T]$ and any $x \in \mathbb{R}^2$ such that $|x - X_i(t)| \geq 4R$.

Proof. By dividing by the vortex intensity we may without loss of generality assume that $\omega_i \geq 0$ and $a_i = 1$ in the following. For notational convenience, we drop the time variable.

The velocity bound is derived from a suitable estimate of the Biot–Savart kernel. In order to separate the singular from the regular part, we set $L = |x - X_i|$ and decompose the velocity into two parts

$$|u_i(x)| \lesssim \int_{B_{L/2}(X_i)} \frac{1}{|x - y|} \omega_i(y) \, dy + \int_{B_{L/2}(X_i)^c} \frac{1}{|x - y|} \omega_i(y) \, dy = I_1 + I_2.$$

The first term is easy to bound because on the domain of integration, it holds that $|x - y| \geq L/2 \gtrsim 1$, leading to $I_1 \lesssim 1$. Concerning the second term, we claim that

$$I_2 \lesssim \|\omega_i\|_{L^p}^{p/2(p-1)} m_i \left(t, \frac{L}{2} \right)^{(p-2)/2(p-1)}, \quad (3.7)$$

from which the bound $I_2 \lesssim 1$ follows via (2.9), (2.10) and (3.6).

The proof of the estimate (3.7) relies on a rearrangement procedure. We start by rewriting I_2 as

$$I_2 = \int_{\mathbb{R}^2} \frac{1}{|x-y|} \zeta_i(y) \, dy,$$

where $\zeta_i = \chi_{B_{\frac{L}{2}}(X_i)^c} \omega_i$. If ζ_i^* is the symmetric-decreasing rearrangement of ζ_i given by

$$\zeta_i^*(y) = \int_0^\infty \chi_{B_{\ell(s)}(0)}(y) \, ds, \quad \text{where } \ell(s) = \frac{|\{\zeta_i > s\}|^{1/2}}{\pi^{1/2}}, \quad (3.8)$$

the term I_2 can be bounded by

$$I_2 \leq \int_{\mathbb{R}^2} \frac{1}{|y|} \zeta_i^*(y) \, dy,$$

see, e.g. theorem 3.4 in [32]. Using the explicit form of the rearrangement in (3.8) and Fubini's theorem, we can rewrite the latter as

$$I_2 \leq \int_0^\infty \int_{B_{\ell(s)}(0)} \frac{1}{|y|} \, dy \, ds = 2\pi \int_0^\infty \ell(s) \, ds.$$

For any $S > 0$, this integral can be further estimated as

$$\begin{aligned} I_2 &\lesssim \int_0^S \ell(s) \, ds + \int_S^\infty \ell(s) \, ds \\ &\leq m_i \left(\frac{L}{2} \right)^{1/2} \int_0^S \frac{1}{s^{1/2}} \, ds + \|\omega_i\|_{L^p}^{p/2} \int_S^\infty \frac{1}{s^{p/2}} \, ds \\ &\lesssim S^{1/2} m_i \left(\frac{L}{2} \right)^{1/2} + \frac{1}{S^{(p-2)/2}} \|\omega_i\|_{L^p}^{p/2}. \end{aligned}$$

Optimizing in S yields (3.7). ■

Starting from this bound on the velocity there is an elementary calculation which shows that T must be bounded below.

Lemma 3.4 ([7]). *Suppose that (3.6) holds. If $\delta \gg R$, then $T \gtrsim 1$.*

As this estimate was already proved in lemma 11 of our previous paper, we skip its proof.

We have completed the argument in the case where (3.6) holds true.

In the *second case*, we assume that

$$m_i(t_*, 2R) = \varepsilon^\alpha \quad \text{for some } t_* \in (0, T), i \in \{1, \dots, N\}, \quad (3.9)$$

and we choose t_* minimal. Note that t_* is indeed positive since $m_i(0, R) = 0$ by our choice of R .

Instead of bounding T from below, it is enough to find a bound for the smaller t_* . We obtain this bound from an estimate on the outer vorticity $m_i(t_*, 2R)$, which in turn relies on an iterative procedure similar to one developed by Marchioro and co-authors, e.g. [5,33]. Differently from these works, however, we start from the hypothesis (2.11) that the components are initially contained in balls with a radius $R \sim 1$, while the argument in [5,33] heavily exploits their assumption that the initial vorticity is strongly concentrated in balls of radius ε . As already mentioned earlier, our analysis also extends to their situation and would produce strong concentration results also in the unbounded vorticity case.

Instead of estimating the outer vorticity function $m_i(t, \rho)$ directly, it is mathematically more convenient to study a smooth variant $\mu_i(t, \rho, \delta R)$, which is defined with the help of a suitable cut-off function. More precisely, we consider a radially symmetric smooth function $\psi = \psi_{\rho, \delta R}$

satisfying

$$\psi(x) = 1 \text{ if } |x| \leq \rho, \quad \psi(x) = 0 \text{ if } |x| > \rho + \delta R \quad (3.10)$$

and

$$|\nabla \psi| \lesssim \frac{1}{\delta R}, \quad |\nabla^2 \psi| \lesssim \frac{1}{(\delta R)^2}, \quad (3.11)$$

and then μ_i is given by

$$\mu_i(t, \rho, \delta R) = \frac{1}{a_i} \int_{\mathbb{R}^2} (1 - \psi_{\rho, \delta R}(x - X_i(t))) \omega_i(t, x) dx. \quad (3.12)$$

It is readily checked that m_i and μ_i satisfy the relation

$$m_i(t, \rho + \delta R) \leq \mu_i(t, \rho, \delta R) \leq m_i(t, \rho). \quad (3.13)$$

In a first step, we derive a differential inequality for μ_i . Here, we assume in addition that $T \lesssim 1$ in order to control the exponential factor in (3.5). Below in the proof of lemma 3.6, we will exploit the differential inequality in order to show that also the opposite bound holds true, $T \gtrsim 1$.

Lemma 3.5. *Suppose that $T \lesssim 1$. Let $i = 1, \dots, N$ and $\rho \in [R, 2R]$ be given. For any $\delta R \in (0, 1)$ and $t \in (0, T)$ it holds*

$$\mu_i(t, \rho, \delta R) \leq \mu_i(0, \rho, \delta R) + \kappa(\varepsilon, \delta R) \int_0^t \mu_i(s, \rho - \delta R, \delta R) ds, \quad (3.14)$$

where

$$\kappa(\varepsilon, \delta R) \lesssim \frac{\varepsilon^2}{(\delta R)^2} + \frac{1}{\delta R}.$$

Proof. As in the previous proofs, we assume without loss of generality that $\omega_i \geq 0$ and $a_i = 1$. Using the fact that ω_i solves the transport equation (2.5), and using the explicit formula (3.4) for the velocity of the centres, we compute via an integration by parts that

$$\begin{aligned} \frac{d}{dt} \mu_i(t, \rho, \delta R) &= \frac{d}{dt} X_i \cdot \int \nabla \psi(x - X_i) \omega_i(x) dx - \int \psi(x - X_i) \partial_t \omega_i dx \\ &= - \iint K(x - y) \cdot \nabla \psi(x - X_i) \omega_i(y) \omega_i(x) dy dx \\ &\quad + \iint (F_i(y) - F_i(x)) \cdot \nabla \psi(x - X_i) \omega_i(y) \omega_i(x) dy dx \\ &= I_1 + I_2. \end{aligned}$$

To simplify the notation slightly, upon a shift in the spatial variables, we will assume that $X_i = 0$ in what follows.

We start with an estimate of I_1 . Because of the symmetry property $K(z) = -K(-z)$, using Fubini we may write

$$I_1 = \frac{1}{2} \iint (\nabla \psi(y) - \nabla \psi(x)) \cdot K(x - y) \omega_i(y) \omega_i(x) dx dy.$$

In the view of the definition of the cut-off function, the integrand is non-zero only if x or y belong to the annulus $B_{\rho+\delta R} \setminus B_\rho$. We split the domain $\mathbb{R}^2 \times (B_{\rho+\delta R} \setminus B_\rho) \cup (B_{\rho+\delta R} \setminus B_\rho) \times \mathbb{R}^2$ that contains the support into the following parts:

$$\begin{aligned} A &= (B_{\rho+\delta R} \setminus B_\rho) \times B_{\rho/2}, \quad B = B_{\rho/2} \times (B_{\rho+\delta R} \setminus B_\rho), \\ C &= ((B_{\rho+\delta R} \setminus B_\rho) \times B_{\rho/2}^c) \cup (B_{\rho/2}^c \times (B_{\rho+\delta R} \setminus B_\rho)), \end{aligned}$$

and we denote the respective contributions of I_1 by I_1^A , I_1^B and I_1^C .

The contributions due to A and B are controlled in a very similar way. For instance, on the set A we know that $\nabla \psi(y) = 0$ and that $|x - y| \geq \rho/2$. In the view of the scaling of the gradient of the

cut-off function in (3.11), it thus follows that

$$|I_1^A| \lesssim \frac{1}{\rho} \frac{1}{\delta R} m_i(t, \rho) \lesssim \frac{1}{\delta R} m_i(t, \rho),$$

by the assumption that $\rho \geq R \sim 1$.

We concentrate now on the remaining set C . Here, we know that both $|x| \geq \rho/2$ and $|y| \geq \rho/2$. For the integrand to be non-zero, we need at least one of $|x|$ or $|y|$ to be larger than ρ . Using the Lipschitz condition on $\nabla \psi$ guaranteed by (3.11), the scaling of the Biot–Savart kernel, the bound on the outer vorticity which follows from the concentration estimate (3.5), and recalling that $\rho \geq R \sim 1$, we thus see that

$$\begin{aligned} |I_1^C| &\lesssim \frac{1}{(\delta R)^2} \iint_C |x - y| |K(x - y)| \omega_i(x) \omega_i(y) \, dx dy \\ &\lesssim \frac{1}{(\delta R)^2} m_i\left(t, \frac{\rho}{2}\right) m_i(t, \rho) \\ &\lesssim \frac{\varepsilon^2}{(\delta R)^2} m_i(t, \rho). \end{aligned}$$

By combining the previous bounds, we conclude that

$$|I_1| \lesssim \left(\frac{\varepsilon^2}{(\delta R)^2} + \frac{1}{\delta R} \right) m_i(t, \rho).$$

Now we turn to the estimate of I_2 . Using the Lipschitz property of the far field F_i on Ω_i guaranteed by lemma 3.1 and the properties of the cut-off function in (3.10) and (3.11), we obtain

$$|I_2| \lesssim \frac{1}{\delta R} \int_{B_{\rho+\delta R} \setminus B_\rho} \int |x - y| \omega_i(y) \, dy \omega_i(x) \, dx.$$

We now use the triangle and Jensen's inequalities and lemma 3.2 to infer

$$\begin{aligned} |I_2| &\lesssim \frac{1}{\delta R} \int_{B_{\rho+\delta R} \setminus B_\rho} |x| \omega_i(x) \, dx + \frac{1}{\delta R} m_i(t, \rho) \int |y| \omega_i(y) \, dy \\ &\lesssim \frac{\rho + \delta R}{\delta R} m_i(t, \rho) + \frac{\varepsilon}{\delta R} m_i(t, \rho) \\ &\lesssim \frac{1}{\delta R} m_i(t, \rho). \end{aligned}$$

Combining the estimates for I_1 and I_2 , integrating in time and using (3.13) yields the thesis. ■

The estimate in (3.14) is the basis for an iteration procedure, which finally yields the desired estimate on T via t_* .

Lemma 3.6. *Suppose that (3.9) holds. If $\varepsilon \ll 1$, then $T \gtrsim 1$.*

Proof. We argue by contradiction and suppose that for any $k, \ell \in \mathbb{N}$ there exists an $\varepsilon \leq 1/k$ such that $T \leq 1/\ell$. In particular, it then holds that $t_* \leq 1/\ell$.

Our proof is based on an iteration of estimate (3.14), in which we choose $\delta R = R/M$ for some integer M that we fix later. To simplify the notation in the following, we write $\mu_i(t, \rho) = \mu_i(t, \rho, R/M)$ in the following estimate.

We start by observing that an iteration over (3.14) yields

$$\begin{aligned} \mu_i(t, \left(2 - \frac{1}{M}\right) R) &\leq C(\varepsilon^2 M^2 + M) \int_0^t \mu_i\left(s, \left(2 - \frac{2}{M}\right) R\right) \, ds \\ &\leq C^{M-1} (\varepsilon^2 M^2 + M)^{M-1} \int_0^t \int_0^{t_1} \dots \int_0^{t_{M-2}} \mu_i(t_{M-1}, R) \, dt_{M-1} \dots dt_2 dt_1, \end{aligned}$$

because the initial value terms drop out thanks to (2.11). Replacing μ_i by the outer vorticities m_i with the help of (3.13), and using the concentration estimate in the form of (3.5), we thus obtain

$$m_i(t, 2R) \leq C^{M-1} (\varepsilon^2 M^2 + M)^{M-1} \frac{t^{M-1}}{(M-1)!} \varepsilon^2,$$

upon choosing C a bit larger if necessary. For t_* given by (3.9) and by the elementary Stirling-type formula $n^n < e^n n!$, the latter yields

$$\varepsilon^{(\alpha-2)/(M-1)} \lesssim \frac{\varepsilon^2 M^2 + M}{M-1} t_*,$$

or, equivalently,

$$\frac{\varepsilon^{(\alpha-2)/(M-1)}}{\varepsilon^2 M + 1} \lesssim t_*,$$

for M large enough. Choosing now M in the interval

$$1 + \frac{\alpha-2}{\log 2} \log \frac{1}{\varepsilon} \leq M \leq \frac{1}{\varepsilon^2}$$

gives on the one hand that $\varepsilon^2 M + 1 \leq 2$, and on the other hand it holds $\varepsilon^{(\alpha-2)/(M-1)} \geq \frac{1}{2}$. Noting that such an M exists provided that ε is small enough (or, equivalently, k large enough, both dependent on γ and p via α) gives that $t_* \geq c$ for some constant $c > 0$. This contradicts the upper bound $t_* \leq 1/\ell$ if $\ell > 1/c$. ■

Data accessibility. This article has no additional data.

Authors' contributions. S.C. and C.S.: investigation; writing—original draft; writing—review and editing. Both authors gave final approval for publication and agreed to be held accountable for the work performed therein.

Competing interests. We declare we have no competing interests.

Funding. This work is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under Germany's Excellence Strategy EXC 2044–390685587, Mathematics Münster: Dynamics Geometry Structure.

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