Exercise 19

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Integració numèrica d'equacions en derivades parcials
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Exercice 1. *Show that the box scheme*

$$\frac{1}{2k}\left[v_m^{n+1}+v_{m+1}^{n+1}-v_m^n-v_{m+1}^n\right]+\frac{a}{2h}\left[v_{m+1}^{n+1}-v_m^{n+1}+v_{m+1}^n-v_m^n\right]=0 \tag{1}$$

for the homogeneous one-way wave equation $u_t+au_x=0$ is accurate of order [2,3].

Resolution. We need to check that:

$$\left|\frac{\mathrm{e}^{kq(\xi)}-g(h\xi)}{k}\right|\leq Ch^r|\xi|^\rho \tag{2}$$

for some constant $C\in\mathbb{R}$ and with $r=2$ and $\rho=3$. Here g is the amplification factor and q is such that:

$$\widehat{u}_t=q(\xi)\widehat{u}$$

Let's compute first the amplification factor g. Let $v_m^n=g^n\mathrm{e}^{\mathrm{i}m\theta}$. Then, substituting this into (1) and using that $\lambda=k/h$, we get the following equation:

$$\left[g+g\mathrm{e}^{\mathrm{i}\theta}-1-\mathrm{e}^{\mathrm{i}\theta}\right]+a\lambda\left[g\mathrm{e}^{\mathrm{i}\theta}-g+\mathrm{e}^{\mathrm{i}\theta}-1\right]=0$$

Factoring this equation we have:

$$\begin{aligned}
(g-1)(1+\mathrm{e}^{\mathrm{i}\theta})&=-a\lambda(g+1)(\mathrm{e}^{\mathrm{i}\theta}-1)\\
(g-1)\frac{\mathrm{e}^{\mathrm{i}\theta/2}+\mathrm{e}^{-\mathrm{i}\theta/2}}{2}&=-a\lambda\mathrm{i}(g+1)\frac{\mathrm{e}^{\mathrm{i}\theta/2}-\mathrm{e}^{-\mathrm{i}\theta/2}}{2\mathrm{i}}\\
(g-1)\cos(\theta/2)&=-a\lambda\mathrm{i}(g+1)\sin(\theta/2)\\
g-1&=-a\lambda\mathrm{i}g\tan(\theta/2)-a\lambda\mathrm{i}\tan(\theta/2)\\
g&=\frac{1-a\lambda\mathrm{i}\tan(\theta/2)}{1+a\lambda\mathrm{i}\tan(\theta/2)}
\end{aligned}$$

Now let's find q. Taking the Fourier transform (on the x variable) of the equation $u_t+au_x=0$ we have:

$$\widehat{u}_t=\widehat{u_t}=-a\widehat{u_x}=-a\mathrm{i}\xi\widehat{u}$$

where the last equality follows from the identity $\widehat{u_x}(\xi)=\mathrm{i}\xi\widehat{u}(\xi)$. So $q(\xi)=-a\mathrm{i}\xi$. Let's study first the Taylor expansion of $g(h\xi)$. Recall that $\tan(x)=x+\frac{x^3}{3}+\mathrm{O}(x^5)$, so:

$$\begin{aligned}
1-a\lambda\mathrm{i}\tan(x)&=1-a\lambda\mathrm{i}x-\frac{a\lambda\mathrm{i}}{3}x^3+\mathrm{O}(x^5)\\
\frac{1}{1+a\lambda\mathrm{i}\tan(x)}&=1-(a\lambda\mathrm{i}\tan(x))+(a\lambda\mathrm{i}\tan(x))^2-(a\lambda\mathrm{i}\tan(x))^3+\cdots\\
&=1-a\lambda\mathrm{i}x-a^2\lambda^2x^2+a\lambda\mathrm{i}\frac{3a^2\lambda^2-1}{3}x^3+\mathrm{O}(x^4)\\
\frac{1-a\lambda\mathrm{i}\tan(x)}{1+a\lambda\mathrm{i}\tan(x)}&=\left[1-a\lambda\mathrm{i}x-\frac{a\lambda\mathrm{i}}{3}x^3+\mathrm{O}(x^5)\right]\left[1-a\lambda\mathrm{i}x-a^2\lambda^2x^2+a\lambda\mathrm{i}\frac{3a^2\lambda^2-1}{3}x^3+\mathrm{O}(x^4)\right]\\
&=1-2a\lambda\mathrm{i}x-2a^2\lambda^2x^2+2a\lambda\mathrm{i}\frac{3a^2\lambda^2-1}{3}x^3+\mathrm{O}(x^4)
\end{aligned}$$

Substituting $x = h\xi/2$ in the latter expression we have:

$$g(h\xi) = 1 - a\lambda\mathrm{i}h\xi - \frac{a^2\lambda^2}{2}h^2\xi^2 + a\lambda\mathrm{i}\frac{3a^2\lambda^2 - 1}{12}h^3\xi^3 + \mathrm{O}(h^4)$$

On the other hand the expansion of $\mathrm{e}^{kq(\xi)} = \mathrm{e}^{-a\lambda\mathrm{i}h\xi}$ is:

$$\mathrm{e}^{-a\lambda\mathrm{i}h\xi} = 1 - a\lambda\mathrm{i}h\xi - \frac{a^2\lambda^2}{2}h^2\xi^2 + \frac{a^3\lambda^3\mathrm{i}}{6}h^3\xi^3 + \mathrm{O}(h^4)$$

Thus:

$$\left|\frac{\mathrm{e}^{kq(\xi)} - g(h\xi)}{k}\right| = \left|\frac{\frac{1}{12}a\lambda\mathrm{i}(1 - a^2\lambda^2)h^3\xi^3 + \mathrm{O}(h^4)}{\lambda h}\right| \leq \frac{1}{12}|a|\left|1 - a^2\lambda^2\right|h^2|\xi|^3 + \mathrm{O}(h^3)$$

So we take $C := \frac{1}{12}|a|\left|1 - a^2\lambda^2\right|$.