The Lorenz system, Poincaré maps and unstable periodic orbits

In this practical you will explore the dynamics of the Lorenz system, which is a very drastic reduction of the Rayleigh-Benard problem of thermal convection. The system is:

$$\begin{aligned}\dot{x} &= \sigma(y-x)\\ \dot{y} &= rx - y - zx\\ \dot{z} &= -bz + xy.\end{aligned}$$

Take the values $\sigma = 10$ and $b = 8/3$. We will explore the range of values $0 \le r \le 25$.

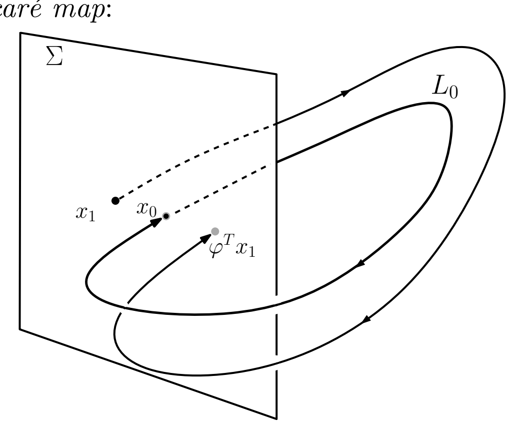
(a) Find all equilibria with a Newton method, continuate them and analize their linear stability as a function of r.

(b) Using a Runge-Kutta time integrator (such as Matlab's `ode45`), explore the phase portrait of the system and confirm the linear stability of the equilibria checking their attracting/repulsing properties.

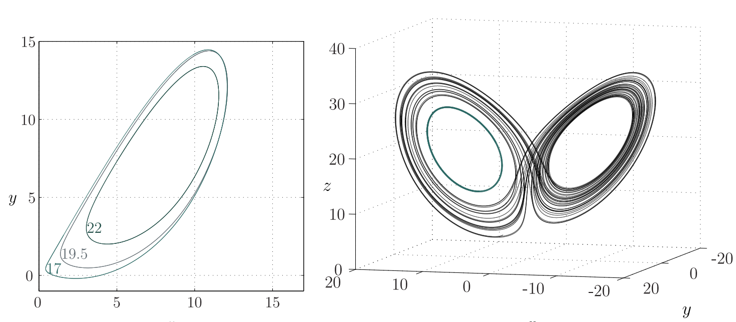
Limit cycles and Poincaré map: by definition, a *limit cycle* is an *isolated* periodic orbit. If the limit cycle is stable (an attractor) it can be found by forward time integration. If the limit cycle is unstable, we cannot approach the limit cycle with time integration[1]. However, these invariant manifolds can be computed in practice by means of the usually termed as *Poincaré map*:
Consider a dynamical system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ and its phase portrait induced by the flow φ^t. If $L_0 \subset \mathbb{R}^n$ is a limit cycle of period T then $\varphi^{t+T}\, x_0 = \varphi^t\, x_0, \forall t \in \mathbb{R}, \forall x_0 \in L_0$. Consider the hypersurface $\Sigma = \{x \in \mathbb{R}^n \mid g(x) = 0\}$ (see figure) with $g(x_0) = 0$ for some $x_0 \in L_0$ satisfying $\nabla g(x_0) \cdot f(x_0) \neq 0$. The map

$$P \equiv \varphi^T|_\Sigma : \Sigma \longrightarrow \Sigma,$$

assigns x_1 to $\varphi^T x_1 = P(x_1)$. The point x_0 is a fixed point (i.e., $P(x_0) = x_0$). To find the unstable cycle you only need to find the zero of the function $Q(x) = x - P(x)$ with a Newton method.



(c) In the Lorenz system there are unstable limit cycles like the ones shown projected in the xy plane for $r = 22, 19.5$ and 17 (left) or $r = 22$ (right). You have to compute them.



[1] An exception is in $\mathbb{R}^2$, where you can integrate backwards in time