Problem 1

a. Clearly, $a * b \in \mathbb{R}$ for any $a, b \in G$. To prove that the group is **closed** under $*$, we just need to show that $a * b = -1$ has no solutions.

Suppose that $\exists a, b \in G$ such that $a * b = -1$.

$$-1 = a * b = a(1 + b) + b \Rightarrow -(1 + b) = a(1 + b) \Rightarrow a = -1 \Rightarrow\Leftarrow$$

Therefore, $a * b = -1$ has no solutions and G is closed under $*$.

We can prove **associativity** easily by considering $a, b, c \in G$:

$$\begin{aligned}(a * b) * c &= (a * b)c + (a * b) + c = (ab + a + b)c + ab + a + b + c \\ &= a(bc + b + c) + a + (bc + b + c) = a(b * c) + a + (b * c) = a * (b * c)\end{aligned}$$

The **identity** element of the group is 0:

$$a * 0 = a \cdot 0 + a + 0 = a$$

The **inverse** of any element $a \in G$ is $-(\frac{a}{a+1})$:

$$a * \left(\frac{-a}{a+1}\right) = a \cdot \left(\frac{-a}{a+1}\right) + a - \left(\frac{a}{a+1}\right) = -\left(\frac{a}{a+1}\right)(a - (a + 1) + 1) = 0$$

The operation $*$ is manifestly **commutative**. Therefore, G is an abelian group.

b. There are two solutions $x = -3, 1$.

$$3 * (x^2 + 2x) = 3(x^2 + 2x) + 3 + x^2 + 2x = 4(x^2 + 2x) + 3 = 15 \to x^2 + 2x - 3 = 0$$

Problem 2

a. By Euclid's division lemma, $a + b = m \cdot n + r$ for $m, r \in \mathbb{Z}, 0 \leq r < n$.

$$\begin{aligned}\overline{a + b} &= \{p \in \mathbb{Z} : \exists q \in \mathbb{Z} \text{ s.t. } p - (a + b) = q \cdot n\} \\ &= \{p \in \mathbb{Z} : \exists q \in \mathbb{Z} \text{ s.t. } p - (m \cdot n + r) = q \cdot n\} \\ &= \{p \in \mathbb{Z} : \exists q \in \mathbb{Z} \text{ s.t. } p - r = q \cdot n\} \\ &= \overline{r} \in \mathbb{Z}_n\end{aligned}$$

Therefore, **closed** under $\oplus$.

Associativity, **commutativity**, and existence of an **identity** element follows directly from $+$.

The **inverse** of an element $\overline{a} \in \mathbb{Z}_n$ is $\overline{n - a} \in \mathbb{Z}_n$. Therefore, G is an Abelian group.

d. Suppose that $(\mathbb{Z}_n \backslash \{\overline{0}\}, \otimes)$ is a group. Then, $\nexists a, b \in \mathbb{Z}^+_{<n}$ such that $a \times b = n$, therefore n is prime.

Suppose that n is prime. Then, $\nexists a, b \in \mathbb{Z}^+_{<n}$ such that $a \times b = n$, meaning that $\mathbb{Z}_n \backslash \{\overline{0}\}$ is closed under $\otimes$. Therefore, $(\mathbb{Z}_n \backslash \{\overline{0}\}, \otimes)$ is a group.

Therefore, $(\mathbb{Z}_n \backslash \{\overline{0}\}, \otimes)$ is a group iff n is prime.

Problem 3

It is a **non-abelian** group. **Associativity** follows directly from matrix multiplication. The **identity** element is the just the canonical identity element for matrices.

Closed:

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u & v \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & u+x & v+xw+z \\ 0 & 1 & w+y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}$$

Inverse

$$\begin{pmatrix} 1 & -u & uw-v \\ 0 & 1 & -w \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & u & v \\ 0 & 1 & w \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Problem 5

a.

$$[A|b] = \left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 2 & 5 & -7 & -5 & -2 \\ 2 & -1 & 1 & 3 & 4 \\ 5 & 2 & -4 & 2 & 6 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 1 & -1 & -1 & 1 \\ 0 & 3 & -5 & -3 & -4 \\ 0 & -3 & 3 & 5 & 2 \\ 0 & -3 & 1 & 7 & 1 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & \frac{2}{3} & 0 & \frac{7}{3} \\ 0 & 1 & -\frac{5}{3} & -1 & -\frac{4}{3} \\ 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & \frac{1}{4} \end{array}\right)$$

Therefore, no solution exists.

b.

$$[A|b] = \left(\begin{array}{ccccc|c} 1 & -1 & 0 & 0 & 1 & 3 \\ 1 & 1 & 0 & -3 & 0 & 6 \\ 2 & -1 & 0 & 1 & -1 & 5 \\ -1 & 2 & 0 & -2 & -1 & -1 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -\frac{3}{2} & \frac{1}{2} & \frac{9}{2} \\ 0 & 1 & 0 & -\frac{3}{2} & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 & \frac{5}{2} & -\frac{5}{2} & -\frac{5}{2} \\ 0 & 0 & 0 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{array}\right)$$

$$\sim \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore, the set of solutions is given by

$$\left\{ \begin{pmatrix} 3+\lambda \\ 2\lambda \\ 0 \\ -1+\lambda \\ \lambda \end{pmatrix} + \alpha_1 \begin{pmatrix} 1 \\ 2 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} : \lambda, \alpha_i \in \mathbb{R} \right\}$$

Problem 6

$$[A|b] = \left(\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right) \sim \left(\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 & -1 \end{array}\right)$$

$$\sim \left(\begin{array}{cccccc|c} 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 \end{array}\right)$$

Therefore, the set of solutions is given by

$$\left\{ \begin{pmatrix} 0 \\ 1-\lambda \\ 0 \\ -2-\lambda \\ 1+\lambda \\ \lambda \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 \\ -1 \\ 0 \\ -1 \\ 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} : \lambda, \alpha_i \in \mathbb{R} \right\}$$

Problem 7

We can rewrite this as $(A - 12)x = A'x = 0$

$$[A'] = \left(\begin{array}{ccc|c} -6 & 4 & 3 & 0 \\ 6 & -12 & 9 & 0 \\ 0 & 8 & -12 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & -\frac{2}{3} & -\frac{1}{2} & 0 \\ 0 & -8 & 12 & 0 \\ 0 & 8 & -12 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -\frac{3}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & -0 & 0 \end{array}\right)$$

Note that A is invertible, so $Ax = 0$ has no solutions. Therefore, the solution to $A'x = 0$ which satisfies $\sum_{i=1}^{3} x_i = 1$ is

$$x = \begin{pmatrix} 3/8 \\ 3/8 \\ 1/4 \end{pmatrix}$$

Problem 9

a. Is a subspace.
b. Is not a subspace since $-(\lambda^2, -\lambda^2, 0) = (-\lambda^2, \lambda^2, 0)$ is not in the set.
c. Is a subspace if $\gamma = 0$.

d. Is not a subspace since $\frac{1}{2}(\xi_1, 1, \xi_3)$ is not in the set.

Problem 10

a.

$$[A|0] = \left(\begin{array}{ccc|c} 2 & 1 & 3 & 0 \\ -1 & 1 & -3 & 0 \\ 3 & -2 & -8 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -\frac{7}{2} & \frac{7}{2} & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore, $x = \lambda \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}, \lambda \in \mathbb{R}$ is a solution to $Ax = 0$, meaning that these vectors are not linearly independent.

b.

$$[B|0] = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & -1 & -2 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore, there are no non-trivial solutions to $Bx = 0$, meaning that these vectors are linearly independent.

Problem 11

$$[A|b] = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 2 & -1 & 4 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Therefore, we can write $\begin{pmatrix} 1 & -2 & 5 \end{pmatrix}$ as the linear combination $-6x_1 + 3x_2 + 2x_3$.

Problem 12

$$[A_{U_1}|0] = \left(\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 1 & 1 & 1 & 0 \\ -3 & 0 & -1 & 0 \\ 1 & 1 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore a basis for U_1 is

$$\left\{ y_1 = \begin{pmatrix} 1 \\ 1 \\ -3 \\ 1 \end{pmatrix}, y_2 = \begin{pmatrix} 2 \\ -1 \\ 0 \\ 1 \end{pmatrix}, y_3 = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \right\}$$

$$[A_{U_2}|0] = \left(\begin{array}{ccc|c} -1 & 2 & -3 & 0 \\ -2 & -2 & 6 & 0 \\ 2 & 0 & -2 & 0 \\ 1 & 10 & -1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

Therefore a basis for U_2 is

$$\left\{ z_1 = \begin{pmatrix} -1 \\ -2 \\ 2 \\ 1 \end{pmatrix}, z_2 = \begin{pmatrix} 2 \\ -2 \\ 0 \\ 0 \end{pmatrix} \right\}$$

We want to fill all x which can be written as $x = \lambda_i y_i = \alpha_i z_i$. In other words, we want to find all non-trivial solutions to the equation $(y_1|y_2|y_3|-z_1|-z_2)v = 0$.

$$[A|0] = \left(\begin{array}{ccccc|c} 1 & 2 & -1 & 1 & -2 & 0 \\ 1 & 1 & 1 & 2 & 2 & 0 \\ -3 & 0 & -1 & -2 & 0 & 0 \\ 1 & 1 & 1 & -1 & 0 & 0 \end{array}\right) \sim \left(\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \frac{5}{3} & 0 \\ 0 & 0 & 0 & 1 & \frac{2}{3} & 0 \end{array}\right)$$

Therefore, the set of non-trivial solutions to $(y_1|y_2|y_3|z_1|z_2)v = 0$ is

$$\left\{ \beta \begin{pmatrix} 1 \\ 0 \\ -\frac{5}{3} \\ -\frac{2}{3} \\ 1 \end{pmatrix} : \beta \in \mathbb{R} \right\}$$

This tells us the allowed values for λ_i and α_i for $x \in U_1 \cap U_2$. From this, we can deduce that a basis for U_1 and U_2 is

$$\left\{ \begin{pmatrix} 8 \\ -2 \\ -4 \\ -2 \end{pmatrix} \right\}$$

Problem 13

We can easily see that $\dim(U_1) = 3 - \dim(A_1) = 3 - 2 = 1$ and $\dim(U_2) = 3 - \dim(A_2) = 3 - 2 = 1$.

$$[A_1|0] = \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & -2 & -1 & 0 \\ 2 & 1 & 3 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$[A_2|0] = \left(\begin{array}{ccc|c} 3 & -3 & 0 & 0 \\ 1 & 2 & 3 & 0 \\ 7 & -5 & 2 & 0 \\ 3 & -1 & 2 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$\therefore \left\{ \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \right\} \text{ is a basis for } U_1 \cap U_2$$

Problem 14

From before, we know that $\dim(A_1) = \dim(A_2) = 2$. It is easy to see that $\left\{ \begin{pmatrix}1\\1\\2\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\3\\1\end{pmatrix} \right\}$ is a basis for U_1, and $\left\{ \begin{pmatrix}3\\1\\7\\3\end{pmatrix}, \begin{pmatrix}-3\\2\\-5\\-1\end{pmatrix} \right\}$ is a basis for U_2.

We want all solutions to $x = \lambda_1 \begin{pmatrix}1\\1\\2\\1\end{pmatrix} + \lambda_2 \begin{pmatrix}1\\-1\\3\\1\end{pmatrix} = \lambda_3 \begin{pmatrix}3\\1\\7\\3\end{pmatrix} + \lambda_4 \begin{pmatrix}-3\\2\\-5\\-1\end{pmatrix}$. In other words, we want to find all non-trivial solutions to the equation

$$\begin{pmatrix}1 & 1 & -3 & 3\\ 1 & -1 & -1 & -2\\ 2 & 3 & -7 & 5\\ 1 & 1 & -3 & 1\end{pmatrix} \lambda = 0$$

$$[A_\lambda|0] = \left(\begin{array}{cccc|c}1 & 1 & -3 & 3 & 0\\ 1 & -1 & -1 & -2 & 0\\ 2 & 3 & -7 & 5 & 0\\ 1 & 1 & -3 & 1 & 0\end{array}\right) \sim \left(\begin{array}{cccc|c}1 & 0 & -2 & 0 & 0\\ 0 & 1 & -1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 0\end{array}\right)$$

This tells us the allowed values for λ_i for $x \in U_1 \cap U_2$. From this, we can deduce that a basis for U_1 and U_2 is

$$\left\{ \begin{pmatrix}3\\1\\7\\3\end{pmatrix} \right\}$$

Problem 17

All columns of A are linearly independent so $\mathrm{rk}(A) = 3$. This means that $\ker(\Phi) = \{0\}$ and $\mathrm{Im}(\Phi)$ has dimension 3.

$$[A|b] = \left(\begin{array}{ccc|c}3 & 2 & 1 & b_1\\ 1 & 1 & 1 & b_2\\ 1 & -3 & 0 & b_3\\ 2 & 3 & 1 & b_4\end{array}\right) \sim \left(\begin{array}{ccc|c}1 & 0 & -1 & b_1 - 2b_2\\ 0 & 1 & 2 & 3b_2 - b_1\\ 0 & 0 & 7 & b_3 + 11b_2 - 4b_1\\ 0 & 0 & -\frac{9}{5} & \frac{3}{5}b_4 + b_1 - 3b_2\end{array}\right)$$

Therefore, the image of Φ is given by

$$\left\{ \begin{pmatrix}b_1\\b_2\\b_3\\b_4\end{pmatrix} : b_i \in \mathbb{R}, b_4 = \frac{5b_1 + 2b_2 - 3b_2}{7} \right\}$$

Problem 19

It is easy to see that $\mathrm{rk}(A) = 3$ so $\dim(\mathrm{Im}(\Phi)) = 3$ and $\dim(\mathrm{Ker}(\Phi)) = 0$. The transformation matrix with respect to the new basis is

$$\tilde{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix}^{-1} A \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 6 & 9 & 1 \\ -3 & -5 & 0 \\ -1 & -1 & 0 \end{pmatrix}$$