# Thermohaline circulation

Víctor Ballester and Victor Botez

Computational Fluid Dynamics
M2 - Applied and Theoretical Mathematics
Université Paris-Dauphine, PSL

April 4, 2024

# Table of contents

# Introduction

# Motivation

- Thermohaline circulation is a global circulation of the ocean that is driven by the density differences due to temperature and salinity.
- It is responsible for the transport of heat and salt around the globe, and it plays a key role in the climate system.

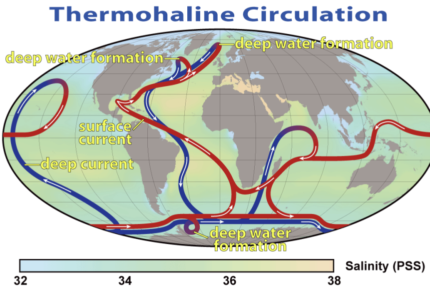


**Figure:** Schematic of the thermohaline circulation.

# Objectives

Our goals are:

- Study the thermohaline circulation in a two-dimensional domain.
- Analyze the stability of the convection patterns.
- Investigate the impact of the initial conditions on the convection patterns.

# Theoretical model

# Governing equations (I)

Navier-Stokes equations and diffusion equations for salinity and temperature :

$$\begin{aligned} \rho\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) &= -\nabla P + \rho\nu\nabla^2\mathbf{u} + \rho\mathbf{g} \\ \frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\mathbf{u}) &= 0 \\ \frac{\partial T}{\partial t} + (\mathbf{u}\cdot\nabla)T &= \kappa_T\nabla^2 T \\ \frac{\partial S}{\partial t} + (\mathbf{u}\cdot\nabla)S &= \kappa_S\nabla^2 S \end{aligned} \tag{1}$$

Taylor expansion of density :

$$\rho = \rho_0(1 - \alpha_T\Delta T + \alpha_S\Delta S) \tag{2}$$

# Governing equations (II)

Navier-Stokes equations :

$$
\begin{aligned}
\left(\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u}\right) &= -\frac{\nabla P}{\rho_0} + \nu\nabla^2\mathbf{u} + \rho_0\left(\alpha_T\Delta T - \alpha_S\Delta S\right) g\mathbf{e_z} \\
\nabla\cdot\mathbf{u} &= 0
\end{aligned}
\tag{3}
$$

Diffusion equations :

$$
\begin{aligned}
\frac{\partial T}{\partial t} + (\mathbf{u}\cdot\nabla)T &= \kappa_T\nabla^2 T \\
\frac{\partial S}{\partial t} + (\mathbf{u}\cdot\nabla)S &= \kappa_S\nabla^2 S
\end{aligned}
\tag{4}
$$

# Boundary conditions

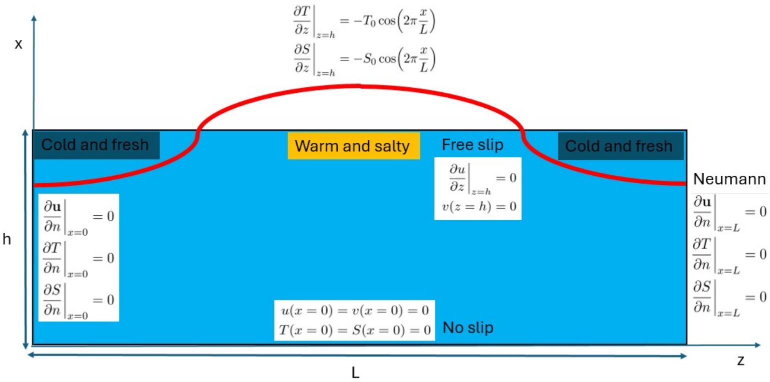


**Figure:** Drawing of the setup used in all simulations.

# Dimensionless equations and control parameters

Navier-Stokes equations in the Boussinesq approximation :

$$
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} &= -\nabla p + \mathrm{Pr}\nabla^2\mathbf{u} + \mathrm{Pr}\mathrm{Ra}\left(T - \frac{1}{R_\rho}S\right)\mathbf{e}_z \\
\frac{\partial T}{\partial t} + (\mathbf{u}\cdot\nabla)T &= \nabla^2 T \\
\frac{\partial S}{\partial t} + (\mathbf{u}\cdot\nabla)S &= \frac{1}{\mathrm{Le}}\nabla^2 S \\
\nabla\cdot\mathbf{u} &= 0
\end{aligned}
\tag{5}
$$

Dimensionless quantities :

$$\mathrm{Pr} = \frac{\nu}{\kappa_T} \tag{6}$$

$$\mathrm{Ra} = \frac{\alpha_T g T_0 h^3}{\nu\kappa_T} \tag{7}$$

$$R_\rho = \frac{\alpha_T T_0}{\alpha_S S_0} \tag{8}$$

$$\mathrm{Le} = \frac{\kappa_T}{\kappa_S} \tag{9}$$

# Numerical methods

# Chorin splitting method

In order to integrate the incompressible Navier-Stokes equations, we use the Chorin splitting method:

1. Solve for $\mathbf{u}^a$ in $\dfrac{\mathbf{u}^a - \mathbf{u}^n}{\Delta t} + \mathbf{u}^n \cdot \nabla \mathbf{u}^n = 0$ using a semi-Lagrangian method.
2. Solve for $\mathbf{u}^*$ in $\dfrac{\mathbf{u}^* - \mathbf{u}^a}{\Delta t} = \mathrm{Pr}\nabla^2 \mathbf{u}^n + \mathrm{PrRa}\left(T^n - \dfrac{1}{R_\rho} S^n\right)\mathbf{e}_z$ using a central differences scheme.
3. Set the boundary conditions for the intermediate velocity field $\mathbf{u}^*$.
4. Solve the Poisson equation for the pressure, $\nabla^2 p^{n+1} = \dfrac{1}{\Delta t}\nabla \cdot \mathbf{u}^*$.
5. Set the boundary conditions for the pressure $p^{n+1}$.
6. Project the pressure to the intermediate velocity to obtain the new velocity at time $t^{n+1}$, $\mathbf{u}^{n+1} = \mathbf{u}^* - \Delta t \nabla p^{n+1}$.
7. Set the boundary conditions for the velocity field $\mathbf{u}^{n+1}$.

# Temperature and salinity evolutions

For each time step, once we gave found the next iterate of the velocity field $\mathbf{u}^{n+1}$, we solve the advection-diffusion equations for the temperature and salinity fields:

1. Solve for $T^a$ in $\dfrac{T^a - T^n}{\Delta t} + \mathbf{u}^{n+1} \cdot \nabla T^n = 0$ and for $S^a$ in $\dfrac{S^a - S^n}{\Delta t} + \mathbf{u}^{n+1} \cdot \nabla S^n = 0$ using a semi-Lagrangian method.
2. Solve for $T^{n+1}$ in $\dfrac{T^{n+1} - T^a}{\Delta t} = \nabla^2 T^n$ and for $S^{n+1}$ in $\dfrac{S^{n+1} - S^a}{\Delta t} = \dfrac{1}{\mathrm{Le}} \nabla^2 S^n$ using a central differences scheme.
3. Set the boundary conditions for the temperature and salinity fields $T^{n+1}$ and $S^{n+1}$.

# Implicit vs explicit

Problems with the explicit method:

- The time step is limited by the stability condition of the heat equation $\partial_t f = \kappa \nabla^2 f$:
$$\kappa \Delta t \left( \frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} \right) \leq \frac{1}{2}.$$
- For us $\kappa = \mathrm{Pr}, 1, \frac{1}{\mathrm{Le}}$, and for water $\mathrm{Pr} \sim 10$, which implies $\Delta t \leq 2.5 \times 10^{-6}$ for $\Delta x = \Delta y = 0.01$!

**Solution:** use an implicit method for the heat equation to make it unconditionally stable!

2. Solve for $\mathbf{u}^*$ in $\dfrac{\mathbf{u}^* - \mathbf{u}^a}{\Delta t} = \mathrm{Pr} \nabla^2 \mathbf{u}^* + \mathrm{Pr}\mathrm{Ra} \left( T^n - \dfrac{1}{R_\rho} S^n \right) \mathbf{e}_z$ using a central differences scheme.

Our scheme is still consistent because at zero order in time $\mathbf{u}^* = \mathbf{u}^{n+1}$.

# Boundary conditions and ghost points

We use the ghost cell method to impose second order boundary conditions to the actual boundary of the domain.

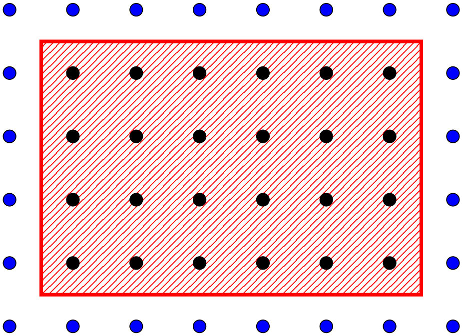

**Figure:** The ghost cell method. The domain is represented in red and the dots represent the grid cells, the black ones being the inner cells and the blue ones, the ghost cells.

# Results

# Convection phase diagram

**Aim:** fixed $\mathrm{Pr}$ and $\mathrm{Ra}$, study the convection patterns in terms of the $\mathrm{Le}$ (measures the ratio of diffusivities) and $R_\rho$ (measures the importance in the density).

We focus our study on a neighbourhood of $R_\rho = 1$, where both temperature and salinity are equally important in the density. We do the study for $\mathrm{Le} \in \{0.01, 0.1, 1, 10, 100\}$.

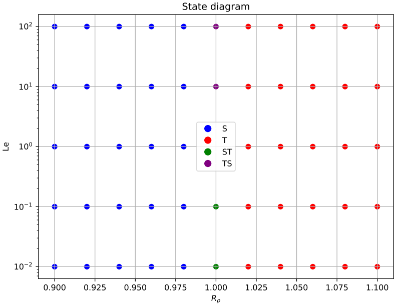


**Figure:** Phase diagram of the convection patterns.

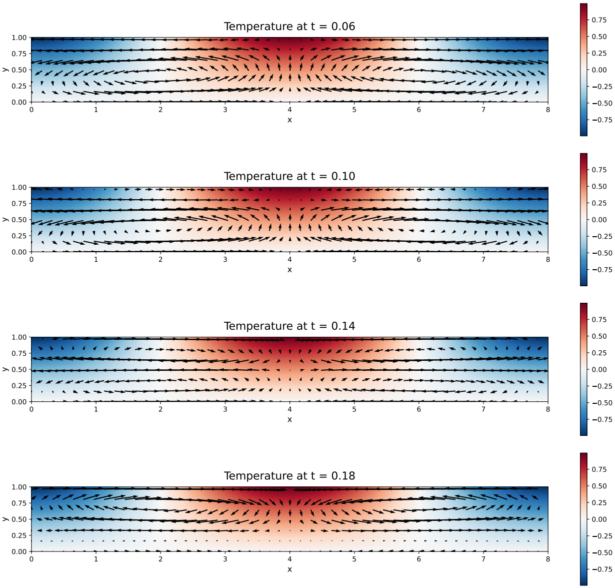


**Figure:** Transition from a temperature convection to a salinity convection.

# Creation of a new set of initial conditions

1. Simulate a system $(R_\rho, Le)$ and wait until it reaches stationarity

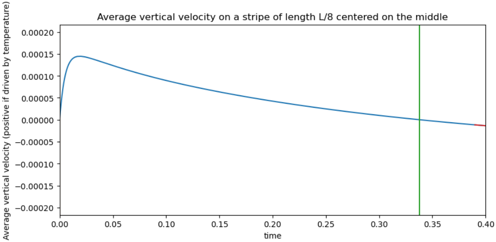


**Figure:** Simulation of $(R_\rho, Le) = (1.01, 100)$. The blue line is the average vertical velocity, the green line is its intersection with $y = 0$.

2. Save the final velocity, temperature and salinity fields
3. Use them as initial conditions for $(R'_\rho, Le')$

# Impact of initial conditions

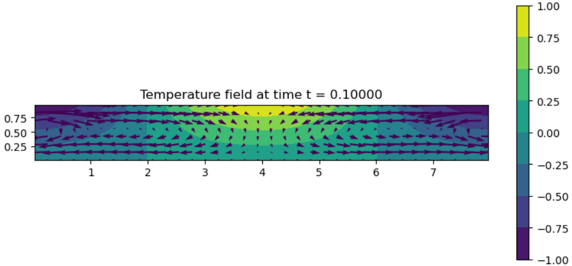


**Figure:** Attempts performed at $Le$ kept constant. Simulation of the system $(R_\rho, Le) = (1, 10)$ with initial condition $(R_\rho, Le) = (5, 10)$.

# Creation of a stable state with four convecting cells

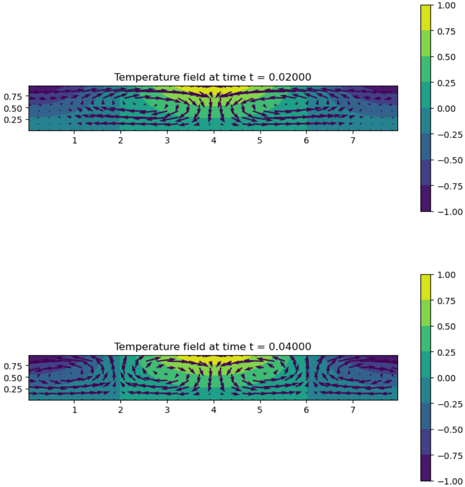


**Figure:** Transition from two convecting cells to four convecting cells. Initial system is $(R_\rho, L_e) = (0.9, 0.25)$ and final system is $(R_\rho, L_e) = (1, 100)$

# Conclusions

# Conclusions

1. Coherent results between explicit and implicit diffusions
2. Phase diagram coherent with our understanding
3. Interesting transitions from one pair of convecting cells to two pairs of convecting cells