

Exercise 19

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Integració numèrica d'equacions en derivades parcials

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Exercise 1. *Show that the box scheme*

$$\frac{1}{2k} [v_m^{n+1} + v_{m+1}^{n+1} - v_m^n - v_{m+1}^n] + \frac{a}{2h} [v_{m+1}^{n+1} - v_m^{n+1} + v_{m+1}^n - v_m^n] = 0 \quad (1)$$

for the homogeneous one-way wave equation $u_t + au_x = 0$ is accurate of order $[2, 3]$.

Resolution. We need to check that:

$$\left| \frac{e^{kq(\xi)} - g(h\xi)}{k} \right| \leq Ch^r |\xi|^\rho \quad (2)$$

for some constant $C \in \mathbb{R}$ and with $r = 2$ and $\rho = 3$. Here g is the amplification factor and q is such that:

$$\widehat{u}_t = q(\xi) \widehat{u}$$

Let's compute first the amplification factor g . Let $v_m^n = g^n e^{im\theta}$. Then, substituting this into (1) and using that $\lambda = k/h$, we get the following equation:

$$[g + ge^{i\theta} - 1 - e^{i\theta}] + a\lambda [ge^{i\theta} - g + e^{i\theta} - 1] = 0$$

Factoring this equation we have:

$$\begin{aligned} (g-1)(1+e^{i\theta}) &= -a\lambda(g+1)(e^{i\theta}-1) \\ (g-1) \frac{e^{i\theta/2} + e^{-i\theta/2}}{2} &= -a\lambda i(g+1) \frac{e^{i\theta/2} - e^{-i\theta/2}}{2i} \\ (g-1) \cos(\theta/2) &= -a\lambda i(g+1) \sin(\theta/2) \\ g-1 &= -a\lambda i g \tan(\theta/2) - a\lambda i \tan(\theta/2) \\ g &= \frac{1 - a\lambda i \tan(\theta/2)}{1 + a\lambda i \tan(\theta/2)} \end{aligned}$$

Now let's find q . Taking the Fourier transform (on the x variable) of the equation $u_t + au_x = 0$ we have:

$$\widehat{u}_t = \widehat{u}_t = -a\widehat{u}_x = -ai\xi \widehat{u}$$

where the last equality follows from the identity $\widehat{u}_x(\xi) = i\xi \widehat{u}(\xi)$. So $q(\xi) = -ai\xi$. Let's study first the Taylor expansion of $g(h\xi)$. Recall that $\tan(x) = x + \frac{x^3}{3} + O(x^5)$, so:

$$\begin{aligned} 1 - a\lambda i \tan(x) &= 1 - a\lambda i x - \frac{a\lambda i}{3} x^3 + O(x^5) \\ \frac{1}{1 + a\lambda i \tan(x)} &= 1 - (a\lambda i \tan(x)) + (a\lambda i \tan(x))^2 - (a\lambda i \tan(x))^3 + \dots \\ &= 1 - a\lambda i x - a^2 \lambda^2 x^2 + a\lambda i \frac{3a^2 \lambda^2 - 1}{3} x^3 + O(x^4) \\ \frac{1 - a\lambda i \tan(x)}{1 + a\lambda i \tan(x)} &= \left[1 - a\lambda i x - \frac{a\lambda i}{3} x^3 + O(x^5) \right] \left[1 - a\lambda i x - a^2 \lambda^2 x^2 + a\lambda i \frac{3a^2 \lambda^2 - 1}{3} x^3 + O(x^4) \right] \\ &= 1 - 2a\lambda i x - 2a^2 \lambda^2 x^2 + 2a\lambda i \frac{3a^2 \lambda^2 - 1}{3} x^3 + O(x^4) \end{aligned}$$

Substituting $x = h\xi/2$ in the latter expression we have:

$$g(h\xi) = 1 - a\lambda i h\xi - \frac{a^2\lambda^2}{2}h^2\xi^2 + a\lambda i \frac{3a^2\lambda^2 - 1}{12}h^3\xi^3 + O(h^4)$$

On the other hand the expansion of $e^{kq(\xi)} = e^{-a\lambda i h\xi}$ is:

$$e^{-a\lambda i h\xi} = 1 - a\lambda i h\xi - \frac{a^2\lambda^2}{2}h^2\xi^2 + \frac{a^3\lambda^3 i}{6}h^3\xi^3 + O(h^4)$$

Thus:

$$\left| \frac{e^{kq(\xi)} - g(h\xi)}{k} \right| = \left| \frac{\frac{1}{12}a\lambda i(1 - a^2\lambda^2)h^3\xi^3 + O(h^4)}{\lambda h} \right| \leq \frac{1}{12}|a||1 - a^2\lambda^2|h^2|\xi|^3 + O(h^3)$$

So we take $C := \frac{1}{12}|a||1 - a^2\lambda^2|$.