Instructions:

- Show your work. Full points are only given for correct answers *with* adequate justification. A correct final answer with missing or substantially incorrect justification will not merit full points on a problem.
- On any question asking you to calculate a number, you may leave your final answer either in combinatorial notation (using factorials and combinations, etc.) or a precise numerical value.
- All graphs are assumed to be simple.

Problem 1. Prove that the sum of the first n odd numbers is n^2.

Proof. We will prove by induction that $1 + \ldots + (2n-1) = n^2$. For $n = 1$ this is clear. Assume it is known for n, then consider

$$1 + \ldots + (2n-1) + (2(n+1)-1) = n^2 + 2n + 1 = (n+1)^2,$$

which proves the induction step. □

Problem 2. Find the solution to the recurrence relation

$$a_n = 4a_{n-1} - 4a_{n-2}$$

for all $n \geq 2$, with initial conditions $a_0 = 3$ and $a_1 = 8$.

Proof. We get that the auxiliary polynomial is

$$t^2 - 4t + 4 = (t-2)^2,$$

which has a repeated root at $t = 2$. Therefore, we know

$$a_n = \alpha 2^n + \beta n 2^n.$$

The initial conditions give the equations

$$\alpha = 3$$
$$2\alpha + 2\beta = 8$$

which gives $\beta = 1$. Therefore,

$$a_n = 3 \cdot 2^n + n2^n.$$

□

Problem 3. Suppose X and Y are disjoint finite non-empty sets and $f : X \to Y$ is a function. Define a bipartite graph $G = (V, E)$ where $V = X \cup Y$ and there is an edge between $x \in X$ and $y \in Y$ if and only if $f(x) = y$.

1. Show that every vertex $x \in X$ has degree 1.

2. Show that f is a bijection if and only if every vertex of G has degree 1.

3. Show that G is connected if and only if $|Y| = 1$.

Remark: Recall that to prove and 'if and only if' statement, you have to prove both directions.

Proof. (1). Because f is a function, for every $x \in X$, there is a unique $y \in Y$ with $f(x) = y$. This means there is a single edge incident to x so x has degree 1.
(2). First suppose f is a bijection. By (1), we only have to show that every $y \in Y$ has degree 1. We know that, because f is surjective, for every $y \in Y$, there is some $x \in X$ with $f(x) = y$. That means that the degree of every $y \in Y$ is at least 1. Additionally, if the degree of y is greater than 1, then there are $x \neq x'$ with $f(x) = y = f(x')$, which is impossible since f is injective. Therefore the degree of each $y \in Y$ is exactly 1.
(3). If $|Y| = 1$, then, because f is a function, every $x \in X$ satisfies $f(x) = y$. Then G is connected since any two $x, x' \in X$ are connected by a path: there's an edge from x to y and an edge from y to x. It follows that any two vertices of the graph are connected by a path so G is connected.
On the other hand, suppose G is connected. Note that if $y \in Y$ is not in the range of f, then $\deg(y) = 0$ so the graph is not connected. So for all $y \in Y$, $\{x \in X : f(x) = y\}$ is non-empty. If $y \neq y'$ are elements of Y, then there are no edges from $\{x \in X : f(x) = y\} \cup \{y\}$ to any vertices outside of this set. Likewise, there are no edges from $\{x \in X : f(x) = y'\} \cup \{y'\}$ to any vertices outside this set. Since both are non-empty, they must be in separate connected components, so G is not connected, a contradiction. Therefore, there cannot be two distinct elements of Y so $|Y| = 1$. □

Problem 4. A *numerical palindrome* is a number that reads the same forwards as it does backwards. For example,

$$1102011$$

is a numerical palindrome.

1. How many 5 digit numerical palindromes are there? (We do not allow the first digit to be 0)
2. How many 6 digit numerical palindromes are there whose digits add up to 20? (We do not allow the first digit to be 0)

Proof. (1). The choice of the first, second, and third digits completely determine the numerical palindrome. There are 9 choices for the first/fifth digit, then 10 choices for both the second/fourth digit, and 10 choices for the third digit. That gives

$$9 \times 10 \times 10 = 900$$

choices.
(2). If the digits of the numerical palindrome add up to 20, then the first three digits add up to 10. So we are solving

$$x_1 + x_2 + x_3 = 10$$

subject to the constraint that $x_1 \geq 1$. Note that the additional constraint that all of the x_i's be less than or equal to 9 is automatically satisfied, except in the unique solution where

$x_1 = 10$, which has to be removed. The total number of solutions to $x_1 + x_2 + x_3 = 9$ with no constraints is

$$\binom{9+3-1}{3-1} = \binom{9+3-1}{9} = 55,$$

so we have $55 - 1 = 54$ 3 digit numerical palindromes whose digits add up to 20. $\square$

Problem 5. Suppose A, B, and C are pairwise disjoint sets with n elements each.

1. The set Ξ consists of triples (f, g, h) of bijections $f : A \to B$, $g : B \to C$, and $h : C \to A$. How many elements does Ξ have?

2. Now assume $n \geq 2$. The set Υ consists of the triples $(f, g, h) \in \Xi$ so that $(h \circ g \circ f)(a) \neq a$ for some $a \in A$. How many elements are there in Υ?

Proof. (1) There are $n!$ bijections from a set of size n to a set of size n so Ξ has $(n!)^3$ elements.
(2) The complement of Υ in Ξ consists of those triples (f, g, h) so that $(h \circ g \circ f)(a) = a$ for all $a \in A$. How many elements are there of this complement? We have $n!$ choices for f, $n!$ choices for g, and then once we have chosen f and g, how many choices do we have for h? We can list the elements of C by $C = \{(g \circ f)(a) : a \in A\}$ since $(g \circ f) : A \to C$ is a bijection. Then we must define h on $(g \circ f)(a)$ so that $h((g \circ f)(a)) = a$; this defines h on all of C. So given f and g, we have a unique choice of h. Therefore, there are $(n!)^2$ elements of the complement of Υ in Ξ so Υ has $(n!)^3 - (n!)^2$ elements. $\square$

Problem 6. Let X be a set.

1. Suppose $f : X \to X$ is a function that satisfies

$$f(f(x)) = x$$

   for all $x \in X$. Show that f is a bijection.

2. If $|X| = 4$, how many functions $f : X \to X$ are there with $f(f(x)) = x$ for all $x \in X$?

Solution.. (1) We will first show f is injective. Suppose $x, y \in X$ and $f(x) = f(y)$. Then $x = f(f(x)) = f(f(y)) = y$ so we have shown that if $f(x) = f(y)$ then $x = y$, so f is injective.
Now we show f is surjective. Pick $y \in X$, we want to find some x such that $f(x) = y$. Setting $x = f(y)$, we have $f(x) = f(f(y)) = y$ as desired.
(2). We divide into cases based on the number of fixed elements. One possibility is that $f(x) = x$ for all $x \in X$ (so all 4 elements are fixed). There is a unique such f so there is 1 function in this case.
Next, if $f(x) = x'$ for some $x' \neq x$, then also $f(x') = x$ so there are at least two elements that are moved by f. So suppose there are two elements that are moved, and two that are

fixed. Then the are are $\binom{4}{2}$ possibilities for the two elements that are swapped, and then the remaining two are fixed, so there are $\binom{4}{2}$ functions in this case.
Finally, we consider the case that all four elements are moved. Then f is determined by a choice of two disjoint two-element subsets of X, since f swaps the pair of elements in each subset. There are $\binom{4}{2}$ choices of the first set and then, having chosen the first two element set, there is a unique choice for the second, since you choose one two-element set and the choice of the complement is determined. But choosing the same two sets in the opposite order results in the same function, so one must divide by 2 to get the correct count. So there are $\binom{4}{2}/2 = 3$ functions with no fixed points.
This shows there are

$$1 + \binom{4}{2} + \binom{4}{2} = 1 + 6 + 3 = 10$$

such functions.
Problem 7. If X is a set, an *equipartition* of X is an equivalence relation on X such that every equivalence class has the same size. In other words, an equivalence relation E on X is an equipartition if $|[x]_E| = |[y]_E|$ for all $x, y \in X$, where $[x]_E = \{z \in X : (x, z) \in E\}$.

1. Show that if $|X| \geq 2$, then there are at least two different equipartitions of X.

2. Suppose $|X| = 9$. How many equipartitions of X are there? *Hint*: First, figure out the possibilities for the cardinality of the equivalence classes. Then count the number of ways of selecting the elements of the classes.

Proof. (1) If $|X| \geq 2$, then $E_0 = \{(x, y) : x, y \in X\}$, the equivalence relation in which all elements of X are equivalent, and $E_1 = \{(x, x) : x \in X\}$, the relation of equality, are distinct equivalence relations, since E_0 has 1 class with $|X|$ many elements and E_1 has $|X|$ many classes each with 1 element.
(2) Because all classes must have the same size, the possibilities for an equipartition of X are: 1 class of size 9 or 9 classes of size 1, as described above, or 3 classes of size 3. To count the number of equivalence relations of size 3, first we consider how to pick 3 disjoint subsets of X, in order. First, we choose a subset Y_1 of size 3, so there are $\binom{9}{3}$ choices. Then we choose a subset Y_2 of size 3 from the remaining 6 elements, so there are $\binom{6}{3}$ choices. Then we have 3 elements remaining which we are forced to put in the third set Y_3. This defines an equivalence relation, whose classes are Y_1, Y_2, and Y_3. Formally, we can define

$$E = \{(x, y) : x, y \in Y_1\} \cup \{(x, y) : x, y \in Y_2\} \cup \{(x, y) : x, y \in Y_3\}.$$

However, if we chose the same sets Y_1, Y_2, Y_3 in a different order, we get the same equivalence relation. So we have overcounted by $3! = 6$. So the number of equipartitions with 3 classes is

$$\frac{\binom{9}{3}\binom{6}{3}}{3!} = 280.$$

Therefore, the final number of possibilities is

$$1 + 1 + \frac{\binom{9}{3}\binom{6}{3}}{3!} = 282.$$

$\square$

Problem 8. Let G be a graph whose vertices consist of pairs (a, b) where a and b are non-negative integers less than or equal to 3. Say that there is an edge between (a, b) and (c, d) if either $|a - c| = 1$ and $b = d$ or $|b - d| = 1$ and $a = c$, where $|x|$ denotes the absolute value of x. In other words, two vertices are connected by an edge if they differ by exactly 1 in one coordinate and agree on the other.

1. What is the degree of $(0, 0)$?
2. What is the degree of $(2, 2)$?
3. Is there an Euler cycle in G?

Proof. (1). The degree of $(0, 0)$ is 2: it has edges to $(1, 0)$ and to $(0, 1)$ and that's it.
(2) The degree of $(2, 2)$ is 4: it has edges to $(1, 2)$ and $(3, 2)$ as well as $(2, 1)$ and $(2, 3)$.
(3). No. The vertex $(0, 1)$, for example, has degree three, since it has edges to $(0, 0)$, $(0, 2)$, and $(1, 0)$. Since G has a vertex of odd degree, there is no Euler cycle. $\square$