# COMP 302 - Classtest 3 Problem Set

## Capture-Avoiding Substitutions

For each expression, identify the bound variables and the free variables. What does the expression evaluate to? These are also implemented in the OCaml file.

### (a)

```
[1/y, 5/b, 3/a, (a+b)/x] let x = y in let y = x in x + y
```

The only free variable is highlighted in red, the rest are bound variables.

$$[1/y,\ 5/b,\ 3/a,\ (a+b)/x]\ let\ x = y\ in\ let\ y = x\ in\ x + y$$

Substituting properly, we obtain:

$$\begin{aligned} let\ x = 1\ in\ let\ y = x\ in\ x + y \\ let\ y = 1\ in\ 1 + y \\ 1 + 1 \\ 2 \end{aligned}$$

### (b)

```
[z/x] let x = 1 in let x = 2 in let x = 3 in x + x
```

Here, the $x$ is bound everywhere and there are no free variables. Evaluating goes as follows:

$$\begin{aligned} let\ x = 1\ in\ let\ x = 2\ in\ let\ x = 3\ in\ x + x \\ let\ x = 2\ in\ let\ x = 3\ in\ x + x \\ let\ x = 3\ in\ x + x \\ 3 + 3 \\ 6 \end{aligned}$$

### (c)

```
[0/x] let y = x in if y < x then x else y
```

The free variable is highlighted in red:

$$[0/x]\ let\ y = x\ in\ if\ y < x\ then\ x\ else\ y$$

Substituting goes as follows:

$$\begin{aligned} let\ y = 0\ in\ if\ y < 0\ then\ 0\ else\ y \\ if\ 0 < 0\ then\ 0\ else\ 0 \\ if\ false\ then\ 0\ else\ 0 \\ 0 \end{aligned}$$

### (d)

```
[7/y, 2/z, 3/x] let x = y in let y = x in let z = x + (x + y) + z in z
```

The free variables are highlighted in red:

$$[7/y,\ 2/z,\ 3/x]\ let\ x = y\ in\ let\ y = x\ in\ let\ z = x + (x + y) + z\ in\ z$$

Substituting goes as follows:

$$
\begin{aligned}
[7/y]\ \mathit{let}\ x = y\ \mathit{in}\ \mathit{let}\ y = x\ \mathit{in}\ \mathit{let}\ z = x + (x + y) + 2\ \mathit{in}\ z \\
\mathit{let}\ x = 7\ \mathit{in}\ \mathit{let}\ y = x\ \mathit{in}\ \mathit{let}\ z = x + (x + y) + 2\ \mathit{in}\ z \\
\mathit{let}\ y = 7\ \mathit{in}\ \mathit{let}\ z = 7 + (7 + y) + 2\ \mathit{in}\ z \\
\mathit{let}\ z = 7 + (7 + 7) + 2\ \mathit{in}\ z \\
\mathit{let}\ z = 23\ \mathit{in}\ z \\
23
\end{aligned}
$$

## Subtyping

1. $$\texttt{int} \leq \texttt{float} \quad \text{by } S\text{-Base}$$

2. $$\frac{\texttt{int} \leq \texttt{float} \quad \texttt{int} \leq \texttt{float}}{\texttt{int} \times \texttt{int} \leq \texttt{float} \times \texttt{float}} \quad \text{by } S\text{-Prod}$$

3. $$\frac{\texttt{int} \leq \texttt{float} \quad \texttt{int} \leq \texttt{float}}{\texttt{float} \to \texttt{int} \leq \texttt{int} \to \texttt{float}} \quad \text{by } S\text{-Fun} \quad (\text{contravariant input, covariant output})$$

4. $$\frac{\dfrac{\texttt{int} \leq \texttt{int} \quad \texttt{int} \not\leq \texttt{even}}{\texttt{int} \times \texttt{int} \not\leq \texttt{even} \times \texttt{int}} \quad \texttt{int} \leq \texttt{float}}{(\texttt{even} \times \texttt{int}) \to \texttt{int} \not\leq (\texttt{int} \times \texttt{int}) \to \texttt{float}} \quad \text{by } S\text{-Fun}$$

5. $$\frac{\dfrac{\texttt{int} \leq \texttt{float} \quad \texttt{int} \leq \texttt{int}}{\texttt{float} \to \texttt{int} \leq \texttt{int} \to \texttt{int}} \quad \texttt{bool} \leq \texttt{bool}}{(\texttt{int} \to \texttt{int}) \to \texttt{bool} \leq (\texttt{float} \to \texttt{int}) \to \texttt{bool}} \quad \text{by } S\text{-Fun}$$

6. $$\texttt{int ref} \not\leq \texttt{float ref} \quad \text{by invariance of references (S-Ref requires equality)}$$

7. $$\frac{\dfrac{\texttt{int} \leq \texttt{int} \quad \texttt{bool} \leq \texttt{bool}}{\texttt{int} \to \texttt{bool} \leq \texttt{int} \to \texttt{bool}} \quad \texttt{even} \leq \texttt{int}}{(\texttt{int} \to \texttt{bool}) \times \texttt{even} \leq (\texttt{int} \to \texttt{bool}) \times \texttt{int}} \quad \text{by } S\text{-Prod}$$