Problem 3

$$\det(A-\lambda) = \begin{vmatrix} 1-\lambda & 0 \\ 1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 \Rightarrow \lambda = 1 \text{ (double root)}$$

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\} \text{ is a basis for the eigenspace associated to } \lambda = 1$$

$$\det(B-\lambda) = \begin{vmatrix} -2-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (\lambda+3)(\lambda-2) \Rightarrow \lambda = -3, 2$$

$$\begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -3 \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \left\{ \begin{pmatrix} 2 \\ -1 \end{pmatrix} \right\} \text{ is a basis for the eigenspace associated to } \lambda = -3$$

$$\begin{pmatrix} -2 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\} \text{ is a basis for the eigenspace associated to } \lambda = 2$$

Problem 8

$$A^T A = \begin{pmatrix} 13 & 12 & 2 \\ 12 & 13 & -2 \\ 2 & -2 & 8 \end{pmatrix} \to x_{\lambda=0} = \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}, x_{\lambda=9} = \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}, x_{\lambda=25} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

Normalizing the eigenvectors, we find the orthonormal basis

$$\left\{ \frac{1}{3} \begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{18}} \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right\}$$

Using this basis,

$$P = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{18}} & -\frac{2}{3} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{18}} & \frac{2}{3} \\ 0 & \frac{4}{\sqrt{18}} & \frac{1}{3} \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$$

$$u_1 = \frac{\frac{1}{\sqrt{2}}}{5} A \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad u_2 = \frac{\frac{1}{\sqrt{18}}}{3} A \begin{pmatrix} 1 \\ -1 \\ 4 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The SVD is then given by

$$A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{18}} & -\frac{1}{\sqrt{18}} & \frac{4}{\sqrt{18}} \\ -\frac{2}{3} & \frac{2}{3} & \frac{1}{3} \end{pmatrix}$$

Problem 9

$$A^T A = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix} \to x_{\lambda=8} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, x_{\lambda=2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Normalizing the eigenvectors, we find the orthonormal basis

$$\left\{ \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\}$$

Using this basis,

$$P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{pmatrix}$$

$$u_1 = \frac{\frac{1}{\sqrt{2}}}{\sqrt{8}} A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad u_2 = \frac{\frac{1}{\sqrt{2}}}{\sqrt{2}} A \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The SVD is then given by

$$A = \begin{pmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Problem 10

$$\hat{A}_1 = \sigma_1 u_1 v_1^T = 5 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} = \frac{5}{2} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

Problem 11

Let λ be a non-zero eigenvalue of $A^T A$. Then $\exists x \in \mathbb{R}^n$ such that $(A^T A)x = \lambda x$. Notice that $(AA^T)(Ax) = A(A^T Ax) = \lambda(Ax)$. Therefore λ is an eigenvalue of AA^T.

Similarly, let λ be a non-zero eigenvalue of AA^T. Then $\exists x \in \mathbb{R}^m$ such that $(AA^T)x = \lambda x$. Notice that $(A^T A)(A^T x) = A^T(AA^T x) = \lambda(A^T x)$. Therefore λ is an eigenvalue of $A^T A$.

Combining these statements, we see that $A^T A$ and AA^T possess the same non-zero eigenvalues.

Problem 12

$$||Ax||_2 = ||U\Sigma V^T x||_2 = ||\Sigma V^T x||_2 \leq ||\sigma_1 V^T x||_2 = \sigma_1 ||V^T x||_2 = \sigma_1 ||x||_2$$

$$\therefore \max_x \frac{||Ax||_2}{||x||_2} = \sigma_1$$