2D Turbulence spreading

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Introduction

Aim: Understand how turbulence spreads in a 2D incompressible fluid when we constantly inject vortices in the domain.

The natural questions that arise are:

- Do the vortices injected spread until reaching the boundaries?
- If so, which profile does the energy and enstrophy distributions follow?

Additionally, we worked on a point vortex model to compare and observe similarities between the two models.

Figure: Turbulence in the atmosphere

Problem setup (Navier-Stokes)

We consider the 2D incompressible Navier-Stokes equations which, after nondimensionalizing, read:

$$\partial_t \boldsymbol{u} + (\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{u} = -\boldsymbol{\nabla} p + \frac{1}{\mathrm{Re}}\Delta \boldsymbol{u} + \boldsymbol{f}$$

$$\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$$

The force is localized in the center of the domain.

- It injects vortices of size $\sim 1/k_\ell$.
- The injection region is a disk of size π/k_r.

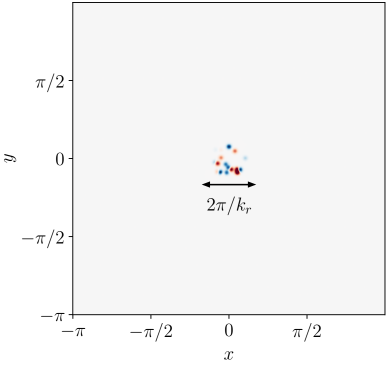


Figure: Forcing region for $k_r = 8$

Thus, there are three parameters that control the dynamics of the system: k_ℓ, k_r and Re.

Goal: Understand the behavior of the system in the limit $\mathrm{Re} \to \infty$ and infinite domain ($k_r \to \infty$).

We carried out two types of simulations:

- Fully parallel simulations: The domain is split among the processors.
- Embarrassingly parallel simulations: Several simulations are run simultaneously, each one in a different processor, in order to produce statistics.

The following table summarizes the values used in the simulations:

k_r \ Re	0.25	0.5	1	2	4	8	16	32	64	128
8	${\color{green}\checkmark}_{512}$	${\color{green}\checkmark}_{512}$	${\color{green}\checkmark}_{512}$	${\color{green}\checkmark}{\color{blue}\checkmark}_{512}$	${\color{green}\checkmark}{\color{blue}\checkmark}_{1024}$	${\color{green}\checkmark}{\color{blue}\checkmark}_{1024}$	${\color{green}\checkmark}{\color{blue}\checkmark}_{1024}$	${\color{green}\checkmark}{\color{blue}\checkmark}_{2048}$	${\color{green}\checkmark}_{2048}$	${\color{green}\checkmark}_{4096}$
16				${\color{blue}\checkmark}_{1024}$	${\color{blue}\checkmark}_{2048}$	${\color{green}\checkmark}{\color{blue}\checkmark}_{2048}$	${\color{green}\checkmark}{\color{blue}\checkmark}_{2048}$	${\color{green}\checkmark}{\color{blue}\checkmark}_{2048}$	${\color{green}\checkmark}_{4096}$	${\color{green}\checkmark}_{4096}$
32				${\color{blue}\checkmark}_{2048}$	${\color{blue}\checkmark}_{4096}$	${\color{green}\checkmark}{\color{blue}\checkmark}_{4096}$	${\color{green}\checkmark}{\color{blue}\checkmark}_{4096}$	${\color{green}\checkmark}{\color{blue}\checkmark}_{4096}$	${\color{green}\checkmark}_{8192}$	${\color{green}\checkmark}_{8192}$
64						${\color{green}\checkmark}_{8192}$	${\color{green}\checkmark}_{8192}$	${\color{green}\checkmark}_{8192}$		

Table: Simulations carried out during the study. ${\color{green}\checkmark}=$ fully parallel simulations, ${\color{blue}\checkmark}=$ embarrassingly parallel simulations.

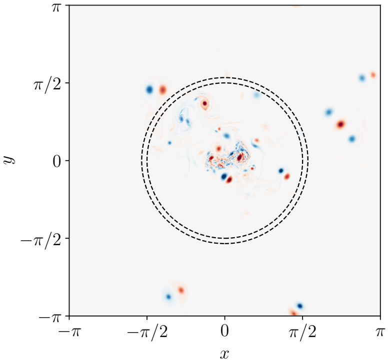
In all the simulations we set $k_\ell = 4k_r$.

The quantities monitored during the simulations are the following:

$$E_r = \sum_{r-\Delta r<\|\boldsymbol{x}\|\leq r} \|\boldsymbol{u}(\boldsymbol{x})\|^2$$

$$\Omega_r = \sum_{r-\Delta r<\|\boldsymbol{x}\|\leq r} |\omega(\boldsymbol{x})|^2$$

which account for the energy and enstrophy in annuli of radius r and width Δr.



We also defined quantities that measures where most of the energy or enstrophy is located:

$$\mathcal{R}_E^2 = \frac{\sum_{\Delta r<r\leq\pi} r^2 E_r}{\sum_{\Delta r<r\leq\pi} E_r} \qquad \mathcal{R}_\Omega^2 = \frac{\sum_{\Delta r<r\leq\pi} r^2 \Omega_r}{\sum_{\Delta r<r\leq\pi} \Omega_r}$$

We call them *mean energy radius* and *mean enstrophy radius*, respectively.

Problem setup (Point vortex)

We consider the simplest model of point vortices in 2D, which is based on the evolution by advection of a set of points where the vorticity ω is singular. At all other points in $\mathbb{R}^2$, the vorticity is zero.

At all instants of time ω can be thought as a sum of Dirac deltas:

$$\omega(\boldsymbol{x}, t) = \sum_{i=1}^{N} \Gamma_i \delta(\boldsymbol{x} - \boldsymbol{x}_i(t))$$

where Γ_i is the circulation of the i-th vortex, and $\boldsymbol{x}_i(t)$ is its position.

A set of ODEs can be derived for $\boldsymbol{x}_i(t)$ by imposing that they satisfy the 2D incompressible Euler equations:

$$\begin{aligned} \partial_t \omega + \boldsymbol{u} \cdot \boldsymbol{\nabla} \omega &= 0 \\ \boldsymbol{\nabla} \cdot \boldsymbol{u} &= 0 \end{aligned}$$

In the point vortex simulations, we input vortices in the center and remove them when they reach the boundaries.

For this model, we monitor the number of vortices N_r in annuli of radius r and width Δr, in order to later on give an evolution of the density of number of vortices in rings of radius r:

$$\rho_r = \lim_{\Delta r \to 0} \frac{N_r}{2\pi r \Delta r}$$

We also monitor an equivalent *mean radius* $\mathcal{R}_N$ weighted with the number of vortices in each annulus:

$$\mathcal{R}_N^2 = \frac{\sum_{\Delta r < r \leq \pi} r^2 N_r}{\sum_{\Delta r < r \leq \pi} N_r}$$

Results

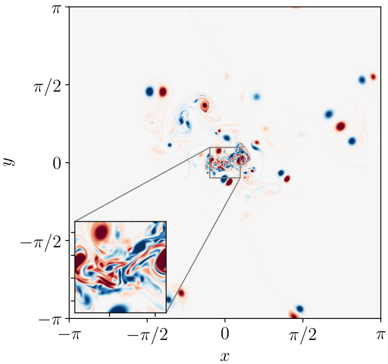


Figure: Navier-Stokes simulation with $k_r = 16$ and $\mathrm{Re} = 128$.

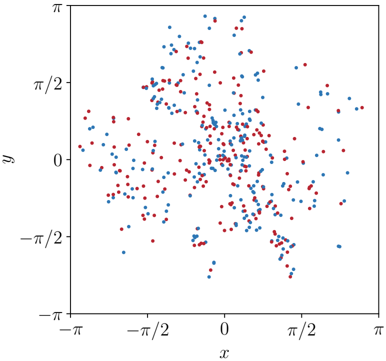
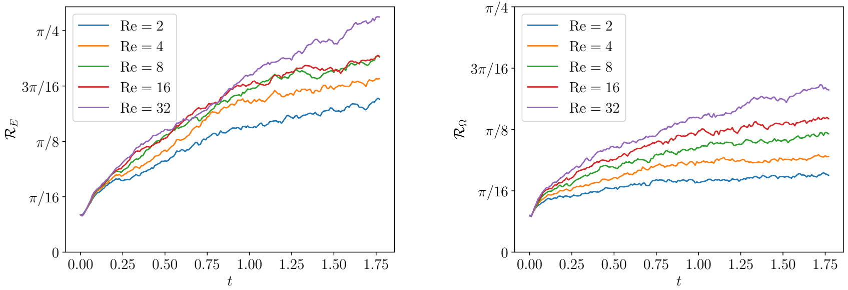


Figure: Point vortex simulation with $k_r = 16$.

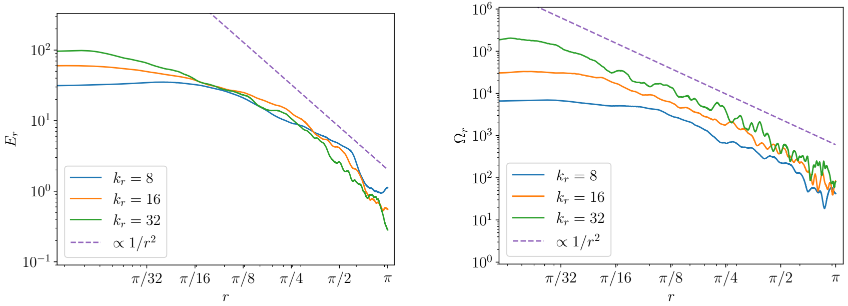
Results (Navier-Stokes)



(a) Energy mean radius (b) Enstrophy mean radius

Figure: Mean energy radius and mean enstrophy radius for several runs varying Re at $k_r = 16$.

- $\mathcal{R}_E$ and $\mathcal{R}_\Omega$ increase with time, which means that vortices spread far from the perturbation region.
- The higher the Reynolds number, the faster the spreading.

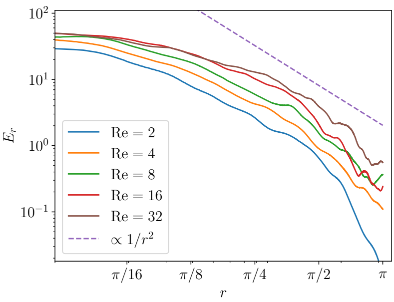


(a) Energy profile

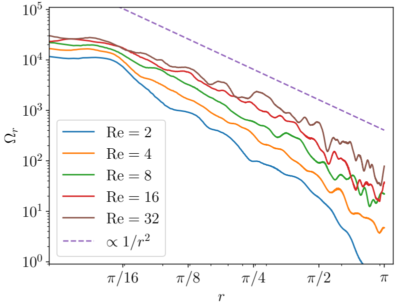
(b) Enstrophy profile

Figure: Energy and enstrophy profiles varying k_r at $t = 1.66$ and $\mathrm{Re} = 32$.

- Energy plots are smoother, while enstrophy plots are more spiky.
- As k_r increases, the apparent power law A/r^2 for the enstrophy extends.



(a) Energy profile



(b) Enstrophy profile

Figure: Energy and enstrophy profiles varying Re at $t = 1.75$ and $k_r = 16$.

- Again, energy plots are smoother than enstrophy plots.
- As Re increases, the magnitudes of both quantities increase due to less dissipation acting on the system.

Results (Point vortex)

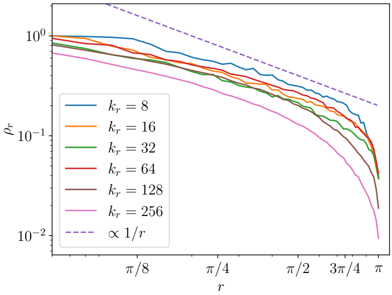


Figure: Density of the number of vortices as a function of the radius.

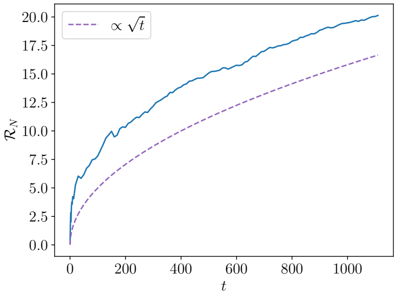


Figure: Mean radius weighted with the number of vortices.

- The density of vortices ρ_r following a power law A/r in the middle range of r implies that the flux of vortices on that region is almost constant.
- We observe a monotonous increase of $\mathcal{R}_N$ with time, which resembles the Navier-Stokes case.

Conclusions

- Energy injected in the center seems to spread across the domain, unlike the 3D case.
- The enstrophy seems to follow a power law A/r^2, that for large k_r, extends to the whole domain.
- When a stationary state is reached for the point vortex model, we observe a constant flux of vortices in the middle range of r.

Future improvements:

- Extend the integration time of the Navier-Stokes simulations.
- Simulate cases for $k_r = 64$ and higher Reynolds numbers.
- Replicate results adding a drag term $-\alpha \boldsymbol{u}$ to the equations. Can we conclude the same?