Numerical Methods for Fluid Dynamics: TD4

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Forewords

Download TD4.tar, and then type :

```
> tar xvf TD4.tar
> cd TD4
> ipython --pylab
```

1 Scalar Advection

We consider a scalar advection problem for a quantity q (which could for example be temperature)

$$\frac{\partial q}{\partial t} = -\mathbf{u} \cdot \boldsymbol{\nabla} q \,. \tag{1}$$

Let's consider the flow $\mathbf{u} = u\,\mathbf{e}_x + v\,\mathbf{e}_y$ steady in tyme in a unit square given by

$$u = \sin^2(\pi x)\,\sin(2\pi y)\,, \qquad v = -\sin(2\pi x)\,\sin^2(\pi y)\,. \tag{2}$$

This flow satisfies $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ and the boundary conditions $u = v = 0$ on the boundaries $x = 0$, $x = 1$ and $y = 0$, $y = 1$.

We consider an initial field q in the form of a localised perturbation. At $t = 0$, this perturbation is centered on $(x_0\,,\,y_0) = (1/2\,,\,3/4)$ and has a caracteristic diameter $\alpha = 0.2$, its analytical expression is

$$q(x, y, t = 0) = \frac{1}{4}\,(1 + \cos(\pi r))\;, \tag{3}$$

$$\text{with} \quad r(x, y) = \min(\sqrt{(x - x_0)^2 + (y - y_0)^2}, \alpha)/\alpha\,. \tag{4}$$

1. A semi-Lagrangian method with bilinear interpolation has been implemented. Adapt the *Advection.py* code so that the spot is advected for a time T one way (already implemented), and then for a time T the other way (i.e. with the reversed flow $u \to -u\,,\ v \to -v$). Compare the final solution at time $2T$ to the initila field.

2. Implement in the function `SemiLag2` a second order semi-Lagrangian scheme using the method we saw during the lecture. Compare both schemes (1st order and 2nd order).

2 The lid-driven cavity

We now want to solve the Navier-Stokes in a driven cavity

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \boldsymbol{\nabla} \mathbf{u} = -\boldsymbol{\nabla} p + \mathrm{Re}^{-1} \Delta \mathbf{u} \,, \qquad \boldsymbol{\nabla} \cdot \mathbf{u} = 0 \,. \tag{5}$$

We rely on the same initial conditions and boundary conditions as last week for TD3, i.e. $\mathbf{u} = \mathbf{0}$ at t=0 and $\mathbf{u} = \mathbf{g}$ on $\partial\Omega$ for $t > 0$.

Let us recall that :

$$\mathbf{g} = \begin{cases} \mathbf{0} & \text{for} \quad (0 \leq x \leq 1,\, y = 0)\,,\ (x = 0,\, 0 \leq y < 1)\,,\ (x = 1,\, 0 \leq y < 1)\,, \\ \mathbf{e}_x & \text{for} \quad (0 \leq x \leq 1,\, y = 1)\,, \end{cases} \tag{6}$$

where $(\mathbf{e}_x, \mathbf{e}_y)$ are the basis vectors. We use a fractional stepping to solve successively for

$$\frac{\mathbf{u}^a - \mathbf{u}^n}{\Delta t} + \mathbf{u}^n \cdot \boldsymbol{\nabla} \mathbf{u}^n = \mathbf{0} \,, \qquad \mathbf{u}^a = \mathbf{g} \quad \text{on} \quad \partial\Omega \,, \tag{7}$$

$$\frac{\mathbf{u}^* - \mathbf{u}^a}{\Delta t} = \frac{1}{\mathrm{Re}} \Delta \mathbf{u}^n \,, \qquad \mathbf{u}^* = \mathbf{g} \quad \text{on} \quad \partial\Omega \,, \tag{8}$$

$$\Delta p^{n+1} = \frac{1}{\Delta t} \boldsymbol{\nabla} \cdot \mathbf{u}^* \,, \qquad \partial_n p^{n+1} = 0 \tag{9}$$

$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^*}{\Delta t} = -\boldsymbol{\nabla} p^{n+1} \,. \tag{10}$$

1. Add the advection step in the *NavierStokes.py* code and compare with the results of last week...
2. Verify that $\boldsymbol{\nabla} \cdot \mathbf{u}^{n+1} = 0$, and
$$\frac{\mathbf{u}^{n+1} - \mathbf{u}^n}{\Delta t} + \mathbf{u}^n \cdot \boldsymbol{\nabla} \mathbf{u}^n = -\boldsymbol{\nabla} p^{n+1} + \frac{1}{\mathrm{Re}} \Delta \mathbf{u}^n \,.$$
3. Are the boundary conditions exactly satisfied ?

DM4 : Lax-Friedrich and Lax-Wendroff

1. Modify the lid driven cavity code (Ex. 2) to use a Lax-Friedrich and then a Lax-Wendroff advection schemes.