Exercise 19

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Exercice 1. *Show that the box scheme*

$$\frac{1}{2k}\left[v_m^{n+1}+v_{m+1}^{n+1}-v_m^n-v_{m+1}^n\right]+\frac{a}{2h}\left[v_{m+1}^{n+1}-v_m^{n+1}+v_{m+1}^n-v_m^n\right]=0$$

for the homogeneous one-way wave equation $u_t+au_x=0$ *is accurate of order [2,3].*

Resolution. We need to show that for $k=\lambda h$ we have:

$$P_{k,h}\phi-P\phi=P_{k,h}\phi=\mathrm{O}(k^2)+\mathrm{O}(h^3)$$

because in this case we have $P\phi=\phi_t+a\phi_x=0$. Here $P_{k,h}$ is the operator of the box scheme. We have that:

$$\begin{aligned}P_{k,h}\phi=\frac{1}{2k}&\left[\phi(t+k,x)+\phi(t+k,x+h)-\phi(t,x)-\phi(t,x+h)\right]+\\&+\frac{a}{2h}\left[\phi(t+k,x+h)-\phi(t+k,x)+\phi(t,x+h)-\phi(t,x)\right]\end{aligned}$$

Now, Taylor-expanding the equation in k and h centered at $\phi=\phi(t,x)$, and omitting the evaluation of the functions at (t,x) to simplify the reading, we have:

$$\begin{aligned}P_{k,h}\phi=\frac{1}{2k}\Bigg[&\phi+k\phi_t+\frac{k^2}{2}\phi_{tt}+\frac{k^3}{6}\phi_{ttt}+\\&+\phi+k\phi_t+h\phi_x+\frac{k^2}{2}\phi_{tt}+\frac{h^2}{2}\phi_{xx}+kh\phi_{tx}+\frac{k^3}{6}\phi_{ttt}+\frac{k^2h}{2}\phi_{ttx}+\frac{kh^2}{2}\phi_{txx}+\frac{h^3}{6}\phi_{xxx}\\&-\phi\\&-\phi-h\phi_x-\frac{h^2}{2}\phi_{xx}-\frac{h^3}{6}\phi_{xxx}\\&+\mathrm{O}(k^4)+\mathrm{O}(k^3h)+\mathrm{O}(k^2h^2)+(kh^3)+\mathrm{O}(h^4)\Bigg]\\+\frac{a}{2h}\Bigg[&\phi+k\phi_t+h\phi_x+\frac{k^2}{2}\phi_{tt}+\frac{h^2}{2}\phi_{xx}+kh\phi_{tx}+\frac{k^3}{6}\phi_{ttt}+\frac{k^2h}{2}\phi_{ttx}+\frac{kh^2}{2}\phi_{txx}+\frac{h^3}{6}\phi_{xxx}\\&-\phi-k\phi_t-\frac{k^2}{2}\phi_{tt}-\frac{k^3}{6}\phi_{ttt}\\&+\phi+h\phi_x+\frac{h^2}{2}\phi_{xx}+\frac{h^3}{6}\phi_{xxx}\\&-\phi\\&+\mathrm{O}(k^4)+\mathrm{O}(k^3h)+\mathrm{O}(k^2h^2)+(kh^3)+\mathrm{O}(h^4)\Bigg]\end{aligned}$$

Which simplifies to:

$$\begin{aligned}P_{k,h}\phi=\phi_t&+\frac{k}{2}\phi_{tt}+\frac{k^2}{6}\phi_{ttt}+\frac{h}{2}\phi_{tx}+\frac{kh}{4}\phi_{ttx}+\frac{h^2}{4}\phi_{txx}\\&+a\left(\phi_x+\frac{h}{2}\phi_{xx}+\frac{h^2}{6}\phi_{xxx}+\frac{k}{2}\phi_{tx}+\frac{k^2}{4}\phi_{ttx}+\frac{kh}{4}\phi_{txx}\right)\\&+\mathrm{O}(k^3)+\mathrm{O}(k^2h)+\mathrm{O}(kh^2)+\mathrm{O}(h^3)+\mathrm{O}(h^4/k)+\mathrm{O}(k^4/h)\end{aligned}$$

Now using that $\phi_t + a\phi_x = 0$, and so $\phi_{tt} + a\phi_{tx} = 0$, $\phi_{tx} + a\phi_{xx} = 0$ and $\phi_{ttx} + a\phi_{txx} = 0$, we have:

$$P_{k,h}\phi = \frac{k^2}{6}\phi_{ttt} + \frac{h^2}{4}\phi_{txx} + a\frac{h^2}{6}\phi_{xxx} + a\frac{k^2}{4}\phi_{ttx} + \mathrm{O}(k^3) + \mathrm{O}(k^2h) + \mathrm{O}(kh^2) + \mathrm{O}(h^3) + \mathrm{O}(h^4/k) + \mathrm{O}(k^4/h)$$

Now using that $a\phi_{xxx} = -\phi_{txx}$ and $a\phi_{ttx} = -\phi_{ttt}$, we have:

$$P_{k,h}\phi = -\frac{k^2}{12}\phi_{ttt} + \frac{h^2}{12}\phi_{txx} + \mathrm{O}(k^3) + \mathrm{O}(k^2h) + \mathrm{O}(kh^2) + \mathrm{O}(h^3) + \mathrm{O}(h^4/k) + \mathrm{O}(k^4/h)$$

Finally, letting $k = \lambda h$ we have:

$$P_{k,h}\phi = -\frac{k^2}{12}\phi_{ttt} + \frac{k^2}{12\lambda^2}\phi_{txx} + \mathrm{O}(h^3) = \mathrm{O}(k^2) + \mathrm{O}(h^3)$$