Kinematic and Dynamic Analysis of a Linkage
Walschaerts Valve Gear

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Introduction

In the context of the subject *Beweging en trillingen (H01N0A)* this report treats the analysis of the linkage in a Walschaerts valve gear, a system that is used in steam locomotives. The linkage that is studied in this report is based on that in figure 1.

In a first section we define all links and joints with their geometric properties in the way we used them for the assignment. We also make the motion analysis of the linkage.

Second is the kinematic analysis which finds the positions, velocities and accelerations of each bar.

The final section reports upon the inverse dynamic analysis which finds the forces and torques on the linkages' joints when a driving torque is applied to the train's wheel.

Contents

1 Definition of the mechanism

Walschaerts valve gear is a linkage that was used in steam locomotives. It connects the steam pistons and the train's wheels in a way that also regulates the steam flow.

Although in real life the pistons are the driving bodies and the wheels is the driven bodies, the assistants recommended us to analyse the mechanism in the opposite way. In this assignment, the driving torque is thus applied to the wheel instead of the pistons.

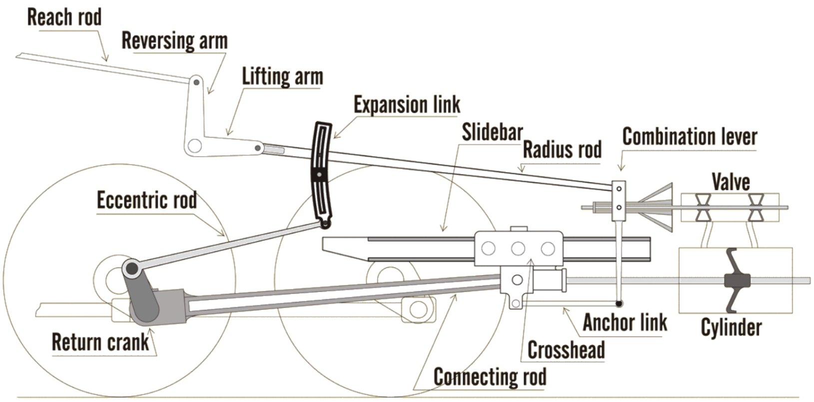


Figure 1: Example of a Walschaerts valve gear. [1]

1.1 Schematic of the mechanism and definition of the parameters

As shown in figure 2, the linkage has twelve bodies.

- Body **(1)** is the train. This is the ground to which other bodies are fixed.
- Body **(2)** is the wheel of the train. Its rotation point (A) is fixed in the origin of the xy-plane. The ends of bars (3) and (12) are eccentrically attached to the wheel with hinges (B) and (C) respectively.
- Bodies **(3)**, **(4)**, **(7)**, **(8)**, **(10)** and **(12)** are straight bars.
- Body **(5)** is massless and is used to make a special joint between bar (4) and bar (7). It is attached to bar (4) with a prismatic joint (I) and to bar (7) with a hinge (H).
- Bar **(6)** is a kinked, L-shaped bar which consists of two parts of 95 cm long that are fixed to each other in an angle of 90 degrees. The two ends of this L-shaped bar are fixed to the ground (1) and to bar (7) with hinges.
- Bodies **(9)** and **(11)** are the pistons of the valve gear. Both pistons are fixed to the ground (1) with prismatic joints ((L) and (P) respectively) that only allow movement in the x-direction. Their dimensions are shown in figure 3.

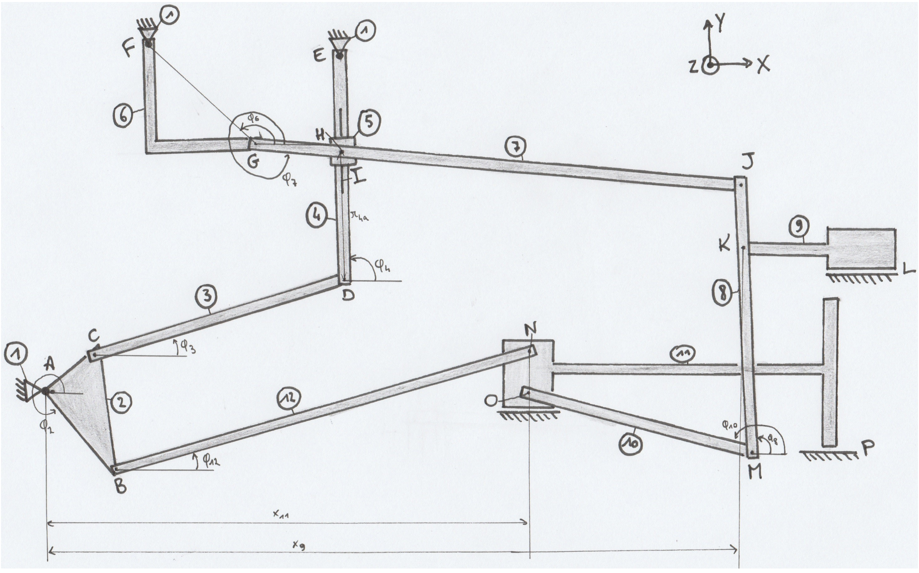


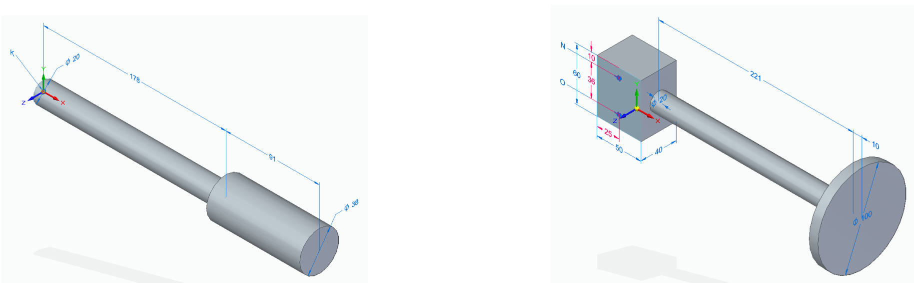
Figure 2: Schematic representation of the studied linkage. The numbers indicate the 12 bodies. The letters indicate the 16 joints. Warning: this figure is not on scale.

The dimensions of the bars and other bodies are defined in tables 1 and 2.

All bodies in this linkage are made of steel which has a density $\rho_{steel} = 7800\ \mathrm{kg/m^3}$ [2].

1.2 Motion analysis

The Walschaerts valve gear is a twelve bar linkage system with sixteen joints. This makes the mobility $M = 3 * (12 - 1) - 16 * 2 = 1$. This means the system has one degree of freedom.



(a) Piston (9).

(b) Piston (11).

Figure 3: Pistons (9) and (11) with their dimensions in cm and with their joints with other bodies.

Bar	Length (cm)
1 between A and E	321
1 between A and F	385
2 between A and C	27
2 between A and B	59
3 between C and D	299
4 between D and F	142
6 between E and G (in a straight line)	135
7 between G and H	100
7 between H and J	452
8 between J and K	30
8 between K and M	146
10 between O and M	158
12 between B and N	559
Y-coordinate of joint K [1]	
Y-coordinate of joint N [2]	

Table 1: Lenghts of the bars and the wheel. Bars' numbers and joints' letters as indicated in figure 2.

Line segment	Angle to the positive x axis (degrees)
[AE]	69
[AF]	37,0188

Table 2: Unvariable angles in the linkage's ground. Joints' letters as indicated in figure 2. Variable angles are calculated in section 2

2 Kinematic analysis

In the kinematic analysis we apply a driving torque to the train's wheel (body (2)) and see how the other bars react.

The driving torque is applied to body (2) in point (A) and makes it turn with a constant angular velocity as defined in equation 2, where $\phi_2(t)$ is the angle that section [AB] of the wheel makes with the positive x axis.

$$\phi_2(t) = \pi * t \ rad \tag{1}$$

$$\frac{\mathrm{d}\phi_2(t)}{\mathrm{d}t} = \pi \ rad/s \tag{2}$$

$$\frac{\mathrm{d}^2\phi_2(t)}{\mathrm{d}t^2} = 0 \ rad/s^2 \tag{3}$$

The kinematic analysis looks for the positions, velocities and accelerations of the other bodies in the linkage. The ten unknowns are $\phi_3(t)$, $\phi_4(t)$, $r_{4a}(t)$ (the length of the part of bar (4) between joint (D) and joint (I)), $\phi_6(t)$, $\phi_7(t)$, $\phi_8(t)$, $x_9(t)$ (the x coordinate of joint (K)), $\phi_{10}(t)$, $x_{11}(t)$ (the x coordinate of joint (N)) and $\phi_{12}(t)$ and their first and second order derivatives. The angles are defined in figure 2.

The analysis is done for 201 time samples of 0,05 s each.

2.1 Position analysis

In order to find the ten unknown positions, a set of ten "loop closure equations" is solved for each time step. A loop closure equation is the sum of all x or y vector components of linkage bodies that form a closed loop. These sums are all equal to zero. This results in a set of ten equations which are listed below in vector form [3].

$$\vec{AB} + \vec{CD} + \vec{DF} - \vec{AF} = 0 \tag{4a}$$

$$-\vec{GE} + \vec{GJ} - \vec{MJ} - \vec{OM} - \vec{NO} - \vec{BN} - \vec{AB} + \vec{AE} = 0 \tag{4b}$$

$$\vec{AC} + \vec{CD} + \vec{DI} - \vec{GH} + \vec{GE} - \vec{AE} = 0 \tag{4c}$$

$$\vec{AC} + \vec{CD} + \vec{DI} + \vec{HJ} - \vec{KJ} - \vec{AK} = 0 \tag{4d}$$

$$\vec{AB} + \vec{BN} - \vec{AN} = 0 \tag{4e}$$

The MATLAB function `loop_closure_eqs.m` lists all loop closure equations. In this case, we get a nonlinear system of equations because of the multiplication of the unknown variables $\phi_4(t)$ and $r_{4a}(t)$. The MATLAB function `fsolve` is used to solve this nonlinear set.

After solving this system for each of the 201 time samples, the MATLAB code returns values for each of the ten unknowns in function of the time. The plots of these ten variables are shown in figure 4.

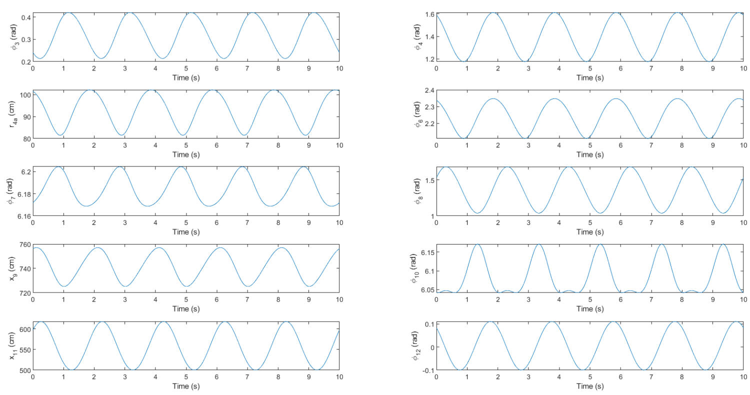


Figure 4: Angles of the bodies (to the positive x axis), distance from joint (D) to (H) and x coordinates of the pistons (according to (A)) in function of time.

[3]These five vector equations define the ten equations because each vector can be split in one x component and one y component.

2.2 Velocity analysis

The velocity analysis finds the ten unknown variables $\dfrac{\mathrm{d}\phi_3(t)}{\mathrm{d}t}$, $\dfrac{\mathrm{d}\phi_4(t)}{\mathrm{d}t}$, $\dfrac{\mathrm{d}r_{4a}(t)}{\mathrm{d}t}$, $\dfrac{\mathrm{d}\phi_6(t)}{\mathrm{d}t}$, $\dfrac{\mathrm{d}\phi_7(t)}{\mathrm{d}t}$, $\dfrac{\mathrm{d}\phi_8(t)}{\mathrm{d}t}$, $\dfrac{\mathrm{d}x_9(t)}{\mathrm{d}t}$, $\dfrac{\mathrm{d}\phi_{10}(t)}{\mathrm{d}t}$, $\dfrac{\mathrm{d}x_{11}(t)}{\mathrm{d}t}$ and $\dfrac{\mathrm{d}\phi_{12}(t)}{\mathrm{d}t}$.

The first order time derivatives of the loop closure equations 4 are now used to find the unknown velocities. They are listed in the MATLAB function `velocity_eqs`. The first order derivative of body (2) is known and is defined in equation 2.

The new set of loop closure equations is now linear because the derivative of the multiplication of $\phi_4(t)$ and $r_{4a}(t)$ contains no multiplication of the now unknown $\dfrac{\mathrm{d}\phi_4(t)}{\mathrm{d}t}$ and $\dfrac{\mathrm{d}r_{4a}(t)}{\mathrm{d}t}$.

The MATLAB code solves this system and returns values for the velocities of the bodies in function of the time. They are plotted in figure 5.

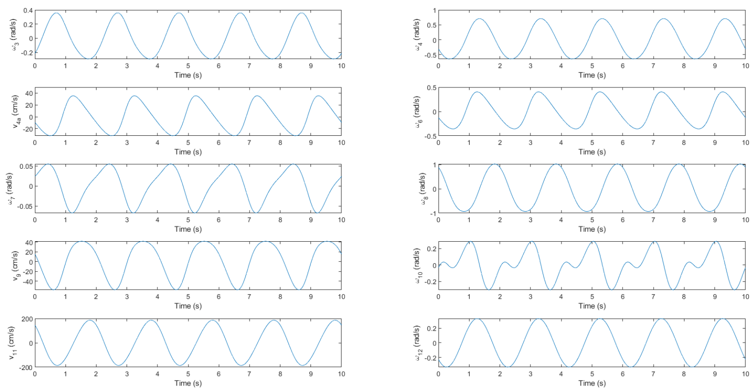


Figure 5: Angular velocities of the bodies (to the positive x axis) and linear velocities of joint (H) (according to (D)) and of the pistons (according to (A)) in function of time.

2.3 Acceleration analysis

The acceleration analysis finds the ten unknown variables $\frac{\mathrm{d}^2\phi_3(t)}{\mathrm{d}t^2}$, $\frac{\mathrm{d}^2\phi_4(t)}{\mathrm{d}t^2}$, $\frac{\mathrm{d}^2 r_{4a}(t)}{\mathrm{d}t^2}$, $\frac{\mathrm{d}^2\phi_6(t)}{\mathrm{d}t^2}$, $\frac{\mathrm{d}^2\phi_7(t)}{\mathrm{d}t^2}$, $\frac{\mathrm{d}^2\phi_8(t)}{\mathrm{d}t^2}$, $\frac{\mathrm{d}^2 x_9(t)}{\mathrm{d}t^2}$, $\frac{\mathrm{d}^2\phi_{10}(t)}{\mathrm{d}t^2}$, $\frac{\mathrm{d}^2 x_{11}(t)}{\mathrm{d}t^2}$ and $\frac{\mathrm{d}^2\phi_{12}(t)}{\mathrm{d}t^2}$.

This happens analogously to the velocity analysis. The MATLAB code now solves a linear set of the second order time derivatives of the ten loop closure equations. These are listed in the MATLAB function `acceleration_eqs`. The plots of the accelerations of the bodies are in figure 6.

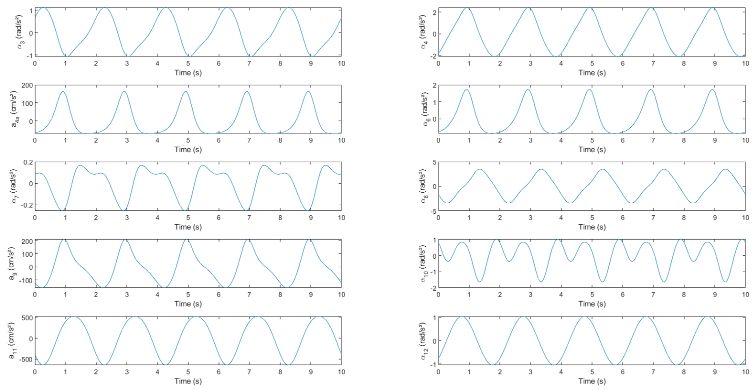


Figure 6: Angular accelerations of the bodies (to the positive x axis) and linear accelerations of joint (H) (according to (D)) and of the pistons (according to (A)) in function of time.

2.4 Checking the results

When writing the equations, there are places where errors can easily sneak in (e.g. notation errors). To check whether the generated results are correct, control calculations have been made. This section will be about checking the kinematics.

2.4.1 Checking the position

To check the position, we calculate the position of the joint (F) in two different ways. First, because (F) is a stationary point, it can be calculated via the given distance and angle from (A):

$$\vec{F_1} = r_{1b} * e^{i*\phi_{AF}} \tag{5}$$

The second way is by following a path of consecutive points in the mechanism:

$$\vec{F_2} = r_{2c} * e^{i*(\phi_2+\phi_A)} + r_3 * e^{i*\phi_3} + r_4 * e^{i*\phi_4} \tag{6}$$

The x and y components of these vectors can be found by taking respectively the real and imaginary part of these complex numbers. The difference between

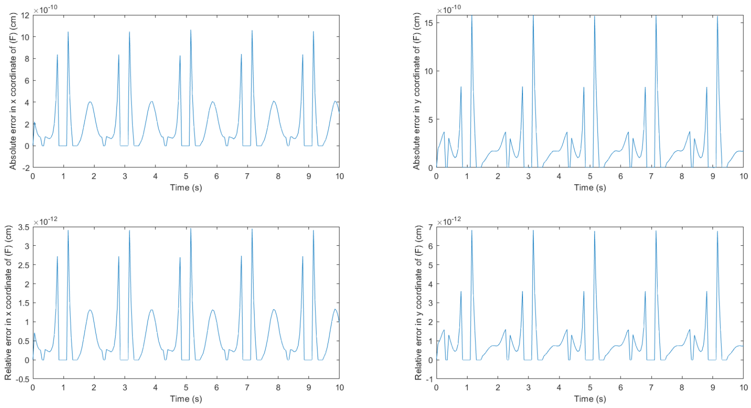


Figure 7: Position errors in coordinates of point (F) in function of time.

$\vec{F_1}$ and $\vec{F_2}$ is plotted in figure 7. The x and y components match well, there is nothing special to report there.

The relative error $1-\vec{F_1}/\vec{F_2}$ takes on values of the order of 10^{-12} and the absolute error $\vec{F_1}-\vec{F_2}$ is of the order of 10^{-10} cm. This is not yet near machine precision, so the error is probably a result of using `fsolve`. The error is small enough to consider this check successful.

We can also visually inspect the position results by watching an animation of the mechanism. In the animation in the attached MATLAB code [4] one can see that point (F) (like (A) and (E)) doesn't move.

2.4.2 Checking the velocity and the acceleration

Checking the speed and acceleration is done in the same way. Because (F) is a stationary point, its velocity and acceleration have to be equal to zero at all times. This means that by plotting the velocity or acceleration of (F) calculated via a path of consecutive points from (A) to (F), one will plot the error.

The velocity of point (F) (via (A)) can be calculated as follows:

$$\vec{v_F} = \vec{\omega_2} \times \vec{AC} + \vec{\omega_3} \times \vec{CD} + \vec{\omega_4} \times \vec{DF} \tag{7}$$

Figure 8 presents the error in velocity of point (F). The x and y components match well, there is nothing special to report there. Except for the two deviating values, this error seems to stay perfectly at 0. After zooming in we can see that this error doesn't perfectly equal 0 but is instead in the order of 10^{-14}cm/s. This is starting to approach machine precision but is still not as precise. Again, the error is a result of using `fsolve`. The error is small enough to consider this check successful.

[4] The animation pops up when running `main.m` with `movie_12bar = 1;` on line 22 of the code

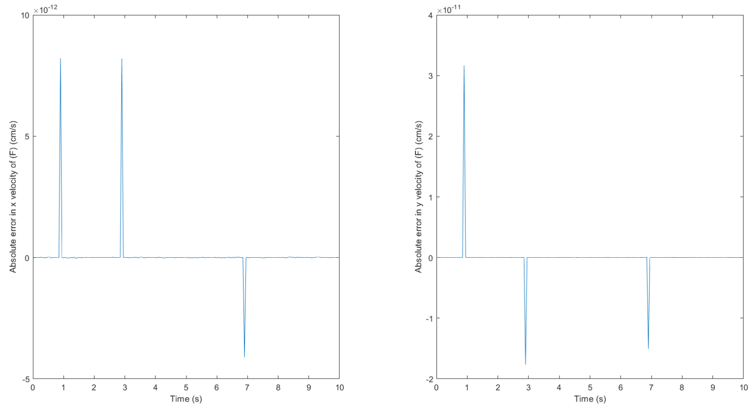


Figure 8: Velocity errors in coordinates of point (F) in function of time.

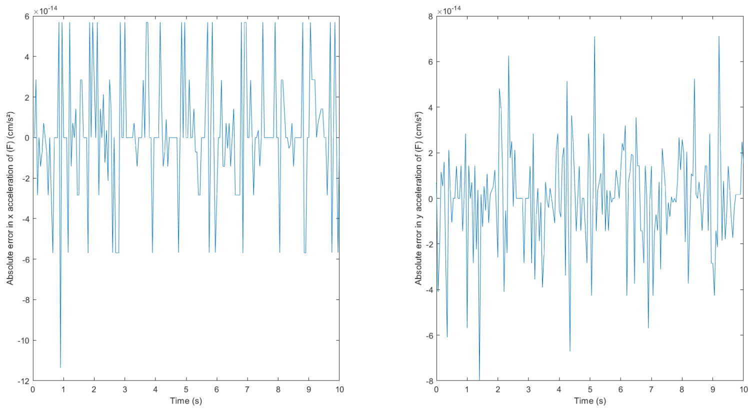


Figure 9: Acceleration errors in coordinates of point (F) in function of time.

The acceleration of point (F) (via (A)) can be calculated as follows:

$$\vec{a_F} = \vec{\omega_2}\times(\vec{\omega_2}\times\vec{AC})+\vec{\alpha_2}\times\vec{AC}+\vec{\omega_3}\times(\vec{\omega_3}\times\vec{CD})+\vec{\alpha_3}\times\vec{CD}+\vec{\omega_4}\times(\vec{\omega_4}\times\vec{DF})+\vec{\alpha_4}\times\vec{DF} \tag{8}$$

Figure 9 shows the acceleration error. Once again the x and y components match so there is nothing special to report. The error here is from the order of 10^{-14}cm/s^2. Like with the position and velocity errors, this isn't machine precision yet but a deviation introduced by `fsolve`. It's small enough to consider this check successful.

3 Dynamic analysis

3.1 Inverse dynamic analysis

The dynamic analysis determines the 32 internal reaction forces or torques and the driving torque on the wheel. These forces and torques are defined in figure 10

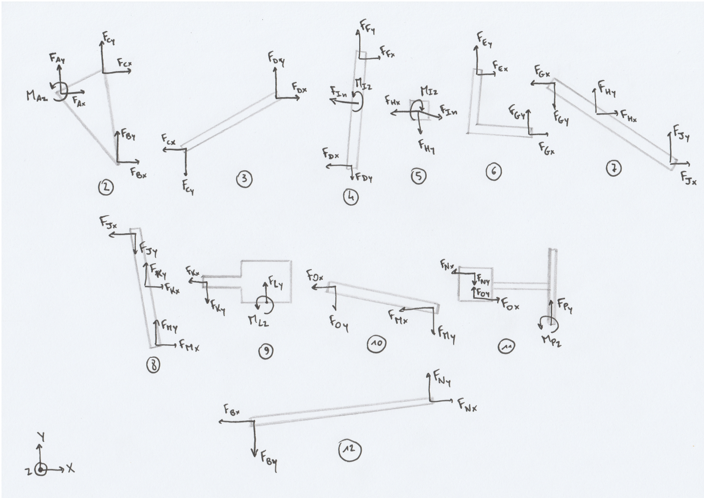


Figure 10: Definition of the internal reactions forces and torques.

In order to calculate these 33 unknown variables, a linear system of 33 equations is set up. Each of the eleven bodies (without the ground (1)) gives three dynamic equations: one for the forces in the x direction, one for the forces in the y direction and one for the torques around the z axis. The 33 equations can be found below:

Body (2):

$$F_{A,x} + F_{B,x} + F_{C,x} = m_2 * a_{2,x} \tag{9a}$$

$$F_{A,y} + F_{B,y} + F_{C,y} = m_2 * a_{2,y} \tag{9b}$$

$$\begin{aligned} -F_{A,x} * |cog_2 A|_y + F_{A,y} * |cog_2 A|_x + M_{A,z} \\ -F_{B,x} * |cog_2 B|_y + F_{B,y} * |cog_2 B|_x \\ -F_{C,x} * |cog_2 C|_y + F_{C,y} * |cog_2 C|_x \\ = I_2 * \alpha_2 \end{aligned} \tag{9c}$$

Body (3):

$$-F_{C,x} + F_{D,x} = m_3 * a_{3,x} \tag{10a}$$

$$-F_{C,y} + F_{D,y} = m_3 * a_{3,y} \tag{10b}$$

$$\begin{aligned} F_{C,x} * |cog_3C|_y - F_{C,y} * |cog_3C|_x \\ -F_{D,x} * |cog_3D|_y + F_{D,y} * |cog_3D|_x \\ = I_3 * \alpha_3 \end{aligned} \tag{10c}$$

Body (4):

$$-F_{D,x} + F_{F,x} - F_{I,n} * cos(\phi_4) = m_4 * a_{4,x} \tag{11a}$$

$$-F_{D,y} + F_{F,y} + F_{I,n} * sin(\phi_4) = m_4 * a_{4,y} \tag{11b}$$

$$\begin{aligned} F_{D,x} * |cog_4D|_y - F_{D,y} * |cog_4D|_x \\ -F_{F,x} * |cog_4F|_y + F_{F,y} * |cog_4F|_x \\ = I_4 * \alpha_4 \end{aligned} \tag{11c}$$

Body (5):

$$-F_{H,x} + F_{I,n} * cos(\phi_4) = m_5 * a_{5,x} = 0 \tag{12a}$$

$$-F_{H,y} - F_{I,n} * sin(\phi_4) = m_5 * a_{5,y} = 0 \tag{12b}$$

$$-M_{I,z} = I_5 * \alpha_5 = 0 \tag{12c}$$

Body (6):

$$F_{E,x} + F_{G,x} = m_6 * a_{6,x} \tag{13a}$$

$$F_{E,y} + F_{G,y} = m_6 * a_{6,y} \tag{13b}$$

$$\begin{aligned} -F_{E,x} * |cog_6E|_y + F_{E,y} * |cog_6E|_x \\ -F_{G,x} * |cog_6G|_y + F_{G,y} * |cog_6G|_x \\ = I_6 * \alpha_6 \end{aligned} \tag{13c}$$

Body (7):

$$-F_{G,x} + F_{H,x} + F_{J,x} = m_7 * a_{7,x} \tag{14a}$$

$$-F_{G,y} + F_{H,y} + F_{J,y} = m_7 * a_{7,y} \tag{14b}$$

$$\begin{aligned} F_{G,x} * |cog_7G|_y - F_{G,y} * |cog_7G|_x \\ -F_{H,x} * |cog_7H|_y + F_{H,y} * |cog_7H|_x \\ -F_{J,x} * |cog_7J|_y + F_{J,y} * |cog_7J|_x \\ = I_7 * \alpha_7 \end{aligned} \tag{14c}$$

Body (8):

$$-F_{J,x} + F_{K,x} + F_{M,x} = m_8 * a_{8,x} \tag{15a}$$

$$-F_{J,y} + F_{K,y} + F_{M,y} = m_8 * a_{8,y} \tag{15b}$$

$$\begin{aligned} F_{J,x} * |cog_8 J|_y - F_{J,y} * |cog_8 J|_x \\ -F_{K,x} * |cog_8 K|_y + F_{K,y} * |cog_8 K|_x \\ -F_{M,x} * |cog_8 M|_y + F_{M,y} * |cog_8 M|_x \\ = I_8 * \alpha_8 \end{aligned} \tag{15c}$$

Body (9):

$$-F_{K,x} = m_9 * a_{9,x} \tag{16a}$$

$$-F_{K,y} + F_{L,y} = m_9 * a_{9,y} \tag{16b}$$

$$\begin{aligned} F_{K,x} * |cog_9 K|_y - F_{K,y} * |cog_9 K|_x \\ +F_{L,y} * |cog_9 L|_x + M_{L,z} \\ = I_9 * \alpha_9 \end{aligned} \tag{16c}$$

Body (10):

$$-F_{M,x} - F_{O,x} = m_{10} * a_{10,x} \tag{17a}$$

$$-F_{M,y} - F_{O,y} = m_{10} * a_{10,y} \tag{17b}$$

$$\begin{aligned} F_{M,x} * |cog_{10} M|_y - F_{M,y} * |cog_{10} M|_x \\ F_{O,x} * |cog_{10} O|_y - F_{O,y} * |cog_{10} O|_x \\ = I_{10} * \alpha_{10} \end{aligned} \tag{17c}$$

Body (11):

$$-F_{N,x} + F_{O,x} = m_{11} * a_{11,x} \tag{18a}$$

$$-F_{N,y} + F_{O,y} + F_{P,y} = m_{11} * a_{11,y} \tag{18b}$$

$$\begin{aligned} F_{N,x} * |cog_{11} N|_y - F_{N,y} * |cog_{11} N|_x \\ -F_{O,x} * |cog_{11} O|_y + F_{O,y} * |cog_{11} O|_x \\ +F_{P,y} * |cog_{11} P|_x + M_{P,z} \\ = I_{11} * \alpha_{11} \end{aligned} \tag{18c}$$

Body (12):

$$-F_{B,x} + F_{N,x} = m_{12} * a_{12,x} \tag{19a}$$

$$-F_{B,y} + F_{N,y} = m_{12} * a_{12,y} \tag{19b}$$

$$\begin{aligned} F_{B,x} * |cog_{12}B|_y - F_{B,y} * |cog_{12}B|_x \\ -F_{N,x} * |cog_{12}N|_y + F_{N,y} * |cog_{12}N|_x \\ = I_{12} * \alpha_{12} \end{aligned} \tag{19c}$$

The masses of the bodies are $m_i = V_i * \rho_{steel}$ with $\rho_{steel} = 7800$ kg/m^3 [2]. The linear accelerations of the bars are calculated by multiplicating the angular accelerations with the distance from the joint to the center of gravity of the bar. All these equations are implemented in the MATLAB function `dynamics_12bar.m`.

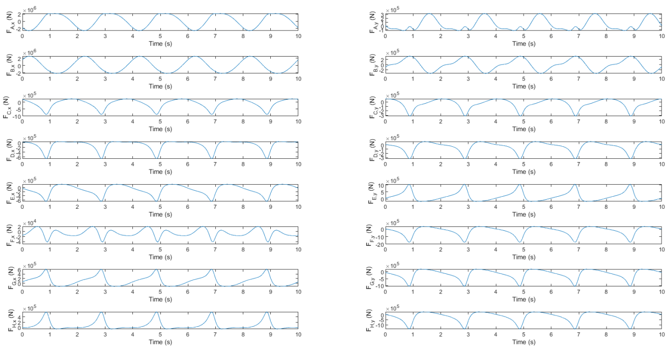


Figure 11: Internal reaction forces and torques in function of time (Part 1/2).

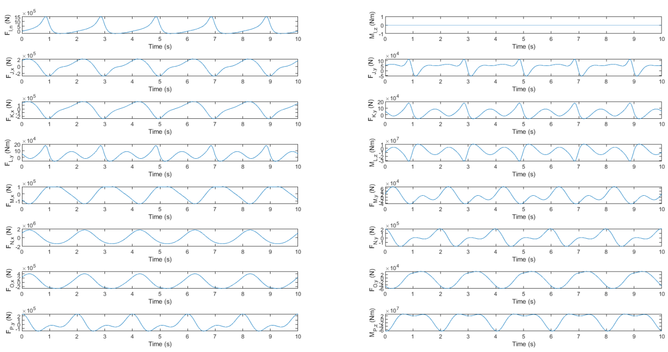


Figure 12: Internal reaction forces and torques in function of time (Part 2/2).

After solving this linear set of equations, the MATLAB code returns values for the 33 forces and torques for each time sample. The plots of these forces and torques in function of the time are listed in figures 11, 12 and 13.

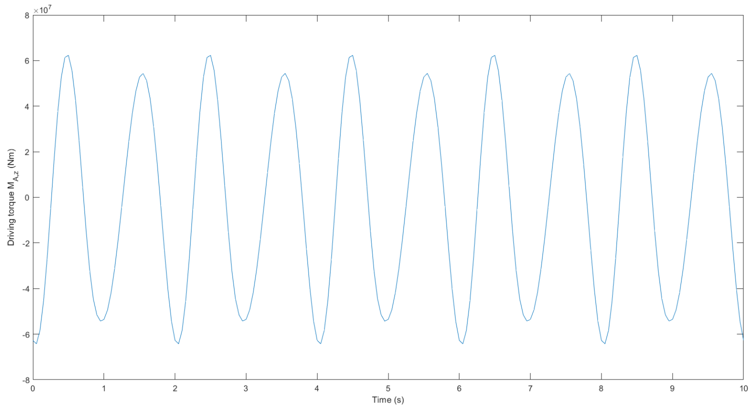


Figure 13: Driving torque in function of time.

3.2 Checking the results

For checking the dynamics we make use of the method of shaking forces, in which we determine the reaction forces that act on the ground. These reaction forces are the opposites of the internal forces F_A, F_F and F_E . The shaking forces are equal to these reaction forces on the ground:

$$F_{shake,x,1} = F_{A,x} + F_{F,x} + F_{E,x} \tag{20}$$

and can also be calculated as follows (for the x components):

$$\begin{aligned} F_{shake,x,2} = a_{2,x} * m_2 + a_{3,x} * m_3 + a_{4,x} * m_4 + a_{5,x} * m_5 \\ + a_{6,x} * m_6 + a_{7,x} * m_7 + a_{8,x} * m_8 + a_{9,x} * m_9 \\ + a_{10,x} * m_{10} + a_{11,x} * m_{11} + a_{12,x} * m_{12} \end{aligned} \tag{21}$$

The y components are calculated analogously.

We make the sum over all linkages except for the ground. The error of the shaking forces $F_{shake,x,1} - F_{shake,x,2}$ is plotted in figure 14. The x and y components match well enough, even though one is one order of magnitude greater than the other. The error is from the order of 10^{-9} Newton for the x component and 10^{-10} Newton for the y component. This error is greater than machine precision, but this is a result of the used math algorithm. The error is small enough to consider this check successful.

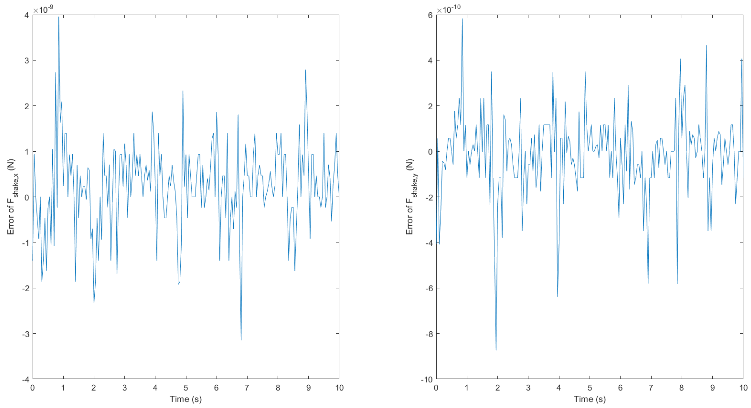


Figure 14: Error of the shaking forces in function of time.

References

[1] Henk Oversloot. Walschaerts and crosshead construction, 2006.

[2] Ames Web. Density of steel, 2013.