COMP 302 - Classtest 3 Problem Set

Capture-Avoiding Substitutions

For each expression, identify the bound variables and the free variables. Which variables will be renamed? What does the expression evaluate to? These are also implemented in the OCaml file.

(a)

```
[1/y, 5/b, 3/a, (a+b)/x] let x = y in let y = x in x + y
```

The only free variable is highlighted in red, the rest are bound variables.

$$[1/y,\ 5/b,\ 3/a,\ (a+b)/x]\ let\ x = y\ in\ let\ y = x\ in\ x + y$$

Renaming properly, we get:

$$[1/y,\ 5/b,\ 3/a,\ (a+b)/x]\ let\ x' = y\ in\ let\ y' = x'\ in\ x' + y'$$

Substituting properly, we obtain:

$$\begin{aligned} let\ x' = 1\ in\ let\ y' = x'\ in\ x' + y' \\ let\ y' = 1\ in\ 1 + y' \\ 1 + 1 \\ 2 \end{aligned}$$

(b)

```
[z/x] let x' = 1 in let x'' = 2 in let x''' = 3 in x''' + x'''
```

Here, the x is bound everywhere and there are no free variables. Renaming and evaluating goes as follows:

$$\begin{aligned} let\ x' = 1\ in\ let\ x'' = 2\ in\ let\ x''' = 3\ in\ x''' + x''' \\ let\ x'' = 2\ in\ let\ x''' = 3\ in\ x''' + x''' \\ let\ x''' = 3\ in\ x''' + x''' \\ 3 + 3 \\ 6 \end{aligned}$$

(c)

```
[0/x] let y = x in if y < x then x else y
```

The free variable is highlighted in red:

$$[0/x]\ let\ y = x\ in\ if\ y < x\ then\ x\ else\ y$$

No renaming takes place, and substituting goes as follows:

$$\begin{aligned} let\ y = 0\ in\ if\ y < 0\ then\ 0\ else\ y \\ if\ 0 < 0\ then\ 0\ else\ 0 \\ if\ false\ then\ 0\ else\ 0 \\ 0 \end{aligned}$$

(d)

```
[7/y, 2/z, 3/x] let x = y in let y = x in let z = x + (x + y) + z in z
```

The free variables are highlighted in red:

$$[7/y,\ 2/z,\ 3/x]\ let\ x = y\ in\ let\ y = x\ in\ let\ z = x + (x + y) + z\ in\ z$$

Renaming properly gives us:

$$[7/y,\ 2/z,\ 3/x]\ let\ x' = y\ in\ let\ y' = x'\ in\ let\ z' = x' + (x' + y') + z\ in\ z'$$

Substituting goes as follows:

$$\begin{aligned}
[7/y]\ let\ x' = y\ in\ let\ y' = x'\ in\ let\ z' = x' + (x' + y') + 2\ in\ z' \\
let\ x' = 7\ in\ let\ y' = x'\ in\ let\ z' = x' + (x' + y') + 2\ in\ z' \\
let\ y' = 7\ in\ let\ z' = 7 + (7 + y') + 2\ in\ z' \\
let\ z' = 7 + (7 + 7) + 2\ in\ z' \\
let\ z' = 23\ in\ z' \\
23
\end{aligned}$$

Subtyping

1. $$\texttt{int} \leq \texttt{float} \quad \text{by } S\text{-Base}$$

2. $$\frac{\texttt{int} \leq \texttt{float} \quad \texttt{int} \leq \texttt{float}}{\texttt{int} \times \texttt{int} \leq \texttt{float} \times \texttt{float}} \quad \text{by } S\text{-Prod}$$

3. $$\frac{\texttt{int} \leq \texttt{float} \quad \texttt{int} \leq \texttt{float}}{\texttt{float} \to \texttt{int} \leq \texttt{int} \to \texttt{float}} \quad \text{by } S\text{-Fun} \quad (\text{contravariant input, covariant output})$$

4. $$\frac{\dfrac{\texttt{even} \not\leq \texttt{int} \quad \texttt{int} \leq \texttt{int}}{\texttt{even} \times \texttt{int} \not\leq \texttt{int} \times \texttt{int}} \quad \texttt{int} \leq \texttt{float}}{(\texttt{even} \times \texttt{int}) \to \texttt{int} \not\leq (\texttt{int} \times \texttt{int}) \to \texttt{float}} \quad \text{by } S\text{-Fun}$$

5. $$\frac{\dfrac{\texttt{float} \leq \texttt{int} \quad \texttt{int} \leq \texttt{int}}{\texttt{float} \to \texttt{int} \leq \texttt{int} \to \texttt{int}} \quad \texttt{bool} \leq \texttt{bool}}{(\texttt{int} \to \texttt{int}) \to \texttt{bool} \leq (\texttt{float} \to \texttt{int}) \to \texttt{bool}} \quad \text{by } S\text{-Fun}$$

6. $$\texttt{int ref} \not\leq \texttt{float ref} \quad \text{by invariance of references (S-Ref requires equality)}$$

7. $$\frac{\dfrac{\texttt{int} \leq \texttt{int} \quad \texttt{bool} \leq \texttt{bool}}{\texttt{int} \to \texttt{bool} \leq \texttt{int} \to \texttt{bool}} \quad \texttt{even} \leq \texttt{int}}{(\texttt{int} \to \texttt{bool}) \times \texttt{even} \leq (\texttt{int} \to \texttt{bool}) \times \texttt{int}} \quad \text{by } S\text{-Prod}$$