2D Turbulence spreading

Víctor Ballester

Supervisors: Alexandros Alexakis (ENS)
Emmanuel Dormy (ENS)

July XXXX, 2024

Introduction

The aim of this work is to understand how turbulence spreads in a 2D incompressible fluid.

This is motivated by the work of Alexakis 2023, where the author studied the spreading of turbulence in a long triply periodic domain. The natural questions that arise are:

- Do the vortices injected in the center of the domain spread til reaching the boundaries?
- If so, which profile does the energy and enstrophy distributions follow?

In this work, we tried to answer these questions in the 2D case.

On top of that, we carried out a numerical simulation of the point vortex model, in order to compare and observe some similarities between the two models.

Problem setup (Navier-Stokes)

We consider the 2D incompressible Navier-Stokes equations, which after adimensionalizing read:

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \boldsymbol{\nabla})\mathbf{u} = -\boldsymbol{\nabla} p + \frac{1}{\mathrm{Re}} \Delta \mathbf{u} + \mathbf{f}$$
$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

where $\mathbf{u}(\mathbf{x}, t)$ is the velocity field, $p(\mathbf{x}, t)$ is the pressure, Re is the Reynolds number, and $\mathbf{f}(\mathbf{x}, t)$ is an external force.

The force is localized in the center of the domain. Particularly, it injects vortices of sizes $\sim 1/k_\ell$ within a disk of size π/k_r. We do that in a way that the net momentum injected is zero.

The Reynolds number is defined in terms of the length scale $1/k_\ell$ and the injection rate controlled by the amplitude of the force.

Thus, there are three parameters that control the dynamics of the system: k_ℓ, k_r and Re. The following table summarizes the values used in the simulations:

k_r \ Re	0.25	0.5	1	2	4	8	16	32	64	128
8	${\color{green}\checkmark}_{512}$	${\color{green}\checkmark}_{512}$	${\color{green}\checkmark}_{512}$	${\color{green}\checkmark}{\color{blue}\checkmark}_{512}$	${\color{green}\checkmark}{\color{blue}\checkmark}_{1024}$	${\color{green}\checkmark}{\color{blue}\checkmark}_{1024}$	${\color{green}\checkmark}{\color{blue}\checkmark}_{1024}$	${\color{green}\checkmark}{\color{blue}\checkmark}_{2048}$	${\color{green}\checkmark}_{2048}$	${\color{green}\checkmark}_{4096}$
16				${\color{blue}\checkmark}_{1024}$	${\color{blue}\checkmark}_{2048}$	${\color{green}\checkmark}{\color{blue}\checkmark}_{2048}$	${\color{green}\checkmark}{\color{blue}\checkmark}_{2048}$	${\color{green}\checkmark}{\color{blue}\checkmark}_{2048}$	${\color{green}\checkmark}_{4096}$	${\color{green}\checkmark}_{4096}$
32				${\color{blue}\checkmark}_{2048}$	${\color{blue}\checkmark}_{4096}$	${\color{green}\checkmark}{\color{blue}\checkmark}_{4096}$	${\color{green}\checkmark}{\color{blue}\checkmark}_{4096}$	${\color{green}\checkmark}{\color{blue}\checkmark}_{4096}$	${\color{green}\checkmark}_{8192}$	${\color{green}\checkmark}_{8192}$
64						${\color{green}\checkmark}_{8192}$	${\color{green}\checkmark}_{8192}$	${\color{green}\checkmark}_{8192}$		

Table: Simulations carried out during the study. ${\color{green}\checkmark}=$ fully parallel simulations, ${\color{blue}\checkmark}=$ embarrassingly parallel simulations.

In all the simulations we set $k_\ell = 4k_r$.

The quantities monitored during the simulations are the following:

$$E_r = \sum_{r-\Delta r<\|\boldsymbol{x}\|\leq r} \|\boldsymbol{u}(\boldsymbol{x})\|^2 \qquad \Omega_r = \sum_{r-\Delta r<\|\boldsymbol{x}\|\leq r} |\omega(\boldsymbol{x})|^2$$

which account for the energy and enstrophy in annuli of radius r and width Δr.

We also defined a quantity that measures where most of the energy or enstrophy is located:

$$\mathcal{R}_E^2 = \frac{\sum_{\Delta r<r\leq\pi} r^2 E_r}{\sum_{\Delta r<r\leq\pi} E_r} \qquad \mathcal{R}_\Omega^2 = \frac{\sum_{\Delta r<r\leq\pi} r^2 \Omega_r}{\sum_{\Delta r<r\leq\pi} \Omega_r}$$

Problem setup (Point vortex)

We consider the most simple model of point vortices in 2D, which is based on the evolution by advection of a set of point where the vorticity ω is singular. In all the other points in $\mathbb{R}^2$, the vorticity is zero.

At all instants of time ω can be thought as a sum of Dirac deltas:

$$\omega(\mathbf{x}, t) = \sum_{i=1}^{N} \Gamma_i \delta(\mathbf{x} - \mathbf{x}_i(t))$$

where Γ_i is the circulation of the i-th vortex, and $\mathbf{x}_i(t)$ is its position.

A set of ODEs can be derived when imposed that they satisfy the 2D incompressible Euler equations:

$$\begin{aligned} \partial_t \omega + \boldsymbol{u} \cdot \boldsymbol{\nabla} \omega &= 0 \\ \boldsymbol{\nabla} \cdot \boldsymbol{u} &= 0 \end{aligned}$$

In the point vortex simulations, we input vortices in the center and remove them when they reach the boundaries.

For this model, we monitor the number of vortices N_r in annuli of radius r and width Δr, in order to later on give an evolution of the linear density of number of vortices in rings of radius r:

$$\rho_N(r) = \lim_{\Delta r \to 0} \frac{N_r}{2\pi r}$$

We also monitor an equivalent *mean radius* $\mathcal{R}_N$ weighted with the number of vortices in each annulus:

$$\mathcal{R}_N^2 = \frac{\sum_{\Delta r < r \leq \pi} r N_r}{\sum_{\Delta r < r \leq \pi} N_r}$$

Results

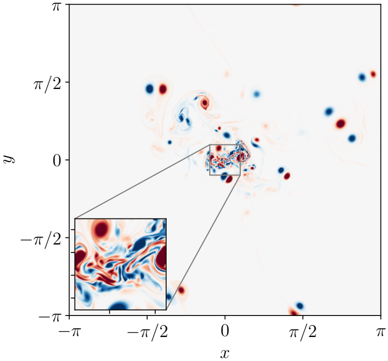


Figure: Navier-Stokes simulation with $k_r = 16$ and $\mathrm{Re} = 128$.

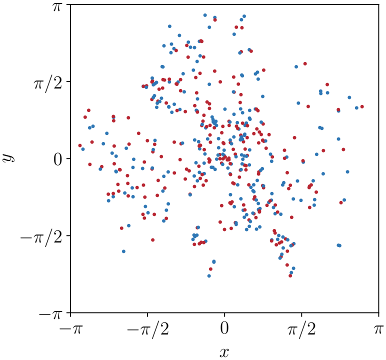
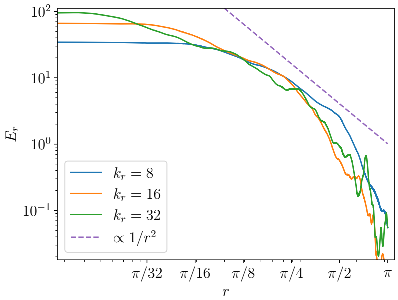
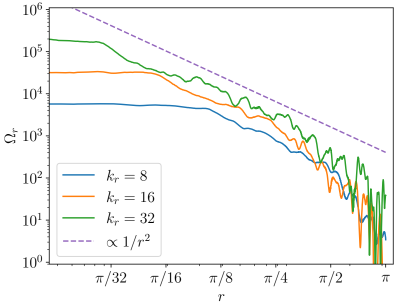


Figure: Point vortex simulation with $k_r = 16$.

Results (Navier-Stokes)



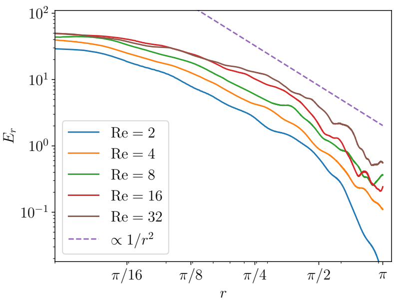
(a) Energy profile



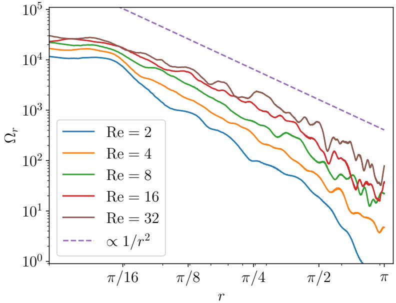
(b) Enstrophy profile

Figure: Energy and enstrophy profiles varying k_r at fixed time and Re $= 32$.

- Energy plots, smoother; enstrophy plots, more spiky.
- As k_r increases, the apparent power law A/r^2 for the enstrophy extends.



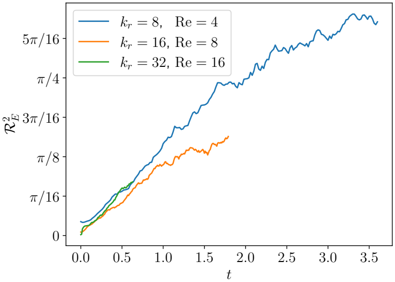
(a) Energy profile



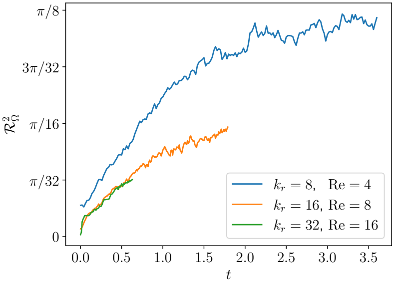
(b) Enstrophy profile

Figure: Energy and enstrophy profiles varying Re at fixed time and $k_r = 16$.

- Again, energy plots are smoother than enstrophy plots.
- As Re increases, the magnitudes of both quantities increase due to less dissipation acting on the system.



(a) Energy mean radius



(b) Enstrophy mean radius

Figure: Mean energy radius and mean enstrophy radius several runs varying simultaneously both k_r and Re.

- $\mathcal{R}_E^2$ and $\mathcal{R}_\Omega^2$ increase with time, which means that vortices spread far from the perturbation region.

Results (Point vortex)

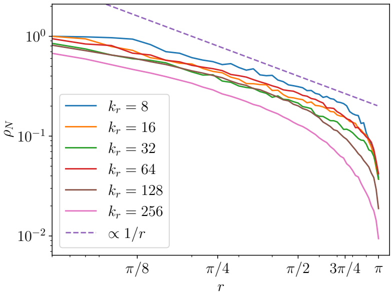


Figure: Density of the number of vortices as a function of the radius.

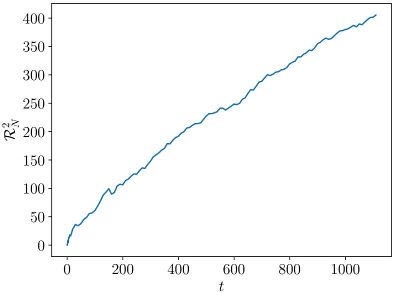


Figure: Mean radius weighted with the #vortices as a function of time.

- The density of vortices $\rho_N(r)$ following a power law A/r in the middle range of r implies that the flux of vortices on that region is almost constant.
- We observe an monotonous increase of $\mathcal{R}_N^2$ with time, which fits well with a linear function.

Conclusions

- Energy injected in the center seems to spread across the domain, regardless of the size of the perturbation region if the Reynolds number is high enough.
- The enstrophy seems to follow a power law A/r^2, that for large k_r and enough time, extends to the whole domain.
- When a stationary state is reached for the point vortex model, we observe a constant flux of vortices in the middle range of r.

Improvements to be done:

- Integrate the Navier-Stokes simulation further in time.
- Simulate the cases for $k_r = 64$ and higher Reynolds numbers.
- Replicate the results adding a drag term $-\alpha \boldsymbol{u}$ to the equations. Can we conclude the same?

Bibliography

Alexakis, A. (2023). "How far does turbulence spread?" In: *Journal of Fluid Mechanics* 977, R1. DOI: 10.1017/jfm.2023.951.