Problem 1

$$\frac{df}{dx} = 3x^2 + 12x - 3 = 3(x^2 + 4x - 1) = 3(x - x_+)(x - x_-) \qquad x_\pm = -2 \pm \sqrt{5}$$

$$\frac{d^2 f}{dx^2}\bigg|_{x=x_\pm} = 6x + 12\bigg|_{x=x_\pm} = 6(x+2)\bigg|_{x=x_\pm} = \pm 6\sqrt{5}$$

Therefore x_+ is a minimum, and x_- is a maximum.

Problem 5

Define $x_2 = -\xi, \overline{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}, \overline{p} = \begin{pmatrix} p_0 \\ p_1 \\ -1 \end{pmatrix}$. Then, the optimization problem becomes a linear program

$$\max_{x \in \mathbb{R}^3} \overline{p}^T \overline{x} \qquad \text{subject to } x \leq \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}$$

Problem 6

The Lagrangian is of the form

$$\mathcal{L}(x, \lambda) = c^T x + \lambda^T (Ax - b)$$

We then find the extrema of the Lagrangian (with respect to x)

$$\frac{\partial \mathcal{L}}{\partial x} = c + A^T \lambda \Rightarrow \mathcal{D}(\lambda) = \mathcal{L}(x, \lambda)\bigg|_{c + A^T \lambda = 0} = -b^T \lambda$$

The associated Lagrangian dual problem is then

$$\max_{\lambda \in \mathbb{R}^5} -\begin{pmatrix} 33 & 8 & 5 & -1 & 8 \end{pmatrix} \lambda \qquad \text{subject to } \lambda \geq 0, c + A^T \lambda = 0$$

Problem 7

The Lagrangian is of the form

$$\mathcal{L}(x, \lambda) = \frac{1}{2} x^T Q x + c^T x + \lambda^T (Ax - b)$$

We then find the extrema of the Lagrangian (with respect to x)

$$\frac{\partial \mathcal{L}}{\partial x} = Qx + c + A^T \lambda \Rightarrow \mathcal{D}(\lambda) = \mathcal{L}(x, \lambda)\bigg|_{x = -Q^{-1}(c + A^T \lambda)} = -\frac{1}{2}(c + A^T \lambda)^T Q^{-1} (c + A^T \lambda) - b^T \lambda$$

The associated Lagrangian dual problem is then

$$\max_{\lambda \in \mathbb{R}^5} \; -\frac{1}{2}(c + A^T\lambda)^T Q^{-1}(c + A^T\lambda) - b^T\lambda \qquad \text{subject to } \lambda \geq 0, Qx + c + A^T\lambda = 0$$

Problem 8
The Lagrangian is given by

$$\mathcal{L}(\omega, \lambda) = \frac{1}{2}\omega^T\omega - \lambda(\omega^T x - 1)$$

We then find the extrema of the Lagrangian (with respect to ω)

$$\frac{\partial \mathcal{L}}{\partial \omega} = \omega - \lambda x \Rightarrow \mathcal{D}(\lambda) = \mathcal{L}(\omega, \lambda)\Big|_{\omega = \lambda x} = -\frac{1}{2}\lambda^2 x^T x + \lambda$$

The associated Lagrangian dual problem is then

$$\max_{\lambda \in \mathbb{R}} \; -\frac{1}{2}\lambda^2 x^T x + \lambda \qquad \text{subject to } \lambda \geq 0, \omega - \lambda x = 0$$

Problem 9
We can derive the convex conjugate function by finding the maximum of the function

$$s^T x - \sum_{i=1}^{d} x_i \log(x_i)$$

with respect to x

$$\frac{\partial}{\partial x_i}(s^T x - \sum_{i=1}^{d} x_i \log(x_i)) = s_i - 1 - \log(x_i)$$

Plugging this into the expression for the convex conjugate function

$$f^*(s) = s^T x - \sum_{i=1}^{d} x_i \log(x_i)\Big|_{x_i = e^{s_i - 1}} = \sum_{i=1}^{d} e^{s_i - 1}$$

Problem 10
We can derive the convex conjugate function by finding the maximum of the function

$$s^T x - \frac{1}{2}x^T A x - b^T x - c$$

with respect to x

$$\frac{\partial}{\partial x}(s^T x - \frac{1}{2}x^T A x - b^T x - c) = s^T - x^T A - b^T$$

Plugging this into the expression for the convex conjugate function

$$f^*(s) = s^T x - \frac{1}{2}x^T A x - b^T x - c\Big|_{x = A^{-1}(s-b)} = \frac{1}{2}(s-b)^T A^{-1}(s-b) - c$$