Octave problems

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Integració numèrica d'equacions en derivades parcials
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Exercise 3. *For values of $x \in [-1, 3]$ and $t \in [0, 2.4]$, compare the following numerical schemes for the one-dimensional wave equation*

$$u_t + u_x = 0$$

with initial condition

$$u(0, x) = \begin{cases} \cos(\pi x)^2 & |x| \leq \frac{1}{2} \\ 0 & \textit{otherwise} \end{cases}$$

and boundary condition $u(t, -1) = 0$. Use spatial time steps of $h = 1/10$, $h = 1/20$, and $h = 1/40$.

a. *Forward-time, backward-space (FTBS) with $\lambda = 0.8$.*

b. *Forward-time, centered-space (FTCS) with $\lambda = 0.8$.*

c. *Lax-Friedrichs (LF) with $\lambda = 0.8$ and $\lambda = 1.6$.*

d. *Leapfrog (L) with $\lambda = 0.8$ and using the forward-time, centered-space scheme for the first step.*

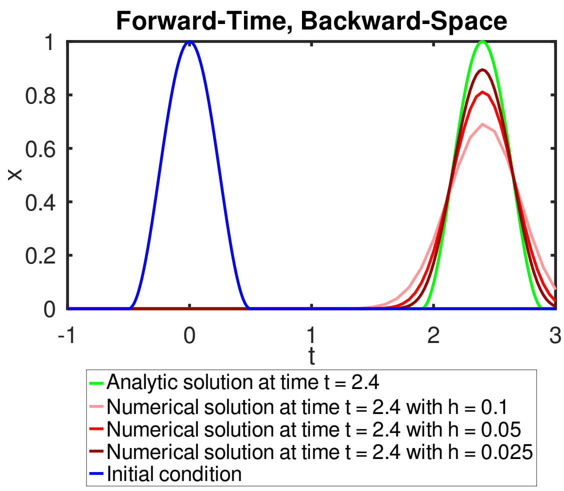
For the schemes b, c, and d, use the numerical boundary condition $v_M^{n+1} = v_{M-1}^{n+1}$.

Resolution. In the next table we expose the error in L^2 norm of all the experiments that we have done.

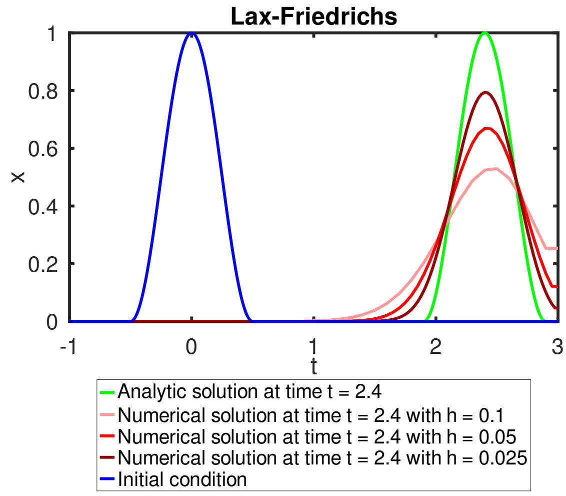
Scheme	$h = 1/10$	$h = 1/20$	$h = 1/40$
FTBS ($\lambda = 0.8$)	0.192501	0.119992	0.068666
FTCS ($\lambda = 0.8$)	26.0543	4337.15	1.63829×10^9
LF ($\lambda = 0.8$)	0.293134	0.206742	0.130863
LF ($\lambda = 1.6$)	22.6263	2274.83	4.01576×10^8
L ($\lambda = 0.8$)	0.163822	0.0417945	0.0114763

Taula 1: Error in L^2 norm for the different schemes.

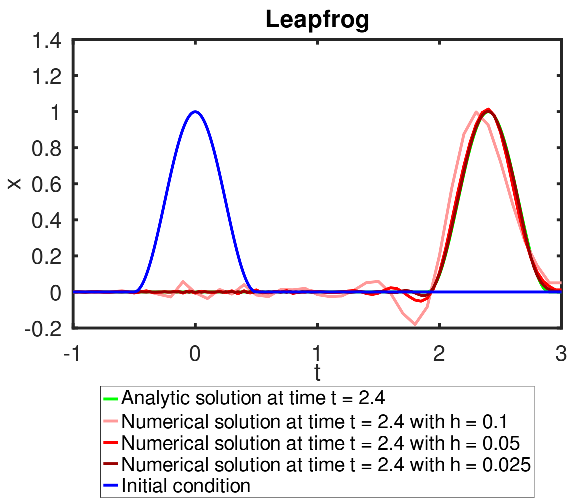
From here we can conclude that the useful schemes are the FTBS with $\lambda = 0.8$, the Lax-Friedrichs with $\lambda = 0.8$ and the Leapfrog with $\lambda = 0.8$. And the other ones are useless as the error seams to approach to infinity as we decrease the step size h. In Fig. 1 we show the solutions of the three convergent methods with the three spatial steps mentioned above.



(a) FTBS with $\lambda = 0.8$.



(b) Lax-Friedrichs with $\lambda = 0.8$.



(c) Leapfrog with $\lambda = 0.8$.

Figura 1: Plot of the analytical and numerical solutions of the convergent schemes

We can do now a further analysis of these three schemes. In the following table we summarize the error and order of convergence of these schemes as we halve the step size h.

h	FTBS			Lax-Friedrichs			Leapfrog		
	Error	Rate	Order	Error	Rate	Order	Error	Rate	Order
1/10	0.192501	-	-	0.293134	-	-	0.163822	-	-
1/20	0.119992	1.60429	0.681933	0.206742	1.41787	0.50373	0.0417945	3.9197	1.97074
1/40	0.068666	1.74747	0.805267	0.130863	1.57984	0.659776	0.0114763	3.64181	1.86466
1/80	0.0371725	1.84723	0.88536	0.0758213	1.72593	0.787377	0.00324644	3.53504	1.82172
1/160	0.0194683	1.90939	0.93311	0.0413688	1.83281	0.87406	0.000942768	3.44352	1.78388
1/320	0.0100036	1.94612	0.960603	0.0217645	1.90075	0.926566	0.00027973	3.37028	1.75287

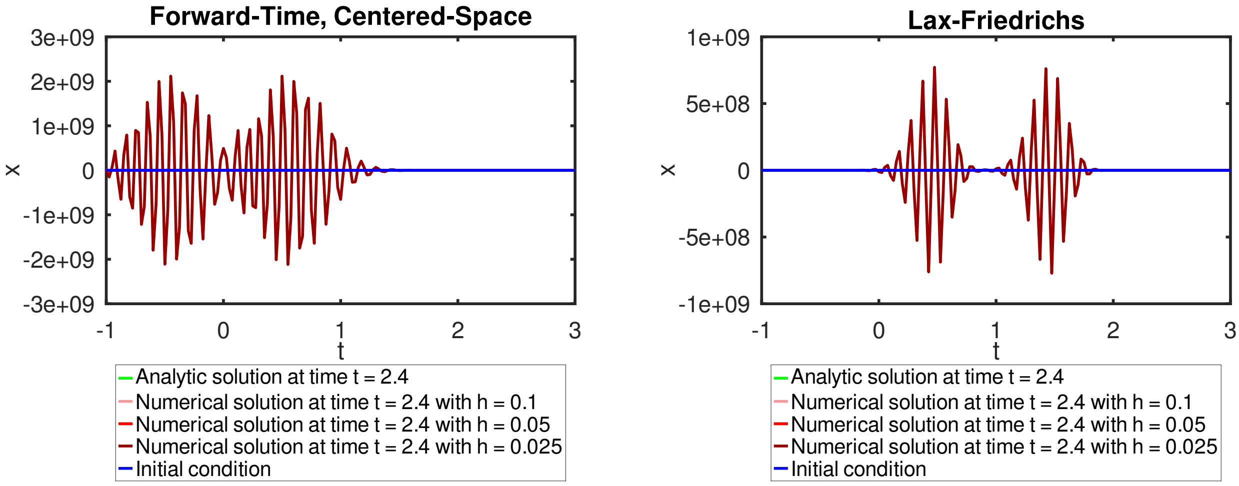
Taula 2: Errors in norm L^2 and orders for the three schemes.

The values in both tables were obtained executing:

```
getResults
```

Observe that the forward-time, backward-space and the Lax-Friedrichs schemes converge as approximately a first-order scheme, whereas the Leapfrog method as a second-order scheme. An interesting fact to note here is that the Leapfrog *loses* convergency as we decrease the stepsize. This is not strange, as it may happen that the numerical order of convergency differ from the theoretical order of convergency, because of propagation of errors.

We add here the plots of the divergent schmes. Observe the oscillating pattern.



(a) FTCS with $\lambda = 0.8$. (b) Lax-Friedrichs with $\lambda = 1.6$.

Figura 2: Plot of the analytical and numerical solutions of the divergent schemes

Exercise 15. *Solve* $u_t + u_x = 0$, $x \in [-1, 1]$, $t \in [0, 1.2]$, *with initial condition* $u(0, x) = \sin(2\pi x)$ *and periodicity* $u(t, -1) = u(t, 1)$. *Use the following schemes:*

a. *Forward-time, backward-space (FTBS) with* $\lambda = 0.8$.

b. *Lax-Wendroff (LW) with* $\lambda = 0.8$.

Demonstrate the first-order accuracy of the FTBS scheme and the second-order accuracy of the LW scheme using time steps of $h = 1/10$, $h = 1/20$, $h = 1/40$ *and* $h = 1/80$. *Do it in both the norm* L^2 *and the norm* L^∞.

Resolution. The next tables summarize all the experiments that we have done. Note that we obtain a first order of approximation with the FTBS scheme and a second order of approximation with the Lax-Wendroff scheme, in both L^2 and L^∞ norms.

	Forward-time, backward-space					
	L^2			L^∞		
h	Error	Rate	Order	Error	Rate	Order
1/10	0.379872	-	-	0.371158	-	-
1/20	0.211259	1.79813	0.846501	0.371158	1.75963	0.815271
1/40	0.111737	1.89068	0.918907	0.21093	1.88856	0.917286
1/80	0.0575043	1.94311	0.958366	0.111688	1.94248	0.957901

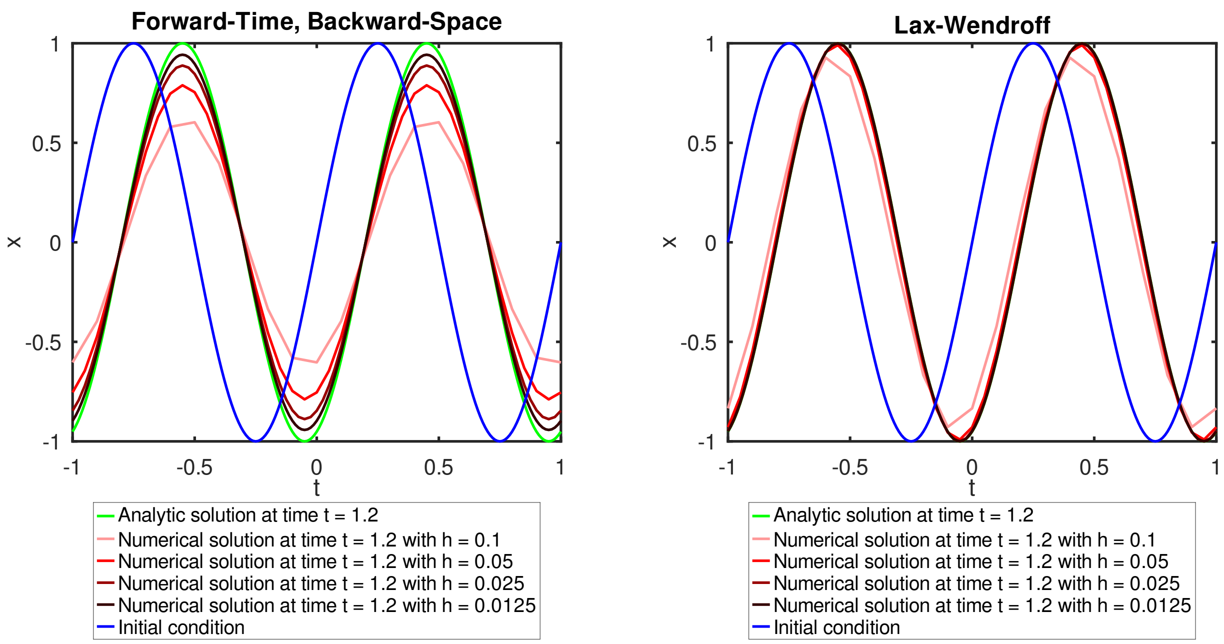
Taula 3: Errors and orders for the FTBS scheme.

	Lax-Wendroff					
	L^2			L^∞		
h	Error	Rate	Order	Error	Rate	Order
1/10	0.169042	-	-	0.166403	-	-
1/20	0.0441959	3.82482	1.93539	0.166403	3.78585	1.92062
1/40	0.0111394	3.96753	1.98824	0.0439539	3.95231	1.9827
1/80	0.00278932	3.9936	1.99769	0.0111211	3.98882	1.99596

Taula 4: Errors and orders for the Lax-Wendroff scheme.

These values were obtained executing:

```
getResults
```



(a) FTBS with $\lambda = 0.8$.

(b) Lax-Wendroff with $\lambda = 0.8$.

Figura 3: Plot of the analytical and numerical solutions of the FTBS and Lax-Wendroff schemes

To get the previous plots, execute the following commands:

```
a=1;x0=-1;x1=1;t1=1.2;lamb=0.8;myplot(a,lamb,0.01,x0,x1,t1,@ftbs,"Forward-Time, Backward-Space");
a=1;x0=-1;x1=1;t1=1.2;lamb=0.8;myplot(a,lamb,0.01,x0,x1,t1,@lw,"Lax-Wendroff");
```

Exercise 28. *For a Dirichlet problem in the unit square with homogeneous boundary conditions, experimentally verify that:*

a. *The second-order finite difference discretization of the Poisson equation leads to a globally convergent scheme with quadratic error in the step size.*

b. *The discretization using the 9-point Laplacian (problem 26) leads to a globally convergent scheme with quartic error in the step size.*

Do it in a computationally inefficient way: construct the system matrix (`help ones`, `help diag`) and solve it using `A\b`, assuming that A is the matrix and b is the right-hand side. What are the reasonable dimensions that you can reach?

Resolution. Recall that the two schemes for the equation $\Delta u = f$ that we need to study are:

$$v_{i+1,j} + v_{i-1,j} + v_{i,j+1} + v_{i,j-1} - 4v_{i,j} = h^2 f_{i,j}$$

$$v_{i+1,j+1} + v_{i+1,j-1} + v_{i-1,j+1} + v_{i-1,j-1} + 4(v_{i+1,j} + v_{i-1,j} + v_{i,j+1} + v_{i,j-1}) - 20v_{i,j} = \\ = \frac{h^2}{2}\left(f_{i+1,j} + f_{i-1,j} + f_{i,j+1} + f_{i,j-1} + 8f_{i,j}\right)$$

In order to compare the numerical solutions with the real one, we will do the study with the function

$$u = xy(1-x)(1-y)\mathrm{e}^{x+y}$$

which is $\mathcal{C}^\infty$ and satisfies the homogeneous boundary condition $u = 0$ in $[0,1]^2$. Moreover, $f = \Delta u = 2xy\mathrm{e}^{x+y}(xy + x + y - 3)$.

The following table summarizes the results of the experiments, which can be obtained by running:

```
getResults(1/10, 6)
```

h	Matrix dim	5-point Laplacian			9-point Laplacian		
		Error L^2	Rate	Order	Error L^2	Rate	Order
1/10	81×81	$1.53509 \cdot 10^{-3}$	-	-	$1.51386 \cdot 10^{-5}$	-	-
1/20	361×361	$5.43297 \cdot 10^{-4}$	2.82551	1.49851	$1.33873 \cdot 10^{-6}$	11.3082	3.4993
1/40	1521×1521	$1.92128 \cdot 10^{-4}$	2.82778	1.49967	$1.18333 \cdot 10^{-7}$	11.3132	3.49994
1/80	6241×6241	$6.79312 \cdot 10^{-5}$	2.82827	1.49992	$1.04594 \cdot 10^{-8}$	11.3136	3.49999
1/160	25281×25281	$2.40176 \cdot 10^{-5}$	2.82839	1.49998	$9.24487 \cdot 10^{-10}$	11.3137	3.5
1/320	101761×101761	$8.49155 \cdot 10^{-6}$	2.82842	1.5	$8.17052 \cdot 10^{-11}$	11.3149	3.50015

Taula 5: Error in L^2 norm for the different schemes.

The time employed in the calculation of all the six solutions using the 5-point Laplacian was 1.71633 seconds, while the time for the 9-point Laplacian was 3.27701 seconds. Note that we defined the matrices as `sparse`, because otherwise we couldn't have achieved such fast times.

Another important fact to note is that we have not achieved the theoretical orders of convergency of 2 and 4, respectively, but 1.5 and 3.5 instead, probably due to propagation of errors.