Problem 1

Symmetric

$$\langle x, y\rangle = x_1y_1 - (x_1y_2 + x_2y_1) + 2x_2y_2 = y_1x_1 - (y_1x_2 + y_2x_1) + 2y_2x_2 = \langle y, x\rangle$$

Bilinear

$\langle \alpha x + \beta y, z\rangle = \alpha(x_1 - (x_1z_2 + x_2z_1) + 2x_2z_2) + \beta(y_1 - (y_1z_2 + y_2z_1) + 2y_2z_2) = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$

Linear in first argument + symmetric $\Rightarrow$ bilinear.

Positive Definite

$$\langle x, x\rangle = x_1^2 - 2x_1x_2 + 2x_2 = (x_1 - x_2)^2 + x_2^2 > 0 \text{ iff } x \neq 0$$

Therefore $\langle \cdot, \cdot \rangle$ is an inner product.

Problem 2

No, this definition is not symmetric

$$\langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \rangle = 7 \qquad \langle \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle = 8$$

Problem 5

Let A be a matrix whose columns are the columns which span U. Notice $A\lambda = 0$ has a solution space of dimension 1, so the dimension of U is 3.

By inspection, we find the following basis for U

$$\left\{ \begin{pmatrix} 0 \\ -1 \\ 2 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \\ 1 \\ -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -3 \\ 4 \\ 1 \\ 2 \\ 1 \end{pmatrix} \right\} \to B = \begin{pmatrix} 0 & 1 & -3 \\ -1 & -3 & 4 \\ 2 & 1 & 1 \\ 0 & -1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$$

The projection matrix is then

$$\mathbf{P}_\pi = B(B^TB)^{-1}B^T = \frac{1}{63}\begin{pmatrix} 30 & -18 & 12 & -9 & -12 \\ -18 & 43 & 4 & 18 & -14 \\ 12 & 4 & 58 & 9 & 7 \\ 9 & 18 & 9 & 9 & 0 \\ -21 & -14 & 7 & 0 & 49 \end{pmatrix}$$

The orthogonal projection of x onto U is then given by

$$\pi_U(x) = \mathbf{P}_\pi x = \begin{pmatrix} 1 \\ -5 \\ -1 \\ -2 \\ 3 \end{pmatrix} \qquad d(x, U) = ||x - \pi_U(x)|| = 2\sqrt{15}$$

Problem 6

The requirement that the the vector $e_2 - \pi_U(e_2)$ be orthogonal to the basis vectors of U tells us that the projection vector is

$$\pi_U(e_2) = \frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \Rightarrow d(e_2, U) = 1$$

Problem 7

Let π be a projection.

$$(\mathrm{id} - \pi)((\mathrm{id} - \pi)(x)) = (\mathrm{id} - \pi)(x) - \pi((\mathrm{id} - \pi)(x)) = (\mathrm{id} - \pi)(x)$$

Therefore $\mathrm{id} - \pi$ is a projection.

Let $\mathrm{id} - \pi$ be a projection.

$$(\mathrm{id} - \pi)(x) = (\mathrm{id} - \pi)((\mathrm{id} - \pi)(x)) = (\mathrm{id} - \pi)(x) - \pi((\mathrm{id} - \pi)(x)) = (\mathrm{id} - \pi)(x) - (\pi - \pi^2)(x)$$

Therefore π is a projection.

Putting these statements together, we find that π is a projection iff $\mathrm{id} - \pi$ is a projection.

The kernel of $\mathrm{id} - \pi$ is clearly just the $\mathrm{Im}(\pi)$. Meanwhile the $\mathrm{Im}(\mathrm{id} - \pi)$ is $\ker(\pi)$; this equality is not as obvious so we show why it holds:

$$x \in \ker(\pi) \Rightarrow x = (\mathrm{id} - \pi)(x) \in \mathrm{Im}(\mathrm{id} - \pi)$$

$$x \in \mathrm{Im}(\mathrm{id} - \pi) \Rightarrow \exists y \in V \text{ s.t. } x = (\mathrm{id} - \pi)(y) \Rightarrow \pi(x) = \pi(\mathrm{id} - \pi)(y) = 0$$

Problem 8

$$\tilde{b}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \rightarrow \tilde{b}_2 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} - \frac{\tilde{b}_1 \tilde{b}_1^T}{||\tilde{b}_1||^2}\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} -4 \\ 5 \\ -1 \end{pmatrix}$$

We thus find that an orthonormal basis is

$$\left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{42}} \begin{pmatrix} -4 \\ 5 \\ -1 \end{pmatrix} \right\}$$