Machine Learning with Graphs
Homework 3

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Question 2: Subgraphs and Order Embeddings (35 points)

Q2.1 Transitivity (8 points)

We say that graph $A = (V_A, E_A)$ is a subgraph of $B = (V_B, E_B)$, denoted $A \subseteq B$, when there exist a bijective function $f : V_A \to V_B'$ where $V_B' \subseteq V_B$ with $|V_B'| = |V_A|$ such that

$$(u, b) \in E_A \Leftrightarrow (f(u), f(v)) \in E_B \tag{1}$$

Let $A \subseteq B$ and $B \subseteq C$ and let f_{AB} and f_{BC} be bijective functions that constitutes the sub-graphs. Then $f_{AC} = f_{AB} \circ f_{BC}$ is also bijective and

$$(u, b) \in E_A \Leftrightarrow (f_{AB}(u), f_{AB}(v)) \in E_B \Leftrightarrow (\underbrace{f_{BC}(f_{AB}}_{f_{AC}}(u)), \underbrace{f_{BC}(f_{AB}}_{f_{AC}}(v))) \tag{2}$$

which proves that $A \subseteq C$

Q2.2 Anti-symmetry (8 points)

We say that graph $A = (V_A, E_A)$ and graph $B = (V_B, E_B)$ are graph-isomorphic, $A = B$, when there exist a bijective function $f : V_A \to V_B$ such that

$$(u, b) \in E_A \Leftrightarrow (f(u), f(v)) \in E_B \tag{3}$$

This is trivially fulfilled since when $A \subset B$ there exits a bijective function f_{AB} and similarly when $B \subset A$ we have bijective f_{BA} which necessitates that $|V_A| = |V_B|$. Using f_{AB} we then have the bijective mapping between the two graphs.

Q2.3 Common subgraph (4 points)

We show both ways.

- $\Rightarrow$ Let $X \subseteq A$ and $X \subseteq B$ then by the assumption of perfectly preserving the embedding constraint, we have $z_X \preceq z_A$ and $z_X \preceq z_B$ therefore $z_X \preceq \min\{z_A, z_B\}$ trivially.
- $\Leftarrow$ Let $z_X \preceq \min\{z_A, z_B\}$ and assume without loss of generality that $z_A \preceq z_B$. Then $z_X \preceq z_A$ since $\min\{z_A, z_B\} = z_A$ so $X \subseteq A$. We also have that $\preceq$ is transitive therefore $z_X \preceq z_A$ and $z_A \preceq z_B$ means that $z_X \preceq z_B$ so $X \subseteq B$.

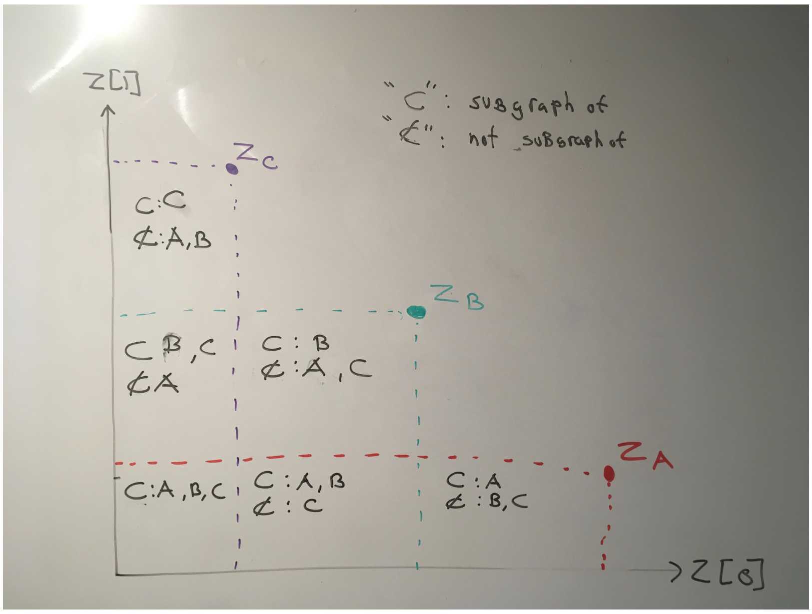
Q2.4 Order embeddings for graphs that are not subgraphs of each other (5 points)

Since $z_B[1] \leq z_A[1]$, along with the given $z_B[0] < z_A[0]$, would imply that $B \subseteq A$, we must have $z_B[1] > z_A[1]$ in which case $z_B \not\preceq z_A$. A similar argument is made for the embeddings of B and C and we get the relation:

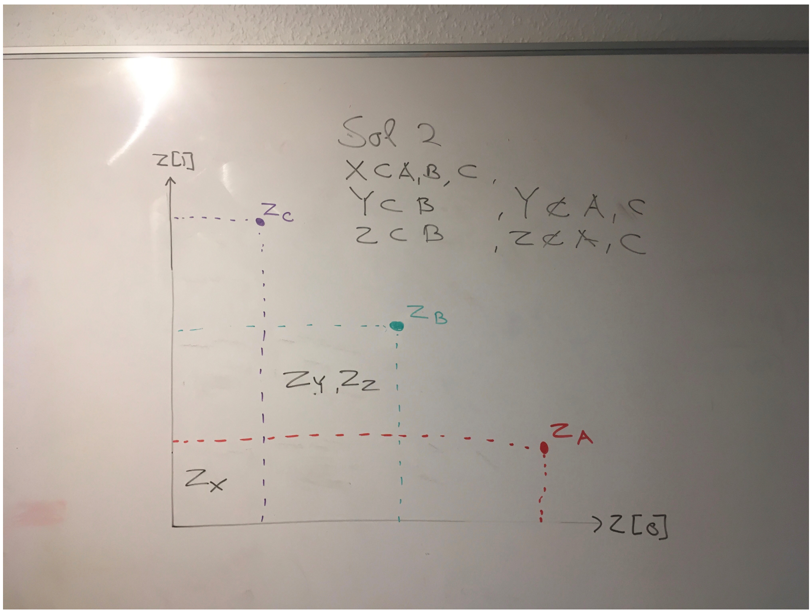
$$z_A[1] < z_B[1] < z_C[1] \tag{4}$$

Q2.5 Limitation of 2-dimensional order embedding space (10 points)

It is helpful to draw the embedding map (with structure according to the previous question) with sub-graph zones highlighted.



For example the zone bottom middle are graphs that are sub-graphs of A, B but not C. Using this map, I see there is just one solution:



such that

- $X \subset A, B, C$
- $Y \subset B$ and $Y \not\subset A, C$
- $Z \subset B$ and $Z \not\subset A, C$.

Which fulfil that $z_X \preceq z_Y$ and $z_X \preceq z_Z$.