

Theory of Statistics - Discrete Distributions

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1 Bernoulli Distribution.

$$X \sim \text{Bernoulli}(p) : 0 \leq p \leq 1 \quad (1)$$

Take some random variable X with a Bernoulli parametric model with success parameter p defined by:

$$X \triangleq \begin{cases} 1 & \text{if success } p, \\ 0 & \text{otherwise.} \end{cases}$$

Then, $X \sim \text{Bernoulli}(p)$ read as; “ X has distribution Bernoulli with parameter p ”. The probability mass function is then defined by:

$$f_X(x; p) \triangleq p^x (1 - p)^{1-x} : x \in \{0, 1\}. \quad (2)$$

2 Binomial Distribution.

$$X \sim \text{Bin}(n, p) : 0 \leq p \leq 1 \quad (3)$$

This is a finite generalisation of a series of Bernoulli random variables with replacement.

Take some random variable X with each observation $x_i \in X : 0 \leq i \leq n$ where n is the number of trials and that each $x_i \sim \text{Bernoulli}(p)$. Then, provided all the observations x_i are independent and since the parameter p is the same (a special case of the Poisson binomial distribution) for all. Then, we claim that the each x_i are “independent and identically distributed” **i.i.d.** Assuming now, we select each x_i with replacement, then there are $\binom{n}{x}$ ways to select x observations from a trial, or sample size of n .

That is, $X \sim \text{Bin}(n, p) : 0 \leq p \leq 1$ and hence has probability mass function defined by:

$$f_X(x; n, p) \triangleq \binom{n}{x} p^x (1 - p)^{n-x} : 0 \leq x \leq n. \quad (4)$$

Clear, this is just an extension of a Bernoulli trial sequentially repeated n times.

3 Geometric Distribution.

$$\boxed{X \sim Geom(p) : 0 \leq p \leq 1} \quad (5)$$

This is a special case result of a Binomial random variable, for when $x = 1$.

Take some random variable X such that $X \sim Bin(n, p)$, given that p is the probability of success. Take $x = 1$ with $n = k$ so that, the probability of first success on the k^{th} trial is given by,

$$f_X(1; k, p) = \binom{k}{1} p^1 (1-p)^{k-1}.$$

Then we have the following special case result as our probability mass function:

$$\boxed{f_X(k; p) \triangleq p(1-p)^{k-1} : k \in \{1, 2, \dots\}.} \quad (6)$$

4 Poisson Distribution.

$$\boxed{X \sim Poisson(\lambda)} \quad (7)$$

This is another special case result of a Binomial random variable, for when p is close to zero and n is very large[Pap91].

Take some random variable Y such that $Y \sim Bin(n, p)$ and fix some parameter $\lambda \doteq np$ where n is known to be large and p is known to be small. Such a case is intuitively a rare event since the probability of success p is small for a large number of trials n . This is known as “the law of rare events”[Sem]. We continue with the assumption that such events are sufficiently independent. Then, we have a limiting case of the Binomial random variable Y with parameter λ .

Hence if,

$$Y \sim Bin(n, p)$$

Substituting for λ and by introduction of an auxiliary random variable X

$$X \sim Bin(n, \frac{\lambda}{n})$$

we derive the probability mass function as follows,

$$\begin{aligned} f_X(x; \lambda) &= \frac{n!}{(n-x)!x!} \left(\frac{\lambda}{n}\right)^x \left(1 - \frac{\lambda}{n}\right)^{n-x} \\ &= \frac{n!}{n^x(n-x)!} \frac{1}{\left(1 - \frac{\lambda}{n}\right)^x} \frac{\lambda^x}{x!} \left(1 - \frac{\lambda}{n}\right)^n \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ and noting the definition of $e^{-\lambda} \doteq \lim_{n \rightarrow \infty} \left(1 - \frac{\lambda}{n}\right)^n$

$$\begin{aligned} \frac{\lambda^x}{x!} \lim_{n \rightarrow \infty} \left(\frac{n!}{n^x(n-x)!} \right) \left(\frac{1}{\left(1 - \frac{\lambda}{n}\right)^x} \right) \left(1 - \frac{\lambda}{n}\right)^n \\ = \frac{\lambda^x}{x!} (1)(1)(e^{-\lambda}) \end{aligned}$$

hence, we have the probability mass function to be:

$$\Rightarrow f_X(x; \lambda) = \frac{e^{-\lambda} \lambda^x}{x!} : x \in \{0, 1, 2, \dots\} \text{ for some fixed } \lambda.$$

Thus, Given a random variable X such that $X \sim \text{Poisson}(\lambda)$ then X has probability mass function:

$$f_X(x; \lambda) \triangleq \frac{e^{-\lambda} \lambda^x}{x!} : x \in \{0, 1, 2, \dots\} \quad (8)$$

5 Hypergeometric Distribution.

$$X \sim \text{Hypergeom}(M, N, n) : \max(0, M + n - N) \leq k \leq \min(M, n) \quad (9)$$

Bernoulli trials without replacement (Binomial is actually a special case of the hypergeometric). Probability mass function is defined as:

$$f_X(x; M, N, n) \triangleq \frac{\binom{M}{x} \binom{N-M}{n-x}}{\binom{N}{n}} : N \in \{1, 2, \dots\} \quad (10)$$

$$m \in \{0, 1, 2, \dots, N\}$$

$$n \in \{1, 2, \dots, N\}$$

References

- [Pap91] Athanasios Papoulis. *Probability, Random Variables, and Stochastic Processes*. McGraw-Hill, New York, NY, USA, 1991.
- [Sem] Ladislaus Semali. Home page.