1 Subgraphs and order embeddings

1.1 Transitivity

We are given that A is a subgraph of B and B is a subgraph of C.
The question is then if A is also a subgraph of C, which will now be shown.
The relations between the graphs can be formulated as below, which says that A is contained in B, and B is contained in C.

- $V_A \subseteq V_B$
- $V_B \subseteq V_C$

If we look at this we see that A is subset of B and B is a subset of C. Hence it can also be written out as this.

$V_A \subseteq V_B \subseteq V_C$

Which in essence says that A is also a subset contained in C.

Hence the Transitivity applies and given the initial information we know that $V_A \subseteq V_C$

1.2 Anti-symmetry

If A is a subgraph of B then it implies that $V_A \subseteq V_B$ meaning that the amount of nodes cannot be greater than B as that the entirety of A can be found in B.
Like wise if B is a subgraph of A then it implies that $V_B \subseteq V_A$ which also means B cannot be greater than A and that the entirety of B can be found in A.

For both of these conditions to uphold it means that they are the same graph i.e. $A = B$, which means that they are graph isomorphic to one another, as neither can be greater than the other.

1.3 Common subgraph

For X to be a subgraph of A it must hold that the embedding is element wise smaller than A i.e. $Z_X \preccurlyeq Z_A$.
Similarly it must hold that if X is a subgraph of B then $Z_X \preccurlyeq Z_B$.
For X to a subgraph of both then it must hold for e.g. a 2 dimensional embedding then the following 4 conditions must hold.

1. $Z_X[0] < Z_A[0]$
2. $Z_X[0] < Z_B[0]$

3. $Z_X[1] < Z_A[1]$
4. $Z_X[1] < Z_B[1]$

If this is made more compact it means that
$Z_X[0] < min\{Z_A[0], Z_B[0]\}$ and $Z_X[1] < min\{Z_A[1], Z_B[1]\}$.

Which is the same as Z_X being element wise smaller than then smallest element in Z_A and Z_B i.e.
$Z_X \preccurlyeq min\{Z_A, Z_B\}$

1.4 Order embeddings for graphs that are nor subgraphs of each other

With the embedding relation given that $Z_A[0] > Z_B[0] > Z_C[0]$ and that they are all non isomorphic and not subgraphs of each other. Then this relations suggests a reverse relationship for the other dimension i.e. $Z_A[1] < Z_B[1] < Z_C[1]$, which ensures that non of the graphs are element wise smaller than another.

1.5 Limitation of 2 dimensional order embedding space

We consider the graph relation of $Z_A[0] > Z_B[0] > Z_C[0]$. From those we can show how graph A,B and C are visually placed in relations to each other, which is also illustrated on figure 1.1.
By using the following assignment we will find a relation that ensures $Z_X \preccurlyeq Z_Y$ and $Z_X \preccurlyeq Z_Z$ by formulated the relations that must apply between X,Y,Z and A,B,C in relation to their 2 dimensional embedding

- Graph X is a subgraph of A, B and C
- Graph Y is a subgraph of B
- Graph Z is a subgraph of B

With this scheme we find that a set of conditions apply to the relation between the graphs. Notably we consider the conditions of X, which can only be placed in the left bottom square on figure 1.1.

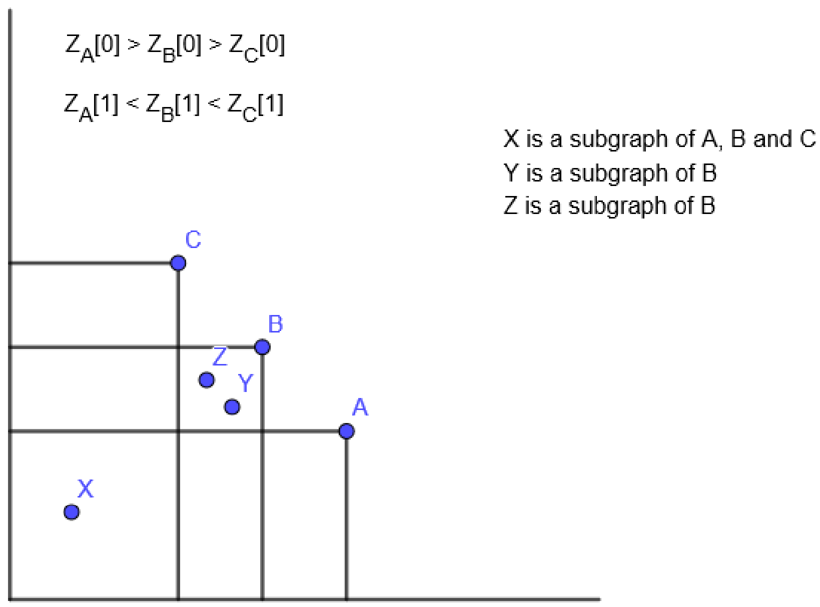

Figure 1.1: Drawing of the embedding space and relations between the graphs A,B,C and the relations between the embedding of X,Y,Z.

With these subgraph relations upheld it means that Z and Y will have their embedding confined for each of their dimensions confined in the following space:

- $Z_C[0] < Z_Y[0], Z_Z[0] < Z_B[0]$
- $Z_A[0] < Z_Y[1], Z_Z[1] < Z_B[0]$

The coordinates of X on the other hand are defined are

- $Z_X[0] < min\{Z_A[0], Z_B[0], Z_C[0]\}$ i.e. $Z_X[0] < Z_C[0]$
- $Z_X[1] < min\{Z_A[1], Z_B[1], Z_C[1]\}$ i.e. $Z_X[1] < Z_C[1]$

Given these different limits on the positions that the embeddings can occur, and the initial assumption of how A,B,C are related we can see that Y and Z will always their embedding placed such that:
$Z_X \preccurlyeq Z_Y$ and $Z_X \preccurlyeq Z_Z$

Thus it can be concluded that Y and Z, will always be embedded such that the relation between X and them is upheld, while X is not necessarily a subgraph of either.

An example of how such graphs could be constructed can be seen on figure 1.2, where Y and Z are subgraphs of B. X is a subgraph of A,B,C. X is however not found as a subgraph in Z and Y at all.
Given how these graphs are related, then Y and Z would always be embedded in the same square as previously defined, while X would just as well be confined to the lower

left. Given these relations, then X would always be embedded such that the $Z_X \preccurlyeq Z_Y$ and $Z_X \preccurlyeq Z_Z$ condition is fulfilled.

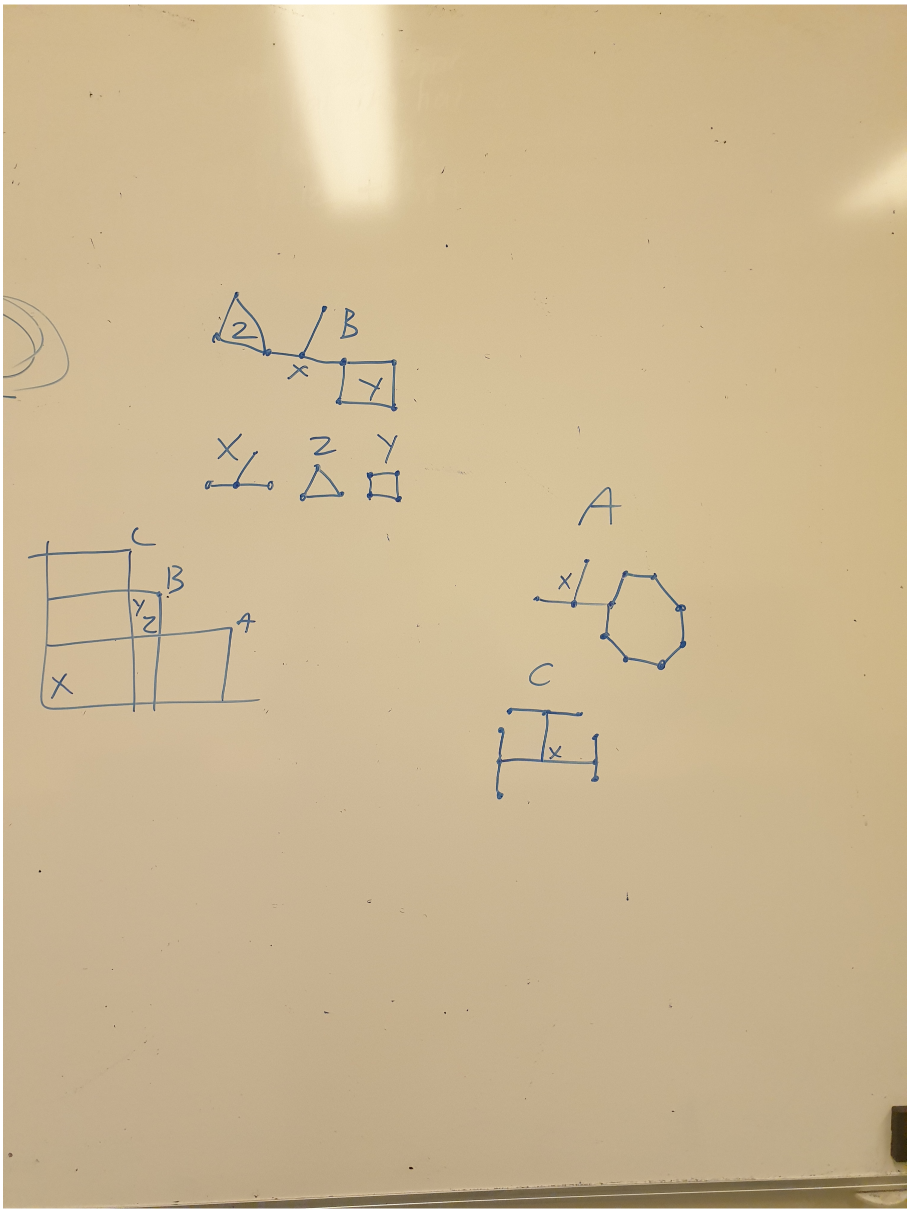

Figure 1.2: Example of the construction, where X is not found in Y and Z, however X is found in A,B,C. Hence X is embedded AS IF it was a subgraph of Y and Z while it is not.